Quantum Field Theory Problems

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Question 1. *Consider the space $\mathcal{D}$ of continuously differentiable complex-valued functions f on $[0,1]$. Consider the operator A on $L^2([0,1])$ with domain $\mathcal{D}$, defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.*

Proof. We want to check if $\langle A\psi | \varphi \rangle = \langle \psi | A\varphi \rangle$. This gives us $\langle i\psi' | \varphi \rangle, \langle \psi | i\varphi' \rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^* \varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^* \varphi dx = \int_0^1 (-i)\psi'^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx) \neq \int_0^1 i\psi^* \varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is D_α, then, evaluating the same integral, we have $\int_0^1 (i\psi')^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^* \varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^* \alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^*\alpha\psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1 - \alpha^*\alpha)i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^*\alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^* \varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. $\square$

Question 2. *Recall the definition of the manifold X_m, the measure λ_m on X_m, and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of $\mathcal{H}$. Finally, recall the definitions of $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that*

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\mathbb{K} \tag{1}$$

where $\mathbb{K}$ is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_\mathbf{p}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_\mathbf{p} w_{\mathbf{p}'}}}[a(p), a^\dagger(p')]$. The first expression then becomes $\int \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{4w_\mathbf{p} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p - p')1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p)f(p) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2w_\mathbf{p}} f(w_\mathbf{p}, \mathbf{p})$, we have the equality

$$\int \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{4w_\mathbf{p} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')] =$$

$$\int \int d\lambda_m(p) d\lambda_m(p') \sqrt{4w_\mathbf{p} w_{\mathbf{p}'}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p)\sqrt{4w_\mathbf{p}^2} f(\mathbf{p})^* g(\mathbf{p})1 = \int d\lambda_m(p)2w_\mathbf{p} f(\mathbf{p})^* g(\mathbf{p})1$.

Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, where 1 is the identity operator on our Fock space.

Now we consider $\int \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}')1$. Integrating once, we find this gives us $\int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p})1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. $\qquad\square$

Question 3. *Consider the theory for massive scalar bosons of mass m. Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation*

$$\frac{\partial \varphi}{\partial t} = i[H_0, \varphi] \tag{2}$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f. Then, since $H_0 = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p})a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$, we have

$$(H_0\varphi)(f) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p})a(\mathbf{p}) \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}'),$$

$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}') \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} w_\mathbf{p} a^\dagger(\mathbf{p})a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{w_\mathbf{p}}{\sqrt{2w_{\mathbf{p}'}}} A, \text{ where}$$

$$A =$$

$$a^\dagger(\mathbf{p})a(\mathbf{p})e^{-i(x,p)}a(\mathbf{p}') + a^\dagger(\mathbf{p})a(\mathbf{p})e^{i(x,p')}a^\dagger(\mathbf{p}') - e^{-i(x,p')}a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) - e^{i(x,p')}a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a(\mathbf{p}')) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p}')a(\mathbf{p}) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')a(\mathbf{p}))$$

$$= e^{-i(x,p')}[a^\dagger(\mathbf{p}), a(\mathbf{p}')]a(\mathbf{p}) + e^{i(x,p')}a^\dagger(\mathbf{p})[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. Also, notice that $[A, B] = AB - BA = (-1)(BA - AB) = -[B, A]$. Thus, A becomes

$$e^{-i(x,p')}(-1)(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')a(\mathbf{p}) + e^{i(x,p')}a^\dagger(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

$$= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{w_\mathbf{p}}{\sqrt{2w_{\mathbf{p}'}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$

$$= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{w_\mathbf{p}}{\sqrt{2w_\mathbf{p}}} (e^{i(x,p)}a^\dagger(\mathbf{p}) - e^{-i(x,p)}a(\mathbf{p}))$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x, p) = tw_\mathbf{p} + \mathbf{x} \cdot \mathbf{p}$, so the time derivative of $e^{\pm i(x,p)} = \pm i w_\mathbf{p} e^{\pm i(x,p)}$. Thus, $\frac{\partial \varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{i w_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}}(-e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived form the commutator. Thus, $\frac{\partial \varphi}{\partial t} = i[H_0, \varphi]$, up to a set of measure zero. $\qquad \square$

