Exercise 3.2

1. Consider the universal set $U = \{x|x\ \textit{is multiple of}\ 2\ \wedge 0 < x \leq 30\}$, $A = \{x|x\ \textit{is a multiple of}\ 6\}$ and $B = \{x : x\ \textit{is a multiple of}\ 8\}$

$$U = \{x|x\ \text{is multiple of}\ 2\ \wedge 0 < x \leq 30\}$$
$$U = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30\}$$

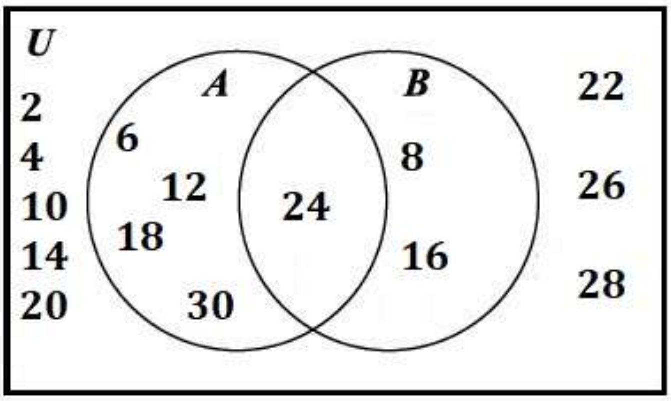
(i) List all elements of sets A and B in tabular form

$$A = \{x|x\ \text{is a multiple of}\ 6\}$$
$$A = \{6,12,18,24,30\}$$
$$B = \{x : x\ \text{is a multiple of}\ 8\}$$
$$B = \{8,16,24\}$$

(ii) Find $A \cap B$

$$A \cap B = \{6,12,18,24,30\} \cap \{8,16,24\}$$
$$= \{24\}$$

(iii) Draw a Venn Diagram



2. Let $U = \{x|x\ \textit{is an integer}\ \wedge 0 < x \leq 150\}$, $G = \{x|x = 2^m\ \textit{for integer}\ m \wedge 0 \leq m \leq 12\}$ and $H = \{x|x\ \textit{is a square}\}$

$$U = \{x|x\ \text{is an integer}\ \wedge 0 < x \leq 150\}$$
$$U = \{1,2,3, \dots ,150\}$$

(i) List all elements of sets G and H in tabular form

$$G = \{x|x = 2^m\ \text{for integer}\ m \wedge\ 0 \leq m \leq 12\}$$
$$G = \{1,2,4,8,16,32,64,128\}$$
$$H = \{x|x\ \text{is a square}\}$$
$$H = \{1,4,9,16,25,36,49,64,81,100,121,144\}$$

Note: For G $\textit{Compute}\ 2^m \leq 150$ and for H $\textit{Compute}\ k^2 \leq 150$

(ii) Find $G \cup H$

$$G \cup H = \{1,2,4,8,16,32,64,128\} \cup \{1,4,9,16,25,36,49,64,81,100,121,144\}$$
$$= \{1,2,4,8,9,16,25,32,36,49,64,81,100,121,128,144\}$$

(iii) Find $G \cap H$

$$G \cap H = \{1,2,4,8,16,32,64,128\} \cap \{1,4,9,16,25,36,49,64,81,100,121,144\}$$
$$= \{1,4,16,64\}$$

3. Consider the set $P = \{x|x\ \textit{is a prime number}\ \wedge 0 < x \leq 20\}$ and $Q = \{x|x\ \textit{is a divisor of}\ 210\ \wedge 0 < x \leq 20\}$

$$P = \{x|x\ \text{is a prime number}\ \wedge 0 < x \leq 20\}$$
$$P = \{2,3,5,7,11,13,17,19\}$$

$$Q = \{x|x \text{ is a divisor of } 210 \wedge 0 < x \leq 20\}$$
$$Q = \{1,2,3,5,6,7,10,14,15\}$$

(i) Find $P \cap Q$

$$\begin{aligned} P \cap Q &= \{2,3,5,7,11,13,17,19\} \cap \{1,2,3,5,6,7,10,14,15\} \\ &= \{2,3,5,7\} \end{aligned}$$

(ii) Find $P \cup Q$

$$\begin{aligned} P \cup Q &= \{2,3,5,7,11,13,17,19\} \cup \{1,2,3,5,6,7,10,14,15\} \\ &= \{1,2,3,5,6,7,10,11,13,14,15,17,19\} \end{aligned}$$

4. Verify the commutative properties of union and intersection for the following pairs of set.
(i) $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 6, 8, 10\}$

