EE376A Final Project

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1 Introduction

Various quantum computing methods have been proposed and are being worked on: trapped atoms in an optical lattice, superconducting qubits, etc. However, the problem of decoherence remains a heavy clound over theorists' and engineers' heads. Even with the existence of quantum error-correcting codes, it is currently unknown by what factor error will increase with large amounts of qubits. Thus the intrinsically fault-tolerant qubits in non-abelian anyonic systems are incredibly exiting.

2 Non-Abelian Anyons

2.1 Statistics Through Bosons & Fermions

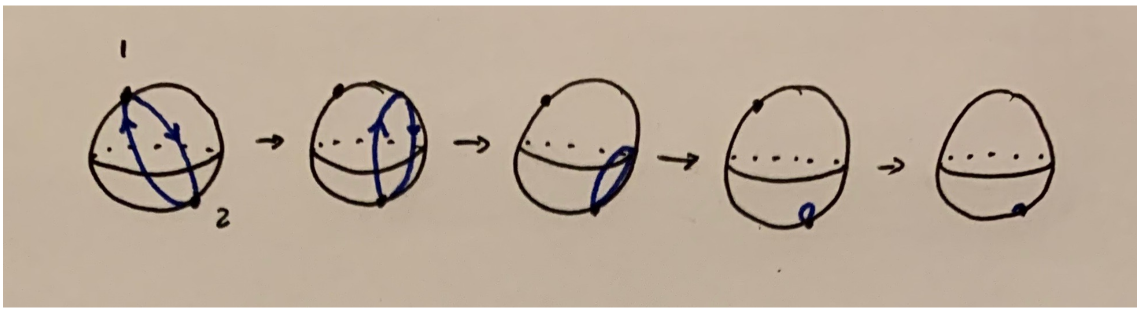
In ordinary 3-dimensional space, there are two types of particles: bosons and fermions. These can be distinguished by their *exchange statistics.*

Definition 1. *Exchange statistics refer to the phase gained by a wavefunction when two identical particles' positions are swapped.*

To figure out the exchange statistics in 3-dimensional space, we consider the space of possible positions (configuration space) of our two particles. All we require about this configuration space is that the particles do not intersect, so there cannot be a cusp/point-like point in our configuration space. Fix one particle at, oh, I don't know, the origin. Since, aside from this, all other points in $\mathbb{R}^3$ are fair game, our configuration space is $\mathbb{R}^3 - \{0\}$.

Remark 1. *For those into topology, notice that the 2-sphere is a deformation-retraction from* $\mathbb{R}^3$*. For those not into topology, use the fact that the map* $\boldsymbol{r} \mapsto \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$ *is continuous, marking an equivalence class in topology and thus this domain of physics.*

We can easily make this space a 2-sphere. Now, swap the particles' positions. Quantum mechanically, this gains a phase. What is the phase? We don't know yet. But let's swap again, so we were back to where we started. If we were to paint the path of one of the identical particles over its two swaps, it would form a circle on the surface of our 2-sphere. But notice that a circle on the surface of a 2-sphere can be smoothly deformed into a point. Thus the path taken by one of the particles (and thus the other, because it also gets exchanged) is topologically (homotopically) equivalent to a stationary path.



This gives us a major hint as to the phase gained by an exchange. If one exchange gives us $\psi \mapsto p\psi$, then two exchanges, due to the particles being identical, gives us $\psi \mapsto p^2\psi$. Since this is equivalent to doing nothing, we have $p^2 = 1$. Thus $p = \pm 1$. Bosons correspond to $p = 1$, fermions correspond to $p = -1$.

2.2 Anyons

Now suppose there is a restriction in your physical system such that your particles are only allowed to move in 2-dimensional space $\mathbb{R}^2$. Our configuration space still doesn't allow intersections, so we can still remove the origin. Other than that, everything is hunky-dory. By the same map above, our configuration space is a 1-sphere (circle). Now do an exchange. $\psi \mapsto p\psi$. We still don't know anything, so let's go again. $\psi \mapsto p^2\psi$. This path is equivalent to one loop around the circle. However, you cannot smoothly deform a circle into a point on the circle, because of that pesky hole, a topological invariant. Thus there's no way we can place any limitations on p; p could be *any*thing. So the particle could have *any* phase. So the particle might as well be called an *any*on. An anyon then corresponds to a particle with any exchange statistics.

NOTE* The wavefunction I have mentioned is not the wavefunction to each anyon. It is the wavefunction of the composite system. This is hard to find in the literature describing these systems, so know this before doing one's own reading.

2.2.1 Interlude for Topology Junkies

The triviality of two exchanges in $\mathbb{R}^3$ can be expressed by the configuration space of one exchange: $(\mathbb{R}^3 - \{0\})/\mathbb{Z}_2 \cong (S^2)/\mathbb{Z}_2 \cong \mathbb{R}P^2$. It is not hard to show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, so the elements in $\pi_1(\mathbb{R}P^2)$ correspond to particles with their statics represented by this group action.

In $\mathbb{R}^2$, the nontriviality becomes painfully obvious: just replace $\mathbb{R}^3$ with $\mathbb{R}^2$ and follow the homeomorphisms/group isomorphisms: $(\mathbb{R}^2 - \{0\})/\mathbb{Z}_2 \cong S^1/\mathbb{Z}_2 \cong \mathbb{R}P$, which, which again with the functor π_1, gives us $\pi_1(\mathbb{R}P) \cong \mathbb{Z}$. (It might seem super extra to refer to π_1 as a functor, but topological quantum computation has functors galore, so this is a sort of primer).

2.3 Topological Degeneracy

In quantum mechanics, distinct states often have distinct energy. However, not always.

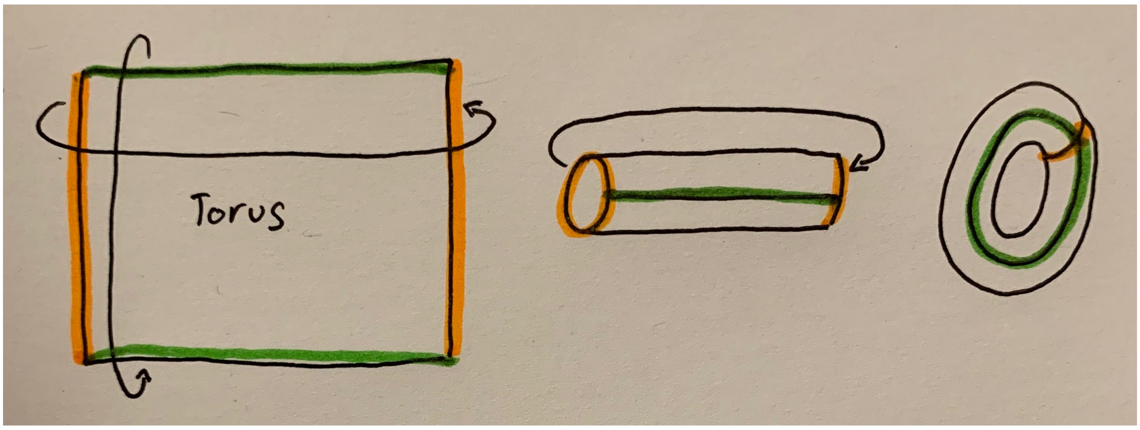
Definition 2. *In systems where there are states that are clearly different but have the same energy, the system is known as degenerate, or having degeneracy.*

Definition 3. *A linear transformation U (could be a matrix, could be an operator on a wavefunction) is unitary if $U * U = UU* = Id$, where $U*$ is the conjugate transpose of U.*

Degeneracy can arise due to the topology of the surface upon which particles act. To demonstrate this, Wen and Niu in [2] considered a system of particles defined on a torus of what is called the Quantum Hall regime.

