11.10 Taylor and Maclaurin Series

Taylor and Maclaurin Series

We start by supposing that *f* is any function that can be represented by a power series

$\boxed{1}$ $$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \quad |x - a| < R$$

Let's determine what the coefficients c_n must be in terms of *f*.

$\boxed{5}$ **Theorem** If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor and Maclaurin Series

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Taylor series of the function *f* at *a* (or **about *a*** or **centered at *a***).

$$\boxed{7} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Maclaurin series

Example

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Taylor and Maclaurin Series

In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree *n* called the
n* th-degree Taylor polynomial of *f* at *a.

Taylor and Maclaurin Series

We have therefore proved the following.

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following Theorem.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

Example

Prove that e^x is equal to the sum of its Maclaurin series.

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

Example 8

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Solution:

Arranging our work in columns, we have

$$
\begin{array}{ll}
f(x) = (1 + x)^k & f(0) = 1 \\
f'(x) = k(1 + x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1 + x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
\vdots & \vdots \\
f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1 + x)^{k-n} & f^{(n)}(0) = k(k-1)\cdots(k-n+1)
\end{array}
$$

Example 8 – *Solution*

cont'd

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**.

Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test.

Example 8 – *Solution*

cont'd

If its nth term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Taylor and Maclaurin Series

It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Taylor and Maclaurin Series

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

Important Maclaurin Series and their Radii of Convergence

Table 1

Multiplication and Division of Power Series

Example 13

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution:

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

Example 13 – *Solution*

cont'd

We multiply these expressions, collecting like terms just as for polynomials:

$$
\begin{array}{rl}
 & 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots \\
\times & \phantom{1 + {}} x \phantom{{} + \tfrac{1}{2}x^2} - \tfrac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \\
+ & - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 - \cdots \\
\hline
 & x + x^2 + \tfrac{1}{3}x^3 + \cdots
\end{array}
$$

Example 13 – *Solution*

cont'd

Thus $$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

Example 13 – *Solution*

cont'd

We use a procedure like long division:

$$
\begin{array}{r@{}l}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots \big) & \overline{x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots} \\
 & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\
\hline
 & \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
 & \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\hline
 & \tfrac{2}{15}x^5 + \cdots
\end{array}
$$

Thus $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Approximating Functions by Polynomials

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

The notation $T_n(x)$ is used to represent the nth partial sum of this series and we can call it as it the nth-degree Taylor polynomial of f at a.

Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i$$

Approximating Functions by Polynomials

$$= f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so T_n can be used as an approximation to f:

$$f(x) \approx T_n(x).$$

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of f at a.

Approximating Functions by Polynomials

Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1.

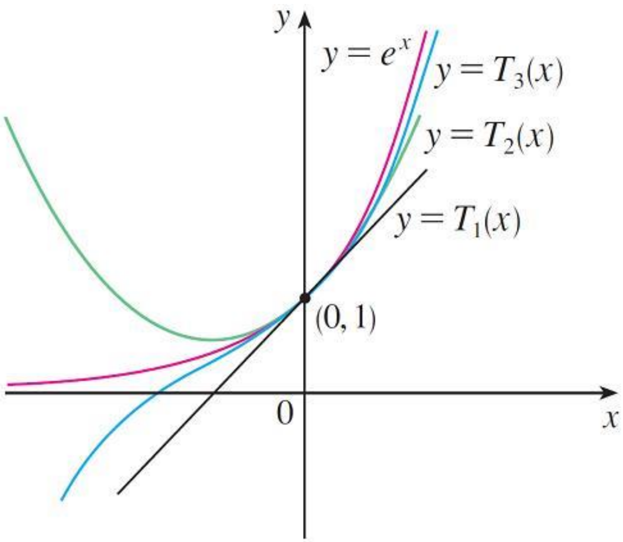

Figure 1

Example 1

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at *a* = 8.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution:

(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$

$$f'(x) = \tfrac{1}{3}x^{-2/3} \qquad f'(8) = \tfrac{1}{12}$$

$$f''(x) = -\tfrac{2}{9}x^{-5/3} \qquad f''(8) = -\tfrac{1}{144}$$

$$f'''(x) = \tfrac{10}{27}x^{-8/3}$$

Example 1 – *Solution*

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x - 8) + \frac{f''(8)}{2!}(x - 8)^2$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x)$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

Example 1 – *Solution*

cont'd

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example.

But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leqslant \frac{M}{3!}|x - 8|^3$$

where $|f'''(x)| \leq M$.

Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leqslant \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Example 1 – *Solution*

cont'd

Therefore we can take $M = 0.0021$. Also $7 \leq x \leq 9$, so $-1 \leq x - 8 \leq 1$ and $|x - 8| \leq 1$.

Then Taylor's Inequality gives

$$|R_2(x)| \leqslant \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leq x \leq 9$, the approximation in part (a) is accurate to within 0.0004.