Lie Algebras & their Relation to Conformal Field Theory

Alec Lau

Contents

1 Topological Groups

A topological group is a topological space G with a group structure such that the multiplication map

$$m : G \times G \to G, \tag{1}$$

$$i : G \to G \tag{2}$$

are continuous. For example, the open subset $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is a topological group equipped with the usual matrix multiplication and matrix inversion.

Example 1. *Given the topology of $M_n(\mathbb{R})$ as the topology of a finite-dimensional $\mathbb{R}-$vector space, we equip $GL_n(\mathbb{R})$ with the subspace topology.*

Example 2. *The special linear group $SL_n(\mathbb{R}) := \{g \in GL_n(\mathbb{R}) | \det g = 1\}$*

Example 3. *The orthogonal group $O(n) := \{g \in GL_n(\mathbb{R}) : \langle gv, gw\rangle = \langle v, w\rangle \forall v, w \in \mathbb{R}^n\}$ This is equivalent to $g^T g = 1\}$. If this is the case, then $\det g = \pm 1$.*

Example 4. *The special orthogonal group is* $SO_n(\mathbb{R}) := \{g \in O(n) | \det g = 1\}$.

Remark 1. *If we think about* $L \in SO_4(\mathbb{R})$ *as a linear transformation on* $\mathbb{R}^4$*, the space of all quaternions,* $L(1)$ *is a nonzero quaternion and we can define a linear map*

$$L' : \mathbb{R}^4 \to \mathbb{R}^4, \tag{3}$$

$$L'(x) \mapsto L(x)L(1)^{-1}, x \in \mathbb{R}^4 \tag{4}$$

L' *is distance preserving due to its being an orthogonal transformation. Since* L' *also fixes the real axis by orthogonality it must map the space of pure quaternions to itself (the space of quaternions with real part equal to 0). If we take the map*

$$\phi : SO_4(\mathbb{R}) \to S^3 \times O(3), \tag{5}$$

$$\phi(L) = (L(1), L') \tag{6}$$

, we have a homeomorphism between $SO_4(\mathbb{R})$ *and* $S^3 \times O(3)$*. This is because* L' *maps the space of pure quaternions to itself and preserves Euclidean norm, and is therefore an element of* $O(3)$*.* $L(1)$ *is obviously a unit quaternion.* ϕ *is injective because it sends the identity element to* $(1, id)$*. The image is also a compact connected subspace of* $S^3 \times O(3)$*. Due to linearity,* L' *and the preimage of* ϕ *are uniquely defined by* $L(1) \in S^3$*, so* ϕ *is surjective. Since* $SO_4(\mathbb{R})$ *is compact and* $S^3 \times SO_3(\mathbb{R})$ *is Hausdorff,* ϕ *must be a homeomorphism. In addition, the map*

$$\psi : S^3 \to O(3) \tag{7}$$

is a homeomorphism, the kernel of which is obtained by restricting ψ *to* S^3 *to the two element group* ± 1*. Thus* $\pi_1(O(3)) \cong \pi(\mathbb{R}P^3)$*, and therefore* $\pi_1(SO_4(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$*.*

Remark 2. $GL_n(\mathbb{R})$ *is not connected, because we have a continuous surjective determinant map* $\det : GL_n(\mathbb{R}) \to (\mathbb{R} - \{0\}) := \mathbb{R}^\times$*.*

Remark 3. $GL_n(\mathbb{R})^+ := \{g \in GL_n(\mathbb{R}) | \det g > 0\}$ *is connected, we have an isomorphism and homeomorphism of topological groups* $GL_n(\mathbb{R}) \cong \mathbb{R}^+ \ltimes SL_n(\mathbb{R})$*. Since* $SL_n(\mathbb{R})$ *is connected,* $GL_n(\mathbb{R})^+$ *is connected.*

1.1 Quotients

For G a topological group and $H \subset G$ a subgroup, the map

$$\pi : G \to G/H \tag{8}$$

where G/H has the quotient topology, maps open sets to open sets.

