Topics in Topology Lecture Notes

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Live-Teχed lectures on 4-manifolds by Prof. Ciprian Manolescu for Math 283a in Spring 2020.

Here are some highlights of the course:

1. There exist smooth 4-manifolds X_1, X_2 such that X_1 is homeomorphic but not diffeomorphic to X_2. Dimension four is the first dimension where this happens.

2. The Thom Conjecture: For a smoothly embedded surface $\Sigma \subset \mathbb{C}P^2$, $[\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}$. Then genus $g(\Sigma) \geq \frac{(d-1)(d-2)}{2}$ (= the genus of algebraic curves in that class). Proved by Kronheimer and Mrowka.

3. The Milnor Conjecture: For $T_{p,q}$ a torus knot, (p twists, q strands), and $\Sigma \subset B^4$ a smoothly properly embedded surface, $\partial\Sigma = \Sigma \cap \partial B^4 = T_{p,q}$, and $g(\Sigma) \geq \frac{(p-1)(q-1)}{2}$. Also proved by Kronheimer and Mrowka.

The original proofs of these used gauge theory; Yang-Mills eqs, Seiberg-Witten eqs. Newer proofs use algebraic topology, namely Khovenov homology.

Contents

1 Classifying 4-manifolds, bilinear forms

A basic problem in 4-dimensional topology is to classify smooth 4-manifolds. This is unfortunately hopeless, on account of the following two theorems:

Theorem 1. *(Atiyah & Rubin, 1955) There does not exist an algorithm which determines whether a given presentation of a group yields the trivial group.*

Theorem 2. *(Markov 1960s) Given a finitely presented group G, there exists a smooth closed 4-manifold X with $\pi_1(X) = G$.*

Thus somehow 4-manifolds are at least as complicated as groups. This works because a compact 4-manifold allows a triangulation which gives finite generators.

Proof. 1. Step 1. $\pi_1(X_1 \# X_2) = \pi_1(X_1) * \pi_1(X_2)$ where $\#$ is the connected sum.

$$\pi_1(X_i) = \pi_1(X_i - B^4) *_{\pi_1(S^3)} \pi_1(B^4) \tag{1}$$

$$= \pi_1(X_i - B^4) \tag{2}$$

Thus $\pi_1(X_1 - B^4) *_{\pi_1(S^3)} \pi_1(X_2 - B^4) = \pi_1(X_1) * \pi_1(X_2)$

2. Step 2. $G = \langle g_1, ..., g_l | r_1, ..., r_r \rangle$. Let $N = (S^1 \times S^3)\#...\#(S^1 \times S^3)$ with l products. $\pi_1(S^1 \times S^3) \cong \mathbb{Z} \Rightarrow \pi_1(N) = \mathbb{Z} * ... * \mathbb{Z} = \langle g_1, ..., g_l \rangle$.

3. Step 3. $r_i (= g_1^5 g_4^{-1} ...)$ represented by a loop $\gamma_i \subset N$. Transversality (1+1<4) implies γ_i can be assumed embedded and disjoint.

4. Do surgery on loops: $\gamma \subset N$ a tubular neighborhood $\cong S^1 \times B^3$:

$$\partial(N - (S^1 \times B^3)) = S^1 \times S^2 = \partial(B^2 \times S^2). \tag{3}$$

Let $\tilde{N} = (N - (S^1 \times B^3))) \cup_{S^1 \times S^2} (B^2 \times S^2)$. Then

$$\pi_1(\tilde{N}) = \pi_1(N - (S^1 \times B^3)) *_{\mathbb{Z}} 1 = \tag{4}$$

$$\pi_1(N) = \pi_1(N - (S^1 \times B^3)) *_{\mathbb{Z}} \mathbb{Z} = \pi_1(N - (S^1 \times B^3)) \tag{5}$$

See 1 Do this for all γ_i simultaneously. Then we get a new 4-manifold M with $\pi_1(M) = \langle g_1, ..., g_l | r_1, ..., r_m \rangle = G$.

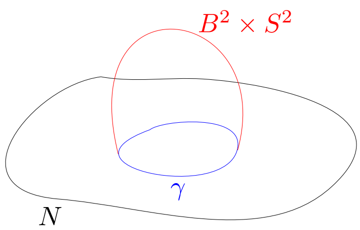


Figure 1: Surgery to remove a nontrivial loop

$\square$

Thus we can construct 4-manifolds with fundamental group equal to any group, and through the previous theorem we can't classify 4-manifolds. In the homework there is a calculation that the Euler characteristic of the resulting manifold via surgery $\chi(M) = 2 - 2l + 2m$.

Since we can't classify 4-manifolds, let's consider a more tractable problem: classify closed, simply connected, smooth 4-manifolds. Notice that trivial fundamental group implies that the smooth 4-manifold is orientable. Furthermore, this classification should work up to diffeomorphism. We could also classify closed, simply connected topological 4-manifolds up to homeomorphism, or, even weaker, homotopy equivalence. Consider X^4 closed oriented, $\pi_1 = 1$. We can abelianize to get $H_1 = 0$, by Poincaré Duality we get $H^3 = 0$ and by the Universal Coefficient Theorem we get $H_3 = 0$. Also $H_4 = H^4 = H_0 = H^0 = \mathbb{Z}$.

$$H_2 = H^2 = Hom(H_2, \mathbb{Z}) \oplus Ext(H_1, \mathbb{Z}) = Hom(H_2, \mathbb{Z}) = \mathbb{Z} \tag{6}$$

which is torsion free. Let $r = b_2(X) = b_2^+(X) + b_2^-(X)$. By Hurewicz, $\pi_2 = H_2$. The cup product gives $Q : H^2 \times H^2 \to H^4 \cong \mathbb{Z}$ (using orientation), given by

$$(a, b) \to (a \smile b)[X] \tag{7}$$

We are given a symmetric, unimodular (det=1) bilinear form $Q : \mathbb{Z}^r \times \mathbb{Z}^r \to Z$. Q is symmetric over $\mathbb{R}$: bilinear forms are classified by rank r and signature σ. Over z it's more complicated to classify Qs.

Definition 1. *A form Q is called even if $Q(a, a) \equiv 0 mod 2, \forall a \in \mathbb{Z}^r$.*

e.g.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

We can see that $Q([x, y], [x, y]) = 2xy$. If Q_1, Q_2 are rank 2 signature 0, they are equivalent over $\mathbb{R}$ not over $\mathbb{Z}$.

Remark 1. *$H^2 \cong H_2$ we can view $Q : H_2(X) \times H_2(X) \to \mathbb{Z}$, $(a, b) \to a \cdot b$ the intersection form.*

Theorem 3. *X^4 smooth implies that every $\alpha \in H_2(X; \mathbb{Z})$ is represented by $[\Sigma]$ for some embedded smooth surface $\Sigma \subset X$.*

Proof. {complex line bundles over X} modulo isomorphism is isomorphic to $H^2(X;\mathbb{Z})$ given by $E \mapsto c_1(E)$, the first chern class. Given $\alpha \in H^2(X;\mathbb{Z})$. $\Sigma = s_a^{-1}(0)$, where s_a is a generic section of L_a a line bundle. $PD[\Sigma] = a$ (Poincaré Dual). Σ is smooth by transversality. $\square$

Hence $[\Sigma_1] \cdot [\Sigma_2]$ make Σ_1, Σ_2 transverse, count intersection points with signs.

Remark 2. *$\pi_2 = H_2$ implies that ever class $\alpha \in H_2$ is represented as the image of $f : S^2 \to X$. Generically one can make f immersed, but not embedded. Thus we have the minimum genus problem: $min\{g(\Sigma)|\Sigma$ embedded, $[\Sigma] = a\}$ = ???*

Here are some examples of smooth closed, simply-connected 4-manifolds.

Example 1. *S^4. Then $Q = 0$.*

Example 2. *$\mathbb{C}P^2$ with complex orientation. $H_2 = \mathbb{Z}$. Then $Q = (1)$.*

Example 3. *$\overline{\mathbb{C}P^2}$ with reverse orientation. $Q = (-1)$.*

Example 4. *$S^2 \times S^2$. $H_2(\mathbb{Z} \oplus \mathbb{Z}), ([S^2 \times pt], [pt \times S^2])$. Then Q equals Pauli X.*

Example 5. *Connected sums $X = X_1 \# X_2$.*

$$Q_x = \begin{pmatrix} Q_{X_1} & 0 \\ 0 & Q_{X_2} \end{pmatrix} = Q_{X_1} \oplus Q_{X_2} \tag{9}$$

Corollary 1. *There is no orientation reversing self-diffeomorphism of $\mathbb{C}P^2$. $\mathbb{C}P^2 \times \overline{\mathbb{C}P^2}$ is not homotopy equivalent to $S^2 \times S^2$.*

Proof. $(1) \neq (-1)$. Pauli Y is not equal to [[1,0][0,-1]] $\square$

Theorem 4. *(Whitehead) X_1, X_2 closed simply connected topological 4-manifolds. Then X_1 is homotopy equivalent to X_2 if and only if $Q_{X_1} \cong Q_{X_2}$ (over $\mathbb{Z}$).*

Proof. (Sketch) Look at generators of $H_2(X) = H_2(X - B^4) = \pi_2(X - B^4)$ because π_1 is trivial. These generators are represented by some maps

$$f_i : S^2 \to X - B^4, i = 1, ..., r \tag{10}$$

where r is is the second Betti number $b_2(X)$. Construct the map

$$f : \wedge_{i=1}^{r} S^2 \to X - B^4 \tag{11}$$

which is an isomorphism on H_*. Thus you can use the relative Hurewicz Theorem to show that this is an isommorphism on π_*. Then via a separate Whitehead theorem f is a homotopy equivalence. Then $X \sim (\vee_{i=1}^{r} S^2 \cup_h e^4, h : S^3 \to \vee_{i=1}^{r} S^2$. Then we claim that

$$\pi_3(\vee_{i=1}^{r} S^2) = \{\text{symmetric } r \times r \text{ matrices on } \mathbb{Z}\} \tag{12}$$

$$h \to Q \tag{13}$$

Then we use the fact that $\pi_3(S^3) = \mathbb{Z}$. This is because, for a map $S^3 \xrightarrow{h} S^2$, we have $L := h^{-1}(x \in S^2)$ is a framed link. Then we get $[h] \in \pi_3(S^2) = \mathbb{Z}$ which is the linking number of L, L' (where $L' := h^{-1}(x'))$. This is called the Pontryagin-Thom construction. Thus for h general:

$$h : S^3 \to \vee_{i=1}^{r} S^2 \tag{14}$$

$$x_i \in S_i^2 \tag{15}$$

$$x_i' \in S_L^2 \tag{16}$$

And

$$Q_{ij} = lk(L_i, L_j') \tag{17}$$

Thus this proof all reduces to algebraic topology. □

Here is a much harder theorem, with homeomorphism:

Theorem 5. *(Freedman 1982)*

1. *For every unimodular symmetric bilinear form Q, there exists a topological, simply connected, closed 4-manifold X^4 with $Q_X \cong Q$.*

2. *If Q is even $(Q(a,a) \equiv 0) mod 2$ for all a, then X is unique up to homeomorphism.*

3. *If Q is odd (not even), then there are exactly 2 homeomorphism types of such X and at most one of them carries a smooth structure (is a smooth manifold).*

Remark 3. *If Q is even, we don't know if it's smooth.*

Corollary 2. *If X is smooth, then X is determined by Q_X up to homeomorphism.*

Thus we have **homeomorphism invariants of simply connected** X^4: $Q_X, KS_X \in \mathbb{Z}/2$ (the Kirby-Siefenmann invariant). In any dimension n we have

$$KS_X \in H^4(X; \mathbb{Z}/2) \tag{18}$$

where X is a topological manifold. If $KS_X \neq 0$, then X is not smooth. The point is then that if Q_X is even, then $KS_X = 0$. If Q_X odd, then $KS_X = 0$ or 1.

Consider then a $Q = (1)$. This is the form of $\mathbb{C}P^2$. There exists a topological 4-manifold, $*\mathbb{C}P^2$ with $Q = 1, KS = 1$, and not smooth. This is a very difficult construction and Freedman does it using Casson handles.

Now we want to consider X^4 smooth, up to diffeomorphism?

Definition 2. *A smooth structure on a topological manifold X is a diffeomorphism equivalence class of smooth manifolds homeomorphic to X.*

Definition 3. *If X is already given on a smooth manifold, we say a smooth structure is* ***exotic*** *if it does not contain X, i.e. $[X']$ where X is homeomorphic to X' but not diffeomorphic to X'.*

Example 6. *For dimension ≤ 3: Every manifold has a unique smooth structure. (dimension 3 was proven by Moise in the 1950s)*

Example 7. *$\mathbb{R}^n$ has a unique smooth structure for $n \neq 4$.*

Example 8. *$\mathbb{R}^4$ has uncountably many smooth structures(!) (Donaldson, Gompf, Taubes, 1980s)*

Example 9. *If X^4 is closed, it can have at most countably many smooth structures.*

Remark 4. *In dimension 4, smooth structures are the same as piecewise linear structures. (One can just think about countably many finite simplicial complexes)*

Every closed X^n for $n \neq 4$ has only finitely many smooth structures. There exist closed X^4 with countably many smooth structures, e.g. $\mathbb{C}P^2 \#^k \overline{\mathbb{C}P^2}, k \geq 2$ (Akhmedov-Park). It is unknown if there exist exotic smooth structures on $S^4, \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, S^2 \times S^2$.

Open Question: Are there infinitely many smooth structures on every X^4 closed and smooth?

S^n has a unique smooth structure $f_n, n = 1, 2, 3, 5, 6$. S^7 has 28 smooth structures (including orientation). In principle one can count smooth structures on $S^n, n \neq 4$.

Conjecture 1. *(4D Smooth Poincaré Conjecture) (SPC4) S^4 has a unique smooth structure, i.e.*

$$X \sim S^4 \Rightarrow X \cong (diff) S^4 \tag{19}$$

Theorem 6. *(4D Topological Poincaré Conjecture)*

$$X \sim S^4 \Rightarrow X \simeq (homeo) S^4 \tag{20}$$

Potential counterexamples to SPC5:

Example 10. *$P :=$ balanced presentation of the trivial group*

$$1 = \langle g_1, ..., g_m | r_1, ..., r_m \rangle \tag{21}$$

the number of generators is equal to the number of relations. From above X_P is obtained from $\#^m(S^1 \times S^3)$ by surgery along loops. ***Some exercises****:*

$$\langle x, y | x^4 y^3 = y^2 x^2, x^6 y^4 = y^3 x^3 \rangle = 1 \tag{22}$$

$$\langle x, y | x^4 = y^5, xyx = yxy \rangle = 1 \tag{23}$$

Then $\pi_1(X_P) = 1, H_1 = H_3 = 0, H_0 = H_4 = \mathbb{Z}$. From the homework, $\chi(X_P) = 2$ in this case, which means that $b_2 = 0, \Rightarrow Q = 0$, and then Freedman proved that $X_P \simeq S^4$.

Example 11. ***(Gluck Twists)*** *(From homework) Suppose we have a knotted embedding*

$$\Sigma \hookrightarrow S^4 \tag{24}$$

Let $\Sigma = S^2$ and V by a tubular neighborhood of Σ, i.e. $\cong S^2 \times D^2$. Then

$$G_\Sigma = (S^4 - V) \cup_\varphi (S^2 \times D^2) \tag{25}$$

$$\varphi : S^2 \times S^1 \to S^2 \to S^1, \tag{26}$$

$$\varphi(x, \theta) = (rot_\theta(x), \theta) \tag{27}$$

From the homwork $H_2(G_\Sigma) = 0$, so therefore $G_\Sigma \simeq S^4$.

Classification of symmetric, unimodular bilinear forms over $\mathbb{Z}$. Over $\mathbb{R}$ it's easy, with rank and signature. (there's a chapter in Serre's "A Course in Arithmetic")

$$Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z} \text{ over } \mathbb{R} \tag{28}$$

$$Q \sim Q' = Q' = AQA^T \tag{29}$$

with rank $(r = b_2(x) = b_2^+(X) + b_2^-(X))$, signature $(\sigma = \sigma(x) = b_2^+(X) - b_2^-(X))$, and parity.

Case:

1. Q is indefinite and odd, then $Q \cong m(1) \oplus n(-1), m, n > 0$ (diagonal)

2. Q is indefinite and even, then

$$Q \cong m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus nE_8, m \geq 0, n \in \mathbb{Z} \tag{30}$$

   where $E_8 = 2I - A$, where A is the adjacency matrix of the following Dynkin diagram:

$$\begin{array}{ccccccccccccc} \cdot & \text{——} & \cdot & \text{——} & \cdot & \text{——} & \cdot & \text{——} & \cdot & \text{——} & \cdot & \text{——} & \cdot \\ & & & & | & & & & & & & & \\ & & & & \cdot & & & & & & & & \end{array}$$

3. If Q is definite, then it's complicated. E.g. $m(1), m(-1); E_8, E_8 \oplus E_8, E_{16}$,...

Remark 5. *Indefinite implies that Q is determined by r, σ, and parity.*

This is not true for definite matrices. E.g.

$$E_8 \oplus \langle 1 \rangle \ncong 9\langle 1 \rangle \tag{31}$$

where odd, $rk.9, \sigma = 9$.

Thus we can ask which Qs appear as Q_X for X^4-closed and simply connected? Freedman proved that all Q, if X is topological. If X is smooth, then what?

Theorem 7. *(Rohlin 1952) X smooth, simply connected, Q_x-even implies $16|\sigma(x)$. (From algebra we know that $8|\sigma(x)$)*

Corollary 3. *(Freedman) There exists an "E_8-manifold" $Q = E_8$. This is topological, but not smoothable. In fact it is not even triangulable. $Q-$even, $KS = 0$.*

What about $E_8 \oplus E_8$? $16|\sigma$, still not smoothable. This leads to the beginning of Gauge Theory:

Theorem 8. *(Donaldson diagonalizability theorem, 1982) For X^4 smooth, closed, simply connected, then Q_X being definite implies Q_X is diagonalizable (over $\mathbb{Z}$), i.e. $\pm m\langle 1\rangle$.*

The original proof used Yang-Mills theory. Newer proofs use Sieberg-Witten thoery or Heegaard Floer homology.

For indefinite forms: for the odd case, $m(1) \oplus n(-1)$ realized by $\#^m \mathbb{C}P^2 * \#^n \overline{\mathbb{C}P^2}$. For the even case,

$$n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8 \tag{32}$$

$K3$ surface $(z_0^4 + z_1^4 + z_2^4 + z_3^4) = 0\} \subset \mathbb{C}P^3$ (Fermat quartic). This is simply connected, and

$$Q_X = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8) \tag{33}$$

$$Q_Y = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2E_8 \tag{34}$$

Connected sums of $\pm K3$s and $(S^2 \times S^2)$s then one can realize

$$Q_X = n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8 \tag{35}$$

for $|m| \leq \frac{2}{3}n$

Conjecture 2. *($\frac{11}{8}$ conjecture) If X^4 smooth, simply connected,*

$$Q_X = n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus m(-E_8) \tag{36}$$

therefore $|m| \leq \frac{2}{3}n$, so $b_2 \leq \frac{n}{8}|\sigma|$.

Using Seiberg-Witten theory and algebraic topology, we have the following theorem.

Theorem 9. *We have $b_2 \leq \frac{10}{8}|\sigma|$ is equivalent to $|m| \leq n$.*

Theorem 10. *(Hopkins-Lin-Shi-Xu, 2019) If $m = 2p \geq 4$, then $n \geq$ the following:*

1. $2p + 2, p = 1, 2, 5, 6$ *(mod 8)*
2. $2p + 3, p = 3, 4, 7$ *(mod 8)*
3. $2p + 4, p = 0$ *(mod 8)*

In summary: A smooth 4-manifold that is closed and simply connected is determined up to homeomorphism by $\sigma, \chi = 2 + b_2 = 2 + b_2^+ + b_2^-$, and type (parity of Q_X). (χ is the Euler number, with is 2 plus the Betti number)

In examples of 4-manifolds from algebraic geometry, we can calculate $b_2^{\pm}$ using characteristic classes.

2 Characteristic Classes

We give a review of characteristic classes.

Definition 4. ***Chern Classes**: Rank r complex vector bundle $E \to X$, where X is a paracompact space.*

$$c_k(E) \in H^{2k}(X; \mathbb{Z}) \tag{37}$$

The geometric interpretation, if X is a manfiold: c_k is Poincaré dual to the locus where $r + 1 - k$ generic sections of E are linearly dependent.

Remark 6. *(aside from the author) A **generic section** is a section s of a bundle $E \to M$ that intersects the zero section transversely. Let $Z(s)$ be the zero locus of s. Since s is transverse* *to the zero section, $Z(s)$ is a submanifold of complex codimension r, the rank of E. This has a fundamental class $[Z(s)] \in H_{n-2r}(M)$. Taking the Poincaré dual of $[Z(s)]$, we get a cohomology class in $H^{2r}(M;\mathbb{Z})$, which is the* ***Euler class*** *$e(E)$, and is the r^{th}* ***Chern class****, so we have*

$$e(E) = c_r(E) := PD([Z(s)]) \in H^{2r}(M;\mathbb{Z}) \tag{38}$$

For $s_1, ..., s_{r-i+1}$ generic sections, we get

$$c_i(E) = PD([Z(s_1, ..., s_{r-i+1})]) \in H^{2i}(M;\mathbb{Z}) \tag{39}$$

We define $c_0(E) = 1$ and have the ***total Chern class*** *of E given by $c(E) := 1 + c_1(E) + ... \in H^*(M;\mathbb{Z})$.*

Properties:

1. $c_0 = 1, c_k = 0$ for $k > r$
2. Functoriality: $f : X \to Y, E \to Y$, we have $f^*c_k(E) = c_k(f^*E)$
3. $c = c_0 + c_1 + ... \in H^*(X;\mathbb{Z}) \Rightarrow c(E \oplus F) = c(E) \smile c(F)$
4. $X = \mathbb{C}P^n$, $E = TX, c(E) = (1+\omega)^{n+1}, \omega \in H^2(\mathbb{C}P^n) = \mathbb{Z}$, where ω is the Poincaré dual of $[\mathbb{C}P^{n-1}]$
5. $c_1(E) = c_1(\wedge^r E)$, where $\wedge^r E$ is the determinant line bundle of E
6. L_1, L_2 - line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$
7. $c_k(E^*) = (-1)^k c_k(E)$

Definition 5. ***Siefel-Whitney Classes****: $E \to X$ rank r real vector bundle. Then $w_k(E) \in H^k(X;\mathbb{Z}/2)$, with properties 1-3 as above, with analogy for $\mathbb{R}P^n$ for 4. In addition,*

1. *If $E-$complex, then $w_{2k-1} = 0, w_{2k} = c_k$ mod 2*
2. *$w_1^{(E)} = 0$ if and only if E is orientable*
3. *If $w_1(E) = 0$, then $w_2(E) = 0$ if and only if E is spinnable oriented:*

E-real, oriented vector bundle of rank r, means that the principal $SO(r)$-bundle clutching function $U_\alpha \cap U_\beta \to SO(r)$. $Spin(r) \xrightarrow{2:1} SO(r)$. $\pi_1(SO(r)) = \mathbb{Z}/2$ for $r \geq 3$; in that case, $spin(r)$ is the universal cover of $SO(r)$. A ***spin structure*** *on E is a lift of E to a $spin(r)$-bundle.*

Definition 6. ***Pontryagin Classes***: *E-real vector bundle, then $E \otimes_{\mathbb{R}} \mathbb{C} = (E \otimes_{\mathbb{R}} \mathbb{C})^* \Rightarrow c_{2k} = 0$ for k odd.*

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X; \mathbb{Z}) \tag{40}$$

Definition 7. ***Euler Class***: *$E \to X$ a real vector bundle, of rank r, then*

$$e(E) \in H^r(X; \mathbb{Z}) = PD[\textit{zero set of a generic section of } E \textit{ when } X \textit{ is a manifold.}] \tag{41}$$

Properties:

1. *$w_r = e$ mod 2*
2. *$E-$complex, $e = c_{r/2} \in H^r(X; \mathbb{Z})$*
3. *X oriented manifold, $E = TX \Rightarrow e(TX)[X] = \chi(X)$ is the Euler characteristic*

For X^4 a closed smooth simply connected manifold. Characteristic classes of TX are $w_i, p_i, e, e \in H^4(X; \mathbb{Z}) = \mathbb{Z}, \chi = 2 + b_2$.

$$p_1(TX) \in H^4(X; \mathbb{Z}) = \mathbb{Z} \tag{42}$$

$$p_1(TX)[X] = 3\sigma(X) \tag{43}$$

by the Hirzebruch signature theorem.
$w_1 = 0$ for X oriented. $w_2 \in H^2(X; \mathbb{Z}_2) = (H^2(X; \mathbb{Z}) \text{ mod } 2)$.

Lemma 1. *w_2 is a characteristic element for Q_X, i.e. $\langle w_2, \alpha \rangle = \langle \alpha, \alpha \rangle$ mod 2, $\forall \alpha \in H^2(X; \mathbb{Z})$.*

Proof. $\alpha = PD[\Sigma]$, $\Sigma \hookrightarrow X$ oriented. Then $TX|_\Sigma = T\Sigma \oplus N\Sigma$, $T\Sigma, N\Sigma =$ *oriented*, $(w_1 = 0)$, $w(T\Sigma) = 1 + w_2(T\Sigma), w(N\Sigma) = 1 + w_2(N\Sigma)$. Then we have

$$w(TX)|_\Sigma = (1 + w_2(T\Sigma))(1 + w_2(N\Sigma)) \tag{44}$$

$$\langle w_2, \alpha \rangle = w_2(TX)[\Sigma] = w_2(T\Sigma)[\Sigma] + w_2(N\Sigma)[\Sigma] \tag{45}$$

$$= e(T\Sigma)[\Sigma] + e(N\Sigma)[\Sigma] \tag{46}$$

and $e(T\Sigma)[\Sigma] = 2 - 2g = 0 \mod 2$. Thus this is equal, for Σ' a small deformation of Σ, to $\langle[\Sigma], [\Sigma']\rangle = \langle\alpha, \alpha\rangle \mod 2$. $\square$

Corollary 4. *Q_X is even if and only if $\langle w_2, \alpha\rangle$ mod 2, for all α, if and only TX-spin (X spin)*

Corollary 5. *If we know w_2, e, p_1, this determines the homeomorphism type of X.*

From algebraic geometry, we consider the following 4-manifolds:
$Z_d = \{(z_0 : z_1 : z_2 : z_3 :) \in \mathbb{C}P^3 | p(z_0, z_1, z_2, z_3) = 0\}$ such that $\frac{\partial P}{\partial z_i} = P = 0, \forall i$ has no nonzero solutions. Then Z_d is a smooth manifold. From the homework, the diffeomorphism type of Z_d depends only on d, not on P. Therefore we can take $P = z_0^d + z_1^d z_2^d + z_3^d$.

1. $d = 1$: It's a linear equation so $\mathbb{C}P^2 \subset \mathbb{C}P^3$
2. $z_0^2 + z_1^2 z_2^2 + z_3^2 = 0$. Let $x = z_0 + iz_1, y = z_0 - iz_1$. Then we have $xy = uv$, so Z_2 is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1 \cong S^2 \times S^2$ via the map $[x : y : u : v] \mapsto ([x : u], [x : v])$
3. $d = 3$. Homework. $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$
4. $d = 4$ is a $K3$ surface.
5. $d \geq 5$ surfaces of general type

Invariants of Z_d. First of all $\pi_1(Z_d) = 1 = \pi_1(\mathbb{C}P^3)$ using the Lefschetz hyperplane theorem. See Milnor's notes on Morse theory. $\mathbb{C}P^3 \hookrightarrow \mathbb{C}P^N$ via Veronese embedding of degree d: $Z_d = \mathbb{C}P^3 \cap$ (a hyperplane). Let $X = Z_d$. $c(TX) =$? For $H \to \mathbb{C}P^3$ a hyperplane line bundle $c_1(H) = h = PD[\mathbb{C}P^2] \in H^2(\mathbb{C}P^3) = \mathbb{Z}$. $NX = H^{\otimes d}|_X, c_1(NX) = c_1(H^{\otimes d})|_X = d\eta$, for $\eta = h|_X \in H^2(X)$. So we do the same as above; decompose via

$$c(T\mathbb{C}P^3|_X) = c(TX)c(NX) = (1 + c_1(TX) + c_2(TX))(1 + d\eta) \tag{47}$$

$$= (1 + \eta)^4 = 1 + 4\eta + 6\eta^2 \tag{48}$$

$$\text{after calculation} = c_1(TX) = (4 - d)\eta, c_2(TX) = (d^2 - 4d + 6)\eta^2 \tag{49}$$

Then we can calculate the Euler class:

$$\chi(X) = e(TX)[X] = c_2(TX)[X] = (d^2 - 4d + 6)(\eta^2[X]) \tag{50}$$

$$= d(d^2 - 4d + 6) \tag{51}$$

because $h[X] = d \in \mathbb{C}P^3$. ($h[X] = \langle[X]$,[hyperplane]$\rangle$).

Further, we have $b_2 = \chi - 2 = d^3 - 4d^2 + 6d - 2$. Then

$$\sigma = \frac{1}{3}p_1(TX)[X], p_1 = -c_2(TX \otimes \mathbb{C}) = -c_2(TX \oplus T^*X) \tag{52}$$

$$c(TX) = 1 + c_1 + c_2, c(T^*C) = 1 - c_1 + c_2 \tag{53}$$

$$p_1 = -2c_2 + c_1^2 \tag{54}$$

This is true for all algebraic surfaces. In our case $\sigma = \frac{d(4-d^2)}{3}$

Finally, for the parity, we have $w_2 = c_1$ mod 2. Therefore $Q(X\text{-spin})$ is even, so equivalently $w_2 = 0$, so equivalently d is even. Thus for example the $K3$ surface, $c_1 = 0$ so it is a Calabi-Yau surface, $b_2 = 22, \sigma = -16, d$ (our intersection form) is even, Q_X-even, indefinite, implying

$$n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus m(-E_8) \tag{55}$$

and therefore $n = 3, m = 2$.

For another algebraic family, consider a hyperplane bundle $H \to \mathbb{C}P^2$. Let s be a generic section of $H^{\otimes 2p}$, zero set $B_p \subset \mathbb{C}P^2$, so a Riemann surface.

$$R_p = \{\xi | \xi^2 = s\} \subset \text{ total space of } H^{\otimes p} \tag{56}$$

These are both bundles over $\mathbb{C}P^2 : R_p \xrightarrow{2:1} \mathbb{C}P^2$, which is 2-to-1 away from the zero set B_p. Thus R_p is the double cover of $\mathbb{C}P^2$ branched over B_p. We can calculate $\pi_1(R_p) = 1, b_2^+ = p^2 - 3p + 3, b_2^- = 2p^2 - 3p + 1; R_p$−spin$\leftrightarrow$ p-odd. Then we have $R_1 = S^2 \times S^2, R_2 = \mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}, R_3 = K3$ surface, $R_p, p \geq 4$ is a surface of general type.