Definition 4. *The Quantum Hall regime is a low temperature 2-dimensional electronic system (e.g. electron gas) with a strong magnetic field normal to the gas. The energy levels of this system are called Landau levels. Physicists are justifiably (extremely so) obsessed with this domain, because a lot of really interesting and/or counterintuitive stuff happens here. Classically, for a current of electrons flowing through a perpendicular magnetic field, by the right-hand rule there is a Lorentz force that causes the flowing electrons to flow to the side, which causes a new current. The resistivity of this current takes on rational values ($\frac{a}{b}, a, b$ are integers) in the Quantum Hall regime, called the Fractional Quantum Hall Effect. The anyonic particle models needed for topological quantum computation either show up or are conjectured to show up in the Fractional Quantum Hall regime for a particular $\frac{a}{b}$: they are point excitations.*

For the Fractional Quantum Hall regime on a torus, just think of a graph with $N_1 \times N_2$ squares, with the upper and lower boundaries identified and the left and right boundaries identified. (the game *Asteroids* is on a torus)



The proof will be (heavily) paraphrased here, as there is not much to gain from going into the nitty-gritty calculations. We note the symmetries of this system, i.e. the translational symmetry going up or down. (The n^{th} square from the bottom is equal to the $(n\%N_1)^{th}$ square from the bottom, and the m^{th} square from the left is equal to the $(m\%N_2)^{th}$ square from the left, due to the fact that this lattice is on a torus). We also have a symmetry when there are k magnetic flux quanta (not too crucial that you know what they are, just that they cause a second symmetry argument) on this lattice, corresponding to the k single-particle states in the Landau levels (energy levels) of our system. Consider k' electrons on this torus. For a $\frac{p}{q}$ resistivity in our system, $k' = \frac{p}{q}k$. Translating our system by N_1/k up or N_2/k to the right is also a symmetry, as well as their integral powers, because the flux quanta are identical and our particles are identical. Call the operator associated to the N_1/k translation T_1, and the operator associated to the N_2/k translation T_2. Note that T_1, T_2 are unitary. Let a ground state (lowest energy state) be an eigenvector for this linear transformation: $T_2\psi_0 = \lambda\psi_0$. It is not hard but not trivial to show with some quantum mechanics that $T_1T_2\psi_0 = T_2T_1e^{-i\pi p/q}\psi_1$. Thus, taking these operators to different powers, we see that there are $q-1$ more such ψ_0 eigenvectors that are distinct from our "original" ψ_0: $\psi_{0,n} = T_1^n\psi_0$, where n is modulo q. Since we defined our ground state (lowest energy state) to be an eigenvector of T_2 (and it follows that it's an eigenvector of T_1), we truly have q distinct ground states (lowest energy states).

This reasoning is followed to show that, for Hall resistivity (inverse of conductance of current induced by the perpendicular magnetic field) $\frac{p}{q}$, the degeneracy is q^g, where g is the number of holes (g for genus) of the surface upon which we are working in.

The last section may have been a bit heavy, but the point is that the number of distinct states that yield the same energy depends on the number of holes on which we are working in. The number of holes is a topological invariant, as smooth deformations cannot create or close up holes. This is very important to understanding when topological quantum computation is able to be realized: quantum mechanics (and physics in general) is often about defining symmetries on a system, and two systems that are topologically equivalent (there exists a smooth mapping between points on these spaces) are, in a sense, symmetric. Degeneracy due to topology, along with a large energy gap from the lowest energy state to the next lowest state (first excited state) yield the notion of a topological order. A topological order isn't well defined, but suffice it to say that the energy gap (a span of energies that, due to the quantum mechanics of the system, are not allowed between the ground state and excited state) allows for nice fault tolerance. The larger the energy gap, the greater the amount of energy needed to add to a particle in the ground state to get to the next space of allowed energies, so the stronger the error needed to actually cause an error.

This sort of topological symmetry is extremely robust, as local perturbations do not affect the overall topology. Hinting at fault tolerance, a perturbation that deforms a donut into the canonical coffee mug doesn't change the topology of the system (both have one hole), so the symmetry remains. See the robustness yet?

2.4 Non-abelian Anyons

Definition 5. *A set of actions is called abelian if, for all A, B in the set, $AB = BA$.*

Unitary transformations are what quantum computation in general is all about. If one has a state, a quantum logic gate (could be a rotation, could be creating a superposition of states (see Hadamard gate), could be a CNOT gate) is represented by a unitary matrix on the (2-dimensional, because it's a bit) qubit space.

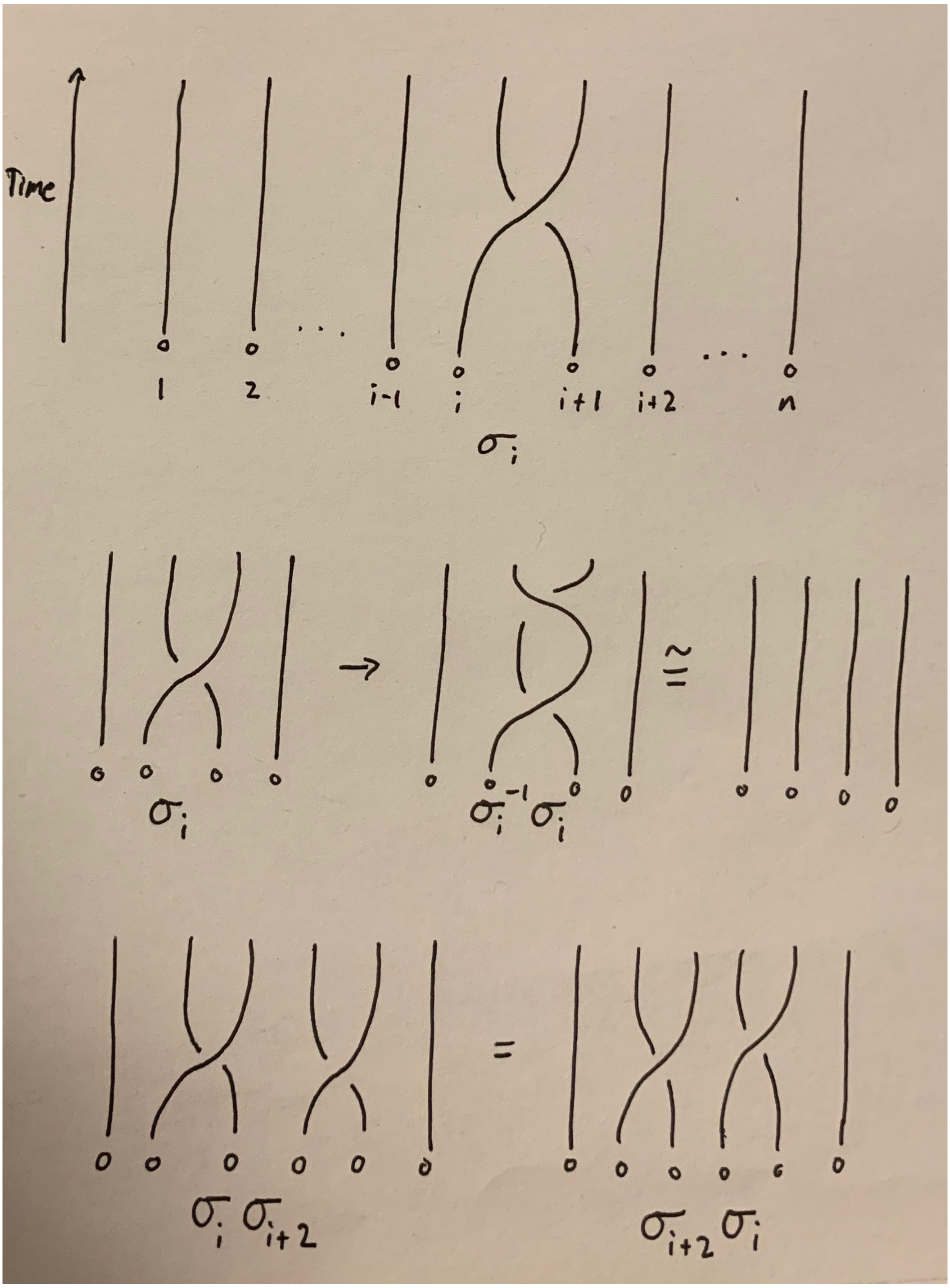
Many operators are non-abelian ($AB \neq BA$) in quantum mechanics, so although one may be used to systems that are abelian, don't take this for granted.

