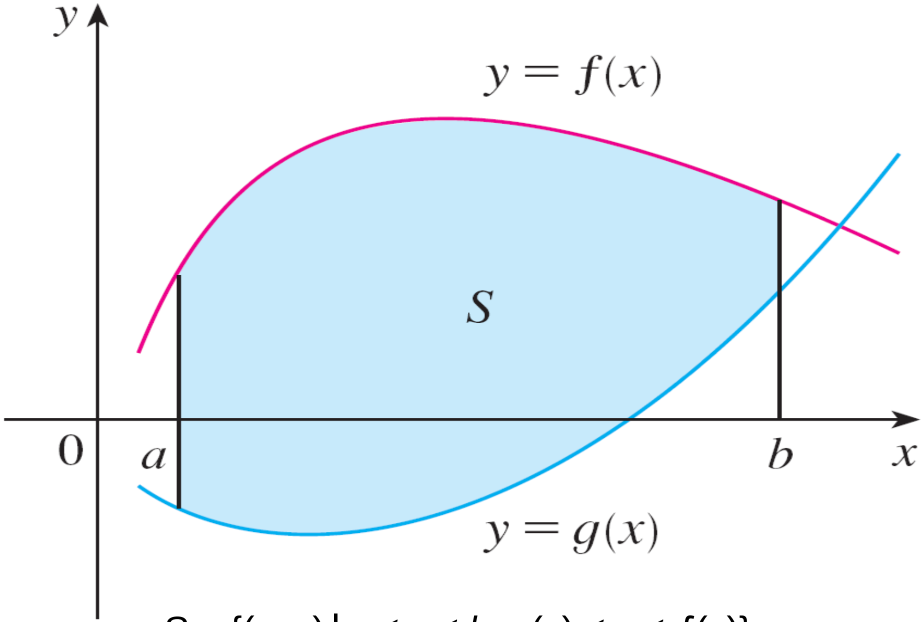
Areas Between Curves

and

Volumes

Areas Between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. (See Figure 1.)