Enriques-Kodaira classification of algebraic surfaces: The Kodaira dimension: let K be the canonical bundle, $p_n = dim(H^0(K^{\otimes n}), n \geq 1$. K is the smallest k such that $\frac{p_n}{n^k}$ is bounded ($k = 0, 1, 2$ in dim 4), or $-\infty$ if all $p_n = 0$. $K = -\infty$: rational or ruled surfaces ($\mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1$). For $K = 0$, we can have $K3, T^4$ (abelian surfaces), Enriques surfaces, hyperelliptic surfaces. For $K = 1$, elliptic surfaces. For $K = 2$, surfaces of general type.

3 Handles & h-cobordism

Why is dimension 4 harder than dimension n, where $n \geq 5$? Say X^n is a smooth, closed n-manifold. Let $f : X \to \mathbb{R}$ be a Morse function. Since it is morse, there are finitely many critical points (assume all critical values are different). $p \in Crit(f)$ be a local model near p.

$$f(x_1, ..., x_n) = -x_1^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2 + c \tag{57}$$

Observe that $f(p) = c, k = \text{index}(p)$. In Morse theory, passing from $Y = X_{\leq c-\epsilon} (= f^{-1}(-\infty, c-\epsilon))$ to $X_{\leq c+\epsilon}$ by adding a k-handle.

In adding handles: take a Y_n, and an embedding $\varphi : S^{k-1} \hookrightarrow \partial Y$ with trivial normal bundle. We have a framing (identification $vS^{k-1} \cong S^{k-1} \times \mathbb{R}^{n-k}$).

$$Y' = Y \cup_{nbhd(\varphi(S^{k-1}))} (D^k \times D^{n-k}) \tag{58}$$

$$\cong Y \cup_{S^{k-1} \times D^{n-k}} (D^k \times D^{n-k}) \tag{59}$$

Y' is obtained from Y by attaching an $n-$dimensional $k-$handle. $\partial Y'$ is obtained from ∂Y by surgery along the attaching sphere, i.e. take out $S^{k-1} \times D^{n-k}$, and glue in $D^k \times S^{n-k-1}$.

From Morse theory, what we get is that every manifold has a decomposition into handles, a *handle decomposition*, and these can be arranged without loss of generality in nondecreasing index. Thus $X = (X_0|X_1|...|X_n)$ where X_i is a union of $i-$handles. For example, the torus has indices under the Morse height function $0, 1, 1, 2$, so T^2 has the handle decomposition given by 2.

Remark 7. *We can read $H_*(X)$ from its handle decomposition $(CW)-$complex. The cores of k-handles are $k-$cells, and $C_k(X)-$gride by k-handles $(h_\alpha^k)_{\alpha \in A}, \partial h_\alpha^k = \sum_\beta \langle h_\alpha^k | h_\beta^{k-1} \rangle h_\beta^{k-1}$, where the bracket is the incidence number: the attaching sphere of $(h_\alpha^k) \cdot$ (belt sphere of h_β^{k-1}).*

Theorem 11. *(Cerf) Every monotone (arranged in nondecreasing index) 2-handle decompositions of X can be related by a sequence of*

1. *handleslides*

2. *creating/cancelling handle pairs. See Figure 4.*

3. *isotopies between levels (in X_i for some i)*

Handleslides, taking the connected sum of a "push-off" of the attaching sphere i.e. identifying another element of the normal bundle other than 0.

Creating/cancelling handle pairs: For a $(k-1)-$handle h_{k-1} and a k-handle (h^k) such that the attaching sphere of h^k is transverse to the belt sphere of h^{k-1} is 1 point. This is the "geometric intersection number."

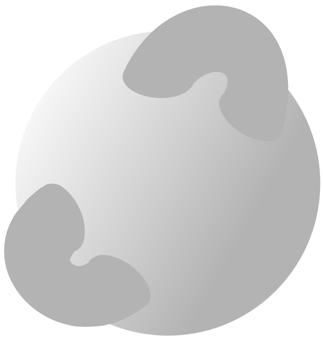


Figure 2: Handle decomposition of T^2

Proof. (SKETCH) Handle decompositions come from Morse functions, so we want to relate two Morse functions $f_0, f_1 : M \to \mathbb{R}$ by a family f_t. Generically, we use 2 types of singularities:

1. birth-death, corresponding to handle cancellation and creation.

2. There exists a trajectory of ∇f between two critical points of the same index correspond to a handleslide.

$\square$

For $n \geq 5$, we have the following theorem:

Theorem 12. *(h-cobordism Theorem, Smale, 1960s) M^n, N^n closed, simply connected, and oriented. W^{n+1} cobordism ($\partial W = (-M) \cup N$) is simply connected. Inclusions of $M \hookrightarrow W, N \hookrightarrow W$ are homotopy equivalences. This is the definition of an $h-$cobordism. Then there exists a diffeomorphism $W \cong M \times [0,1]$. In particular, $M \cong N$.*

Corollary 6. *(High dimensional Poincaré Conjecture) $n \geq 6$, $\Sigma^n \sim S^n$ (homotopy equivalent) implies $\Sigma^n \simeq S^n$ homeomorphic.*

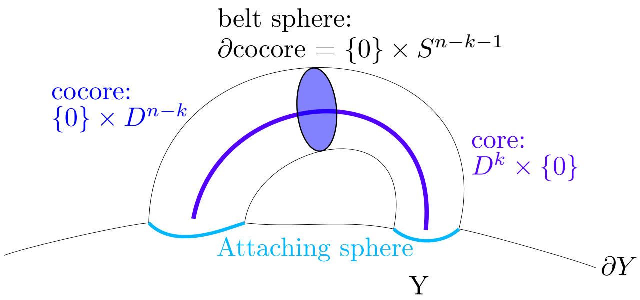


Figure 3: Handle core, cocore, and belt sphere.

Proof. Σ' is an h-cobordism from S^{n-1} to S^{n-1}. S^n is an h-cobordism from B^n to B^n, with the middle being $S^{n-1} \times [0,1]$. Every homeomorphism of S^{n-1} extends radially to B^n (NOT true for diffeomorphisms). $\square$

Remark 8. *The topolocial Poincaré Conjecture is true in all dimensions. The h-cobordism theorem fails in dimension 4.*

Proof. (PROOF SKETCH OF H-COBORDISM) Choose a morse function $f : W \to [0,1], f^{-1}(0) = M, f^{-1}(1) = N$. We arrange the critical points in increasing order, and get a handle decomposition. The goal is to eliminate all handles, so that $W \cong M \times [0,1]$. We can eliminate the 0-handles (balls). W is in this lens a graph with vertices 0-handles and edges 1-handles. In removing 0-handles, we take loops in the graph and create 1-handles. Notice that handles of complementary dimensions are similar $(D^k \times D^{n+1-k}, f \leftrightarrow -f)$. Dually we can eliminate $(n+1)-$handles. For W_j made of $j-$handles, $H_*(W, M) = 0, C_*(W, M)$ generated by handles; acycle, free ex. over $\mathbb{Z}$. Up to

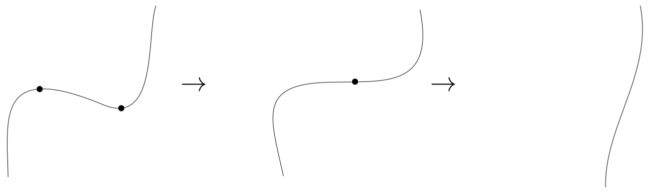


Figure 4: Cancelling pairs of critical points (handles).

isomorphism, it decomposes into $\oplus(\mathbb{Z} \xrightarrow{id} \mathbb{Z})$ $(h^k_\alpha \to h^{k-1}_\alpha)$ generated by elementary basis changes

$$e_i \to e_i \pm e_j \tag{60}$$

$$e_j \to e_j \tag{61}$$

the above move corresponds to handleslides. $\langle h^k_\alpha | h^{k-1}_\beta \rangle = 1$ algebraically. We want to make the geometric incidence number $= 1$, so we want to cancel $(h^k_\alpha, h^{k-1}_\beta)$. $P^{k-1} \cap Q^{n-k+1}$ inside Z^n =level set. $[P] \cdot [Q] = 1$; we want $P \cap Q = 1$ point. We want to cancel $(+,-)$ pairs of intersections: see Figure 5. The disc bounded by P and the $(+,-)$ loop is a Whitney disk. If $\pi_1(Z) = 1$ and $2+2<n$, if there exists an embedded Whitney disc, we can get rid of $(+,-)$, and eventually we get $P \cap Q =$ 1 point. If $\pi_1(Z) = 1$, it is okay if there don't exist 1-handles and n-handles. (these can be traded bor 3-handles and $(n-2)$-handles) $\square$

The problem in dimension four is that we can find immersed, but not recn. embedded Whitney disks. Freedman proved that one can get rid of transverse double points in Whitney discs by adding an "infinite tower of handles" (Casson handle).

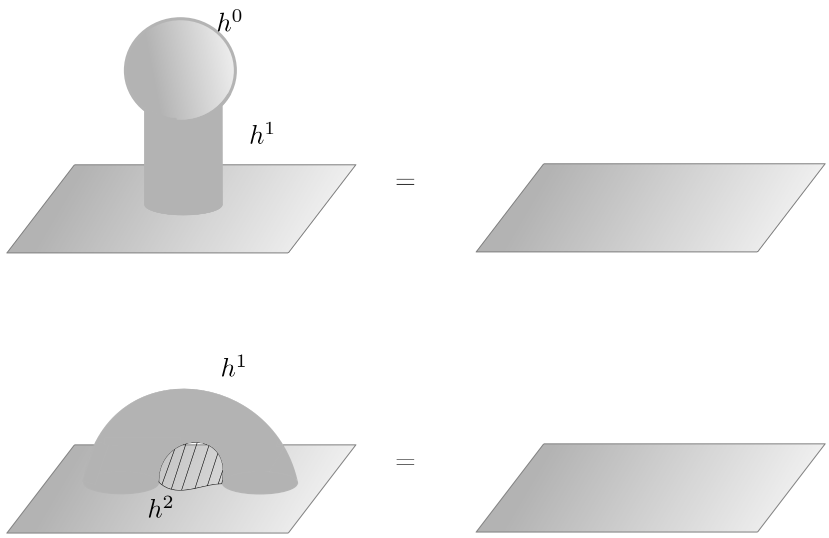


Figure 5: Cancellation of handles.

Theorem 13. *(Topological h-cobordism theorem in dimension 4, Freedman, 1982) Same hypothesis, M^4, N^4, W^5 implies that $W \simeq M \times [0,1]$, $M \simeq N$. (homeomorphic)*

Corollary 7. *Topological 4d Poincaré Conjecture.*

Theorem 14. *(Wall) If M^4, N^4 are smooth, closed, and simply connected, $Q_M \cong Q_N$ implies that M, N are h-cobordant.*

Proof. $\sigma(M) = \sigma(N)$ imply M, N are cobordant. There exists $W^5, \partial W = (-M) \cup N$. Do some surgeries on W to get an h-cobordism to get rid of $H_1(W, M)$ $\square$

Therefore Wall and Freedman's h-cobordism implies that smooth, simply connected closed 4-manifolds are determined by Q_M up to homeomorphism.

Corollary 8. *Topological 4d Poincaré Conjecture.*

Remark 9. *There exist examples of exotic smooth 4-manifolds $M \simeq N, M \not\cong N, Q_M \cong Q_N \Rightarrow M, N$ are h-cobordant. This just says that the smooth h-cobordism fails in*

4 Heegaard splitting & Kirby Diagrams

Recall that if we have a closed, connected, smooth manifold n-manifold, and a morse function $f : X^n \to \mathbb{R}$, we can get a handle decomposition. We can assume there is only one min (0-handle) and only one max (n-handle). This is because 0-handles are just balls, and we can homotope these to points. k-handles behave similarly to $(n-k)$-handles: $D^k \times D^{n-k}$ via the association $f \leftrightarrow -f$.

Handle decompositions in dimension 3: for $X^3, [0-h|1-h|2-h|3-h]$. For the first two sections $[0-h|1-h|$, we have a handlebody of genus g. This is given by the manifold H_g such that $\partial H_g = \Sigma_g$, the surface with genus g. H_g is also given by the $g-$fold boundary connected sum of $\natural^g(S^1 \times D^2)$, where we identify disks on their boundary.

Remark 10.

$$\partial(X_1 \natural X_2) = (\partial X_1)\#(\partial X_2) \tag{62}$$

Writing $X = H_g \cup_{\Sigma_g} H'_g$ is a **Heegaard splitting**.

For handle decomposition in dimension 4, we get Kirby diagrams. For $X^4 = [0-h|1-h|2-h|3-h|4-h] = [X_0|X_1|X_2|X_3|X_4]$; $X_0 \cong X_4 \cong B^4$. A 0-handle is given by B^4, as well as a 4-handle. A 1-handle is given by $D^3 \times [0,1]/(D^3 \times \{0\} \sim D^3 \times \{1\}) \cong D^3 \times S^1$.

A 2-handle is given by $X_0 \cup X_1 = \natural^k(S^1 \times D^3)$ attached by $\partial(X_0 \cup X_1) = \#^k(S^1 \times S^2)$. In drawing the 1-handles, we draw a picture of the handle decomposition in 4-space in terms of the attaching spheres. The 1-handle is going to be drawn in $S^3(\mathbb{R}^3)$ as some 2-spheres that are identified (you have a handle joining them).

For 2-handles, we take the boundary of $X_0 \cup X_1$ (which is $\#^k(S^1 \times S^2)$) and add a 2-disc attached along a 1-dimensional attaching sphere (and add a neighborhood of the 2-disc, as it is a 2-handle). The attaching sphere can be any knot: for $K \subset \#^k(S^1 \times S^2)$. For more 2-handles, we have an attaching link $L \subset \#^k(S^1 \times S^2)$. See Figure 6 where in this diagram we assume the left spheres are identified with the right handles, so in this diagram we have 2 1-handles and a few 2-handles. We also need the link to be framed: $\nu(L) \cong \sqcup(S^1 \times D^3)$. The twisting of the framing of L can be thought of as a new link L' being the boundary of $\nu(L)$. The framing of a link $L \subset S^3$ is specified by an integer $m \in \mathbb{Z}$ on each component of L. m here is $lk(L, L') = [S] \circ [L'], \partial S = L$. Then the attaching circles for 2-handles are curves in $\mathbb{R}^3$, and we can specify framings by integers, as before.

So far we have our Kirby diagram as $X_0 \cup X_1 \cup X_2$, specified by some link in $\mathbb{R}^3$ where some

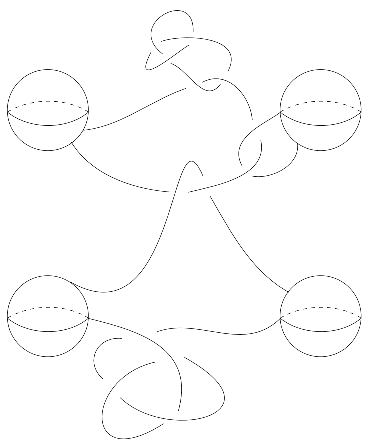


Figure 6: Attaching link.

components have integers and some dots (dotted components should be unlinks, in dotted notation)

$$[X_0|X_1|X_2|X_3|X_4] \tag{63}$$

$$S^3, \#^k(S^1 \times S^2), \#^l(S^1 \times S^2), S^3 \tag{64}$$

for boundaries. We also need, after doing surgery on the link $L, \#^k(S^1 \times S^2)$, we get $\#^l(S^1 \times S^2)$ for some l. If $\partial(X_0 \cup X_1 \cup X_2) = \#^l(S^1 \times S^2)$, the attaching the 3-handles is automatic!

Theorem 15. *(Laudenbach-Poenara) Every self-diffeomorphism of $\#^l(S^1 \times S^2)$ extends to a self-diffeomorphism of a boundary connected sum of $(S^1 \times D^3)$.*

Thus $|X_3 \cup X_4]$ attaches to $[X_0 \cup X_1 \cup X_2|$ via φ along the boundary $\partial = \#^l(S^1 \times S^2)$. Then we just write on the diagram "$\cup l$ 3-handles," and sometimes "$\cup$ 4-handles." This gives a Kirby diagram for X^4. We also have Kirby diagrams for manifolds with boundary. For X^4 with $\partial X^4 = Y^3$, we have $X^4 = [X_0|X_1|X_2|X_3|$. If $\partial X_0 \cup X_1 \cup X_2) = M\#(\#^l(S^1 \times S^2))$ we could attach l 3-handles. Any framed link and some 1-handles specifies some $X^4, \partial X^4 = Y$ with no 3-handles.

Remark 11. *For D_1, D_2 Kirby diagrams, we write $D_1 \sim D_2$ if the 4-manfiolds are diffeomorphic, and $D_1 \sim^\partial D_2$ if the boundaries are diffeomorphic.*

Example 12. *S^4= 0-handle$\cup$ 4-handle, or $[0-h|1-h|2-h|4-h] \to [0-h|4-h]$, or $[0-h|2-h|3-h|4-h]$.*

Example 13. *$(S^1)_1\cup$ a 4-handle is $\mathbb{C}P^2$. For $h : \mathbb{C}P^2 \to \mathbb{R}$ a morse function, there are 3 critical points of index 0,2,4. $\partial\mathbb{C}P^2 = L(1,1) = S^3$.*

Example 14. *$(S^1)_{-1}\cup$ a 4-handle is $\overline{\mathbb{C}P^2}$.*

Example 15. *For the hopf link with each circle labeled 0, this is $S^2 \times S^2$. For a morse function $h : S^2 \to \mathbb{R}$ is the height, we can project onto each factor $f = h \circ \pi_1 + h \circ \pi_2$ and get 4 critical points, of index $0, 2, 2, 4$.*

Remark 12. *$H_*(X)$ can be read from the diagram $C_k(X)$ is generated by $k-$handles. ∂(2-handle) is $\sum$(incidence number)*(1-handles), where the incidence number is how many times it goes through S^ks. This is counted by signs.*

If we just have 2-handles, $H_2(X)$ is generated by 2-handles, i.e. link components $L_1, ..., L_k$. $Q_X : Q_{ij} = lk(L_i, L'_j)$.

Example 16. *For the Hopf link with 0 on each component,*

$$Q_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{65}$$

Example 17. *$K3$ surface: $Q_X = 2(-E_8) \oplus 3Q$ where Q is*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{66}$$

$b_2 = 16 + 32 + 22$. For an elliptic surface given by

$E_2 = K3$. (1 on a number of strands (in the box in the diagram) means a positive full twist, and a -1 is a twist the other way. n in the box is just n full twists)

Definition 8. *A morse function $f : M \to \mathbb{R}$ is called **perfect** if the number of critical points of f is $\sum b_i(M)$ i.e. $\partial = 0$ is the Morse complex (with $\mathbb{Q}$ coefficients.*

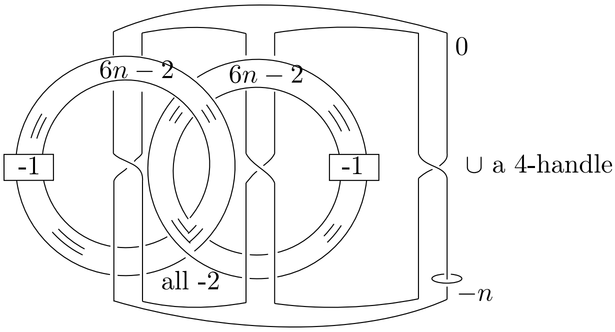


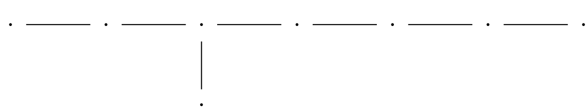
Figure 7: Kirby diagram for an elliptic surface.

Conjecture 3. *X^4 a simply connected, closed, smooth 4-manifold. Does every such X admit a perfect morse function? I.e., does it have a handlebody decomposition with only 2-handles? (and 1 0-handle and 1 4-handle?) Ciprian suspects no.*

Corollary 9. *(Freedman's Theorem) All unimodular symmetric bilinear forms Q appear as Q_X for a closed simply connected topological 4-manifold, but not necessarily for a smooth one (Donaldson diagonalizability).*

They do, however, appear as Q_X, for a smooth 4-manifold with boundary: $Q = Q_{ij}$ pick any framed link $:= \cup L_1$ with $lk(L_i, L_j') = Q_{ij}$. Then you get a Kirby diagram for X^4 with only 2-handles and $Q_X = Q$. Recall from the homework that this means that ∂X is an integral homology sphere.

Example 18. *For E_8 the adjacency matrix for*



Corresponds to . Here X^4 is called "E_8 plumbing." $\partial X^4 = P$, the Poincaré homology sphere, with representation the trefoil knot with label 1.

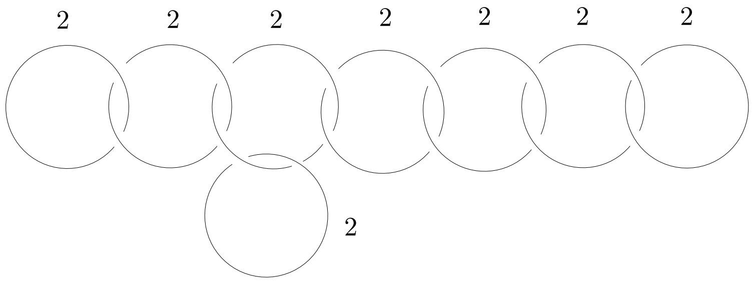


Figure 8: Corresponding link.

5 Kirby Calculus

A surgery diagram for a 3-manifold Y is a Kirby diagram for X^4 with only 2-handles, $\partial X = Y$.

Theorem 16. *(Lickorish, Wallace) Every closed oriented 3-manifold admits a surgery diagram. (Every 3-manifold is integral surgery on a link $L \subset S^3$, i.e. it has a surgery diagram)*

The proof depends on another theorem that we're not going to prove, which is that every such Y^3 appears as ∂X^4, for X^4 a closed, smooth 4-manifold. (Rokhlin)

Proof. Draw a Kirby diagram for X^4. Replace 1-handles with 0-framed 2-handles. We can also replace with 3-handles (turn it upside down). $\square$

Example 19. *The zero set $= S^3$. The unknot with an integer n is the lens space $-L(n, 1)$.*

Example 20. *The trefoil knot with label 1 is -Poincaré sphere.*

Example 21. *The Borromean rings (3 linked rings such that the removal of one results in to unlinks) with labels 0,0,0 is the 3-torus T^3.*

The point of this is that both 3-manifolds and 4-manifolds can be expressed in terms of links.

Remark 13. *We can read the homology of Y $H_*(Y)$ from the surgery diagram. Recall we do this is the Kirby diagram: the generators were the handles. For 3-manifolds, we have a framed link as a surgery diagram, which also gives us a 4-manifold X with $\partial X = Y$, made of 2-handles, so $H_1 = H_3 = 0$, and $H_2(X)$ is generated by 2-handles. Then we take the long exact sequence*

$$... \to H_2(X) \to H_2(X, \partial X) \to H_1(Y) \to H_1(X) \tag{67}$$

$$... \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^r \to H_1(Y) \to 0 \tag{68}$$

where Q is the linking matrix. Thus $H_1(Y)$ is just the cokernel of the linking matrix Q. Thus you know $H_2(Y) = H^1(Y)$ which is the free part of $H_1(Y)$, by the universal coefficient theorem since $H_0(Y)$ has no torsion. For example for the trefoil knot with framing 1 just has the matrix $\langle 1 \rangle$ which has no cokernel so the Poincaré sphere is a homology sphere.

The next question is in how many ways we can express them.

Theorem 17. *(Cerf's Theorem) Any two handle decompositions of the same X^n are related by a sequence of handelslides, handle cancellations/creations, and isotopies.*

We apply this to Kirby diagrams for 4-manifolds:

Theorem 18. *(Kirby) Any two Kirby diagrams for some X^4 are related by a sequence of the following Kirby moves:*

1. *Isotopies and Reidemeister moves on links; see Figure 9.*
2. *Handleslides of 1-handles; see Figure 10.*
3. *Handleslides of 2-handles; see Figures 11 and 12. (we follow the framing of the second knot) The new framing for K_1 is $n_1 + n_2 \pm 2lk(K_1, K_2)$.*
4. *Handle cancellations/creations. There are two kinds. They have to be between consecutive handles. Therefore we consider them between 1-handles and 2-handles, and 2-handles and 3-handles. We want the incidence number to be geometrically 1. We thus need to have the attaching sphere of the 2-handle intersect the belt sphere of the 1-handle at exactly one point. Then we can kill the 1-handle and the 2-handle. For 2-handles and 3-handles, if we have an unknot labeled 0 and a 3-handle, we can just delete them.*

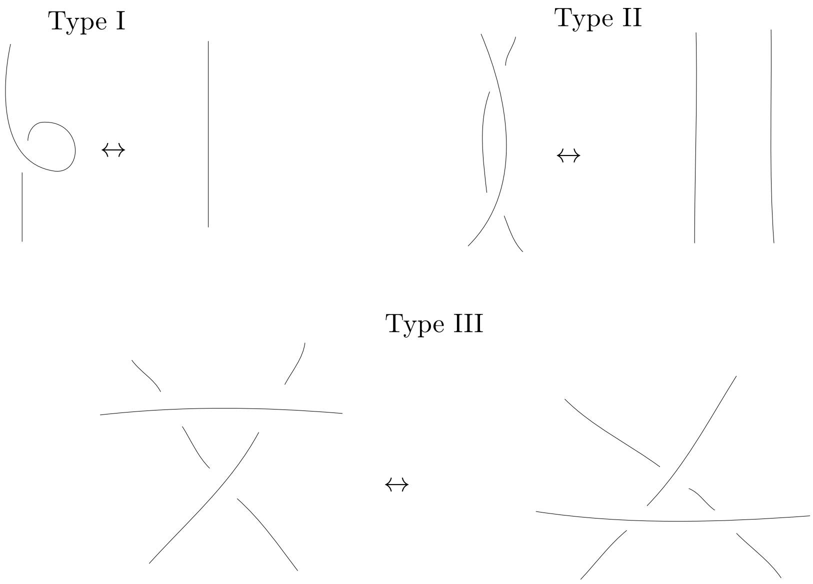


Figure 9: Reidemeister moves.

Remark 14. *The link without its frame twisting along the axis for something like the trefoil knot is not always the 0-framing.* $lk(K, K') =$ ***writhe*** *of the diagram. In the case of the trefoil knot it's -3.*

So if the two knots are linked with each other, handlesliding one over the other picks up some more linking numbers depending on how many times it links with said knot. There is also a sign, depending on the orientation of the links.

Remark 15. *We don't need to show handleslides of 3-handles because we don't have 3-handles.*

Remark 16. *In dotted notation, there is 1 more move (dotted notation is just instead of drawing the spheres of the handle that we've identified, we draw a circle with a dot on it). Our move is sliding a 2-handle under a 1-handle; see Figure 13.*

Playing with these moves is called Kirby calculus.

Example 22. *Given a Hopf link with one knot (circle) labeled 2 and the other 0, we do a handleslide a 2-curve over the 0-curve in Figure 14. Under this move we've reversed orientation. Now we figure out the framings. The linking number is* $lk(K_1, K_2) = \frac{1}{2}(-\sum x + \sum y)$*, where* x *is the crossing*

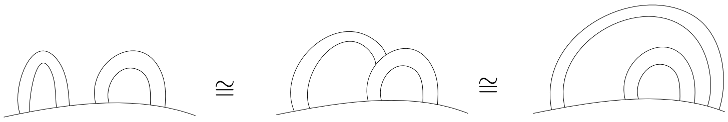


Figure 10: A handleslide of 1-handles.

from the right over the other, and y is the opposite: left crosses over the right. Under this move, the linking number of this result is $2+0\pm2$. *In this case, the orientation is reversed, so we subtract 2, resulting in 0. Then, after some Reidemeister moves, we get the same Hopf link but with each circle labeled 0, which is just* $S^2 \times S^2$.

Example 23. *Suppose we have the same Hopf link but with labels 0 and 1. Under the handleslides, we get a linking number of 0+1-2=-1. After Reidemeister moves, we get 2 unlinked circles of number -1 and 1. This result is* $\overline{\mathbb{C}P^2}\#\mathbb{C}P^2$

In general, if one has a Hopf link with numbers 0 and p, the result is $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ if p is odd, and $S^2 \times S^2$ if p is even. More generally, if one has numbers p and q, it's not a closed 4-manifold. If you look at the linking form, the linking form is

$$\begin{pmatrix} p & 1 \\ 1 & q \end{pmatrix} \tag{69}$$

, so the determinant of Q is $pq-1$. This means generally that $\partial X \neq S^3$, because we can compute H_2 via the cokernel of Q.

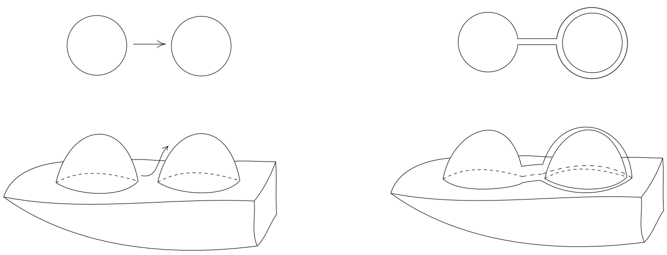


Figure 11: A 2-handleslide producing a nontrivial link.