Question 4. *In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude*

$$\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | S | \mathbf{p_1} \rangle \tag{3}$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!} \int_{\mathbb{R}} d^4x : \varphi(x)^4 : + \mathcal{O}(g^2)$. We then have

$$\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | S | \mathbf{p_1} \rangle = \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | \mathbf{p_1} \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4x \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | : \varphi(x)^4 : | \mathbf{p_1} \rangle + \mathcal{O}(g^2)$$

$$= \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | \mathbf{p_1} \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4x \langle 0 | a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1}) | 0 \rangle + \mathcal{O}(g^2)$$

For the first term, we notice that $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | \mathbf{p_1} \rangle = \langle 0 | a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4})a^\dagger(\mathbf{p_1}) | 0 \rangle$. Applying the first two operators we get either ground state back if $\mathbf{p_1} = \mathbf{p_4}$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4} | \mathbf{p_1} \rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0 | a(\mathbf{p})\varphi(x) | 0 \rangle = \frac{e^{i(x,p)}}{\sqrt{2w_\mathbf{p}}}$, $\langle 0 | \varphi(x)a^\dagger(\mathbf{p}) | 0 \rangle = \frac{e^{-i(x,p)}}{\sqrt{2w_\mathbf{p}}}$. $\langle 0 | a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1}) | 0 \rangle = \langle 0 | a(\mathbf{p_2})\varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p_3})\varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p_4})\varphi(x) | 0 \rangle \langle 0 | a^\dagger(\mathbf{p_1})\varphi(x) | 0 \rangle$. This expression is equal to $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16 w_{\mathbf{p_2}} w_{\mathbf{p_3}} w_{\mathbf{p_4}} w_{\mathbf{p_1}}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want

to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8-1)!!$ diagrams, and 4! diagrams of this type. Thus we have 4! $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$. Thus we have
$\langle\mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|S|\mathbf{p_1}\rangle = (-ig(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}}) + \mathcal{O}(g^2)$. $\qquad\square$

Question 5.   *1. Derive Maxwell's equations as the Euler-Lagrange equations of the action*

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4}$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard from by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory.

2. *Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so*

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{5}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{6}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2); S = E \times B \tag{7}$$

Peskin & Schroeder 2.1.

Proof. 1. Let's first calculate $F^{\mu\nu}$. Given our identification with E^i and B^i, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8}$$

Treating A_ν as our dynamical variables, we take

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{9}$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \right] = 0 \tag{10}$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{4}(2\partial_\mu A_\nu \partial_\mu A_\nu - 2\partial_\nu A_\mu \partial_\mu A_\nu) \right] = 0 \tag{11}$$

$$\partial_\mu \left[-\frac{1}{4}(4\partial_\mu A_\nu - 4\partial_\nu A_\mu) \right] = 0 \tag{12}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{13}$$

With the identification $F^{0i} = -E^i, F^{ij} = -\epsilon^{ijk}B^k$, we have $-\frac{\partial E}{\partial t} - \partial_i \epsilon^{ijk} B^k = 0$. Because I always forget the Levi-Civita symbols, we recall that

$$\epsilon^{ijk}\partial_j v_k = (\nabla \times v)^i \tag{14}$$

and thus $-\frac{\partial E}{\partial t} + \epsilon^{jik}\partial_i B^k = 0$, or

$$\nabla \times B = \frac{\partial E}{\partial t} \tag{15}$$

2. With this construction, we have

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{16}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\gamma)} \partial^\nu A_\gamma - \mathcal{L}\delta^{\mu\nu} + \partial_\lambda F^{\mu\lambda} A^\nu \tag{17}$$

$$= -F^{\mu\gamma} \partial^\nu A_\gamma + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} + \partial_\lambda (F^{\mu\lambda} A^\nu) \tag{18}$$

$$= F^{\mu\iota}(\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} - \partial_\lambda F^{\lambda\mu} A^\nu \tag{19}$$

$$= F^{\mu\iota} F_\iota^\nu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} - (0) A^\nu \tag{20}$$

This is now a viable energy-momentum tensor. We now $\hat{T}^{00}$ and $\hat{T}^{0i}$:

$$\hat{T}^{00} = F^{0\iota} F_\iota^0 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{21}$$

$$= E^\iota E_\iota + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{22}$$

We then have

$$\langle,\rangle = tr(\overline{F^{\mu\nu}} F^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \tag{23}$$

$$= tr\left(\begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix} \right) \tag{24}$$

