

## 11.2

# Series

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# Series

In general, if we try to add the terms of an infinite sequence  $\{a_n\}_{n=1}^{\infty}$  we get an expression of the form

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$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

# Series

In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1.

So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum.

# Series

We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

These partial sums form a new sequence  $\{s_n\}$ , which may or may not have a limit.

# Series

If  $\lim_{n \rightarrow \infty} s_n = s$  exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series  $\sum a_n$ .

**2 Definition** Given a series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ , let  $s_n$  denote its  $n$ th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence  $\{s_n\}$  is convergent and  $\lim_{n \rightarrow \infty} s_n = s$  exists as a real number, then the series  $\sum a_n$  is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number  $s$  is called the **sum** of the series. If the sequence  $\{s_n\}$  is divergent, then the series is called **divergent**.

# Series

Thus the sum of a series is the limit of the sequence of partial sums.

So when we write  $\sum_{n=1}^{\infty} a_n = s$ , we mean that by adding sufficiently many terms of the series we can get as close as we like to the number  $s$ .

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

# Series

We summarize the results of Example 2 as follows.

**4** The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if  $|r| < 1$  and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If  $|r| \geq 1$ , the geometric series is divergent.

# Example 9

Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent.

**Solution:**

For this particular series it's convenient to consider the partial sums  $s_2, s_4, s_8, s_{16}, s_{32}, \dots$  and show that they become large.

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$



# Example 9 – *Solution*

cont'd

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{4}{2}$$

## Example 9 – *Solution*

cont'd

Similarly,  $s_{32} > 1 + \frac{5}{2}$ ,  $s_{64} > 1 + \frac{6}{2}$ , and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that  $s_{2^n} \rightarrow \infty$  as  $n \rightarrow \infty$  and so  $\{s_n\}$  is divergent.

Therefore the harmonic series diverges.

# Series

**6 Theorem** If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

The converse of Theorem 6 is not true in general.

If  $\lim_{n \rightarrow \infty} a_n = 0$ , we cannot conclude that  $\sum a_n$  is convergent.

# Series

**8 Theorem** If  $\sum a_n$  and  $\sum b_n$  are convergent series, then so are the series  $\sum ca_n$  (where  $c$  is a constant),  $\sum (a_n + b_n)$ , and  $\sum (a_n - b_n)$ , and

$$(i) \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

$$(ii) \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$(iii) \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

# The Integral Test and Estimates of Sums

**The Integral Test** Suppose  $f$  is a continuous, positive, decreasing function on  $[1, \infty)$  and let  $a_n = f(n)$ . Then the series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if the improper integral  $\int_1^{\infty} f(x) dx$  is convergent. In other words:

(i) If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent.

(ii) If  $\int_1^{\infty} f(x) dx$  is divergent, then  $\sum_{n=1}^{\infty} a_n$  is divergent.

# Example

Test the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$  for convergence or divergence.

# The Integral Test and Estimates of Sums

The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is called the ***p*-series**.

**1** The *p*-series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if  $p > 1$  and divergent if  $p \leq 1$ .