Sample Test

Test 1

sols

Print your name:

Solve each one of the following questions. Write your solutions clearly and show all your calculations in the spaces provided for each problem. You may use a basic calculator but you may not use any device with internet access (i.e. phone, computer, tablet and so on).

1. (1 pt) Find the exact value for the average of $f(x) = \sqrt{x}$ over the interval $[0, 4]$.

$$\frac{1}{4}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{1}{4}\cdot\frac{2}{3}\cdot 2^3 = \boxed{\frac{4}{3}}$$

2. Consider the region S bounded by the x-axis, the graph of $y = cos^2(x)$, with $0 \le x \le \pi/4$. Solve the following four problems.

2.(a) (1 pt) Set up a definite integral that is equal to the area of the region S.

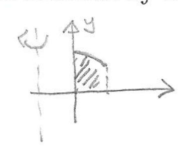
$$\boxed{\int_0^{\pi/4} \cos^2 x\,dx} = \int_0^{\pi/4}\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx = \frac{1}{2}\cdot\frac{\pi}{4} + \left[\frac{1}{4}\sin(2x)\right]_0^{\pi/4} = \boxed{\frac{\pi}{8}+}$$

2.(b) (1 pt) Evaluate the integral you wrote for [2.(a)]

$\int$ Done above.

2.(c) (1 pt) Set up a definite integral that is equal to the volume of the solid obtained by revolving the region S about $x = -1$. **Do not evaluate this integral.**

$$2\pi\int_0^{\pi/4} (x+1)\cos^2(x)\,dx$$



2.(d) (1 pt) Find the volume of the solid obtained by revolving the region S about $y = -3$.

$$\pi\int_0^{\pi/4}\left[(\cos^2 x+3)^2 - 3^2\right]dx = \pi\int_0^{\pi/4}(\cos^4 x + 6\cos^2 x)\,dx =$$

$$= \pi\int_0^{\pi/4}\left[\left(\frac{1+\cos(2x)}{2}\right)^2 + 6\left(\frac{1+\cos(2x)}{2}\right)\right]dx =$$

$$= \pi\int_0^{\pi/4}\left[\frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)+3+3\cos 2x\right]dx =$$

$$= \pi\left\{\frac{13}{4}\cdot\frac{\pi}{4} + \left[\frac{7}{4}\sin 2x\right]_0^{\pi/4} + \frac{1}{4}\int_0^{\pi/4}\frac{1+\cos(4x)}{2}\,dx\right\}$$

$$= \frac{13}{16}\pi^2 + \frac{7\pi}{4} + \frac{\pi^2}{32} + \frac{\pi}{32}\cancel{\left[\sin(4x)\right]_0^{\pi/4}}_{0} = \boxed{\frac{27}{32}\pi^2 + \frac{7\pi}{4}}$$

3. (2 pts) Evaluate the integral $\int x cos(2x) dx$ by using integration by parts. ($\int udv = uv - \int vdu$). (Use C for the constant of integration.)

$u = x \qquad dv = \cos(2x)\,dx$

$du = dx \qquad v = \frac{1}{2}\sin(2x)$

$\int x\cos(2x)\,dx = \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

4. (2 pts) Evaluate the integral $\int 7x^3 \ln(x) dx$, using integration by parts. Set $u = \ln(x)$ and $dv = 7x^3$.(Use C for the constant of integration.)

$du = \frac{1}{x}\,dx$

$v = \frac{7}{4}x^4$

$\int 7x \ln(x)\,dx = \frac{7}{4}x^4 \ln(x) - \frac{7}{4}\int x^3\,dx$

$= \frac{7}{4}x^4 \ln(x) - \frac{7}{16}x^4 + C$

5. (2 pts) Evaluate the integral $\int cos^3(x) \sin^3(x)\, dx$.
Use the trigonometric identity $cos^2(x) + sin^2(x) = 1$.