This is equal to $2(B^2 - E^2)$. Thus we have that $\hat{T}^{\mu\nu} = E^2 + \frac{1}{4}2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $T^{\hat{0}i}$, we have

$$T^{\hat{0}i} = F^{0j}F^i_j + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{25}$$

$$= E^j\epsilon_{jik}B^k g^{mi} + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{26}$$

$$= \mathbf{E} \times \mathbf{B} = \mathbf{S} \tag{27}$$

□

Question 6. *Consider the the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is*

$$S = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \tag{28}$$

(a) *Find the conjugate momenta to $\phi(x), \phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is*

$$H = \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \tag{29}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) *Diagonlize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.*

(c) *Rewrite the conserved charge*

$$Q = \int d^3x \frac{i}{2}(\phi^*\pi^* - \pi\phi) \tag{30}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) *Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1,2$. Show that there are now four conserved charges, one given by the*

generalization of part (c), and the other three given by

$$Q^i = \int d^3x \frac{i}{2}(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b) \tag{31}$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Peskin & Schroeder, 2.2.

Proof. (a) We have that $p(\mathbf{x}) = \frac{\partial L}{\partial\dot\phi(\mathbf{x})} = \frac{\partial}{\partial\dot\phi(\mathbf{x})}\int d^4x(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi) = \frac{\partial}{\partial\dot\phi(\mathbf{x})}\int d^4x(\partial_\mu\phi^* g_\nu^\mu\partial^\nu\phi - m^2\phi^*\phi) = \frac{\partial}{\partial\dot\phi(\mathbf{x})}\int d^4x(\frac{\partial\phi^*}{\partial t}\frac{\partial\phi}{\partial t} - \nabla\phi^*\cdot\nabla\phi - m^2\phi^*\phi) = \frac{\partial}{\partial\dot\phi(\mathbf{x})}\int d^4x(\dot\phi^*\dot\phi - \nabla\phi^*\cdot\nabla\phi - m^2\phi^*\phi)$. Thus, $\pi = \dot\phi^*$. Similarly, $\pi^* = \dot\phi$. Since ϕ, ϕ^* are the dynamical variables, the canonical commutation relations are

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{32}$$

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0 \tag{33}$$

from quantization of the Klein-Gordon field given in the textbook. Given the equation for the Hamiltonian, we have

$$H = \int d^3x[\sum_{a,b}\pi_i(\mathbf{x})\dot\phi_i(\mathbf{x}) - \mathcal{L}] \tag{34}$$

$$= \int d^3x[\pi^*\dot\phi^* + \pi\dot\phi - \mathcal{L}] \tag{35}$$

$$= \int d^3x[\dot\phi\dot\phi^* + \dot\phi^*\dot\phi - \mathcal{L}] \tag{36}$$

$$= \int d^3x[2\dot\phi\dot\phi^* - \dot\phi\dot\phi^* + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi] \tag{37}$$

$$= \int d^3x[\dot\phi\dot\phi^* + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi] \tag{38}$$

$$= \int d^3x[\pi^*\pi + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi] \tag{39}$$

We want to compute $i\frac{\partial \phi}{\partial t}$ via the Heisenberg Equation of Motion, so we calculate $[\phi, H]$.

$$i\frac{\partial \phi}{\partial t} = [\phi, H] \tag{40}$$

$$= [\phi(x'), \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \tag{41}$$

$$= \int d^3x[\phi(x'), \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{42}$$

$$= \int d^3x([\phi(x'), \pi^*\pi] + [\phi, \nabla\phi^* \cdot \nabla\phi] + m^2[\phi, \phi^*\phi]) \tag{43}$$

$$= \int d^3x\delta^{(3)}(x' - x)i\pi^*(x) \tag{44}$$

$$= i\pi^*(x) \tag{45}$$

$$i\frac{\partial \pi^*}{\partial t} = [\pi^*, H] \tag{46}$$

$$= [\pi^*(x'), \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \tag{47}$$

$$= \int d^3x[\pi^*(x'), \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{48}$$

$$\text{(integrating by parts)} = \int d^3x([\pi^*(x'), \pi^*\pi] + [\pi^*(x'), \phi^*(-\nabla^2 + m^2)\phi]) \tag{49}$$

$$= \int d^3x\delta^{(3)}(x' - x)(-i)(-\nabla^2 + m^2)\phi(x) \tag{50}$$

$$= i(\nabla^2 - m^2)\phi \tag{51}$$

Since $i\frac{\partial \phi}{\partial t} = i\pi^*$ and $i\frac{\partial \pi^*}{\partial t} = i(\nabla^2 - m^2)\phi$, so $\frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2)\phi$, which is the Klein-Gordon equation.

