Differential Topology Problems

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All uncredited problems written by Prof. Ralph Cohen.

Question 1. *Let $\pi : \tilde{X} \to X$ be a covering space. Let Φ be a smooth structure on X. Prove that there is a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ so that $\pi : (\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion.*

Proof. First we have to show that $\tilde{X} \to X$ is a topological manifold. Since π is a local homeomorphism, $\tilde{X}$ is locally Euclidean. Let p_1, p_2 be distinct points such that $\pi(p_1) = \pi(p_2) \in U \subset X$, for U an evenly covered open subset of X. Then the components of $\pi^{-1}(U)$ containing p_1, p_2 are, by definition of a covering space, disjoint open subsets of $\tilde{X}$. If $\pi(p_1) \neq \pi(p_2)$, since X is a manifold, there exist disjoint subsets that contain $\pi(p_1), \pi(p_2)$. These map under π^{-1} to disjoint open subsets of $\tilde{X}$. Thus $\tilde{X}$ is Hausdorff. For second-countable-ness, we are inspired by Proposition 4.40 in Lee. We check first that each fiber of π is countable. For $x \in X$ and an arbitrary point $p \in \pi^{-1}(x)$. We consider a map β from $\pi_1(X, x)$ to $\pi^{-1}(x)$. Since the fundamental group of a topological manifold is countable, if we can show surjectivity of such a map, we're done. Choose a homotopy class $[f] \in \pi_1(X, x)$ of an arbitrary loop $f : [0, 1] \to X$ with $f(0) = f(1) = x$. From the path-lifting property of covering spaces, there is a lift of f given by $\tilde{f} : [0, 1] \to \tilde{X}$ starting at p_0. The Monodromy Theorem for covering spaces shows that $\tilde{f}(1) \in \pi^{-1}(x)$ depends only on the path class of f. Thus set β such that $\beta[f] = \tilde{f}(1)$. Since the components of $\tilde{X}$ are path-connected, for any point $p \in \pi^{-1}(x)$, there is a path $\tilde{g}$ in $\tilde{X}$ from p_0 to p, and then $f = \pi \circ \tilde{f}$ is a loop in X such that $p = \beta[f]$. The set of all evenly covered open subsets is an open cover of X, and thus has a countable subcover $\{U_i\}$. $\pi^{-1}(U_i)$ has one point in each fiber over U_i, so $\pi^{-1}(U_i)$ has countable components. All components of the form $\pi^{-1}(U_i)$ are thus countable and an open cover of $\tilde{X}$. Since the components are second-countable, $\tilde{X}$ is second-countable. Thus $\tilde{X}$ is a topological manifold.

For Φ a smooth structure on $X \xleftarrow{\pi} \tilde{X}$, we choose any point $x \in X$ such that there exist two neighborhood pairs $U_1, U_2, V_1, V_2 \subset X$ such that $x \in U_1 \cap U_2, x \in V_1 \cap V_2$ and $\pi^{-1}(U_1) \neq \pi^{-1}(U_2) \subset \tilde{X}$ and V_1, V_2 are the domains of charts ψ_1, ψ_2, respectively, in Φ. Since π is continuous and maps U_1, U_2 homeomorphically, $\pi^{-1}(x) \in \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$. Since $\psi_1, \psi_2 \in \Phi$, $\psi_1 \circ \psi_2^{-1}$ is smooth. Now we define a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ by composing the charts in Φ with π. To simplify notation, we call $\tilde{U}_i = \pi^{-1}(U_i \cap V_i)$,$\phi_i = \psi_i|_{U_i \cap V_i}$ $i = 1, 2$:

$$\tilde{\psi}_i : \tilde{X} \to \mathbb{R}^n \tag{1}$$

$$\tilde{\psi}_i(\tilde{U}_i) = \phi_i \circ \pi(\tilde{U}_i) \tag{2}$$

See Figure 1. $(\tilde{U}_i, \tilde{\psi}_i)$ are charts of $\tilde{X}$ because $\tilde{U}_i, U_i \cap V_i$, and $V_1 \cap V_2$ are open, and the maps that

