NP

The two major causes for exponential running time are

(a) the intrinsic difficulty of the problem

(b) the output is exponential w.r.t. input.

We are primarily interested in (a).

P: Class of problems that can be solved in poly-time.

NP: Class of problems that can be solved in poly-time by a non-deterministic Turing machine. The solution size is bounded by a polynomial of input.

∴ It can be verified in P.

<u>Polynomial time reducibility</u>: We say that a problem A is polynomial time reducible to another problem B, i.e. $A \leq_p B$ if ∃ an algorithm that maps an instance I of A to an instance f(I) of B such that I is a Yes (No) instance of A if f(I) is a Yes (No) instance of B.

Observe that $|f(I)|$ ~~can be~~ cannot be exponential w.r.t. $|I|$.

<u>NP-Hard</u>: NP-Hard problems are the hardest problems w.r.t. NP.

⇒ Solving a NP-Hard problem yields solution to the problems in NP. However, these problems ∉ NP.

<u>Ex</u>: K^{th} Largest Subset problem:

Instance: A finite set A, ∀a∈A a size $s(a) \in Z^+$ and two integers ≥ 0, $B \leq \sum_{a \in A} s(a)$ and $K \leq 2^{|A|}$.

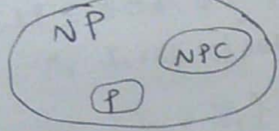
Question: Do K distinct subsets of A ∃ s.t. for such a subset A' $\sum_{a \in A'} s(a) \leq B$.

The NP-Complete problem PARTITION $\leq_p$ K^{th} Largest Subset. However, the solution set of K^{th} Largest Subset is not bounded by a polynomial of input size ⇒ K^{th} Largest Subset ∉ NP.

<u>Partition</u>: Instance: A finite set A and a size $s(a) \in Z^+$ ∀ a∈A.

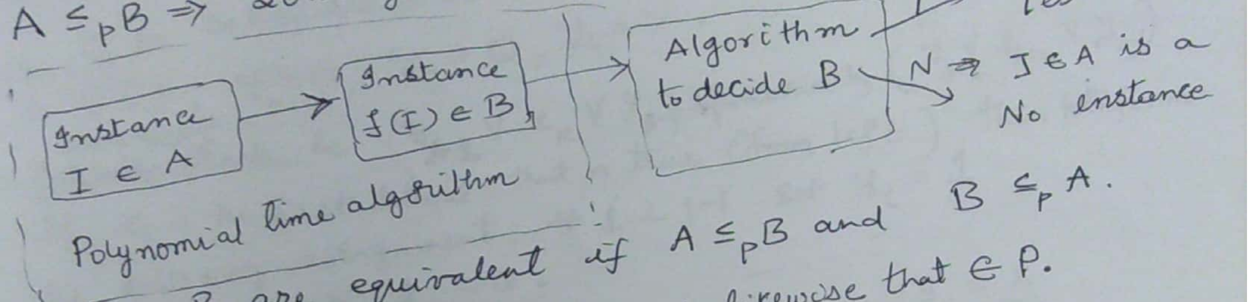
Question: $\exists A' \subseteq A$ s.t. $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$

The accepted [illegible]



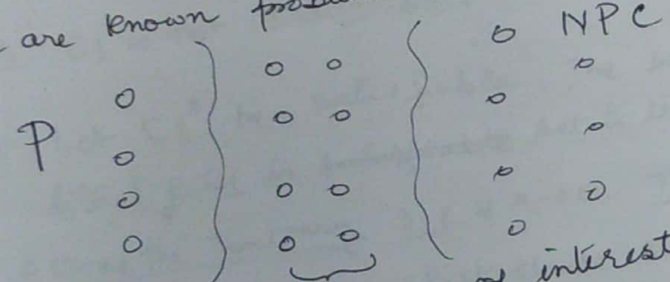
$P \subset NP$
$NPC \subset NP$
$P \cap NPC = \phi$.

