

7.4

Integration of Rational Functions by Partial Fractions

Integration of Rational Functions by Partial Fractions

$$\int \frac{x + 5}{x^2 + x - 2} dx = \int \left(\frac{2}{x - 1} - \frac{1}{x + 2} \right) dx$$

$$= 2 \ln |x - 1| - \ln |x + 2| + C$$

Integration of Rational Functions by Partial Fractions

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q . Such a rational function is called *proper*.

Integration of Rational Functions by Partial Fractions

We know that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$.

Integration of Rational Functions by Partial Fractions

The division statement is

1

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

Example 1

Find $\int \frac{x^3 + x}{x - 1} dx$.

Solution:

Integration of Rational Functions by Partial Fractions

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Integration of Rational Functions by Partial Fractions

Case I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).

Integration of Rational Functions by Partial Fractions

In this case the partial fraction theorem states that there exist constants A_1, A_2, \dots, A_k such that

$$\boxed{2} \quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

Example 2

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$.

Integration of Rational Functions by Partial Fractions

Case II $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \quad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Integration of Rational Functions by Partial Fractions

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

Example 4

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx.$

Integration of Rational Functions by Partial Fractions

Case III $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

9

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

Integration of Rational Functions by Partial Fractions

For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in (9) can be integrated by completing the square (if necessary) and using the formula

10

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 6

Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx.$

Integration of Rational Functions by Partial Fractions

Case IV $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

$$(11) \quad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by using a substitution or by first completing the square if necessary.

Example 8

Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx.$



Rationalizing Substitutions

Example 9

Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.