Commutative Property of Union: $A \cup B = B \cup A$

$$\begin{aligned} L.H.S &= A \cup B \\ &= \{1,2,3,4,5\} \cup \{4,6,8,10\} \\ &= \{1,2,3,4,5,6,8,10\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cup A \\ &= \{4,6,8,10\} \cup \{1,2,3,4,5\} \\ &= \{1,2,3,4,5,6,8,10\} \end{aligned}$$

Hence
$$L.H.S = R.H.S$$

Commutative Property of Intersection: $A \cap B = B \cap A$

$$\begin{aligned} L.H.S &= A \cap B \\ &= \{1,2,3,4,5\} \cap \{4,6,8,10\} \\ &= \{4\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cap A \\ &= \{4,6,8,10\} \cap \{1,2,3,4,5\} \\ &= \{4\} \end{aligned}$$

Hence
$$L.H.S = R.H.S$$

(ii) N, Z

Let $A = N = \{1,2,3, \dots\}$ and $B = Z = \{0, \pm1, \pm2, \pm3, \dots\}$

Commutative Property of Union: $A \cup B = B \cup A$

$$\begin{aligned} L.H.S &= A \cup B \\ &= \{1,2,3, \dots\} \cup \{0, \pm1, \pm2, \pm3, \dots\} \\ &= \{0, \pm1, \pm2, \pm3, \dots\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cup A \\ &= \{0, \pm1, \pm2, \pm3, \dots\} \cup \{1,2,3, \dots\} \\ &= \{0, \pm1, \pm2, \pm3, \dots\} \end{aligned}$$

Hence
$$L.H.S = R.H.S$$

Commutative Property of Intersection: $A \cap B = B \cap A$

$$\begin{aligned} L.H.S &= A \cap B \\ &= \{1,2,3, \dots\} \cap \{0, \pm1, \pm2, \pm3, \dots\} \\ &= \{1,2,3, \dots\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cap A \\ &= \{4,6,8,10\} \cap \{0, \pm1, \pm2, \pm3, \dots\} \end{aligned}$$

Alternate:

Let, $A = N$ and $B = Z$

Union

$$A \cup B = B \cup A$$
$$N \cup Z = Z \cup N$$
$$Z = Z$$

Hence Proved

Intersection

$$A \cap B = B \cap A$$
$$N \cap Z = Z \cap N$$
$$N = N$$

Hence Proved

$$= \{1,2,3,\ldots\}$$

Hence

$$L.H.S = R.H.S$$

(iii) $\boldsymbol{A = \{x|x \in R \wedge x \geq 0\}, B = R}$

Commutative Property of Union: $\boldsymbol{A \cup B = B \cup A}$

$$\begin{aligned} L.H.S &= A \cup B \\ &= \{x|x \in R \wedge x \geq 0\} \cup R \\ &= R \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cup A \\ &= R \cup \{x|x \in R \wedge x \geq 0\} \\ &= R \end{aligned}$$

Hence

$$L.H.S = R.H.S$$

Commutative Property of Intersection: $\boldsymbol{A \cap B = B \cap A}$

$$\begin{aligned} L.H.S &= A \cap B \\ &= \{x|x \in R \wedge x \geq 0\} \cap R \\ &= \{x|x \in R \wedge x \geq 0\} \end{aligned}$$

$$\begin{aligned} R.H.S &= B \cap A \\ &= R \cap \{x|x \in R \wedge x \geq 0\} \\ &= \{x|x \in R \wedge x \geq 0\} \end{aligned}$$

Hence

$$L.H.S = R.H.S$$

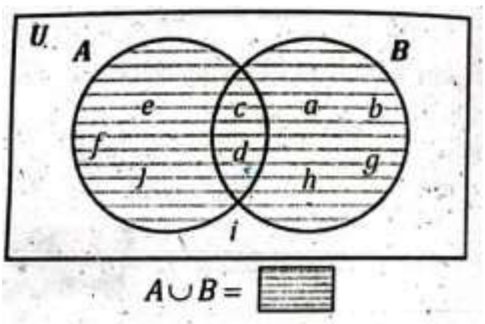
5. Let $\boldsymbol{U = \{a,b,c,d,e,f,g,h,i,j\}}$, $\boldsymbol{A = \{a,b,c,d,g,h\}}$, $\boldsymbol{B = \{c,d,e,f,j\}}$, Verify De Morgan's Laws for these sets. Draw Venn Diagram.