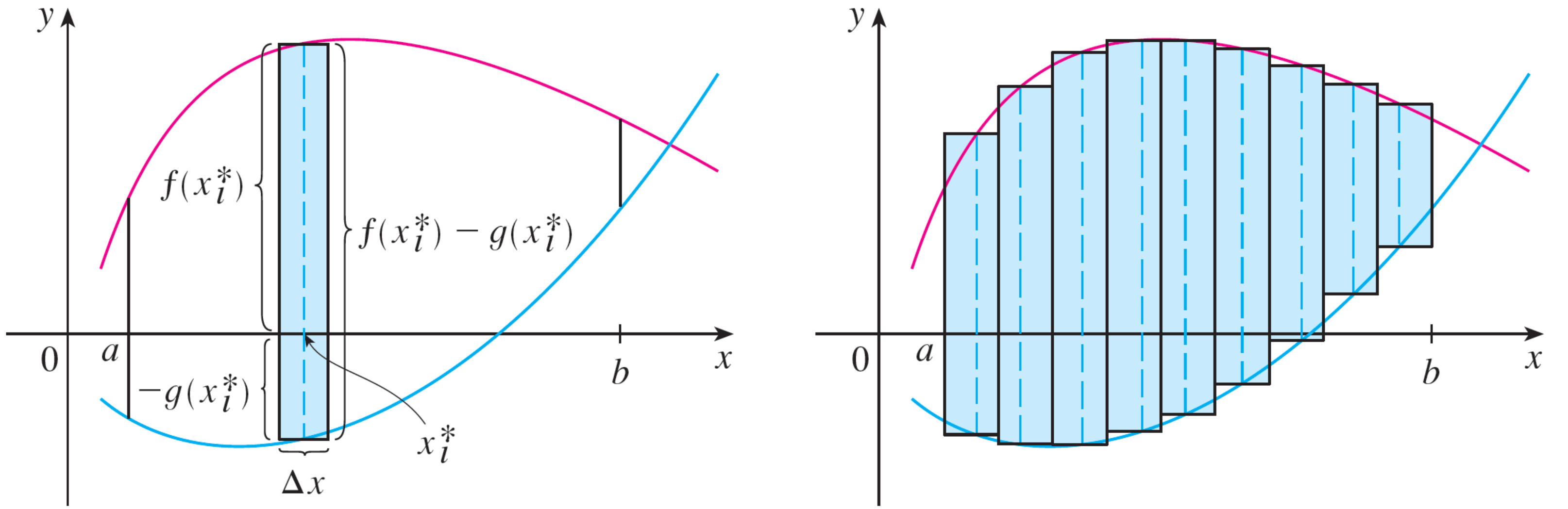


$S = \{(x, y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$

Figure 1

Areas Between Curves

We divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i*) - g(x_i*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i* = x_i$.)



(a) Typical rectangle

(b) Approximating rectangles

Figure 2

Areas Between Curves

The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

is therefore an approximation to what we intuitively think of as the area of *S*.

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** *A* of the region *S* as the limiting value of the sum of the areas of these approximating rectangles.

Areas Between Curves

$$\boxed{1} \qquad A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

Therefore we have the following formula for area.

$\boxed{2}$ The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

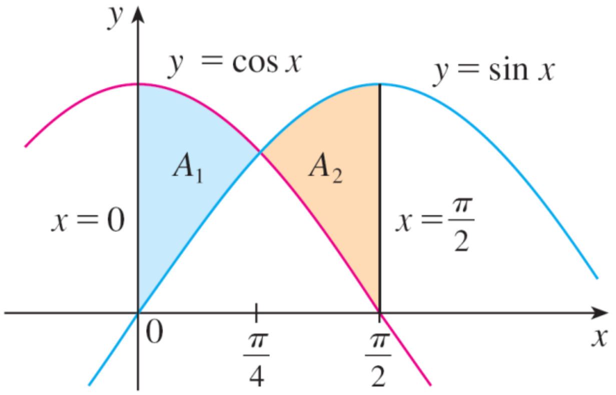
$$A = \int_a^b [f(x) - g(x)]\,dx$$

Areas Between Curves

$\boxed{3}$ The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.



Areas Between Curves

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see Figure 13), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

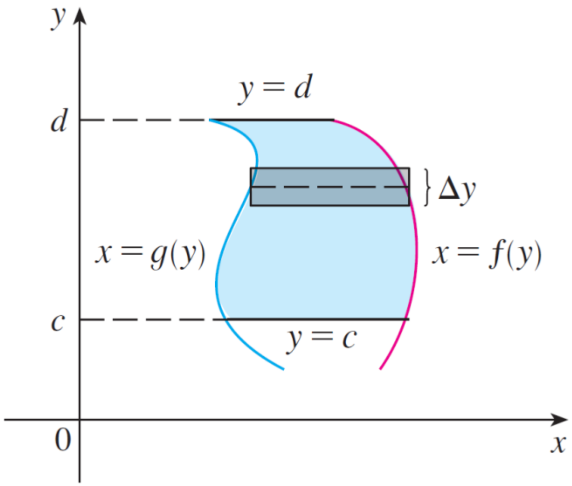
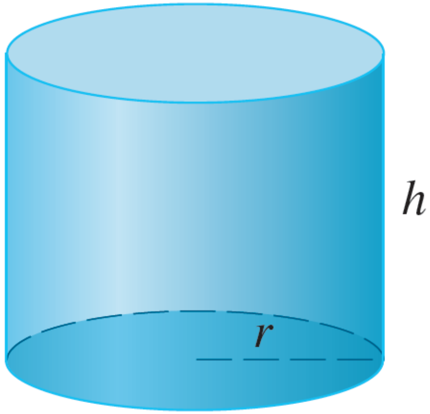


