Beautiful Problems

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1 Bounding Steps of the Euclidean Algorithm

Proof. Looking at the Euclidean algorithm, we see that the worst case step is if the fewest multipliers are applied, so this points us toward using the Fibonacci sequence to bound this. First we prove that, in Euclid's algorithm, we always have $a_n \geq f_{n+2}$. We use induction on n. Note that the results hold when $n = 0$ and $n = 1$; when $n = 0$ we want $a_0 \geq f_2 = 1$, which is trivial as a_0 is a positive integer. When $n = 1$ we want $a_1 \geq f_3 = 2$, which holds because $a_1 > a_0$. This last inequality holds because a_0 is the residue of a_2 modulo a_1 (or, in the special case a_1 and a_0 are the numbers we begin with, we assume they are different). Now suppose $n \geq 2$. Then a_{n-2} is given us the residue of a_n when divided by a_{n-1}; $a_n = qa_{n-1} + a_{n-2}$. Since $a_n > a_{n-1}$, we have $q \geq 1$ and thus $a_n \geq a_{n-1} + a_{n-2} \geq f_{n+1} + f_n = f_{n+2}$ by induction hypothesis.

Thus if it takes more than 45 divisions, then in the setup we have either x or y is equal to a_n with $n \geq 46$. But according to google we see that $f_{48} > 2^{32}$, thus $a_n \geq f_n + 2 \geq f_{48} > 2^{32}$ which contradicts the assumption. □

2 Ramanujan and String Theory

Proof. a) Notice that the modes of y(x, t) are identical (up to normalization) to the eigenfunctions of the infinite square well. The key property we'll exploit is that these functions are orthonormal. Then

$$K = \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)\int_0^L dx 2\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L}) \tag{1}$$

$$= \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)2L\delta(nm) \tag{2}$$

$$= \frac{1}{2}M\sum_{n=1}^{\infty}\dot{\xi}_n(t)^2 \tag{3}$$

Using a very similar calculation, we have

$$U = \frac{1}{2}T\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\xi_n(t)\xi_m(t)\frac{n\pi}{L}\frac{m\pi}{L}\int dx 2\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} \tag{4}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{Tn^2\pi^2}{L}\xi_n(t)^2 \tag{5}$$

Thus we see that the total energy looks like that of infinitely many harmonic oscillators, each with mass M and with the n^{th} oscillator having frequency $\omega_n = \sqrt{\frac{T\pi^2}{M:}}n = \omega_1 n$

b) Using a separation of variables argument, we assume for now that the solution to the time-independent solution Ψ has the form $\Psi(\xi_1,\xi_2) = X(\xi_1)Y(\xi_2)$. Plugging this into the TISE and dividing by Ψ, we get

$$-\frac{1}{X}\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial\xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 = E - (-\frac{1}{Y}\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial\xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2) \tag{6}$$

Following the standard separation of arguments process, we conclude that this is equal to a constant E, and calling $E = E_1 + E_2$, with

$$-\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial\xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 X = E_1 X \tag{7}$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial\xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2 Y = E_2 Y \tag{8}$$

Thus $X(\xi_1$ and $Y(\xi_2)$ are harmonic oscillator eigenstates with energies that add up to E. Thus we can have the eigenstates be

$$\Psi_{n_1,n_2}(\xi_1,\xi_2) \propto H_{n_1}(\sqrt{\frac{M\omega_1}{\hbar}}\xi_1)H_{n_2}(\sqrt{\frac{M\omega_2}{\hbar}}\xi_2)\exp(-\frac{M}{2\hbar}(\xi_1^2\omega_1 + \xi_2^2\omega_2)) \tag{9}$$

with energy

$$E_{n_1,n_2} = \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2}) \tag{10}$$

c) For N decoupled harmonic oscillators, the previous result generalizes:

$$E_{n_1,\ldots,n_N} = \sum_{i=1}^{N} \hbar\omega_i(n_i + \frac{1}{2}) \text{ for } n_i \in \mathbb{Z}\forall i \tag{11}$$

