Refresher on Algebraic Topology

Alec Lau

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Note: These are the only problems I had saved before I decided to make a page to preserve this stuff. Because there is so much machinery in algebraic topology, I have saved some of these problems as a refresher for whenever I return to it, as said machinery is very forgettable, beautiful though it may be. Also, these problems were done open-book and open-past-homework with Hatcher's "Algebraic Topology" textbook, so there may be some steps I skipped over. In proving that homology groups are the abelianization of homotopy groups, I didn't want to bust out any category theory, so whatever.

All problems were written by Professor Ralph Cohen, saxophone extraordinaire.

Question 1. *Is every covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$ isomorphic to a product of covering spaces $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to \mathbb{R}P^2 \times \mathbb{R}P^3$, where $p_1 : \tilde{X}_1 \to \mathbb{R}P^2$ and $p_2 : \tilde{X}_2 \to \mathbb{R}P^3$? Why or why not?*

Proof. Let Y be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$. If there exists an isomorphism $f : Y \to \tilde{X}_1 \times \tilde{X}_2$, then from the relations $p_1 = p_2 f, p_2 = p_1 f$, it follows that $(p_1 \times p_2)_* \pi_1(\tilde{X}_1 \times \tilde{X}_2) \cong p_*(Y)$ due to the induced isomorphisms. Since $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, we know from the product topology that a map $f : Y \to \mathbb{R}P^2 \times \mathbb{R}P^3$ is continuous if and only if the maps $g : Y \to \mathbb{R}P^2, h : Y \to \mathbb{R}P^2$ defined by $f = g \times h$ are continuous. Therefore a loop in $\mathbb{R}P^2 \times \mathbb{R}P^3$ is equivalent to a pair of loops in $\mathbb{R}P^2$ and $\mathbb{R}P^3$, and furthermore a homotopy on the product space is equivalent to a pair of homotopies on the corresponding components. Thus there exists a bijection $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3)$ given by $[f] \mapsto ([g], [h])$. Trivially, this is a group homomorphism, and by bijectivity an isomorphism:

$$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3). \tag{1}$$

Next we compute the homology groups of $\mathbb{R}P^n, n \in \{2,3\}$.

$\mathbb{R}P^n$ is topologized as the quotient space $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$, so we can thus restrict to vectors of length 1, so $\mathbb{R}P^n = S^n/(v \sim v)$. Thus $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an $n-$cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. By induction on n, $\mathbb{R}P^n$ has a CW structure $e^0 \cup e^1 \cup ... \cup e^n$ with one cell e^i in each dimension $i \leq n$. To compute the boundary map d_k we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1} \tag{2}$$

with q the quotient map. The map $q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence $\text{deg}q\varphi = \text{deg}(1) + \text{deg(-1)} = 1 + (-1)^k$, and so the boundary maps d_k is either 0 or multiplication by 2, depending on whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is even} \tag{3}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is odd} \tag{4}$$

$$\tag{5}$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} \text{ if } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 \text{ if } k = \text{ odd}, 0 < k < n \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

Thus we know that $H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. We know from homework 4 that H_1 is the abelianization of π_1 (since $\mathbb{R}P^n$ is path-connected and nonempty), but a group with cardinality 2 must be isomorphic to $\mathbb{Z}_2$ to have the group axioms still hold. Thus, $\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, and are manifolds, and so are locally path-connected semi-locally simply-connected as well. Thus we can apply the Galois Correspondence Theorem to say that, for every subgroup H of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there is an isomorphism class of covering spaces Y such that $p_*(Y) \cong H$, therefore all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ have their fundamental groups as subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the trivial subgroup (0,0), and groups with generator (1,0), (0,1), and (1,1). I claim that there is no pair of maps $p_1 : \tilde{X}_1 \to \mathbb{R}P^2, p_2 : \tilde{X}_2 \to \mathbb{R}P^3$ such that $(p_1 \times p_2)_*(\tilde{X}_2 \times \tilde{X}_3) \cong \{(0,0),(1,1)\}$. We see that $|\{(0,0),(1,1)\}| = 2$. By Lagrange's Theorem for any subgroup of this group must have cardinality that divides 2. Since 2 is prime, the only subgroup in it has cardinality 1 and is thus is the trivial subgroup. Thus, if we have $\pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2) \cong \{(0,0),(1,1)\}$, a subgroup $\pi_1(\tilde{X}_1) \times 0 \cong 0$ and another subgroup $0 \times \pi_1(\tilde{X}_2) \cong 0$. Thus both $\pi_1(\tilde{X}_1), \pi_1(\tilde{X}_2)$ are trivial subgroups, and it is impossible for $f((0,0)) \mapsto (1,1)$ if f is an isomorphism. If, without loss of generality $\pi_1(\tilde{X}_1) \times 0$ were the whole group, this has cardinality 2, but the group is $\{(0,0),(1,0)\}$, a different subgroup corresponding to a different covering space. Thus, $\{(0,0),(1,1)\} \ncong \pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2)$, so not all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ are isomorphic to a product of each summand's covering space. □

