Polar Coordinates

This axis corresponds to the positive *x*-axis.

If *P* is any other point in the plane, let *r* be the distance from *O* to *P* and let θ be the angle (usually measured in radians) between the polar axis and the line *OP* as in Figure 1.

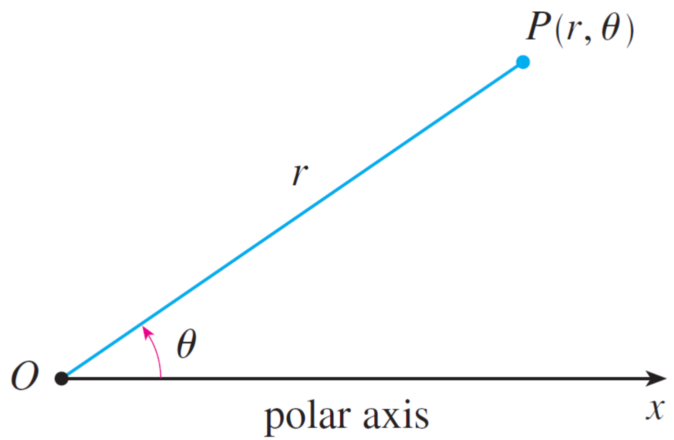


Figure 1

Polar Coordinates

Convention:

$(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

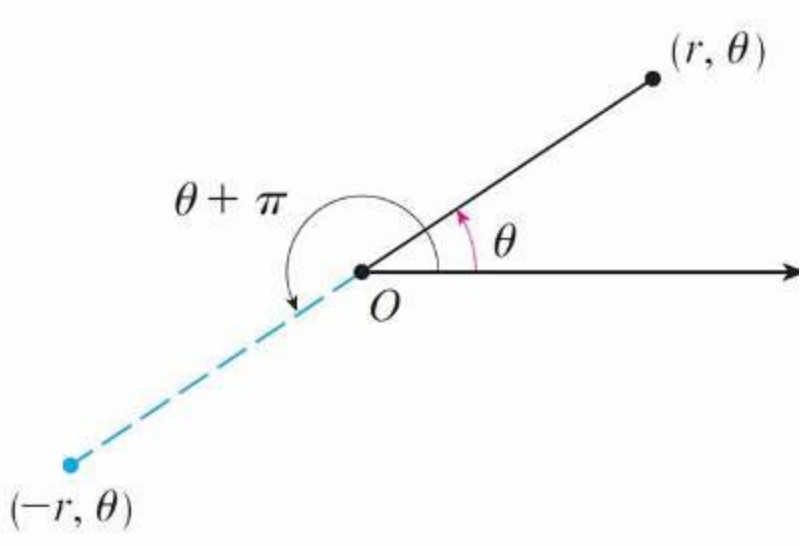


Figure 2

Examples

Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

Solution:

