π value estimation

H.I. Park C. Hwang C.O. Hwang

Lab meeting presentation: 15th Sep 2017

Chord Summation With 2^n-polygon

- Method Briefing
- Results of Method
- Error Analysis

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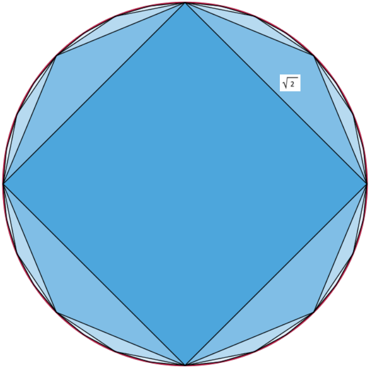
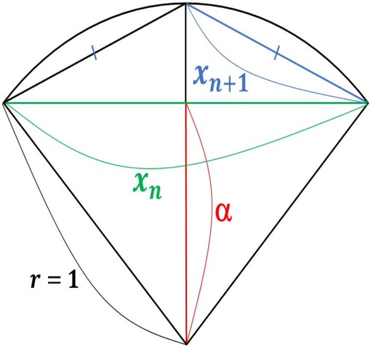
Method Briefing

- Let x_n be the side length of 2^{n+1}-polygon





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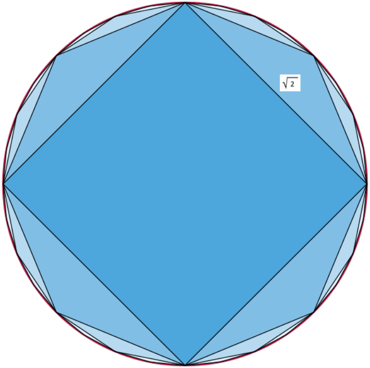

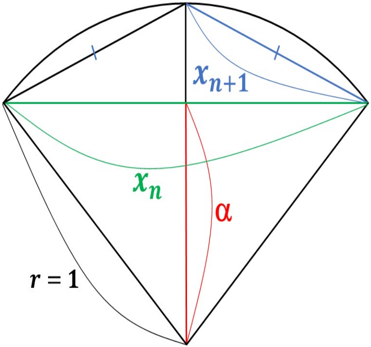

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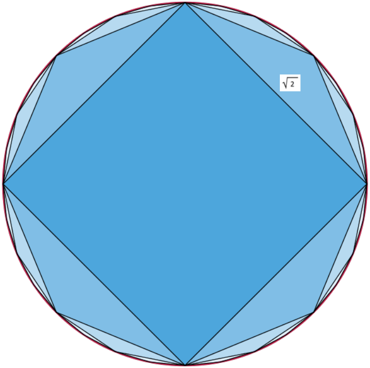

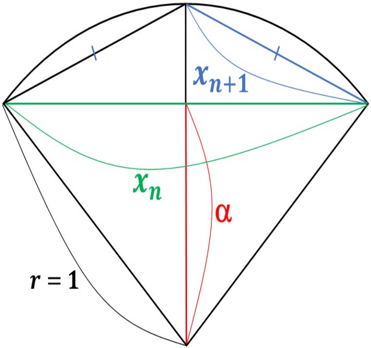

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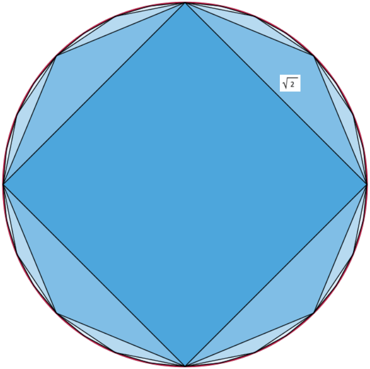
Method Briefing. 2

- Let y_n be the chord summation of 2^{n+1}-polygon: estimate of π



- $y_1 = 2^1\sqrt{2}$ and $y_n = 2^n \times x_n$
- $y_2 = 2^2\sqrt{2-\sqrt{2}}$; $y_3 = 2^3\sqrt{2-\sqrt{2+\sqrt{2}}}$; *and so on...*
- $\lim_{n\to\infty} y_n = \lim_{n\to\infty} 2^n x_n = \pi$