Anyons are not the complete description for an anyonic system. We also need to know what kind of anyons can arise in our physical system, and what happens when we fuse two anyons together or split an anyon apart. If we have two anyons, and fuse them together, we might get a different anyon. Label this splitting by $a \times b = \sum_c N^c_{a,b} c$, where a, b, c are called labels, species, or even topological charge in some sources. The $N^c_{a,b}$ coefficients for every anyon label c tell us the probability that a fused with b gives us c; if $N^c_{a,b} = 0$, then a cannot fuse with b and still give c as a result. For abelian anyons, their fusion gives a unique label:$a \times b = \sum_{c'} \delta_{c'c}$. For non-abelian anyons, fusion can be thought of as a quantum mechanical measurement: there is no determinism for what label the labels a and b fuse to. This is the measurement proposed in the measurement proposed in the topological quantum computation model. The constants N^c_{ab} are called the fusion rules of the model, and the types of topological charges that can arise are called the label set of the model.

Example 1. *A proposed model that yields universal quantum computation (one can construct any quantum gate from this model) are Fibonacci Anyons:*
The label set is just two labels: $\{1, \tau\}$.
The fusion rules are: $1 \times \tau = \tau \times 1 = \tau, \tau \times \tau = 1 + \tau$. *Here the plus sign means that there are two possibilities (called fusion channels)*

3 Computation

Since all the action with topological quantum computation happens with anyons, and anyons only live in 2-dimensional domains, anyons allow a really convenient visualization for time evolution: since they are confined to two dimensions, the time dimension can be the z axis on a 3D graph. Think of the following diagram as following the worldlines of anyons:



The operation σ_i swaps the i^{th} strand with the $(i+1)^{th}$ strand. Its inverse is denoted σ_i^{-1}. We also have equivalence relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i-j| \geq 2 \tag{1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{2}$$

This second condition is called the **Yang-Baxter Equation**, and has great importance here, in statistical mechanics, pure algebra, and low-dimensional topology, as we will soon see. We encourage you to prove the Yang-Baxter equation for braids via doodling.

These σ_i generate a mathematical group known as the braid group. We can also represent these σ_i as unitary matrices, and they act as gates on our wavefunction.

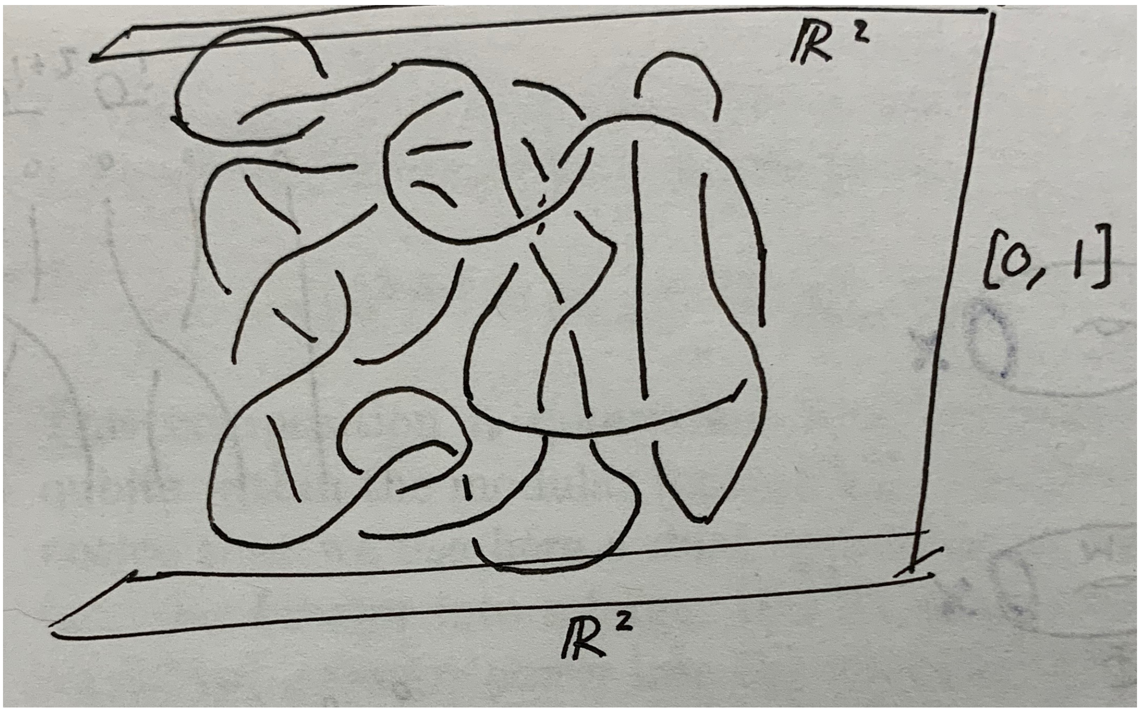
Remark 2. *Associating a group's elements as matrices is an entire field of math called represenation theory. This arises because groups are hard to study, but linear algebra is less hard.*