(b) Since ϕ satisfies the Klein-Gordon equation, and, in the same way, so does ϕ^*, we take the Fourier transform to gain more insight into $\nabla^2\phi$:

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}) \Rightarrow \tag{52}$$

$$[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi(\mathbf{p}) = 0, \ [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi^*(\mathbf{p}) = 0 \tag{53}$$

We write ϕ in terms of two real valued scalar free fields ψ_1, ψ_2, of which we already know the theory:

$$\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \ \phi^* = \frac{\psi_1 - i\psi_2}{\sqrt{2}} \tag{54}$$

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Since ψ_1, ψ_2 are independent free fields, both must satisfy the harmonic oscillator equation:

$$\tfrac{1}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 = 0, \ \tfrac{\pm i}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow$$

$$[\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 = 0, \ [\tfrac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow$$

$$\omega_1 = \sqrt{p_1^2 + m^2}, \ \omega_2 = \sqrt{p_2^2 + m^2}$$

Since the frequencies of the oscillators have independent momentums and ϕ is not hermitian, we create two different creation and annihilation operators:

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \frac{i}{\sqrt{2\omega_i}}p_i, \ a_i^\dagger = \sqrt{\frac{\omega_i}{2}}q_i - \frac{i}{\sqrt{2\omega_i}}p_i, \ i \in \{1,2\} \tag{55}$$

where $q_1 = \phi, q_2 = \phi^*, p_1 = \pi, p_2 = \pi^*$, with the notation in the spirit of Peskin and Schroeder. These creation operators, given their frequencies, represent creating two different particles with mass m. From the theory of a real-valued scalar free field, we know that

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \tag{56}$$

$$\phi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \tag{57}$$

The two different operators ensure that ϕ is not hermitian. From above we know that $\pi = \dot{\phi}^*, \pi^* = \dot{\phi}$, and, using our real-valued scalar free field as reference, we have

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{\omega_\mathbf{p}}{2}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \tag{58}$$

$$\pi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \tag{59}$$

Finally, we rewrite our Hamiltonian in terms of our operators:

$$H = \int d^3\mathbf{x}(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \tag{60}$$

$$= \int d^3\mathbf{x}(\int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}')e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} - a_1(\mathbf{p})a_2(\mathbf{p}')e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \tag{61}$$

$$- a_2^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}')e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})a_2(\mathbf{p}')e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}\} \tag{62}$$

$$+ \int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}[-i\mathbf{p}a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + i\mathbf{p}a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}] \tag{63}$$

$$\cdot [i\mathbf{p}'a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - i\mathbf{p}'a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}] \tag{64}$$

$$+ m^2\int\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}')e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} + a_1^\dagger(\mathbf{p})a_2^\dagger(\mathbf{p}')e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \tag{65}$$

$$+ a_2(\mathbf{p})a_1(\mathbf{p}')e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} + a_2(\mathbf{p})a_2^\dagger(\mathbf{p}')e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\}) \tag{66}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3}(\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \tag{67}$$

$$+ \frac{p^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\} \tag{68}$$

$$+ \frac{m^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\}) \tag{69}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3}[\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \tag{70}$$

$$+ \frac{p^2+m^2}{2\omega_\mathbf{p}}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\}] \tag{71}$$

where in (70) the middle two terms have a positive sign because we subtract $\mathbf{p}\cdot\mathbf{p}' = \mathbf{p}\cdot(-\mathbf{p})$. Furthermore, since $\mathbf{p} \neq -\mathbf{p}$, we can commute our operators. Thus we have the Hamiltonian as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\{a_1a_1^\dagger + a_2a_2^\dagger\} \tag{72}$$

Since this Hamiltonian is constructed purely out of constants and operators whose eigenvectors are momentum eigenstates, our Hamiltonian is now diagonalized. The indices 1 and 2 represent the two particles of mass m.

