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Practical work on filter banks

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The files related to this practical work are available on the *site pédagogique* of TSIA201.

1 Two-channel filter bank

A two-channel filter bank is defined by the diagram in figure 1.

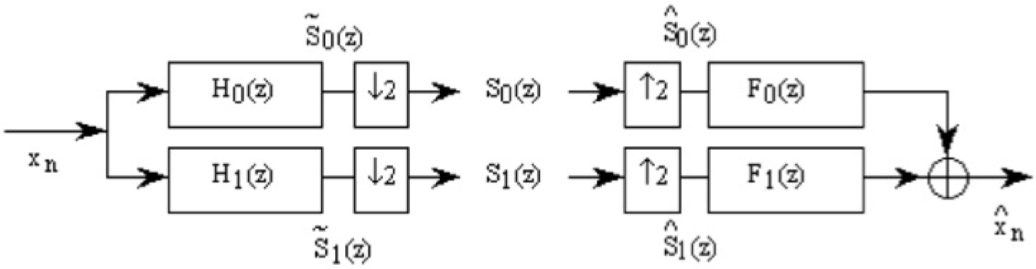


Figure 1 – General diagram of a 2-channel filter bank.

We remind that the aliasing cancellation (AC) conditions of a two-channel filter bank are $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, and that its transfer function (TF) is $T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z))$.

1.1 CQF filter bank

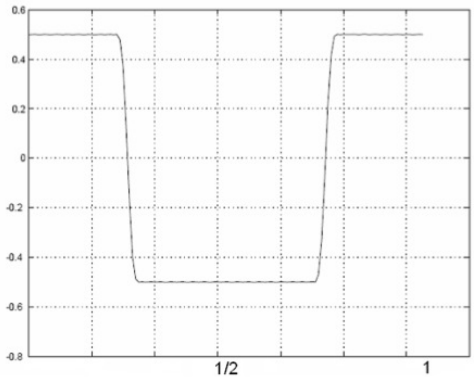
Moreover we assume that H_0 and H_1 are *conjugate quadrature filters* (CQF) : N is even and $H_1(z) = -z^{-(N-1)}\widetilde{H}_0(-z)$, where $\tilde{H}_0(z) = H_0^*(\frac{1}{z})$. The equations (AC) and (CQF) imply that $\forall k \in \{0, 1\}$, $F_k(z) = z^{-(N-1)}\widetilde{H}_k(z)$: the analysis and synthesis filters are said *paraconjugate* (PC).

Finally, we assume that $H_0(z)$ is a *symmetric power* (SP) filter : $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 2c$. Then the equations (TF), (CQF), (PC) and (SP) imply that $T(z) = cz^{-(N-1)}$: the CQF filter bank guarantees perfect reconstruction.

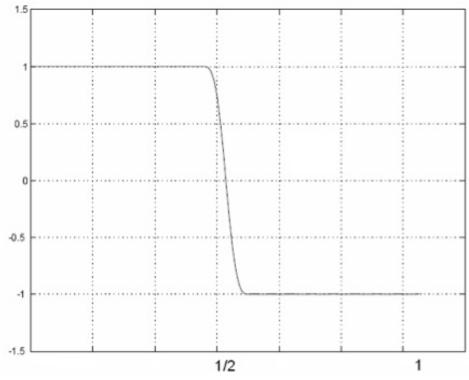
1.2 Synthesis of a half-band filter

In other respects, if we let $G(e^{2i\pi\nu}) = e^{-2i\pi\nu(N-1)}G_R(\nu)$ where function $G_R(\nu) = \widetilde{H}_0(e^{2i\pi\nu})H_0(e^{2i\pi\nu})$ is 1-periodic and nonnegative, then $g(n)$ is a causal and symmetric filter of length $2N-1$, thus of type I. The (SP) condition is then equivalent to $G_R(\nu) + G_R(\nu + \frac{1}{2}) = 2c$: we say that G is a *half-band* filter. We can therefore write $G(z)$ in the form $G(z) = c(V(z^2) + z^{-(N-1)})$, where $v(n)$ is a causal and symmetric filter of length N, thus of type II.

Finally, if we want that $H_0(z)$ be a low-pass filter, then $G(z)$ must be low-pass and half-band, which implies that $V(z)$ is a nearly all-pass filter : $V(e^{2i\pi\nu}) = e^{-2i\pi\nu\frac{N-1}{2}}V_R(\nu)$, where $V_R(\nu)$ is 2-periodic and $V_R(\nu) = 1\ \forall\nu \in [-\nu_c, \nu_c]$ with ν_c close to $\frac{1}{2}$, but $V_R(\frac{1}{2}) = V_R(-\frac{1}{2}) = 0$. The relationship between functions $G_R(\nu)$ and $V_R(\nu)$ is illustrated in figure 2.