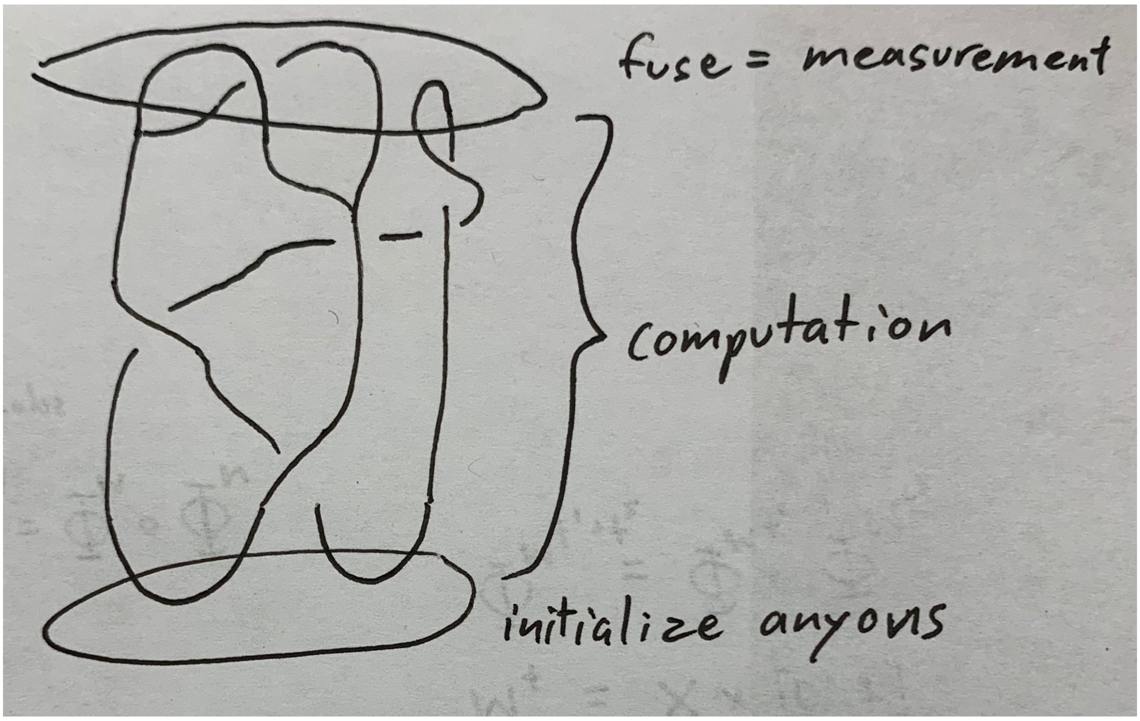
These worldlines are topologically known as braids, and a whole branch of topology is dedicated to studying braids. We will now refer to exchanging anyons as braiding anyons.

Suppose we have N degenerate states, and 1 anyon $\psi_n, 1 \leq n \leq N$ in each degenerate state. The wavefunction is $\psi := \psi_1\psi_2 ... \psi_n ... \psi_m ... \psi_N$. Exchanging anyon m with n adds a unitary transformation to the wavefunction. $\psi \to U_{nm}\psi$. Exchanging ψ_a with ψ_b gives another unitary transformation: $\psi \to U_{ab}\psi$. If $U_{nm}U_{ab} \neq U_{ab}U_{nm}$, then this anyonic system is non-abelian i.e. we have non-abelian anyons. If there were no degeneracy (degeneracy 1, 1 state per ground state energy), then U_{nm}, U_{ab} would just be phases and thus be abelian. Thus for anyons to be non-abelian we need degeneracy of at least 2.

Since we create anyons from vacuum, split them, and fuse them at the end of a computation, the worldlines actually form what is called a *tangle*:

Definition 6. *A tangle is a collection of a certain number of circles and a certain number of arcs in $\mathbb{R}^2 \times [0, 1]$. This definition becomes much more clear when viewing the diagram.*





4 Categories

Let's go through some ego-massaging abstraction.

Definition 7. *A category is defined as a set of objects and a set of morphisms between those objects. For each object there is an identity morphism from the object to itself, and the morphisms are associative. (e.g. the morphism from A to B is denoted here as m_{AB}: $m_{AB}(m_{BC}m_{CD}) = (m_{AB}m_{BC})m_{CD}$)*

An example of a category diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr] \arrow[rr, bend left] \arrow[rrr, bend left] & B \arrow[r] & E \arrow[r] & F \\
C \arrow[urr, bend right] & D \arrow[l] \arrow[u] \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Definition 8. *A functor is a mapping F between categories C and D that associates to each object $X \in C$ an object $F(X) \in D$, and each morphism $m_{AB} : A \to B \in C$ to $F(m_{AB}) : F(A) \to F(B) \in D$. We also have the conditions $F(Id_X) = Id_{F(X)}, F(m_{AB} \circ m_{BC}) = F(m_{AB}) \circ F(m_{BC})/$. One can essentially think of a functor as a map between categories that preserves structure.*

This may seem completely unnecessary, but it adds further understanding to things you already know (you can definitely come up with examples of categories on your own) and it a good way to express a lot of concepts in topological quantum computation.

Definition 9. *A tensor category is a category C with a bifunctor $\otimes : C \times C \to C$ defined by $X \times Y \mapsto X \otimes Y$. This is often also called a monoidal category because under $\otimes$ the set of the objects becomes a monoid.*

The $\otimes$ here is called the tensor product, and, to paraphrase the words of the great professor Patrick Hayden, it's "literally the dumbest way to put things together," inthat the only rules associated with the tensor product are basic associativity and other mathematical bookkeeping that is very intuitive and is just there to ensure nothing goes wrong. Actually, a great way to gain intuition for a tensor category is to consider an anyonic system as a tensor category.

4.1 Anyons as Tensor Categories

As Rowell puts it in [4], tensor categories are a good model for anyonic systems. An anyon by itself doesn't tell one much. What matters is its interaction with other anyons. Similarly, an object in a tensor category is determined by the morphisms to other objects in the category. This is because, in quantum mechanics, particles are only defined by their interaction with other particles and how they react to measurements.

Definition 10. *A simple object in a category is an object that cannot be written as a tensor product of other nontrivial objects.*

Definition 11. *A semisimple category is a category where all objects can be written as a direct sum of simple objects. E.g. $X = Y \oplus Z = Z, Y$, where $\otimes$, the direct sum, can be thought of as the disjoint union of Y, Z.*

Definition 12. *A tensor category is rigid if each object X has a dual object X^* such that $1 \to X \otimes X^*$ is a morphism that satisfies natural conditions.*

And now for how a modular tensor category models an anyonic system:

simple object→anyon

dual simple object→antiparticle

label→topological charge of anyon

tensor product→fusion

fusion rules→fusion rules

tangle→anyon trajectories

There are a couple more correspondences, but to explain those is beyond a basic introduction to category theory. A computation is as follows:

For the anyons: create anyon pairs from the system (often called "from vacuum"), braid the anyons, and measure the anyon types of pairs of neighboring anyons.

For the category: Take a morphism from the space of morphisms from 1 to $X \otimes X \otimes ... \otimes X = X^{\otimes n}$ for a simple object X. Braid the simple objects according to a category braiding, defined as $c_{Y,Z} : Y \otimes Z \to Z \otimes Y$. Finally, implement a morphism from the space of morphisms from $X^{\otimes n}$ to 1.

Remark 3. *A braided category not necessarily has $c_{Y,Z} \circ c_{Z,Y} = Id$. This is called a braided category because, as we can see from the braid group, braiding two strands and the braiding the strands again gives you a twist, not the identity.*

5 Topological Quantum Field Theory

Speaking of functors, we can introduce the notion of a topological quantum field theory.