Theorem 19. *Two surgery diagrams represent the same 3-manifolds if and only if they are related by the following moves:*

1. *Reidemeister moves*
2. *Handleslides*
3. *Blow-up/blow-down: Having an unknot with ± 1 is the same via ∂ as $\emptyset$. This is the same as $\pm\mathbb{C}P^2$.*

Recall that a Heegaard diagram consists of a surface Σ of genus g, and some alpha curves $\alpha_1, ..., \alpha_g$ simple closed curves on Σ that span $\mathbb{Z}^g \subset \mathbb{Z}^{2g} = H_1(\Sigma, \mathbb{Z})$ and beta curves $\beta_1, ..., \beta_g$ that satisfy the exact same properties. This represents a 3-manifold

$$Y^3 = \Sigma \cup (\cup_i D^2_{\alpha_i}) \cup (\cup_i D^2_{\beta_i}) \cup B^3_\alpha \cup B^2_\beta \tag{70}$$

(discs with boundary the curve in the subscript)

Example 24. *For the torus on the left in Figure ???, the red curve is the alpha curve and the blue curve is the beta curve. This represents S^3, because if we put the standard torus inside S^3 and we*

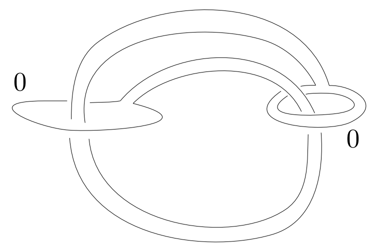


Figure 12: A 2-handleslide.

fill in the disc in the red curve to get a solid torus and fill in the disc on the blue curve we get S^3 *back. For the torus of the right, the red and blue can be thought of as the same curve, so filling in the discs and identifying the curves gives us* S^2*, and the transverse side gives* $S^1 \times S^2$.

These diagrams give a **Heegaard splitting**:

$$\#^k(S^1 \times S^3) = Y^+_{k,g} \cup Y^-_{k,g} \tag{71}$$

a decomposition of $\#^k(S^1 \times S^3)$ into two handlebodies.

6 Trisections of 4-manifolds

(Gay, Kirby, 2012)

Definition 9. *For* X^4 *closed, smooth, connected, for* $0 \le k \le g$*, a* ***(g,k)-trisection*** *of* X *is a decomposition* $X = X_1 \cup X_2 \cup X_3$ *such that, for all* $i \in \{1,2,3\}$*, there exists a diffeomorphism* $\varphi_i : X_i \to \natural^k(S^1 \times B^3)$ *such that* $\varphi_i(X_i \cap X_{i+1}) = Y^-_{k,g}, \varphi_i(X_i \cap X_{i-1}) = Y^+_{k,g}$.

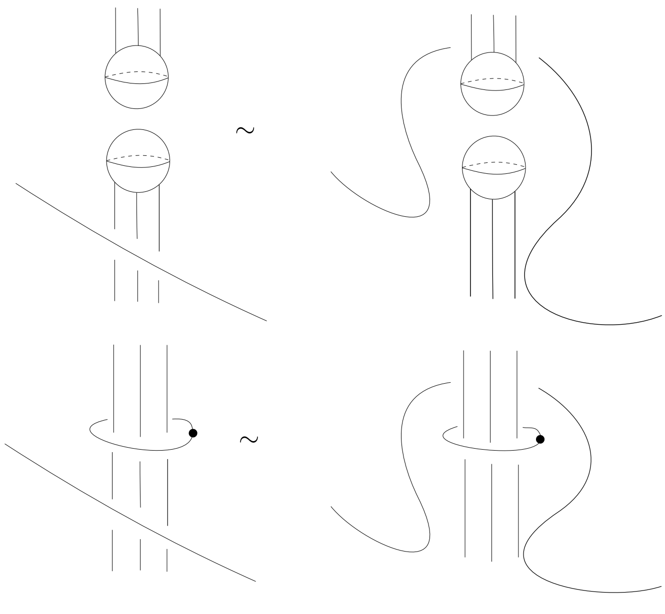


Figure 13: Sliding a 2-handle under a 1-handle.

A **tristection diagram** is like a Heegaard diagram. For a surface Σ_g and 3 sets of curves that each span $\mathbb{Z}^g \subset H_1(\Sigma)$

$$(\alpha_1, ..., \alpha_g) \to X_1, \tag{72}$$

$$(\beta_1, ..., \beta_g) \to X_2, \tag{73}$$

$$(\gamma_1, ..., \gamma_g) \to X_3 \tag{74}$$

such that $(\alpha, \beta), (\alpha, \gamma), (\beta, \gamma)$ are all Heegaard diagrams for $\#^k(S^1 \times S^2)$, and in fact they should represent the splitting $Y^+_{k,g} \cup Y^-_{k,g}$.

Remark 17. *From the homework we know that $\chi(X) = 2+g-3k$. This means that k is determined by g, and g is fixed mod 3 for a given X. Thus the terminology is often a "genus g trisection of X."*

Example 25. *For $g = 0, k = 0$. Our surface of genus zero is S^2, and we have no curves on it. Thus in the middle we have S^2, the 3 boundaries are B^3, and all X_i are B^4. Thus $\partial X_i = S^3$, the two segments on the diagram. So $\partial(X_1 \cup X_2) = S^3$, and adding $B^4 = X_3$ to S^3 yields S^4. See the*

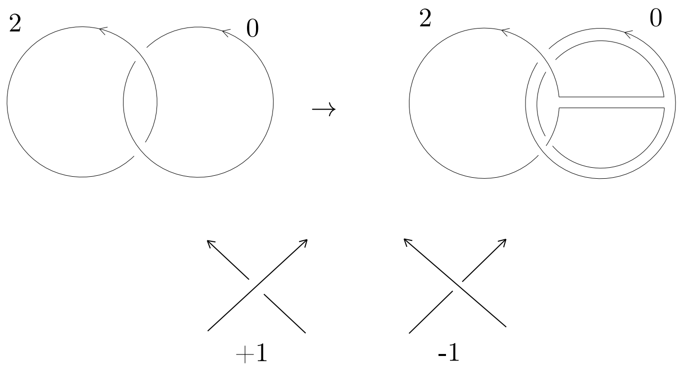


Figure 14: Kirby calculus on a Hopf link, and contributions to the linking number.

top diagram in Figure 15.

Example 26. *For $\Sigma_g = T^2$, so $g = 1$, and $k = 1$. Thus we have a torus with an alpha curve, a beta curve, and a gamma curve. There is an S^1 direction where nothing happens, so we have S^1 crossed the previous example but with one dimension lower, i.e. each segment is B^2, $X_i = B^3$, and the middle is S^1. See the middle of Figure 15. Thus we have $S^1 \times S^3$.*

Example 27. *For $X = \mathbb{C}P^2$. We use the structure as a toric variety. Toric varieties come with a moment map $f : \mathbb{C}P^2 \to \mathbb{R}^2$ which is a combination of two morse functions on $\mathbb{C}P^2$:*

$$f([z_0 : z_1 : z_2]) = (\frac{|z_0|^2}{\sum |z_i|^2}, \frac{|z_1|^2}{\sum |z_i|^2}) \tag{75}$$

The image of f is an isoceles triangle with coordinates (0,0), (1,0), and (0,1). The preimage of a point on the long side is a circle, and the preimage of the point p in the middle is a torus. We then divide the triangle into three parts Q_i, with common boundary p. Let $X_i = f^{-1}(Q_i)$. See the lowest part of Figure 15.

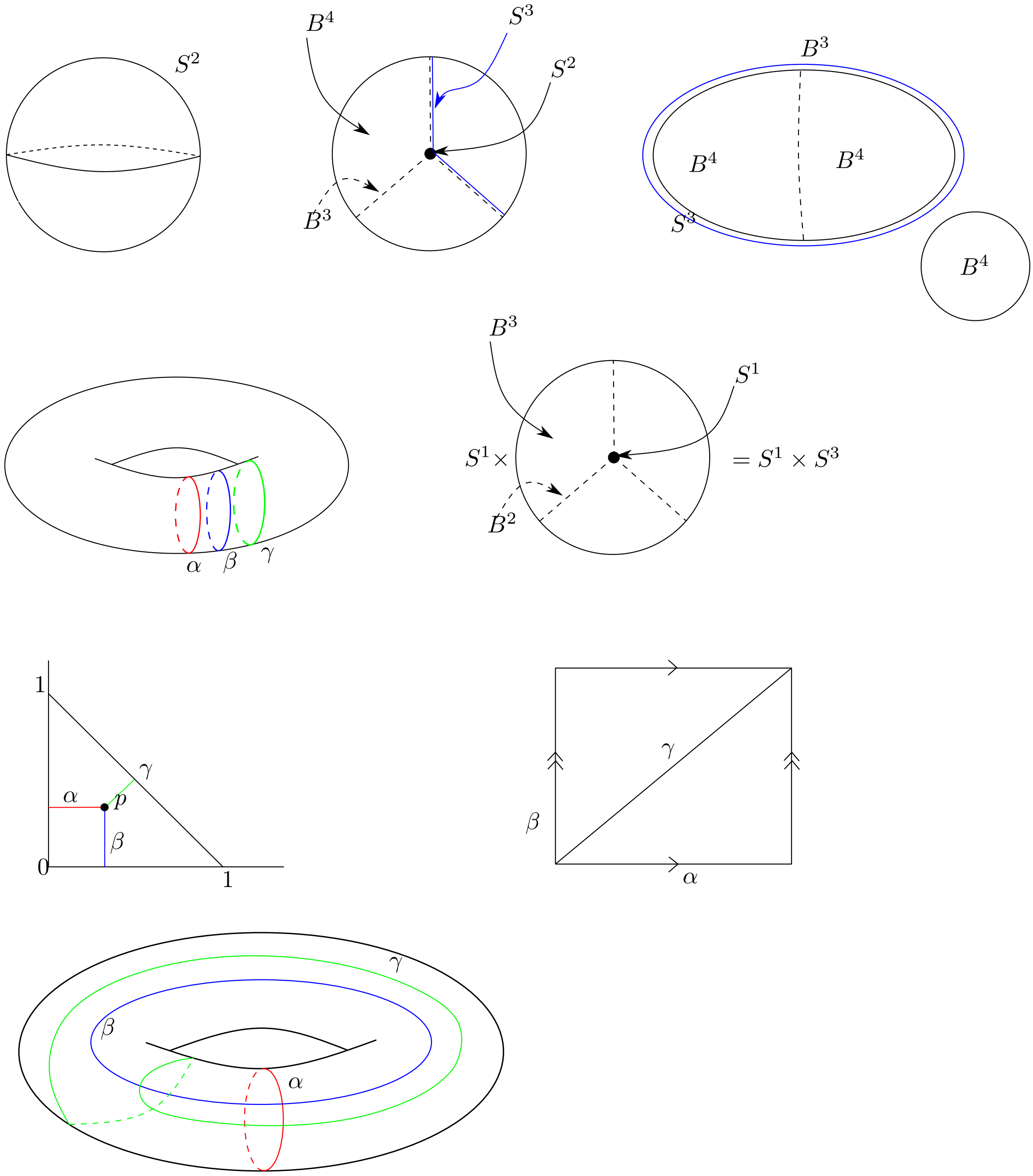


Figure 15: Trisection diagrams for $S^4, S^1 \times S^3$, and $\mathbb{C}P^2$.

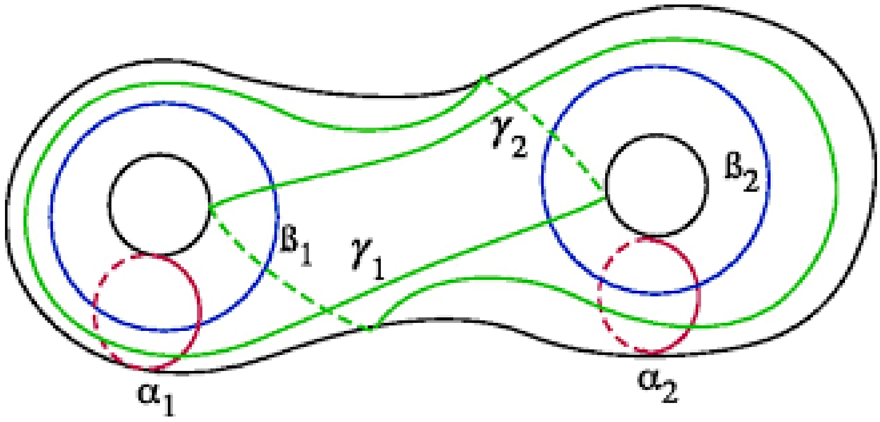


Figure 16: Trisection diagram for $S^2 \times S^2$.

Example 28. *$S^2 \times S^2$, we have $g = k = 2$. See Figure 16*

Heegaard diagrams exist because one starts with a handlebody decomposition i.e. a morse function on a 3-manifold, and similarly Kirby diagrams come from morse functions on a 4-manifold.

Theorem 20. *(Gay, Kirby) Every closed, smooth, connected, oriented 4-manifold admits a trisection. (A trisection specificies an orientation. For example $\overline{\mathbb{C}P^2}$ one puts the diagonal in the other direction)*

Proof. (SKETCH) Choose a generic compact "Morse" 2-function $f : X \to B^2$. Local models: for

1. Generic points: $f = submersion$, one has $(t, x, y, z) \to (t, x)$
2. Folds: $(t, x, y, z) \to (t, \pm x^2 \pm y^2 \pm z^2)$
3. Cusps: $(t, x, y, z) \to (t, x^3 - tx \pm y^2 \pm z^2)$

A morse 2-function is a family $f_t : X \to \mathbb{R}$ of ordinary functions, so the fold corresponds to curves of critical points, cusps correspond to birth/death singularities. For example, we can have the image of a manifold given by Figure 17 They play witih moves on the Cerf graphic They show that one can arrange so that the cerf graphic looks like 18 $f^{-1}(0) = \Sigma_g$. The trisection boundary segment intersects the arcs of the Cerf diagram. With each intersection, the preimage is a surface of of genus 1-less. Thus the preimage of the trisection boundary segment is a handlebody of genus g. The preimages of the three parts give X_i in a trisection. □

Theorem 21. *(Gay, Kirby) Any two trisection diagrams of the same X^4 are related by a sequence of diffeomorphisms, $\alpha-$handleslides, $\beta-$handleslides, $\gamma-$handleslides, and stabilizations, i.e. a connected sum with a genus 3 surface representing S^4 (Figure ???, a $g = 3, k = 1$)*

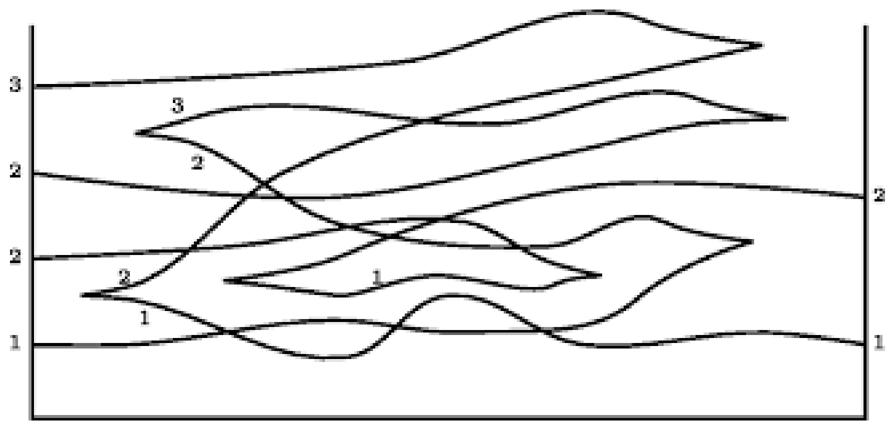


Figure 17: A cerf diagram of critical values.

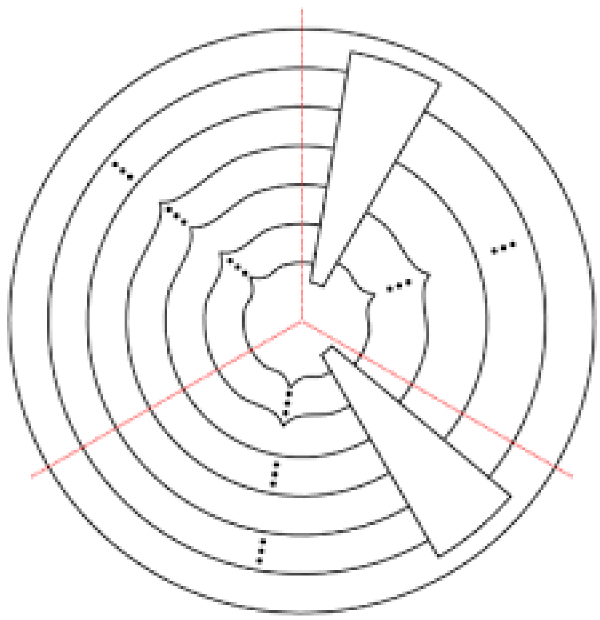


Figure 18: Cerf diagram, rearranged. There are k arcs without cusps and $g-k$ arcs with cusps.

S4 diagram

In summation we can represent 4-manifolds with trisection diagrams or Kirby diagrams. At this point in the class Ciprian said "Next we're going to do the serious stuff," referring to Seiberg-Witten gauge theory.

7 Spin and $Spin^c$ structures

Motivation: Say we want to find a square root of the Laplacian $\Delta = -\sum(\frac{\partial}{\partial x_i})^2 : C_0^\infty(\mathbb{R}^n, \mathbb{C}^m) \to C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$. This is a self-adjoint operator:

$$\int_{\mathbb{R}^n} \langle \Delta\varphi, \psi\rangle = \int_{\mathbb{R}^n} \langle \varphi, \Delta\psi\rangle \tag{76}$$

We want $D = \sum A_i \frac{\partial}{\partial x_i}$ self-adjoint such that $D^2 = \Delta$:

$$\sum A_i^2 \frac{\partial^2}{\partial x_i^2} + \sum(A_iA_j + A_jA_i)\frac{\partial^2}{\partial x_i \partial x_j} = \Delta \tag{77}$$

$$\Rightarrow \sum A_i^2 = -Id, \sum(A_iA_j + A_jA_i) = 0, A_i^* = -A_i \tag{78}$$

Definition 10. *The **Clifford algebra** is the real algebra generated by elements A_i such that $A_i^2 = -I, A_iA_j + A_jA_i = 0, \forall i \neq j$.*

Thus we want a representation of the Clifford algebra.

Definition 11. *Let H be a real n-dimensional inner product space. A **Clifford module** for H is a Hermitian complex vector space V with a Clifford multiplication, i.e. a map $\gamma : H \to End(V)$ such that*

1. *$||e|| = 1$ implies $\gamma(e)^2 = -1$*
2. *$e_1 \perp e_2$ implies $\gamma(e_1)\gamma(e_2) + \gamma(e_2)\gamma(e_1) = 0$*
3. *$\gamma(e)^* = -\gamma(e)$*

D is called the Dirac operator.

Remark 18. *A Clifford module is a skew-Hermitian representation of the Clifford algebra.*

Theorem 22. *1. If $n = 2k$, there is a unique finite-dimensional irreducible Clifford module (S, γ) up to isomorphism with $dim_{\mathbb{C}} S = 2^k$*

2. *If* $n = 2k+1$ *there are exactly 2 such irreducible modules up to isomorphism* $(S, \gamma), (S, -\gamma), dim_{\mathbb{C}} S = 2^k$.

Example 29. *In* $n = 3$, H *has a basis* e_1, e_2, e_3, $S = \mathbb{C}^2$.

$$\gamma(e_i) = B_i, \tag{79}$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{80}$$

These are the Pauli matrices. These satisfy $B_i^2 = -1$, *they all anticommute. This gives modules* $(S, \pm\gamma)$, *the Clifford modules in dimension 3, and everything is isomorphic to them or a direct sum of them.*

Example 30. *In* $n = 4$, H *has a basis* e_1, e_2, e_3, e_4, $S = \mathbb{C}^4 = S^+ \oplus S^-, dim_{\mathbb{C}} S^{\pm} = 2$.

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix}, i = 1, 2, 3, 4, \tag{81}$$

where $B_4 = Id$.

Definition 12. *A spin*^{c} ***stucture*** *on an* n*-dimensional oriented Riemannian manifold* X *is a Hermitian bundle* $S \to X$ *(complex bundle with hermitian metric) with a bundle map* $\rho : TX \to End(S)$ *(linear map varying continuously) such that* $\forall x \in X, (S_x, \rho_x : T_x X \to End(S_x)$ *is isomorphic to one of the irreducible Clifford modules for* $T_x X$.

So TX plays the role of the inner product (real) space, and the standard irreducible Clifford module to be realized by this Hermitian bundle.

Example 31. *In dimension 3, a spin*^{c} *structure is a Hermitian bundle* $S \to X$ *of rank 2,* $\rho : TX \to End(S)$ *such that there exists an orthonormal basis at each* x *for* $T_x X$ *and Hermitian basis for* S *such that* $\rho(e_i) = B_i$. *In other words, it's a principal-*$U(2)$ *bundle with a compatibility condition with* Tx.

Example 32. *When* $n = 4$, *we have two Hermitian bundles* S^+, S^- *of rank 2, and with map* $\rho : TX \to Hom(S^+, S^-)$ *such that there exists an orthonormal, Hermitian basis where* $\rho(e_i) = B_i, i = 1, 2, 3, 4$.

Remark 19. *The category of Clifford modules is semisimple, so everything is a direct sum of irreducibles.*

Remark 20.

$$\det B_i = 1 : \det(S^+) \xrightarrow{\cong} \det(S^-) \tag{82}$$

$$\wedge^2 S^+ \to \wedge^2 S^- \tag{83}$$

For the notation

$$\begin{tikzcd} TX \arrow[r, "\rho"] \arrow[dr, "\gamma"'] & Hom(S^+, S^-) \arrow[d] \\ & Hom(S, S) \end{tikzcd}$$

and $\det \rho(e) = 1, \forall e, ||e|| = 1$. $L = \wedge^2 S^+ = \wedge^2 S_-$ *is a complex line bundle determined by the bundle associated to* (S, γ).

We let the **class** of (S, γ) be $c_1(\det L) = c_1(S^+) = c_1(S^-) \in H^2(X; \mathbb{Z})$.

Remark 21. *Warning:* $c_1(S) = c_1(S^+ \oplus S^-) = 2c_1(S^{\pm})$.

Remark 22. *At some* $x \in X$, $Aut(S_x, \gamma_x) = ?$ *(preserving the hermitian structure).*

$$\begin{matrix} S^+ & & S^- \\ \circlearrowright & & \circlearrowright \\ A^+ & & A^- \end{matrix}$$

$B_i A^+ = A^- B_i, \forall i$. *If* $B_4 = Id$, *we get* $A^+ = A^- = A \in U(2)$, $AB_iA^{-1} = B_i$, $A \in Z(U(2)) = S^1 = \{e^{i\theta} \cdot Id\}$

An alternate view of spin-c structures: Recall $Spin(n) \xrightarrow{2:1} SO(n)$, with X Riemannian and oriented. TX has an $SO(n)$ structure group (i.e. $Fr(TX)$ = principal $SO(n)$-bundle). Think of a spin structure as a lift to a spin(n)-bundle.

$$Spin^c(n) = Spin(n) \times_{\mathbb{Z}/2} U(1) - \{(g, e^{i\theta}) \in Spin(n) \times U(1)\}/(g, e^{i\theta}) \sim (\tau(g), -e^{i\theta}) \tag{84}$$

So we have a fibration $U(1) \to Spin^c(n) \to SO(n) = Spin(n)/(\mathbb{Z}/2)$.

Example 33. *For* $n = 3$, $Spin(2) = SU(2) = S(\mathbb{H})$ *(the sphere in the quaternionic space). There exists an* h *such that* $(x \to hxh^{-1}), x \in Span(i, j, k) \subset \mathbb{H}$ *(the imaginary parts).* $Spin^c(3) = SU(2) \times_{\mathbb{Z}/2} U(1) = U(2)$, *because the* $U(1)$ *part corresponds to the determinant of some* $A \in U(2)$

(can take $\det(A)^{\frac{1}{2}} \cdot I$*)), and then you multiply by* $\det(A)^{\frac{1}{2}} \cdot A) \in SU(2)$*.* $\det(A)^{\frac{1}{2}}$ *is only defined up to* ± 1*. (spin*^{c} *structures on* X^3 *are hermitian rank 2 vector bundles with compatibility with* TX*, i.e. a* $U(2)$*-bundle with compatibility* $\rho : TX \to End(S)$*))*

Example 34. *In dimension 4,* $Spin(4) = SU(2) \times SU(2) = S(\mathbb{H}) \times S(\mathbb{H})$*, with the map* $x \to h_1 x h_2^{-1}, x \in \mathbb{H}$*.*

$$Spin^c(4) = (SU(2) \times SU(2)) \times_{\mathbb{Z}/2} U(1) \subset (SU(2) \times_{\mathbb{Z}/2} U(2)) \times (SU(2) \times_{\mathbb{Z}/2} U(1)) = U(2) \times U(1) \tag{85}$$

$$= \{(A, B) \in U(2) \times U(2) | \det(A) = \det(B)\} \tag{86}$$

So $Spin^c(4)$ *structures are hermitian rank 2 vector bundles* $S^{\pm}$ *with* $\det(S^{\pm}) = L$ *with compatibility with* TX *given by* γ*.*

For X^4 a smooth, simply connected, oriented, closed 4-manifold. Recall that X admits a spin structure if and only if $\omega_2(TX) = 0$ if and only if Q_X is even.

Proposition 1. *Any such* X *admits a spin*^{c} *structure. The space of spin*^{c} *structures on* X *is an affine space modeled on* $H^2(X, \mathbb{Z})$*., i.e. for* $s_0, s_1 \in Spin^c(X), s_0 - s_1$ *is well-defined in* $H^2(X, \mathbb{Z})$*, i.e.* $Spin^c(X) \cong H^2(X; \mathbb{Z})$*, but not canonically, with ?* $\to 0$*.*

Proof. (For $\pi_1(X) = 0$. For existence: say we have an atlas $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that on each U_α we have a trivialization $TX|_{U_\alpha} = \mathbb{R}^4 \times U_\alpha$, where $S_\alpha = S|_{U_\alpha}$ is the standard clifford module. On $U_\alpha \cap U_\beta$ we have

$$\varphi_{\alpha\beta} : U_{\alpha\beta} \to Aut(S, \gamma) = S^1 \tag{87}$$

which is a map $S_\alpha \to S_\beta$ over $U_{\alpha\beta}$ On $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$\varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta} : U_{\alpha\beta\gamma} \to S^1 \tag{88}$$

If we can arrange $\varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta} = 1$ we would get a *spin*^{c} structure on X. We run into a road-block here, though. If we have a Cech 2-cocycle $[\varphi] \in H^2(X, C^\infty S^1)$ (the structure sheaf), with

$C^\infty S^1(U) = C^\infty(U, S^1)$ with

$$0 \to \mathbb{Z} \to C^\infty\mathbb{R} \to C^\infty S^1 \to 0 \tag{89}$$

$$H^2(X, C^\infty\mathbb{R}) \to H^2(X, C^\infty S^1) \to H^3(X;\mathbb{Z}) \tag{90}$$

$$0 \to H^2(X, C^\infty S^1) \to 0 \tag{91}$$

where the left zero is because $C^\infty\mathbb{R}$ has partitions of unity, and the right 0 is because $0 = H_1(X;\mathbb{Z})$ (via Poincaré duality) because $\pi_1(X) = 1$. Thus $[\varphi] = 0$.

In classifying these: Suppose $(S,\gamma), (S',\gamma')$ are two *spin*^{c} structures. On U_α we get an isomorphism

$$\psi_\alpha : (S,\gamma)|_{U_\alpha} \to (S',\gamma')|_{U_\beta}, \psi_\alpha\psi_\beta^{-1} : U_{\alpha\beta} \to Aut(S,\gamma) = S^1 \tag{92}$$

gives a 1-cocycle in $H^1(X, C^\infty S^1)$. Then we have

$$0 \cong H^1(X; C^\infty\mathbb{R}) \to H^1(X, C^\infty S^1) \xrightarrow{\cong} H^2(X;\mathbb{Z}) \to H^2(X, C^\infty\mathbb{R}) \cong 0 \tag{93}$$

$\square$

Remark 23. *X^4 admits a spin^c structure, even if $H_1(X) \neq 0$ (homework 3). In any dimension, if the spin^c structure on a manifold X^n exists, they are an affine space over $H^2(X;\mathbb{Z})$. There exist 6-manifolds without spin^c structures.*

Remark 24. *For X^4 we have a map $c_1 : Spin^c(X) \to H^2(X;\mathbb{Z})$ with $(S,\gamma) \mapsto c_1(S^\pm) = c_1(\det(S^\pm))$. For $s \in Spin^c(X), c_1(E) = h \in H^1(X;\mathbb{Z})l$, we have*

$$\det(S^\pm \otimes E) = \det(S^\pm) \otimes E^2 \tag{94}$$

$$c_1(S^\pm \otimes E) = c_1(S^\pm) + 2h \tag{95}$$

If $H^2(X;\mathbb{Z})$ has no 2-torsion (e.g. if $\pi_1(X) = 1$, then the spin^c structure is determined by its class, $c_1(s) = c_1(S^\pm)$; i.e. $c_1 : Spin^c(X) \to H^2(X;\mathbb{Z})$ is injective. In the homework, $Image(c_1) =$

$\{k \in H^2(X;\mathbb{Z}) | k \mod 2 = w_2(TX)\}$, *i.e.* $\langle k, a\rangle \equiv \langle a, a\rangle \mod 2, \forall a$.

$$Char(X) = \langle characteristic\ elements \rangle \tag{96}$$

If, for example,

1. *X-spin, $w_2 = 0$, then $Spin^c(X) \cong Char(X) = 2H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{Z})$*
2. *$X = \mathbb{C}P^2, H^2(X;\mathbb{Z}) = \mathbb{Z}$, Char(X) $= 2\mathbb{Z} + 1 \subset \mathbb{Z}$. ($Q_X = \langle 1 \rangle$)*

If we have V an inner product space over $\mathbb{R}$ and an orientation, i.e. a trivialization of the determinant; so if we have an orthonormal basis $\{e_i\}$ the volume for $e_1 \wedge ... \wedge e_n, n = dim(V)$. The **Hodge star** operator $* : \wedge^k V \to \wedge^{n-k} V$ such that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_{vol}, \forall \alpha, \beta \in \wedge^k V$. If $I \subseteq \{1, 2, ..., n\}$,

$$V_I = e_{i_1} \wedge ... \wedge e_{i_k} \tag{97}$$

is a basis for $\wedge^k V$. $*V_I = \pm V_{\tilde{I}}$ where $\tilde{I} = \{1, 2, ..., n\} - I$. For example, if $k = 2$, $dim(V) = 4$. $dim(\wedge^2 V) = \binom{4}{2} = 6$, so

$$* : \wedge^2 V \to \wedge^2 V, *^2 = 1 \Rightarrow \tag{98}$$

$$\wedge^2 V = \wedge^+ V \oplus \wedge^- V \tag{99}$$

where $\wedge^\pm$ are the ± 1 eigenspaces of *. A basis for $\wedge^+ V$ is:

$$e_1 \wedge e_2 + e_3 \wedge e_4, \tag{100}$$

$$e_1 \wedge e_3 - e_2 \wedge e_4, \tag{101}$$

$$e_1 \wedge e_4 + e_2 \wedge e_3, \tag{102}$$

For $\wedge^- V$ is

$$e_1 \wedge e_2 - e_3 \wedge e_4, \tag{103}$$

$$e_1 \wedge e_3 + e_2 \wedge e_4, \tag{104}$$

$$e_1 \wedge e_4 - e_2 \wedge e_3, \tag{105}$$

If X^4 is a Riemannian manifold with an orientation, $\wedge^2 T^*X = \wedge^+ T^*X \oplus \wedge^- T^*X$. Hodge theory tells us that $H^2(X;\mathbb{R}) = \{\text{harmonic 2-forms on } X\} = \{\omega \in \Omega^2(X) | d\omega = 0 = d^*\omega\}$ where $d^* = -*d*$. Thus * gives, for an automorphism on harmonic 2-forms gives $H^2(X,\mathbb{R}) = H^+ \oplus H^-$ ($H^{\pm 1}$ are the eigenspaces of H) of dimension $b_2(X) = b_2^+(X) + b_2^-(X)$.