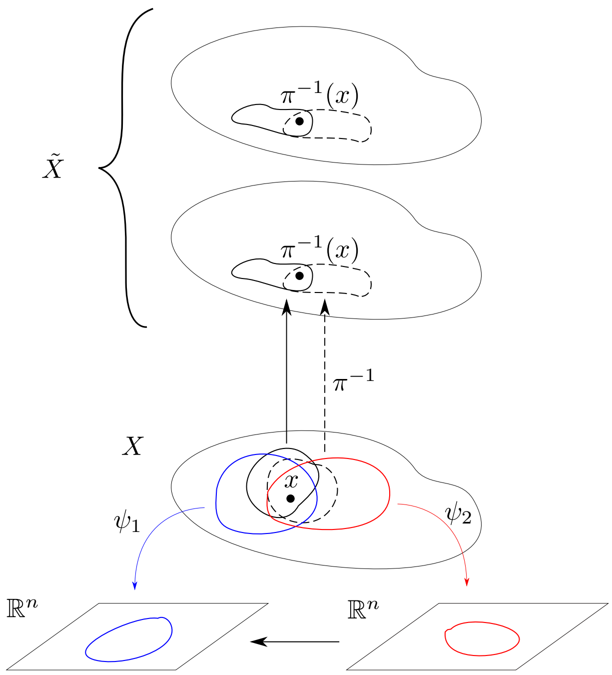

Figure 1: Charts on $\tilde{X}$

compose these charts are homeomorphisms: $\psi_i|_{V_1 \cap V_2}$ is still a homeomorphism. Now we need to check that the transition maps for these charts are smooth:

$$\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1} = (\phi_1 \circ \pi) \circ (\phi_2 \circ \pi)^{-1} \tag{3}$$

$$= (\psi_1 \circ \pi) \circ \pi^{-1} \circ \phi_2 \tag{4}$$

On $V_1 \cap V_2$, we have $\pi|_{V_1 \cap V_2} \circ \pi^{-1}|_{V_1 \cap V_2} = Id|_{V_1 \cap V_2}$. The identity map is smooth, so our transition map is then $\psi_1 \circ \psi_2^{-1}$, which we know is smooth. By combining our maximal smooth atlas Φ with surjective π, we have thus created a smooth atlas $\tilde{\Phi}$ on $\tilde{X}$. It remains to show that this smooth atlas is maximal. There is no such chart $(\tilde{W}, \tilde{\phi})$ not contained in this atlas, because $\pi(\tilde{W})$ is an open subset of X, and thus has an open cover $\{U_\alpha, \psi_\alpha\}$ of charts in the maximal smooth atlas of X:

$$\tilde{\phi}(\tilde{W}) = (\cup_\alpha \psi_\alpha)|_{(\cup_\alpha U_\alpha)\cap\pi(\tilde{W})} \circ \pi(\tilde{W}) \tag{5}$$

Since $\tilde{\phi}$ can be written in this way, $(\tilde{W}, \tilde{\phi})$ is contained in this smooth atlas. Thus this smooth atlas is maximal, and $\pi(\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion, as the charts with properly shrunken domains have exactly one chart in X. □

Question 2. *Consider the DeRham homomorphism*

$$\int : \Omega^k(M) \to C^k(M; \mathbb{R}) \tag{6}$$

for each k. Prove that $\int$ is a map of cochain complexes. That is,

$$\int d\omega = \delta(\int \omega) \tag{7}$$

where $\delta : C^k(M;\mathbb{R}) \to C^{k+1}(M;\mathbb{R})$ is the singular coboundary operator.