5.0.1 Extreme TLDR for Quantum Field Theory

Quantum mechanics does not account for relativity. This is sad, because both theories are extremely well-developed. The Feynman path-integral in quantum mechanics associates a probability that a particle will travel from one point to another in a certain amount of time. However, for a particle to travel an incredibly large distance in a short amount of time has a very small but very nonzero probability according to the path-integral formulation. But for large distances, one might have to travel faster than the speed of light to make it in time. This is forbidden in relativity. The best attempt to reconcile this is to think of quantum mechanical particles as excitations of a particular quantum field e.g. the field (the physical object, not area of study) of quantum electrodynamics describes photons and electrons, the field of quantum chromodynamics describs quarks and gluons, etc. This makes sense intuitively: effects of excitations are local in fields, and locality should work nicely with the bounds of relativity.

In a quantum field theory, the field of the theory takes eats a function and spits out an operator. Position and momentum become operators too, and to create a particle in the field one applies a creation operator. To annihilate a particle, one applies an annihilation operator. For those with experience in quantum mechanics, this is very similar to the quantization of a classical harmonic osciallator.

Definition 13. *A correlation function in quantum field theory is an amplitude for the propagation of a particle or excitations or scattering of particles. Basically all one needs to know now is they are what people care about.*

Basically a quantum field theory has local operators about which you can study correlation operators.

Definition 14. *A Gauge Group is a set of actions one can take on a system that preserves the system's symmetry. This set forms a group in abstract algebra, hence the name. For example, $SO(n)$ is the group of orthogonal matrices with determinant n. One may remember from a standard linear algebra course that $SO(n)$ corresponds to the rotations of an n-sphere. Thus $SO(n)$ is the Gauge group of rotational symmetries.*

5.1 Topolgical Quantum Field Theory

Definition 15. *An n-manifold M^n is a topological space (space with topology, which is a set of sets that allows for continuity, convergence, all sorts of analytical stuff to be defined) that locally is homeomorphic to $\mathbb{R}^n$. That is, for each point $x \in M^n$, there exists some $\epsilon > 0$ such that an n-dimensional sphere centered at x with radius ϵ can be smoothly mapped to such a sphere in $\mathbb{R}^n$ and vise-versa. E.g. an n-sphere, a torus. Something that is not a manifold is the space of two circles joined at a point (the wedge sum of two circles). The point is what screws things up.*

One most likely has an idea of a field as a function defined in all of space and maybe time that takes in a position and time and spits out a number, like electric or magnetic fields. But if you think about it, that field is described on $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$ when time dependence is added, or $\mathbb{R}^{2,1}$ if we are confined to a 2-dimensional infinite plane with time dependence. Notice that these are manifolds too. $\mathbb{R}^3$ is certainly locally homeomorphic to $\mathbb{R}^3$: use the identiy homeomorphism. Why can't we define a quantum field theory on an arbitrary manifold? We were working with weird topology in discussing topological degeneracy.

Turns out we can. Topological quantum field theory is the effective theory in many condensed matter systems, including the Fractional Quantum Hall regime. This is essentially the idea behind a topological quantum field theory. We just need a few rules to make sure everything works out.

Definition 16. *An orientation on a manifold can be (handwavingly) thought of as a distiction of sides of a manifold. For example, if one thinks about an arrow pointing outward on the outside of a sphere, one can put an inward pointing arrow on the inside of a sphere. You can move arrows around all you want on the manifold, you'll never get it to point the other way. On the other hand, take a Mobius strip. You can certainly change the direction of an arrow just by going around a strip. It isn't possible to orient a mobius strip, but it is possible to orient a sphere/torus. Most often orientability allows topological arguments to be made.*

Another good question to ask to motivate the geometry of a quantum field theory is whether we can construct a particular quantum field theory from parts. We can probe all of the degrees of freedom hidden in the quantum field theory. We can take different basic building blocks and cut and glue them together, and use this to excite and study all the possible degrees of freedom of our quantum field theory.

To start, we are concerned with a quantum field theory in $n+1$ dimensions: n dimensional space plus time. Thus, we take for some n manifold and associate it with a vector space of states with an inner product (henceforth referred to as a Hilbert space, with coefficients in the complex numbers $\mathbb{C}$): $X^n := X \to \mathcal{H}_X$. But $\mathcal{H}_X$ cannot just be any ol' Hilbert space. It depends a lot on the geometry of X. Our Hilbert space is, after all, the space of states (wavefunctions) $\Psi(\phi_X)$ on our field space. Oftentimes there are infinitely many states, so our Hilbert space $\mathcal{H}_X$ is usually infinite-dimensional, but if we consider the space of ground states $\mathcal{H}_{X,0}$ becomes finite-dimensional a lot of times. This occurs in a topological context, so a major motivation is what happens to the hilbert space when the topology of X changes.

The formal definition of a topological quantum field theory is a quantum field theory on a space where the correlations are independent on the metric of the space, and thus are topological.

5.1.1 Topological Quantum Field Theory (more explicit)

This next section is not too necessary and will only be useful for those with a background in quantum mechanics. One may skip this section and go right to the "Topological Quantum Field Theory as a Functor" section, although it leaves the notion of a topological quantum field theory as a rather abstract toy. Here we introduce some well-deserved axioms of a topological quantum field theory:

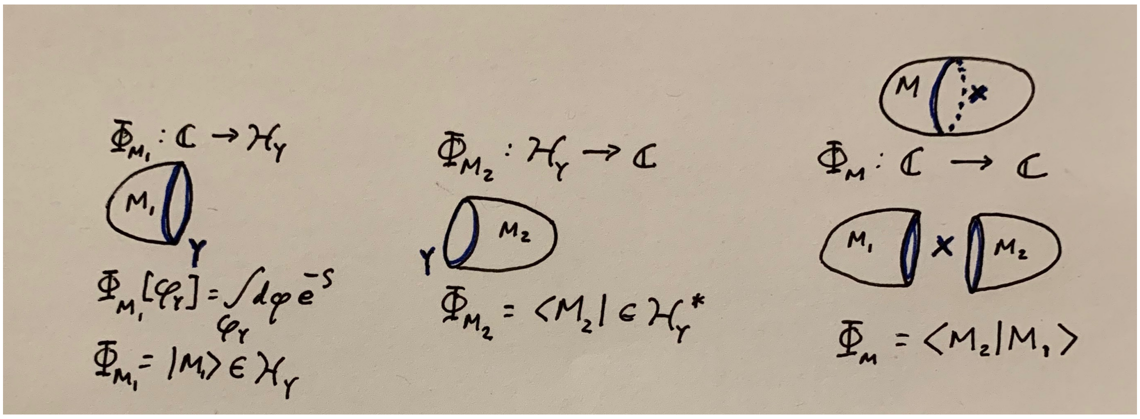
$$\mathcal{H}_{\emptyset} = \mathbb{C} \tag{3}$$

$$\mathcal{H}_{-X} = \mathcal{H}_X^* \tag{4}$$

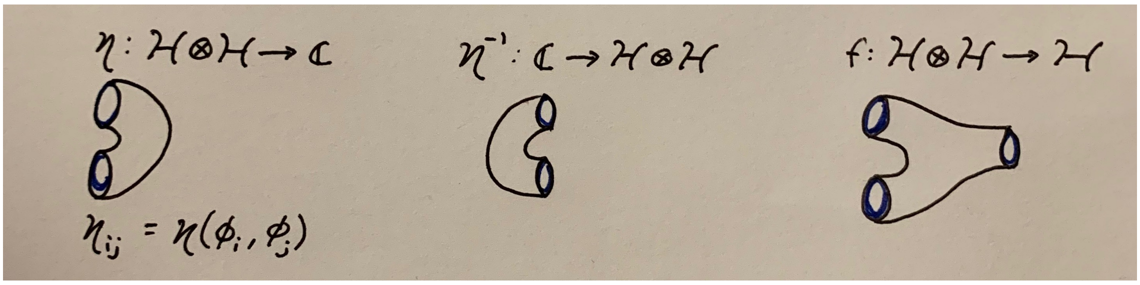
$$\mathcal{H}_{X \amalg Y} = \mathcal{H}_X \otimes \mathcal{H}_Y \tag{5}$$

where $\emptyset$ is empty space, $-X$ is X with the opposite orientation, * is the dual vector space, and $\amalg$ is the disjoint union. This tensor is a vector space tensor, very similar to the category tensor.

