Quantum Field Theory Problems

Alec Lau

Question 1. *Consider the space $\mathcal{D}$ of continuously differentiable complex-valued functions f on [0,1]. Consider the operator A on $L^2([0,1])$ with domain $\mathcal{D}$, defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.*

Proof. We want to check if $\langle A\psi|\varphi\rangle = \langle\psi|A\varphi\rangle$. This gives us $\langle i\psi'|\varphi\rangle, \langle\psi|i\varphi'\rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^*\varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^*\varphi dx = \int_0^1 (-i)\psi'^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx) \neq \int_0^1 i\psi^*\varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is D_α, then, evaluating the same integral, we have $\int_0^1 (i\psi')^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^*\varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^*\alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^*\alpha\psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1-\alpha^*\alpha)i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^*\alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^*\varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. □

Question 2. *Recall the definition of the manifold X_m, the measure λ_m on X_m, and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of $\mathcal{H}$. Finally, recall the definitions of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that*

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\not{1} \tag{1}$$

where $\not{1}$ is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_\mathbf{p}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}}[a(p), a^\dagger(p')]$. The first expression then becomes $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p-p')1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2w_\mathbf{p}} f(w_\mathbf{p}, \mathbf{p})$, we have the equality

$$\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}}}f(\mathbf{p})^*g(\mathbf{p}')[a(p),a^\dagger(p')] =$$
$$\int\int d\lambda_m(p)d\lambda_m(p')\sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}}f(\mathbf{p})^*g(\mathbf{p}')[a(p),a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p)\sqrt{4w_{\mathbf{p}}^2}f(\mathbf{p})^*g(\mathbf{p})1 = \int d\lambda_m(p)2w_{\mathbf{p}}f(\mathbf{p})^*g(\mathbf{p})1$. Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3\mathbf{p}}{(2\pi)^3}f(\mathbf{p})^*g(\mathbf{p})1$, where 1 is the identity operator on our Fock space.

Now we consider $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}f(\mathbf{p})^*g(\mathbf{p}')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')1$. Integrating once, we find this gives us $\int \frac{d^3\mathbf{p}}{(2\pi)^3}f(\mathbf{p})^*g(\mathbf{p})1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}),a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. $\square$

Question 3. *Consider the theory for massive scalar bosons of mass m. Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation*

$$\frac{\partial\varphi}{\partial t} = i[H_0,\varphi] \tag{2}$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f. Then, since $H_0 = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}}dx^4f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}')+e^{i(x,p)}a^\dagger(\mathbf{p}'))$, we have

$$(H_0\varphi)(f) = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})\int_{\mathbb{R}^{1,3}}dx^4f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}'+e^{i(x,p)}a^\dagger(\mathbf{p}'),$$
$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}}dx^4f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}'+e^{i(x,p)}a^\dagger(\mathbf{p}')\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})$$

Thus we have

$$[H_0,\varphi](f) = \int_{\mathbb{R}^{1,3}}dx^4f(x)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}}A, \text{ where}$$
$$A =$$
$$a^\dagger(\mathbf{p})a(\mathbf{p})e^{-i(x,p)}a(\mathbf{p}')+a^\dagger(\mathbf{p})a(\mathbf{p})e^{i(x,p')}a^\dagger(\mathbf{p}')-e^{-i(x,p')}a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})-e^{i(x,p')}a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a(\mathbf{p}')) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}')-a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))$$

Because $[a(\mathbf{p}),a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}),a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p}')a(\mathbf{p})-a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))+e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}')-a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')a(\mathbf{p}))$$
$$= e^{-i(x,p')}[a^\dagger(\mathbf{p}),a(\mathbf{p}')]a(\mathbf{p})+e^{i(x,p')}a^\dagger(\mathbf{p})[a(\mathbf{p}),a^\dagger(\mathbf{p}')]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. Also, notice that $[A,B] = AB - BA = (-1)(BA - AB) = -[B,A]$. Thus, A becomes

$$\begin{aligned} &e^{-i(x,p')}(-1)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')a(\mathbf{p}) + e^{i(x,p')}a^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}') \\ &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p})) \end{aligned}$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p})) \\ &= \textstyle\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}}}}(e^{i(x,p)}a^\dagger(\mathbf{p}) - e^{-i(x,p)}a(\mathbf{p})) \end{aligned}$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x,p) = tw_{\mathbf{p}} + \mathbf{x}\cdot\mathbf{p}$, so the time derivative of $e^{\pm i(x,p)} = \pm i w_{\mathbf{p}} e^{\pm i(x,p)}$. Thus, $\frac{\partial\varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}}(-e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived form the commutator. Thus, $\frac{\partial\varphi}{\partial t} = i[H_0,\varphi]$, up to a set of measure zero. □

Question 4. *In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude*

$$\langle \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4 | S | \boldsymbol{p}_1\rangle \tag{3}$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!}\int_{\mathbb{R}} d^4x : \varphi(x)^4 : + \mathcal{O}(g^2)$. We then have

$$\begin{aligned} \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|S|\mathbf{p_1}\rangle &= \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \tfrac{ig}{4!}\textstyle\int_{\mathbb{R}} d^4x \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}| : \varphi(x)^4 : |\mathbf{p_1}\rangle + \mathcal{O}(g^2) \\ &= \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \tfrac{ig}{4!}\textstyle\int_{\mathbb{R}} d^4x \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle + \mathcal{O}(g^2) \end{aligned}$$