Question 2. *Let $X = \mathbb{R}P^2 \wedge S^3$ and $Y = \mathbb{R}P^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.*

Proof. From the above, we know that the chain complex for $\mathbb{R}P^3$ is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \tag{7}$$

With any G coefficients, this becomes

$$0 \to G \xrightarrow{0} G \xrightarrow{2} G \xrightarrow{0} G \to 0 \tag{8}$$

so we have

$$H_0(\mathbb{R}P^3; G) \cong G, H_1(\mathbb{R}P^3; G) \cong G/2G, H_2(\mathbb{R}P^3; G) \cong 0, H_3(\mathbb{R}P^3; G) \cong G, \tag{9}$$

$$H_0(\mathbb{R}P^2; G) \cong G, H_1(\mathbb{R}P^2; G) \cong G/2G, H_2(\mathbb{R}P^2; G) \cong 0 \tag{10}$$

For $n > 0$ take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\tilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\tilde{H}_0(S^n) = 0$ By induction on n, starting with the case of S^0, we see that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$. Thus $H_1(S^3; G) \cong H_2(S^3; G) \cong 0, H_3(S^3) \cong G)$, due to the equivalence of $\tilde{H}_n$ and H_n for $n > 0$. Since $S^3, \mathbb{R}P^2$ are both path-connected and nonempty, $S^3 \vee \mathbb{R}P^2$ is path-connected and nonempty. By definition, $H_0(S^3 \vee \mathbb{R}P^2) = C_0(S^3 \vee \mathbb{R}P^2)/\mathrm{Im}\partial_1$ since $\partial_0 = 0$. Define a homomorphism $\epsilon : C_0(S^3 \vee \mathbb{R}P^2) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. This is obviously surjective since $S^3 \vee \mathbb{R}P^2$ is nonempty. Ker $\epsilon =$ Im ∂_1 since $S^3 \vee \mathbb{R}P^2$ is path-connected, and thus ϵ induces an isomorphism.

We conclude that $H_0(S^3 \vee \mathbb{R}P^2; G) \cong G$. Since reduced homology is the same as homology relative to a basepoint, we know that, for $n > 0$,

$$\tilde{H}_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3) \oplus H_n(\mathbb{R}P^2) \tag{11}$$

Thus we have $H_1(S^3 \vee \mathbb{R}P^2; G) \cong G/2G, H_2(S^3 \vee \mathbb{R}P^2; G) \cong 0, H_3(S^3 \vee \mathbb{R}P^2; G) \cong G$. These are the same (up to isomorphism) homology groups as $\mathbb{R}P^3$.

In calculating cohomology for any G coefficients, we notice that $H^n(X; G) \cong \mathrm{Hom}(H_n(X; \mathbb{Z}), G) \oplus \mathrm{Ext}(H_{n-1}(X; \mathbb{Z}), G)$.

Lemma 1. *Hom*$(\mathbb{Z}, G) \cong G$, *Hom*$(G/2G, G) \cong 0$

Proof. By mapping 1 to each element of G, we get a cardinality of G. Since this is a homomorphism, the structure of the image is preserved, and $fg(n) = f(n) \star g(n)$, where $\star$ is the group operation of G. Since every element of the group is hit in the image, and the composition of these homomorphisms is mapped to the group operation, we have all elements of G following the same structure of G, and thus is isomorphic to G.

In order for there to be a nontrivial homomorphism, orders of elements must match from $G/2G$ to G. However, this is not the case, as we mod out by $2G$, so no generator of $G/2G$ has the same order as an element in G. Thus, the only homomorphism possible is the trivial homomorphism. $\square$

Using this, we calculate cohomology for $\mathbb{R}P^3$, and, using the rules of Ext on page 195 of Hatcher, we find

$$H^0(\mathbb{R}P^3;G) \cong G, H^1(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z},G) \cong 0 \oplus 0 \cong 0, \tag{12}$$

$$H^2(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z}_2,G) \cong 0 \oplus G/2G \cong G/2G, \tag{13}$$

$$H^3(\mathbb{R}P^3;G) \cong G \oplus 0 \tag{14}$$

We also have

$$H^n(S^3 \vee \mathbb{R}P^2;G) \cong \mathrm{Hom}(H_n(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \oplus \mathrm{Ext}(H_{n-1}(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \tag{15}$$

Since the homology groups are isomorphic, the cohomology groups are isomorphic as well.