8 Seiberg-Witten Equations

With the Hodge star $* : \Omega^k(M) \to \Omega^{n-k}(M)$ for M a closed, oriented, dimension n, Riemannian manifold. With the exterior differential on forms $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with De Rham cohomology $H^k(M;\mathbb{R}) = ker(d)/Im(d)$. If one has a Riemannian metric, one can define the adjoint

$$d^* = \pm * d* : \Omega^{k-1}(M) \to \Omega^k(M) \tag{106}$$

Theorem 23. *(Hodge Decomposition Theorem) We can decompose k-forms $\Omega^k(M)$ as $Im(d) \oplus \mathcal{H}^k(M) \oplus Im(d^*)$ where the first two terms are the kernel of d and the last two terms are the kernel of d^*.*

$$\mathcal{H}^k(M) = \langle harmonic\ k-forms \rangle = \langle \omega | (d+d^*)\omega = 0 \rangle = \langle \omega | \Delta\omega = 0 \rangle \tag{107}$$

where $\Delta = dd^* + d^*d$.

For X^4 we have $* : \Omega^2 \to \Omega^2, *^2 = 1$. We can decompose Ω^2 as $\Omega^2_+ \oplus \Omega^2_-$ as the $\pm$ eigenspaces of $*$, with projections

$$\Pi^\pm : \Omega^2 \to \Omega^2_\pm, \Pi^\pm = \frac{1 \pm *}{2}, \Pi^+ + \Pi^- = id \tag{108}$$

From the homework we have

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^2_+ \tag{109}$$

$$d^+ = \Pi^+ \circ d \tag{110}$$

In cohomology this is

$$H^0 \xrightarrow{d} H^1 \xrightarrow{d^+} H^2_+ \tag{111}$$

where H^2_+ is the $+1$ eigenspace of $*$ acting on harmonic 2-forms $\mathcal{H}^2(X)$.

Remark 25. *$dim(H^2_+) = b^+_2(X)$ with respect to the intersection form.*

From the homework, if we have a 1-form $\alpha \in \Omega^1, d^*\alpha = 0, d^+\alpha = 0$, then $\alpha \in \mathcal{H}^2$, so $d\alpha = 0$, so we know some part of the cohomology just from this half-complex.

Suppose X^4 has some *spin*^{c} structure $(S, \gamma : TX \to Hom(S, S))$, and we know that

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix} \tag{112}$$

From our Riemannian metric we have that $TX \cong T^*X$ through g. $\gamma(e_i)$ anticommute (it's a Clifford algebra), so γ extends to a map from the wedge product $\wedge^k TX$, and we can complexify it to endomorphisms of S:

$$\gamma : \wedge^k TX \otimes \mathbb{C} \to End(S) \tag{113}$$

with $\gamma(e^{i_1} \wedge ... \wedge e^{i_k}) = \gamma(e^{i_1})...\gamma(e^{i_k})$.

Lemma 2. *$\Lambda^2_+ \subset \wedge^2 TX$ acts trivially on S^- and $\gamma : \Lambda^2_+ \to \overline{su}(S^+)$ is an isomorphism. (a 1-form takes S^+ to S^- and vise versa. A 2-form is a linear combination of wedges of 1-forms, so it takes $S^\pm$ to $S^\mp$ and back to $S^\pm$) $\overline{su}(S^+) = \langle A \in End(S^+) | A^* = -A, tr(A) = 0\rangle$. ($\overline{su}(2)$'s fiber is 3-dimensional and thus so is the lie algebra and so is Λ^2_+)*

Proof. Check in a local orthonormal basis, e.g. $e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2_+$ acts on S^- gives $B_1(-B_2^*) + B_3(-B_4^*) = 0$. We can similarly check for the other elements and do the same for S^+, and this gives the basis for skew-hermitian matrices. $\square$

Corollary 10. *For $\omega \in \Omega^2_+(X)$, this gives a section $\gamma(\omega) \in \Gamma(\overline{su}(S))$*

In bundles, we want to be able to have connections.

Remark 26. *"E-valued k-forms," for a vector bundle $E \to X$, for $\Omega^k(X; E) = \Gamma(\Lambda^k T^*X \otimes E)$ (k-forms with values in E are given by sections in $\Lambda^k w T^*X \otimes E$).*

Definition 13. *If X is a smooth manifold, $E \to X$ is a vector bundle, a* ***connection*** *A on E is an operator $\nabla A : \Gamma(E) \to \Gamma(T^*X \otimes E) = \Omega^1(X; E)$ (E-valued 1-forms) such that it satisfies the*

Leibniz rule

$$\nabla_A(fs) = df \otimes s + f(\nabla_A s), \forall f : X \to \mathbb{R}, s \in \Gamma(E) \tag{114}$$

so this gives a way of differentiating sections of a bundle.

Remark 27. *If A, B are connections, their difference doesn't satisfy the Leibniz rule, because $(\nabla_A - \nabla_B)(fs) = f((\nabla_A - \nabla_B)s)$, so $\nabla_A - \nabla_B \in \Gamma(Hom(E, T^*X \otimes E)) = \Gamma(T^*X \otimes E \otimes E^*) = \Omega^1(X; End(E))$. Thus one can think of the difference of two connections is a section on a bundle.*

Thus the space of connections on E is a space, but there is no "0" connection, so it is an affine space over $\Omega^1(X, End(E))$ (just like we described *spin*^{c} connections as an affine space over $H^2_+(X; E)$).

Definition 14. *Suppose E is a Hermitian complex vector bundle (complex vector bundle with a Hermitian form). Then a connection A on E is called **unitary** if, in addition to the Leibniz rule, if*

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle, \forall s, t \in \Gamma(E) \tag{115}$$

If A, B are unitary, with $\nabla = \nabla_A - \nabla_B$,

$$\langle \nabla_S, t\rangle + \langle s, \nabla t\rangle = 0 \Rightarrow \tag{116}$$

$$\nabla \in \Omega^1(X; \overline{u}(E) \subset End(E)) \tag{117}$$

∇ is skew-Hermitian $A = -A^*$. Unitary connections on E are an affine space over $\Omega^1(X; \overline{u}(E))$.

Definition 15. *If X has a spin^c structure (S, γ), a spin^c **connection** $\nabla = \nabla_A$ on S is a unitary connection such that*

$$\nabla_A(\gamma(v)s) = \gamma(v)\nabla_A s + \gamma(\nabla^{LC} v)s, \forall v \in \Gamma(TX)\textit{(vector fields)}, s \in \Gamma(S) \tag{118}$$

(∇^{LC} is the Levi-Civita connection) so γ is a way vector fields act on spinors. γ goes from TX to endomorphisms on s. This gives us a new multiplication, known as "Clifford multiplication."

For $A, B - spin^c$ connections, $\nabla = \nabla_A - \nabla_B$, $\nabla(\gamma(v)s) = \gamma(v)\nabla S$.

$$\nabla S = \omega \circ s, \omega \in \Omega^1(X; End(S,\gamma) \cup \overline{u}(E)) \tag{119}$$

From before $End(S,\gamma) = \{A | AB_i = B_i A, \forall i\} = \{z \cdot I | z \in i\mathbb{R}\}$. The space of $spin^c$ connections is an affine space over $\Omega^1(X; i\mathbb{R}) = i\Omega^1(X;\mathbb{R})$.

Definition 16. *The **curvature** of a connection A is*

$$\Omega^0(X;E) d_A \Omega^1(X;E) \xrightarrow{d_A} \Omega^2(X;E) \tag{120}$$

$$= \Gamma(E) \xrightarrow{\nabla_A} \Gamma(T^*X;E) \to \Gamma(\Lambda^2 T^*X \otimes E) \tag{121}$$

$F_A = d_A \cdot d_A \in \Omega^2(X; End(E)), F_A(fs) = fF_A(s)$

We have the following properties:

1. $c_1(E) = [\frac{1}{2\pi} tr(F_A)] \in Im(H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}))$.
2. For A a connection on E, we get a trace connection A^τ on $\det(E)$. $F_{A^\tau} = tr(F_A)$.

If one has X^4 with a $spin^c$ stucture, with $A = spin^c$ connection, then the curvature $F_A \in \Omega^2(X; i\mathbb{R})$, with $F_A = \frac{1}{2} F_{A^\tau}, L = \det(S^+) = \det(S^-), A^\tau =$connections in L.

$$F_A^+ = \Pi^+ \circ F_A \in \Omega^2_+(X; i\mathbb{R}) \cong_\gamma \Gamma(\overline{su}(S^+)) \tag{122}$$

If you have a spinor $\Phi \in \Gamma(S)$, you can get an element $(\Phi\Phi^*)_0 \in \Gamma(\overline{su}(S^+))$, with $A = \Phi\Phi^* \in \Gamma(S \otimes S^*) = \Gamma(End(S))$. On can think of this as a map

$$A : \Psi \to \Phi\langle \Phi, \Psi \rangle, A^* = -A \tag{123}$$

$A_0 =$ the trace-free part of A, i.e. $A - \frac{1}{2}(tr(A))I$. Let $\sigma(\Phi) = \gamma^{-1}(i\Phi\Phi^*)_0 \in \Omega^2_+(X; i\mathbb{R})$.

$F_A^+ = \sigma(\Phi) \in \Omega^2_+(X; i\mathbb{R})$ is one of the Seiberg-Witten equations.

Definition 17. *For (X^4, g), with a $(S,\gamma) = spin^c$ structure, $A = spin^c$ connection,*

$$D_A := r(S) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S) \xrightarrow{g} r(TX \otimes S) \xrightarrow{\gamma} r(S) \tag{124}$$

D_A *is the* ***Dirac operator****, sometimes denoted* $/D_A, /(\partial_A)$.

Remark 28.

$$D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix} \tag{125}$$

$$D_A^+ : r(S^+) \to r(S^-) \tag{126}$$

Example 35. *For a Euclidean space* $X = \mathbb{R}^4$*,* g *is the Euclidean metric,* $S^+ = S^- = \mathbb{C}^2$ *trivial. Then*

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix} = A_i \tag{127}$$

$D_A(S) =?$ *Well,* S *is a section* $S : \mathbb{R}^4 \to \mathbb{C}^2$*, take* A *to be the trivial connection. Then*

$$s \xrightarrow{\nabla_A} ds \xrightarrow{g} \sum e^i \frac{\partial s}{\partial x^i} \xrightarrow{\gamma} (\sum A_i \frac{\partial}{\partial x^i})s = D_A(S) \tag{128}$$

$D_A^2 = \sum_{i,j} A_i A_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = -\sum \frac{\partial^2}{\partial x_i^2} = \Delta$

On an arbitrary $X, g(S, \gamma), A :$ we have the **Weitzenböck formula**

$$D_A^2 \Phi = \nabla_A^* \nabla_A \Phi + \frac{s}{4}\Phi + \frac{1}{2}\gamma(F_A)\Phi \tag{129}$$

where $\nabla_A^* \nabla_A$ is the Laplacian for spinors, and the rest are curvature terms, s is the scalar curvator of X, coming from the fact that the riemannian metric may be nontrivial.

8.1 The Equations

For $(A, \Phi), A = spin^c$ connection and Φ is a positive spinor, i.e. an element of $\Gamma(S^+)$.

$$D_A^+ \Phi = 0 \tag{130}$$

$$F_A^+ = \sigma(\Phi) \tag{131}$$

8.2 Results

Remark 29. *These are gauge invariant; For a Gauge group* $G = \Gamma(Aut(S,\gamma)) = C^\infty(X, S^1)$*, if we have* $u \in G : u \cdot \Phi \in \Gamma(S)$*. For* $u \cdot A = A - u^{-1}du$*.*

$$u(\nabla_A(u^{-1}\Phi)) = u(u^{-1}\nabla_A\Phi + d(u^{-1})\Phi) = \nabla_A\Phi + ud(u^{-1})\Phi \tag{132}$$

$$= -\,u^{-1}du \tag{133}$$

with $uu^{-1} = 1 \Rightarrow d(uu^{-1}) = 0$*. With* $du \in \Omega^1(X; i\mathbb{R})$ *(e.g.* $u = e^f, f : X \to i\mathbb{R}$*, e.g. if* $\pi_1 X = 1$*, all* $u : X \to S^1$ *are of this form),* $u^{-1}du = df$*.*

Remark 30. *The gauge group is infinite-dimensional.*

As an exercise: if $SW(A,\Phi) = 0$, then $SW(U \cdot (A,\Phi)) = 0$. "The Seiberg-Witten equations are gauge invariant." by counting solutions to the SW equations gives SW invariants of the 4-manifold.

Definition 18. *Let* A_0 *be some fixed spin*^{c} *connection. We say* A *is in the* ***Coulomb gauge*** *with respect to* A_0 *if* $d^*(A - A_0) = 0$*.*

Everything can be put in a Coulomb gauge by applying some $u \in G$:

$$\Omega^1 = Im(d) \oplus \mathcal{H}^1 \oplus Im(d^*) \tag{134}$$

applying $v \in G$ means that one can change A by df. Thus solutions to the SW equations modulo gauge are going to be the same as SW equations in the Coulomb gauge modulo solutions that don't lift to $\mathbb{R}$, i.e. $H^1(X;\mathbb{Z})$:

$$\{\text{sol.s to SW eq.s}\}/G = \{\text{sol.s to SW eq.s in Coulomb gauge}\}/H^1(X;\mathbb{Z}) \times S^1 \tag{135}$$

where S^1 represents constant gauge transformations.

In summation, we have, for X a closed, smooth, oriented manifold, with g a Riemannian metric, and $s \in Spin^c(X)$. For simplicity, make the non-essential assumption that $\pi_1(X) = 1$. The Seiberg-Witten equations are, for a pair (A,Φ) of a *spin*^{c} connection and $\Phi \in \Gamma(S^+)$ is a positive spinor,

$$D_A^+\Phi = 0 \tag{136}$$

$$F_A^+ = \gamma^{-1}((\Phi\Phi^*)_0) \tag{137}$$

Definition 19. *Let $\mathcal{M}_{SW}$ be the moduli space $\{(A, \Phi) satisfying\ SW\}/G = \{(A, \Phi)$ satisfying SW eqs., $d^*(A - A_0) = 0\}/S^1$, where S^1 are the constant gauge transformations.*

What makes the SW equations special?

Theorem 24. *$\mathcal{M}_{SW}$ is compact. (in the C^∞ topology)*

Proof. Suppose we have a solution (A, Φ). The Weitzenböck formula is

$$D_A^2 \Phi = \nabla_A^* \nabla_A \Phi + \frac{s}{4} \Phi + \frac{1}{2} \gamma(F_A) \Phi \tag{138}$$

There is a Hermitian metric on S, so we can take the Hermitian inner product:

$$\nabla \langle \Phi, \Phi \rangle = \langle \nabla_A \Phi, \Phi \rangle + \langle \Phi, \nabla_A \Phi \rangle = 2Re(\langle \nabla_A \Phi, \Phi \rangle) \tag{139}$$

because this is a *spin*^{c} connection (it satifies the Leibniz rule) and we have a Hermitian inner product. This is then equal to

$$\frac{1}{2} \Delta |\Phi|^2 = \frac{1}{2} d^* d |\Phi|^2 = d^*(Re\langle \nabla_A \Phi, \Phi \rangle \tag{140}$$

$$= - * d * Re\langle \nabla_A \Phi, \Phi \rangle = Re(\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle - \langle \nabla_A \Phi, \nabla_A \Phi \rangle) \leq \rangle \nabla_A^* \nabla_A \Phi, \Phi \tag{141}$$

$$= \langle D_A^2 \Phi, \Phi \rangle - \frac{s}{4} \langle \Phi, \Phi \rangle - \langle \gamma(F_A) \Phi, \Phi \rangle \tag{142}$$

$$= - \frac{s}{4} \langle \Phi, \Phi \rangle - \langle \gamma(F_A) \Phi, \Phi \rangle \tag{143}$$

$(\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle - \langle \nabla_A \Phi, \nabla_A \Phi \rangle$ is not 0, because it's a pointwise equality. We do have $\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle_{L^2} = \langle \nabla_A \Phi, \nabla_A \Phi \rangle_{L^2})$ $D_A^2 \Phi = 0$, and recall that Ω_-^2 acts trivially on S^+, so $\gamma(F_A^+) \Phi = (\Phi \Phi^*)_0 \Phi$. Then we have

$$= - \frac{s}{4} |\Phi|^2 - \frac{1}{2} \langle (\Phi \Phi^*)_0 \Phi, \Phi \rangle \tag{144}$$

In a unitary basis at some $x \in X, \Phi = \begin{pmatrix} t \\ 0 \end{pmatrix}, t = |\Phi|$, and

$$\Phi\Phi^* = \begin{pmatrix} |t|^2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{145}$$

$$(\Phi\Phi^*)_0 = \frac{1}{2}\begin{pmatrix} |t|^2 & 0 \\ 0 & -|t|^2 \end{pmatrix} \tag{146}$$

$$\Rightarrow \langle(\Phi\Phi^*)_0\Phi, \Phi\rangle = \frac{1}{2}|\Phi|^4 \tag{147}$$

Thus we get

$$\frac{1}{2}\Delta|\Phi|^2 \leq -\frac{s}{4}|\Phi|^2 - \frac{1}{2}|\Phi|^4 \tag{148}$$

Since X is compact, there exists some point $x \in X$ such that $|\Phi|$ is maximal. $0 \leq \Delta|\Phi|^2 \Rightarrow \Phi = 0$ or $|\Phi|^2 \leq -\frac{s}{2}$ (in particular, if $s \geq 0$, then $|\Phi| = 0$). This is a pointwise bound on Φ, so we can integrate to get L^p bounds on Φ. We also know that $F_A^+ = \gamma^{-1}((\Phi\Phi^*)_0)$ means we can get bounds on F_A^+; recall that we can always write $A = A_0 + ia, a \in \Omega^1(X;\mathbb{R})$. Then the curvature changes by applyind d^+ to this a: $F_A^+ = F_{A_0}^+ + id^+a$. Now we have a bound on d^+a. We also know, via the Coulomb condition, $d^*a = 0$. We have the maps

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega_2^+ \tag{149}$$

so $0 = H^1(X,\mathbb{R}) = \{a|d^+a = 0, d^*a = 0\} = ker(d^+)/Im(d)$. Then

$$d^* + d^+ : \Omega^1 \to \Omega_2^+ \oplus (\Omega_0/\mathbb{R}) \tag{150}$$

is injective, Fredholm (linear elliptic operator), and a bound on $(d^+ + d^*)a$ gives a bound on a via an elliptic estimate in some Sobolev norm. From elliptic bootstrapping we get C^∞ bounds on both a, Φ. This tells us that $\mathcal{M}_{SW}$ is compact. $\square$

There are two types of solutions to the SW equations: reducible ($\Phi = 0$) and irreducible

$(\Phi \neq 0)$. We can think of them in terms of the action of S^1:$e^{i\theta} \in S^1 \cdot (\Lambda, \Phi) \to (A, e^{i\theta}\Phi)$. For irreducible solutions the S^1-action is free.

Reducible solutions $\Phi = 0, D_A\Phi = 0, F_A^+ = 0, F_{A_0}^+ + d^+a = 0$. Then $d^+a = -F_{A_0}^+, d^*a = 0$. But we know that $d^+ + d^*$ is injective, so there exists 0 or 1 reducible solutions.

Our goal now is to count irreducible solutions. It would be nice if $\mathcal{M}_{SW}$ is a manifold.

Definition 20.

$$\tilde{SW} : Conn \oplus \Gamma(S^+) \to \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-) \oplus (\Omega^0(X)/\mathbb{R}) \tag{151}$$

$$(A, \Phi) \to (F_A^+ - \gamma(\Phi\Phi^*)_0, D_A^+\Phi, d^*(A - A_0)) \tag{152}$$

where $\Omega^0(X)/\mathbb{R} = Im(d^*)$. *The above map is the* ***Seiberg-Witten map****.*

Then $\mathcal{M}_{SW} = \tilde{SW}^{-1}(0)/S^1$ is a manifold by transversality, if the Seiberg-Witten map is indeed transverse.

$$d\tilde{SW}_{(A,\Phi)} = (d^+ + \langle\Phi, \cdot\rangle, D^+_{A_0}, ..., d^*) \tag{153}$$

Together these form a linear elliptic operator, which is Fredholm between suitable spaces. index$(d\tilde{SW}) = dim(ker) - dim(coker) \in \mathbb{Z}$, which is invariant under deformation. From the **Atiyah-Singer index theorem**, we have

$$index(d\tilde{SW}) = \frac{c_1(s)^2 - \sigma(x)}{4} - b_2^+(X) + b_1(X) \tag{154}$$

Definition 21. *The* ***Perturbed SW eq.s*** *are*

$$\tilde{SW}(A, \Phi) = (\eta, 0, 0), \eta \in \Omega^2_+(X; i\mathbb{R}) \tag{155}$$

$\tilde{SW}^{-1}(\eta, 0, 0)$ *is still compact.*

Theorem 25. *(Transversality theorem, Sard's theorem in infinite dimensions) For a generic* η, $\tilde{SW}^{-1}(\eta, 0, 0)$ *is a smooth manifold, of dimension* $index(d\tilde{SW})$. *So if this map is surjective, then it's a regular value, and the dimension of the cokernel is 0, and the dimension of the manifold is the dimension of the kernel of* $d\tilde{SW}$, *which is the index. The Transversality theorem tells you that it suffices to consider these kinds of perturbations.*

This follows from the generalization of Sard's theorem to infinite dimensions.

We still have a problem. Even if $\mathcal{M}_{SW}$ is a manifold, we want to divide by S^1, and normally if the action is free, we can divide and still get a manifold. But this action is not quite free because we have those pesky reducibles. How many reducibles are there to these perturbed SW equations?

$$\tilde{SW}(A, \Phi)|_{\Phi=0} = (\eta, 0, 0) \tag{156}$$

gives us $F_A^+ = \eta$, and $F_{A_0}^+ + d^+ a = \eta$. $d^* a = 0, d^+ a = \eta - F_{A_0}^+$.

$$\Omega^0 \to \Omega^1 \xrightarrow{d^+} \Omega^2_+ \tag{157}$$

with cohomology H^0, H^1, H^2_+. "Reducibles exist if and only if $\eta - F_{A_0}^+ \in Im(d^+)$" is a codimension $b^2_+(X)$ condition. If $b^2_+(X)$ is zero, then we're in trouble, because we'll always get reducibles. The conclusion is then: If $b^2_+(X) > 0$, for generic η there are no reducibles. Then $\mathcal{M}_{SW,\eta} = \mathcal{M}_{SW}(X, s, g, \eta) = \tilde{SW}^{-1}(\eta, 0, 0)/S^1$ is a smooth compact manifold of dimension

$$d := \frac{c_1(S)^2 - \sigma}{4} - b_2^+ + b_1 - 1 \tag{158}$$

Recall $\sigma = b_2^+ - b_2^-, \chi = 2 - 2b_1 + b_2^+ + b_2^-$, so

$$d = \frac{c_1(S)^2 - (3\sigma + 2\chi)}{4} \tag{159}$$

so we have a very nice space of solutions. We would like to count the solutions, so we would like this manifold to be 0-dimensional. If $d = 0$, $\mathcal{SW}, \eta = \langle \text{fin. many points} \rangle$, and $SW_X(s, g, \eta) = \#\mathcal{SW}, \eta$, where the $\#$ is counting with signs; to fix orientations, we need to choose a homology orientation on X, i.e. orient $H^0(X) \oplus H^1(X) \oplus H^2_+(X)$.

Remark 31. *$SW_X(s, g, \eta) = \#\mathcal{SW}, \eta$ are called the **Seiberg-Witten invariants**, which can also be defined when $d > 0$ (we need d to be even), but in all known examples, $SW_X = 0$ for $d > 0$.*

Definition 22. *X is called **of simple type** if $SW_X(S) = 0 \forall s$ with $d > 0$.*

Conjecture 4. *(Witten) All X^4 are of simply type. This is known to be true for symplectic 4-manifolds (e.g. this contains all complex projective surfaces: $\mathbb{C}P^2, S^2 \times S^2, K3, \ldots$ all are surfaces in projective space, and we can pull back the Fubini-Study form on $\mathbb{C}P^n$ and get a symplectic form on the manifold).*

Theorem 26. *If $b_+^2(X) \geq 2$, then $SW_X(S, g, \eta)$ is independent of generic (g, η). We need this condition; before we assumed it was at least 1 so there were no reducibles. Now we assume that it is at least 2 and get the proof:*

Proof. (g_0, η_0), then $(g_1, \eta_1) \in Met \times \Omega_+^2$. Interpolate by a family (g_t, η_t), and look at the moduli space of solutions in this family: $\cup_{t\in[0,1]}\mathcal{M}_{SW}, (g_t, \eta_t)$ is a smooth manifold of dimension $d+1 = 1$. And we can arrange that it's smooth and compact, but we need to avoid reducibles in a 1-parameter family; the codimension b_+^2 condition implies that we need $b_+^2 > 1$, implying that we have a 1-manifold with boundary $(-\mathcal{M}_{SW}(g_0, \eta_0)) \cup (\mathcal{M}_{SW}(g_1, \eta_1))$ □

Remark 32. *Our notation states $SW_X(S), s \in spin^c(X)$. We assume that $b_2^+ \geq 2, d = 0$.*

Remark 33. *When $b_2^+ = 1$, there exists a wall of perturbations where reducibles exist. There are 2 chambers of perturbations. We have invariants $SW_X^+(S), SW_X^-(S), SW_X^+ - SW_X^- = \pm 1$.*

Recall that the space of $spin^c(S)$ is an affine space over $H^2(X;\mathbb{Z})$. When $\pi_1(X) = 1$, we have a map

$$c_1 : Spin^c(X) \to H^2(X;\mathbb{Z}) \tag{160}$$

that is injective.

$$Spin^c(X) \cong \{k \in H^2(X;\mathbb{Z}) | \langle k, a\rangle \equiv \langle a, a\rangle \mod 2, \forall a\} = Char(x) \tag{161}$$

Then we can define the Seiberg-Witten invariants of X as

$$SW_X : Char(X) \to \mathbb{Z} \tag{162}$$

where $SW_X(k) = SW_X(S), k = c_1(S)$.

Definition 23. *K is called a* ***basic class*** *if $SW_X(k) \neq 0$.*

9 Properties of Seiberg-Witten Invariants

$SW_X : Char(X) \subset H^2(X;\mathbb{Z}) \to \mathbb{Z}$

1. There are only finitely many basic classes. (a stronger version of compactness), i.e. $SW_X(s) = 0$ for all but finitely many s.

2. If X admits a metric of positive scalar curvature, then the Seiberg-Witten invariant is 0, due to the Weitzenböck formula; because $s \geq 0$, $\Phi = 0$ so there are no irreducibles.

3. $SW_X(-s) = (-1)^{b_2^+(X)-b_1(X)+1} SW_X(s)$

4. If $X = X_1 \# X_2, b_2^+(X_i) \geq 2$, then $SW_X = 0$

5. If X is of simple type (Seiberg-Witten invariants are nonzero only when the expected dimension is zero) with basic classes $K_i, i = 1, ..., s$, then $X' = X \# \overline{\mathbb{C}P^2}$ has basic classes $\{K_i \pm E | i = 1, ..., s\}, E \in H^2(\mathbb{C}P^2; \mathbb{Z})$ is a generator, and $SW_{X'}(K_i \pm E) = \pm SW_X(K_i)$. (called the blow-up formula)

6. If X is a complex projective surface, then $SW_X(\pm c_1(TX)) = \pm 1$ (sign doesn't matter because of symmetry)

7. (Taubes) If X is symplectic, then we can choose an almost-complex structure J combatibly with the symplectic structure, and (TX, J) is contractible, and furthermore $SW_X(\pm c_1(TX, J)) = \pm 1$. (This generalizes (6): $X \hookrightarrow \mathbb{C}P^n, \omega_{FS} = i^*\omega$ is a symplectic form on X) (Pullback of the Fubini-Study form)

8. **Adjunction Inequality:** If $\Sigma \subset X$ is an embedded, oriented closed surface, $[\Sigma]^2 \geq 0, [\Sigma] \neq 0$, then there exists K-basic class (SW invariant is nonzero) on X, $2g(\Sigma) - 2 \geq [\Sigma]^2 + |K \cdot [\Sigma]|$. Furthermore, if X is of simple type and $g(\Sigma) \neq 0$, the inequality is also true if $[\Sigma]^2 < 0$, so basically it's almost always true.

We have some "fun" applications: existence of exotic smooth structures in dimension 4.