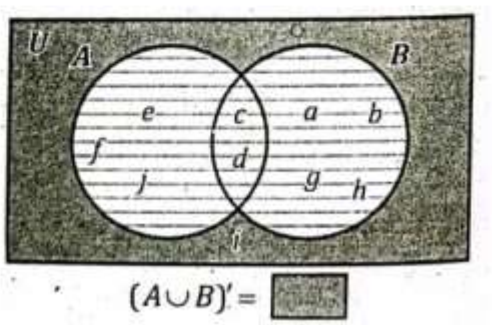
De Morgan's Laws: (i) $\boldsymbol{(A \cup B)' = A' \cap B'}$ (ii) $\boldsymbol{(A \cap B)' = A' \cup B'}$

(i) $\boldsymbol{(A \cup B)' = A' \cap B'}$

$$\begin{aligned} A \cup B &= \{a,b,c,d,g,h\} \cup \{c,d,e,f,j\} \\ &= \{a,b,c,d,e,f,g,h,j\} \end{aligned}$$

$$\begin{aligned} L.H.S &= (A \cup B)' \\ &= U - (A \cup B) \\ &= \{a,b,c,d,e,f,g,h,i,j\} - \{a,b,c,d,e,f,g,h,j\} \\ L.H.S &= \{i\} \end{aligned}$$



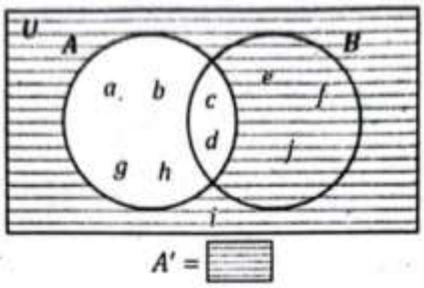


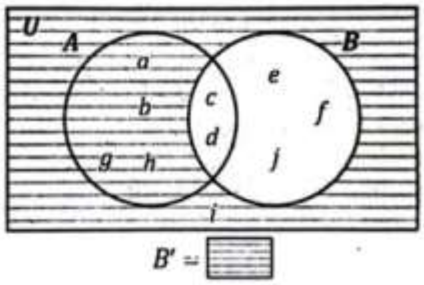
$$\begin{aligned} A' &= U - A \\ A' &= \{a,b,c,d,e,f,g,h,i,j\} - \{a,b,c,d,g,h\} \\ A' &= \{e,f,i,j\} \end{aligned}$$

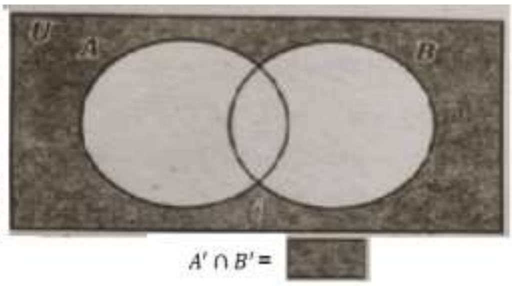
$$B' = U - B$$

$$B' = \{a, b, c, d, e, f, g, h, i, j\} - \{c, d, e, f, j\}$$
$$B' = \{a, b, g, h, i\}$$

$$R.H.S = A' \cap B'$$
$$= \{e, f, i, j\} \cap \{a, b, g, h, i\}$$
$$R.H.S = \{i\}$$



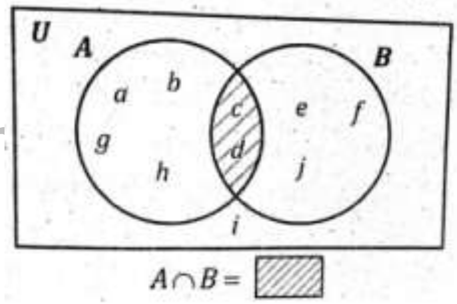


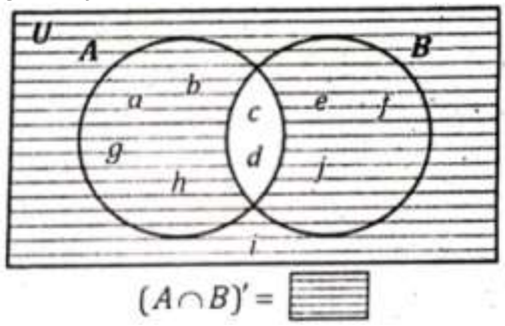


(i) $(\boldsymbol{A} \cap \boldsymbol{B})' = \boldsymbol{A}' \cup \boldsymbol{B}'$

$$A \cap B = \{a, b, c, d, g, h\} \cap \{c, d, e, f, j\}$$
$$= \{c, d\}$$

$$L.H.S = (A \cap B)'$$
$$= U - (A \cap B)$$
$$= \{a, b, c, d, e, f, g, h, i, j\} - \{c, d\}$$
$$L.H.S = \{a, b, e, f, g, h, i, j\}$$