If we can prove $H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) \not\cong H^*(\mathbb{R}P^3;\mathbb{Z}_2)$, then this means the spaces are not homotopy equivalent. Plug in $G = \mathbb{Z}_2$. From Example 3.8 in Hatcher, we have $H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$, and $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^4)$, where $|\alpha| = |\beta| = 1$. Suppose we have $\beta \in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$. Then $\beta \smile \beta = \beta^2 \in H^2(\mathbb{R}P^3;\mathbb{Z}_2)$, and $\beta^2 \smile \beta = \beta^3 \in H^3(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and $\beta^3 \neq 0$. Noticing that $H^1(\mathbb{R}P^2 \vee S^3) \cong H^1(\mathbb{R}P^2) \oplus H^1(S^3)$, we have $(\alpha, 0) \in H^1(\mathbb{R}P^2 \vee S^3)$. Suppose there is an isomorphism $f : \mathbb{R}P^2 \vee S^3 \to \mathbb{R}P^3$. Then for $\alpha' = f(\beta) \in H^1(\mathbb{R}P^2;\mathbb{Z}_2)$, and for $a = f(\beta^2) \in H^2(\mathbb{R}P^2;\mathbb{Z}_2)$. We now have $(\alpha', 0) \smile (a, 0) = (\alpha' a, 0) \in H^3(\mathbb{R}P^2 \vee S^3)$. But $H^3(\mathbb{R}P^2;\mathbb{Z}_2) \cong 0$, so $\alpha' a = 0$. But since f is a ring isomorphism, then $f(\beta^3 \neq 0) = \alpha' a = 0$. Since f maps a nonzero element to 0, it cannot be an isomorphism, so the cup product structures of $\mathbb{R}P^2 \vee S^3$ and $\mathbb{R}P^3$ are not homotopically equivalent. $\square$

Question 3. *Let M^n be a closed, path connected, orientable manifold. Let $x \in U \subset M$ where U*

is an open neighborhood homeomorphic to $\mathbb{R}^n$*. Consider the "pinch map,"* $p : M^n \to S^n$ *defined as the composition*

$$p : M^n \xrightarrow{quotient} M^n/(M^n - U) \xrightarrow{homeo} S^n \tag{16}$$

Show that

$$p_* : H_n(M^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z}) \tag{17}$$

is an isomorphism.

Proof. Since M is closed and path-connected, we can apply Theorem 3.26 in Hatcher to conclude that $H_n(M) \cong H_n(M, M - x)$. Hatcher states that the isomorphism between the two factors though $H_n(M, M - U)$ for U any neighborhood in M containing x. This is because the homomorphism $i_* : H_n(M, M - U) \to H_n(M, M - x)$ induced by inclusion is bijective, since $X - U$ is a deformation retract of $X - x$. By excision, $H_n(X, X - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x)$. Since $\mathbb{R}^n$ is contractible, this is isomorphic to $H_{n-1}(\mathbb{R}^n - U)$. The second map is isomorphic for any $x \in U$, because $\mathbb{R}^n - U$ and $\mathbb{R}^n - x$ deformation retract onto a sphere centered at x. Thus $H_n(M) \cong H_n(M, M - U)$.

We notice that $(M, M - U)$ is a good pair; Simply take an open cover ϵ-thick covering the boundary of U, and add this to $M - U$. Because $U \cong \mathbb{R}^n$, the overlap of our covering with U can easily be deformation retracted until we are left with $M - U$. Since $(M, M - U)$ is a good pair, we can have a neighborhood V be a neighborhood of U in M that deformation retracts onto U. We have a commutative diagram

