5 Integrals

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we write

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Example

If *f* is the function whose graph is shown in Figure 1 and $g(x) = \int_0^x f(t)\,dt$, find the values of *g*(0), *g*(1), *g*(2), *g*(3), *g*(4), and *g*(5).

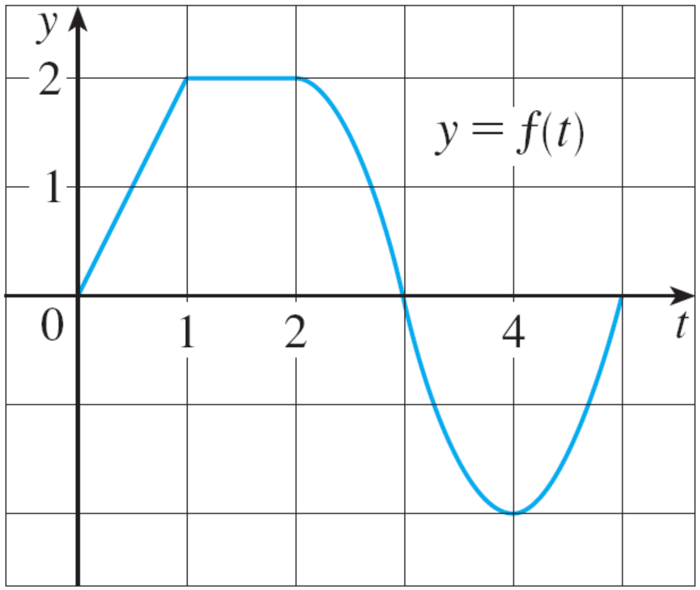

Figure 1

Indefinite Integrals

We write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we regard an indefinite integral as representing an entire *family* of functions.

Also

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

1 Table of Indefinite Integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \dfrac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

Example

Evaluate

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt.$$