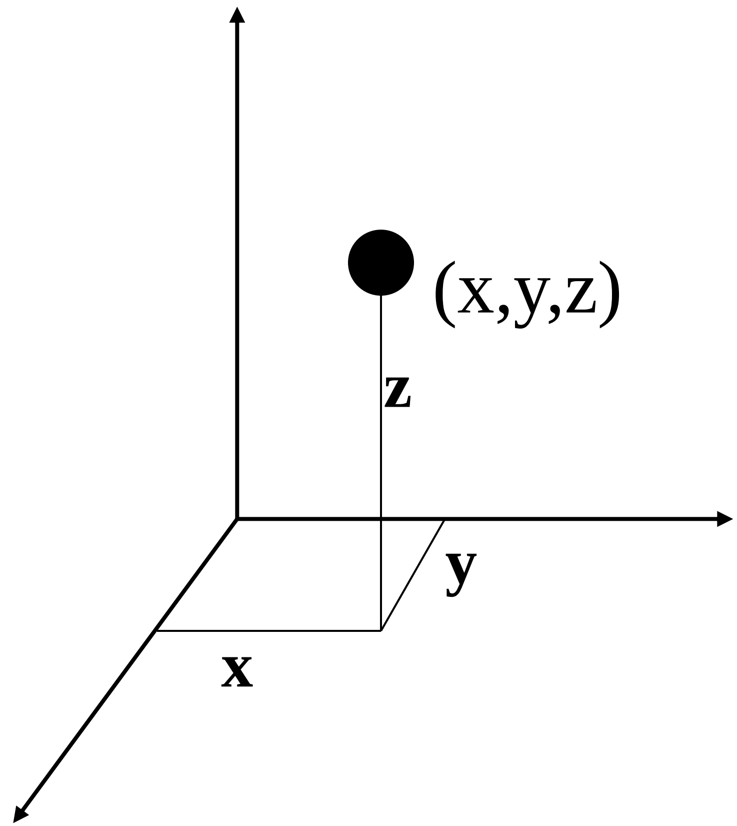
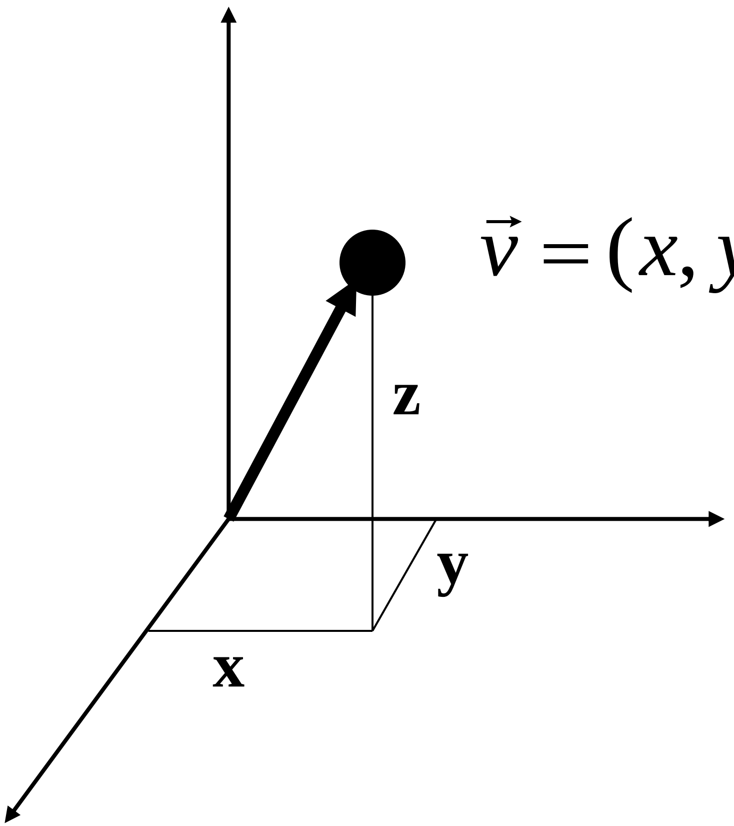
Linear Algebra Review