Proof.

$$\pi^{-1}(\pi(\Omega)) = \cup_{h\in H}(\Omega \cdot h) \tag{9}$$

for Ω open. The RHS is open, and this completes the proof. □

Example 5. *We have an isomorphism:*

$$\mathbb{R}/\mathbb{Z} \cong S^1 \tag{10}$$

$$t \mapsto e^{i2\pi t} \tag{11}$$

2 Lie Groups

A *Lie Group* is a topological group with a C^∞ manifold structure so that m and i are are smooth. A *symplectic form* ψ on a finite-dimensional vector space V is an alternating bilinear form on V, $\psi : V \times V \to k$ that is nondegenerate, aka an isomorphism $V \cong V^*$ given by $v \mapsto \psi(v, \cdot)$.

Example 6. *The symplectic group of a symplectic space* (V, ψ) *is*

$$Sp(\psi) := \{g \in GL_n(V) : \psi(gv, gw) = \psi(v, w), \forall v, w \in \psi\} \tag{12}$$

Definition 1. *Given a representation of a Lie group G $\rho : g \to GL_n(V)$, the **character** of (V, ρ) is given by $\chi_V(g) = \chi_\rho(g) := tr(\rho(g))$.*

For V, V' two representations of G, we have

$$\chi_{V\otimes V'}(g) = \mathrm{tr}(\rho(g) \otimes \rho'(g)) \tag{13}$$

$$= \mathrm{tr}(\rho(g)) \cdot \mathrm{tr}(\rho'(g)) \tag{14}$$

$$= \chi_V(g) \cdot \chi_{V'}(g). \tag{15}$$

For a G representation V, the dual representation V^* is given by

$$(g \cdot l)(v) = l(\rho(g^{-1})(v)) \tag{16}$$

Thus

$$\chi_{V^*} = \mathrm{tr}(\rho(g^{-1})^*) = \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}(\rho(g))} \tag{17}$$

3 Root System

A *hyperplane* is a plane is a plane with dimension 1 less than its ambient space, e.g. sheets in $\mathbb{R}^3$, a line in a sheet, etc.

Let E be a finite-dimensional Euclidean vector space with the Euclidean inner product $(\cdot, \cdot)$. A *root system* Φ is a set of nonzero vectors called *roots* in E such that:

The roots span E.

The only scalar multiples of a root $\alpha \in \Phi$ are α and $-\alpha$.

$\forall \alpha \in \Phi$, Φ is closed under reflection through the hyperplane perpendicular to α.

For any roots $\alpha, \beta \in \Phi$, the projection of β through the span of α is either an integer or half-integer multiple of α.

4 Weyl Group

The *Isometry Group* on a metric space is the set of distance-preserving bijective maps from the space to itself, with composition the group operation.

Example 7. *The isometry group of S^2 is $O(3)$. (group of $n \times n$ matrices where $M^T M = M M^T = Id$)*

The *Weyl Group* is a subgroup of the isometry group of the root system, generated by reflections of the hyperplanes orthogonal to the roots.

5 Lie Algebras

A *Lie Algebra* is an algebra $\mathfrak{g}$ where the binary operation is bilinear,

$[ax + by, z] = a[x, z] + b[y, z], [x, ay + bz] = a[x, y] + b[x, z]$

alternative,

$[x, x] = 0, \forall x \in \mathfrak{g}$

and satisfies the *Jacobi Identity*

$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

Remark 4. *We can show that $[x, y] = -[y, z]$ by considering $[x + y, x + y]$.*

Example 8. $\mathfrak{g} = \{X \in Mat(n, \mathbb{C}) | \forall t \in \mathbb{R}, e^{tX} \in G\}$*, for G a lie group.*

Example 9. *$SL(n, \mathbb{R})$, the matrix group with real entries and determinant 1, has lie algebra $n \times n$ matrices with real values and trace 0.*

5.0.1 Physics Flash

The angular momentum operators have the same commutation relations as $\mathfrak{so}(3)$ of $SO(3) < O(3)$, where $SO(3)$ is the group of othorgonal matrices of determinant 1.

