CHAPTER 10

HIGHER ORDER DIFFERENTIAL EQUATIONS

In Chapter 9, we studied methods of solving special types of linear and nonlinear differential equations of the first order. In this chapter, we shall consider systematic methods for the solution of certain classes of differential equations of orders more than one.

LINEAR DIFFERENTIAL EQUATIONS

(10.1) Definition. A linear differential equation of order n in the dependent variable y and the independent variable x is of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x) \qquad (1)$$

where $a_0(x)$, $a_1(x)$, ..., $a_{n-1}(x)$, $a_n(x)$ and $F(x)$ are functions of the independent variable x only and $a_0(x)$ is not identically zero. Using primes, (1) is also written as

$$a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

We shall first study equations of the type

$$a_0\, y^{(n)} + a_1\, y^{(n-1)} + \dots + a_{n-1}\, y' + a_n\, y = F(x) \tag{2}$$

where $a_0, a_1, \dots, a_{n-1}, a_n$ are real constants. The equation (1) is with variable coefficients while (2) is with constant coefficients.

In order to solve (2), we shall first consider the equation

$$a_0\, y^{(n)} + a_1\, y^{(n-1)} + \dots + a_{n-1}\, y' + a_n\, y = 0 \tag{3}$$

The coefficient of $y^{(n)}$ may be made 1 by dividing throughout by a_0. The differential equation (3) is called **homogeneous** linear differential equation of order n. [The use of the word **homogeneous** here is quite different from the one already mentioned in **9.12**]. If $F(x)$ is not identically zero then (2) is called **nonhomogeneous** and (3) is called the **associated homogeneous equation of** (2).

(10.2) Definition. If $y_1(x), y_2(x), \dots, y_m(x)$ are m functions of an independent variable x and $c_1, c_2, \dots, c_m$ are constants, then

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_m y_m(x)$$

is called a **linear combination** of $y_1(x), y_2(x), \dots, y_m(x)$. We usually write $y_1, y_2, \dots, y_m$ instead of $y_1(x), y_2(x), \dots, y_m(x)$ when it is clear from the context that $y_1, y_2, \dots, y_m$ are functions of x.

As in vector spaces, the m functions $y_1, y_2, \dots, y_m$ are called **linearly dependent** if and only if, there exist constants $c_1, c_2, \dots, c_m$, at least one of which is non-zero, such that

$$c_1 y_1 + c_2 y_2 + \dots + c_m y_m = 0$$

The functions $y_1, y_2, \dots, y_m$ are called **linearly independent** if and only if, they are not linearly dependent, i.e., if and only if

$$c_1 y_1 + c_2 y_2 + \dots + c_m y_m = 0$$

implies $$c_1 = c_2 = \dots = c_m = 0$$

(10.3) Before investigating a solution of (2) of (**10.1**), we state the following facts :

(i) If $y_1, y_2, \dots, y_m$ are m solutions of (3), then any linear combination

$$c_1 y_1 + c_2 y_2 + \dots + c_m y_m$$

is also a solution of (3); $c_1, c_2, \dots, c_m$ being arbitrary constants.

(ii) Every homogeneous linear n th-order differential equation

$$a_0\, y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n\, y = 0$$

has n linearly independent solutions $y_1, y_2, \dots, y_n$.

(iii) If $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of (3), then any linear combination

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is the general solution of (3), $c_1, c_2, \ldots, c_n$ being arbitrary constants.

(iv) Let y_p be any particular solution of (2), i.e., y_p does not contain any constant then $y_c + y_p$ is the general solution of (2).

(i), (iii) and (iv) can be easily checked by actual substitutions in (3) and (2). Proof of (ii) is beyond the scope of this book.

Thus to find the general solution of (2), we have to find an independent set of n solutions of (3) so as to determine y_c and a particular solution y_p of (2) and then

$$y = y_c + y_p \tag{4}$$

is the general solution of (2). In the general solution (4) of (2), y_c is called the **Complementary Function** (**C. F.**) and y_p is called the **Particular Integral** (**P. I.**)

These statements are also true for the equation (1) of (**10.1**) with variable coefficients.