A point



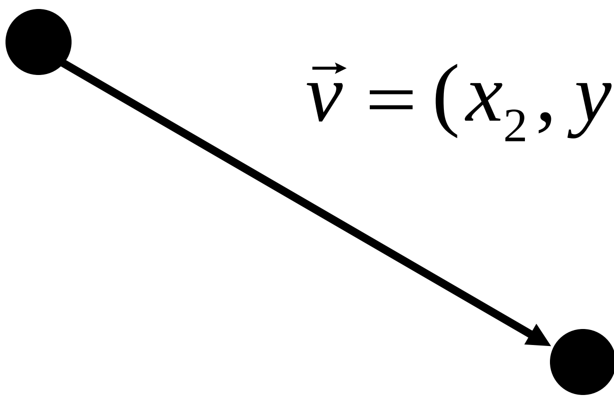
A vector



$$\vec{v} = (x, y, z)$$

Point + vector = Point

A=(x1,y1,z1)

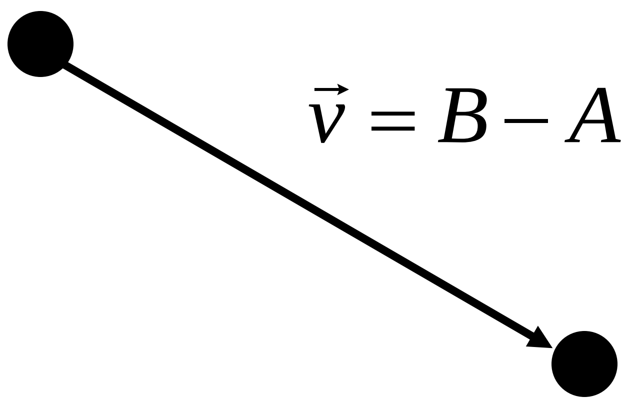


$$\vec{v} = (x_2, y_2, z_2)$$

$$B = A + \vec{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$

Point – point = vector

A=(x1,y1,z1)



$$\vec{v} = B - A = (x_2, y_2, z_2) - (x_1, y_1, z_1)$$

B=(x2,y2,z2)

Vector arithmetic

$$\vec{v}_1 = \begin{bmatrix} v_{1x} & v_{1y} & v_{1z} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} v_{2x} & v_{2y} & v_{2z} \end{bmatrix}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} v_{1x} + v_{2x} & v_{1y} + v_{2y} & v_{1z} + v_{2z} \end{bmatrix}$$

$$\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} v_{1x} - v_{2x} & v_{1y} - v_{2y} & v_{1z} - v_{2z} \end{bmatrix}$$

$$-\vec{v}_1 = \begin{bmatrix} -v_{1x} & -v_{1y} & -v_{1z} \end{bmatrix}$$

$$s\vec{v}_1 = \begin{bmatrix} sv_{1x} & sv_{1y} & sv_{1z} \end{bmatrix}$$

Vector norm

- The norm (length) of a vector is:

$$\left|\vec{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- Unit vector (norm=1.0)

$$\frac{\vec{v}}{\left|\vec{v}\right|}$$

Scalar product

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \sum a_i b_i$$

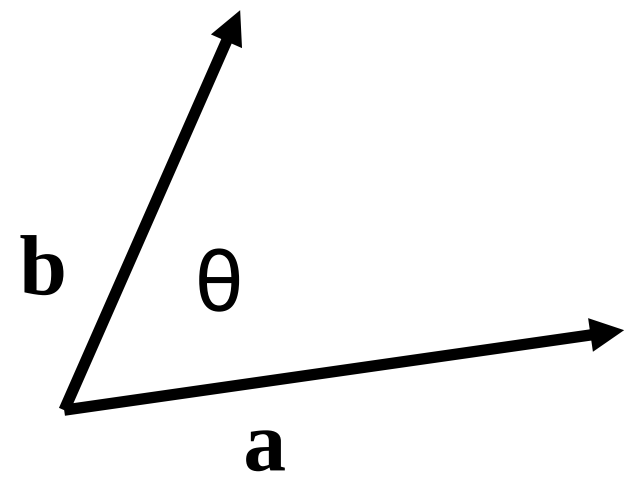
$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$$