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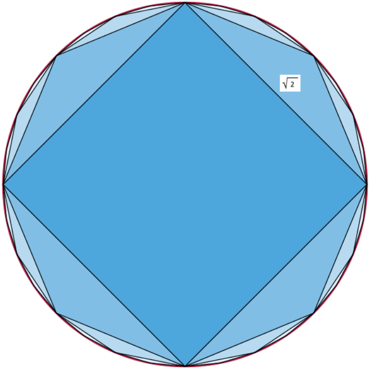

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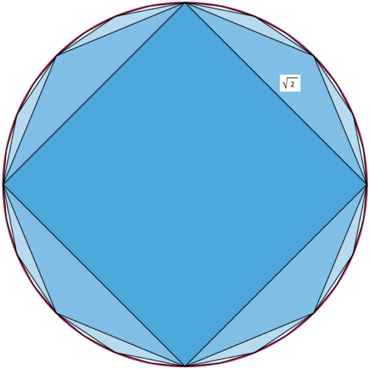

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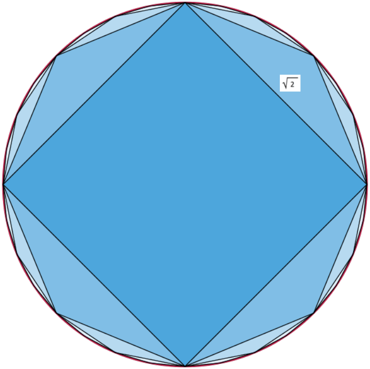

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Results.

- Let e_n be an error defined as $e_n = \pi - y_n$
- Let $\hat{x}_1 = 1.4142135623$ (orange)
- Let $x_1 = \sqrt{2}$ (blue). Exact value from Mathematica.

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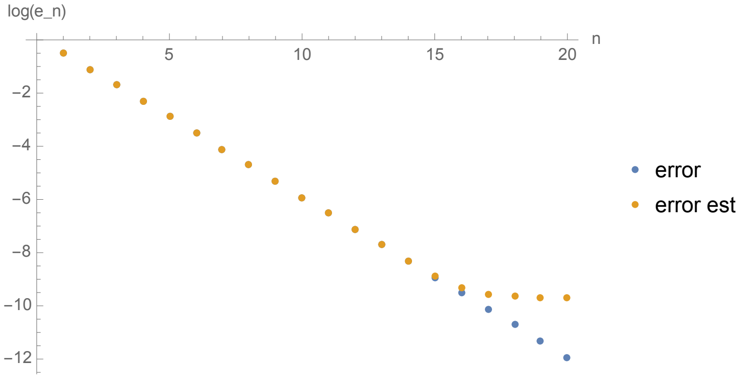

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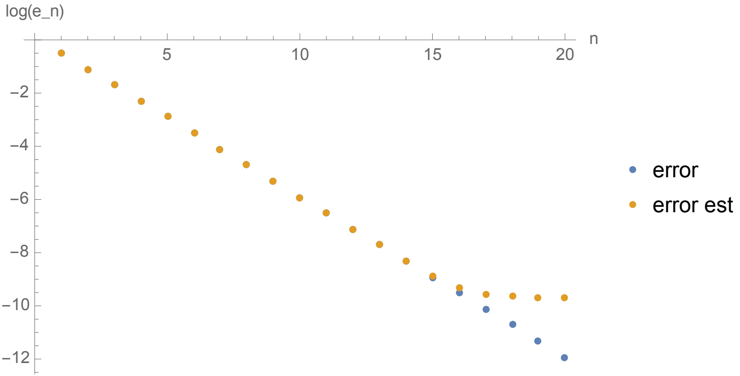

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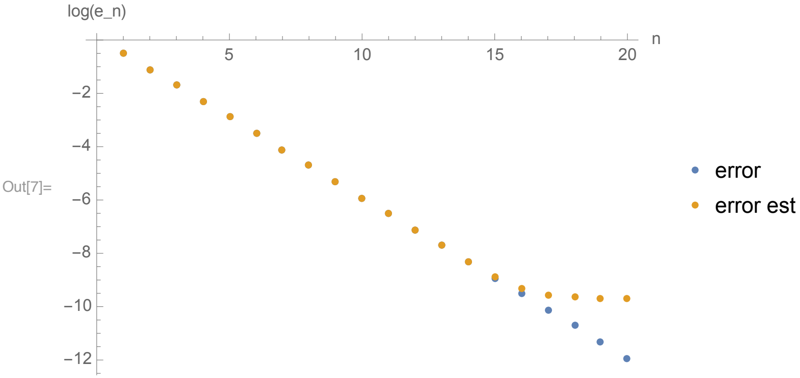
- 