5.1 Representations of Lie Algebras

Given a vector space V, $\mathfrak{gl}(V)$ is the lie algebra of linear automorphisms on V, with the bracket given by $[X, Y] = XY - YX$.

A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V) \tag{18}$$

A representation is called faithful if its kernel is is 0. We can construct a representation for any Lie algebra $\mathfrak{g}$:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{19}$$

$$ad(x)(y) \mapsto [x, y] \tag{20}$$

This representation is called the *adjoint representation*.

5.2 Semi-Simple Lie Algebras

A lie algebra is *simple* if it is not abelian and has no non-trivial ideals. (recall a left ideal is a subset of our algebraic object such that the subset is closed under addition, scalar multiplication, and LEFT multiplication from another vector)

A lie algebra is *semi-simple* if it can be expressed as a (is isomorphic to a) direct sum of simple lie algebras.

In this way one can think of a simple lie algebra as being one-dimensional.

The *Cartan Subalgebra* is the maximal abelian ideal of our lie algebra (generated by all $H^i, i = 1, ..., r$ such that $[H^i, H^j] = 0$) denoted $\mathfrak{h}$.

6 Root System of a Semi-Simple Lie Algebra

The remaining generators of $\mathfrak{g}$ are chosen such that $[H^i, E^\alpha] = \alpha^i E^\alpha$, (recall angular momentum operators/spin operators in quantum mechanics) and α^i is a component of the vector $\alpha = (\alpha^1, ..., \alpha^r)$. α is called a root and E^α is the corresponding ladder operator. α is therefore a nonzero element of $\mathfrak{h}^*$. The roots are the nonzero weights for the adjoint representation. These roots form, if you can believe it, a root system.

Remark 5. *Because $\mathfrak{h}$ is the maximal subalgebra, of $\mathfrak{g}$, the roots are non-degenerate.*

In the adjoint representation we have

$$ad(H^i)E^\alpha = \alpha^i E^\alpha \mapsto H^i |\alpha\rangle = \alpha^i |\alpha\rangle$$

The one-to-one correspondence with between the states $|\alpha\rangle$ and E^α reflect the nondegenerate character of roots. In this representation, the zero eigenvalue has degeneracy r for each state $|H^i\rangle$. The adjoint representation is a representation using the Lie algebra itself as the vector space. We define an inner product using the Killing Form

$$\tilde{K}(X,Y) = Tr(adXadY) \tag{21}$$

$$K(X,Y) = \frac{1}{2g}Tr(adXadY) \tag{22}$$

where g is the dual coxeter number of the algebra.

6.1 Weights

For an arbitrary representation, a basis $\{|\lambda\rangle\}$ can always be found such that $H^i |\lambda\rangle = \lambda^i |\lambda\rangle$ eigenvalues exist. These eigenvalues build the vector $\lambda = (\lambda^1, ..., \lambda^r)$ called a *weight*. The *weight space* with weight λ is a subspace of V:

$$V_\lambda = \{v \in V : \forall H^i \in \mathfrak{h}, H \cdot v = \lambda(H^i)v\} \tag{23}$$

where $\lambda(H^i)$ is the weight associated to V_λ. A weight of the representation is a linear functional λ such that the corresponding V_λ is nonzero. The nonzero elements of the weight space are *weight vectors.*

(This is to say the weight vectors are the simulataneous eigenvectors for the action of the elements of $\mathfrak{h}$ with weight $\lambda = (\lambda^1, ..., \lambda^r)$)

A simple root can be written as a linear combination of a set of fundamental weights (weights that satisfy $\omega_i(H^{\alpha_j}) = \delta_{ij}$)

6.2 Examples

Let $E_{jk} \in \mathbb{C}^{3\times 3}$ be the 3×3 matrix whose j, k^{th} element is equal to 1 and the rest are equal to 0, for $j, k \in \{1, 2, 3\}$.