Example 36. $X_1 = K \# \overline{\mathbb{C}P^2}, X_2 = \#^3 \mathbb{C}P^2 \#^{20} \overline{\mathbb{C}P^2}$. *Both are simply connected. What are their intersection forms?* $2(-E_8) \oplus 3\sigma_x \oplus \langle -1 \rangle$, *and* $3\langle 1 \rangle \oplus 20\langle -1 \rangle$. *They are both odd (and therefore non-spin), indefinite, so it is determined by* σ, χ *(in fact diagonal). Thus by Freedman* X_1 *is homeomorphic to* X_2*. What about diffeomorphic?* $SW_{K3}(0) = \pm 1$ *because it's a complex projective surface (property 6).* $K3$ *is Calabi-Yau, so* $c_1(TK3) = 0$ *(Recall that* $Z_d \subset \mathbb{C}P^3$ *gives* $c_1 = (4-d)h, Z_4 = K3$*). Then we apply the Blow-up formula (property 5) to get* $SW_{X_1}(E) = \pm 1$*. However,* $SW_{X_2} = 0$*, because of property 2:* X_2 *has positive scalar curvature. By a theorem of Schoen and Yau, if* M_1, M_2 *are manifolds of dimension* ≥ 3 *and admit positive scalar curvature, then so does the connected sum. We apply this to the Fubini-Study metric on* $\mathbb{C}P^2, \overline{\mathbb{C}P^2}$*. This implies* $SW_{X_2} = 0$*. Or we can use property 4:* $X_2 = (2\mathbb{C}P^2) \# (\mathbb{C}P^2 \#^{20} \overline{\mathbb{C}P^2})$*. Each side has* $b_2^+ \geq 1$*, and therefore* SW_{X_2} *is 0. Notice we couldn't have done this with* X_1*, because it doesn't split into two parts of* $b_2^+ \geq 1$*. Therefore* $SW_{X_1} \neq SW_{X_2} \Rightarrow X_1, X_2$ *are not diffeomorphic.*

Example 37. *$X_1 = Z_5 \subset \mathbb{C}P^3$ vs. $X_2 = 9\mathbb{C}P^2\#^{44}\overline{\mathbb{C}P^2}$. We can compute that $Q_{X_1} = 9\langle 1\rangle \oplus 44\langle -1\rangle = Q_{X_2}$, so X_1, X_2 are homeomorphic. Then we apply property 6: $SW_{X_1} \neq 0$, and then through either 2 or 4 $SW_{X_2} = 0$, so X_1, X_2 are not homeomorphic.*

There are other applications.

Theorem 27. *If X^4 is closed and symplectic, then there is no decomposition $X = X_1\#X_2$ with $b_2^+(X_i) > 0, i = 1, 2$.*

Theorem 28. *There exist (simply connected) almost complex 4-manifolds that are not symplectic. An easy example for non-simply connected manifolds is $S^1 \times S^3$. There $H_2 = 0$, whereas a symplectic form must have a nontrivial 2nd homology class.*

Recall the definition of an almost-complex structure.

Definition 24. *$J \in End(TX)$ with $J^2 = -Id$, and (TX, J) is a complex bundle.*

From the homework we know that X^4 has an almost complex structure (with $\pi_1(X^4) = 1$) if and only if $b_2^+(X)$ is odd. We proved this using characteristic classes.

Example 38. *$X = \#^3\mathbb{C}P^2, b_2^+ = 3$ gives us that X is almost complex. $X = \mathbb{C}P^2\#^2\mathbb{C}P^2$ (or using positive scalar curvature), so $SW_X = 0$ gives us that X is not symplectic.*

Proof. (SKETCH OF PROPERTY 3) Symmetry: for a *spin*^{c} structure $(S, \gamma : TX \to End(S) = S \otimes S^* = End(S^*))$ which gives a conjugate *spin*^{c} structure (S^*, γ). Since this is a complex bundle we have $S^* \cong \overline{S}$ using the Hermitian metric. We have

$$c_1(S^+) = -c_1(\overline{S^+}) \tag{163}$$

via properties of chern classes. Then we have a one-to-one correspondence of solutions of SW equations for (S, γ) with solutions of SW equations for (S^*, γ). □

Proof. (SKETCH OF PROPERTY 4) Connected sums: $X = X_1\#X_2$. We stretch the metric when we add a cylinder $S^3 \times [-T,T], T \to \infty$ (called "neck stretching"). In the limit,

$$\tilde{\mathcal{M}}_{SW}(X) \cong \tilde{\mathcal{M}}_{SW}(X_1) \times \tilde{\mathcal{M}}_{SW}(X_2), \tag{164}$$

$$\tilde{\mathcal{M}}_{SW}(X) = \{(A,\phi)| \text{ satisfy SW}, d^*(A-A_0)=0\} \tag{165}$$

$$\mathcal{M}_{SW}(X) = \tilde{\mathcal{M}}_{SW}(X)/S^1 \tag{166}$$

If we want to think of $\mathcal{M}_{SW}(X)$ itself, it is an S^1-bundle over $\mathcal{M}_{SW}(X_1) \times \mathcal{M}_{SW}(X_2)$. So the solution on X is a tripli (solution on X_1, solution on X_2, and a gluing parameter in S^1. Recall that the expected dimension of $\mathcal{M}_{SW}(X)$, i.e. $d(X,s)$, is $\frac{c_1(S)^2-\sigma}{4} - (b_2^{-b_1+1})$. and what we have is

$$d(X,s) = d(x_1,s_1) + d(x_2,s_2) + 1 = 0 \tag{167}$$

$$\Rightarrow \text{one of } d(X_i,s_i) \text{ must be } < 0 \tag{168}$$

Therefore there are no irreducible solutions for such (X_i,s_i), so $SW_X = 0$. □

Proof. (SKETCH OF PROPERTY 5) Blow-up Formula $b_2^+(\overline{\mathbb{C}P^2}) = 0$ means that reducible solutions exist generically. We get exactly one reducible solution on $(\overline{\mathbb{C}P^2}, \pm E)$ even though $d(\overline{\mathbb{C}P^2}, \pm E) = -1$ (because we divided by S^1, the action of S^1 is trivial on a point), and we can pair it with irreducibles on some random (X,s), and get irreducibles on $(X\#\overline{\mathbb{C}P^2}, s \pm E)$. □

Proof. (SKETCH OF PROPERTY 6) We can interpret solutions to SW equations on a complex surface as divisors (complex curves) $s = c_1(TX) \Rightarrow$ empty curve, given $SW_X = \pm 1$. □

Proof. (SKETCH OF PROPERTY 7) On symplectic manifolds, via Taubes, SW are Gromov-Witten invariants; SW equations count J-holomorphic curves. □

Proof. (SKETCH OF FIRST PART OF PROPERTY 8) Adjunction inequality, part a): For $\Sigma \hookrightarrow X, [\Sigma]^2 \geq 0, [\Sigma] \neq 0, SW_X(k) \neq 0 \Rightarrow 2g(\Sigma) - 2 \geq [\Sigma]^2 + |k \cdot [\Sigma]|$. It suffices to prove $k \cdot [\Sigma] + [\Sigma]^2 \leq 2g(\Sigma) - 2$ because we can replace k by $-k$ using symmetry. It also suffices to prove it assuming $[\Sigma]^2 = 0$, because we can reduce to this case using the blow-up formula:

$$\tilde{X} = X\#\overline{\mathbb{C}P^2}\tilde{\Sigma} = \Sigma\#\overline{\mathbb{C}P^1} \subset \overline{\mathbb{C}P^2} \tag{169}$$

so we get a surface of the same genus. We get K-basic if and only if $K - E$ is basic. $[\tilde{E}]^2 = [\Sigma]^2 + [\overline{\mathbb{C}P^1}]^2 = [\Sigma]^2 - 1$. If adjunction is true in $\tilde{X}$, then

$$2g(\tilde{\Sigma}) - 2 \geq [\tilde{\Sigma}]^2 - 1 + K \cdot [\Sigma] - E \cdot [\overline{\mathbb{C}P^1}] \tag{170}$$

$$\Rightarrow 2g(\Sigma) - 2 \geq K \cdot [\Sigma] \tag{171}$$

So we blow up until we get the self-intersection point is equal to 0, and apply adjunction there. We claim that $[\Sigma]^2 = 0 \Rightarrow K \cdot [\Sigma] \leq 2g(\Sigma) - 2$. Well, if $[\Sigma]^2 = 0$, then we have a neighborhood of Σ isomorphic to $\Sigma \times D^2 \subset X$.

*We know that $SW_X(K) \neq 0$, so $\mathcal{M}_{SW}(X, S, g) \neq 0, \forall$ metrics g on X (for *any* metric on X, because nonzero is an open condition). Otherwise we can choose (g_t, η_t) given by generic metrics and perturbations, giving us $(g, 0)$; $\mathcal{M}_{SW}(X, s, g_t, \eta_t) \neq 0$, there exists a solution for (g_t, η_t). Take the limit as $t \to 0$ to get a solution for $(g, 0)$ (use compactness in families).*

So there exists a solution (A, Φ) so SW equations on X, metric g, $\eta = 0$. Then we take a neighborhood of our surface $\Sigma \times D^2$. Then we insert a cylinder (stretch the neck) $[0, R] \times S^1 \times \Sigma$ meeting the rest of the manifold $X - (\Sigma \times D^2)$. This gives us a Riemannian metric g_R, which is the product metric on the cylinder $[0, R] \times S^1 \times \Sigma$. On sigma we'll make the metric with constant curvature such that volume(Σ) is equal to 1. $\square$

We prove the Adjunction Inequality:

Proof. We reduced to the case $[\Sigma]^2 = 0$. We want to show that $2g(\Sigma) - 2 \geq K \cdot [\Sigma]$. $[\Sigma]^2 =$ means that we have a tubular neighborhood of the form $D^2 \times \Sigma$. Then we take X as $[D^2 \times \Sigma | [0, R] \times \Sigma \times S^1 - (D^2 \times \Sigma)]$. This gives a metric g_R. On the cylinder it is the product metric, and on Σ we have constant curvature (can talk about Gaussian curvature or scalar curvature), and volume $=$ 1. We use the fact that K is a basic class, i.e. the SW equations have solutions for any metric and any perturbation. For this metric and no perturbation, there exists an irreducible solution (A, ϕ) to the SW equations. From the proof of compactness, we got that $|\phi|^2 \leq -\frac{s}{2}$, where s is the scalar curvature. One of the quations is $F_A^+ = \gamma^{-1}((\phi\phi^*)_0)$; at a point, $\Phi = \begin{pmatrix} t \\ \\ 0 \end{pmatrix}, t = |\phi|$ gives us

$$(\phi\phi^*)_0 = \frac{1}{2}\begin{pmatrix} t^2 & 0 \\ & \\ 0 & -t^2 \end{pmatrix} \tag{172}$$

so $|(\phi\phi^*)_0| = \frac{1}{4}t^4 \leq \frac{1}{4}(\frac{s}{2})^2 = \frac{s^2}{16}$ and thus $|F_A^+|^2 \leq \frac{s^2}{8}$.

$$|F_A^+|^2 \leq \frac{s^2}{8} \Rightarrow \int_x |F_A^+|^2 dvol \leq \frac{1}{8}\int s^2 dvol \tag{173}$$

From Chern-Weil theory, $c_1(S^+) = [\frac{i}{2\pi}F_A]$

Lemma 3. *For $\alpha \in \Omega^2(X^4)$ closed, then $[\alpha]^2 = ||\alpha^+||^2_{L^2} - ||^-||^2_{L^2}, \alpha^\pm =^(\alpha) \in \Omega^2_\pm$.*

Proof.

$$[]^2 = \int_x \alpha \wedge \alpha = \int (\alpha^+ + \alpha^-) \wedge (\alpha^+ + \alpha^-) \tag{174}$$

$$= \int \alpha^+ \wedge \alpha^+ + \int \alpha^- \wedge \alpha^- + 2\int \alpha^- \wedge \alpha^+ \tag{175}$$

Since $\int \alpha^+ \wedge \alpha^- = -\int \alpha^+ \wedge *\alpha^- = \langle \alpha^+, \alpha^- \rangle = 0$, we get

$$= \int \alpha^+ \wedge *\alpha^+ - \int \alpha^- \wedge *\alpha^- + 0 \tag{176}$$

$$= \langle \alpha^+, \alpha^+ \rangle_{L^2} - \langle \alpha^-, \alpha^- \rangle_{L^2} \tag{177}$$

□

We apply this to $\frac{i}{2\pi}F_A$:

$$4\pi^2 c_1(S^+)^2 = \int |F_A^+|^2 dvol - \int |F_A^-|^2 dvol \tag{178}$$

Since

$$\int_x |F_A|^2 dvol = \int |F_A^+|^2 dvol + \int |F_A^-|^2 dvol \tag{179}$$

$$= 2\int |F_A^+|^2 dvol - 4\pi^2 c_1(S^+)^2 \tag{180}$$

$$\leq (\frac{1}{4}\int s^2 dvol) + const \tag{181}$$

$\frac{1}{4}\int s^2 dvol$ is some constant, i.e. independent of R. Then we can say

$$\frac{1}{4}\int s^2 dvol = const + R\frac{1}{4}\int_\Sigma s_\Sigma^2 \tag{182}$$

The Gauss-Bonnet formula gives

$$\int_\Sigma K = 2\pi(2g-2) \Rightarrow \tag{183}$$

$$K = 2\pi(2g-2) \tag{184}$$

(since the volume of Σ is 1). Thus we have $\frac{s}{\Sigma} = 2 * 2\pi(2g-2)$. Then our two facts

$$\int_X |F_A|^2 dvol \leq (\frac{1}{4}\int s^2 dvol) + const, \frac{1}{4}\int s^2 dvol = const + R(2\pi(2g-2))^2 \tag{185}$$

give us

$$\int_X |F_A|^2 dvol \leq const + r(2\pi(2g-2))^2 \tag{186}$$

This integral is $\geq$ the integral on the cylinder,

$$\int_{S^1\times\Sigma\times[0,R]} |F_A|^2 dvol \leq \int_X |F_A|^2 dvol \tag{187}$$

which is just $\geq R(\int_\Sigma F_A)^2$ via Cauchy-Schwartz. Apply Chern-Weil to the surface, and we get

$$R(2\pi\langle c_1(S^+), [\Sigma]\rangle)^2 \tag{188}$$

Divide by R and take $R \to \infty$, we get

$$2\pi\langle c_1(S^+), \Sigma\rangle \leq 2\pi(2g-2) \Rightarrow \langle c_1(S^+), \Sigma\rangle \leq 2g \cdot 2. \tag{189}$$

$\square$

Remark 34. *This can be extended to $[\Sigma]^2 < 0$, assuming X is of simple type.*

Theorem 29. *(Adjunction Formula) Suppose X^4 has an almost complex structure J. Suppose we have a $\Sigma \subset X$ that is $J-$holomorphic, i.e. $J^*(T\Sigma) = T\Sigma$. Then*

$$2g(\Sigma) - 2 = [\Sigma]^2 - c_1(TX, J)[\Sigma] \tag{190}$$

Proof. $TX|_\Sigma = T\Sigma \oplus \nu\Sigma$. So

$$c_1(TX)[\Sigma] = c_1(T\Sigma) + c_1(\nu\Sigma)[\Sigma] = \chi(E) + [\Sigma]^2 \tag{191}$$

since $c_1(T\Sigma), \chi(\Sigma) = 2 - 2g(\Sigma)$. $\square$

This applies to Σ a complex curve $\subset X$, a complex projective surface, or Σ is a J-holomorphic curve $\subset X$ a symplectic 4-manifold.

Theorem 30. *(Symplectic Thom Conjecture, proved by Ozsváth-Szabó, 1998) For X^4 symplectic, and $\Sigma \subset X$ is symplectic (i.e. $\omega|_\Sigma$ is a volume form). Then Σ is genus minimizing in its homology class.*

Proof. If $\Sigma = S^2$, there's nothing to prove. Note that $[\omega]^2 = [\omega \wedge \omega] > 0 \Rightarrow b_2^+(X) > 0$. Assume $b_2^+(X) \geq 2$, so the SW invariants are well-defined. It is a theorem that X-symplectic implies that X is of simple type, so we can apply adjunction. From Taubes, if X is symplectic, then $K = -c_1(TX)$ is a basic class ($SW_X(K) = \pm 1$). Also, there exists a compatible J such that Σ is a J-holomorphic curve. If $S \subset X$ is another surface $[S] = [\Sigma]$, then by Adjunction we know that

$$2g(S) - 2 \geq [\Sigma]^2 + K \cdot [\Sigma] = 2g \tag{192}$$

by the Adjunction inequality and the Adjunction formula, respectively. This implies that $g(\Sigma) \leq$

$g(S)$. This argument can be extended to $b_2^+(X) = 1$, e.g. $X = \mathbb{C}P^2$. Here we have to be careful to consider SW in a particular chamber. $\square$

Theorem 31. *(Thom Conjecture, proved by Kronheimer and Mrowka in 1994). For $S \subset \mathbb{C}P^2$ smoothly embedded, $[S] = d[\mathbb{C}P^1]$, then*

$$genus(S) \geq \frac{(d-1)(d-2)}{2} \tag{193}$$

with $\frac{(d-1)(d-2)}{2}$ the genus of an algebraic curve of degree d, which we'll call Σ.

Proof. Apply the above. $2g(\Sigma - 2 = [\Sigma]^2 + K \cdot [\Sigma] = d^2 - 3d$ $\square$

Corollary 11. *(Local Thom Conjecture) If $\Sigma \subset \mathbb{C}^2$ is an affine smooth curve, then Σ is locally genus minimizing, i.e. for $B \subset \mathbb{C}^2$ a ball, $\partial B \pitchfork \Sigma$, and another surface $S \subset B$ is a surface such that $\partial B \cap S = \partial B \cap \Sigma$, then $g(S|_B) \geq g(\Sigma|_B)$.*

Proof. Compactify Σ to $\overline{\Sigma} \subset \mathbb{C}P^2$. $\overline{\Sigma}$ may not be smooth but we can deform it to $\overline{\Sigma}_\epsilon$ smooth. We can isotope S to S_ϵ (same genus) such that $S_\epsilon \cap \partial B = \overline{\Sigma}_\epsilon \cap \partial B$. The Thom Conjecture says that $g(S_\epsilon \cup (\overline{\Sigma}_\epsilon - B)) \geq g(\overline{\Sigma}_\epsilon) \Rightarrow g(S_\epsilon|_B) \geq g(\overline{\Sigma}_\epsilon|_B)$, which is equal to $g(S|_B) \geq g(\Sigma|_B)$. $\square$

Definition 25. *For any $K \subset S^3$ knot, the **4-ball genus** or **slice genus** of K is*

$$g_s(K) = min\{g(S) | S \subset B^4, \partial S = S \cap \partial B = K\} \tag{194}$$

for S smoothly and properly embedded.

Definition 26. *K is a **slice knot** if $g_S(K) = 0$. i.e. bounds $D^2 \subset B^4$.*

Example 39. *See Figure 19. (In four dimensions this is embedded, because we can have different colors where the disk intersects itself)*

Corollary 12. *If K arises as a transverse intersection $\partial B \cap S, S \subset \mathbb{C}^2$ is an affine curve. Then $g_s(K) = g(S|_B)$.*

Example 40. *Torus knots $T_{p,q}, p, q \geq 1, gcd(p,q) = 1$. Written explicitly in coordinates, if we*

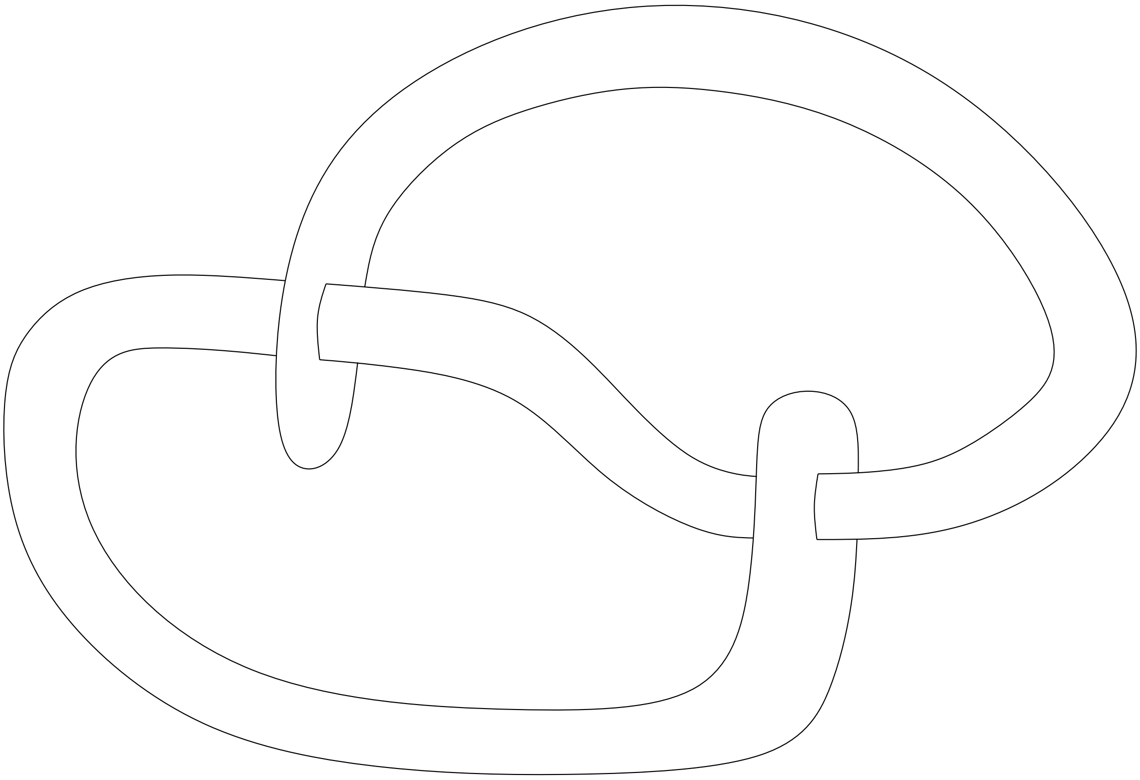


Figure 19: A slice knot.

have

$$S_0 = \{x^p - y^q = 0\} \subset \mathbb{C}^2 \tag{195}$$

then $T_{p,q} = S_0 \cap B(\sqrt{2}) \subset T^2 = \{x||x| = 1\}\{y||y| = 1\}$. *If* $(x,y) \in S_0$*, then* $|x| = |y|$*, and* $(x,y) \in \partial B(\sqrt{2}), x^p = y^q \Rightarrow x = e^{iq\theta}, y = e^{ip\theta}, \theta \in [0, 2\pi]$*, so* $(x,y) \in T$. S_0 *as a singularity at 0, but we can deform it so* S_ϵ*, e.g.* $S_\epsilon = \{x^p - y^q = \epsilon\}$. *This is smooth, so*

$$S_\epsilon \cap \partial B(\sqrt{2}) \tag{196}$$

which is isotopic to $T_{p,q}$. *What is* $g(S_\epsilon \cap B(\sqrt{2}))$*? We can compute this using the Riemann-Hurwitz formula. We look at the* ($q:1$ *cover) map*

$$S_\epsilon \to \mathbb{C} \tag{197}$$

$$(x,y) \mapsto x \in \mathbb{C} \tag{198}$$

This is a $q:1$ cover, but with p branch points: when $x^p = \epsilon$, there exist a unique preimage $(y=0)$. By the ***Riemann-Hurwitz formula****,* $\chi(S_\epsilon \cap B) = q\chi(D^2) - p(q-1)$. *This is* $p+1-pq$. $\chi(S_\epsilon \cap B) = 1 - 2g(S_\epsilon \cap B)$, *so*

$$g(S_\epsilon \cap B) = \frac{(p-1)(q-1)}{2} \tag{199}$$

The Local Thom Conjecture implies the

Theorem 32. *(Milnor Conjecture, proved by Kronheimer and Mrowka, 1993, 1 year before SW equations, so using Yang-Mills)*

$$g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2} \tag{200}$$

10 Knots

Suppose we have $K \subset S^3$ a knot.

Definition 27. *The* ***unknotting number*** $u(K)$ *is minimal number of crossing changes needed to turn K into the unknot.*

e.g. $u(K) = 0$ if and only if K is the unknot. The trefoil knot has $u(K) = 1$. (trefoil is $T_{2,3}$)

Lemma 4. $g_s(K) \leq u(K)$. *(The slice genus is less than or equal to the unknotting number)*

Proof. If we have k crossing changes from K to U, we get a surface $\subset B^4, \partial\Sigma = K, g(\Sigma) = k$. Going from one crossing to its resolution corresponds to the surface's saddle point. □

Exercise: The standard diagram of $T_{p,q}$ can be unknotted in $\frac{(p-1)(q-1)}{2}$ moves.

Corollary 13. $u(T_{p,q} = \frac{(p-1)(q-1)}{2}$

Proof. $\frac{(p-1)(q-1)}{2} = g_s(T_{p,q})$ via the Milnor Conjecture, and this is $\leq u(K) \leq \frac{(p-1)(q-1)}{2}$. □

Which knots arise as a surface $S \cap \partial B^4$ where $S \subset \mathbb{C}^2$ is affine (algebraic), and $S \pitchfork B^4$? To answer this we need some notions.

Definition 28. *A* ***braid*** *is a closed loop in* $Conf_n(\mathbb{R}) = \{\underline{x} \subset \mathbb{R}^2 | Card(\underline{x}) = n\}$ *starting and ending at* $\{(1,0),(2,0),...,(n,0)\}$.

Remark 35. *Every braid is a composition of crossings.*

Definition 29. $B_n = \pi_1(Conf_n(\mathbb{R}^2))$ *is the* ***braid group*** *- generated by* $\sigma_i, i = 1, ..., n-1$*. The relations are*

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2, \tag{201}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{202}$$

Definition 30. *A braid is called* ***positive*** *if it's a product of* $\sigma_i s$ *only (not their inverses).*

Definition 31. *A braid is* ***quasi-positive*** *if it's of the form*

$$\prod_{k=1}^{m} w_k \sigma_i w_k^{-1}, \tag{203}$$

$$w_k \in B_n \text{ is any word} \tag{204}$$

If we have a braid $b \in B_n$, we can take its closure $\hat{b}$, where we close its strands. See the top of Figure 20. This gives a link. In the case of the diagram, it's a knot.

Theorem 33. *Every link is the closure of a braid.*

Definition 32. *A knot (or link) is* ***braid positive*** *(resp.* ***quasipositive****) if it's the closure of a positive (resp. quasipositive) braid.*

Example 41. *The torus knots* $T_{p,q}$ *are braid positive.*

Theorem 34. *(Rudolph, Boileau-Orevkov)* $K \subset S^3$ *is of the form* $S\cap$*, where* $S \subset \mathbb{C}^2$ *is an algebraic curve,* $S \pitchfork \partial B$ *if and only if* K *is quasipositive. In fact, if* $K = \hat{b}$*, where* b *is quasipositive:* $b = \prod_{k-1}^{m}(w_k \sigma_i w_k^{-1})$ *then* K *bounds a complex curve of genus* $\frac{m-n+1}{2}$*.*

Corollary 14. *(above theorem and the Local Thom Conjecture):* $g_s(K) = \frac{m-n+1}{2}$ *for quasipositive* K*.*

Remark 36. *For braid positive knots, one can show that this is also* $u(K) = g_s(K) = \frac{m-n+1}{2}$*. For general quasipositive knots,* $u(K)$ *may be bigger than* $g_s(K)$*. The* 8_{20} *knot is quasipositive, slice* ($g_s(K) = 0$)*, but* $u(K) = 1$*.*

See the bottom of Figure 20 for the 820 knot. See: Knot Info, Knot Atlas.

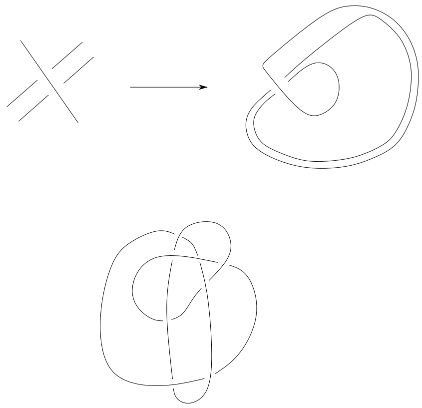


Figure 20: The braid closure and 820 knot.

11 Donaldson's Diagonalizability Theorem

Theorem 35. *(Donaldson's Diagonalizability Theorem) If X^4 smooth, closed, simply-connected, and Q_X is definite, then Q_X is diagonal.*

Remark 37. *This was originally proved by Donaldson in 1982 using Yang-Mills theory.*

Proof. (Seiberg-Witten version) Let's say Q_X is negative definite (otherwise switch orientations). Then $b_2^+(X) = 0$, so we cannot avoid reducibles among solutions to SW equations. Picka metric g and a $Spin^c$ structure, $c_1(S^+) = K \in Char(X)$.

$$\tilde{\mathcal{M}}_{SW} = \{(A, \phi) | d^*(A - A_0) = 0\} \tag{205}$$

$$\mathcal{M}_{SW} = \tilde{\mathcal{M}}_{SW}/S^1 \tag{206}$$

By transversality, for a generic perturbation η, $\tilde{\mathcal{M}}_{SW}$ is a smooth manifold of dimension $\frac{K^2-\sigma}{4} - b^+ + b_1 = \frac{K^2+b_2}{2}$, since $b_2(X) = -\sigma(X) = b_2^-(X)$ (negative because it's negative-definite) $:= d+1$, where d is the expected dimension of $\mathcal{M}_{SW}$. We have some reducibles. How many? Well, a reducible is $(A,\phi), \phi = 0$, so all we need to satisfy is $F_A^+ = \eta$, $F_{A_0}^+ + d^+(A - A_0) = \eta, A - A_0 :=$ $a \in \Omega^1(X; i\mathbb{R})$. Thus we need to solve

$$d^+a = \eta - F_{A_0}^+ \tag{207}$$

We have the sequences

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^2_+ \tag{208}$$

$$H^0 \to H^1 \to H^2_+ \tag{209}$$

$H^1 = H^1_+ = 0$. The $d^+a = \eta - F_{A_0}^+$ has a solution which is unique mod gauge. $(d^*a = 0)$, and $ker(d^+ + d^*) = H^1 = 0$, so there exists a unique reducible x. The local model in $\tilde{M}_{SW}$ near x where the S^1 action is trivial and everywhere else it is free. There is a diffeomorphism taking this to the origin in $\mathbb{C}^m$:

$$e^{i\theta}(z_1, ..., z_m) \to (e^{i\theta}z_1, ..., e^{i\theta}z_m) \tag{210}$$

So when we divide by S^1 in $\mathcal{M}_{SW}$, we get a cone on $S(\mathbb{C}^m)/S^1 = \mathbb{C}P^{m-1}$, with x as the point. So when we divide by gauge, this is why we don't want reducibles.