$\int \cos^3 x\,(1-\cos^2 x)\sin(x)\,dx = -\int u^3(1-u^2)\,du$

$u = \cos x$

$du = -\sin(x)\,dx$

$= -\frac{1}{4}u^4 + \frac{1}{6}u^6 + C$

$= \boxed{-\frac{1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C}$

6. (2 pts) Write out the form of the partial fraction decomposition of the function. Determine the numerical values of the coefficients. Show all your calculations.

$$\frac{x^5 + x^3 + 2x^2 - x + 1}{x^3 + x}.$$

$$\frac{x^5 + x^3 + 2x^2 - x + 1}{x^3 + x} = x^2 + \frac{2x^2 - x + 1}{x(x^2+1)} = \boxed{x^2 + \frac{1}{x} + \frac{x-1}{x^2+1}}$$

$$\frac{2x^2 - x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + Bx^2 + Cx}{x^2+1}$$

$2 = A + B$

$-1 = C$

$1 = A$; $B = 1$

7. (2 pts) Evaluate the integral

$$\int \frac{2x-3}{x^2+2x+1}\,dx.$$

(Remember to use $\ln(|u|)$ where appropriate. Use C for the constant of integration.)

$$\frac{2x-3}{(x+1)^2} = \frac{2}{x+1} - \frac{5}{(x+1)^2}$$

$$\boxed{2\ln|x+1| + \frac{5}{x+1} + C}$$

8. (2 pts) Evaluate the integral (a and b represent two numbers).

$$\int_a^b \frac{x^2+x+2}{(x+2)(x+1)^2}\,dx.$$

First decompose the rational function $\frac{x^2+x+1}{(x+1)(x+2)^2}$ into partial fractions then evaluate the integral. Show all your calculations. Use the back of this page if you need extra space.

$$\frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{x^2+x+2}{(x+2)(x+1)^2}$$

$$\circledast\ A(x+1)^2 + B(x+1)(x+2) + C(x+2) = x^2+x+2$$

$\boxed{C = 2}$

$\boxed{A = 4}$

$\circledast$ at $x=0$

$4 + 2B + 4 = 2$

$\boxed{B = -3}$

$$\int \left(\frac{4}{x+2} - \frac{3}{x+1} + \frac{2}{(x+1)^2}\right)dx = 4\ln|x+2| - 3\ln|x+1| - \frac{2}{x+1} + C$$

$$\boxed{4\ln|b+2| - 3\ln|b+1| - \frac{2}{b+1} - 4\ln|a+2| + 3\ln|a+1| + \frac{2}{a+1}}$$

8. (2 pts) Evaluate the integral (a and b represent two numbers).

$$\int_a^b \frac{x^2+x+1}{(x+1)(x+2)^2}dx.$$

First decompose the rational function $\frac{x^2+x+1}{(x+1)(x+2)^2}$ into partial fractions then evaluate the integral. Show all your calculations.

$$\frac{x^2+x+1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$A(x+2)^2 + B(x+1)(x+2) + C(x+1) = x^2+x+1$$

$$-C = 4-2+1 = 3 \quad \boxed{C=-3}$$

$$\boxed{A = 1}$$

$$4+2B-3=1$$

$$\boxed{B=0}$$

$$= \frac{1}{x+1} - \frac{3}{(x+2)^2}$$

$$\int_a^b \frac{x^2+x+1}{(x+1)(x+2)^2}dx = \boxed{\ln\left|\frac{b+1}{a+1}\right| + \frac{3}{b+2} - \frac{3}{a+2}}$$

9. (1 pt) The integral $\int_0^1 \pi\,(x^3 - x^4)dx$ represents the volume of a solid obtained by revolving a region R about the y-axis. Describe R.

$$2\pi\int_0^1 x\left(\frac{x^2-x^3}{2}\right)dx$$

R is the region bounded by the x-axis, $x=0$, $x=1$ and $y=\frac{x^2-x^3}{2}$

$\boxed{\text{answers may vary}}$

10. (2 pts) Consider the definite integral $\int_{\frac{\sqrt{2}}{3}}^{\frac{1}{3}} \frac{dx}{x^2\sqrt{9x^2-1}}$. Write the definite integral (in the new variable t) obtained from the given one by using the substitution $3x = sec(t)$.

$$\cos(t) = \frac{1}{3x}$$

$$3\,dx = \sec(t)\tan(t)\,dt$$

$$\frac{1}{3}\int \frac{\sec t \tan t\, dt}{\frac{\sec^2 t}{9}\cdot \tan(t)} = 3\int \cos t\, dt = 3\sin(t) + C$$

$$= 3\sqrt{1-\frac{1}{9x^2}} + C$$

$$= \frac{1}{x}\sqrt{9x^2-1} + C$$

Ans: $\boxed{3\int_{\pi/4}^{0} \cos t\, dt}$