

11.8

Power Series

Power Series

A **power series** is a series of the form

$$\boxed{1} \quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n 's are constants called the **coefficients** of the series.

A power series may converge for some values of x and diverge for other values of x .

Power Series

For instance, if we take $c_n = 1$ for all n , the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Power Series

The following theorem says that this is true in general.

4 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x - a)^n$, there are only three possibilities:

- (i) The series converges only when $x = a$.
- (ii) The series converges for all x .
- (iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

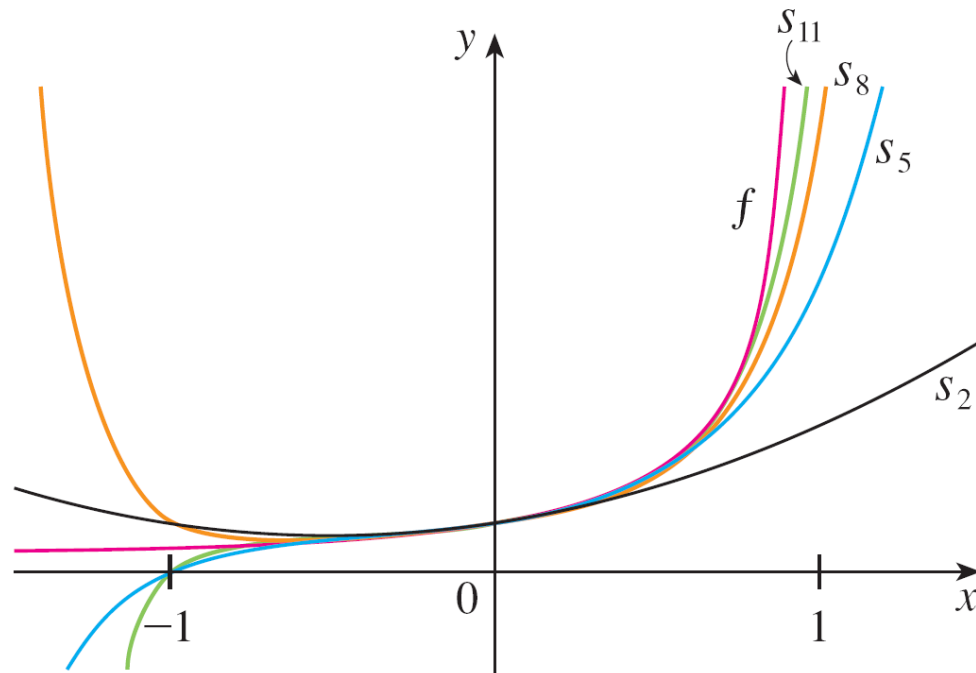
Power Series

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n! x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

Representations of Functions as Power Series

1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$



$f(x) = \frac{1}{1-x}$ and some partial sums

Example

Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.



Differentiation and Integration of Power Series

Differentiation and Integration of Power Series

2 Theorem If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

$$(i) \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

$$(ii) \quad \int f(x) dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \cdots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R .