HOMOGENEOUS LINEAR EQUATIONS

(**10.4**) Consider the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{1}$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are real constants. To find a solution of (1) we shall try a judicious guess. The differential equation (1) requires a function y with the property that if y and its various derivatives are each multiplied by constants a_j and the resulting products are then added, the result will equal zero. This can only happen if a function is such that its various derivatives are constant multiples of itself. The exponential function $y = e^{mx}$, m being a constant, has such properties. Here we have

$$\frac{dy}{dx} = m e^{mx}, \quad \frac{d^2y}{dx^2} = m^2 e^{mx}, \quad \ldots, \quad \frac{d^n y}{dx^n} = m^n e^{mx}$$

Substituting in (1) we have

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1}\, m\, e^{mx} + a_n e^{mx} = 0$$

or
$$e^{mx}\left(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n\right) = 0$$

Since $e^{mx} \neq 0$, we have

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad (2)$$

Thus, $y = e^{mx}$ is a solution of (1) if and only if m is a solution of (2). Equation (2) is called the **characteristic** (or **auxilliary**) **equation** of the given differential equation (1). Observe that (2) can be obtained from (1) by merely replacing the k th derivative in (1) by m^k ($k = 0, 1, \ldots, n$). Three cases arise according as the roots of (2) are.

(I) real and distinct

(II) real and repeated

(III) complex.

Case I. Distinct Real Roots

Let $m_1, m_2, \ldots, m_n$

be n distinct real roots of (2). Then $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are n distinct solutions of (1). These n solutions are linearly independent. Hence the general solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Case II. Repeated Real Roots

In equation (1), writing $D \equiv \dfrac{d}{dx}$, we have

$$\left(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n\right) y = 0$$

or
$$[f(D)]\, y = 0$$

where
$$f(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n .$$

Note that if we write D for m in the characteristic equation (2) then it is the same as $f(D) = 0$. If $m_1, m_2, \ldots, m_n$ are the roots of $f(D) = 0$, then (1) may be written as

$$(D - m_1)(D - m_2) \ldots (D - m_n)\, y = 0$$

Let the root m_1 be repeated twice say $m_2 = m_1$. Then the general solution of (1) is

$$\begin{aligned} y &= (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \\ &= c_0 e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \end{aligned}$$

which contains only $n-1$ arbitrary constants and, therefore, is not the general solution. To obtain the general solution we proceed as follows :

The part of the general solution of (1) corresponding to the twice repeated root m_1 of (2) is solution of

$$(D-m_1)^2 y = 0$$

i.e., of

$$(D-m_1)(D-m_1)y = 0 \tag{3}$$

Let $(D-m_1)y = v$

Then (3) becomes

$$(D-m_1)v = 0$$

or

$$\frac{dv}{dx} - m_1 v = 0$$

Separating the variables,

$$\frac{dv}{v} = m_1\, dx$$

Therefore, $\ln v = m_1 x + k$, where k is a constant.

or

$$v = c_2 e^{m_1 x}$$

Replacing v we obtain

$$(D-m_1)y = c_2 e^{m_1 x}$$

or

$$\frac{dy}{dx} - m_1 y = c_2 e^{m_1 x} \tag{4}$$

which is a linear equation of order one.

$$\text{Its I. F.} = \exp\left(\int (-m_1)\,dx\right) = e^{-m_1 x}$$

Multiplying (4) by $e^{-m_1 x}$, we get

$$\frac{d}{dx}\left(ye^{-m_1 x}\right) = c_2$$

or

$$ye^{-m_1 x} = c_2 x + c_1$$

or

$$y = (c_1 + c_2 x)e^{m_1 x}$$

is the part of the general solution corresponding to the repeated root m_1. The general solution of (1) is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

In the same manner, if the characteristic equation (2) has the triple real root m_1, the corresponding part of the general solution of (1) is the solution of

$$(D - m_1)^3 y = 0$$

Proceeding as before, we can easily find

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$$

as the part of the general solution corresponding to this triple root m_1.