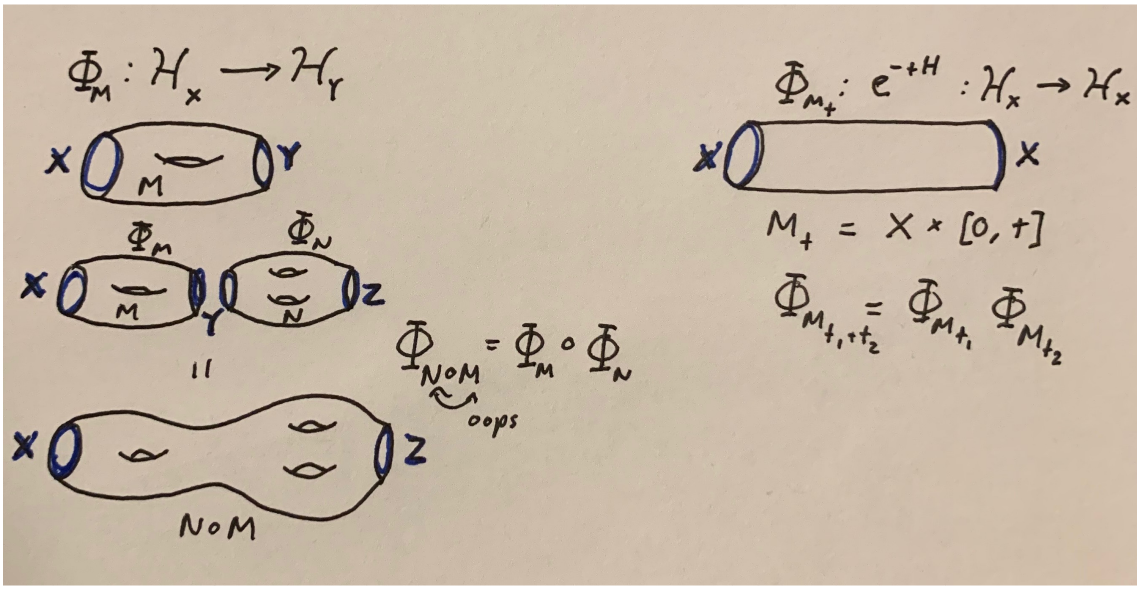
We can create a field by taking a manifold with one boundary $\mathcal{H}_Y$ and defining the field on the manifold's boundary. This is going from empty space to a state in $\mathcal{H}_Y$ $\Phi_{M_1} : \mathbb{C} \to \mathcal{H}_Y$, $\Phi_{M_1} = |M_1\rangle \in \mathcal{H}_Y$. A quantum state on Y is a wavefunction on the space of fields living on the boundary of M_1. We compute the path-integral over M_1 with the boundary condition that the field on the boundary of M_1 (Y) is ϕ_Y. The value of the path integral over M_1 with our boundary condition is the wavefunction at that certain point in field space. We can also define the inner product this way. Notice that M_2 has opposite orientation, as per our well-chosen axioms. We can compute the partition function on the closed manifold resulting from this by cutting the manifold in two parts, let's say on X: this is creating a state in $\mathcal{H}_{M_1}$ and a state on M_2 which is in the dual hilbert space $\mathcal{H}^*_{M_1}$. Thus by computing the inner product, you are computing the partition function:



We can also have bilinear forms $\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ and other forms (e.g. $f : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$):



The partition function is essentially saying that you can cut the (closed) manifold somewhere, and vary over all possible boundary conditions. We can also do this with an arbitrary operator $\Phi_M : \mathcal{H}_X \to \mathcal{H}_Y$, for example time evolution on the right, with a gluing axiom to be defined shortly:



For *really* physics-y people, the wavefunctions for supersymmetric quantum mechanics on a manifold Σ with fermions added to it are differential forms on our manifold, $\mathcal{H}_\Sigma = \Omega^*(\Sigma)$ and the ground states are harmonic forms. The partition function is the Witten index.

5.1.2 Topological Quantum Field Theory as a Functor

In this illustration we have chosen $n = 1$ for the ease of drawing. We thus, through our operators, construct $n + 1 = 2$ manifolds with our 1-manifold states to be the boundaries of these manifolds. This actually creates a tensor category called the *Cobordism category.*

Definition 17. *A cobordism is an equivalence relation between manifolds. For n-manifolds A, B, a cobordism is an $(n+1)$-manifold whose boundary is the disjoint union of A and B. A cobordism category is a tensor (aka monoidal) category describing these cobordisms. The official definition is a bit delicate and complicated, but the intuition is what matters. It is easy to see that this is indeed a tensor category.*

Notice that, in order for a gluing between cobordisms to exist, the boundary over which we want to glue must match for both cobordisms. This gives us the notion of a label for gluing. Think anyonic fusion here; we've chosen to use the word label for a reason.

Definition 18. *An $(n+1)$-dimensional topological quantum field theory is a tensor (aka monoidal) functor from a cobordism category to a suitably chosen tensor category of vector spaces.*

The 'suitably' here refers to the choice of Hilbert spaces so there is indeed a nice correspondence, as in the example.

6 Other objects of Importance

6.1 Quantum Groups

Disclaimer* This is a brief overview of some very heavy algebra. One can easily skip this section knowing that quantum groups are abstract algebraic objects that can generate solutions to the Yang-Baxter equation, have a deep connection to braids and tangles, and the matrix representations of these quantum groups form tensor categories. We have decided to include this because it was a large part of our reading for this project.

Recall the Yang-Baxter equation in the braid group: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. This is of great importance in statistical mechanics, and solutions to this equation are highly nontrivial. There is another way to express this equation, using the non-categorical tensor product: Say c acts on $V \otimes V$ for a vector space V with coefficients in field k (just a number system like $\mathbb{R}$ or $\mathbb{C}$). c is a solution to the Yang-Baxter equation if:

$$(c \otimes Id)(Id \otimes c)(c \otimes Id) = (Id \otimes c)(c \otimes Id)(Id \otimes c) \tag{6}$$

holds in $V \otimes V \otimes V$.

There are truly brilliant approaches to classifying solutions to this equation, them being the construction of abstract algebraic objects called Quantum Groups. Different authors of papers refer to different things by a 'quantum group.' Here a quantum group will be denoted by what is called a Hopf Algebra. There are many axioms of a Hopf algebra, but we will only mention a few, because they are of most interest in anyonic theory and the others require diving down a rabbit hole of quantum algebra to define.

