Surgery Theory in Lagrangian Immersions

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Contents

1 Surgery Operations

One set of techniques to generate manifolds is through surgery. The surgery we consider is, for a closed $(k+l)-$manifold, we remove $S^k \times D^l$ and glue $D^{k+1} \times S^{l-1}$ along the boundary caused by removing $S^k \times D^l$.

Example 1. *The* $(k+l)$*-sphere.*

First we write the sphere as the boundary of the product of discs.

$$\begin{aligned} S^{k+l} &= \partial(D^{k+1} \times D^l) & (1)\\ &= \partial D^{k+1} \times D^l \cup D^{k+1} \times \partial D^l & (2)\\ &= S^k \times D^l \cup D^{k+1} \times S^{l-1} & (3)\\ \text{Post-surgery:} \quad &\to D^{k+1} \times S^{l-1} \cup D^{k+1} \times S^{l-1} & (4)\\ &= S^{k+1} \times S^{l-1} & (5) \end{aligned}$$

See 1. This action trivializes the homotopy class $S^n \to M$ in $\pi_n(M)$.

2 An explicit Lagrangian Immersion & Surgery

Definition 1. *For a symplectic manifold M and a lagrangian submanifold L, an immersion $i : L \hookrightarrow M$ is a* ***Lagrangian Immersion****.*

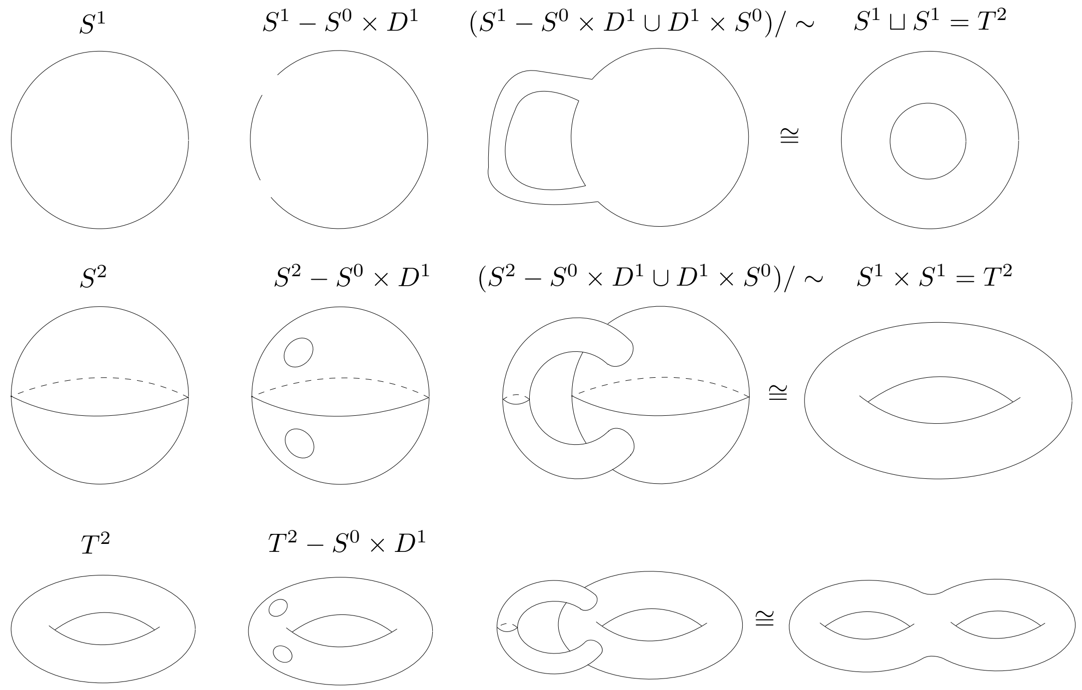


Figure 1: Surgeries of common manifolds. One can easily see that, through this kind of surgery, one can generate surfaces of any genus.