Proof. We start inductively. We want to show that the following diagram commutes:

$$\begin{array}{ccc}
\Omega^0(M) & \xrightarrow{\int} & C^0(M;\mathbb{R}) \\
\downarrow d & & \downarrow \delta \\
\Omega^1(M) & \xrightarrow{\int} & C^1(M;\mathbb{R})
\end{array}$$

We have that $f \in \Omega^0(M)$ is just a C^∞ function on M to $\mathbb{R}$. Consider a singular chain element $\sigma : \Delta^0 \to M$ in $C_0(M)$. We have

$$\int_\sigma f = f(\sigma(\Delta^0)) \in \mathbb{R} \tag{8}$$

Thus $\int f$ is clearly an element of $\mathrm{Hom}(C_0(M), \mathbb{R}) = C^0(M;\mathbb{R})$. Now let σ denote the singular chain element $\sigma : [0,1] \to M$. Now we take the boundary homomorphism δ of this element in the following way:

$$\delta(\int f)(\sigma) = (\int f)(\partial\sigma) \tag{9}$$
$$= (\int f)(\sigma(1) - \sigma(0)) \tag{10}$$
$$= (\int f)(\sigma(1)) - (\int f)(\sigma(0)) \tag{11}$$
$$= f(\sigma(1)) - f(\sigma(0)) \tag{12}$$

Now we take our same $f \in \Omega^0(M)$ and take d of f to obtain $df = f'(t)dt \in \Omega^1(M)$. Taking the De Rham homomorphism of this 1-form gives us, for $\sigma \in C_1(M)$,

$$(\int df)(\sigma) = \int_\sigma df \xrightarrow{\text{Stokes' Theorem}} \int_{\partial\sigma} f = f(\sigma(1)) - f(\sigma(0)) \in \mathbb{R} \tag{13}$$

Now we proceed with the inductive step, which is to prove that this diagram commutes:

$$\begin{tikzcd}
\Omega^n(M) \arrow[r, "\int"] \arrow[d, "d"] & C^n(M;\mathbb{R}) \arrow[d, "\delta"] \\
\Omega^{n+1}(M) \arrow[r, "\int"] & C^{n+1}(M;\mathbb{R})
\end{tikzcd}$$

We proceed in the exact way as before: for $\omega \in \Omega^n(M)$, we take $(\int \omega)(\sigma)$, for $\sigma : \Delta^n \to M$. $\delta(\int \omega)(\sigma) = (\int \omega)\partial\sigma \sum_i (-1)^i (\int f)(\sigma)|[v_0, ..., \hat{v}_i, ..., v_n] \in \mathbb{R}$. In the other direction of the diagram, we have $d\omega$, then $(\int d\omega)(\sigma)$. This is equal to $\int_\sigma d\omega = \int_{\partial\sigma} \omega$ through Stokes' Theorem. This is then equal to the same thing: $\sum_i (-1)^i (\int f)(\sigma)|[v_0, ..., \hat{v}_i, ..., v_n] \in \mathbb{R}$. Thus $\int d\omega = \delta \int \omega$. $\square$

Question 3. *Suppose $P^p \to M^n$ and $Q^q \to M^n$ are smoothly embedded closed submanifolds of M^n, which we also assume is closed. Suppose further that the submanifolds intersect transversely: $P^p \pitchfork Q^q$. Let $\nu_P \to P$ be the normal bundle of P^p in M^n, and let $P^p \to \eta_P$ be a tubular neighborhood.*

1. *Show that the restriction of ν_P to $P^p \cap Q^q$,*

$$(\nu_P)_{P^p \cap Q^q} \to P^p \cap Q^q \tag{14}$$

   is isomorphic to the normal bundle of $P^p \cap Q^q$ in Q^q.

2. *Show that the space of $\eta_P \cap Q^q$ is a tubular neighborhood of $P^p \cap Q^q$ in Q^q.*

Proof. 1. Call our smooth embeddings $f : P^p \to M^n, g : Q^q \to M^n$. Since $P^p \pitchfork Q^q$, we have $Df_x(T_x P^p) \oplus Dg_x(T_x Q^q) = T_x M^n$, for all $x \in P^p \cap Q^q$. Thus we have the diagram

$$\begin{tikzcd}
\nu_p \arrow[d, "i_{P^p \cap Q^q}"'] \arrow[dr] & \nu_M \arrow[l, "i_p"'] \arrow[dr] \arrow[drr, bend left, "i_q"] & & \\
\nu_p|_{P^p \cap Q^q} \arrow[dr] & P^p \arrow[r, "f"] & M^n & \nu_q \arrow[dl] \\
& P^p \cap Q^q \arrow[u, "\cup"] \arrow[r, "\subset"] & Q^q \arrow[u, "g"'] &
\end{tikzcd}$$

where the square in the middle is commutative. We examine the pullback bundle of ν_q by the inclusion $i : P^p \cap Q^q \to Q^q$. We have, for the diagram

$$\begin{tikzcd}
i^*\nu_q \arrow[r] \arrow[d] & \nu_q \arrow[d, "\pi|_{Q^q}"] \\
P^p \cap Q^q \arrow[r, "i"] & Q^q
\end{tikzcd}$$