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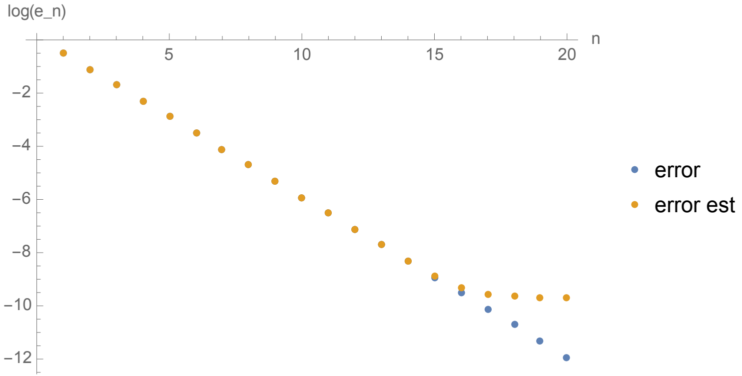

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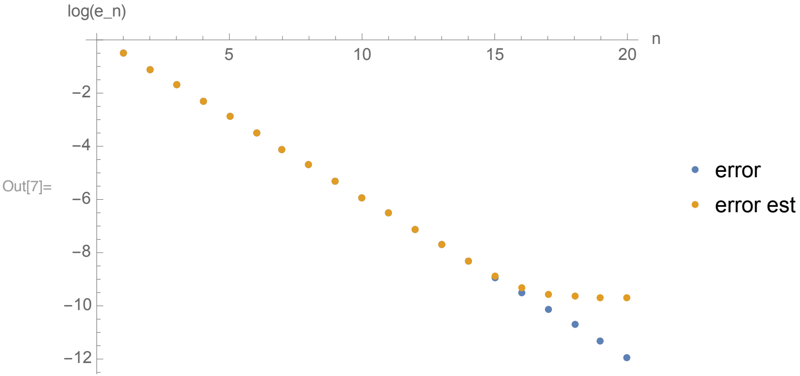
- 

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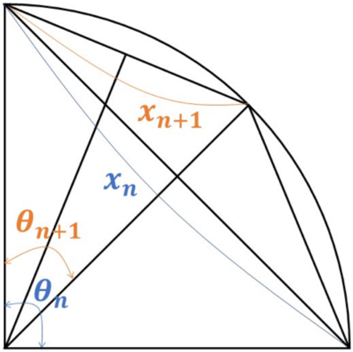
Results. Theoretical Error

- It is observed that $e_{n+1}/e_n \approx 0.25$



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- $\frac{e_n}{e_{n+1}} = \frac{\pi - 2^n x_n}{\pi - 2^{n+1} x_{n+1}} = \frac{\pi - 2^n \times 2\sin(\theta_n/2)}{\pi - 2^{n+1} \times 2\sin(\theta_{n+1}/2)}$

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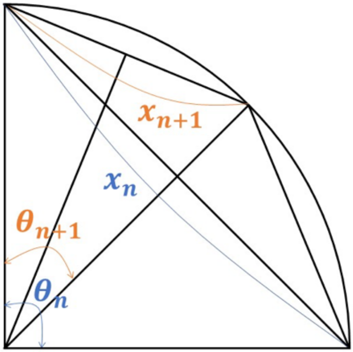

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- Using Taylor series: $\sin(\theta) \approx \theta - \frac{\theta^3}{3!} = 0.785 - 0.0807 + (0.0025\ldots)$
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Error analysis. Why deviation?