If the characteristic equation (2) has the real root m_1 occuring k times, then the part of the general solution of (1) corresponding to the k-fold repeated root m_1 is

$$y = (c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1})\, e^{m_1 x}.$$

Case III. Complex Roots

Suppose the characteristic equation has the complex number $a + ib$ as a non-repeated root. Since coefficients of (1) are real, the conjugate complex number $a - ib$ is also a non-repeated root. The corresponding part of the general solution is

$$y = k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x}$$

where k_1 and k_2 are arbitrary constants.

i.e.,
$$\begin{aligned} y &= e^{ax}\left[k_1 e^{ibx} + k_2 e^{-ibx}\right] \\ &= e^{ax}\left[k_1(\cos bx + i \sin bx) + k_2(\cos bx - i \sin bx)\right] \\ &= e^{ax}\left[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx\right] \\ &= e^{ax}\left[c_1 \sin bx + c_2 \cos bx\right] \end{aligned}$$

where $c_1 = i(k_1 - k_2)$, $c_2 = k_1 + k_2$ are two arbitrary constants.

If $a + ib$ and $a - ib$ are conjugate complex roots, each repeated k times, then the corresponding part of the general solution of (1) may be written as

$$\begin{aligned} y = e^{ax}\big[&(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1})\sin bx \\ &+ (c_{k+1} + c_{k+2} x + \dots + c_{2k} x^{k-1})\cos bx\big]. \end{aligned}$$

Example 1. Solve: $(D^2 + 4D + 3)y = 0$

Solution The characteristic equation is

$$D^2 + 4D + 3 = 0$$

with roots

$$D = -1, -3$$

Hence the general solution of the given equation is

$$y = c_1 e^{-x} + c_2 e^{-3x}.$$

Example 2. Solve: $(D^3 - 5D^2 + 7D - 3)y = 0$

Solution. The characteristic equation is

$$D^3 - 5D^2 + 7D - 3 = 0 \qquad (1)$$

$D = 1$ is a solution of this equation. The other two roots can be found from the second factor of

$$(D - 1)(D^2 - 4D + 3) = 0$$

Hence all roots of (1) are $D = 1, 1, 3.$

The general solution is

$$y = (c_1 + c_2 x)e^x + c_3 e^{3x}.$$

Example 3. Solve: $(D^3 - D^2 + D - 1)y = 0$

Solution. The characteristic equation is

$$D^3 - D^2 + D - 1 = 0$$

By inspection, $D = 1$ is a root of this equation.

Now, $\quad D^3 - D^2 + D - 1 = (D - 1)(D^2 + 1) = 0$

Hence the other two roots are

$$D = \pm i = a + ib \text{ with } a = 0, b = 1$$

The general solution is

$$y = c_1 e^x + e^{0x}(c_2 \sin x + c_3 \cos x)$$

or

$$y = c_1 e^x + c_2 \sin x + c_3 \cos x.$$

Example 4. Solve: $(D^2 + D - 12)y = 0,$

where $y(2) = 2, y'(2) = 0$

Solution. The characteristic equation is

$$D^2 + D - 12 = 0$$

with roots

$$D = 3, -4$$

Hence the general solution of the given equation is

$$y = c_1 e^{3x} + c_2 e^{-4x}$$

Since the initial conditions are given at $x = 2$, we rewrite the general solution in the form

$$y = k_1 e^{3(x-2)} + k_2 e^{-4(x-2)}$$

where $\quad k_1 = c_1 e^6, \; k_2 = c_2 e^{-8}$

$$y(2) = 2 = k_1 + k_2 \qquad (1)$$

$$y' = \frac{dy}{dx} = 3k_1 e^{3(x-2)} - 4k_2 e^{-4(x-2)}$$

$$y'(2) = 0 = 3k_1 - 4k_2 \qquad (2)$$

Solving (1) and (2), we get

$$k_1 = \frac{8}{7}, \; k_2 = \frac{6}{7}$$

Hence the solution of the differential equation satisfying the given conditions is