For the lie group $SL_3(\mathbb{C})$ we define for the lie algebra $H_{jk} = E_{jj} - E_{kk}$

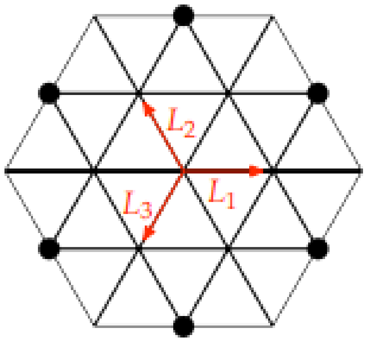
$$\mathfrak{h} = \mathbb{C}H_{12} + \mathbb{C}H_{13} + \mathbb{C}H_{23} \tag{24}$$

is a Cartan subalgebra. Define L_i be the following:

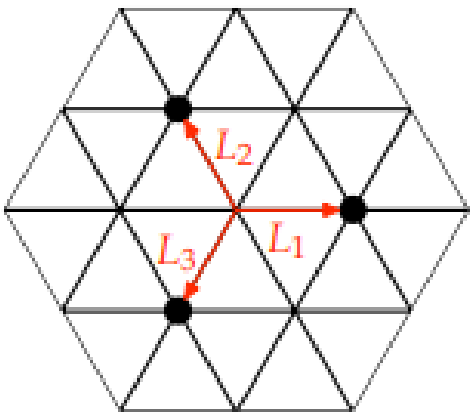
$$L_i : \mathfrak{h} \to \mathbb{C} \tag{25}$$

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \mapsto x_i \tag{26}$$

With this notation we can see that $[H_{jk}, E_{rs}] = ((L_r - L_s)(H_{jk}))E_{rs}$, so $L_r - L_s$ are the roots of the representation, drawn with a dot.



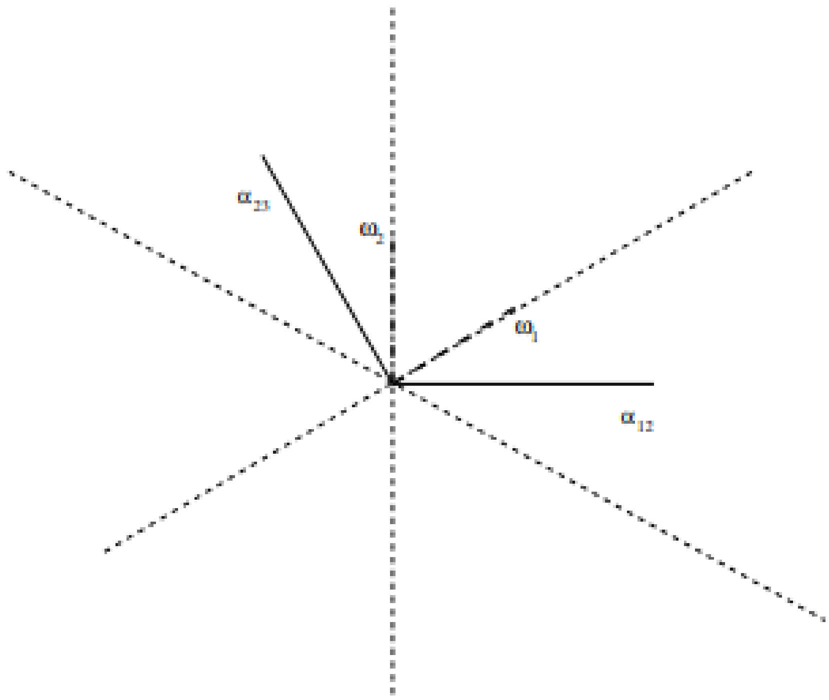
Letting e_i be the i^{th} unit vector, we see that $H_{jk}e_i = \delta_{ji}e_i - \delta_{ki}e_i = L_i(H_{jk})e_i$, so the weights are L_i, shown in the diagram below with a dot:



(images from Clara Loh's notes: "Representation theory of Lie algebras")

Here's another example explaining these diagrams:

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2 \\ \alpha_2 &= -\omega_1 + 2\omega_1 \end{aligned}$$



Simple Roots and Fundamental Weights for $SU(3)$

(image from notes of Peter Woit)