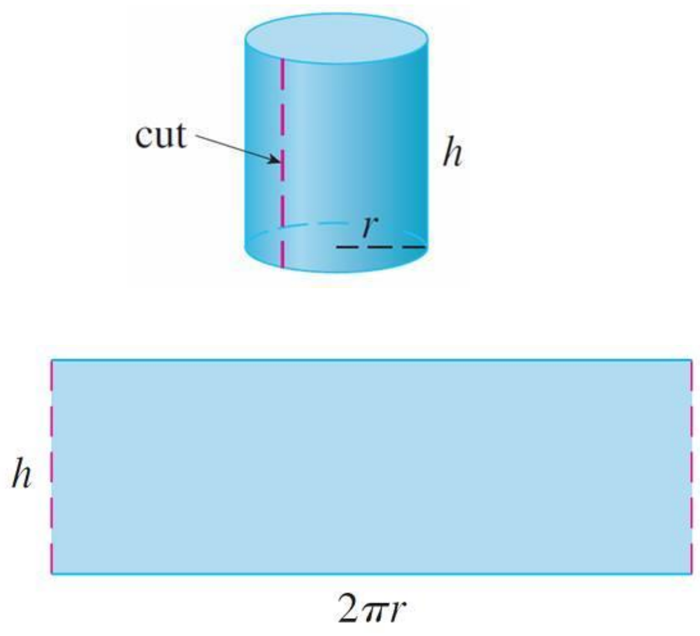
8.2

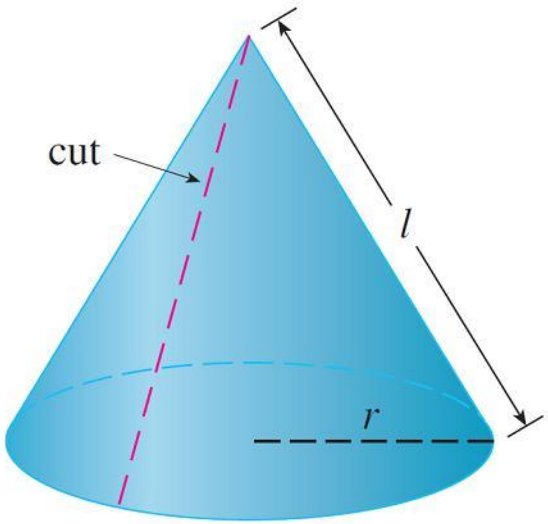
Area of a Surface of Revolution

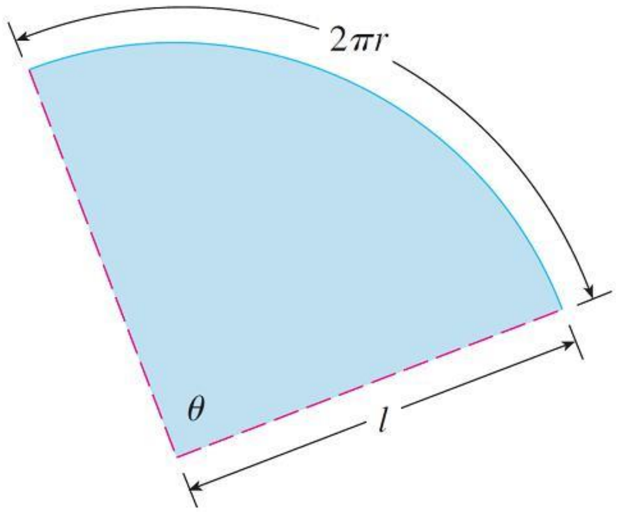
Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution.