★ $A \leq_p B \Rightarrow$ Solving B yields solution to A.



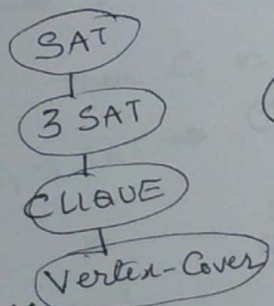
★ A and B are equivalent if $A \leq_p B$ and $B \leq_p A$.
and likewise that ∈ P.

There are known problems ∈ NPC



Problems of interest that do not have polynomial algorithms and they are not shown to be in NPC.

The chain of reductions of NPComplete problems.



Reductions are transitive.
$(SAT \leq_p 3SAT) \land (3SAT \leq_p CLIQUE) \Rightarrow SAT \leq_p CLIQUE$

NP-Complete: A problem X is NPcomplete if any $Y \in NP$ reduces to X i.e. $\forall Y \in NP$ $Y \leq_p X$.

NP-Complete reductions.

① SAT $\le$ 3 SAT. $F = (x_1 \vee x_2 \vee x_3 \vee x_4 \vee \neg x_5) \wedge (x_2 \vee \bar{x}_3 \vee x_4 \vee \bar{x}_5) \wedge ($

The first clause of F is transformed into i.e $C_1 \in F$ is now $C_1^* \in F^* =$

$(x_1 \vee x_2 \vee y_1) \wedge (\bar{y}_1 \vee x_3 \vee y_2) \wedge (\bar{y}_2 \vee x_4 \vee x_5)$ ~~$\wedge (\bar{y}_3 \vee x_5 \vee y_4)$~~

If F has 'n' literals then F^* has $3n$-literals.

Let C_1 be satisfiable where $x_3 = 1$ and $x_1 = x_2 = x_4 = x_5 = 0$ $\rightarrow$ First satisfied literal.

$\Rightarrow y_1 = 1, \; y_2 = 0, \; y_3 = 0, \; y_4 = X.$

Let a 3CNF clause be $(\bar{y}_{k-2} \vee x_k \vee y_{k-1})$ preceded by $(\bar{y}_{j-2} \vee x_{k-1} \vee y_{j-1})$

If x_k is the first literal that is true (from left) then perform the following assignment: $\forall i < j-1$ set $y_i = 1$

$\forall i \geq j-1$ set $y_j = 0$.

IIrly $C1$ is not satisfiable. Set $y_1 = 0$. $\forall i > 1$ $y_i = \underline{X}$.

The we will show that ≥ 1

$\Leftarrow$ Let $C1^*$ be satisfiable.

literal $\in C1$ is ~~satisfiable~~ set to true.

Assume the contrary i.e $\forall x_i = 0$. Then $y_1 = 1$

$y_2 = 1$ —— $\therefore$ The last clause will be $(\bar{y}_p \vee x_p \vee x_{p-1})$

where $y_p = 1$, $x_p = x_{p-1} = 0$. $\therefore$ It evaluates to zero

($\leftrightarrow$) $\therefore \geq 1$ literals must be true.

$\Leftarrow$ C_1^* is not satisfiable. Then C_1 is also not satisfiable

By contraposition C_1 is SAT $\Rightarrow$ C_1^* is SAT.

	4	5	6	— —	K	x_i
No. of literals						$\bar{y}_{i-2}$
	1	2	3	— —	K−3	y_{i-1}
No. of y's					$2(K-3)$	

y_i & $\bar{y}_i$

$(x_1 \vee x_2 \vee y_1) - \wedge (\bar{y}_{i-2} \vee x_i \vee y_{i-1}) \text{ --- } \wedge (\bar{y}_{p-3} \vee x_{p-1} \vee x_p)$

Reduction is clearly polynomial in fact it has time complexity of $O(n)$ ~~time~~. $I \rightarrow f(I)$.

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