

7.1

Integration by Parts

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts*.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integration by Parts

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)] dx = f(x)g(x)$$

or
$$\int f(x)g'(x) dx + \int g(x)f'(x) dx = f(x)g(x)$$

We can rearrange this equation as

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$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

Formula 1 is called the **formula for integration by parts**.

Integration by Parts

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)dx$ and $dv = g'(x)dx$, so, by the Substitution Rule, the formula for integration by parts becomes

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$$\int u \, dv = uv - \int v \, du$$

Example 1

Find $\int x \sin x \, dx$.

Integration by Parts

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Evaluating both sides of Formula 1 between a and b , assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

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$$\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) dx$$

Use trigonometric identities to integrate certain combinations of trigonometric functions.

Find $\int \sin^5 x \cos^2 x \, dx$.

Evaluate $\int_0^{\pi} \sin^2 x \, dx.$

Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

- (a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx\end{aligned}$$

Then substitute $u = \sin x$.

- (b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx\end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

- (c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Find $\int \tan^6 x \sec^4 x \, dx$