(a) $\frac{1}{2c}G_R(\nu) - \frac{1}{2} = \frac{1}{2}V_R(2\nu)$



(b) $V_R(\nu)$

FIGURE 2 – Synthesis of $v(n)$ by the Rémez method

1. Try to directly synthesize the half-band and low-pass filter $g(n)$ by using the Rémez method. By construction this filter satisfies Chebyshev's alternance property. But does the *half-band* condition rigorously hold ? (one coefficient out of two of $g(n)$ must be rigorously zero, except the middle coefficient)
2. Then try to modify the filter $g(n)$ obtained in this way, by manually zeroing the coefficients which should be zero. Does Chebyshev's alternance property still rigorously hold ?
3. Thus, we see that the only way of synthesizing a low-pass filter $g(n)$, which is exactly half-band and which exactly respects Chebyshev's alternance property, consists in first synthesizing the nearly all-pass filter $v(n)$ by using the Rémez method, from which the filter $g(n)$ can then be derived. You may use the following syntax :

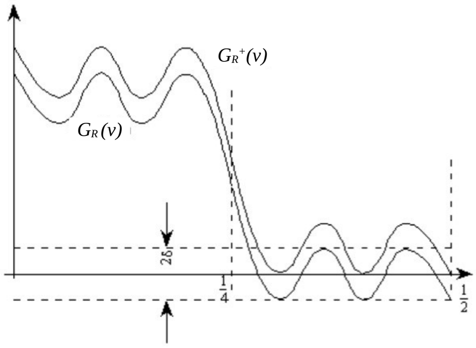
   — Matlab function `firpm` : `[v,delta] = firpm(odd order,2*[0,nu_c,.5,.5],[1 1 0 0])`

   — Python function `remez` : `v = scipy.signal.remez(even length,[0,nu_c,.5,.5],[1 0])`

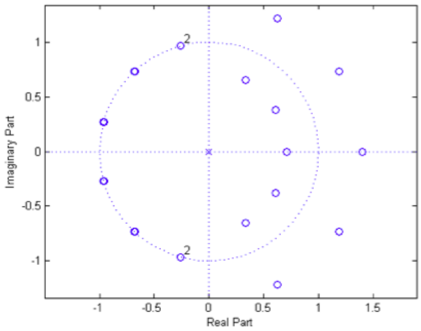
1.3 Raising and factorization of the half-band filter

We then want to determine the filter $h_0(n)$ by exploiting the relation $G_R(\nu) = \widetilde{H}_0(e^{2i\pi\nu})H_0(e^{2i\pi\nu})$.

4. Is the function $G_R(\nu)$, associated to the previously synthesized filter $g(n)$, nonnegative at all frequencies ? (you will look into its oscillations in the stop-band). Is it thus possible to factorize $G_R(\nu)$ in the form $G_R(\nu) = \widetilde{H}_0(e^{2i\pi\nu})H_0(e^{2i\pi\nu})$?
5. In order to get a filter $G^+(z)$ associated to a function $G^+_R(\nu)$ that is entirely nonnegative, we will raise $G(z)$ by defining $g^+(n) = g(n) + \varepsilon\,\delta_0(n - (N - 1))$, which is equivalent to $G^+_R(\nu) = G_R(\nu) + \varepsilon$ (cf. figure 3a). Thus if ε is chosen strictly greater than the amplitude of the oscillations of $G_R(\nu)$ in the stop-band, we get $G^+_R(\nu) > 0\ \forall\nu \in \mathbb{R}$. Moreover, the filter $g^+(n)$ obtained in this way still satisfies the *half-band* condition (for a slightly different value of c, which does not affect the perfect reconstruction condition).
6. $G^+(z)$ is a linear phase filter which has $2N-2$ roots, $N-1$ of which are located inside the unit circle, and the $N-1$ other ones are located outside the unit circle (cf. figure 3b). The factorization $G_R(\nu) =$



(a) Raising of the half-band filter.



(b) Factorization of the half-band filter.

