Differential geometry quick reference sheet

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1 Coordinate transformations

If

$$\begin{aligned} \mathscr{T} : M &\to N \\ (\theta, \phi) &\mapsto (u, v) \end{aligned}$$

is a mapping between two manifolds, then we can write the metric tensor of N in terms of the coordinates on M the following way:

$$\begin{aligned} g_{ab} &= J(\mathscr{T})^T g_{\mu\nu} J(\mathscr{T}) \\ &= \begin{pmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \end{pmatrix}^T g_{\mu\nu} \begin{pmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \end{pmatrix} \end{aligned}$$

This also gives metrics induced by embeddings.

The volume form transforms via

$$\mathscr{T}^* du \wedge dv = \sqrt{\det(J(\mathscr{T}))} d\theta \wedge d\phi$$

sometimes, with a pseud-riemannian metric, we'll stick a negative sign inside the square root.

Through coordinate charts, these formulae make calculating the metric and volume form doable.

Example 1. Suppose

$$\begin{aligned} u &= Ef(\theta)\cos(\phi) \\ v &= Ef(\theta)\sin(\phi) \end{aligned}$$

and the metric tensor on N is $d\mu \otimes d\mu + d\nu \otimes d\nu$. Then this metric tensor in terms of θ, ϕ is

$$\begin{aligned} g' &= \begin{pmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \end{pmatrix} \\ &= \begin{pmatrix} E\frac{\partial f}{\partial \theta}\cos(\phi) & E\frac{\partial f}{\partial \theta}\sin(\phi) \\ -Ef\sin(\phi) & Ef\cos(\phi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E\frac{\partial f}{\partial \theta}\cos(\phi) & -E\frac{\partial f}{\partial \theta}\sin(\phi) \\ Ef\sin(\phi) & Ef\cos(\phi) \end{pmatrix} \\ &= E^2(\frac{\partial f}{\partial \theta}^2 d\theta \otimes d\theta + f^2 d\phi \otimes d\phi) \end{aligned}$$

How does the volume form transform? Well,

$$\mathscr{T}^* du \wedge dv = E^2 f \frac{\partial f}{\partial \theta}$$

Example 2. Consider the embedding $S_r^2 \to \mathbb{R}^3$ given by

$$\begin{aligned} (\theta, \phi) \mapsto x_1 &= r\cos\theta\sin\phi \\ x_2 &= r\sin\theta\sin\phi \\ x_3 &= r\cos\theta \end{aligned}$$

Then we have

$$\begin{aligned} g_{induced} &= \begin{pmatrix} -r\sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ r\cos\theta\cos\phi & r\sin\theta\cos\phi & -r\sin\phi \end{pmatrix} \begin{pmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & -r\sin\phi \end{pmatrix} \\ &= r^2\sin\phi d\theta \otimes d\theta + r^2 d\phi \otimes d\phi \end{aligned}$$

Example 3. We find the volume form on S^n. In stereographic projection coordinates, we have

$$\mathscr{T} : \mathbb{R}^n \to S^n \backslash \{0\} \tag{1}$$

$$(x_1, ..., x_n) \mapsto (\frac{2x_1}{1 - \sum_j x_j^2}, ..., \frac{2x_n}{1 - \sum_j x_j^2}, \frac{-1 + \sum_j x_j^2}{1 + \sum_j x_j^2}) \tag{2}$$

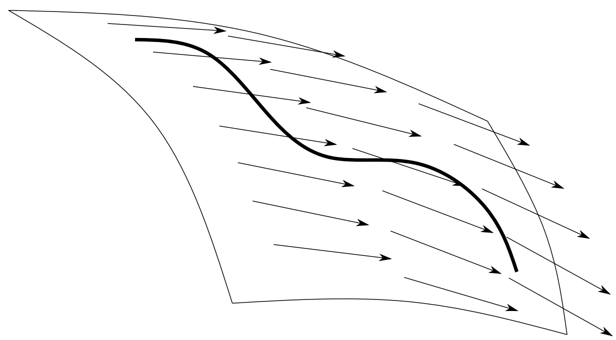

Figure 1: How does this vector field change along this curve?