(c) This is just plugging in our values for momentum and position and integrating, like the

previous problem. We have

$$Q = \int d^3\mathbf{x} \frac{i}{2} (\phi^* \pi^* - \pi\phi) \tag{73}$$

$$= \int d^3\mathbf{x} \frac{i}{2} \int d^3\mathbf{p} \int d^3\mathbf{p}' \frac{1}{(2\pi)^6} \frac{1}{2} [(-i)(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}) \tag{74}$$

$$- i(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}})] \tag{75}$$

$$= \int d^3\mathbf{p} \frac{1}{(2\pi)^3} \frac{1}{4} ([a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \tag{76}$$

$$+ [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) - a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]) \tag{77}$$

$$= \frac{1}{2} \int d^3\mathbf{p} \frac{1}{(2\pi)^3} [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \tag{78}$$

This means that this theory has two particle types: one created by $a_1^\dagger(\mathbf{p})$ and one created by $a_2^\dagger(\mathbf{p})$. In examining $[Q, a_i^\dagger]\,|n\rangle$ for some state n-particle state $|n\rangle$, we can deduce the charge. It is easy to see that $[a_1, a_2] = 0, [a_i, a_i^\dagger] = 1$ since ψ_1, ψ_2 are independent fields. Thus

$$[Q, a_1^\dagger] = a_1^\dagger, [Q, a_2^\dagger] = -a_2^\dagger \tag{79}$$

This means that the charges are valued at 1 unit for particles created by $a_1^\dagger$ and -1 for particles created by $a_2^\dagger$].

(d) For two complex scalar fields, the lagrangian is then

$$\mathcal{L} = \partial_\mu \phi_1^* \partial^\mu \phi_1 - m^2 \phi_1^* \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 - m^2 \phi_2^* \phi_2 \tag{80}$$

We then have

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \Delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Delta\phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1^*)} \Delta\phi_1^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2^*)} \Delta\phi_2^* - \mathcal{J}^\mu \tag{81}$$

$$= \partial^\mu \phi_1^* \Delta\phi_1 + \partial^\mu \phi_2^* \Delta\phi_2 + \partial^\mu \phi_1 \Delta\phi_1^* + \partial^\mu \phi_2 \Delta\phi_2^* - \mathcal{J}^\mu \tag{82}$$

$$\rightarrow Q = \int d^3x (\pi_1^* \Delta\phi_1 + \pi_2^* \Delta\phi_2 + \pi_1 \Delta\phi_1^* + \pi_2 \Delta\phi_2^*) \tag{83}$$

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If we set

$$\Phi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{84}$$

we rewrite our theory as

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi \tag{85}$$

$$Q = \int d^3x (\dot{\Phi}^\dagger \Delta \Phi + (\Delta \Phi)^\dagger \dot{\Phi}) \tag{86}$$

The symmetry of this lagrangian is

$$\Phi \mapsto M\Phi \tag{87}$$

for $M \in U(2)$. We know that this system should have $U(1)$ symmetry from the above problem. Using the $\det : U(n) \to U(1)$ map, we have a short exact sequence

$$SU(2) \to U(2) \to U(1) \tag{88}$$

giving us $U(2) = SU(2) \times U(1)$. For this reason, since $U(1)$ is just a complex number, each conserved charge from this symmetry has the commutation relations of $SU(2)$. In order to put this into a continuous symmetry picture, we exponentiate an element $\sigma \in SU(2)$ is a factor $i(\alpha_1, \alpha_2)$ and take $\alpha_1, \alpha_2 \to 1$:

$$\Phi \mapsto e^{i(\alpha_1, \alpha_2)\sigma} \Phi \tag{89}$$

$$\Delta \Phi \mapsto i\sigma \Phi \tag{90}$$

$$\Delta \Phi^* \mapsto -i\sigma \Phi \tag{91}$$

$SU(2)$ is generated by the Pauli matrices, so we have conserved charges

$$Q^i = i \int d^3x (\dot{\Phi}^\dagger \sigma^i \Phi - \Phi^\dagger \sigma^i \dot{\Phi}) \tag{92}$$

$$= i \int d^3x (\phi_a^* \sigma_{ab}^i \pi_b^* - \pi_a \sigma_{ab}^i \phi_b) \tag{93}$$

Generalizing to n independent identical complex scalar fields, we let $\Phi = (\phi_1, ..., \phi_n)^T$, and our symmetry becomes $U(n) = SU(n) \times U(1)$, meaning the charges we get are of the same form as what we got, but replacing the σ^i with $n-$dimensional skew-hermitian matrices.