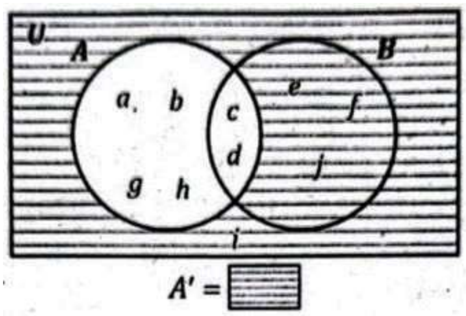


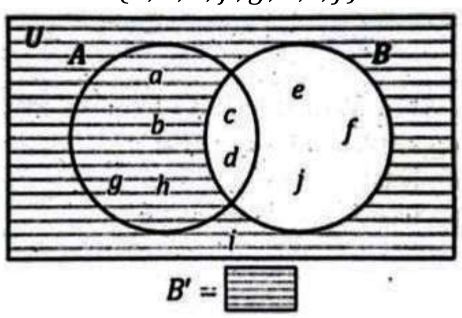


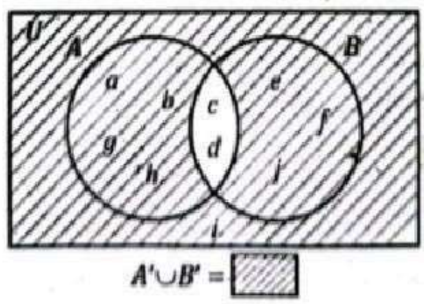
$$A' = U - A$$
$$A' = \{a, b, c, d, e, f, g, h, i, j\} - \{a, b, c, d, g, h\}$$
$$A' = \{e, f, i, j\}$$

$$B' = U - B$$
$$B' = \{a, b, c, d, e, f, g, h, i, j\} - \{c, d, e, f, j\}$$
$$B' = \{a, b, g, h, i\}$$

$$R.H.S = A' \cup B'$$
$$= \{e, f, i, j\} \cup \{a, b, g, h, i\}$$
$$R.H.S = \{a, b, e, f, g, h, i, j\}$$







HENCE Proved

$$L.H.S = R.H.S$$

6. If $U=\{1,2,3,\dots,20\}$ and $A=\{1,3,5,\dots,19\}$, verify the followings.
(i) $A\cup A'=U$

$$\begin{aligned} A' &= U-A \\ &= \{1,2,3,\dots,20\}-\{1,3,5,\dots,19\} \\ A' &= \{2,4,6,\dots,20\} \end{aligned}$$

$$\begin{aligned} L.H.S &= A\cup A' \\ &= \{1,3,5,\dots,19\}\cup\{2,4,6,\dots,20\} \\ &= \{1,2,3,\dots,20\} \\ &= U \\ &= R.H.S \end{aligned}$$

(ii) $A\cap U=A$

$$\begin{aligned} L.H.S &= A\cap U \\ &= \{1,3,5,\dots,19\}\cap\{1,2,3,\dots,20\} \\ &= \{1,3,5,\dots,19\} \\ &= A \\ &= R.H.S \end{aligned}$$

(iii) $A\cap A'=\emptyset$

$$\begin{aligned} A' &= U-A \\ &= \{1,2,3,\dots,20\}-\{1,3,5,\dots,19\} \\ A' &= \{2,4,6,\dots,20\} \end{aligned}$$

$$\begin{aligned} L.H.S &= A\cap A' \\ &= \{1,3,5,\dots,19\}\cap\{2,4,6,\dots,20\} \\ &= \{\ \} \\ &= \emptyset \\ &= R.H.S \end{aligned}$$

7. In a class of 55 students, 34 like to play cricket and 30 like to play hockey. Also, each student likes to play at least one of the two games. How many students like to play both games?

Let

$$C = Set\ of\ students\ who\ like\ cricket$$
$$H = Set\ of\ students\ who\ like\ hockey$$

According to question: $n(C\cup H)=55,\ n(C)=34,\ n(H)=30,\ n(C\cap H)=?$

Using Inclusion-Exclusion Principle,

$$\begin{aligned} n(C\cup H) &= n(C)+n(H)-n(C\cap H) \\ 55 &= 34+30-n(C\cap H) \\ 55 &= 64-n(C\cap H) \\ n(C\cap H) &= 64-55 \\ \boldsymbol{n(C\cap H)} &= \boldsymbol{9\ students} \end{aligned}$$

8. In a group of 500 employees, 250 can speak Urdu, 150 can speak English, 50 can speak Punjabi, 40 can speak Urdu and English, 30 can speak both English and Punjabi and 10 can speak Urdu and Punjabi. How many can speak all the three languages?