- Because we used approximate value of $\sqrt{2} = 1.41421\ldots$

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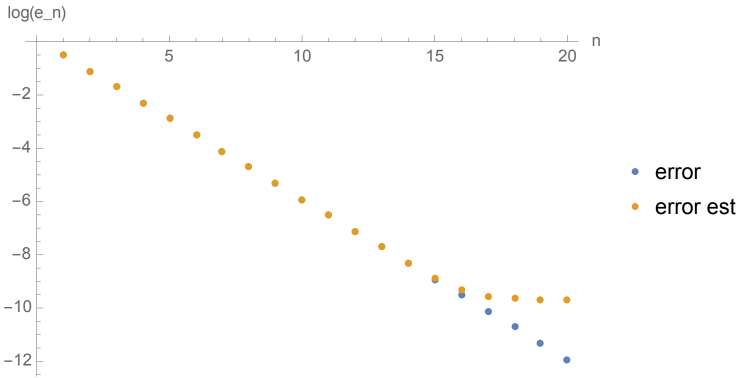

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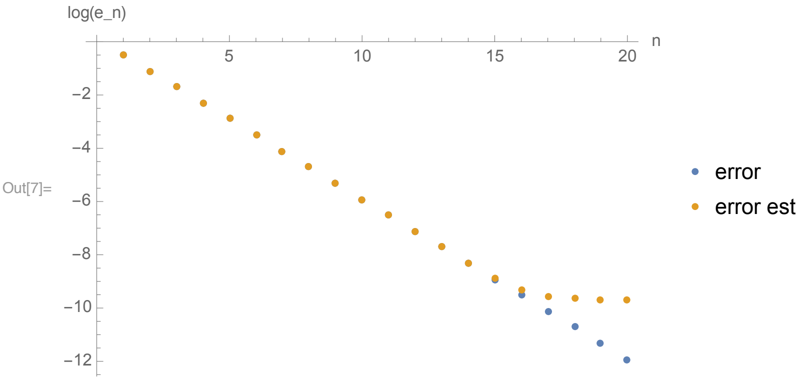

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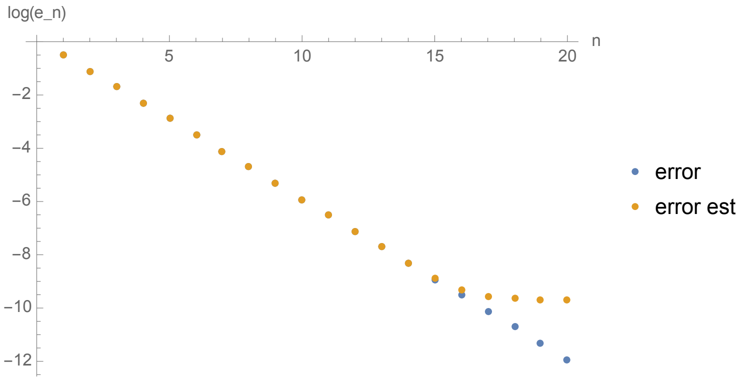

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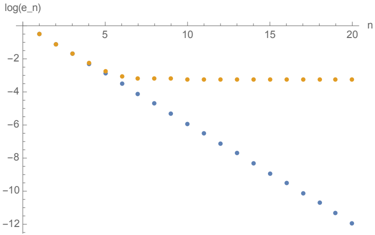

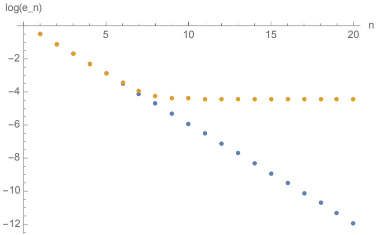

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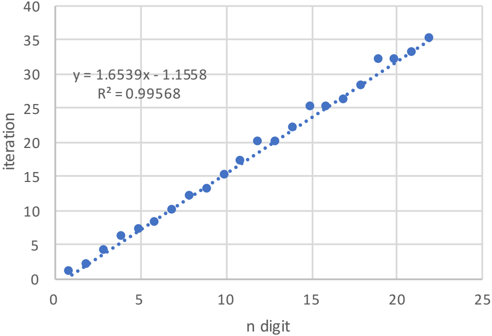

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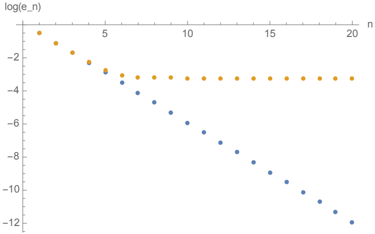