d) Taking the limit as $N \to \infty$, from the previous part we have the energy spectrum as

$$\begin{aligned} E_{n_1,n_2,\ldots} &= \sum_{k=1}^{\infty} \hbar\omega_k(n_k + \frac{1}{2}) &(12)\\ &= \sum_{k=1}^{\infty} \hbar k\omega_1(n_k + \frac{1}{2}) &(13)\\ &= \hbar\omega_1 \sum_{k=1}^{\infty} kn_k + \frac{\hbar\omega_1}{2}\sum_{k=1}^{\infty} k &(14)\\ &= \hbar\omega_1(-\frac{1}{24} + \sum_{k=1}^{\infty} kn_k) &(15)\end{aligned}$$

e) The term $\sum_{k=1}^{\infty} kn_k$. For an arbitrary positive integer m, we can get degeneracy depending on our choices for n_k. Thus $\sum_{k=1}^{\infty} kn_k = p(m)$. Thus

$$\frac{1}{\eta(q)} \sim \sum_{\text{energies } E} (\text{number of states of energy } E) q e^{\frac{E}{\hbar\omega_1}} \tag{16}$$

□

3 The Wave Equation on a Riemannian Manifold

Proof. Expanding $(\nabla^2 f)(X,Y)$, we have this equal to $(\nabla(\nabla f))(X,Y)$ by definition. By the definition of ∇, we have this equal to $(\nabla_X(\nabla f))(Y)$. From class and the footnote, we have

$$\nabla_X[\nabla f(Y)] = (\nabla_X(\nabla f))(Y) + \nabla f(\nabla_X Y) \Rightarrow \tag{17}$$

$$(\nabla_X(\nabla f))(Y) = \nabla_X[\nabla f(Y)] - \nabla f(\nabla_X Y) \tag{18}$$

Again by the definition of ∇, the right-hand side then equal to $\nabla_X[\nabla_Y f] - \nabla_{\nabla_X Y} f$, which, again from class and the footnote, is $\nabla_X[Yf] - (\nabla_X Y)f$. Thinking of $[Yf]$ as another function, we have this equal to $X(Yf) - (\nabla_X Y)f$. Thus $(\nabla^2 f)(X,Y) = X(Yf) - (\nabla_X Y)f$.

When we define $\Delta f = \sum_{i,j}^{n}(g^{-1})^{ij}(\nabla^2 f)(\partial_i, \partial_j)$, we can use our formula proved above to obtain $\sum_{i,j}^{n}(g^{-1})^{ij}(\partial_i(\partial_j f) - (\nabla_{\partial_i}\partial_j)f) = \sum_{i,j}^{n}(g^{-1})^{ij}\partial_i(\partial_j f) - (g^{-1})^{ij}(\nabla_{\partial_i}\partial_j)f$. Now we examine the second term, $\sum_{i,j}^{n} -(g^{-1})^{ij}(\nabla_{\partial_i}\partial_j)f$. We will drop the summation in front and use notation with the understanding that repeated indices are summed over. This term is equal, by definition, to

$-(g^{-1})^{ij}\Gamma^k_{ij}\partial_k f$. We have

$$-(g^{-1})^{ij}\Gamma^k_{ij}\partial_k f = -(g^{-1})^{ij}(\frac{1}{2}(g^{-1})^{kl}(\partial_i g_{j,l} + \partial_j g_{il} - \partial_l g_{ij}))\partial_k f \tag{19}$$

$$= \frac{1}{2}(-(g^{-1})^{ij}(g^{-1})^{kl}\partial_i g_{jl} - (g^{-1})^{ij}(g^{-1})^{kl}\partial_j g_{il} + (g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij})\partial_k f \tag{20}$$

$$= (\frac{-1}{2}(-g^{-1})^{ij}(g^{-1})^{kl}\partial_i g_{jl} + \frac{-1}{2}(g^{-1})^{ij}(g^{-1})^{kl}\partial_j g_{il})\partial_k f + \frac{1}{2}((g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij})\partial_k f \tag{21}$$

