Analytic Number Theory Problems

Alec Lau

All mistakes are to be emailed to aszlau@gmail.com. All problems written by Prof. Kannan Soundararajan and Prof. Andrew Glanville's "Multiplicative Number Theory" textbook, unless otherwise marked.

Contents

1 The Prime Number Theorem

1.1 Different Forms of the Prime Number Theorem

Question 1. *Given the conjecture*

$$\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x \tag{1}$$

where

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m \text{ for } p \text{ prime and } m \geq 1 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

and the conjecture

$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x} \tag{3}$$

Use partial summation to prove that (1) and (3) are equivalent and both are equivalent to the conjecture

$$\theta(x) := \sum_{p \leq x} \log(p) = x + o(x) \tag{4}$$

Definition 1. ***Partial Summation:*** *Given a sequence $a_n \in \mathbb{C}$ and a function $f : \mathbb{R} \to \mathbb{C}$, set $S(t) = \sum_{k \le t} a_k$, it is easy to conclude that*

$$\sum_{n=A+1}^{B} a_n f(n) = S(B)f(B) - S(A)f(A) - \sum_{n=A}^{B-1} S(n)(f(n+1) - f(n)) \tag{5}$$

and, if f is continuously differentiable on $[A, B]$, then

$$\sum_{A<n\le B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt \tag{6}$$

Proof. We start with (3). Notice that, if we choose

$$a_n = \begin{cases} 1 \text{ if } n = p \text{ for p prime} \\ 0 \text{ otherwise} \end{cases} \tag{7}$$

and $f(x) = \log x$, then

$$\begin{aligned} \theta(x) &= \sum_{n \le x} a_n f(n) && (8) \\ &= (\sum_{n\le x} a_n) \log x - \int_2^x (\sum_{n\le t} a_n)(\log t)' dt && (9) \\ &= (\sum_{p\le x} 1) \log x - \int_2^x (\sum_{p\le t} 1)\frac{1}{t} dt && (10) \\ &= \pi(x) \log x - \int_2^x \pi(t)\frac{1}{t} dt && (11) \\ &\sim \frac{x}{\log x} \log x - \int_2^x \frac{t}{t \log t} dt && (12) \\ &\sim x - \int_2^x \frac{1}{\log t} dt && (13) \\ &\sim x + (-li(x)) && (14) \end{aligned}$$

It remains to prove that $(-li(x)) \in o(x)$. Thus we examine the asymptotic behavior of $-li(x)/x$. By L'Hospital's Rule, we have

$$\lim_{x\to\infty} \frac{-li(x)}{x} = \lim_{x\to\infty} -\frac{1}{\log x} \tag{15}$$

Since $\log x$ diverges, we have this limit equal to 0, so (4) is true if and only if $\pi(x) \sim \frac{x}{\log x}$.

Now we move on to the conjecture in (1). ∎

1.2 Adding reciprocals

Note: my version of the paper has $\sum_{n\leq x}^{N} \frac{1}{N}$. I'm pretty sure the denominator should be n, as that sum is just 1.

Question 2. *Prove that for any integer $N \geq 1$,*

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + 1 - \int_{1}^{N} \frac{\{t\}}{t^2} dt \tag{16}$$

Deduce that, for any real $x \geq 1$,

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{17}$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{N\to\infty} (\sum_{n=1}^{N} \frac{1}{n} - \log N) = 1 - \int_{1}^{\infty} \frac{\{t\}}{t^2} dt \tag{18}$$

Note that, for $t \in \mathbb{R}$, $[t]$ is the integral part of t, and $\{t\}$ is the rest of t.

Proof. We use partial summation again. Let $f(x) = \frac{1}{x}$ and $a_n = 1$. Thus, by partial summation, we have

$$\sum_{n\leq x} \frac{1}{n} = [N]\frac{1}{N} + \log 1 + \int_{1}^{N} t\frac{1}{t^2} dt \tag{19}$$

$$= [N]\frac{1}{N} + \int_{1}^{N} \frac{1}{t^2}(t - \{t\}) dt \tag{20}$$

$$= 1 + \int_{1}^{N} \frac{t}{t^2} dt - \int_{1}^{N} \frac{\{t\}}{t^2} dt \tag{21}$$