We have $i^*\nu_q = \{(q, v_q) \in P^p \cap Q^q \times V_q | i(q) = \pi_{Q^q}(v_q)\}$. We have that $\nu_p|_{P^p \cap Q^q}$ consists of those very v_q, since the q points in $i^*\nu_q$ are also elements of $P^p \cap Q^q$. Thus we can associate every normal vector in $\nu_p|_{P^p \cap Q^q}$ with a vector in ν_q over $P^p \cap Q^q$.

2. There really is not much to do here. Since a tubular neighborhood is diffeomorphic to a neighborhood of the normal bundle, we need only consider a neighborhood of the normal bundle of P^p when restricted to $P^p \cap Q^q$. Since η_p is (up to diffeomorphism) a neighborhood of a tubular neighborhood of P^p, we have that $\eta_p \cap Q^q$ consists of $P^p \cap Q^q$ and $\eta_p|_{P^p \cap Q^q}$. Since this is ν_p restricted to $P^p \cap Q^q$, we know from the previous problem that it is isomorphic to a neighborhood of the normal bundle of $P^p \cap Q^q$ in Q^q, i.e. a tubular neighborhood of $P^p \cap Q^q$ in Q^q.

□

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Question 4. *Let $\alpha_0 < \alpha_1 < ... < \alpha_n$ be $(n + 1)$ distinct nonzero real numbers. Consider $g : \mathbb{R}^{n+1} \to \mathbb{R}$ given by*

$$g(x_0, ..., x_n) = \alpha_0 x_0^2 + ...\alpha_n x_n^2 \tag{15}$$

and let f be the restriction of g to the sphere S^n. Show that $f : S^n \to \mathbb{R}$ is Morse with $2(n + 1)$

non-degenerate critical points. Find all critical points of f and compute their index, i.e. the number of negative eigenvalues of the Hessian $Hf(x)$.

Proof. Since f is restricted to S^n, we can, without loss of generality, substitute x_i^2 for $1 - x_0^2 - ... - x_{i-1}^2 - x_{i+1}^2 - x_n^2$ in our equation for g:

$$f(x_0, ..., x_n) \mapsto (\alpha_0 - \alpha_i)x_0^2 + ... + (\alpha_n - \alpha_i)x_n^2 + \alpha_i, \tag{16}$$

where there is no term with x_i. We have thus condensed Df down to a map of n coordinates. Taking the derivative of f now, we get

$$Df = (2(\alpha_0 - \alpha_i)x_0, 2(\alpha_1 - \alpha_i)x_1, ..., 2(\alpha_n - \alpha_i)x_n), \tag{17}$$

Where the i^{th} index is eliminated. If we take $x_i = \pm 1$, all other x_ks must be zero to be in the sphere, so this makes Df the zero vector, making the points when $x_i = \pm 1$ both 2 critical points. Since we were doing this without loss of generality, we can repeat this for all $n + 1$ points. Since each $(n + 1)$ x_k can be 1 or -1, we have $2(n + 1)$ critical points.

We observe that the Hessian of f is the matrix where the $i^{th}j^{th}$ entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}$. We notice that this is $\delta_{kj}2(\alpha_j - \alpha_i)$, so the Hessian is a diagonal matrix with $2(\alpha_k - \alpha_i)$ as its diagonal entries, in order. Remember that all α are distinct. Since none of these are equal to zero, the determinant of the Hessian must be $\prod_{k=0, k \neq i}^{n} (\alpha_k - \alpha_i) \neq 0$. Because det(H($f$)) is nonzero and independent of coordinates, all critical points are non-degenerate, so f is morse.