For the first term, we notice that $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4})a^\dagger(\mathbf{p_1})|0\rangle$. Applying the first two operators we get either ground state back if $\mathbf{p_1} = \mathbf{p_4}$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0|a(\mathbf{p})\varphi(x)|0\rangle = \frac{e^{i(x,p)}}{\sqrt{2w_{\mathbf{p}}}}$, $\langle 0|\varphi(x)a^\dagger(\mathbf{p})|0\rangle = \frac{e^{-i(x,p)}}{\sqrt{2w_{\mathbf{p}}}}$. $\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle = \langle 0|a(\mathbf{p_2})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_3})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_4})\varphi(x)|0\rangle\langle 0|a^\dagger(\mathbf{p_1})\varphi(x)|0\rangle$. This expression is equal to $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8-1)!!$ diagrams, and 4! diagrams of this type. Thus we have 4! $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$. Thus we have $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|S|\mathbf{p_1}\rangle = (-ig(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}}) + \mathcal{O}(g^2)$. □

Question 5. *Derive Maxwell's equations as the Euler-Lagrange equations of the action*

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4}$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard from by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory. Peskin & Schroeder 2.1.

Proof. First we calculate $F^{\mu\nu}$. Given our identification with E^i and B^i, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{5}$$

We notice that

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)}F^{\alpha\beta} = (\delta^\gamma_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\delta^\gamma_\beta)F^{\alpha\beta} = F^{\gamma\lambda} - F^{\lambda\gamma}, \tag{6}$$

with

$$F^{\lambda\gamma} = (\partial_\lambda A_\gamma - \partial_\gamma A_\lambda) = -(\partial_\gamma A_\lambda - \partial_\lambda A_\gamma) = -F^{\gamma\lambda} \tag{7}$$

Similarly, we have

$$F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} = F_{\alpha\beta}(\delta^{\gamma\alpha}\delta^{\lambda\beta} - \delta^{\lambda\alpha}\delta^{\gamma\beta}) = F^{\gamma\lambda} - F^{\lambda\gamma} = 2F^{\gamma\lambda} \Rightarrow \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\gamma A_\lambda)} = \frac{\partial F_{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)}F^{\alpha\beta} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} = (-\frac{1}{4})4F^{\gamma\lambda} \tag{9}$$

$$\frac{\partial\mathcal{L}}{\partial A_\lambda} = 0 \tag{10}$$

Thus, by the Euler-Lagrange Equations, we have

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}) = \frac{1}{4}\partial_\mu F^{\mu\nu} = 0 \tag{11}$$

Writing out the sum over μ and substituting $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$, we get

$$-\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} \tag{12}$$

$$= -\frac{\partial\mathbf{E}}{\partial t} - \partial_1\epsilon^{1jk}B^k - \partial_2\epsilon^{2jk}B^k - \partial_3\epsilon^{3jk}B^k \tag{13}$$

$$= -\frac{\partial\mathbf{E}}{\partial t} + \partial_1\epsilon^{j1k}B^k + \partial_2\epsilon^{j2k}B^k + \partial_3\epsilon^{j3k}B^k \tag{14}$$

$$= -\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = 0 \tag{15}$$

Maxwell's Equations are

$$\mu_0\mathbf{J} = -\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} \tag{16}$$

For a system with no sources, $\mathbf{J} = 0$, so this comes down to the Euler-Lagrange equations found above.

In constructing the energy-momentum tensor, we examine the usual formula in Peskin & Schroeder:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\gamma)}\partial^\nu A_\gamma - \mathcal{L}\delta^{\mu\nu} \tag{17}$$

$$= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} \tag{18}$$

where $g^{\alpha\beta}$ is our Minkowski metric. We see that, from the above assertion that $F^{\mu\nu} = -F^{\nu\mu}$, so this is not a symmetric tensor. Thus we add $\partial_\lambda(F^{\mu\lambda}A^\nu)$ to $T^{\mu\nu}$. This term is antisymmetric in its first two indices, and thus is divergenceless, so adding this gives us the same globally conserved momentum and energy. Writing this out, we have

$$\widehat{T^{\mu\nu}} := T^{\mu\nu} + \frac{1}{4}\partial_\lambda(F^{\mu\lambda}A^\nu) \tag{19}$$

$$= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (\partial_\lambda F^{\mu\lambda})A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \tag{20}$$

$$= F^{\mu\iota}(\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (-\partial_\lambda F^{\lambda\mu})A^\nu \tag{21}$$

$$= F^{\mu\iota}F^\nu_\iota + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (-0)A^\nu \tag{22}$$

This is obviously symmetric in indices, and so is now a viable energy-momentum tensor. We now check $\widehat{T^{00}}$ and $\widehat{T^{0i}}$:

$$\widehat{T^{00}} = F^{0\iota}F^0_\iota + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{23}$$

$$= E^\iota E_\iota + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{24}$$

Where, I assume, the second product is the Frobenius inner product of this tensor. In examining this inner product, we have

$$\langle\ ,\ \rangle = \mathrm{tr}(\overline{F^{\mu\nu}}F^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{25}$$

$$= \mathrm{tr}(\begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix}) \tag{26}$$

This is equal to $2(B^2 - E^2)$. Thus, we have that $\widehat{T^{\mu\nu}} = E^2 + \frac{1}{4}2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $\widehat{T^{0i}}$, we have

$$\widehat{T^{0i}} = F^{0j}F^i_j + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{27}$$

$$= E^j\epsilon_{jik}B^k g^{mi} + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{28}$$

$$= \mathbf{E} \times \mathbf{B} = \mathbf{S} \tag{29}$$

$\square$