Let $\mathcal{M}^*_{SW} = \mathcal{M}_{SW} - \{x\}$ be the irreducible locus. This is a smooth manifold of dimension $d = 2m - 1$. We can take a line bundle $L \to \mathcal{M}^*_{SW}$, with

$$L_{[(A,\phi)]} = \{(A, z\phi) | z \in \mathbb{C}\}, \tag{211}$$

$$L|_{\mathbb{C}P^{m1}} = \text{ tautological line bundle} \tag{212}$$

But $c_1(L) = u \in H^2(\mathcal{M}^*_{SW})$, with $u \xrightarrow{res\mathbb{C}P^{m-1}}$generator of $H^2(\mathbb{C}P^{m-1}$, so $u^{m-1}[\mathbb{C}P^{m-1}] = 1$. But $\mathbb{C}P^{m-1}$ bounds a cycle in $\mathcal{M}^*_{SW}$, so $[\mathbb{C}P^{m-1}] = 0 \in H_{2m-2}(\mathcal{M}^*_{SW})$, which is almost a contradiction! (assuming $m > 0$) If $m < 0$, the $\mathcal{M}^*_{SW} =, \tilde{\mathcal{M}_{SW}} = \{x\}$, so we avoid this contradiction. The conclusion is that m must always be negative, so

$$m = \frac{K^2 + b_2}{8} \leq 0 \tag{213}$$

i.e. we need $K^2 \leq -b_2, \forall K \in Char(X) = \{K \in H^2 | K \cdot a \equiv a^2 \mod 2, \forall a\}$. So now we're down to algebra.

$$Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z} \tag{214}$$

such that $K^2 + r \leq 0, \forall K \in Char$. There is a theorem by Elkies that says that if Q is a symmetric unimodular bilinear form with this property, then Q is diagonal, i.e. $Q = r\langle -1 \rangle$. Therefore Q_X is diagonal. □

Remark 38. *The above proof works whenever $H^1(X) = 0$. By killing off generators of $H_1(X;\mathbb{R})$ by surgery on loops, we can reduce to the case $H^1(X) = 0$; we get*

Theorem 36. *For X^4 smooth, closed; Q_X-definite, then Q_X is diagonal.*

12 $K3$ Admits Infinitely Many Smooth Structures

Definition 33. ***Fintushel-Stern knot surgery*** *(in dimension 4): find a torus $T \subset X = K3$ a nondegenerate elliptic fiber, $[T]^2 = 0, [T] \neq 0$. A neighborhood of $T \cong T \times D^2 \subset X, \partial nbhd(T) = T \times S^1 = T^3$. Let*

$$X_K = (X - nbhd(T)) \cup_{T^3} (S^1 \times (S^3 - nbhd(K))), \tag{215}$$

$$K \subset S^3 \text{ is any knot} \tag{216}$$

the gluing is such that

$$* \times \partial D^2 \mapsto * \times (longitude\ of\ K) \tag{217}$$

By Mayer-Vietoris and Seifert-Van Kampen, we can compute that

$$\pi_1(X_K) = \pi_1(X) = 1, Q_{X_K} \cong Q_X \tag{218}$$

Freedman showed that X_K, X are therefore homeomorphic. Write the Seiberg-Witten invariants

$$SW_X : Char(X) \to \mathbb{Z} \tag{219}$$

as a formal power series,

$$\sum_{K \in Char(X)} SW_X(K) e^k \tag{220}$$

(where the e^k is just a formal way to keep track of what characteristic elements we have).

E.g. for $X = K3, c_1(X) = 0$, this is a Calabi-Yau. $SW_X(0) = 1$ for symplectic manifolds/Kähler surfaces. In fact, $SW_{K3} = 1$.

Theorem 37. *(Fintushel-Stern) $SW_{X_K} = SW_X \cdot \Delta_K(t), t = e^{2[T]}$, where $\Delta_K(t)$ is the Alexander polynomial of the knot K.*

Definition 34. *For $\Delta(t)$ of the positive crossing, this is equal to $\Delta(t)$ of the negative crossing + $(t^{1/2} - t^{-1/2})\Delta_{nocrossing}(t)$, and $\Delta_0(t) = 1$ is the **Alexander Polynomial**. (see Skein relations)*

Example 42. $\Delta_{trefoil}(t) = t - 1 + t^{-1}$.

For example $X = K3$, $SW_{X_K} = \Delta_K(t)$; so for K the trefoil, $SW = e^{2[T]} - 1 + e^{-2[T]}$. Thus $SW(X_{trefoil}, s)$ is:

1. 1 if $s = \pm 2[T]$
2. -1 if $s = 0$
3. 0 otherwise.

Corollary 15. *If $\Delta_{K_1}(t) \neq \Delta_{K_2}(t)$, then X_{K_1} and X_{K-2} are not diffeomorphic.*

Some properties of $_K(t)$ are:

1. $\Delta_K(t) = \Delta_K(t^{-1})$
2. $\Delta_K(1) = 1$
3. Every polynomial with these properties appears as $\Delta_K(t)$ for some K.

An open question is if we can have $\Delta_{K_1}(t) = \Delta_{K_2}(t)$, but X_{K_1} and X_{K_2} are not diffeomorphic.

We know that $X_K \cong X_{m(k)}$, where $m(K)$ is the knot with diagram the mirror reflection of that of K, $X_{K_1 \# K_2} \cong X_{K_1 \# m(K_2)}$.

Definition 35. *K is prime if $K \neq K_1 \# K_2$ with K_1, K_2 are not the unknot.*

Theorem 38. *(Gordon-Luecke) If K_1, K_2 are prime knots, $\pi_1(S^3 - K_1) = \pi_1(S^3 - K_2) \Rightarrow K_1 = K_2$ or $m(K_2)$.*

An open question is if $\pi_1(S^3 - K)$ determines the diffeomorphism type of X_K. If so, then simply connected 4-manifolds are at least as complicated as knots.

13 Furuta's $\frac{10}{8}$ Theorem

Theorem 39. *($\frac{10}{8}$ Theorem, Furuta) Let X^4 be smooth, closed, simply connected, spin (Q_X even). Then $b_2(X) \geq \frac{10}{8}|\sigma(X)|$, i.e.*

$$Q_X = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2p(-E_8) \tag{221}$$

for $m \geq 2p$.

If Q_X is even then $0 \in Char(X), \exists s \in Spin^c(X), c_1(s) = 0$. The $Spin^c$ structure coming from the spin structure. Then $s = \overline{s}$, to the SW equations are invariant under conjugation (they're always invariant under conjugation)

$$(A, \phi) \xrightarrow{j} (-A, \overline{\phi}) \tag{222}$$

(taking complex conjugate, because it's in complex space) and symmetry under S^1 constant gauge transformation gives

$$(A, \phi) \xrightarrow{e^{i\theta}} (A, e^{i\theta}\phi) \tag{223}$$

By the spin structure, the SW equations have a $Pin(2)$ symmetry:

$$Pin(Z) = S^1 \cup j \cdot S^1 \subset \mathbb{C} \oplus j = \mathbb{H} \tag{224}$$

since $je^{i\theta} = e^{-i\theta}j$.

$$\tilde{SW} : \Gamma(S^+) \oplus Conn \to \Gamma(S^-) \oplus \Omega^2_+ \oplus \Omega^0/\mathbb{R} \tag{225}$$

$$\tilde{SW}(\phi, A) = (\not{D}_A\phi, F_A^+ - \gamma^{-1}(\phi\phi^*)_0, d^*(A - A_0)) \tag{226}$$

where the term is via the Coulomb gauge, and $Conn \cong \Omega^1$.

$$\tilde{SW} : \mathbb{H}^\infty \oplus \tilde{\mathbb{R}}^\infty \to \mathbb{H}^\infty \oplus \tilde{\mathbb{R}}^\infty, \tag{227}$$

$$Pin(Z) : \mathbb{H} \to \mathbb{H} \tag{228}$$

by left multiplication. $Pin(Z)$ acts on $\tilde{\mathbb{R}}$ trivially on S^1, and j acts by -1. In the SW equations, we don't really want to work with smooth sections, rather we want to work in Sobolev completions.

Let $E \to M$ be a vector bundle with an inner product and connection, and M a Riemannian manifold. $C^\infty(E)$ are smooth sections, $L^2_k(E)$ is the k^{th} Sobolev completion, and $\varphi \in L^2_k(E)$ satisfies $\varphi, \nabla\varphi, ..., \nabla^k\varphi \in L^2$. $L^2_k(E)$ is a Hilbert space, with

$$||\varphi||^2_{L^2_k} = ||\varphi||^2 + ||\nabla\varphi||^2 + ... + ||\nabla^k\varphi||^2 \tag{229}$$

Remark 39. *Note that different metrics and connections give rise to equivalent norms.*

In each case,

$$\tilde{SW} : L^2_k(S^+ \oplus T^*X) \to L^2_{k-1}(S^- \oplus \Lambda^2_+ T^*X \oplus \underline{\mathbb{R}})/\mathbb{R} \tag{230}$$

both of these are Hilbert spaces; let's call them H and H'. $\tilde{SW} = l + c$, where l is linear and c is comprised of quadratic and constant terms. $l = d_{A_0}\tilde{SW} = (\not{D}^+_{A_0}, d^+ + d^*)$. Recall that $index(l) = dim(ker(l)) - dim(coker(l)) = \frac{c_1(S)^2 - \sigma}{4} - b_2^+$. $index(\not{D}_{A_0}) = \frac{c_1(S)^2-\sigma}{4} = -\frac{\sigma}{4}$; $index(d^+ + d^*) = -b_2^+$.

Definition 36. ***Finite-dimensional approximation*** *(Furuta) is the following:*

Since $\tilde{SW} : H \to H'$. Choose a sequence of finite-dimensional subspaces

$$coker(l) \subset V_n \subset V_{n+1} \subset ... \subset H' \tag{231}$$

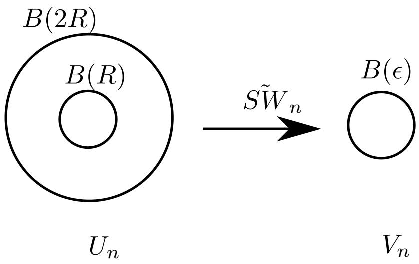


Figure 21: Approximating $\tilde{SW}$ on a bounded set.

Let $U_n = l^{-1}(V_n) \subset H$.

$$\tilde{SW}_n = l + p_{V_n} c : U_n \to V_n \tag{232}$$

where p_{V_n} *is the* L^2-*orthogonal projection to* V_n. *This is the finite-dimensional approximation to* $\tilde{SW}$.

Recall that the main property of the SW equations is that $\tilde{SW}^{-1}(0)$ is compact.

Theorem 40. *There exists* $R, \epsilon > 0$, *such that* $\forall n$*»0, if* $x \in U_n$ *satisfies* $||x|| < 2R$ *and* $|\tilde{SW}_n(X)| < \epsilon$, *then* $||x|| < R$.

The idea is that $\tilde{SW}_n$ is a good approximation to $\tilde{SW}$ on bounded sets like $B(2R)$. See Figure 21. (make sure R is big enough to contain the actual moduli space to the SW equations and a little bit more, so the things that are almost solutions are within epsilon of the solution). From the theorem, we get a map

$$\tilde{SW}_n^+ : B(2R)/\partial B(2R) \to B(\epsilon)/\partial B(\epsilon) \tag{233}$$

$$x \mapsto \tilde{SW}_n(X) \text{ if } |\tilde{SW}(X)| < \epsilon, \tag{234}$$

$$x \mapsto * \text{ otherwise} \tag{235}$$

This way we get a map between spheres that is $Pin(2)-$equivariant,

$$\tilde{SW}_n^+ : U_n^+ \to V_n^+ \tag{236}$$

as $Pin(2)$-representations: $V_n = \mathbb{H}^a \oplus \tilde{\mathbb{R}}^b, U_n = l^{-1}(V_n) = \mathbb{H}^{a-\frac{\sigma}{16}} \oplus \tilde{\mathbb{R}}^{b-b_2^+}$, because, in finite dimensions, $index(l) = dimU_n - dimV_n = dim(ker(l)) - dim(coker(l))$. If

$$Q_X = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2p(-E_8), \tag{237}$$

$$\sigma = -16p, b_2^+ = m \tag{238}$$

gives us a $Pin(2)$-equivariant map

$$f : (\mathbb{H}^{a+p} \oplus \mathbb{R}^{b-m})^+ \to (\mathbb{H}^a \oplus \tilde{\mathbb{R}}^b)^+ \tag{239}$$

For $a := A - A_0$, what is $\tilde{SW}|_{(0 \oplus \tilde{\mathbb{R}}^\infty)^+} : (0, a) \to (0, (d^+ + d^*)a)$ linear, so we get $f|_{(0 \oplus \tilde{\mathbb{R}}^{b-m})^+} = inclusion(\tilde{\mathbb{R}}^{b-m})^+ \hookrightarrow (\tilde{\mathbb{R}}^b)^+$, where the restriction is to an S^1-invariant subspace of $Pin(2)$. Furuta proved, using $Pin(2)$-equivariant K-theory that the existence of such a map implies $m \geq 2p+1$, assuming $p > 0$, and proved a bit more, namely "$m \geq 2p$" : $\frac{10}{8}-$Theorem.

Remark 40. *Hopkins-Lin-Shi-Xu (2018), using $Pin(2)-$equivariant stable homotopy theory, showed that a $Pin(2)$ map with the above properties exists if and only if:*

1. $m \geq 2p + 2, p \equiv 1, 2, 5, 6 \mod 8$
2. $m \geq 2p + 3, p \equiv 3, 4, 7 \mod 8$
3. $2p + 4, p \equiv 0 \mod 8$

This is thus the best we can do using Seiberg-Witten theory. The $\frac{11}{8}-$Conjecture remains open: $m \geq 3p$.

14 Exotic structures on $\mathbb{R}^4$

Let $X = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$. $Q_X = \langle 1 \rangle \oplus 9\langle -1 \rangle$. Notice Q_X is indefinite and odd, so $Q_X = (-E_8) \oplus \langle -1 \rangle \oplus \langle 1 \rangle$. Let the $\langle 1 \rangle$ be the span of some $\alpha \in H_2(X;\mathbb{Z}), \alpha^2 = 1$. Donaldson's Theorem implies that α cannot be represented by a smoothly embedded sphere. Recall that every homology class can be represented by a surface, but here it cannot be genus 0. Otherwise $nbhd(\alpha) = D^2$-bundle over S^2 of Euler number 1 (because the self-intersection is 1). This is just equal to $\mathbb{C}P^2 - B^4, \partial nbhd(S) = S^3$. But if this were true then we could have $X = X' \# \mathbb{C}P^2, Q_{X'} = (-E_8) \oplus \langle -1 \rangle$. This is a definite intersection form, but not diagonal (see Homework 1), so this violates Donaldson's theorem. Freedman theory says that α can be represented by a topological sphere, which is a Casson handle $\cup D^2$. Call this sphere $\Sigma \subset X$. This has the property that a neighborhood U of Σ smoothly embeds in $\mathbb{C}P^2$. Furthermore, U is homeomorphic to $\mathbb{C}P^2 - B^4$, which is a topological sphere.

Lemma 5. *$Z = \mathbb{C}P^2 - \Sigma$ is homeomorphic to $\mathbb{R}^4$*

Proof. (Freedman's theorem for open 4-manifolds) Use $\pi_1(Z) = 1, H_* = 0$ implies Z is contractible (using Mayer-Vietoris and Seifert-Van Kampen). We need also that Z is simply connected at infinity, i.e

Definition 37. *$\forall c \subset Z$ compact, there exists $D \subset Z$ compact $C \subset D$ such that $\pi_1(Z - D) \to \pi_1(Z - C)$ is trivial means that Z is **simply-connected at infinity**.*

In our case, $Z \cup U$ is homeomorphic to $\mathbb{C}P^2 - B^4 - \mathbb{C}P^1 = S^3 \times (0,1)$, which is the "end of Z," which is contractible. □

Lemma 6. *Z is not diffeomorphic to $\mathbb{R}^4$.*

Proof. Assume it is. In $\mathbb{R}^4$, every compact subset can be surrounded by a standardly embedded S^3. Our exotic $\mathbb{R}^4$ will not have this property. Let $K = \mathbb{C}P^2 - U$. This is compact. We get that this is surrounded by S. Get $X'' = (X - nbhd(\Sigma)) \cup B^4$. This is a smooth simply-connected 4-manifold with $Q_{X''} = (-E_8) \oplus \langle -1 \rangle$ which is definite but not diagonal, which contracts Donaldson's theorem. □

14.1 Uncountability

In fact there are uncountably many exotic $\mathbb{R}^4$s. We found an exotic $\mathbb{R}^4$ $\mathbb{ER}^4 \subset \mathbb{C}P^2$ as above. Let $h : \mathbb{R}^4 \xrightarrow{\cong} \mathbb{ER}^4$ be a homeomorphism. Let $h(B^4(\rho)) = \mathbb{ER}^4_\rho$.

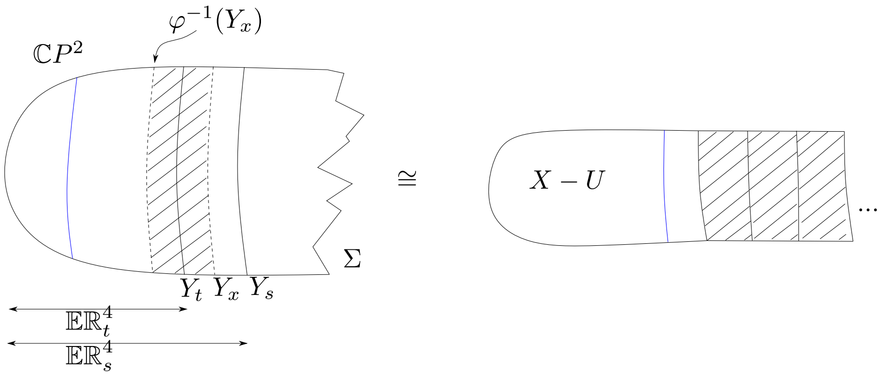


Figure 22: φ diffeomorphism.

Theorem 41. *(Taubes) There exists a $\rho_0 > 0$ such that for all $s > t > \rho_0$,*

$$\mathbb{ER}^4_s \not\cong \mathbb{ER}^4_t \tag{240}$$

Proof. (SKETCH) Suppose $\varphi : \mathbb{ER}^4_t \to \mathbb{ER}^4_s$ is a diffeomorphism. Let $h(S(\rho)) = Y_\rho$. $\mathbb{ER}^4 = \mathbb{C}P^2 - \Sigma$. Pick $x \in (t, s)$, and consider $\varphi^{-1}(Y_x)$. See diagram 22. (From here we can construct a smooth 4-manifold with periodic ends. Call this periodic manifold W) We then have $Q_X = Q_{X-u} = (-E_8) \oplus (-1)$. Taubes proved (using Yang-Mills theory) a version of Donaldson's theorem for smooth manifolds with periodic ends. Therefore Q_W should have been diagonal. $\square$

Remark 41. *There is no known proof with Seiberg-Witten theory. We cannot prove it for manifolds with periodic ends.*

15 Khovanov Homology

Khovanov homology is a combinatorial invariant of knots $K \subset \mathbb{R}^3$. This is used to give new proofs of

1. Milnor Conjecture (cf. Rasmussen)
2. Thom Conjecture (cf. Lambert-Cole)
3. Existence of exotic $\mathbb{R}^4$ (cf. Rasmussen-Gompf)

It may also give a possible approach to disproving the smooth 4d Poincaré Conjecture.

Let $K \subset \mathbb{R}^3$ be an oriented link. Let D be a planar diagram of the link, with over-crossings and under-crossings.

Theorem 42. *(Reidemeister) Two diagrams represent the same link up to isotopy if and only if they are related by a sequence of Reidemeister moves.*

OUTLINE: If we have a diagram D, we get a bigraded complex

$$C(D) = \oplus_{i,j\in\mathbb{Z}} C^{ij}(D) \tag{241}$$

$$d : C^{i,j}(D) \to C^{i+1,j}(D) \tag{242}$$

which is a cochain complex. Here i is the homological grading, and j is the quantum (Jones) grading. Here we have $d^2 = 0$. There is a theorem that $Kh^{*,*}(L) := H^{*,*}(C(D)) = \oplus_{i,j} Kh^{i,j}(L)$ is invariant under the Reidemeister moves; hence it's an invariant of L. Think of this as a bigraded abelian group.

Remark 42. *Since the d map goes up an index it's really cohomology, but people are tired of saying 'cohomology' so they just call it homology. Furthermore, 'Khovanov' is pronounced 'Hovanov,' so it really should be called 'Hovanov Cohomology.'*

$$\chi(Kh^{*,*}(L)) = \sum_{i,j} (-1)^i q^j rk(Kh^{i,j}(L)) = \tilde{J}_L(q) \in \mathbb{Z}[q, q^{-1}] \tag{243}$$

where

Definition 38. *$\tilde{J}_L(q) \in \mathbb{Z}[q, q^{-1}]$ is called the* ***unnormalized Jones polynomial****:*

$$\tilde{J}_L(q) = (q + q^{-1})J_L(q^2), t = q^2 \tag{244}$$

$$J_L(t) = \text{ the Jones polynomial} \tag{245}$$

characterized by $J_0 = 1, tJ_{-cross} - t^{-1}J_{+cross} = (t^{1/2} - t^{-1/2})J_{nocrossing}$ (Skein relations).

Example 43. *Consider the unkot L with diagram the infinity sign. $tJ_L - t^{-1}J_L = (t^{1/2} - t^{-1/2}J_{2circles})$, so*

$$J_{2circles} = \frac{t - t^{-1}}{t^{1/2} - t^{-1/2}} = t^{1/2} + t^{-1/2} \tag{246}$$

Example 44. *Let L be the hopf link. $J_L = t^{5/2} + t^{1/2} = q^5 + q$.*

Example 45. *Let L be the trefoil. $J_L = t + t^3 - t^4 \Rightarrow q^2 + q^6 - q^8$.*

$$\tilde{J}_L(q) = \chi(Kh) = (q + q^{-1})(q^2 + q^6 - q^8) \tag{247}$$

$$= q + q^3 + q^5 - q^9 \tag{248}$$

Let M be a graded abelian group (think "Jones grading"). $M\{L\} = M$ with the grading shifted upward by l. For

$$C^0 \to C^1 \to C^2 \to ... \tag{249}$$

be a cochain complex (think "homological grading"). $C[s] = C$ with homological grading shifted upward by s:

$$C[s]^k = C^{k-s} \tag{250}$$

(this is the opposite convention from some sources)

Definition 39. *(following Bar-Natan, "On Khovanov's categorification of the Jones polynomial") Let D be a diagram for some oriented link K with n crossings. Let the number of $\pm$ crossings be $n_\pm$. For example, for D the hopf link, both crossings are positive, i.e. $n_+ = 2$. Regardless of*

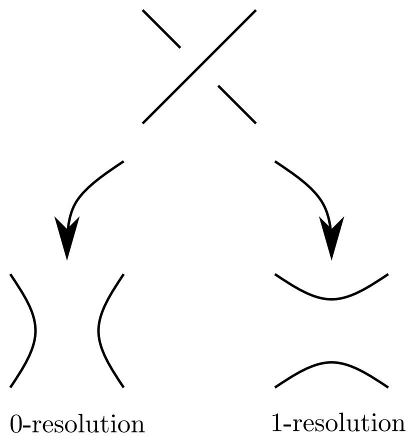


Figure 23: Resolution of a crossing

orientation, any crossing looks like a negative crossings (for positive crossings rotate by $90°$. *We get either a 0-resolution or a 1-resolution. See Figure 23. In* D *we can resolve all crossings in* 2^n *ways. Resolutions correspond to words* $\alpha \in \{0,1\}^n$ *giving a* ***cube of resolutions****. See Figure 24. Edges in the hypercube:* $\xi \in \{0,1,*\}^n, \xi = (\xi_1, ..., \xi_n), \exists$ *a unique* j *with* $\xi_j = *$.

Let $V = \mathbb{Z} \oplus \mathbb{Z}$ *be spanned by* V_+, V_-. *The Jones grading* $gr(V_\pm) = \pm 1$. *For every* $\alpha \in \{0,1\}^n$, *we get*

$$V_\alpha(D) := V^{\otimes k}\{|\alpha|\}, |\alpha| = \sum \alpha_i \tag{251}$$

where k *is the number of circles in* D_α.

$$[[D]]^r = \oplus_{\alpha, |\alpha|=r} V_\alpha(D) \tag{252}$$

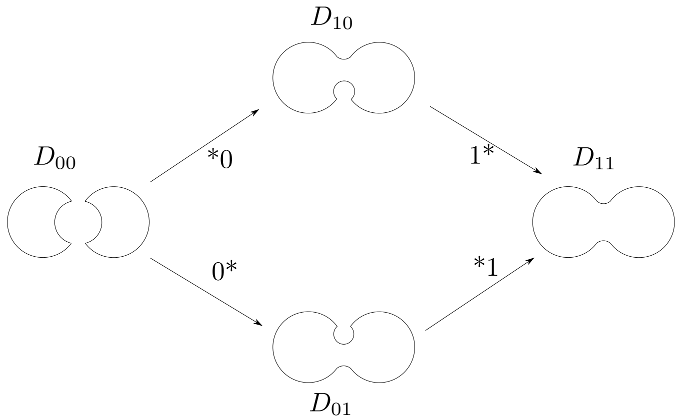


Figure 24: Cube of resolutions for a Hopf link.

The ***Khovanov complex*** *is given by*

$$C^{*,*}(D) = ([[D]]^*[-n_-]\{n_+ - 2n_-\}, d) \tag{253}$$

where the square brackets are the shift in homological grading, and the curly brackets are the shift in Jones grading. For L the hopf link, we have

$$\begin{tikzcd}
 & V\{1\} \arrow[dr, "d_{1,*}"] & & \\
V^{\otimes 2} \arrow[ur, "d_{*,0}"] \arrow[dr, "d_{0,*}"'] \arrow[ddd] & & V^{\otimes}\{2\} \arrow[ddd] & \\
 & V\{1\} \arrow[ur, "d_{*,1}"'] \arrow[dd] & & \\
 & & & \\
([[D]]^0 \arrow[r] & [[D]]^1 \arrow[r] & [[D]]^2)\{2\} \arrow[r, "="] & C^{*,*}(D)
\end{tikzcd}$$

The differentials are given by the gluing map m taking 2 circles to 1 circle, and the cutting map Δ taking 1 circle to 2 circles. See Figure 25. These give

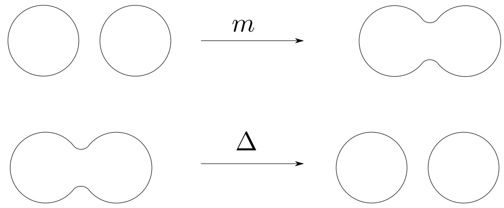


Figure 25: Gluing and cutting maps.

$$V_+ \otimes V_+ \xrightarrow{m} V_+,\, V_+ \xrightarrow{\Delta} V_+ \otimes V_- \tag{254}$$

$$V_+ \otimes V_- \xrightarrow{m} V_-,\, V_- \xrightarrow{\Delta} V_- \otimes V_- \tag{255}$$

$$V^{\otimes 2} \xrightarrow{m} V,\, V \xrightarrow{\Delta} V^{\otimes 2} \tag{256}$$

$$V_- \otimes V_+ \xrightarrow{m} V_- \tag{257}$$

$$V_- \otimes V_- m 0 \tag{258}$$

Tensor these with the identity on all other components. d_ξ *for an edge* ξ.

$$(-1)^\xi := (-1)^{\sum_{i<j} \xi_i} \tag{259}$$

where j *is the location of * in* ξ. *For example,* $*0 \to sign(\pm 1), *1 \to +1$. *The* ***differential on***

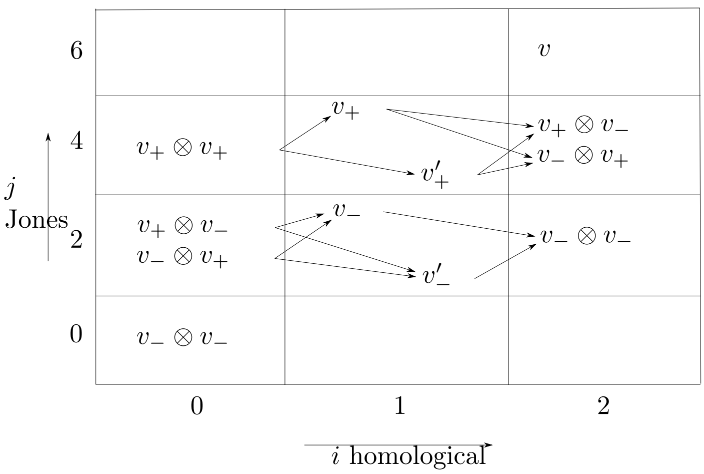


Figure 26: Complexes for the Hopf link.

the Khovanov complex *is*

$$d^r : [[D]]^r \to [[D]]^{r+1} \tag{260}$$

$$d^r = \sum_{\xi} (-1)^{\xi} d\xi \tag{261}$$

where ξ starts at $\alpha, |\alpha| = r$.

Example 46. *In our example of the Hopf link, see 26. These are the complexes for our example. Taking the homology $H = \frac{ker(d)}{Im(d)}$, we get*

		Z	6
		Z	4
Z			2
Z			0
0	1	2	

This is the Khovanov homology of the hopf link $Kh(L)$.

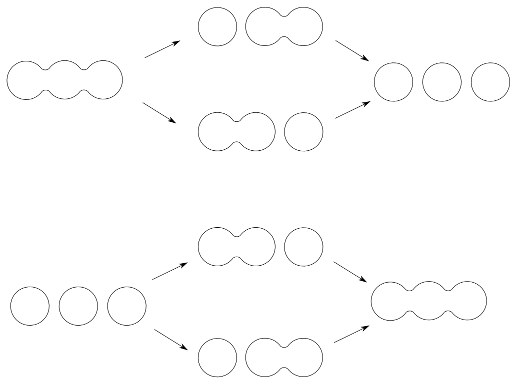


Figure 27: $d^2 = 0$.