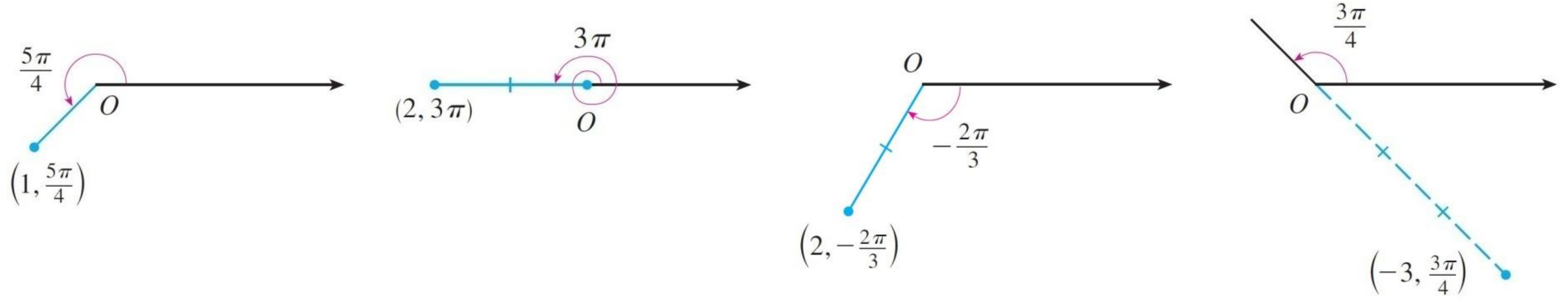
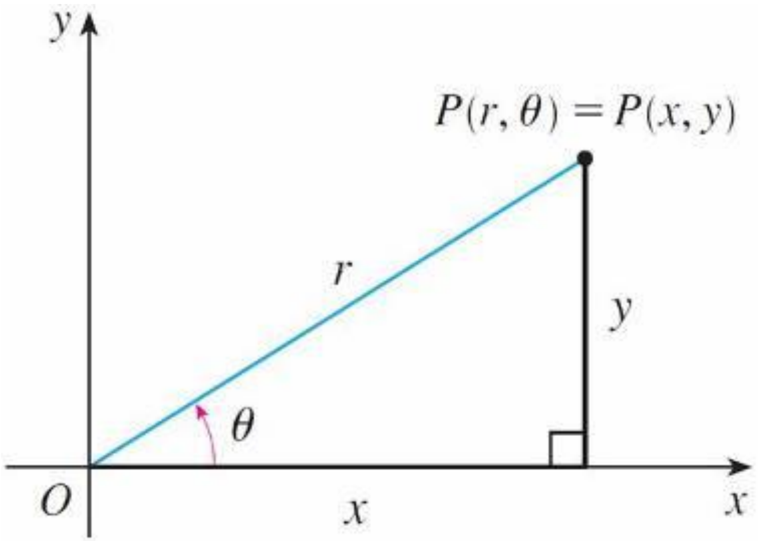


Figure 3

Polar Coordinates: Connection with Cartesian coordinates



$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

1

$$x = r\cos\theta \qquad y = r\sin\theta$$

2

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Example

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example

Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

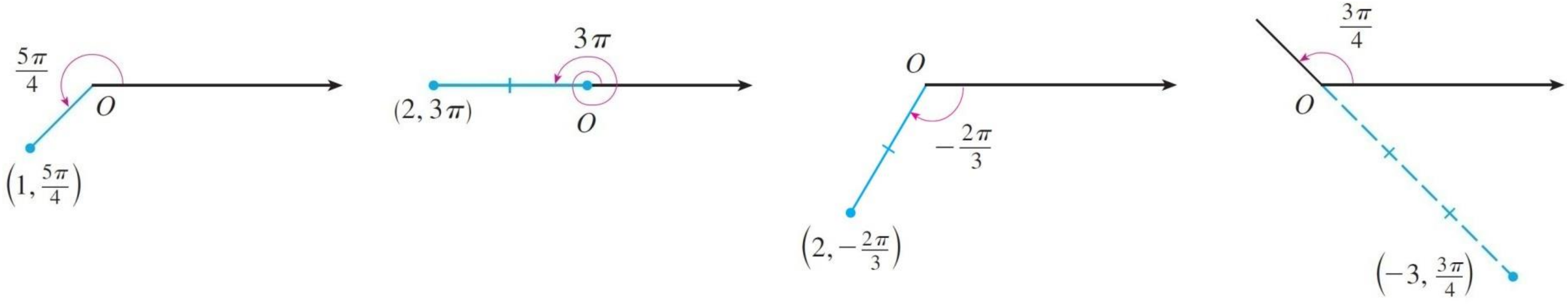


Figure 3

10.4 Areas and Lengths in Polar Coordinates

Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

$$\boxed{1} \qquad A = \tfrac{1}{2} r^2 \theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