Angle between two vectors

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right)$$

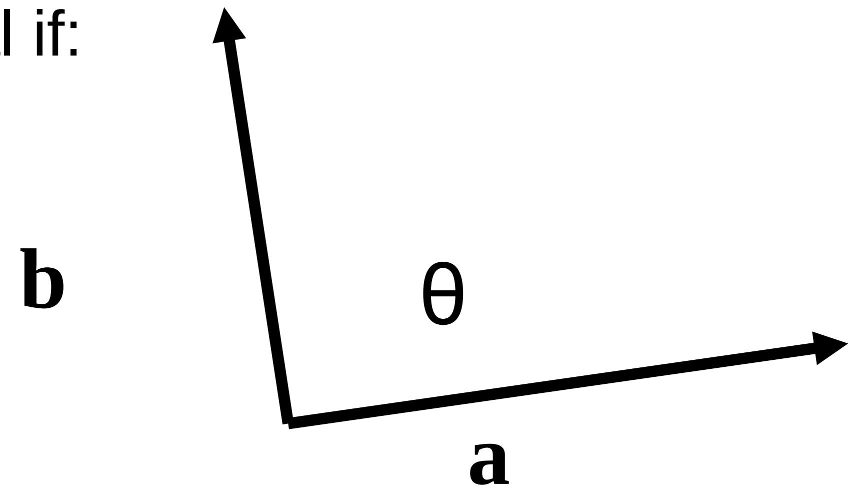
$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right)$$



Orthogonal vectors

Two vectors are orthogonal if:

$$\vec{a} \cdot \vec{b} = 0$$



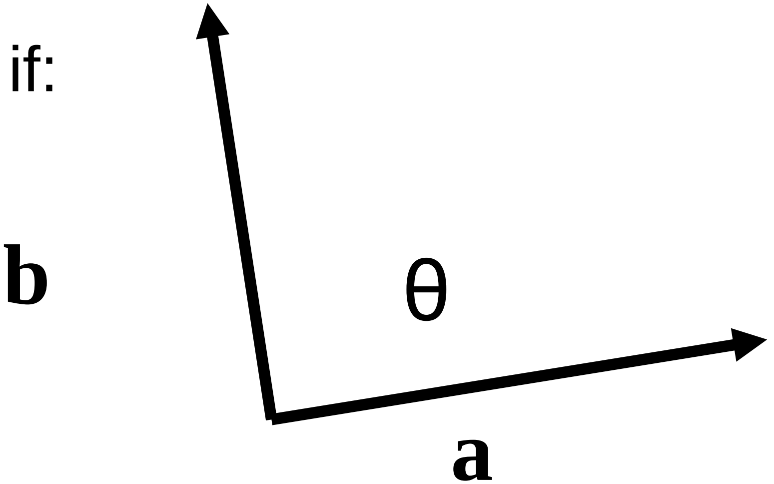
Orthonormal vectors

Two vectors are orthonormal if:

