11.4 The Comparison Tests

The Comparison Tests

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

Example 1

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

The Comparison Tests

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

The Comparison Tests

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example 3

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Alternating Series

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Alternating Series

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$\text{(i)}\ \ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n \to \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$