Figure 13

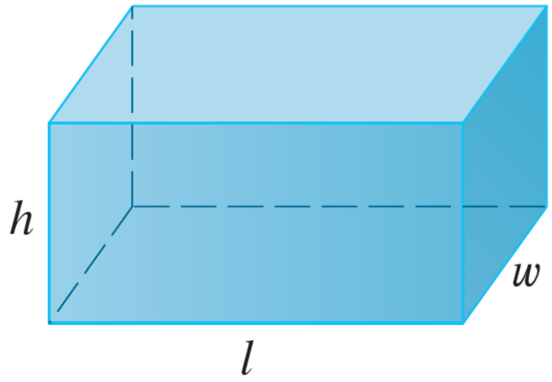
Volumes of Solids

A circular cylinder with base a circle of radius *r*, the cylinder has volume $V = \pi r^2 h$

A rectangular box with base a rectangle of length *l* and width *w*, the rectangular box (also called a *rectangular parallelepiped*) has volume $V = lwh$



(b) Circular cylinder $V = \pi r^2 h$

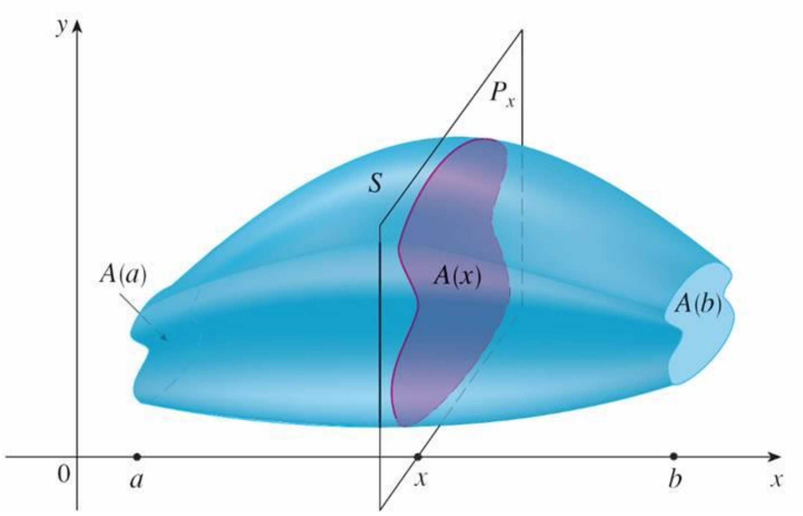


(c) Rectangular box $V = lwh$

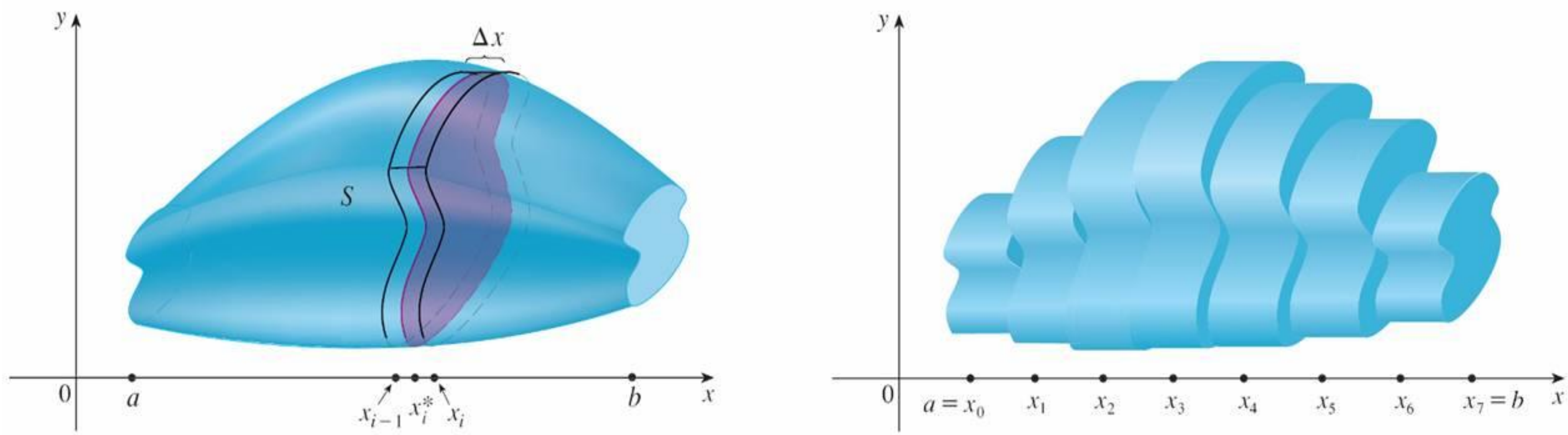
Volumes

Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leq x \leq b$. (Think of slicing S with a knife through x and computing the area of this slice.)