Lemma 7. *In general, $d^2 = 0$.*

Proof. We prove this on a case-by-case basis. See Figure 27. $\square$

Theorem 43. *Khovanov homology is invariant under Reidemeister moves.*

Definition 40. *For (C, d) a complex, $C' \subset C$ is a* ***subcomplex*** *if $d(C') \subseteq'$, leading to C/C' the* ***quotient complex****.*

$$0 \to C' \to C \to C/C' \to 0 \tag{262}$$

$$... \to H^*(C') \to H^*(C) \to H^*(C/C') \to H^{*+1}(C') \to ... \tag{263}$$

Lemma 8. *1. If $H^*(C') = 0$, (C' is* ***acyclic****), then $H^*(C) \cong H^*(C/C')$*

2. If $H^*(C/C') = 0$, then $H^*(C) \cong H^*(C')$

Proof. (Of Reidemeister invariance of Khovanov homology) See Figure 28 and Figure 29 for Reidemeister moves 1 and 2. $R3$ is an exercise (canceling acyclic complexes and subcomplexes). $\square$

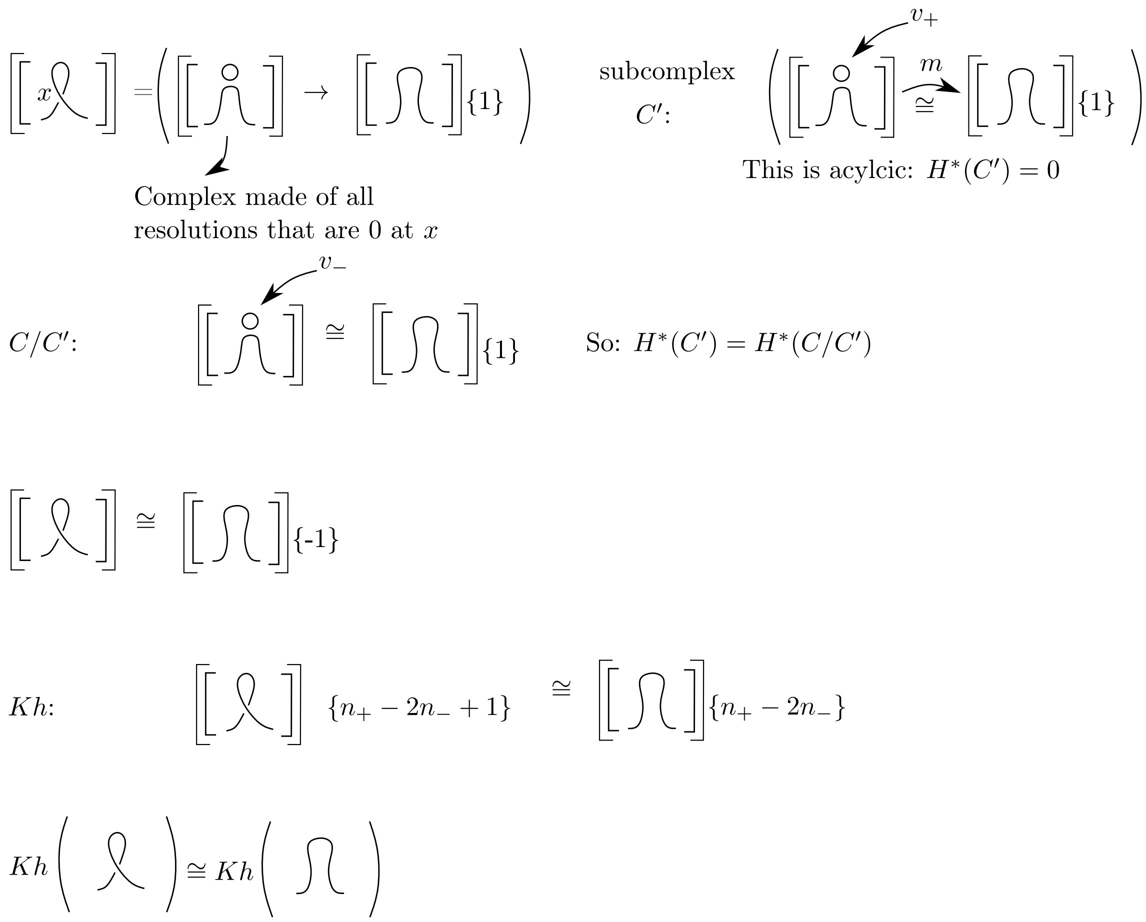


Figure 28: Proof of Reidemeister move 1 invariance of Khovanov homology.

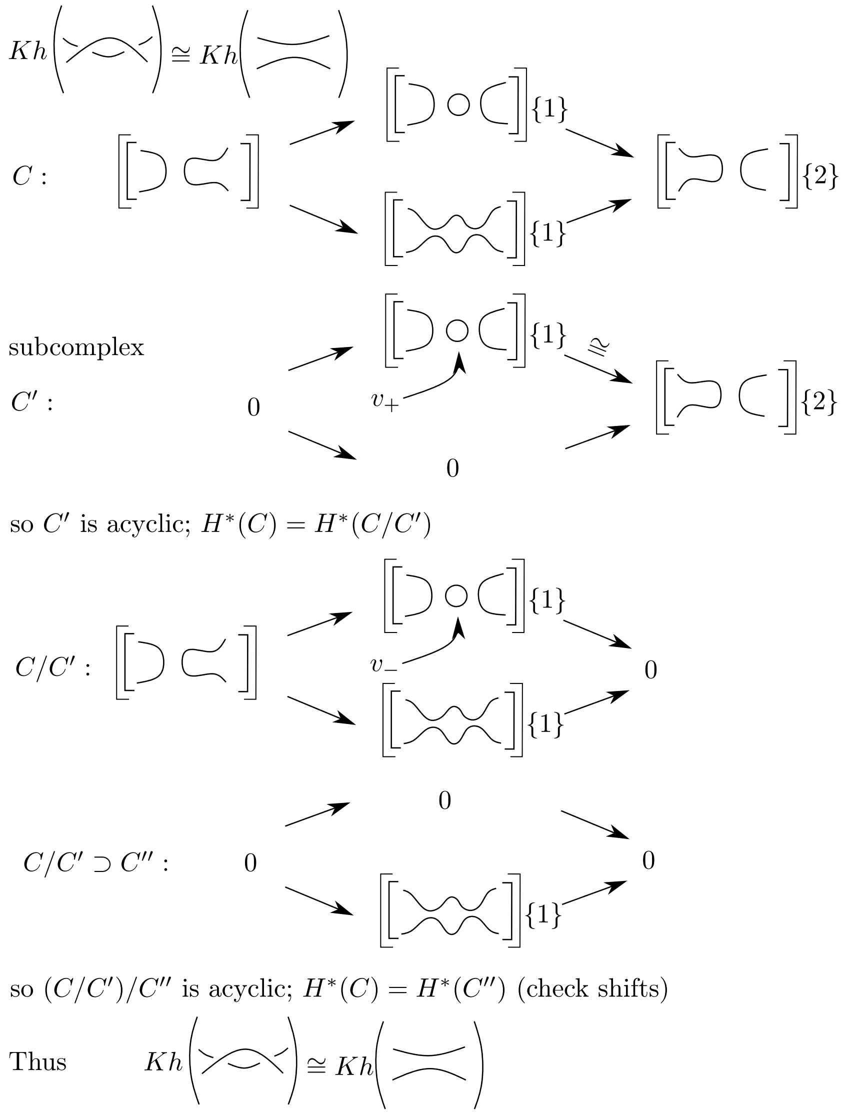


Figure 29: Proof of Reidemeister move 2 invariance of Khovanov homology.

15.1 Topological Quantum Field Theory & Lee Homology

What was essential in the proof that Kh is a link invariant? We need: a unit $1 \in V$ for M, $\epsilon : V \to \mathbb{Z}$ a counit for Δ, with

$$\epsilon(V_+) = 0 \tag{264}$$

$$\epsilon(V_-) = 1 \tag{265}$$

so for:

1. m is a commutative, associative multiplication,
2. Δ a cocommutative, coassociative comultiplication, and
3. the Frobenius law

$$\Delta \circ m = (m \otimes 1) \cdot (1 \otimes \Delta) \tag{266}$$

Definition 41. *This is called a* ***Frobenius Algebra****.*

A Frobenius Algebra gives a 1+1-dimensional topological quantum field theory, inthat a functor associating a closed 1-manifold with an abelian group, and a cobordism with a homomorphism.

Remark 43. *Every closed 1-manifold is a disjoint union of circles. See Figure 30.*

This satisfy the properties in Figure 31.

Remark 44. *To get a homological invariant of knots (like Kh), we need V to be a Frobenius algebra, of rank 2: $V \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Example 47. *For Kh, we could write $V_+ = 1, V_- = x, V = H^*(S^2) = \mathbb{Z}[x]/(x^2)$. We also have*

$$\Delta(1) = 1 \otimes x + x \otimes 1, \epsilon(1) = 0 \tag{267}$$

$$\Delta(x) = x \otimes x, \epsilon(x) = 1 \tag{268}$$

Example 48. *Lee's deformation of Khovanov homology: from $V = \mathbb{Z}[x]/(x^2 - t)$ over $R = \mathbb{Z}[t]$; (we've been talking about abelian groups, but we could use any module over a commutative ring).*

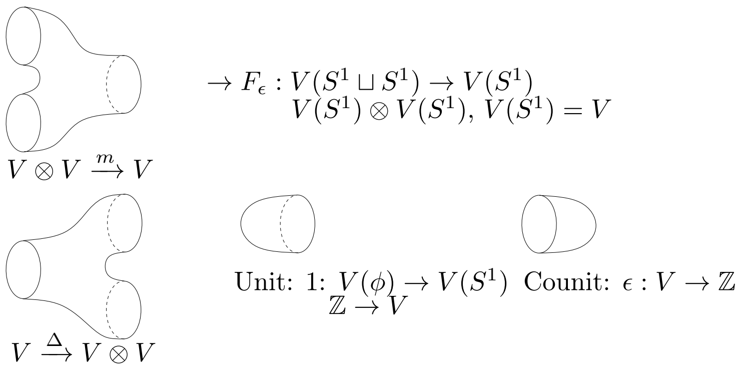


Figure 30: $(1+1)$-topological quantum field theory.

We have 1 and ϵ as before, and

$$\Delta(1) = 1 \otimes x + x \otimes 1 \tag{269}$$

$$\Delta(x) = x \otimes x + t(1 \otimes 1) \tag{270}$$

$$m(V_+ \otimes V_-) = V-, m(V_+ \otimes V_+) = V_+ \tag{271}$$

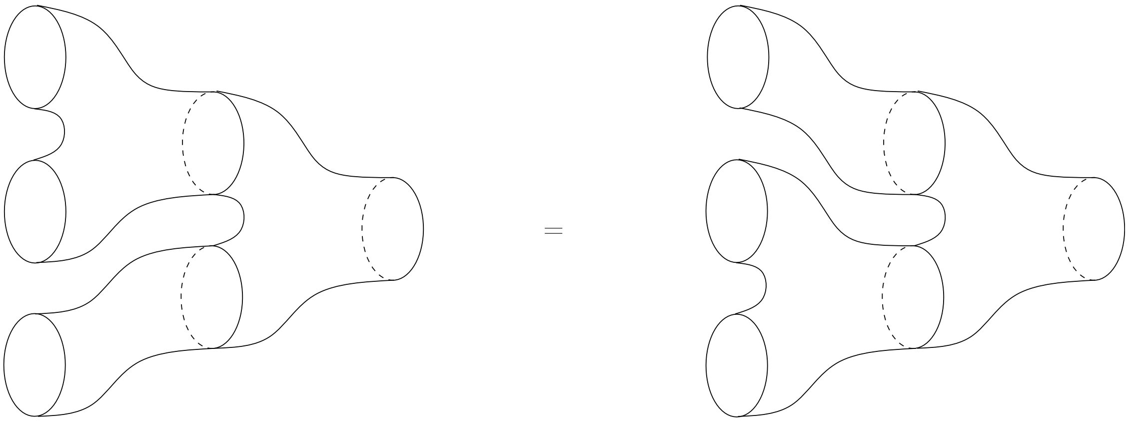
$$m(V_- \otimes V_+) = V_-, m(V_- \otimes V_-) = tV_+ \tag{272}$$

$$\Delta(V_+) = V_+ \otimes V_- + V_- \otimes V_+ \tag{273}$$

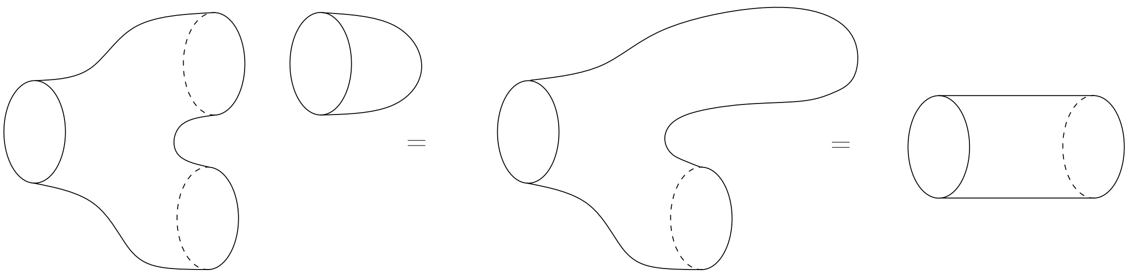
$$\Delta(V_-) = V_- \otimes V_- + tV_+ \otimes V_+ \tag{274}$$

so we get a complex $C'(D)$ of $\mathbb{Z}[t]$-modules. For $t = 0$, this is $C(D)$, the Khovanov complex, and

m is associative:



$\Delta \circ (\epsilon \otimes Id)$



$\Delta \circ m = (m \otimes Id)(Id \otimes \Delta)$

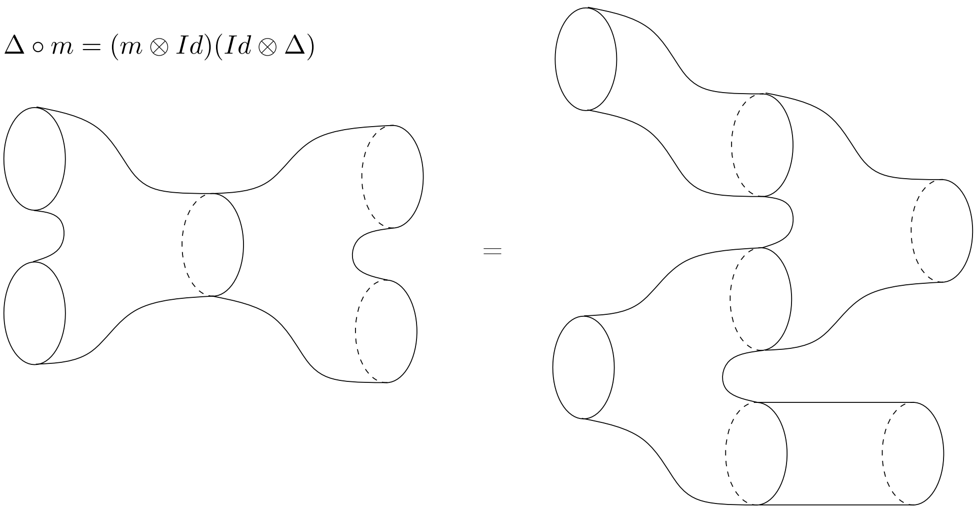


Figure 31: $(1+1)$-topological quantum field theory properties.

for $t = 1$, $C_{Lee}(D)$ a Lee complex. For D a link diagram,

$$C(D) \xrightarrow{H_*} Kh(K) \text{ Khovanov homology (over Z)} \tag{275}$$

$$C_{Lee}(D) \xrightarrow{H_*} Lee(K) \text{ Lee homology (over Z)} \tag{276}$$

$$C'(D) \xrightarrow{H_*} Kh'(K) \text{ Kohvanov-Lee homology (over}\mathbb{Z}[t]) \tag{277}$$

The differential on $C'(D)$ is $d + t\Phi$; d changes (i, j) by $(1, 0)$, and Φ changes (i, j) by $(1, 4)$, where the indices are (homological, quantum). We have $d, \Phi : C \to C$ with $d^2 = 0, (d + \Phi)^2 = 0, \Phi^2 = 0, d\Phi + \Phi d = 0$. This is a filtration on C given by $-q$:

$$\begin{tikzcd} \ldots \subset & C^{q\geq j+1} \arrow[loop below, "d+\Phi"] & \subset & C^{q\geq j} \arrow[loop below, "d+\Phi"] & \subset \ldots \end{tikzcd}$$

giving a filtered complex, giving a spectral sequence, i.e. a collection of page = complexes $(E^r, d^r), d^r \circ d^r, E^{r+1} = H^*(E^r, d^r), E^1(C^*, d), E^2(H^*(E^1), \Phi^*) = (Kh(K), \Phi^*), E^4, E^4, \ldots$ This converges to $E^\infty = H^*(C^d + \Phi)$. In our case, spectral sequences from $E^2 = Kh(K)$ result in $E^\infty = Lee(K)$. Thus changes in grading are:

$$d^1 = d, (i, j) \to (i + 1, j) \tag{278}$$

$$d^2 = \Phi^*, (i, j) \to (i + 1, j + 4) \tag{279}$$

$$\vdots \tag{280}$$

$$d^n, (i, j) \to (i + 1, j + 4(n - 1)) \tag{281}$$

Example 49. *For the trefoil, see Figure 32. since Φ^* maps $(2, 5)$ to $(3, 9)$, and removes them.*

From now on let's work in $\mathbb{Q}$ coefficients, i.e. $Kh(K)$ will denote $Kh(K) \otimes \mathbb{Q}$.

Example 50. *The Lee homology has $E^3 = E^\infty$, seen in Figure 33.*

Theorem 44. *$Lee(L) \otimes \mathbb{Q} = \mathbb{Q}^{2^{l-1}}$, where $l = \#$ components of l. (e.g. for a knot, $Lee \cong \mathbb{Q} \otimes \mathbb{Q}$).*

Proof. Over $\mathbb{Q}$, we can define a new basis for V, namely $a = V_+ + V_-, b = V_- - V_+$. The Lee

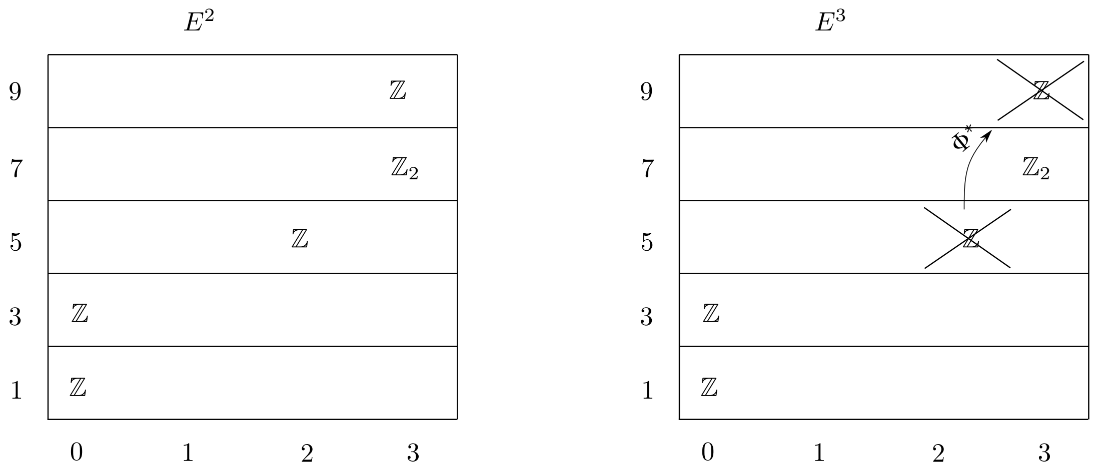


Figure 32: E^2, E^3 pages in this spectral sequence.

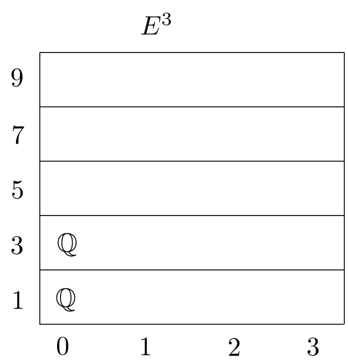


Figure 33: $E^3 = E^\infty$ page in Lee homology.

operations are

$$m : a \otimes a \to 2a \tag{282}$$

$$m : a \otimes b, b \otimes a \to 0 \tag{283}$$

$$m : b \otimes b \to -2b \tag{284}$$

$$\Delta : a \to a \otimes a \tag{285}$$

$$\Delta : b \to b \otimes b \tag{286}$$

We claim that $Lee(L)$ is generated by the following "canonical generators": for σ an orientation of L (there are 2^l such σ). This gives D_σ an oriented resolution (at all crossings). For $C \in D_\sigma, \tau(C) \in \mathbb{Z}_2$, where $\tau(C)$ is the number of circles separating C from ∞, plus 0 if C is oriented counterclockwise, and 1 if it's oriented clockwise, mod 2. In generatal let $s_\sigma = \otimes g_C, C \in D_\sigma$, where g_C is a if $\tau(C) = 0, b$ if $\tau(C) = 1$. This is an element in the Lee complex. E.g. $s_\sigma = a \otimes b \otimes a$ for the trefoil. They are cycles: $(d + \Phi)s_\sigma = 0$, because circles labeled the same (both a or both b) in D_σ do not touch at crossings. So $[S_\sigma] \in Lee(E)$. Put an inner product on $C_{Lee}(D)$ by making the generators $a \otimes a \otimes b \otimes ...$ on an orthonormal basis.

$$Lee(D) = H^*(C_{Lee}(D)) = ker(d + \Phi)/Im(d + \Phi) \cong ker(d + \Phi) \cap ker(d + \Phi)^* \tag{287}$$

where the * is the adjoint with respect to the inner product making the generators an orthonormal basis, with

$$(d + \Phi)^* : a \otimes a \to a \tag{288}$$

$$b \otimes b \to b \tag{289}$$

$$a \to 2a \otimes a \tag{290}$$

$$b \to -2b \otimes b \tag{291}$$

$$rest \to 0 \tag{292}$$

We have $(d + \Phi)^* s_\sigma = 0$, so $s_\sigma \in ker(d + \Phi) \cap ker(d + \Phi)^*$. Thus $dim(Lee(D)) \geq 2^l$.

To prove that $dim(Lee(D)) = 2^l$, we proceed by induction on the number of corssings in D. For a resolution $D \to D_0, D_1$, we have $C_{Lee}(D_1) \subset C_{Lee}(D)$ a subcomplex. So we have a short exact sequence

$$0 \to C_{Lee}(D_1) \to C_{Lee}(D) \to C_{Lee}(D_0) \to 0 \tag{293}$$

inducing a long exact sequence

$$... \to Lee((D_1) \to Lee(D) \to Lee(D_0) \to Lee(D_1) \to ... \tag{294}$$

We have two cases:

1. The two strands at x are from different components of L; D_0, D_1 have $l-1$ components. By induction, $dim(Lee(D_0)) = dim(Lee(D_1)) = 2^{l-1}, dim(Lee(D)) \leq dim(Lee(D_0)) + dim(Lee(D_1)) = 2^{l-1} + 2^{l-1} = 2^l$, so $dim(Lee(D)) = 2^l$.

2. The two strands are from the same component. Then D_0 has l components, D_1 has $l+1$ components, or vise versa.

$$... \to Lee(E) \to Lee(D_0) \xrightarrow{i} Lee(D_1) \to ... \tag{295}$$

   We can check that the canonical generators map to half of the cannonical generators under i, so $dim(Lee(D)) = dim(coker(i)) = 2^l$.

□

15.2 Rasmussen's Invariant

Recall the Milnor Conjecture. For a knot $K \subset S^3$,

Theorem 45. *(Milnor's Conjecture) If $K = T_{p,q}$ with $p, q > 0$ coprime, then $g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.*

This was proved by Kronheimer-Mrowka in 1993, using Yang-Mills theory. We gave a proof with Seiberg-Witten theory based on the adjunction inequality. Here we give a combinatorial proof due to Rasmussen (2004), so we can avoid analysis.

$T_{p,q}$ can be unknotted with $\frac{(p-1)(q-1)}{2}$ crossing changes. Then we get a surface Σ with $\partial\Sigma =$

$T_{p,q}$, with $g(\Sigma) = \frac{(p-1)(q-1)}{2}$, so therefore $g_s(K) \leq \frac{(p-1)(q-1)}{2}$. Rasmussen defined an invariant $s(K) \in 2\mathbb{Z}$ from Khovanov-Lee homology such that:

1. $|s(K)| \leq 2g_s(K)$,
2. $s(T_{p,q}) = (p-1)(q-1)$

These two facts, combined with $g(\Sigma) = \frac{(p-1)(q-1)}{2}$ give a proof of the Milnor Conjecture. We will work with coefficients in $\mathbb{Q}$. Khovanov homology as we looked at it was in terms of $\mathbb{Z}$, but we'll just tensor everything with $\mathbb{Q}$. If we have D a diagram for a knot K, we have $(C(D), d)$ the Khovanov complex with differential d leading to Khovanov homology $Kh(K)$. We also can add Φ to get Lee homology: $C_{Lee}(D) = (C(D), d + \Phi) \xrightarrow{H_*} Lee(K) \cong \mathbb{Q} \oplus \mathbb{Q}$ (for a knot). Because we have this double complex, we have a spectral sequence $Kh(K) \to \mathbb{Q} \oplus \mathbb{Q}$. Recall that $C(D)$ has a quantum grading $gr(q,j)$ and a homological grading $gr(i)$. d changes (i,j) by $(1,0)$, and Φ changes (i,j) by $(1,4)$. So $C_{Lee}(D)$ has a filtration

$$... \supset C_{Lee}^{q \geq i}(D) \supseteq C_{Lee}^{q \geq i+1}(D) \supseteq ... \supseteq 0 \tag{296}$$

and at some point to the left we get the whole Lee complex $C_{Lee}(D)$. This is going to be $C_{Lee}(D) = C_{Lee}^{q \geq -N}(D)$ (and $C_{Lee}^{q \geq N}(D)$) for some $N >> 0$. What about at the level of homology? For homology let's define $I_j = image(H^*(C_{Lee}^{q \geq j} \to H^*(C_{Lee}))) \subset Lee(D)$, so these are the terms that come from the filtration level j. Then

$$Lee(D) = I_{-N} \supseteq I_{-N+1} \supseteq ... \supseteq I_N = 0 \tag{297}$$

$$Lee(D) \cong \oplus_j (I_j / I_{j+1}) \tag{298}$$

This gives a grading on $Lee(D)$.

Definition 42. *If we have $s \in C_{Lee}(D)$, we can talk about $q(s)$ the q-grading of s. $q(s)$ is the maximal grading*

$$q(s) = max\{j | s \in C_{Lee}^{q \geq j}(D)\} \tag{299}$$

$$[s] \in Lee(D), q([s]) = max\{j | q(x) = j, x \in [s]\} \tag{300}$$

Notice that $q(s)$ may be different from $q([s])$.

Proposition 2. *If K is a knot, then the two generators of $Lee(K) \cong \mathbb{Q} \oplus \mathbb{Q}$ are in gradings $s-1$*

and $s+1$ for some $s \in 2\mathbb{Z}$.

Definition 43. *$s = s(K)$ is the **Rasmussen invariant**.*

Proof. Recall that $Lee(K)$ is generated by canonical generators $[s_o], [s_{\overline{o}]}$, where o is the orientation of K and $\overline{o}$ is the opposite orientation. Notice that $C_{Lee}(D)$ is supported only in odd quantum gradings. (For a link it's in $n \equiv l \mod 2$, where l is the number of components). The quantum grading $q(V_-) = -1, q(V_+) = +1$. Define $C_{Lee,even}(D)$ be the part generated by elements with $q \equiv l \mod 4$, and $C_{Lee,odd}(D) = q \equiv l+2 \mod 4$. Recall that d, Φ preserve $q \mod 4$, so $d+\Phi$ preserves these gradings, i.e. $C_{Lee}(D) \cong C_{Lee,even}(D) \oplus C_{Lee,odd}(D)$ since $d+\Phi$ is an automorphism on these summands. Thus $Lee(K) = Lee_{even}(K) \oplus Lee_{odd}(K)$. Define $\iota : C_{Lee}(D) \to C_{Lee}(D)$, with

1. $\iota = +1$ on $C_{Lee,even}$
2. $\iota = -1$ on $C_{Lee,odd}$

So for $x \in C_{Lee}(D)$, we have

$$x = (\frac{x+\iota(x)}{2}) + (\frac{x-\iota(x)}{2}) \tag{301}$$

in $C_{Lee,even}$ and $C_{Lee,odd}$, respectively. Let $i : V \to V, i(V_-) = V_-, i(V_+) = -V_+, \Rightarrow \iota = \pm i^{\otimes n}$. We had $a = V_- + V_+$ and $b = V_- - V_+$, with $i(a) = b, i(b) = a$. So now if we look at s_o vs. $s_{\overline{o}}$, both come from $D_o = D_{\overline{o}}$ the oriented resolution (the oriented resolution is the same as that of the opposite resolution). This gives $i([s_o]) = \pm \pm [s_{\overline{o}]}$. This gives

$$s_o = (\frac{[s_o]+[s_{\overline{o}]}}{2}) + (\frac{[s_o]+[s_{\overline{o}]}}{2}) \tag{302}$$

where one is even and one is odd, so $[s_o + s_{\overline{o}]}$ is even and $[s_o + s_{\overline{o}]}$ is odd, or vise versa. Then the two $\mathbb{Q}$s in $\mathbb{Q} \oplus \mathbb{Q} = Lee(K)$ live in different $q \mod 4$ gradings. Say the gradings are $s_{max} > s_{min}$, $s_{max} - s_{min} \equiv 2 \mod 4$ Observe that $q([s_o]) = q([s_{\overline{o}]) = s_{min}}$. To show $s_{max} - s_{min} = 2$, consider the complex $C_{Lee}(D')$, where D' results from adding a twist to D (see Figure 34). We take the

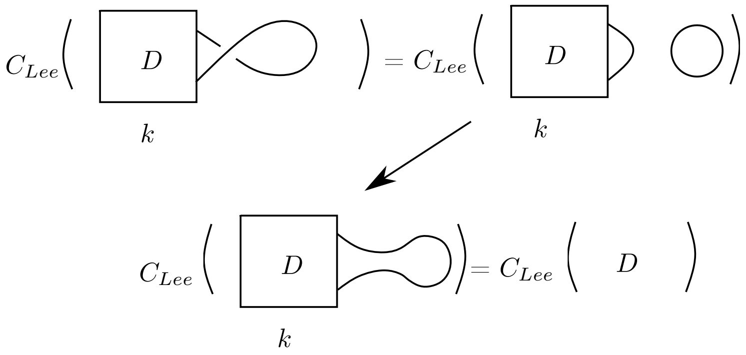


Figure 34: Adding a twist to a knot in the Lee complex.

resolution of D' to get D plus a circle or D plus the other resolution, which is just D. This gives

$$0 \to C_{Lee}(D) \to C_{Lee}(D') \to C_{Lee}(D \cup \circ) \to 0 \tag{303}$$

$$\ldots \to Lee(K) \to Lee(K) \to Lee(K \cup \circ) \xrightarrow{\partial} Lee(K) \to \ldots \tag{304}$$

Recall that $Lee(K \cup \circ) = Lee(K) \otimes V$. The canonical generators of $C_{Lee}(K)$ are s_a, s_b according to their label near x of either a or b. Without loss of generality, let $q(s_a - s_b) = s_{max}$ and $q(s_a + s_b) = s_{min}$. We need to check explicitly that $\partial([s_a - s_b] \otimes [a]) = [s_a]$. For this we just look at the explicit definition of the generators. If this is true we get that $q([s_a - s_b] \otimes a) \leq q([s_a])_{+1}$. This is because the ∂ map is a boundary map, and decreases grading by 1. This is equal to $s_{max} - 1 \leq s_{min} + 1 \Rightarrow s_{max} \leq s_{min} + 2, s_{max} - s_{min} \equiv 2 \mod 4$, giving us $s_{max} = s_{min} + 2$. □

Proposition 3. $|s(K)| \leq 2g_s(K)$.