$$\begin{tikzcd}
H_n(M,U) \arrow[r] \arrow[d, "q_*"] & H_n(M,V) \arrow[d, "q_*"] & H_n(M-U,V-U) \arrow[l] \arrow[d, "q_*"] \\
H_n(M/U,U/U) \arrow[r] & H_n(M/U,V,U) & H_n(M/U-U/U,V/U-U/U) \arrow[l]
\end{tikzcd}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (M, V, U) the groups $H_n(V, U)$ are zero for all n, because a deformation retraction of V onto U gives a homotopy equivalence of pairs $(V, U) \cong (U, U)$, and $H_n(U, U) = 0$. The deformation retraction of V onto U induces a deformation retraction of V/U onto U/U so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of U. From the commutativity of the diagram it follows that the left q_* map is an isomorphism. We see then that $H_n(M, M-U) \cong H_n(M/(M-U), (M-U)/(M-U)) \cong H_n(M/(M-U))$. Thus $H_n(M) \cong H_n(M/(M-U))$. Since $M/(M-U)$ is homeomorphic to S^n, they better have isomorphic homology groups. Call the isomorphism induced by the homeomorphism between $H_n(M/(M-U))$ and $H_n(S^n)$ g, and the isomorphism between $H_n(M)$ and $H_n(M/(M-U))$, shown via the diagram. Now we have that $p_* := f \circ g$, a composition of isomorphisms, so $p_* : H_n(M) \to H_n(S^n)$ is an isomorphism. $\square$

Question 4. *One of the most famous mathematics problems of the* 20^{th} *century was the "Poincaré Conjecture". In 1900 Poincaré claimed that any closed, simply connected, 3-dimensional manifold is homotopy equivalent to the sphere* S^3. *This claim was proved in 2003 by G. Perelman. In this problem we ask you to prove a weaker, but related statement. Namely, prove that if* M^3 *is a closed, simply connected manifold, then there is a map* $g : M^3 \to S^3$ *that induces an isomorphism in homology groups in all dimensions.*

Proof. Define $\tilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$. The map $\mu_x \mapsto x$ defines a two-to-one surjection, and, due to everything being nice and manifold-y, we see that $\tilde{M}$ is a two-sheeted covering space of M^3.

Since M^3 is simply connected, $\tilde{M}$ has either one or two components since it is a two-sheeted covering space of M^3. If it has two components, they are each mapped nomeomorphically to M^3 by the covering projection, so M^3 is orientable, being homreomorphic to a component of the orientable manifold $\tilde{M}$. Thus M^3 is orientable, and $H_0(M^3) \cong \mathbb{Z}$ because simply connected implies nonempty and path-connected. Now since M^3 is a closed and orientable manifold, we can use Poincare duality. Also from Theorem 3.26 (part c), $H_i(M^3) \cong 0, i > 3$. Because M^3 is simply-connected, $\pi_1(M^3) \cong 0 \cong H_1(M^3)$. $H^1(M^3) \cong \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \cong 0$. By Poincare duality, $H_2(M^3) \cong H^1(M^3) \cong 0$. Also by Poincare Duality, $H_3(M^3) \cong H^0(M^3) \cong \mathrm{Hom}(H_0(M^3), \mathbb{Z}) \oplus \mathrm{Ext}(0, \mathbb{Z}) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$. To sum this up, we have $H_0(M^3) \cong H^3(M^3) \cong \mathbb{Z}, H_1(M^3) \cong H_2(M^3) \cong 0 \cong H_i(M^3), i > 3$. These are the exact homology groups of S^3, so let g be the isomorphism between their homology groups. $\square$

Question 5. *Is $(S^2 \times S^4) \wedge S^8$ homotopy equivalent to a compact closed manifold? Explain.*