$\square$

Question 7. *Evaluate the function*

$$\langle 0| \phi(x)\phi(y) |0\rangle = D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \tag{94}$$

for $(x - y)$ spacelike so that $(x - y)^2 = -r^2$, explicitly in terms of Bessel functions.

Proof. We have

$$D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \tag{95}$$

$$= \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^2}{\sqrt{p^2 + m^2}} e^{ipr\cos\theta} \tag{96}$$

θ is the angle between p and $(x - y)$, which also works for the conversion to spherical coordinates. We then have

$$D(x - y) = \frac{1}{(2\pi)^2} \int_0^\infty \frac{dp\, p^2}{\sqrt{p^2 + m^2}} \int_0^\pi d\theta \sin\theta \left(\sum_{n=-\infty}^\infty J_n(pr) e^{in\theta} \right) \tag{97}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty \frac{dp\, p^2}{\sqrt{p^2 + m^2}} \int_0^\pi d\theta \sin\theta \left(J_0(pr) + 2 \sum_{n=1}^\infty i^n J_n(pr) \cos(n\theta) \right) \tag{98}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty \frac{dp\, p^2}{\sqrt{p^2 + m^2}} \left[2J_0(pr) + 2 \sum_{n=1}^\infty i^n J_n(pr) \frac{\cos n\pi + 1}{1 - n^2} \right] \tag{99}$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{dp\, p^2}{\sqrt{p^2 + m^2}} \left[J_0(pr) + \sum_{n=1}^\infty J_{2n}(pr) \frac{2}{1 - 4n^2} \right] \tag{100}$$

As in the book, the integrand has branch cuts on the imaginary axis starting at $p = \pm im$, so we

push the contour up to wrap around the upper branch cut. With $\rho = -ip$, we get

$$D(x - y) = \frac{-i}{2\pi^2} \int_m^\infty d\rho \frac{-\rho^2}{\rho^2 - m^2} [J_0(i\rho r) + \sum_{n=1}^\infty J_{2n}(i\rho r) \frac{2}{1 - 4n^2}] \tag{101}$$

□

Question 8. *Recall the Lorentz commutation relations,*

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \tag{102}$$

1. *Define the generators of rotations and boosts as*

$$L^i = \frac{1}{2} \epsilon^{ijk} J^{jk}, \ K^i = J^{0i}, \tag{103}$$

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written $\Phi \to (1 - i\theta \cdot L - i\beta \cdot \mathbf{K})\Phi$. Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk} L^k$.) Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \ and \ \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K}) \tag{104}$$

commute with one another and separately satisfy the commutation relations of angular momentum.

2. *The finite-dimensional representations of the rotation group correspond precisely to the allowed values for angular momentum: integers or half-integers. The result of part (a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \sigma/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R given by*

$$\psi_L \to (1 - i\theta \cdot \frac{\sigma}{2} - \beta \cdot \frac{\sigma}{2})\psi_L; \tag{105}$$

$$\psi_R \to (1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})\psi_R; \tag{106}$$

3. The identity $\sigma^T = -\sigma^2 \sigma \sigma^2$ allows us to rewrite the ψ_L transformation in the unitarily equivalent form

$$\psi' \to \psi'(1 + i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}), \tag{107}$$

where $\psi' = \psi_L^T \sigma^2$. Using this law, we can represent the object that transforms as $(\frac{1}{2}, \frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformation law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} \tag{108}$$

Show that the object V^μ transforms as a 4-vector.