Performing surgery on the lagrangian submanifold removes a double point of a lagrangian immersion. We will show an explicit instance of this. Define the map

$$S_y(q,a) = \frac{a^3}{3} - aQ(q) + ay, \tag{6}$$

for $q \in \mathbb{R}^n, a \in \mathbb{R}, y \in \{-1,1\}$, and Q a nondegenerate quadratic form. Define a lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}$ as the kernel of $\frac{\partial S_y}{\partial a}$:

$$L_y := \{(q,a) \in \mathbb{R}^n \times \mathbb{R} | \frac{\partial S}{\partial a} = a^2 - Q(q) + y = 0\} \tag{7}$$

Define a lagrangian immersion by the map

$$f_y : L_y \to \mathbb{C}^n \tag{8}$$
$$(q,a) \mapsto q + i\frac{\partial S}{\partial q} = q - ia\frac{\partial Q}{\partial q} \tag{9}$$

Changing the index of the quadratic form gives us examples of surgeries of all indices.

To see this, suppose $q \in \mathbb{R}^2$ and Q is positive definite. Furthermore, we diagonalize Q so that

$Q(q) := Q_1x^2 + Q_2y^2$. When we consider L_-, we get a double point of our lagrangian immersion. To see this, we project to the x axis in 2. One can see that flipping y removes a double point.

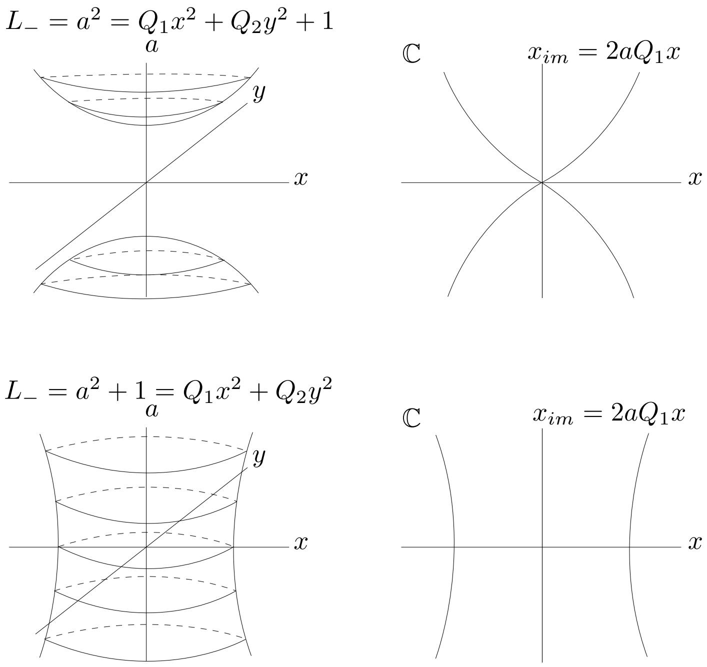


Figure 2: Immersion of L_- into $\mathbb{C}^2$. There is a double point at the origin of the graph $(x, 2aQ_1x)$.

This corresponds to a surgery, shown in 2. Thus, by adding a handle of index 1, we get rid of the double point.

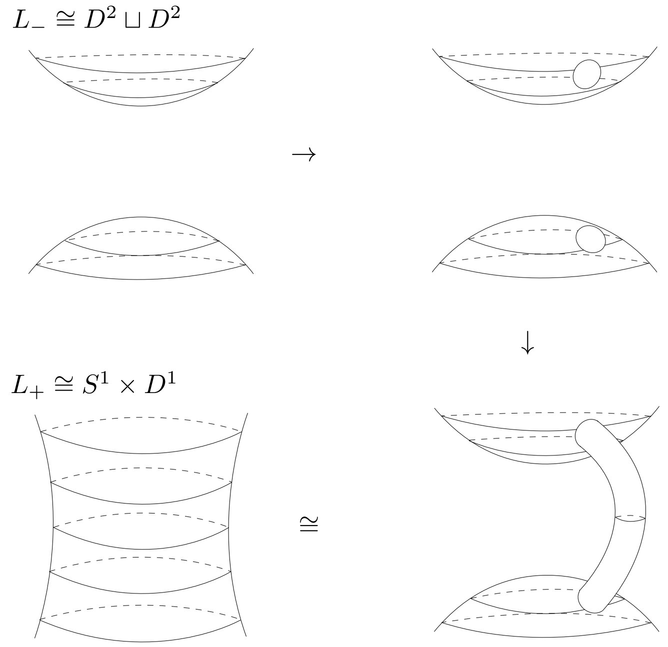
3 Lagrangian Cobordisms

Definition 2. *Two manifolds M and M' are **cobordant** if there exists a manifold W such that $\partial W = M \sqcup M'$. The triple $(W; M, M')$ is called a **cobordism**.*