Thus, knowing the metric on $\mathbb{R}^n$, we have

$$g = \frac{4(dx_1 \otimes dx_1 + ... + dx_n \otimes dx_n)}{1 + \sum_j x_j^2} \tag{3}$$

$$dV = \sqrt{\det g} dx_1 \wedge ... \wedge dx_n = (\frac{2}{1 + \sum_j x_j^2})^n dx_1 \wedge ... \wedge dx_n \tag{4}$$

2 Connections for dummies

2.1 Motivation

Vector fields are cool. Vector fields just assign vectors to every point in the manifold. In fact, general fields are cool - assigning some object to every point in the manifold. It turns out that a vector field is just a section of a vector bundle (go figure). Why not generalize to fiber bundles instead? Then we can deal with cooler stuff like tensor fields.

Suppose we want to find out how a vector field changes along a curve. This has practical applications, and it's just a nice thing to figure out about your vector field (see 1). To find this out, we need to *connect* different fibers, hence the name.

When talking about change along something, we'll want our way of connecting fibers to behave like a derivative, so we'll need some behavior involving the chain rule (Leibniz rule). Thus we'll want to deal with smooth vector fields, i.e. smooth sections

Curves are all well and good, but it's easier to think in terms of sections again -

after all, a section can give a curve, and, who knows, maybe we want to know how it changes with respect to another vector field, not just restricted to a 1-dimensional submanifold.

2.2 Affine connections

With affine connections, the fiber bundle we work with is the tangent bundle TM. Denote the space of smooth sections of TM by $\Gamma(TM)$. An **affine connection** is then a bilinear map

$$\begin{aligned} \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ X \times Y &\mapsto \nabla_X Y \end{aligned}$$

such that, for all smooth functions $f \in C^\infty(M, \mathbb{R})$ and all sections $X, Y \in \Gamma(TM)$,

$$\begin{aligned} \nabla_{fX} Y &= f \nabla_X Y, \\ \nabla_X (fY) &= df(X)Y + f \nabla_X Y \end{aligned}$$

The X in the above definition represents the change that we're applying to Y. Y is the vector field whose change in the X directions is what we'd like to measure. Thus if we want to measure a different change in Y, this different change better be $C^\infty(M, \mathbb{R})-$linear. Since we want the change of Y, we better have that argument obey the Leibniz rule: (the change of fY needs to take into account the change of f and the change of Y in this way).

2.3 Connections on a vector bundle

Let $\pi : E \to M$ be a (smooth) vector bundle. A connection on E is a $\mathbb{R}-$linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that, for any $\sigma \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$,

$$\nabla(f\sigma) = f\nabla\sigma + df \otimes \sigma$$

This is because any connection ∇ must take in a smooth section, and gives out a rule for how it must be "differentiated." This rule, is a particular field on M depending on what "direction" we're differentiating the input section in. The values in that field take in a tangent vector $v \in TM$ and spit out a vector in E, hence the scary-looking tensor product in there.

In other words, got a vector field E? Input it into a connection ∇ and you'll get an output that takes a vector field input. Enter in a vector field X to that output, and you'll see how E changes along X. This is called the **covariant derivative along X**.

2.4 Connections on a principal bundle

Let $\pi : P \to M$ be a (smooth) principal $G-$bundle. A **principal G-connection** on P is a differential 1-form on P with values in $\mathfrak{g}$, with some nice properties. The connection is an element

$$\omega \in \Omega^1(P, \mathfrak{g}) \cong C^\infty(P, T^*P \otimes \mathfrak{g})$$

such that

$$\frac{d}{dt}[g \exp(tX) g^{-1}]_{t=0}(g \cdot \omega) = \omega,$$

and if $\xi \in \mathfrak{g}$ and X_ξ is the vector field on P associated to ξ by differentiating the G action on P, then $\omega(X_\xi) = \xi$.

That first line is similar to what we worked with in the connection on a vector bundle. The second line just means that the connection (ω) must be $G-$equivariant (it better be, since we're talking about principal $G-$bundles here). The last note just means that we should be able to recover elements of the lie algebra when we want to "covariantly differentiate" a vector field based said element of the lie algebra.