Proof. Let $a_i \in H^i(S^2;\mathbb{Z}), b_i \in H^i(S^4;\mathbb{Z})$ be generators of their cohomology groups. From the definition of the external cup product we have $p_1^*(a) \smile p_2^*(b) \in H^*(X \times Y; R)$, for p_1, p_2 projection maps. For $H^0(S^2 \times S^4) \cong \mathbb{Z}$ because this is space and path-connected. Let p_1^*, p_2^* be induced homomorphisms from the projection $S^2 \times S^4 \to S^2, S^2 \times S^4 \to S^4$, respectively. We have $p_1^*(a_1) \smile p_2^*(b_1) = 0 \smile 0 = p_1^*(a_1) \smile 0 = 0 \smile p_2^*(b_1) = 0 \in H^1(S^2 \times S^4;\mathbb{Z})$. We also have $p_1^*(a_2) \smile p_2^*(b_2) = p_1^*(a_2) \smile p_2^*(0) \in H^2(S^2 \times S^4;\mathbb{Z})$ as a generator for H^2. The other nonzero generator b_4, when cupped with another generator $a_i, i \neq 2$ is $0 \smile p_2^*(b_4) \in H^4(S^2 \times S^4;\mathbb{Z})$, which is the generator of H^4. For all other combinations when $a_i \neq a_2, b_j \neq 4$ we have trivial H^{i+j}. With $0 \neq p_1^*(a_2) \smile p_2^*(b_4) \in H^6(S^2 \times S^4;\mathbb{Z})$, we conclude that $H^i(S^2 \times S^4;\mathbb{Z}) \cong \mathbb{Z}$ (has a single generator infinite with $\mathbb{Z}$ coefficients) when $i = 0, 2, 4, 6$ and trivial otherwise. We might think we would run into trouble with $p_1^*(a_2) \smile p_1^*(a_2)$, but because this is the pullback of the generator $H^2(S^2)$ under p_1, and in $H^2(S^2), a_2 \smile a_2 = 0$, this still holds in $H^2(S^2 \times S^4)$. By the Kunneth Formula in Hatcher, we have $H^*(S^2 \times S^4; R) \cong \mathbb{Z}[a_2]/(a_2^2) \otimes_R \mathbb{Z}[a_4]/(a_4^2), |a_2| = 2, |a_4| = 4$.

For $\tilde{H}^*((S^2 \times S^4) \vee S^8)$ (we need not worry about H^0 since the space is nonempty and path-connected), we use the fact from Hatcher that $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong \tilde{H}^*(S^2 \times S^4) \oplus \tilde{H}^*(S^8)$. For $\tilde{H}^*(S^8)$, we know that $H_i \cong H^i \cong \mathbb{Z}$ for $i = 0, 8$, and $\cong 0$ if else. Thus our cohomology ring is $\tilde{H}^*(S^8;\mathbb{Z}) \cong \mathbb{Z}[b]/(b^2), |b| = 8$. Thus we have $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong [\mathbb{Z}[a_2]/(a_2^2) \otimes \mathbb{Z}[a_4]/(a_4^2)] \oplus \mathbb{Z}[b]/(b^2), |a_2| = 2, |a_4| = 4, |b| = 8$.

Any manifold homotopically equivalent to $(S^2 \times S^4) \vee S^8$ must be an 8-manifold. From Theorem 3.26 in Hatcher, if a manifold is not oriented, then $H_8(M;\mathbb{Z}) \cong 0 \Rightarrow H^8(M;\mathbb{Z}) \cong 0$, which cannot be possible, as $H^8((S^2 \times S^4) \vee S^8)$ is nontrivial. Thus a manifold that is homotopy equivalent must be oriented, since in that case $H^8(M;\mathbb{Z}) \cong H^8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. Oriented closed manifolds satisfy Poincare Duality. If a closed manifold were to be homotopy equivalent to $(S^2 \times S^4) \vee S^8$, since the latter is path-connected the former better be path-connected. Suppose that $(S^2 \times S^4) \vee S^8$ satisfies Poincare Duality. Consider the fundamental homology class $[M] \in H_8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. From the Poincare Duality Theorem 3.30 in Hatcher, we know that, for $\alpha \in H^2(M)$, where α is a generator, $[M] \frown \alpha$ generates $H_6(M)$, since $D(\alpha) = [M] \frown \alpha$ is an isomorphism.

Examining the cap product, we have

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|[v_0, ..., v_k])\sigma|[v_k, ..., v_{k+l}]) \tag{18}$$

$$= \psi(\sigma|[v_0, ..., v_k])\psi(\sigma|[v_k, ..., v_{k+l}]) = (\varphi \smile \psi)(\sigma) \tag{19}$$

Thus, $\psi([M] \frown \alpha) = (\alpha \smile \beta)([M])$, where $\psi \in H^6(M)$ is the generator. Since $[M] \frown \alpha$ is a generator, and ψ is a generator homomorphism, $\psi([M] \frown \alpha)$ is a generator for the ring we are in (here we are using $\mathbb{Z}$). Thus, for $\beta \in H^4((S^2 \times s^4) \vee S^8)$ the generator, $1 = \psi([M] \frown \alpha) = (\alpha \smile \psi)([M]) = (\alpha \smile (\alpha \smile \beta))([M]) = 0([M]) = 0$, a contradiction. This is because $\alpha \smile \alpha = 0$ from the ring structure we derived earlier. Because of this, Poincare Duality is not satisfied, so any orientable or otherwise closed manifold has a different ring structure and therefore is not homotopy equivalent. $\square$