$$= 1 + \log N - \log 1 - \int_{1}^{N} \frac{\{t\}}{t^2} dt \tag{22}$$

$$= \log N + 1 - \int_{1}^{N} \frac{\{t\}}{t^2} dt \tag{23}$$

For any *real* x, we have, through partial summation,

$$\sum_{n\leq x} \frac{1}{n} = [N]\frac{1}{N} + \int_{1}^{N} t\frac{1}{t^2} dt \tag{24}$$

$$= \frac{x - \{x\}}{x} + \log N - \int_{1}^{N} \frac{\{t\}}{t^2} dt \tag{25}$$

$$= \log N + 1 - \frac{\{x\}}{x} + \int_{N}^{\infty} \frac{\{t\}}{t^2} dt - \int_{1}^{\infty} \frac{\{t\}}{t^2} dt \tag{26}$$

$$= \log N + \gamma - \frac{\{x\}}{x} + \int_{N}^{\infty} \frac{\{t\}}{t^2} dt \tag{27}$$

It remains to prove that $\frac{\{x\}}{x}$ and $\int_N^\infty \frac{\{t\}}{t^2}dt$. Starting with the former, we see that since $\{x\} < 1$, we have that $|\frac{\{x\}}{x}| < \frac{1}{x}$, so $\frac{\{x\}}{x} \in O(\frac{1}{x}$. Similarly, we have

$$|\int_N^\infty \frac{\{t\}}{t^2}dt| \leq \int_N^\infty |\{t\}||\frac{1}{t^2}|dt \leq \int_N^\infty \frac{1}{t^2}dt \in O(\frac{1}{x}) \tag{28}$$

Thus we conclude

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{29}$$

□

1.3 log$N$!

Question 3. *For an integer $N \geq 1$, show that*

$$\log N! = N\log N - N + 1 + \int_1^N \frac{\{t\}}{t}dt \tag{30}$$

Using that $\int_1^x (\{t\} - 1/2)dt = (\{x\}^2 - \{x\})/2$, show that

$$\int_1^N \frac{\{t\}}{t}dt = \frac{1}{2}\log N - \frac{1}{2}\int_1^N \frac{\{t\} - \{t\}^2}{t^2}dt \tag{31}$$

Conclude that $N! \sim C\sqrt{N}(N/e)^N$, where you can take as fact that

$$C = \exp(1 - \frac{1}{2}\int_1^\infty \frac{\{t\} - \{t\}^2}{t^2}dt) = \sqrt{2\pi} \tag{32}$$

Proof. From rules of logarithms, we have $\log N! = \log(N(N-1)...(2)(1)) = \log N + \log(N-1) + ... + \log 2 + \log 1$. We use partial summation once again. Let $a_n = 1$, and $f(x) = \log x$. From the partial summation formula given by (6), we have

$$\log N! = N\log N - 0 - \int_1^N (\sum_{n\leq t} 1)\frac{dt}{t} \tag{33}$$

$$= N\log N - \int_1^N \frac{[t]}{t}dt \tag{34}$$

$$= N\log N - \int_1^N \frac{t - \{t\}}{t}dt \tag{35}$$

$$= N\log N - \int_1^N dt + \int_1^N \frac{\{t\}}{t}dt \tag{36}$$

$$= N\log N - N + 1 + \int_1^N \frac{\{t\}}{t}dt \tag{37}$$

As for the next part, we notice (38):

$$
\begin{aligned}
\int_1^N \frac{\{t\}}{t} dt &= \int_1^N \frac{\{t\} - \frac{1}{2} + \frac{1}{2}}{t} dt &(38)\\
&= \int_1^N \frac{1}{t}(\{t\} - \frac{1}{2}) dt + \int_1^N \frac{1}{2t} dt &(39)\\
&= \frac{1}{t}\frac{\{t\}^2 - \{t\}}{2}|_1^N - \int_1^N \frac{1}{2}\frac{\{t\}^2 - \{t\}}{-t^2} dt + \frac{1}{2}\log N + \frac{1}{2}\log 1 &(40)\\
&= 0 + \int_1^N \frac{1}{2}\frac{\{t\}^2 - \{t\}}{t^2} dt + \frac{1}{2}\log N &(41)\\
&= \frac{1}{2}\log N - \int_1^N \frac{1}{2}\frac{\{t\} - \{t\}^2}{t^2} dt &(42)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\log N! &= N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^N \frac{\{t\} - \{t\}^2}{t^2} dt &(43)\\
&= N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^\infty \frac{\{t\} - \{t\}^2}{t^2} dt + \frac{1}{2}\int_N^\infty \frac{\{t\} - \{t\}^2}{t^2} dt &(44)\\
&= N\log N - N + \frac{1}{2}\log N + \log C + \frac{1}{2}\int_N^\infty \frac{\{t\} - \{t\}^2}{t^2} dt &(45)
\end{aligned}
$$