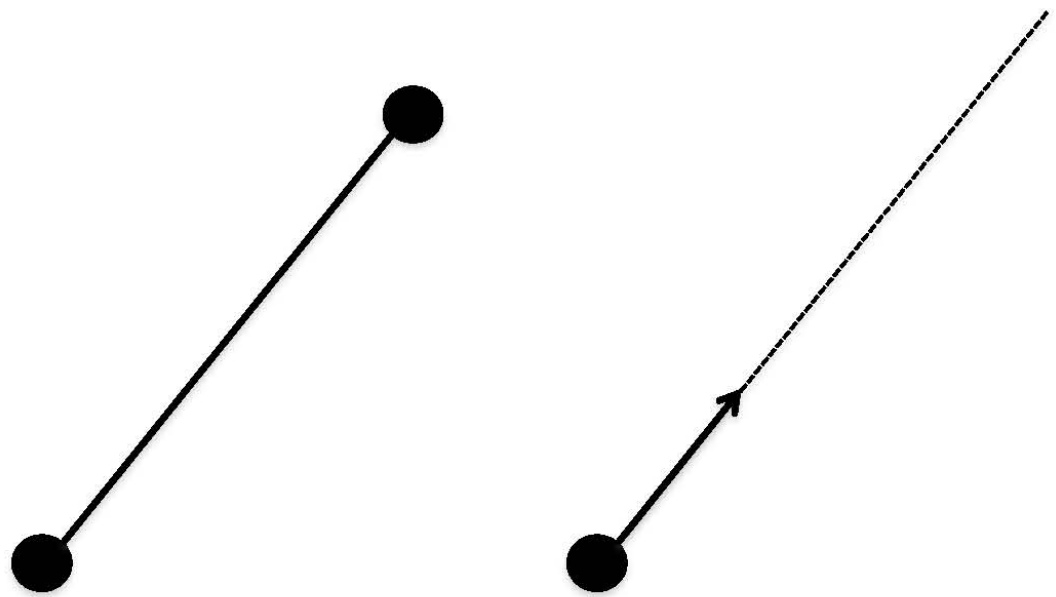
$$\vec{a} \cdot \vec{b} = 0$$

$$|\vec{a}| = 1$$

$$|\vec{b}| = 1$$



Lines



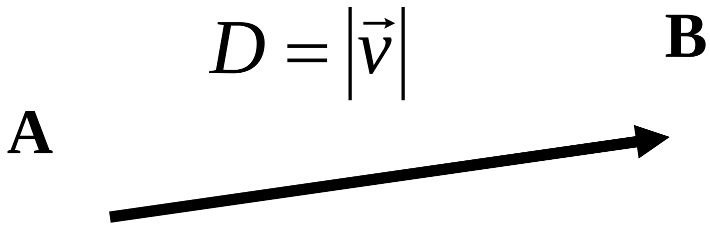
Line segment:
two endpoints

Ray (half-line):
origin + direction vector
$P = O + t * d, t > 0$

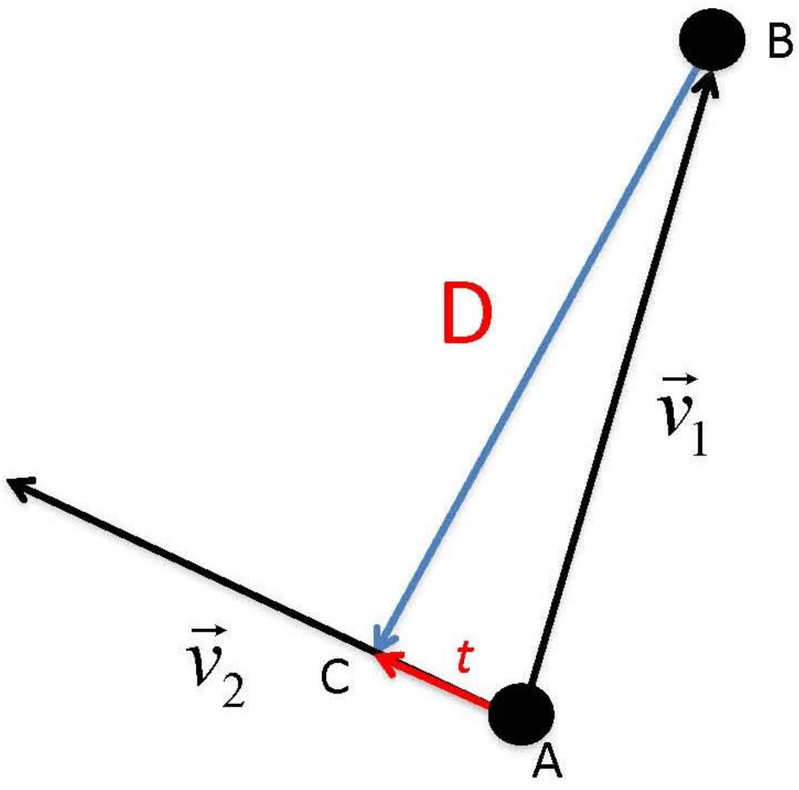
Distance between points

$$\vec{v} = B - A$$

$$D = \left|\vec{v}\right|$$



Distance from point to line



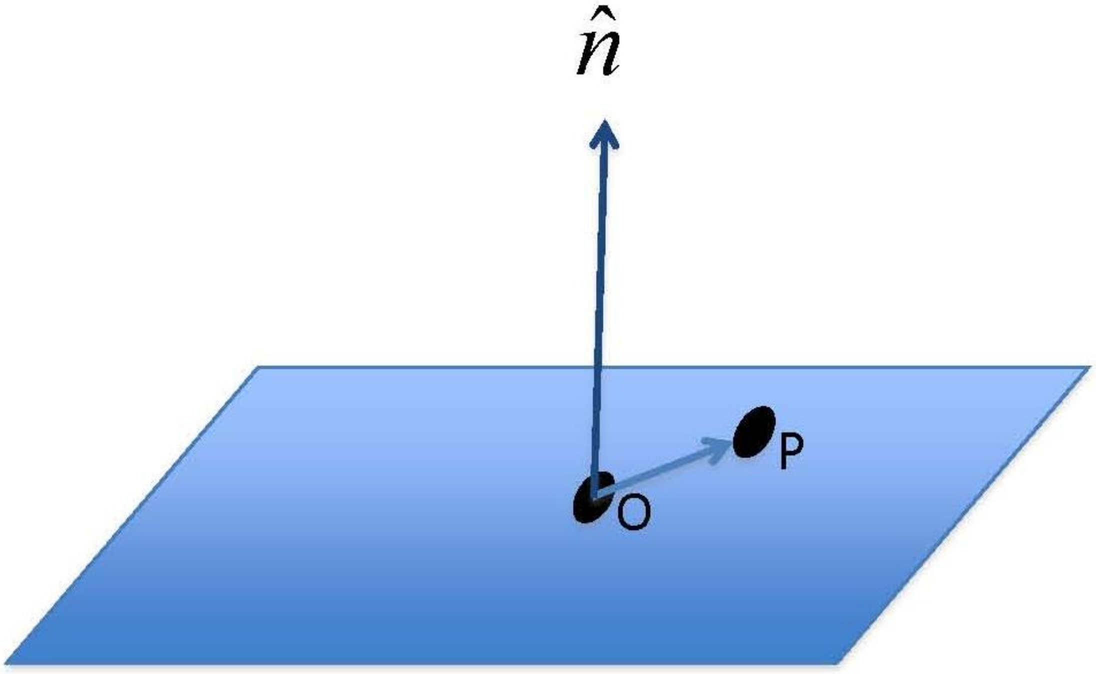
$$\vec{v} = B - A$$

$$t = \vec{v}_1 . \frac{\vec{v}_2}{\left|\vec{v}_2\right|}$$