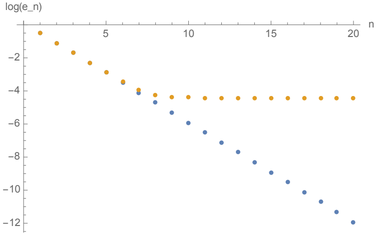

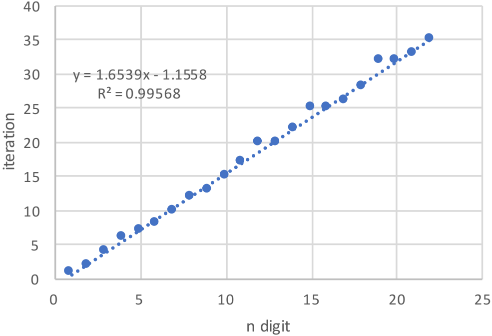

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- Let $\hat{x}_1$ be an approximate value of $\sqrt{2}$ up to k digit.
- Let $\delta = \sqrt{2} - \hat{x}_1$
- Example: $\hat{x}_1 = 1.414$, $\delta = 2.... \times 10^{-4}$
- $1.0 \times 10^{k+1} < \delta < 10.0 \times 10^{k+1}$
- $\hat{x}_{n+1} = f(\hat{x}_n) = \sqrt{2 - \sqrt{4 - \hat{x}_n^2}}$
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- $x_3 = f(x_2) = f(\hat{x}_2 + f'(\hat{x}_1)\delta) \approx f(\hat{x}_2) + f'(\hat{x}_2)f'(\hat{x}_1)\delta = \hat{x}_3 + f'(\hat{x}_2)f'(\hat{x}_1)\delta$
- .... and so on until
- $x_5 = \hat{x}_5 + f'(\hat{x}_4)f'(\hat{x}_3)f'(\hat{x}_2)f'(\hat{x}_1)\delta$
- Define $\alpha = f'(\hat{x}_4)f'(\hat{x}_3)f'(\hat{x}_2)f'(\hat{x}_1)\delta = const$
- $2 \times f'(\hat{x}_5) = 1.0008$ and $2 \times f'(\hat{x}_n) \to 1$ for further iteration of n
- $y_5 = \hat{y}_5 + 32\alpha$
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- .... and so on until
- $x_5 = \hat{x}_5 + f'(\hat{x}_4)f'(\hat{x}_3)f'(\hat{x}_2)f'(\hat{x}_1)\delta$
- Define $\alpha = f'(\hat{x}_4)f'(\hat{x}_3)f'(\hat{x}_2)f'(\hat{x}_1)\delta = const$
- $2 \times f'(\hat{x}_5) = 1.0008$ and $2 \times f'(\hat{x}_n) \to 1$ for further iteration of n
- $y_5 = \hat{y}_5 + 32\alpha$
- $y_6 = \hat{y}_6 + 64f'(\hat{x}_5)\alpha = \hat{y}_6 + 32\alpha$ and so on...

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- $e_n = \pi - y_n$
- using the fact that $y_n = \hat{y}_n + 32\alpha$ for $n > 5$
- $\hat{e}_n = e_n + 32\alpha$
- Theoretical error is given by the following formula:

$$e_n = e_1 \left(\frac{1}{4}\right)^{n-1}$$

- $\frac{32\alpha}{e_n} < 0.2$ for $\hat{e}_n$ to be valid..

$$iteration\ n = \left[log_4 \left(\frac{e_1}{128\alpha} \right) \right]$$

- where $\alpha = 0.0883 \times (5 \times 10^{-k+1})$

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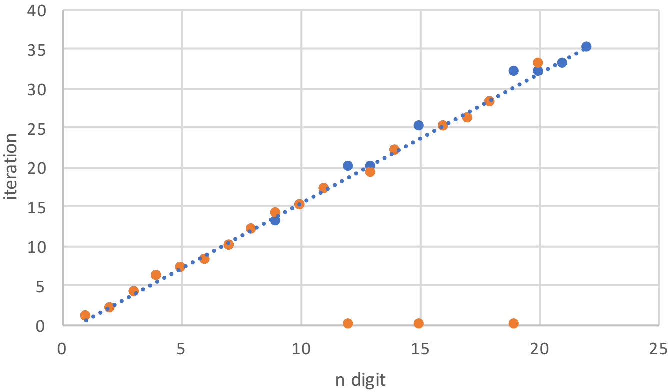
Error analysis.

- Orange: modelling as shown
- Blue: Observation.
- Peaks: 0 digit of $\sqrt{2} = 1.4142135623730950488016887242096980$



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