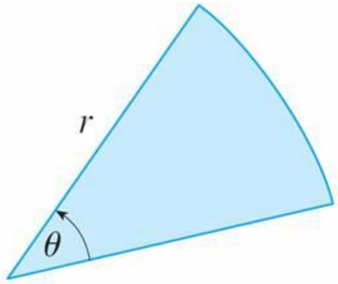


Figure 1

Areas and Lengths in Polar Coordinates

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

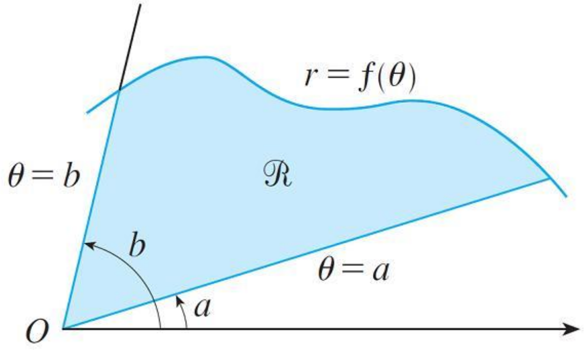


Figure 2

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$.

Areas and Lengths in Polar Coordinates

The rays $\theta = \theta_i$ then divide $\mathscr{R}$ into *n* smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the *i*th subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the *i*th region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

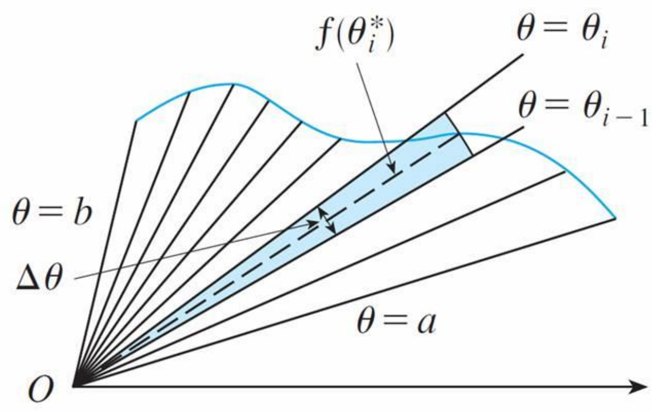


Figure 3

Areas and Lengths in Polar Coordinates

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta$$

and so an approximation to the total area A of $\mathcal{R}$ is

2 $$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

4 $$A = \int_a^b \tfrac{1}{2} r^2 \, d\theta$$

Example 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

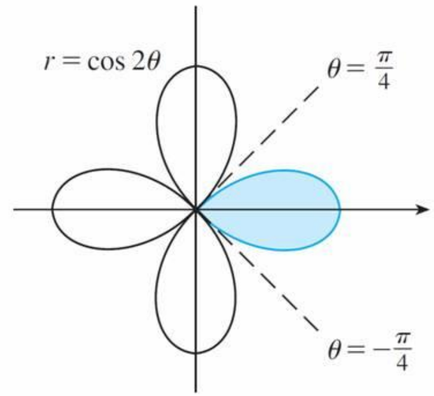


Figure 4

Example 1 – *Solution*

cont'd

Therefore Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta$$

$$= \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \int_{0}^{\pi/4} \cos^2 2\theta \, d\theta$$

Arc Length

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

Arc Length

Assuming that f' is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

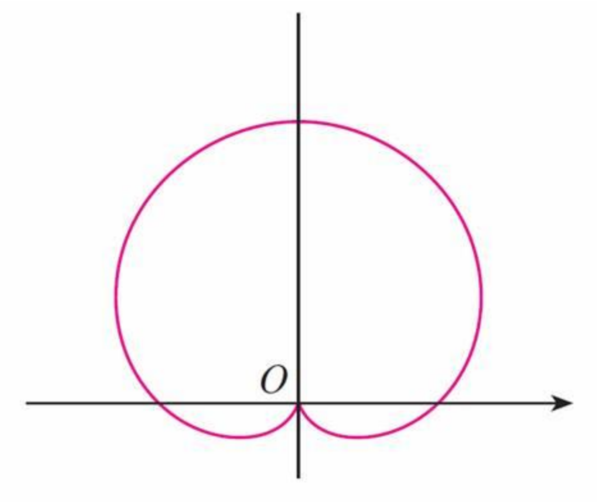
5

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example 4

Find the length of the cardioid $r = 1 + \sin\,\theta$.

Solution:



5

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$