A *bialgebra* is a vector space H over a field k (has coefficients in k, k acting like $\mathbb{R}$ or $\mathbb{C}$)

$$\eta : k \to H$$
$$\mu : H \otimes H \to H$$
$$\Delta : H \to H \otimes H$$
$$\epsilon : H \to k$$

where natural associativity rules apply.

Definition 19. *An antipode of a bialgebra is a map* $S : H \to H$ *if* $H \xrightarrow{\Delta} H \otimes H \xrightarrow{S \otimes Id} H \otimes H \xrightarrow{\mu} H = H \xrightarrow{\Delta} H \otimes H \xrightarrow{Id \otimes S} H \otimes H \xrightarrow{\mu} H = \eta \circ \epsilon$.

For those familiar with quantum mechanics, quantum groups are representations (q-deformations for the theorists) of semi-simple Lie algebras, in particular the Lie algebras of operators in conformal field theories. Quantum groups, because of their solutions to the Yang-Baxter equation, and the fact that they generate tensor categories, have deep connections to the category of tangles and thus to tangles and thus to braids.

The quantum group action on a Hilbert space for our quantum system is the following:

η creates an anyon from vacuum

μ fuses two anyons together

δ splits an anyon into two others

ϵ annihilates an anyon

The Hilbert space on which this acts is the image of the functor of our topological quantum field theory. This is the use of quantum groups in topological quantum computation, and is hard to pin down.

6.2 The Jones Polynomial

Definition 20. *A knot is a tangle where there is only 1 circle and no arcs. Google images "math knots" and you'll gain all the intuition you need.*

Knot theory is a big deal in low-dimensional topology. Classification of knots gets very complicated, and one celebrated approach has been the Jones Polynomial.

Definition 21. *The Jones Polynomial is the assignment of any knot to a polynomial, of the form* $a_{n,1}t^n + a_{n,2}t^{n/2} + a_{n-1,1}t^{n-1} + a_{n-1,2}t^{(t-1)/2} + ... + a_{-m,1}t^{-m}$ *i.e. the variable in the polynomial is* $t^{1/2}$ *and can have negative power, and* $a_{n,i}$ *are integer coefficients.*

Determining this polynomial is built up from local topological invariants, and has great power in classifying knots up to topological invariance. A primitive r^{th} root of unity is given by $e^{2i\pi/r}$, because $(e^{2i\pi/r})^r = e^{2i\pi} = 1$, and r is the smallest power that achieves this.

7 BIG DEAL!

Evaluating the Jones polynomial at a particular root of unity is $\#P$-hard if $r \neq 1, 2, 3, 4, 6$. (Notice that 5 is not among this set of r). Edward Witten showed that, for an arbitrary knot, one can find the Jones polynomial in a topological quantum field theory called the Chern-Simons theory, with gauge group $SO(2)$ (2×2 unitary matrices with determinant 1) by calculating certain expectation values of the theory. Since this is $\#P$-hard, if we have a system whose physics match a non-abelian topological parameter, this may be used as a computer to solve NP and $\#P$-hard problems. The topological quantum field theories in $SU(2)$ Chern-Simons theory responsible for such miraculous insight are constructed using quantum groups.

The majority of our project was learning about these tensor categories. Specifically we looked at $SU(2)$-Chern-Simons topological quantum field theories in [7]. In general, for non-abelian simple objects in our special anyonic tensor categories we associate the aforementioned matrix representations of the braid group, henceforth referred to as braid representations. The BQP-completeness of the anyonc model they create and the $\#P$-hardness of their topological invariants are deeply related to each simple object's braid representation. The most interesting result we came across was an explicit computational model...

8 An explicit computational model from a Topological Quantum Field Theory

Here we report on an explicit computational model based on the Chern-Simons topological quantum field theory at the fifth root of unity (do you know why? look in the section above if you have no idea) ($e^{2i\pi/5}$) with gauge group $SU(2)$ in [8]. We also do an explicit calculation severely handwaved by the authors.

Our label set for such a category, by the rules is $\{0, 1, 2, 3\}$ (to clarify these are the labels of the boundaries of the cobordisms). Call the functor from the cobordism category to the vector space category V. There are a few more axioms from this theory to construct the model:

The **Gluing Axiom** is an equivlance relation between associated vector spaces:

$$V(X \cup_\gamma Y) = \oplus_{\text{over all consistent labels} l} V(X, l) \otimes V(Y, l)$$

. This is just a formal way of ensuring we cannot glue manifolds that don't have the same boundary conditions (labels). We have one more axiom from our TQFT:

V(a sphere with 3 punctures, boundary labels a, b, c) is equivalent (isomorphic) to $\mathbb{C}$ if:

1) $a + b + c$ is even,

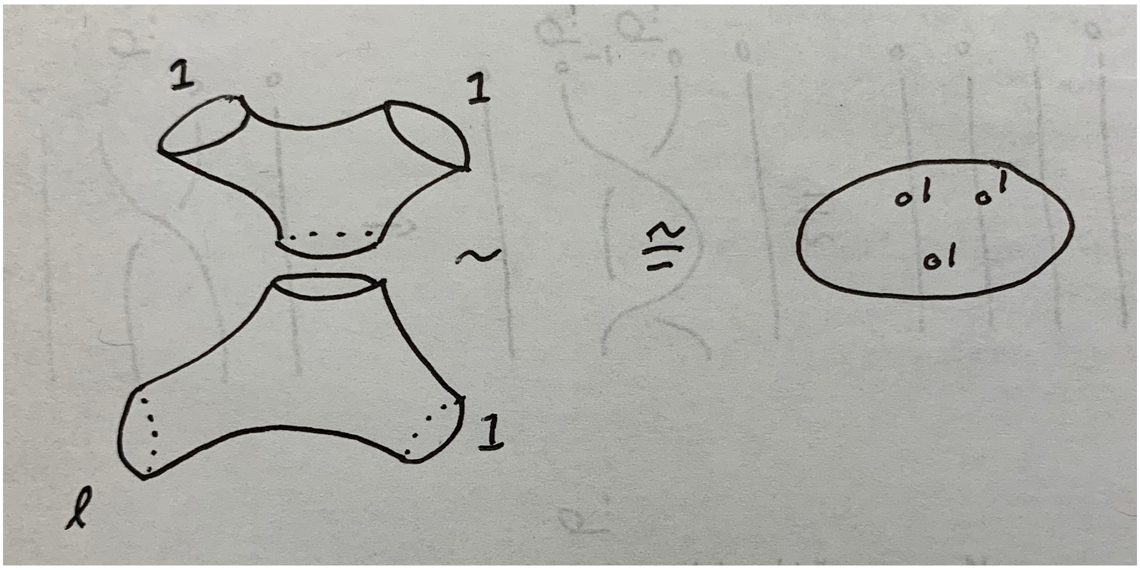
2) $a \leq b + c, b \leq a + c, c \leq a + b$,

3) $a + b + c \leq 2(r - 2) = 4$ (recall $r = 5$)

otherwise the the hilbert space is 0.