As we have shown before, the critical points are $(\pm 1, 0, ..., 0), (0, \pm 1, ...0), ..., (0, 0, ..., \pm 1)$ (where all indeces that are not ± 1 are zero). Since $\alpha_0 < \alpha_1 < ... < \alpha_n$, and the entries of the Hessian can then be $2(\alpha_0 - \alpha_i), 2(\alpha_1 - \alpha_i)$, etc., we can determine how many are negative entries. For the critical points $(0, ..., \pm 1, ..., 0)$, the number of indeces less than i have negative values, as $\alpha_i >$ than all of those indeces' αs. The determinant of this Hessian is simply $2^{n-1}(\alpha_0 - \alpha_i)(\alpha_1 - \alpha_i)...(\alpha_n - \alpha_i)$, where all $(\alpha_k - \alpha_i), \forall k < i$ is negative. The eigenvalues of this Hessian are then all values such that each one of these terms summed with the corresponding eigenvalue is 0, making the determinant zero. For the $(\alpha_k - \alpha_i)$ factors of the determinant, since they are negative, the eigenvalue to make this factor zero must be negative, as $(\alpha_k - \alpha_i - \lambda_k)$ for $\lambda_k < 0$ is positive. In conclusion, for the critical points where the i^{th} entry is ± 1 and all others are zero, there are i negative eigenvalues. $\square$

Question 5. *Let M^m be a C^∞ closed manifold, and let $N^n \subset M^m$ be a smooth embedded submanifold, where N^n is also assumed to be compact with no boundary. We say that N^n can be "moved off of itself" in M if a tubular neighborhood η of N^n with retraction map $\rho : \eta \to N^n$ admits a section $\sigma : N^n \to \eta$ that is disjoint from N. That is, $N^n \cap \sigma(N^n) = \subset \eta \subset M$.*

1. *Suppose the dimensions of the manifolds satisfy $2n < m$. Prove that N^n can be moved of itself in M.*

2. *To see that the dimension requirement above is necessary in general, show that $\mathbb{R}P^1 \subset \mathbb{R}P^2$ cannot be moved off of itself.*

Proof. 1. Denote the embedding of N^n into M^m by e. By Proposition 8.10 in the book, for any choice of $\varepsilon > 0$, we can choose an embedding $\tilde{e}$ isotopic to e such that, for any $x \in N^n, ||e(x) - \tilde{e}(x)|| < \varepsilon$ and $\tilde{e}(N^n) \cap N^n =$. Thus we can choose ε small enough that, for any $x \in N^n, ||e(x) - \tilde{e}(x)||$ is such that $\tilde{e}(N^n)$ is within the tubular neighborhood η, and $\tilde{e}(N^n) \cap N^n =$. This is because the only transversal intersection of two $n-$dimensional submanifolds of an $m-$dimensional submanifold with $2n < m$ is the empty intersection. We note that $\tilde{e} \circ e^{-1}$ is continuous, with image in η, so the ρ map is such that $\rho \circ \tilde{e} \circ e^{-1} = \tilde{Id}$, where $\tilde{Id}$ is a diffeomorphism of N^n. Thus $\tilde{e} \circ e^{-1}$ is a section that is disjoint from N^n, and so N^n can be moved off itself.

2. We treat $\mathbb{R}P^n$ as $S^n/\sim$, where $x \sim -x$. Thus when we embed $\mathbb{R}P^1$ into $\mathbb{R}P^2$, we require that the image of the embedding be an equator of $S^2/\sim$. (The reason it must be an equator is that if it weren't, the image would cease to have $x \sim -x$) Thus we think about an equator of S^2 under this quotient relation. Embedding another $\mathbb{R}P^1$ into $\mathbb{R}P^2$ yields two equators in $S^2/\sim$. Two equators of S^2 must intersect at two points, and these two points must be antipodal points. However, under our quotient relation, these two points are the same point. Thus the self-intersection number mod 2 of the embedding of $\mathbb{R}P^1$ into $\mathbb{R}P^2$ is 1. Therefore, another embedding of $\mathbb{R}P^1$ cannot be isotoped away from itself in that their intersection is .

$\square$