$$C = A + t . \frac{\vec{v}_2}{\left|\vec{v}_2\right|}$$

$$D = \left|C - B\right|$$

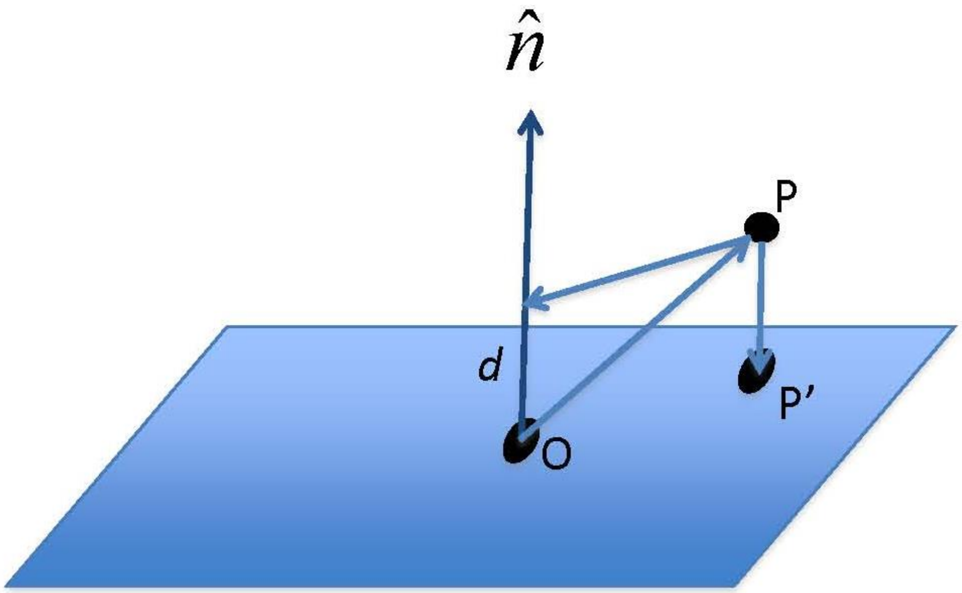
Plane equation



$$(P - O).\vec{n} = 0$$

$$|\vec{n}| = 1$$

Closest distance from point to plane

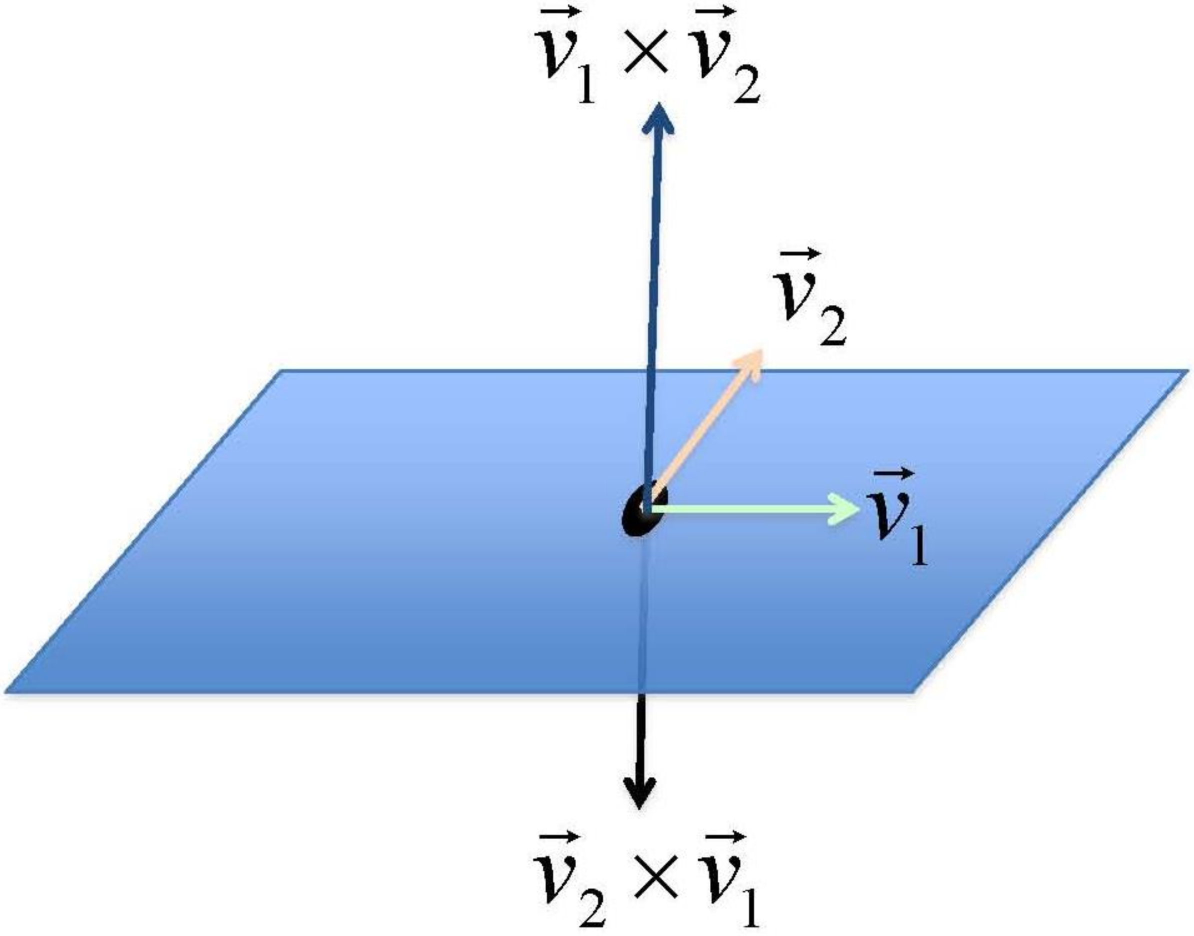


$$(P - O).\vec{n} = d$$

$$P^{'} = P - d\vec{n}$$

$$|\vec{n}| = 1$$

Cross product



Cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{bmatrix}$$

Matrix operations

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Associative: (AB)C = A(BC)

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Non-commutative: AB ≠ BA

Determinant

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Det[A] = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$Det[A] = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$Det[A] = a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$

Matrix inverse

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Matrix inverse

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{Det[A]} \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