Let

$$U = Set\ of\ employees\ who\ can\ speak\ Urdu$$
$$E = Set\ of\ employees\ who\ can\ speak\ English$$
$$P = Set\ of\ employees\ who\ can\ speak\ Punjabi$$

According to question: $n(E\cup U\cup P)=500,\ \ n(U)=250,\ \ n(E)=150,\ \ n(P)=50,\ \ n(E\cap U)=40,$ $n(E\cap P)=30,\ n(U\cap P)=10,\ n(E\cap U\cap P)=?$

Using Inclusion-Exclusion Principle,

$$n(E \cup U \cup P) = n(E) + n(U) + n(P) - n(E \cap U) - n(E \cap P) - n(U \cap P) + n(E \cap U \cap P)$$
$$500 = 150 + 250 + 50 - 40 - 30 - 10 + n(E \cap U \cap P)$$
$$500 = 370 + n(E \cap U \cap P)$$
$$500 - 370 = n(E \cap U \cap P)$$
$$130 = n(E \cap U \cap P)$$
$$\boldsymbol{n(E \cap U \cap P) = 130\ employees}$$

9. In sports event, 19 people wear blue shirts, 15 people wear green shirts, 3 wear blue and green shirts, 4 wear a cap and blue shirts, 2 wear a cap and green shirts. The total number of people with either a blue or green shirt or a cap is 34. How many people are wearing caps?

Let

$$B = Set\ of\ people\ wearing\ blue\ shirts$$
$$G = Set\ of\ people\ wearing\ green\ shirts$$
$$C = Set\ of\ people\ wearing\ caps$$

According to question: $n(B) = 19$, $n(G) = 15$, $n(B \cap G) = 3$, $n(C \cap B) = 4$, $n(C \cap G) = 2$, $n(B \cup G \cup C) = 34$, Assume $n(E \cap U \cap P) = 0$ (no one wears all three), $n(C) =?$

Using Inclusion-Exclusion Principle,

$$n(B \cup G \cup C) = n(B) + n(G) + n(C) - n(B \cap G) - n(B \cap C) - n(G \cap C) + n(B \cap G \cap C)$$
$$34 = 19 + 15 + n(C) - 3 - 4 - 2 + 0$$
$$34 - 19 - 15 + 3 + 4 + 2 = n(C)$$
$$9 = n(C)$$
$$\boldsymbol{n(C) = 9\ people}$$

10. In a training session, 17 participants have laptops, 11 have tablets, 9 have laptops and tablets, 6 have laptops and books and 4 have both tablets and books. Eight participants have all three items. The total number of participants with laptops, tablets or books is 35. How many participants have books?

Let

$$L = Set\ of\ participants\ with\ laptops$$
$$T = Set\ of participants\ with\ tablets$$
$$B = Set\ of\ participants\ with\ books$$

According to question: $n(L) = 17$, $n(T) = 11$, $n(L \cap T) = 9$, $n(L \cap B) = 6$, $n(T \cap B) = 4$, $n(L \cap T \cap B) = 8$, $n(L \cup T \cup B) = 35$, $n(B) = ?$

Using Inclusion-Exclusion Principle,

$$n(L \cup T \cup B) = n(L) + n(T) + n(B) - n(L \cap T) - n(L \cap B) - n(T \cap B) + n(L \cap T \cap B)$$
$$35 = 17 + 11 + n(B) - 9 - 6 - 4 + 8$$
$$35 - 17 - 11 + 9 + 6 + 4 - 8 = n(C)$$
$$18 = n(C)$$
$$\boldsymbol{n(C) = 18\ participants}$$

11. A shopping mall has 150 employees labelled 1 to 150, representing the Universal set U. The employees fall into the following categories:

- **Set A: 40 employees with a salary range of $30k$ - $40k$, labelled from 50 to 89.**
- **Set B: 50 employees with a salary range of $50k$ - $80k$, labelled from 101 to 150.**
- **Set C: 60 employees with a salary range of $100k$ -$150k$, labelled from 1 to 49 and 90 to 100**

(a) Find $(A' \cup B') \cap C$

$$U = \{1,2,3,4, \dots ,150\}$$
$$A = \{50,51,52, \dots ,89\}$$
$$B = \{101,102,103, \dots ,150\}$$
$$C = \{1,2,3, \dots ,49,90,91,92, \dots ,100\}$$