Proof. 1. First we calculate $[L^i, L^j]$. We assume that the metric is $(+, -, -, -)$, as per the convention in Peskin and Schroeder.

$$[L^i, L^j] = \frac{1}{4}(\epsilon^{imn} J^{mn} \epsilon^{jlk} J^{lk} - \epsilon^{jlk} J^{lk} \epsilon^{imn} J^{mn}) \tag{109}$$

$$= \frac{1}{4} \epsilon^{imn} \epsilon^{jlk} [J^{mn}, J^{lk}] \tag{110}$$

$$= \frac{1}{4} \epsilon^{imn} \epsilon^{jlk} i(g^{nl} J^{mk} - g^{ml} J^{nk} - g^{nk} J^{ml} + g^{mk} J^{nl}) \tag{111}$$

$$= \frac{1}{4} \epsilon^{imn} \epsilon^{jlk} i(-\delta^{nl} J^{mk} + \delta^{ml} J^{nk} + \delta^{nk} J^{ml} - \delta^{mk} J^{nl}) \tag{112}$$

$$= i\frac{1}{4}(\epsilon^{iml} \epsilon^{jkn} J^{mk} + \epsilon^{imn} \epsilon^{jmk} J^{nk} + \epsilon^{imn} \epsilon^{jln} J^{ml} + \epsilon^{imn} \epsilon^{jml} J^{nl}) \tag{113}$$

(rewrite indices) $= i(\epsilon^{iml} \epsilon^{jkl} J^{mk})$ \hfill (114)

$$= i(\delta_{ij} \delta_{mk} - \delta_{ik} \delta_{jm}) J^{mk} \tag{115}$$

$$= i(\delta_{jj} J^{mm} - J^{ji}) \tag{116}$$

$$= -iJ^{ji} \tag{117}$$

Since $J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$, we have $J^{ij} = -J^{ji}$. Notice that $L^k = \frac{1}{2} \epsilon^{kij} J^{ij}$, so $J^{ij} = -\epsilon^{ijk} L^k$, so $[L^i, L^j] = iJ^{ij} = i\epsilon^{ijk} L^k$.

Similarly, we look at $[K^i, K^j] = [J^{0i}, J^{0j}]$.

$$[K^i, K^j] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) \tag{118}$$

$$= i(-J^{ij} + \delta^{ij}J^{00}) \tag{119}$$

$$= -iJ^{ij} \tag{120}$$

$$= -i\epsilon^{ijk}L^k \tag{121}$$

Now we examine $[L^i, K^j]$:

$$[L^i, K^j] = [\frac{1}{2}\epsilon^{imk}J^{mk}, J^{0j}] \tag{122}$$

$$= \frac{1}{2}\epsilon^{imk}[J^{mk}, J^{0j}] \tag{123}$$

$$= \frac{i}{2}\epsilon^{imk}(g^{k0}J^{mj} - g^{m0}J^{kj} - g^{kj}J^{m0} + g^{mj}J^{k0}) \tag{124}$$

$$= \frac{i}{2}\epsilon^{imk}(\delta_{k0}J^{mj} - \delta_{m0}J^{kj} + \delta_{kj}J^{m0} - \delta_{mj}J^{k0}) \tag{125}$$

$$= \frac{i}{2}\epsilon^{imk}(\delta_{kj}J^{m0} - \delta_{mj}J^{k0}) \tag{126}$$

$$= \frac{i}{2}(\epsilon^{imj}J^{m0} - \epsilon^{ijk}J^{k0}) \tag{127}$$

$$= \frac{i}{2}(\epsilon^{imj}J^{m0} + \epsilon^{ikj}J^{k0}) \tag{128}$$

$$= i\epsilon^{imj}J^{m0} \tag{129}$$

$$= -i\epsilon^{imj}J^{0m} \tag{130}$$

$$= i\epsilon^{ijm}K^m \tag{131}$$

Knowing this, the commutation relations for $\mathbf{J}_\pm$ are:

$$[J_+^i, J_+^j] = [\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j + iK^j)] \tag{132}$$

$$= \frac{1}{4}[L^i, L^j] + \frac{i}{4}[L^i, K^j] + \frac{i}{4}[K^i, L^j] - \frac{1}{4}[K^i, K^j] \tag{133}$$

$$= \frac{1}{4}i\epsilon^{ijk}L^k + \frac{i}{4}i\epsilon^{ijk}K^k - \frac{i}{4}i\epsilon^{jik}K^k + \frac{1}{4}i\epsilon^{ijk}L^k \tag{134}$$

$$= \frac{1}{2}i\epsilon^{ijk}(L^k + iK^k) \tag{135}$$

$$[J_-^i, J_-^j] = [\frac{1}{2}(L^i - iK^i), \frac{1}{2}(L^j - iK^j)] \tag{136}$$