Taking the exponent of both sides, we get

$$
N! = N^N \cdot \frac{1}{e^N}\sqrt{N} \cdot C\sqrt{e}^{\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2} dt} \tag{46}
$$

Now we examine the integral in the exponent. First we split it up.

$$
\begin{aligned}
\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2} dt - \frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2} dt \le |\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2} dt| - |\frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2} dt| &\le \frac{1}{2}\int_N^\infty |\{t\}||\frac{1}{t^2}| dt - \frac{1}{2}\int_N^\infty |\{t\}^2|\frac{1}{t^2} dt &(47)\\
&\le \frac{1}{2}(\int_N^\infty \frac{1}{t^2} dt - \int_N^\infty \frac{1}{t^2} dt) &(48)
\end{aligned}
$$

It is easy to see that the limit as N approaches infinity the value of this integral converges to 0, so

$$
\begin{aligned}
N! &\sim N^N \cdot \frac{1}{e^N}\sqrt{N} \cdot C\sqrt{e}^0 \Rightarrow &(49)\\
N! &\sim C\sqrt{N}(N/e)^N &(50)
\end{aligned}
$$

□

Definition 2. *The **Riemann Zeta Function** is given by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1} \tag{51}$$

1.4 The Riemann Zeta Function

Note: My copy of the problem has a later part. In another copy of the book, this part is labeled as a problem which requires complex analysis, something I haven't learned yet, so I'm skipping that for now.

Question 4. *Prove that for $\Re(s)>1$,*

$$\zeta(s) = s\int_1^{\infty} \frac{[y]}{y^{s+1}} dy = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{y\}}{y^{s+1}} dy \tag{52}$$

Proof. We use partial summation again. We see that, for $a_n = 1, f(x) = \frac{1}{x^s}$, we have $\zeta(s) = \sum_1^{\infty} a_n f(n)$, so, using the usual partial summation formula,

$$\zeta(s) = \sum_1^{\infty} a_n f(n) = \lim_{N\to\infty} \sum_1^{N} a_n f(n) \tag{53}$$

$$= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - \int_1^N [y]\frac{1}{y^s} dy] \tag{54}$$

$$= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - [N]\frac{1}{N^s} + [1]\frac{1}{1^s} + s\int_1^N [y]\frac{1}{y^{s+1}} dy] \tag{55}$$

$$= \lim_{N\to\infty} [s\int_1^N [y]\frac{1}{y^s} dy] \tag{56}$$

$$= s\int_1^{\infty} [y]\frac{1}{y^s} dy \tag{57}$$

We write this final integral in a different way:

$$s\int_1^{\infty} [y]\frac{1}{y^s} dy = \lim_{N\to\infty} s\int_1^N \frac{y - \{y\}}{y^{s+1}} dy \tag{58}$$

$$= s\int_1^N \frac{y}{y^{s+1}} dy - s\int_1^N \frac{\{y\}}{y^{s+1}} dy \tag{59}$$

$$= \lim_{N\to\infty} [s\int_1^N \frac{1}{y^s} dt - s\int_1^N \frac{\{y\}}{y^{s+1}} dy] \tag{60}$$

$$= \lim_{N\to\infty} [-s\frac{1}{s-1}(\frac{1}{t^{s-1}}|_1^N) - s\int_1^N \frac{\{y\}}{y^{s+1}} dy] \tag{61}$$

$$= \lim_{N\to\infty} [-\frac{s}{s-1}(\frac{1}{N^{s-1}} - \frac{1}{1^{s-1}}) - s\int_1^N \frac{\{y\}}{y^{s+1}} dt] \tag{62}$$

Since $Re(s) > 1$, we have $Re(s) - 1 > 0$, so, evaluating the limit, we get that this expression is equivalent to

$$\frac{-s}{s-1}(0-1) - s\int_1^N \frac{\{y\}}{y^{s+1}}dt = \frac{s}{s-1} - s\int_1^N \frac{\{y\}}{y^{s+1}}dt \tag{63}$$

□