In examining the first two terms, since the indices are dummy indices, we can relabel indices and rewrite their sum as $-(g^{-1})^{ij}(g^{-1})^{kl}(\partial_i g_{jl})\partial_k f$. We know from the first hint that this is equal to $(\partial_i(g^{-1})^{jk})\partial_k f$, so we now have

$$= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{2}(g^{-1})^{ij}(g^{-1})^{kl}\partial_l g_{ij}\partial_k f \tag{22}$$

$$= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{2}(\partial_l \log(\det g))\partial_k f \tag{23}$$

$$\tag{24}$$

as per the second hint. Then,

$$= (\partial_i(g^{-1})^{jk})\partial_k f + (\partial_l \log(\sqrt{\det g}))\partial_k f \tag{25}$$

$$= (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{\sqrt{\det g}}(\partial_l \sqrt{\det g})\partial_k f \tag{26}$$

$$\tag{27}$$

Putting this all together, we have

$$\Delta f = (g^{-1})^{ij}\partial_i(\partial_j f) + (\partial_i(g^{-1})^{jk})\partial_k f + \frac{1}{\sqrt{\det g}}(g^{-1})^{ij}(\partial_l \sqrt{\det g})\partial_k f \tag{28}$$

We now show that this is equal to $\frac{1}{\sqrt{\det g}}\partial_i((g^{-1})^{ij}\sqrt{\det g}\partial_i f)$. Since ∂_i is a derivation, we can use the 'product rule' and apply ∂_i to each of the three terms:

$$\frac{1}{\sqrt{\det g}}\partial_i((g^{-1})^{ij}\sqrt{\det g}\partial_j f) = \frac{1}{\sqrt{\det g}}((\partial_i(g^{-1})^{ij})\sqrt{\det g}\partial_j f + (g^{-1})^{ij}(\partial_i \sqrt{\det g})\partial_j f + (g^{-1})^{ij}\sqrt{\det g}\partial_i\partial_j f) \tag{29}$$

$$= (\partial_i(g^{-1})^{ij})\partial_j f + \frac{1}{\sqrt{\det g}}(g^{-1})^{ij}(\partial_i \sqrt{\det g})\partial_j f + (g^{-1})^{ij}\partial_i(\partial_j f) \tag{30}$$

By relabeling our dummy indices, we see that this is the result we want. □

4 Algebraic Topology with Statistics of Particles

Proof. Starting with $\mathbb{R}^3$, suppose we have two identical particles. We fix one particle at the origin and look at the configuration space of the other particle. We cannot have the two particles in the same place, because of the Pauli Exclusion principle. Other than that, we can have the other particle go to any nonzero point in $\mathbb{R}^3$. In doing a single particle exchange twice, we have the path of the particle during this process is a loop. We have that $\pi_1(\mathbb{R}^3 - \{0\}) \cong \pi_1(S^2)$, because $\mathbb{R}^3 - \{0\}$ can be continuously deformed into S^2 through the map $x \mapsto \frac{x}{|x|}$. Thus $\pi_1(\mathbb{R}^3 - \{0\}) \cong \pi_1(S^2) \cong 0$. Thus, two exchanges give the identity operator. In operator language, this means that, for $\hat{A}$ the exchange operator, $\hat{A}^2\psi = 1\psi$. Thus the eigenvalues of $\hat{A}$ must be equal to 1 or -1, corresponding to bosons and fermions.

