Test 2

Name:

Solve each one of the following questions. Write your solutions clearly and show all your calculations in the spaces provided for each problem. You may use a basic calculator but you may not use any device with internet access (i.e. phone, computer, tablet and so on).

1. (10pts) Find the exact area of the of the surface obtained by rotating the curve $y = x^2$ with $0 \leq x \leq 1$ about the y-axis.

$$2\pi \int_0^1 x \sqrt{1+(2x)^2}\, dx \qquad u = 1+4x^2 \qquad du = 8x\,dx$$

$$2\pi \int_1^5 \frac{1}{8}\sqrt{u}\, du = \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^5 = \frac{\pi}{6}\left[5\sqrt{5}-1\right]$$

2. (10pts) Evaluate the integral or show that it diverges: $\int_2^\infty \frac{t}{\sqrt{-1+t^2}} dt$

$u = -1+t^2 \qquad du = 2t\,dt$

$$\int_2^M \frac{t}{\sqrt{-1+t^2}}\,dt = \frac{1}{2}\int_3^{M^2-1} \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}\left\{\left[M^2-1\right]^{3/2} - 3^{3/2}\right\}$$

$$\lim_{M\to\infty} \frac{1}{3}\left\{(M^2-1)^{3/2} - 3^{3/2}\right\} = \infty$$

$$\boxed{\int_2^\infty \frac{t}{\sqrt{t^2-1}}\,dt \text{ diverges}}$$

3. (10pts) At what points on the curve $x = t^2 + 3t$ *and* $y = t^3 - t + 1$ does the tangent line have slope equal to ½? Write exact answers. Do not simplify.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{2t+3} = \frac{1}{2}$$

$$6t^2 - 2 - 2t - 3 = 0$$

$$6t^2 - 2t - 5 = 0$$

$$t = \frac{1 \pm \sqrt{31}}{6}$$

at $t = \frac{1+\sqrt{31}}{6}$: $\left(\left(\frac{1+\sqrt{31}}{6}\right)^2 + \frac{1+\sqrt{31}}{2} \,;\, \left(\frac{1+\sqrt{31}}{6}\right)^3 - \frac{1+\sqrt{31}}{6} + 1\right)$

at $t = \frac{1-\sqrt{31}}{6}$: $\left(\left(\frac{1-\sqrt{31}}{6}\right)^2 + \frac{1-\sqrt{31}}{2} \,;\, \left(\frac{1-\sqrt{31}}{6}\right)^3 - \frac{1-\sqrt{31}}{6} + 1\right)$

$$y - \left(\left(\frac{1+\sqrt{31}}{6}\right)^3 - \frac{1+\sqrt{31}}{6} + 1\right) = \frac{1}{2}\left(x - \left(\frac{1+\sqrt{31}}{6}\right)^2 - \frac{1+\sqrt{31}}{6}\right) \leftarrow \text{tangent line}$$

$$y - \left[\left(\frac{1-\sqrt{31}}{6}\right)^3 - \frac{1-\sqrt{31}}{6} + 1\right] = \frac{1}{2}\left(x - \left(\frac{1-\sqrt{31}}{6}\right)^2 - \frac{1-\sqrt{31}}{6}\right) \leftarrow \text{tangent line}$$

4. (10pts) Find the exact length of the polar curve $r = e^{2\theta}, 0 \le \theta \le \pi$.

$$\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta = \sqrt{5} \int_0^{\pi} e^{2\theta}\, d\theta$$

$$= \frac{\sqrt{5}}{2}\left[e^{2\pi} - 1\right]$$

5. (10pts) Eliminate the parameter to find a Cartesian equation for the curve $x = 2t + 1$ and $y = t^2 - 5$.

$$t = \frac{x-1}{2} \qquad y = \frac{x^2 - 2x + 1}{4} - 5 = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{19}{4}$$

$$\boxed{y = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{19}{4}}$$

6. (10pts) Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter $x = t^2\cos(t), y = t\sin(t)$; $t=\pi$.

$$x(\pi) = -\pi^2 \qquad \frac{dy}{dx} = \frac{\sin t + t\cos t}{2t\cos t - t^2 \sin t}\bigg|_{t=\pi} = \frac{-\pi}{-2\pi} = \frac{1}{2}$$

$$y(\pi) = 0$$

$$y = \frac{1}{2}(x + \pi^2)$$

7. (40pts) Decide whether each of the following series converges or diverges. Explain clearly your answer.

a. $\sum_{n=1}^{\infty} \frac{n^2+n}{(2n+1)^3}$ diverges. Comparison test (Limit comparison test)

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n} \frac{\frac{n^2+n}{(2n+1)^3}}{\frac{1}{n}} = \frac{1}{8} > 0$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^4+9}}$.

$$\sum_{1}^{\infty} \frac{1}{n^{2}}$$ $p = 2 > 1$

Convergent (p-series)

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{3n^4+9}}}{\frac{1}{\sqrt{n^4}}} = \frac{1}{\sqrt{3}} > 0 \quad \therefore \text{ Convergent}$$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+n}$

alternating series. From the alternating Series test we conclude it converges.

$$\lim_{n} \frac{1}{n^2+n} = 0 \quad \text{and} \quad \left(\frac{1}{n^2+n}\right)_n \text{ is decreasing} \quad (n+1)^2+(n+1) > n^2+n$$

$$\frac{1}{(n+1)^2+(n+1)} < \frac{1}{n^2+1}$$

d. $\sum_{n=1}^{\infty} \boldsymbol{sin}\left(\frac{1}{n^3+1}\right)$

$$\sin\left(\frac{1}{n^3+1}\right) \leq \frac{1}{n^3+1} \leq \frac{1}{n^3}$$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $\therefore$ The comparison test implies that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3+1}\right)$ also converges.

Bonus. (5pts) Decide the convergence or divergence of the series $\sum_0^\infty (n+1)\, 7^{-n}$.

$$\frac{n+1}{7^n} \leq \frac{2^n}{7^n} = \left(\frac{2}{7}\right)^n, \; n \geq 1$$

$\sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n$ is a geometric series.

The comparison test implies that $\sum_{n=1}^{\infty} \frac{n+1}{7^n}$ converges.

Then $\sum_{n=0}^{\infty} \frac{n+1}{7^n}$ also converges.