FIGURE 3 – Raising and factorization of the half-band filter

$\widetilde{H}_0(e^{2i\pi\nu})H_0(e^{2i\pi\nu})$ has several solutions, among which we will choose the one with minimal phase : construct $H_0(z)$ as the N sample-long filter whose $N-1$ roots are equal to the ones of $G^+(z)$ located inside the unit circle, which guarantees the relation $G_R(\nu) = \widetilde{H}_0(e^{2i\pi\nu})H_0(e^{2i\pi\nu})$. For this purpose, you will note that the coefficients of the impulse response $h_0(n)$ and those of the impulse response $g^+(n)$ satisfy the following relation :

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & h_0(0) \\ 0 & 0 & 0 & \dots & 0 & h_0(0) & h_0(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & h_0(0) & h_0(1) & \dots & h_0(N-4) & h_0(N-3) & h_0(N-2) \\ h_0(0) & h_0(1) & h_0(2) & \dots & h_0(N-3) & h_0(N-2) & h_0(N-1) \end{bmatrix} \begin{bmatrix} h_0(0) \\ h_0(1) \\ \vdots \\ h_0(N-1) \end{bmatrix} = \begin{bmatrix} g^+(0) \\ g^+(1) \\ \vdots \\ g^+(N-1) \end{bmatrix}$$

The filter h_0 can therefore be calculated iteratively by using a fixed point algorithm that you will code and whose convergence you will test (at each iteration, you will solve a triangular linear system with N equations and N unknowns, whose matrix can be obtained using the function `hankel`). For the initialization, the choice $h_0 = [1, 0, \dots, 0]^\top$ guarantees the convergence to the desired solution (i.e. to a minimal phase filter). After each iteration, make sure to renormalize h_0 so that $\|h_0\|_2^2 = g^+(N-1)$.

1.4 Implementation of the CQF filter bank

7. Implement the CQF filter bank described above.
8. Test the perfect reconstruction on the provided signals.

2 Transmultiplexer

We will implement the transmultiplexer represented in figure 4 by means of the previous filters (if you were not able to finish exercise 1, you can use the simpler filters $F_0(z) = z^{-1}$, $F_1(z) = 1$, $H_0(z) = 1$ and $H_1(z) = z^{-1}$).

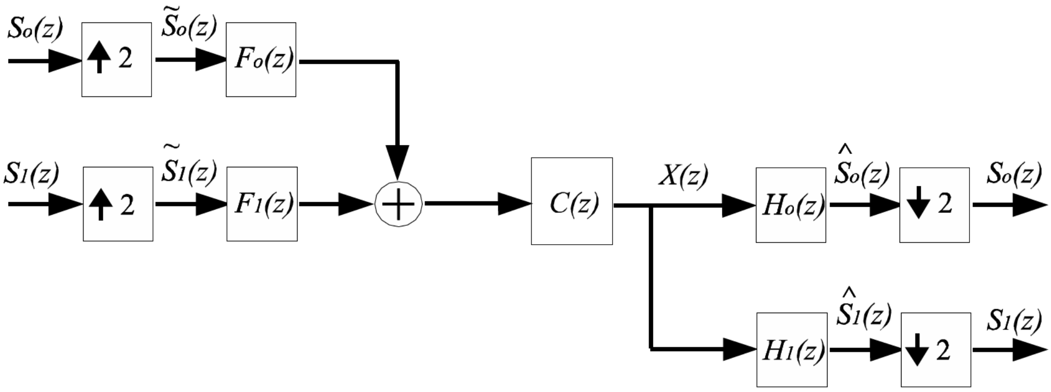


FIGURE 4 – Transmultiplexer.

1. As shown in the course, the transmultiplexer guarantees perfect reconstruction at the output when $C(z) = z^{-1}$. Mix two speech sounds and listen to the signal $X(z)$. Then check that the signals $S_0(z)$ and $S_1(z)$ are retrieved at the right outputs.
2. From now on, filter $C(z)$ will represent the transfer function of a transmission channel between the encoder and the decoder. It is uniformly equal to 1 if the channel is transparent, but in general, the transmission channel is imperfect, and its transfer function $C(z)$ is not constant. In order to simplify, let us assume that $C(z)$ is of the form $1 - \alpha z^{-1}$. In order to keep the perfect reconstruction property at the output of the transmultiplexer, we will have to introduce just after $C(z)$ a causal filter $D(z)$ such that $C(z)D(z) = dz^{-n_0}$, where n_0 is an odd number.
   (a) How to choose $D(z)$ if $\alpha = 0.9$? Implement and test the solution.
   (b) What problem do we encounter if $\alpha = 1.2$? Propose an approximate solution, then implement and test it.
3. If the channel transfer function $C(z)$ is unknown, propose and implement a method for estimating it from the output signals, by choosing appropriate filter $D(z)$ and input signals s_0 and s_1.