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Function heirarchy.

log-fn $\underset{\alpha}{<}$ poly-fn $\underset{\alpha}{<}$ Exp-fn

(i) $\log n = o(n)$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot 1} = 0 \checkmark$$

(ib) $(\log n)^k = o(n)$.

$$\lim_{n\to\infty} \frac{(\log n)^k}{n} = \lim_{n\to\infty} \frac{k(\log n)^{k-1}}{n \cdot 1}$$

$$= \lim_{n\to\infty} \frac{k(k-1)(\log n)^{k-2}}{1 \cdot n}$$

$$\cdots \lim_{n\to\infty} \frac{k(k-1)(k-2)\cdots 1}{n} = 0$$

Because $\underline{k! = O(1) \text{ for } k = O(1)}$.

(ic) $(\log n)^k = o(n^{\epsilon})$

Similar to (ib)

$$\lim_{n\to\infty} \frac{(\log n)^k}{n^{\epsilon}} = \frac{k(\log n)^{k-1}}{n\, \epsilon\, n^{\epsilon-1}} = \frac{k(\log n)^{k-1}}{\epsilon n^{\epsilon}}$$

$$= \lim_{n\to\infty} \frac{k(k-1)(\log n)^{k-2}}{\epsilon^2 n^{\epsilon-1} \cdot n} = \frac{k(k-1)(\log n)^{k-2}}{\epsilon^2 n^{\epsilon}}$$

$$= \lim_{n\to\infty} \frac{k(k-1)-2 \log n}{\epsilon^{k-2} n^{\epsilon}} = \lim_{n\to\infty} \frac{k!}{\epsilon^k n^{\epsilon-1} n}$$

$$= \lim_{n\to\infty} \frac{k!}{\epsilon^k n^{\epsilon}} = 0$$

(iia) $n = o(\alpha^n)$ $\quad \alpha > 1$. ✓

$$\Rightarrow \lim_{n\to\infty} \frac{n}{\alpha^n} = 0 \quad \underline{\underline{\text{i.e}}} \lim_{n\to\infty} \frac{1}{\alpha^n \ln\alpha} = 0$$

$\because \alpha^n \to \infty$ as $n \to \infty$,

Note that $\frac{d}{dn}\alpha^n = \alpha^n \ln\alpha = e^n$ if $\alpha = e$.

(iib) $n^k = o(\alpha^n)$.

$$\Rightarrow \lim_{n\to\infty} \frac{n^k}{\alpha^n} = \lim_{n\to\infty} \frac{k n^{k-1}}{\alpha^n \ln\alpha}$$

$$= \lim_{n\to\infty} \frac{k(k-1) n^{k-2}}{\alpha^n (\ln\alpha)^2} \cdots = \lim_{n\to\infty} \frac{k \cdot k-1 \cdots 1}{\alpha^n (\ln\alpha)^k}$$

both $k!$ and $(\ln\alpha)^k$ are $O(1)$

$$\therefore \lim_{n\to\infty} \frac{n^k}{\alpha^n} = 0 \quad \text{as } \underline{\underline{\alpha^n}} \to \infty$$

The results establish the following. logarithmic functions are asymptotically smaller than polynomial functions and polynomial functions are asymptotically smaller than exponential functions.

Log-fn: $\sum a_i (\log n)^i$

Poly-fn: $\sum a_i n^i$

Exp-fn: $\sum a_i \alpha_i^n$

Pivot or critical instruction.

The instruction that corr. to asymptotically largest function.

If I_1 executes $n(n-1)/2$ times and I_2 ,, $2n$ times then I_1 is critical

If I_1 executes $n(n-1)/2$ times I_2 ,, $2n$ times and I_3 ,, n^2 times then I_3 is critical (because it has larger coefficient compared to I_1).

In some instances we count a specific instruction. E.g. in matrix multiplication we count the number of executions of scalar multiplications (even though comparable (asymptotically same) number of additions also are executed.

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Counting instructions in loops.

Q. Derive $f(n)$ the number of times the following print statement is executed. $f(n)$: function of n.

```
for (i = 1 .. n)
   for (j = i+1 .. n)
      for (k = j+1 .. n)
         print (i, j, k)
```

When $i = 1$ corr. to

$j = 2$ Range of k is $3 .. n$ $\rightarrow n-2$ values

3 ,, ,, $4 .. n$ $\rightarrow n-3$ values

Thus corr $i = 1$ we have

$$\sum_{i=1}^{n-2} i = (n-1)(n-2)/2 .$$

The above expression on R.H.S Corr. to 'n', we must repeat the same for $n-1, n-2$ -- ——

Thus, total number of instructions are

$$\sum_{i=1}^{n} (i-2)(i-1)/2$$

$$= \sum_{i=1}^{n} (i^2 - 3i + 2)/2$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + 2n\right)$$

$$= \frac{1}{12}\left(n(n+1)(2n+1) - 9n(n+1) + 12n\right)$$

$$= \frac{1}{12}\left(n(n+1)(2n+1-9) + 12n\right)$$

$$= \frac{1}{12}\left(n(n+1)(2n-8) + 12n\right)$$

$$= \frac{1}{12} \cdot 2\left(n(n+1)(n-4) + 6n\right)$$

$$= \frac{1}{6} n\left((n+1)(n-4) + 6\right)$$

$$= \frac{1}{6} n\left(n^2 + 2 - 3n\right)$$

$$= \frac{1}{6} n(n-1)(n-2)$$