7 Techniques of Integration

7.3 Trigonometric Substitution

Trigonometric Substitution

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\left|\cos \theta\right|$$

Trigonometric Substitution

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$

Solution:

Let $x = 3 \sin \theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 3 \cos \theta \, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2\theta}$$

$$= \sqrt{9 \cos^2\theta}$$

$$= 3\,|\cos \theta|$$

$$= 3 \cos \theta$$

(Note that $\cos \theta \geq 0$ because $-\pi/2 \leq \theta \leq \pi/2$.)

Example 1 – *Solution*

cont'd

Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

Example 3

Find $\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx.$

Example

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.