In $\mathbb{R}^2$, the same map gives us $\pi_1(\mathbb{R}^2 - \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$. Thus $A^2\psi \ncong 0$, so we can allow any eigenvalue of A to be the statistics of the identical particles. □

5 A Surjection between Groups

Proof. We have two generators for G that we shall define as $h : (a, b) \to (-a, b)$ and $j : (a, b) \to (a, -b)$. It is not hard to see that $h^2 = j^2 = e$, the identity of G. There are other rotation operators as generators, but the subgroup generated by rotation operators has torsion, so the image of any homomorphism in H for this subgroup must also be finite. G is infinite on account on two generators f, g with $f : (a, b) \mapsto (a + 1, b), g : (a, b) \mapsto (a, b + 1)$. It is easy to see that the subgroup generated by these two generators is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This is infinite, so the problems that could arise from the finiteness of H come from these. Label our surjective homomorphism by φ. Observe that

$$\varphi(hf^x) = \varphi(h)\varphi(f^x) \tag{31}$$

$$\varphi(hf^{-x}) = \varphi(f^x h) = \varphi(f^x)\varphi(h) = \varphi(h)\varphi(f^x) = \varphi(hf^x) \tag{32}$$

Thus $\varphi(hf^x) = \varphi(hf^{-x})$, so $\varphi(h)\varphi(f^x) = \varphi(h)\varphi(f^{-x})$. Left multiplying by $\varphi(h)^{-1}$, we find that $\varphi(f^x) = \varphi(f^{-x}) \Rightarrow \varphi(f)^x = \varphi(f)^{-x}$, so $\varphi(f) = \varphi(f)^{-1}$.

Now notice $\varphi(f) = \varphi(f)^{x+1-x} = \varphi(f)^x\varphi(f)\varphi(f)^{-x} = \varphi(f)^{2x+1}$ for any $x \in \mathbb{Z}$. Thus all odd powers of $\varphi(f)$ map to $\varphi(f)$, and all even powers of $\varphi(f)$ map to the identity of H. Thus H is finite. □

6 Infinite Integers

Proof. In the spirit of the above problems, we want the first digit to satisfy $\overline{a}^3 = 7$. An $a \in \mathbb{Z}/[10]\mathbb{Z}$ that satisfies this is 3, so we let 3 be the first digit of z. Thus, we take $3^3 = 27$. In order to make the second digit equal to 0, we need the second digit in z to be $2 \times \overline{7} = \overline{4}$. This formula [digit]$\times\overline{7}$ as the next digit in z is due to the fact that we want the digit in question of our number to be congruent to 0. This sets the digit of the cube of the current z equal to 0. To illustrate this, consider this calculation:

We have $43^3 = 79507$. We want our next digit $\overline{a}$ to satisfy $(\overline{a} \times 10^2 + 43)^3 \equiv 7 \mod (1000)$. Cubing this out, we find our number equal to $a^3 10^6 + 3a^2 10^4 43 + 3a 10^2 43^2 + 43^3$. Let us denote the digit value we're trying to eliminate by d. We only need to consider the latter two terms, as we only care about the third digit for now (if we can't have this equal to zero now, we're screwed). We have now $a554700 + 79507$ (We can see that this new factor of a always has 7 as the last nonzero digit). Thus, we need $a \times 7 = \overline{5}$ in order for this sum to have a third digit of 0. If we set a equal to $\overline{7} \times 7 \times d$, we get $s := \overline{9} \times d$ so that $\overline{s} + \overline{d} = \overline{0}$. Thus, by setting $\overline{a}$ equal to $\overline{7d}$, we get the cube of $\overline{a} \times 10^n$+[our known digits for z, of $n-1$ digits] cancels out our nth digit in the cube.

Example:

$$3^3 = 27 \Rightarrow 2 \times \overline{7} = 4 \rightarrow$$
$$43^3 = 79507 \Rightarrow 5 \times \overline{7} = 5 \rightarrow$$
$$543^3 = 160103007 \Rightarrow 3 \times \overline{7} = 1 \rightarrow$$
$$\vdots$$
$$9244222170515433^3 = 789970954588247434055666517000000000000000007^*$$
$$\vdots$$

*This value is courtesy of Wolfram Alpha

Following this algorithm yields a z satisfying $z^3 = 7 \in R$. Thus, such a z exists. $\square$