Area of a Surface of Revolution

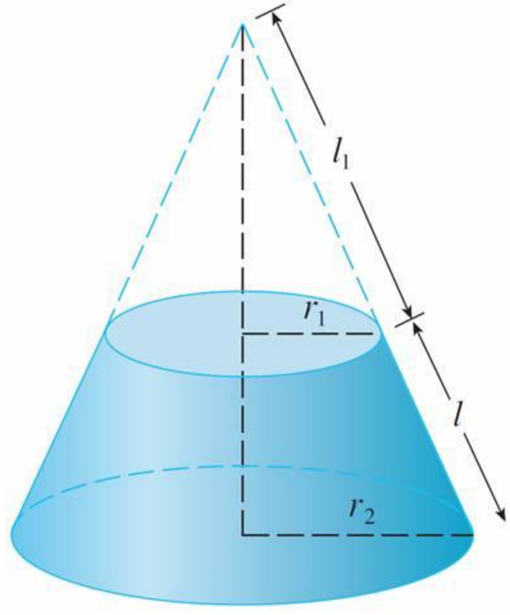




$$A = \tfrac{1}{2} l^2 \theta = \tfrac{1}{2} l^2 \left(\frac{2\pi r}{l} \right) = \pi r l$$

Area of a Surface of Revolution

Find the surface area of



$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)\, l_1 + r_2 l]$

From similar triangles we have

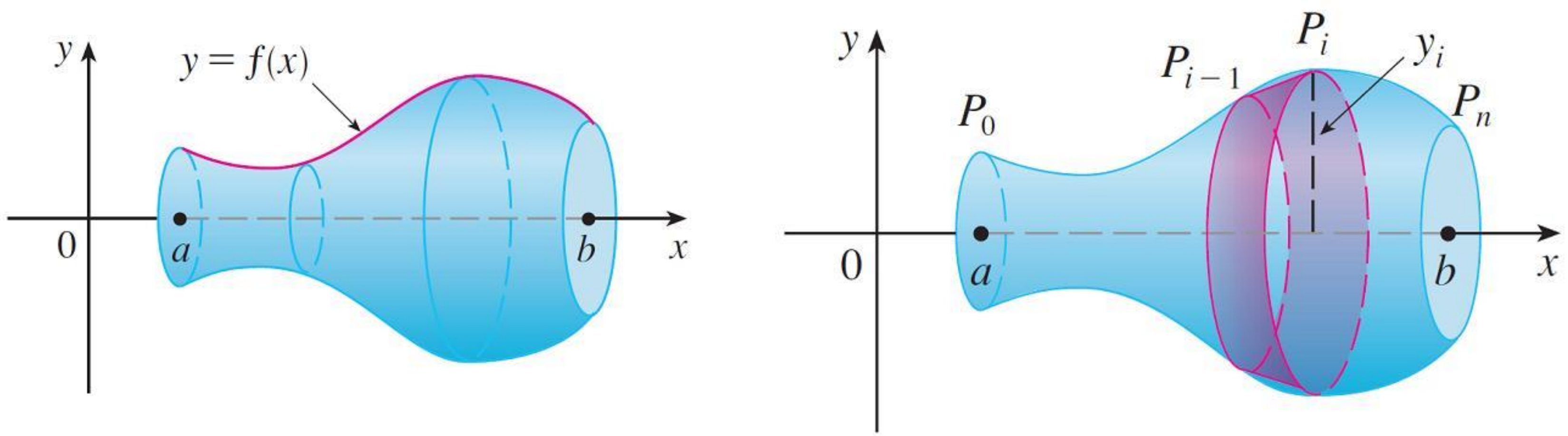
$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$ or $(r_2 - r_1)l_1 = r_1 l$

$$\boxed{A = 2\pi r l}$$

$r = \frac{1}{2}(r_1 + r_2)$

Area of a Surface of Revolution

Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where f is positive and has a continuous derivative.



(a) Surface of revolution

(b) Approximating band

Figure 4

Area of a Surface of Revolution

In order to define its surface area, we divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as we did in determining arc length.

If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve.

The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis.

Area of a Surface of Revolution

The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

As in the proof, We have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$.

Area of a Surface of Revolution

When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous. Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} \left| P_{i-1}P_i \right| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

$$\sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx}$$

Area of a Surface of Revolution

With the Leibniz notation for derivatives, this formula becomes

$$\boxed{5} \qquad S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for surface area becomes

$$\boxed{6} \qquad S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of a Surface of Revolution

Now both Formulas 5 and 6 can be summarized symbolically, using the notation for arc length, as

7 $$\boxed{S = \int 2\pi y \, ds}$$

For rotation about the *y*-axis, the surface area formula becomes

8 $$\boxed{S = \int 2\pi x \, ds}$$

where, as before, we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Example 1

The curve $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$, is an arc of the circle $x^2 + y^2 = 4$.

Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

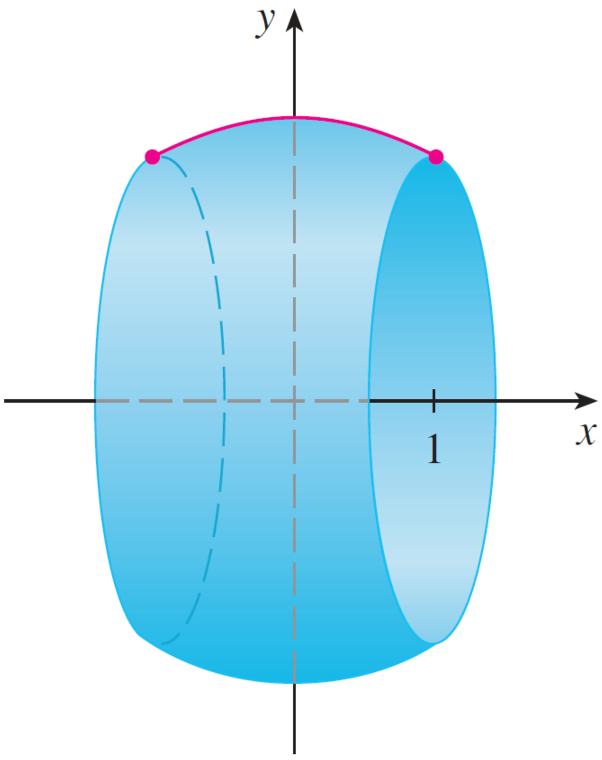


Figure 6

Example 3

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

10.1 Curves Defined by Parametric Equations

Curves Defined by Parametric Equations

Imagine that a particle moves along the curve *C* shown in Figure 1. It is impossible to describe *C* by an equation of the form $y = f(x)$ because *C* fails the Vertical Line Test.

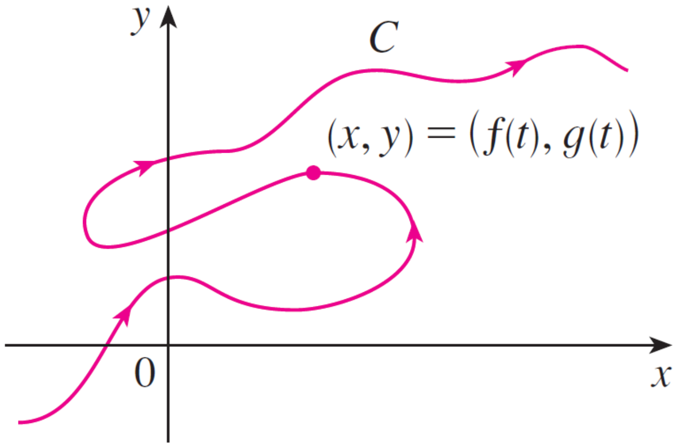


Figure 1

Suppose that *x* and *y* are both given as functions of a third variable *t* (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**).

Example

Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad\qquad y = t + 1$$

Solution:

Each value of t gives a point on the curve, as shown in the table.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

Example – *Solution*

cont'd

For instance, if $t = 0$, then $x = 0$, $y = 1$ and so the corresponding point is (0, 1).

In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

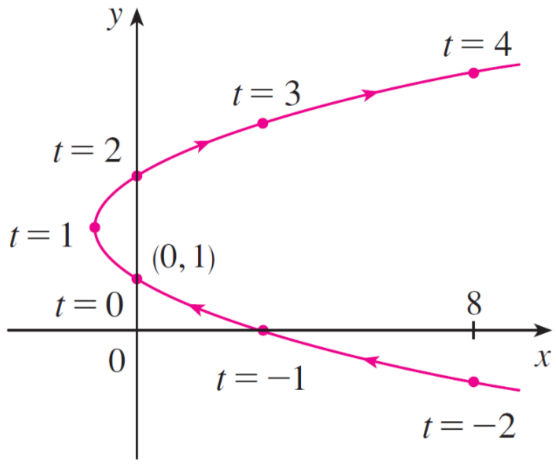


Figure 2