Proof. We use the functoriality of Khovanov-Lee homology under link cobordisms, i.e. $\Sigma \subset \mathbb{R}^3 \times [0,1]$ with $\partial\Sigma = (-L_0) \cup L_1$. We want to construct a map $F_\Sigma : Kh(L_0) \to Kh(L_1)$. By Morse theory, we can split Σ into index 0, index 1, and index 2 critical points, corresponding to minima, saddles, and maxima. In terms of diagrams, we can represent the cobordism as a movie of diagrams:

D_0 as a diagram for L_0 to D_1 a diagram for L_1. There is a sequence of moves given by Reidemeister moves $R1, R2, R3$ and Morse moves, $M1 :\to \circ, M2$ a map between saddles, and $M3 : \circ \to$. To each move we associate a map on Kh. To Reidemeister moves, these are isomorphisms from the proof of invariance. To the Morse moves, we have

$$\to \circ \tag{305}$$

$$1 \to v_+ \tag{306}$$

$$\circ \xrightarrow{\epsilon} \tag{307}$$

$$v_- \to 1 \tag{308}$$

$$v_+ \to 0 \tag{309}$$

and the saddle maps either m or Δ. from the Frobenius algebra. As an aside, the defines a map F_Σ, which is well-defined up to ± 1 as an invariant on Σ, i.e. it doesn't depend on the decomposition of Σ (up to isotopy). This is a theorem of Khovanov and independently of Jacobsson. The same thing works for Lee homology:

$$F_{Lee,\Sigma} : Lee(L_0) \to Lee(L_1) \tag{310}$$

Proposition 4. *If Σ is an oriented cobordism from L_0 to L_1 such that every component of Σ has a boundary component on L_0, then $F_{\Sigma,Lee}([s_{o/L_0}])$ is a nonzero multiple of $[s_{o/L_1}]$ (o is the orientation on Σ).*

The proof of this is to check under each Reidemeister and Morse move explicitly. Hence, if Σ is a connected cobordism between knots K_0, K_1, then $F_{Lee,\Sigma} : \mathbb{Q}\oplus\mathbb{Z} \to \mathbb{Q}\oplus\mathbb{Q}$ is an isomorphism. Say Σ has minimal genus, i.e. $g_s(K)$. How does $F_{Lee,\Sigma}$ change the grading. We claim that it changes the homological grading by 0 and the quantum grading by at least $\chi(\Sigma)$. Let $x \in Lee(K) - \{0\}$ be a class of grading $s_{max} = s+1$. Then what we get is that $q(F_{\Sigma'}(x)) \geq q(x) + \chi(\Sigma')$. $F_{\Sigma'}(x) \in Lee(\circ) = V$ (Lee homology of the unknot), and $q(x)+\chi(\Sigma') = s+1-2g_s(K)$. Therefore $s \leq 2g_s(K)$. By taking the mirror of K $m(K)$ (we reflect), this bounds the surface $\overline{\Sigma}$ (Σ with the opposite orientation, so same genus). $s(m(K)) = -s(K)$, so $-s(K) \leq 2g_s(K)$. Thus $|s| \leq 2g_s(K)$. $\square$

We need one more thing to prove the Milnor conjecture.

Proposition 5. *$s(T_{p,q}) = (p-1)(q-1)$.*

We generalize this to the calculation of s for positive knots.

Definition 44. *$K \subset \mathbb{R}^3$ is **positive** if it has a diagram with all crossings being positive (left crosses over to the right).*

For a positive crossing, the 0-resolution is the oriented resolution $D \to D_o$, where o is the orientation of K.

Proposition 6. *If K has a positive diagram D with n positive crossings, and D_0 consists of k circles, then the Rasmussen invariant $s(K) = n + 1 - k$.*

This would give a very explicit way of computing the Rasmussen invariant, as least for positive knots.

Proof. Recall that s is the q-grading of a combination of the canonical generators:

$$s = \frac{q([s_o] + [s_{\bar{o}}])_{+q([s_{\bar{o}}]-[s_{\bar{o}}])}}{2} \tag{311}$$

One of $[s_o] \pm [s_{\bar{o}]}$ is in degree $s-1$, the other in degree $s+1$ (in filtration degree on the homology). But $q([s_o]) = q([s_{\bar{o}}])_{=s-1}$ because it's a linear combination of something in filtration degree $s-1$ and something deeper in filtration in degree $s+1$, and the linear combination is still in degree $s-1$, as this is the lowest we can go. $q([s_0]) = max\{q(x)|x$ homologous to $s_0\}$. $x = s_o + d\alpha$. For example, for the trefoil, the positive resolution is the resolution giving the center circle. Label the inner circle a and the outer circle b. $s_o = a \otimes b$, where $a = v_+ + v_-, b = -v_+ + v_-$. s_o lives in the lowest homological grading.

$$\begin{tikzcd}
 & 01 \arrow[dr] & \\
00 \arrow[ur] \arrow[dr] & & 11 \\
 & 10 \arrow[ur] &
\end{tikzcd}$$

On the left side of the cube is the 0-resolution. So $Im(d) = 0$ in $[[D_o]]$, therefore s_0 is the unique class homologous to s_o. Therefore $q([s_o]) = q(s_o)$. s_o is a product of as and bs: $s_o = (v_+ \pm v_-) \otimes (v_+ \pm v_-) \otimes ...$ has the same filtration degree as $v_- \otimes v_- \otimes ...$ So $q(s_o)$ is -1 k times, since we had k circles in the resolution, plus n $(n_+ - 2n_-), n_- = 0, n_+ = n$. This gives $q(s_o) = n - k = s - 1$, so $s = n + 1 - k$. $\square$

Example 51. *For $T_{p,q}$ (which is positive), $n = pq$ crossings, $k = p + 1$ circles with the oriented resolution. This gives $s(T_{p,q}) = pq - (p+q) + 1 = (p-1)(q-1)$. This concludes the combinatorial proof of Milnor's Conjecture.*

Let's also give a new proof of the existence of exotic smooth structures on $\mathbb{R}^4$. This will involve Khovanov homology, and will not involve Gauge theory. It does, however, involve several results from Freedman.

- Step 1: Prove the existence of a topologically slice knot that isn't smoothly slice.

Definition 45. *K is (smoothly)* ***slice*** *if there exists a smoothly, properly embedded disc $D \hookrightarrow B^4$ such that $\partial D = K \subset S^3$ (i.e. $g_s(K) = 0$).*

Remark 45. *Every K bounds a topologically embedded disc $D \hookrightarrow B^4$: the cone on K: Cone(K).*

Definition 46. *K is* ***topologically slice*** *if there exists a continuous embedding $\varphi : (D^2 \times D^2, \partial D^2 \times D^2) \to (D^4, \partial D^4 = S^3)$ such that $\varphi(\partial D^2 \times 0) = K$. $\varphi(D^2 \times 0)$ is some $D \subset B^4, \partial D = K$. D has a "flat neighbrohood" which is kind of like a tubular neighborhood, topologically.*

Now comes the "black box:"

Theorem 46. *(Freedman) If K satisfies $\Delta_k(t) = 1$ (the Alexander polynomial of K is 1), then K is topologically slice.*

Example 52. *The Whitehead double $Wh(K)$ of any knot K. $Wh(K)$ is: take K and a translate of K (0-framing) called K' with $lk(K, K') = 0$. This is a link, but somewhere replace this with a clasp. See Figure 35. we don't want to simply join them together because we want the linking number to be 0, hence the twists. As an exercise, $\Delta_{Wh(K)} = 1$ for any K. $s(Wh(T_{2,3})) = 2$ implies that $Wh(T_{2,3})$ is top slice, not smoothly slice. Thus $|s| \leq 2g_s$ (s is an obstruction to slice-ness).*

Remark 46. *$s(Wh(m(T_{2,3}))) = 0$ where the mirror yields the left-handed Trefoil. It is unknown if $Wh(m(T_{2,3}))$ is slice or not. All the invariants that would obstruct slice-ness vanish, and we know it's topologically slice. It's known that the Whitehead doubles of all positive torus knots are not slice.*

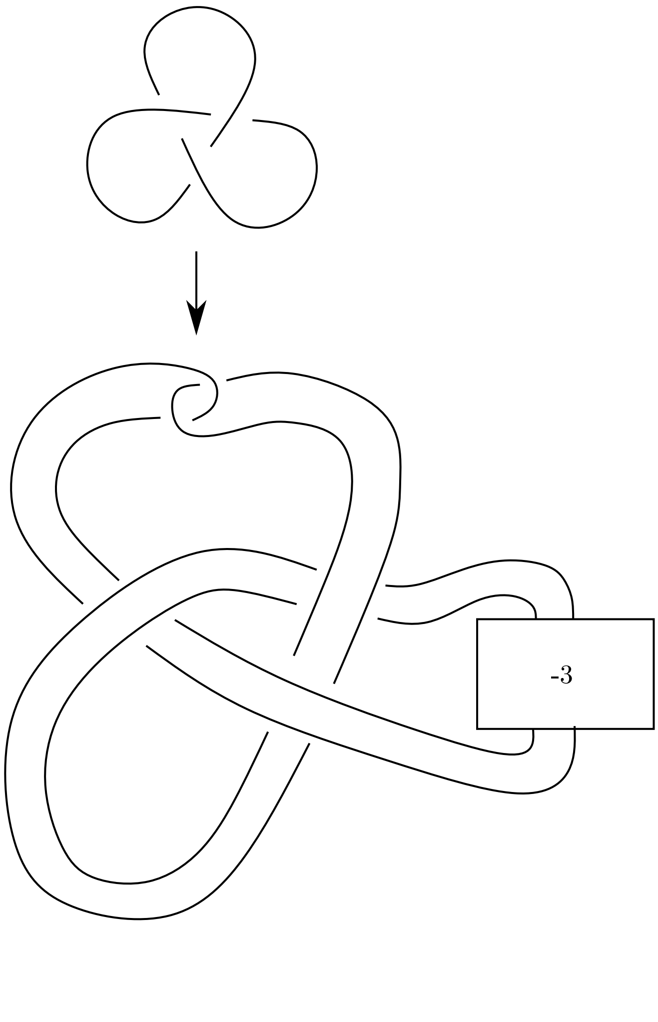


Figure 35: The Whitehead double of the trefoil $T_{2,3}$.

- Step 2: The Trace Embedding Lemma.

Proposition 7. *(Trace Embedding Lemma) $K \subset S^3$ is slice if and only if $X_0(K)$ embeds smoothly in S^4, where*

Definition 47. *$X_0(K)$ is the "trace of the 0-surgery on K", namely the a 0-handle union a cobordism from S^3 to $S^3_0(K)$ (the 0-surgery on K), given by attaching a 2-handle along K of framing 0. $\partial X_0(K) = S^2_0(K)$.*

Example 53. *If K is the unknot, $S^3_0(K) = S^1 \times S^2, X_0(K) = (D^2 \times S^2)$.*

Proof. The forward direction first: If K is slice, then $K = \partial D, D \subset B^4$, then $S^4 = X_0(K) \cup (B^4 - nbhd(D))$. We get that $X_0(K)$ embeds into S^4.

Conversely (slightly more tricky): Construct an embedding of $F : S^2 \to X_0(K)$. $F = (cone) \cup (disc)$. This is smooth away from p, the point of the cone. Suppose we have $i : X(K) \hookrightarrow S^4$ a smooth embedding. Then $i \circ F : S^2 \hookrightarrow S^4$ is an embedding, smooth away from p. Take $S_\epsilon(p), S_\epsilon(p) \cap (i \circ F(S^2)) = K$, because a neighborhood on p looks locally like a cone on K. We get that there exists a smoothly embedded disc $D \subset S^4 - S_\epsilon(p) = B^4, \partial D = K$, so K is slice. $\square$

Similarly, we can prove another proposition:

Proposition 8. *K is topologically slice if and only if there exists a flat embedding of $X_0(K) \hookrightarrow S^4$ (a topological embedding with a collar neighborhood).*

- Proof of the existance of exotic $\mathbb{R}^4$.

Let K be a topolically slice knot that isn't slice. $S^4 = B^4 \cup B^4 = X_0(B^4) \cup (B^4 - nbhd(D))$ where D is a topologically flat disc with $\partial D = K$. Let $Z = S^4 - \{x\} - int(X_0(K)) = \mathbb{R}^4 - int(X_0(K))$. This is an open (non-compact) topological 4-manifold with boundary. A theorem of Freedman says that open 4-manifolds admit smooth structures. (He managed to get a smooth structure everywhere except for one point on a closed 4-manifold. That point "just goes crazy" - Manolescu, 2020). Let's give Z a smooth structure. ∂Z is homeomorphic (and hence diffeomorphic to $S^3_0(K)$. The smooth structure on Z is not the one coming from its embedding in B^4! (because our disc is not smoothly embedding). We get a diffeomorphism $\varphi : \partial Z \to S^3_0(K)$. This is because in dimension 3 homeomorphism implies diffeomorphism. This is a hard theorem from the 1950s by Moisin(?). Let $R = Z \cup_\varphi X_0(K)$. Each piece is smooth, and they're glued in a smooth way. But if we glue them in a smooth way, we get a smooth manifold, homeomorphic to $\mathbb{R}^4$ (show it's homotopy equivalent, i.e.

contractible, and this follows from Mayer-Vietoris and Seifert-Van Kampen). So we get a manifold homeomorphic to $\mathbb{R}^4$. The point is that $X_0(K)$ embeds smoothly in R. We also know that K is not slice, so K does not embed smoothly into ordinary $\mathbb{R}^4$ by the Trace Embedding Lemma. This shows that R is not diffeomorphic to $\mathbb{R}^4$, so R is an exotic $\mathbb{R}^4$.

Can Khovanov homology also help us detect exotic closed 4-manifolds? We also have the Smooth Poincaré Conjecture in dimension 4 (SPC4):

Conjecture 5. *If X^4 is homotopy equivalente (hence homeomorphic by Freedman) to S^4, then it is diffeomorphic to S^4.*

We have an equivalent formulation: W^4-smooth with $\partial W = S^3$, and W is contractible i.e. homotopy equivalent to B^4, then W is diffeomorphic to B^4. This is equivalent because we can take $X \to W = X - B^4$, and $W \to X = W \cup_{S^3} B^4$.

The strategy for disproving SPC4 (suggested by Freedman-Gompf-Morrison-Walker, 2009): Find a knot $K \subset S^3$ such that K bounds a smooth disc D in a contractible W with $\partial W = S^3$. but $s(K) \neq 0$ (hence K is not slice, so $W \neq B^4$, so W is an exotic B^4.

Remark 47. *There exists potential counter examples to SPC4 (e.g. Gluck twists on S^4), $X \simeq S^4$, take $W = X - B^4$. Assume W has a Kirby diagram (or, better said, handle decomposition) with no 3-handles. Turn X upside down (map n-handles to $(4-n)$-handles. So the attaching circles for 2-handles are knots$\subset S^3$, and bound smooth discs in W (cores of the 2-handles). So far, all K coming from this gave $s(K) = 0$.*

Remark 48. *There are invariants similar to s coming from Seiberg-Witten theory, Yang-Mills theory, Heegaard Floer theory, but they cannot tell the difference between slice-ness in B^4 and slice-ness in a homotopy ball B^4.*

An open question is: If $K \subset S^3 = \partial W^4$, W smooth and contractible, and $\Sigma \hookrightarrow W$ smooth and proper, with $\partial \Sigma = K$, do we have $|s(K)| \leq 2g(\Sigma)$? We know it's true for $W = B^4$. If it's true in general, then the FGMW strategy fails.

15.3 The Freedman-Gompf-Morrison-Walker Strategy Fails for Gluck Twists

Theorem 47. *(Manolescu-Marengon-Sarkar-Willis, 2019) If $K \subset S^3 = \partial W$, W a Gluck twist on B^4, and a smoothly and properly embedded surface $\Sigma \hookrightarrow W, \partial \Sigma = K$, then $|s(k)| \leq 2g(\Sigma)$.*

Corollary 16. *The FGMW strategy (find a knot that bounds a disc in a homotopy B^4 but not in B^4 because $s \neq 0$) for disproving $SPC4$ fails for Gluck twists.*

Definition 48. *For $\Sigma \cong S^2, \Sigma \hookrightarrow S^4$, G_Σ is a Gluck twist on Σ homotopy equivalent to S^4, but it is not known if it is diffeomorphic to S^4. For N a neighborhood of $\Sigma \cong D^2 \times S^2$, with*

$$\varphi : \partial N \to \partial N \tag{312}$$

$$(e^{i\theta}, z) = (e^{i\theta}, rot_\theta(z)) \tag{313}$$

we have $G_\Sigma = (S^4 - N) \cup_\varphi N$ is a Gluck twist.

Remark 49. *φ is the generator of $\pi_1(S)(3)) = \pi_1(\mathbb{R}P^3) = \{S^1 \to Rot(S^2)\} = \mathbb{Z}_2$. If $\psi(e^{i\theta}, z) = (e^{i\theta}, rot_\theta(z))$, then $\psi \sim \varphi$ implies $G_\Sigma = (S^4 - N) \cup_\psi N$.*

To represent a Gluck twist in Kirby diagrams, choose a Morse function $f : S^4 \to \mathbb{R}, N = f^{-1}(-\infty, 0]$. Let $h : S^2 \to \mathbb{R}$ is the standard height function, and $\pi : N \cong S^2 \times D^2 \to S^2$ be the projection map. Let $f_N = h \circ \pi + |z|^2$, where $|z|$ is the norm in D^2. Then a Kirby diagram for S^4, where Σ comes from a 0-framed 2-handle, and then other handles are attached.

Remark 50. *The 1-handles can be assumed to be away from Σ.*

$[\Sigma] \cdot [\Sigma] = 0$ because $H_2(S^4) = 0$. From $\varphi(e^{i\theta}, z) = (e^{i\theta}, rot_\theta(z))$, the strands passing through the disc are rotated. Thus a Kirby diagram for G_Σ is a full ± 1 twist for all strands passing through the disc.

Proposition 9. *$G_\Sigma \# \mathbb{C}P^2 \cong \mathbb{C}P^2$ and $G_\Sigma \# \overline{\mathbb{C}P^2} \cong \overline{\mathbb{C}P^2}$ (diffeomorphic).*

Recall that we can pass strands inside a disjoint 1-framed disc by giving a full twist of the strands through the disc (see handleslides).

Proof. $G_\Sigma \# \mathbb{C}P^2$: see Figure 36. For $G_\Sigma \# \overline{\mathbb{C}P^2} \cong \overline{\mathbb{C}P^2}$, we do the same thing but with a $+1$ twist instead of a -1 twist. $\square$

Theorem 48. *(Manolescu-Marengon-Sankar-Willis 2, 2019) For $W = \overline{\mathbb{C}P^2} - B^4$, $K \subset \partial W = S^3$, with smoothly and properly embedded surface $\Sigma \subset W$, $\partial \Sigma = K$, $[\Sigma] = 0 \in H_2(W, \partial W) = H_2(\overline{\mathbb{C}P^2}) = \mathbb{Z}$. Then the Rasmussen invariant $s(K) \leq 2g(\Sigma$.*

Proof. See Figure 37. For N a neighborhood of $\overline{\mathbb{C}P^1}, \partial N = S^3 \xrightarrow{\pi} S^2 = \overline{\mathbb{C}P^1}$ is the Hopf fibration.

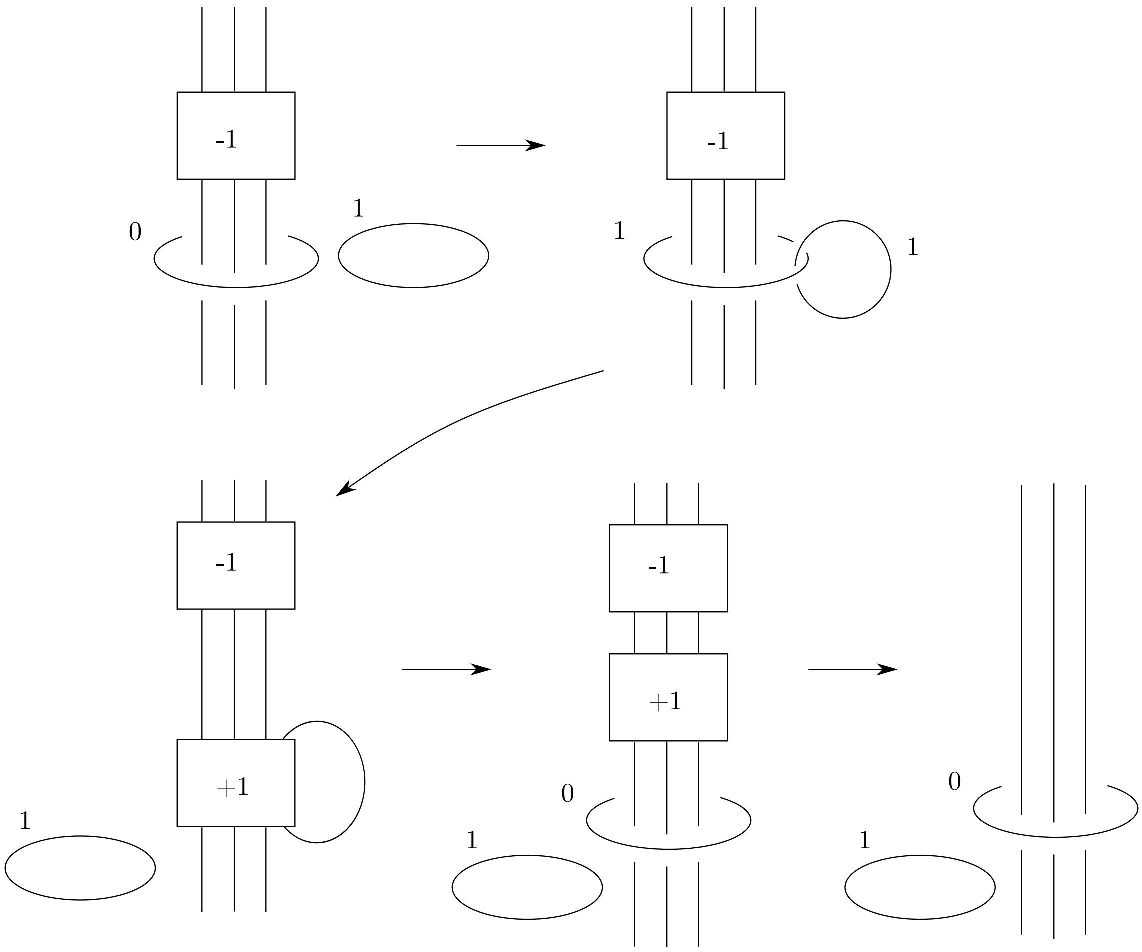


Figure 36: $G_\Sigma\#\mathbb{C}P^2$ Kirby diagram sequence.

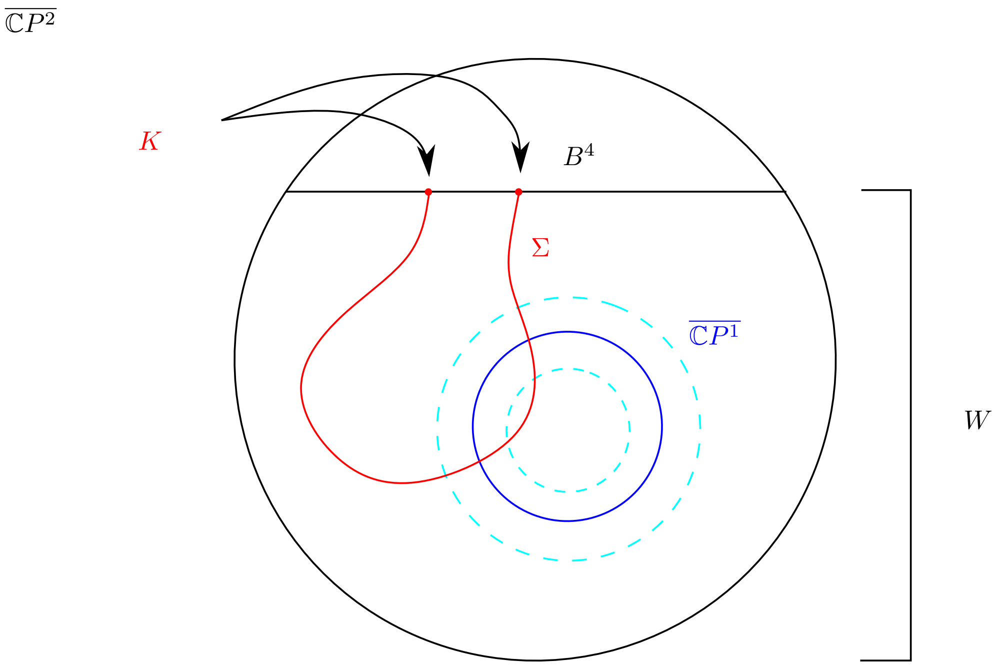


Figure 37: Manolescu-Marengon-Sankar-Willis 2, 2019

$\overline{\mathbb{C}P^1} = N \cup (S^3 \times [0,1]) \cup B^4$. $[\Sigma] \cdot [\overline{\mathbb{C}P^1}] = 0$. Say $\Sigma \pitchfork \overline{\mathbb{C}P^1}$ in $2p$ points. Then $\Sigma \cap N$ is a union of $2p$ discs, and $\Sigma \cap \partial N$ is a union of $2p$ circles, which are the fibers on the Hopf fibration. These are $L_{p,p} \subset S^3$, the torus link $T_{2p,2p}$ with p strands oriented one way, and p oriented the other way. One can define Rasmussen's s invariant for links $dimLee(L) = 2^l, l = \#$ of components. For $Kh(L) \to Lee(L)$, there exists canonical generators $s_\sigma, s_{\overline{\sigma}}$, where σ is the orientation of L. Let

$$s(L) = \frac{q([s_\sigma + s_{\overline{\sigma}}]) + q([s_\sigma - s_{\overline{\sigma}}])}{2} \in \mathbb{Z} \tag{314}$$

We have a cobordism in $S^3 \times [0,1]$ from K to $L_{p,p}$ of genus $g(\Sigma)$. Functoriality under cobordisms (as in Rasmussen's proof of the Milnor Conjecture) gives $s(K) - 2g(\Sigma) + 1 - 2p \leq s(L_{p,p})$. Then we calculate that $s(L_{p,p} = 1 - 2p$, adn then $s(K) \leq 2g(\Sigma)$. □

Remark 51. *We could study the behavior of s with respect to surfaces in any 4-manifold made of 2-handles if we could compute s of the cables (parallel families) of the link of attaching circles for 2-handles.*

Theorem 49. *Take the same hypothesis of Manolescu-Marengon-Sankar-Willis - 2, but with $\mathbb{C}P^2$ instead of $\overline{\mathbb{C}P^2}$; then we get $-s(K) \leq 2g(\Sigma)$.*

Proof. Apply Manolescu-Marengon-Sankar-Willis - 2 to $m(K)$, which bounds $\overline{\Sigma}$ in $\overline{\mathbb{C}P^2}$; $s(m(k)) = -s(K)$. □

This theorem and Manolescu-Marengon-Sankar-Willis - 2 and the proposition that Gluck twists trivialize when connected-summed to $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$. These imply the first theorem by Manolescu-Marengon-Sankar-Willis. W is a Bluck twist on B^4, $\Sigma \subset W, \partial\Sigma = K$. Then $W = G_S - B^4$, where S is a surface in S^4. $G_S \# \mathbb{C}P^2 = \mathbb{C}P^2$ implies $W \# \mathbb{C}P^2 = (\mathbb{C}P^2 - B^4), G_S \# \overline{\mathbb{C}P^2} = \overline{\mathbb{C}P^2}$ implies $W \# \overline{\mathbb{C}P^2} - (\overline{\mathbb{C}P^2} - B^4)$. From Theorem 2 we get $s(K) \leq 2g(\Sigma)$ (in $\overline{\mathbb{C}P^2}$, and from the last theorem $-s(K) \leq 2g(\Sigma)$ (in $\mathbb{C}P^2$. Therefore $|s(K)| \leq 2g(\Sigma)$. (Note that $[\Sigma] = 0$ because $H_2(W) = 0$).

Any remaining hope for disproving SPC4 using the FGMW strategy?

Proposition 10. *Suppose we had two knots K, K' with $S^3_0(K) \cong S^3_0(K')$ with K slice and K'-not slice, because $s(K') \neq 0$. Then SPC4 is false.*

If $S^3_0(K)$ is the result of 0-surgery, and a Kirby diagram represents the 4-manifold $X_0(K)$ "trace of the 0-surgery" then $\partial X_0(K) = S^3_0(K)$.

Example 54. *For K the unknot u, $S^3_0(u) = S^1 \times S^2$, and $X_0(u) = D^2 \times S^2$*

Proof. If K is slice then there exists a disc $D \hookrightarrow_{smooth} B^4, \partial D = K$. Then $X_0(K)$ with $B^4 - nbhd(D)$ yields S^4. This is called the "trace embedding." We know that $\partial X_0(X) = S^3_0(K) = S^3_0(K') = \partial X_0(K')$. Then we replace the union to S^4 by $X := X_0(K') \cup (B^4 - nbhd(K)) \sim S^4$. This is homotopy equivalent, and hence by Freedman homeomorphic to S^4.

Lemma 9. *(Trace Embedding Lemma) For K' not slice, $X_0(K')$ cannot be smoothly embedded in S^4. However, $X_0(K') \hookrightarrow_{smooth} X$. Then X is not diffeomorphic to S^4.*

□

Additional reading: (one of the most exciting results of Khovanov homology) Proof of the Thom Conjecture without gauge thoery (Lambert-Cole, 2018).

Conjecture 6. *(Thom) $\Sigma \subset \mathbb{C}P^2$ smoothly, with $[\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$. Then $g(\Sigma) \geq \frac{1}{2}(d-1)(d-2)$.*

This was first proved by Kronheimer-Mrowka using Seiberg-Witten theory. The ingredients in Lambert-Cole's proof are:

1. Trisections: choose a trisection of $\mathbb{C}P^2$ for which Σ is in "bridge position."
2. Contact geometry: make $\Sigma \cap Y_i$ (in the trisection) be transverse to the standard contact structure in $\#^k(S^1 \times S^2)$.
3. Khovanov homology: the slice-Bennequin inequality for transverse knots, which can be proved using Kh. In fact it is equivalent to Milnor's Conjecture for torus knots.