The cross-sectional area $A(x)$ will vary as x increases from a to b.



Volumes



$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

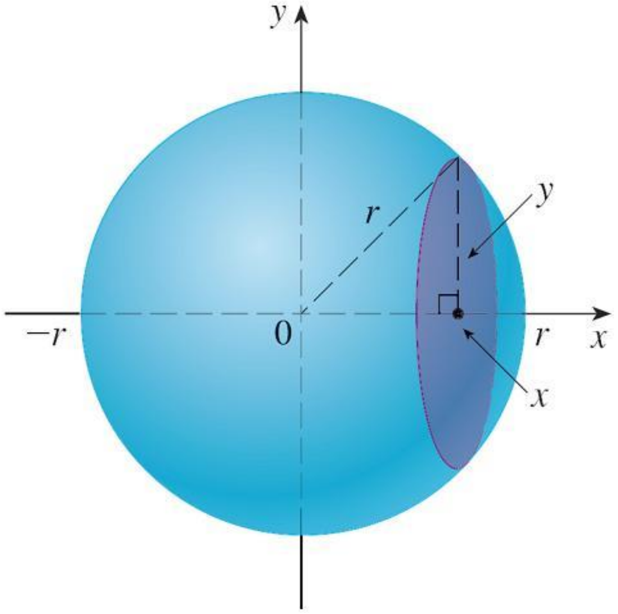
Example 1

Show that the volume of a sphere of radius *r* is $V = \frac{4}{3}\pi r^3$.

The cross-sectional area is

$A(x) = \pi y^2$

$= \pi(r^2 - x^2)$



$$V = \int_{-r}^{r} A(x)\,dx$$

Figure 4

Volumes

- If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figure 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

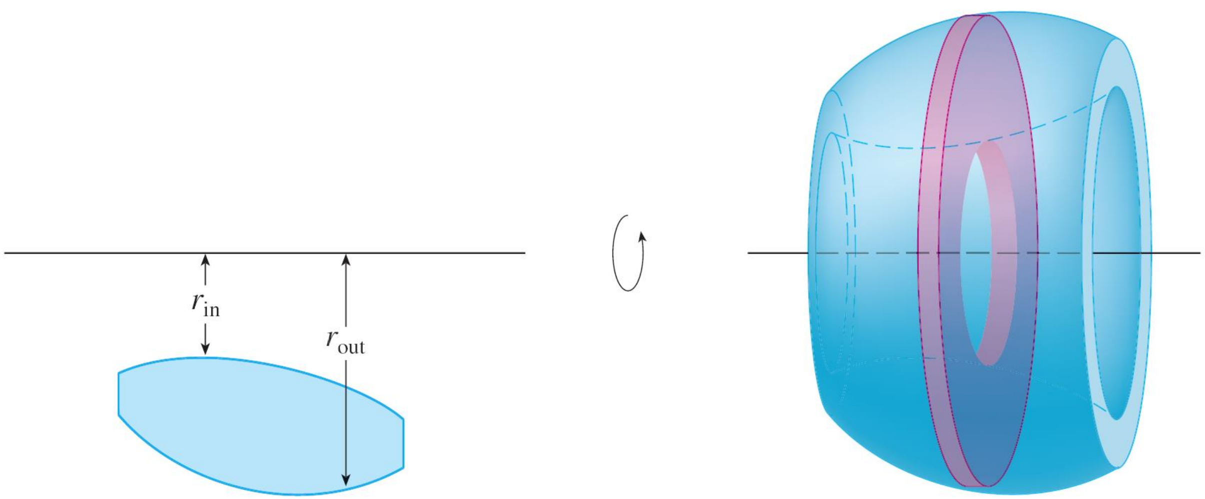


Figure 10