$$= \frac{1}{4}[L^i, L^j] - \frac{i}{4}[L^i, K^j] - \frac{i}{4}[K^i, L^j] - \frac{1}{4}[K^i, K^j] \tag{137}$$

$$= \frac{1}{4}i\epsilon^{ijk}L^k - \frac{i}{4}i\epsilon^{ijk}K^k + \frac{i}{4}i\epsilon^{jik}K^k + \frac{1}{4}i\epsilon^{ijk}L^k \tag{138}$$

$$= \frac{1}{2}i\epsilon^{ijk}(L^k - iK^k) \tag{139}$$

$$[J_+^i, J_-^j] = [\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j - iK^j)] \tag{140}$$

$$= \frac{1}{4}[L^i, L^j] - \frac{i}{4}[L^i, K^j] + \frac{i}{4}[K^i, L^j] + \frac{1}{4}[K^i, K^j] \tag{141}$$

$$= \frac{1}{4}i\epsilon^{ijk}L^k - \frac{i}{4}i\epsilon^{ijk}K^k - \frac{i}{4}i\epsilon^{jik}K^k - \frac{1}{4}i\epsilon^{ijk}L^k \tag{142}$$

$$= 0 \tag{143}$$

2. In the $(j_+, j_-) = (\frac{1}{2}, 0)$ case, we have $\mathbf{J}_+ = \frac{\sigma}{2}$ in the spin$-\frac{1}{2}$ representation of the rotation group. To ensure this, we set $\mathbf{L} = \sigma, \mathbf{K} = -i\sigma$. Similarly, for the (j_+, j_-) case, we set $L = \sigma, \mathbf{K} = i\sigma$. The Lorentz operators are given by $exp[-i\frac{1}{2}(\theta\mathbf{L} \pm i\beta\mathbf{K})]$ for $\mathbf{J}_\pm$. Taylor expanding for infinitesimal θ, β, and neglecting >1 order terms of θ and β, the infinitesimal transformations are given by

$$\Phi \rightarrow (1 - i\theta \cdot \frac{\sigma}{2} \mp \beta \cdot \frac{\sigma}{2})\Phi \tag{144}$$

3. Notice that

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu \sigma_\mu \tag{145}$$

Thus the field is given by $\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2$, and the field transforms by

$$\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2 \to (1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2})\psi_R V^\mu \sigma_\mu \psi_L^T \sigma^2 (1 + i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}) \qquad (146)$$

Expanding this out, we have this equal to

$$(1 - \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j)(V^0 + V^i\sigma^i)(1 + \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j) = (1 - \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j)V^0 \qquad (147)$$

$$+ V^0(1 + \frac{i}{2}\theta^j\theta^j + \frac{1}{2}\beta^j\theta^j) \qquad (148)$$

$$+ (V^i\sigma^i - \frac{i}{2}\theta^i V^i\sigma^j\sigma^i + \frac{1}{2}\beta^j V^i\sigma^j\sigma^i) \qquad (149)$$

$$+ (V^i\sigma^i + \frac{i}{2}V^i\theta^j\sigma^i\sigma^j + \frac{1}{2}V^i\beta^j\sigma^i\sigma^j) \qquad (150)$$

$$= (1 + \beta^j\sigma^j)V^0 \qquad (151)$$

$$+ 2V^i\sigma^i + \frac{1}{2}\beta^j V^i\{\sigma^j, \sigma^i\} + \frac{i}{2}\theta^j V^i[\sigma^i, \sigma^j] \qquad (152)$$

$$= (1 + \beta^j\sigma^j)V^0 \qquad (153)$$

$$+ 2V^i\sigma^i + \frac{1}{2}\beta^j V^i 2\delta_j^i + \frac{i}{2}\theta^j V^i 2i\epsilon^{ijk}\sigma^k \qquad (154)$$

$$= (1 + \beta^j\sigma^j)V^0 \qquad (155)$$

$$+ (2\sigma^j - \theta^j\epsilon^{ijk}\sigma^k + \beta^i)V^i \qquad (156)$$

This is a Lorentz boost, so $V^0 + V^i = V^\mu$ is a Lorentz 4-vector.

$\square$