A surgery on a manifold M to M' determines a cobordism, as demonstrated by 3.

Definition 3. *The **trace** of the surgery removing $S^k \times D^l \subset M$ is the cobordism $(W; M, M')$ obtained by attaching $D^{k+1} \times D^l$ to $M \times I$ at $S^k \times D^l \times \{1\} \subset M \times \{1\}$. In fact, M and M' are cobordant if and only if M' can be achieved by performing a finite number of surgeries on M.*

Lagrangian cobordisms (induced by surgery) are of special focus in symplectic geometry. For example, lagrangian cobordisms induce exact sequences in the Fukaya cateory.



Definition 4. *An A_∞-category is an associative category in the following sense: For morphisms between objects L_0, L_d denoted $CF^*(L_0, L_d)$, we can decompose this morphism as*

$$CF^*(L_0, L_d) \cong CF^*(L_{d-1}, L_d) \otimes CF^*(L_{d-2}, L_{d-1}) \otimes ... \otimes CF^*(L_0, L_1) \tag{10}$$

and the associativity is a higher homotopy equivalence without bound on the degree of homotopies.

Definition 5. *For a symplectic manifold (M, ω), the* ***Fukaya category*** *is the category where the objects are lagrangian submanifolds of M, and the morphisms are Floer Chain Complexes $CF^*(L_0, L_1)$, which, if L_0, L_1 intersect transversely, is the module generated by intersection points $L_0 \cap L_1$, viewed as the set of morphisms from L_0 to L_1.*

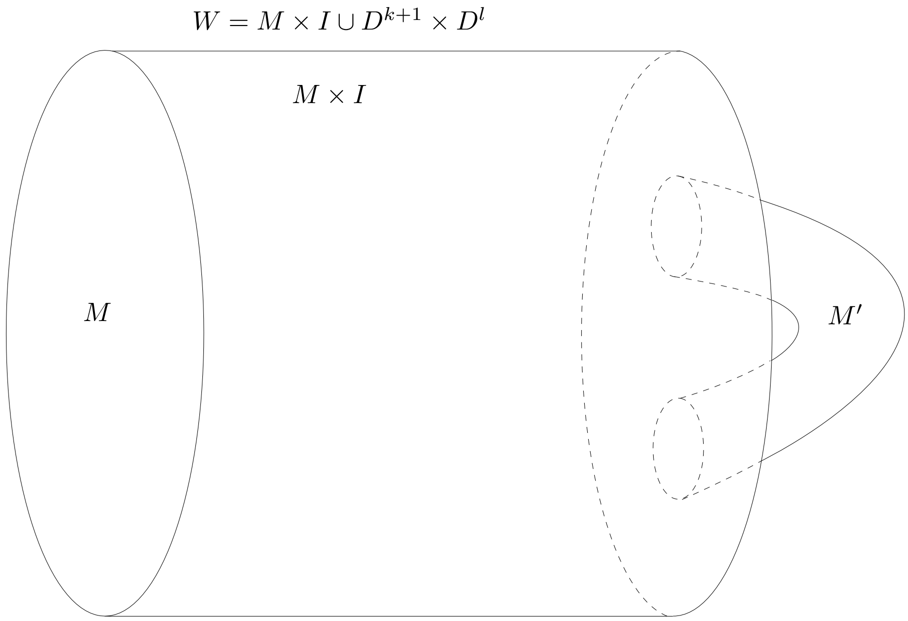


Figure 3: The cobordism generated by a surgery.