6 Applications of Integration

Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

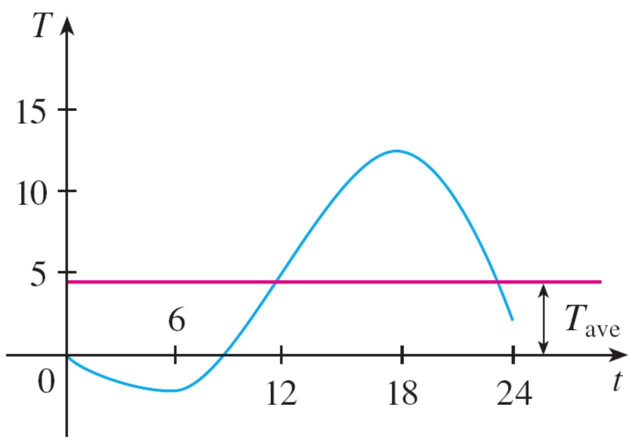

Figure 1

Average Value of a Function

In general, let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$.

Then we choose points $x_1*, \ldots, x_n*$ in successive subintervals and calculate the average of the numbers $f(x_1*), \ldots, f(x_n*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.)

Average Value of a Function

Since Δx = (b – a)/n, we can write n = (b – a)/Δx and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*) + \cdots + f(x_n^*)\right]\Delta x$$

$$= \frac{}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values.

Average Value of a Function

The limiting value is

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Average Value of a Function

The following theorem says that this is true for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus.

Average Value of a Function

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions *f*, there is a number *c* such that the rectangle with base [*a*, *b*] and height *f*(*c*) has the same area as the region under the graph of *f* from *a* to *b*. (See Figure 2.)

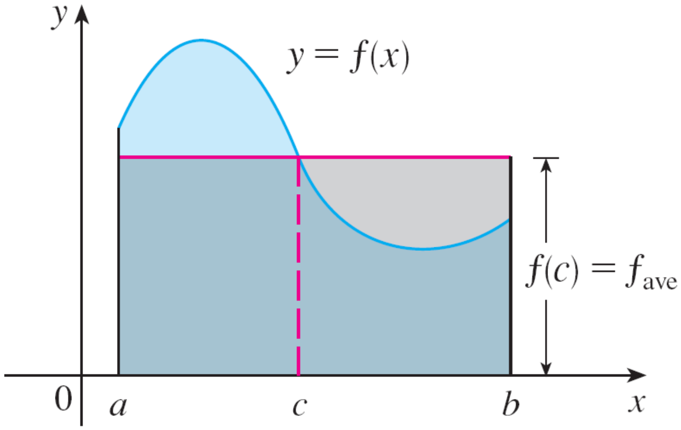

Figure 2

You can always chop off the top of a (two-dimensional) mountain at a certain height and use it to fill in the valleys so that the mountain becomes completely flat.