Matrix inverse

Some properties:

$$A^{-1}A = AA^{-1} = I$$

$$C = AB \Rightarrow C^{-1} = B^{-1}A^{-1}$$

Matrix transpose

Transpose

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Properties

$$(AB)^T = B^T A^T$$
$$(A^{-1})^T = (A^T)^{-1}$$

Matrix and vectors

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

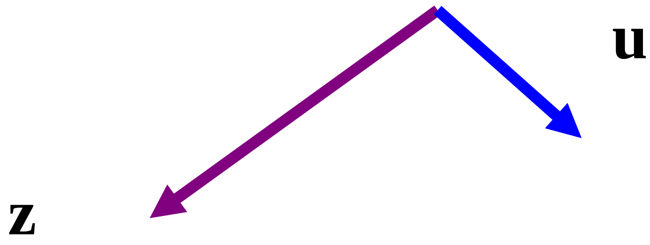
$$A.v = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} . \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_{21} + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_3 + a_{32}v_2 + a_{33}v_3 \end{pmatrix}$$

How to create an orthonormal basis

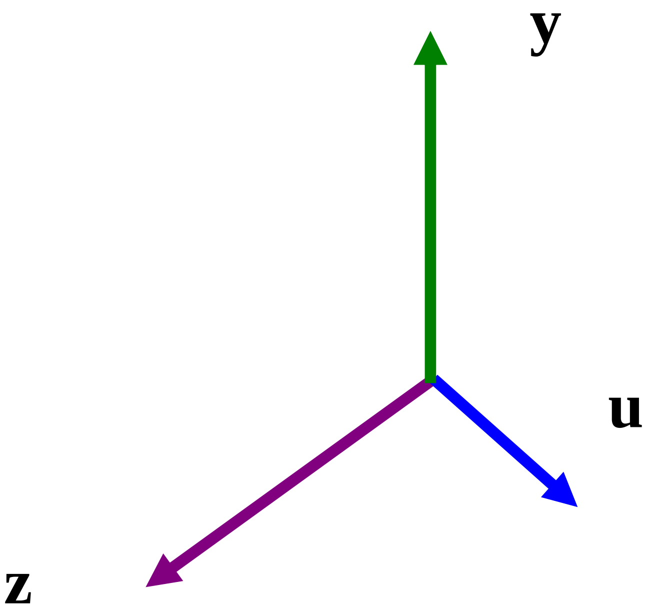
- Given two non-collinear and non-orthogonal vectors u, z
- Normalize (divide the vector by its norm) u and z
- Compute the cross product: $y = z \times u$
- Normalize y
- Compute the cross product: $x = y \times z$
- Normalize x

How to create an orthonormal basis

Normalize u and z

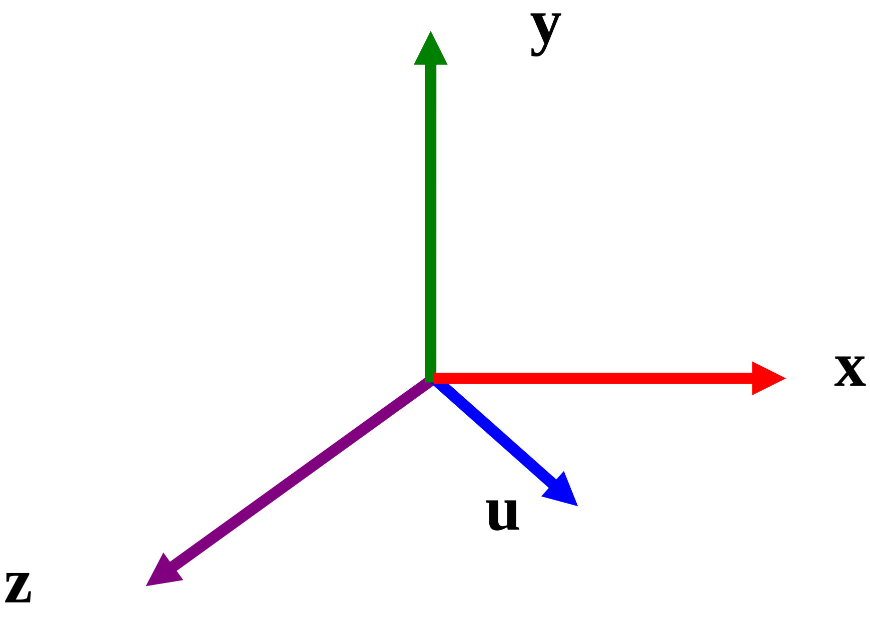


How to create an orthonormal basis



Compute $y = z \times u$
Normalize y

How to create an orthonormal basis



$x = y \times z$

Normalize x