$$y = \frac{8}{7} e^{3(x-2)} + \frac{6}{7} e^{-4(x-2)}.$$

Example 5. Solve: $(D^2 + 4D + 5) = 0, \; y(0) = 1, \; y'(0) = 0$

Solution. The characteristic equation is

$$D^2 + 4D + 5 = 0$$

which has roots $\quad D = -2 \pm i$

The general solution is

$$y = e^{-2x}(c_1 \sin x + c_2 \cos x)$$

Applying the given conditions, we have

$$y(0) = 1 = c_2$$

$$y' = -2e^{-2x}(c_1 \sin x + c_2 \cos x) + e^{-2x}(c_1 \cos x - c_2 \sin x)$$

$$y'(0) = 0 = -2c_2 + c_1, \quad \text{giving } c_1 = 2$$

Hence the required solution is

$$y = e^{-2x}(2 \sin x + \cos x).$$

Example 6. Solve : $(D^3 - 3D^2 + 4)y = 0$,

$$y(0) = 1,\ y'(0) = -8,\ y''(0) = -4$$

Solution. The characteristic equation is

$$D^3 - 3D^2 + 4 = 0$$

or $$(D + 1)(D^2 - 4D + 4) = 0$$

Therefore, $$D = -1, 2, 2$$

The general solution is

$$y = c_1 e^{-x} + e^{2x}(c_2 + c_3 x)$$

Now, $$y' = -c_1 e^{-x} + 2e^{2x}(c_2 + c_3 x) + c_3 e^{2x}$$

$$y'' = c_1 e^{-x} + 4e^{2x}(c_2 + c_3 x) + 4c_3 e^{2x}$$

Applying initial conditions, we get

$$y(0) = 1 = c_1 + c_2 \qquad (1)$$

$$y'(0) = -8 = -c_1 + 2c_2 + c_3 \qquad (2)$$

$$y''(0) = -4 = c_1 + 4c_2 + 4c_3 \qquad (3)$$

Multiplying (2) by -4 and adding to (3), we have

$$28 = 5c_1 - 4c_2 \qquad (4)$$

Multiplying (1) by 4 and adding to (4) yields

$$9c_1 = 32 \quad \text{or} \quad c_1 = \frac{32}{9}$$

Therefore, $c_2 = 1 - \dfrac{32}{9} = -\dfrac{23}{9}$ and $c_3 = \dfrac{6}{9}$

The required solution is

$$y = \frac{32}{9} e^{-x} + e^{2x}\left(\frac{-23}{9} + \frac{6}{9}x\right)$$

$$= \frac{1}{9}\left[32e^{-x} - 23e^{2x} + 6xe^{2x}\right].$$

EXERCISE 10.1

Solve :

1. $(9D^2 - 12D + 4)y = 0$
2. $(75D^2 + 50D + 12)y = 0$
3. $(D^3 - 4D^2 + D + 6)y = 0$
4. $(D^3 + D^2 + D + 1)y = 0$
5. $(D^3 - 6D^2 + 12D - 8)y = 0$
6. $(D^3 - 6D^2 + 3D + 10)y = 0$
7. $(D^3 - 27)y = 0$
8. $(4D^4 - 4D^3 - 3D^2 + 4D - 1)y = 0$
9. $(D^4 + 2D^3 - 2D^2 - 6D + 5)y = 0$
10. $(D^4 - 5D^3 + 6D^2 + 4D - 8)y = 0$
11. $(D^4 - 4D^3 - 7D^2 + 22D + 24)y = 0$
12. $(D^4 + 4)y = 0$
13. $(D^4 - D^3 - 3D^2 + D + 2)y = 0$
14. $(16D^6 + 8D^4 + D^2)y = 0$
15. $(D^4 + 6D^3 + 15D^2 + 20D + 12)y = 0$
16. $(D^2 + 8D - 9)y = 0;\ y(1) = 1,\ y'(1) = 0$
17. $(D^2 + 6D + 9)y = 0;\ y(0) = 2,\ y'(0) = -3$
18. $(D^2 + 6D + 13)y = 0;\ y(0) = 3,\ y'(0) = -1$
19. $(D^3 - 6D^2 + 11D - 6)y = 0,\ y(0) = 0 = y'(0),\ y''(0) = 2$
20. $(D^4 - D^3)y = 0;\ y(0) = y'(0) = 1,\ y''(1) = 3e,\ y'''(1) = e.$