Math 273

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Question 1. *Consider the space $\mathcal{D}$ of continuously differentiable complex-valued functions f on [0,1]. Consider the operator A on $L^2([0,1])$ with domain $\mathcal{D}$, defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1?*

Proof. We want to check if $\langle A\psi|\varphi\rangle = \langle\psi|A\varphi\rangle$. This gives us $\langle i\psi'|\varphi\rangle, \langle\psi|i\varphi'\rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^*\varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^*\varphi dx = \int_0^1 (-i)\psi'^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx) \neq \int_0^1 i\psi^*\varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is D_α, then, evaluating the same integral, we have $\int_0^1 (i\psi')^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^*\varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^*\alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^*\alpha\psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1-\alpha^*\alpha)i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^*\alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^*\varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. $\square$

Question 2. *Recall the definition of the manifold X_m, the measure λ_m on X_m, and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of $\mathcal{H}$. Finally, recall the definitions of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that*

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\not\Vdash \tag{1}$$

where $\not\Vdash$ is the identity operator on the Fock space.

Proof. Integrating this operator in Schwartz space, we have $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_\mathbf{p}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}}[a(p), a^\dagger(p')]$. The first expression then becomes $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p-p')1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2w_\mathbf{p}} f(w_\mathbf{p}, \mathbf{p})$, we have the equality

$$\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')] =$$
$$\int\int d\lambda_m(p) d\lambda_m(p') \sqrt{4w_\mathbf{p}w_{\mathbf{p}'}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p)\sqrt{4w_\mathbf{p}^2} f(\mathbf{p})^* g(\mathbf{p})1 = \int d\lambda_m(p) 2w_\mathbf{p} f(\mathbf{p})^* g(\mathbf{p})1$. Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, where 1 is the identity operator on our Fock space.

Now we consider $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')1$. Integrating once, we find this gives us $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. □

Question 3. *Consider the theory for massive scalar bosons of mass m. Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation*

$$\frac{\partial\varphi}{\partial t} = i[H_0, \varphi] \tag{2}$$

Proof. Suppose we have a Schwartz function f. Then, since $H_0 = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p}) a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}'))$, we have

$$(H_0\varphi)(f) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p}) a(\mathbf{p}) \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)} a(\mathbf{p}' + e^{i(x,p)} a^\dagger(\mathbf{p}'),$$
$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)} a(\mathbf{p}' + e^{i(x,p)} a^\dagger(\mathbf{p}') \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_\mathbf{p}}{\sqrt{2w_{\mathbf{p}'}}} A, \text{ where}$$
$$A =$$
$$a^\dagger(\mathbf{p})a(\mathbf{p})e^{-i(x,p)}a(\mathbf{p}') + a^\dagger(\mathbf{p})a(\mathbf{p})e^{i(x,p')}a^\dagger(\mathbf{p}') - e^{-i(x,p')}a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) - e^{i(x,p')}a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a(\mathbf{p}')) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p}')a(\mathbf{p}) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')a(\mathbf{p}))$$
$$= e^{-i(x,p')}[a^\dagger(\mathbf{p}), a(\mathbf{p}')]a(\mathbf{p}) + e^{i(x,p')}a^\dagger(\mathbf{p})[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$. Also, notice that $[A, B] = AB - BA = (-1)(BA - AB) = -[B, A]$. Thus, A becomes

$$e^{-i(x,p')}(-1)(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')a(\mathbf{p}) + e^{i(x,p')}a^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_\mathbf{p}}{\sqrt{2w_{\mathbf{p}'}}}(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$
$$= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{w_\mathbf{p}}{\sqrt{2w_\mathbf{p}}}(e^{i(x,p)}a^\dagger(\mathbf{p}) - e^{-i(x,p)}a(\mathbf{p}))$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x, p) = tw_\mathbf{p} + \mathbf{x}\cdot\mathbf{p}$, so the time derivative of $e^{\pm i(x,p)} = \pm iw_\mathbf{p}e^{\pm i(x,p)}$. Thus, $\frac{\partial\varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}}(-e^{-i(x,p)}a(\mathbf{p}')+ e^{i(x,p)}a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived form the commutator. Thus, $\frac{\partial\varphi}{\partial t} = i[H_0, \varphi]$, up to a set of measure zero. $\square$

Question 4. *In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude*

$$\langle \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4|S|\boldsymbol{p}\rangle \tag{3}$$

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!}\int_\mathbb{R} d^4x : \varphi(x)^4 : +\mathcal{O}(g^2)$. We then have

$$\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|S|\mathbf{p_1}\rangle = \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_\mathbb{R} d^4x\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}| : \varphi(x)^4 : |\mathbf{p_1}\rangle + \mathcal{O}(g^2)$$
$$= \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_\mathbb{R} d^4x\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle + \mathcal{O}(g^2)$$

For the first term, we notice that $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4})a^\dagger(\mathbf{p_1})|0\rangle$. Applying the first two operators we get either ground state back if $\mathbf{p_1} = \mathbf{p_4}$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0|a(\mathbf{p})\varphi(x)|0\rangle = \frac{e^{i(x,p)}}{\sqrt{2w_\mathbf{p}}}$, $\langle 0|\varphi(x)a^\dagger(\mathbf{p})|0\rangle = \frac{e^{-i(x,p)}}{\sqrt{2w_\mathbf{p}}}$. $\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle = \langle 0|a(\mathbf{p_2})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_3})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_4})\varphi(x)|0\rangle\langle 0|a^\dagger(\mathbf{p_1})\varphi(x)|0\rangle$. This expression is equal to $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8-1)!!$ diagrams, and $4!$ diagrams of this type. Thus we have $4!$ $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$. Thus we have $\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|S|\mathbf{p_1}\rangle = (-ig(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}}) + \mathcal{O}(g^2)$. □