Back to some computer science. We want our state space to be $(\mathbb{C}^2)^{\otimes k}$ for k qubits. Each qubit has dimension 2, one basis vector for 0 and one for 1. Let's start with our surrounding space. A disk is as good as any. Call this disk's boundary label 0. What goes inside the disk? Well, we want anyons, so we might as well add some points to our disc. We want multiple anyons per qubit, as braiding the qubit will yield single and double-qubit gates. Let's try 3 anyons per qubit, so we want to add $3k$ points to the disk. We also want a manifold for each qubit, so scoop up three qubits and put this on a subdisk with boundary label unknown, glued to the bigger disk. From the second axiom (the one with the sphere with 3 points), we know that a sphere with three punctures has a certain structure. We want, for this qubit, 2 of the $\otimes_{\text{over all consistent labels}}$ summands to make this 2 dimensional. Thus we tune the unknown label of the qubit subdisk in order for the dimensions to work out.

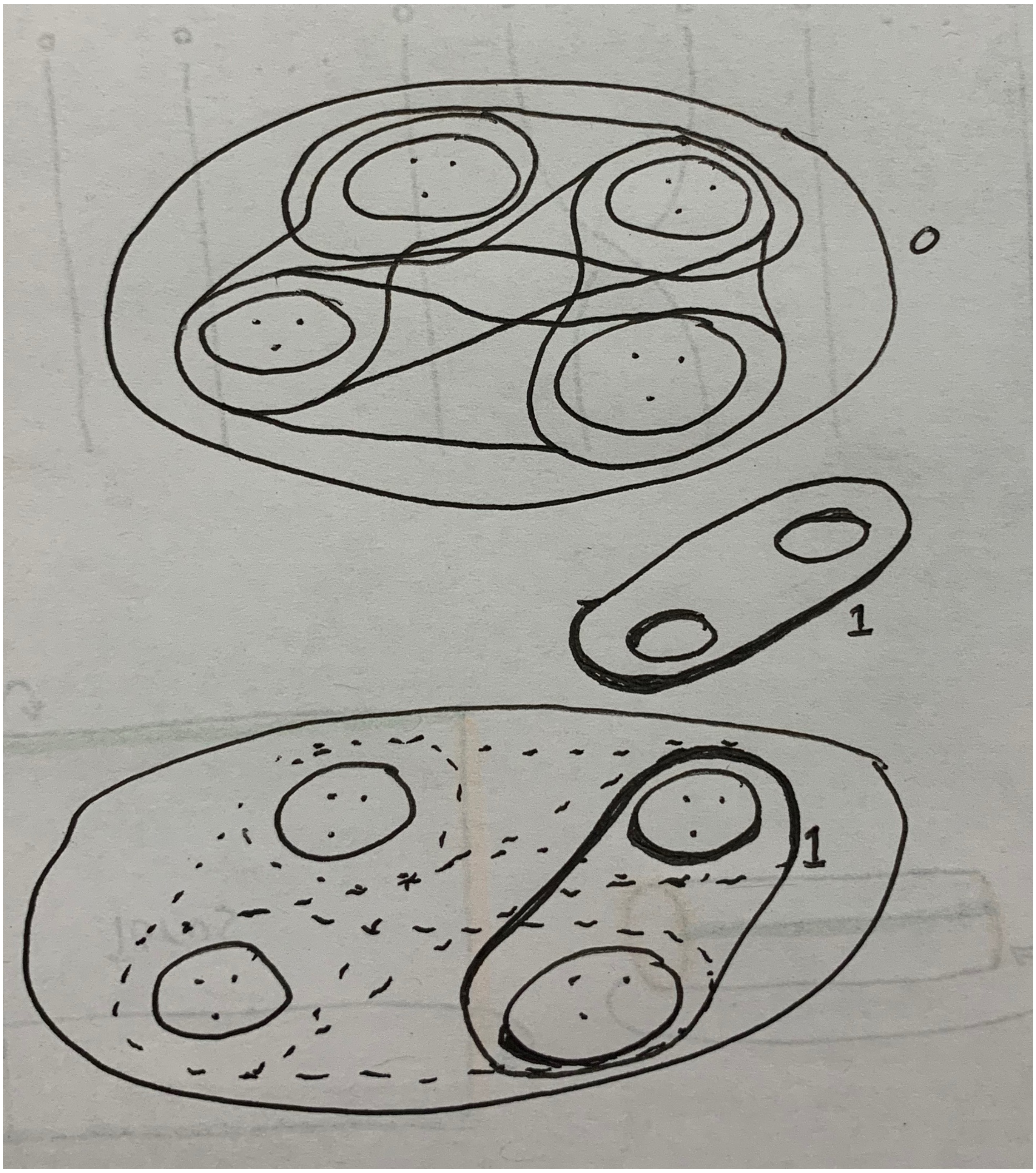
Notice that a disk with 3 punctures is homeomorphic to a sphere with 4 punctures. Notice further that a sphere with 4 punctures is homeomorphic to two spheres with 3 punctures glued together at one of their punctures: (see picture)



Choose the labels of the 3 points in our qubit disk to be 1 each (as labeled in the diagram) We determine what labels go on the boundary over which we are gluing, and we find, using the 3 conditions above, that $l = 1$ gives 2 labels to glue over that give a nonzero vector space (check on your own that the labels are 0 and 1), and thus:

$$V(\text{disk with 3 points with all boundary labels 1}) \cong \mathbb{C}^2$$

Thus we have our qubits! We act on these qubits by the matrix representation of the braid group with 3 strands (one per anyon) on the associated vector space. Let's not get ahead of ourselves, though. We also want 2-qubit gates. By a similar calculation, we put $\binom{k}{2}$ disks on our surrounding disk; one for every possible pair of qubit disks. (see picture) In determining the labels we use the same reasoning.



On each of these 2-qubit disks associate a 2-qubit gate by the braid representation for 6 strands. The representation acts on the vector space associated with this manifold. The paper further constructs the gate through tensor products of the manifolds we do not want to act on, but the idea is the same. Using a physical system, we have chosen an effective quantum field theory, and used this to create requirements for this physical system to be a quantum computer. Now, we state an important theorem achieved in the paper:

Theorem 1. *There is a constant $C > 0$ such that, for all unitary quantum gates $g : \mathbb{C}_i^2\mathbb{C}_j^2 \to \mathbb{C}_i^2\mathbb{C}_j^2$, there is a braid b of length $\leq l$ generators $_i$ so that*

$$||w_0\rho^0(b) - g \oplus Id_1|| + ||w_2\rho^2(b) - g \oplus Id_4|| \leq \epsilon > 0$$

for some unit complex numbers w_0, w_2 (these are quantum mechanical phases), whenever ϵ satisfies

$$l \leq C\frac{1}{\epsilon^2}$$

This theorem states that, although we can only approximate an exact gate g with braid group representations acting on our 2-qubit space, this approximation can get arbitrarily fine:

$$||\text{braid approx. - desired gate}|| + ||\text{braid approx. - desired gate}|| \leq \text{arbitrarily small number}$$

This is just another illustration of the convenience of fault tolerance, and depending on topology for this fault tolerance. A good final conclusion for this "report" is the following result, once again from [8]:

Suppose we have the exact quantum circuit model. Suppose we have a problem instance $M \in BQP$ solved by a circuit of length poly(length(M)). Label our model we constructed by $CS5$, for Chern-Simons at the 5^{th} root of unity. With probability ≥ 34, $CS5$ correctly solve the problem instance M. The number of anyons (points) needed and the length of the braiding exceeds the length of C by at most a multiplicative poly(log(length(M))) factor.

9 Conclusion

Though topological quantum computation may require lots of math, its intrinsic reliance on topological encoding of information makes it a prodigious contender for a realization of a practical quantum computer.

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