

Quantum Field Theory Problems

Alec Lau

Question 1. Consider the space \mathcal{D} of continuously differentiable complex-valued functions f on $[0, 1]$. Consider the operator A on $L^2([0, 1])$ with domain \mathcal{D} , defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.

Proof. We want to check if $\langle A\psi | \varphi \rangle = \langle \psi | A\varphi \rangle$. This gives us $\langle i\psi' | \varphi \rangle, \langle \psi | i\varphi' \rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^* \varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^* \varphi dx = \int_0^1 (-i)\psi'^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx \neq \int_0^1 i\psi^* \varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is \mathcal{D}_α , then, evaluating the same integral, we have $\int_0^1 (i\psi')^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^* \varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^* \alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^* \alpha \psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1 - \alpha^* \alpha) i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^* \alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^* \varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. \square

Question 2. Recall the definition of the manifold X_m , the measure λ_m on X_m , and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of \mathcal{H} . Finally, recall the definitions of $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \mathbb{K} \quad (1)$$

where \mathbb{K} is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}') [a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_{\mathbf{p}}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} [a(p), a^\dagger(p')]$. The first expression then becomes $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p - p') 1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2w_{\mathbf{p}}} f(w_{\mathbf{p}}, \mathbf{p})$, we have the equality

$$\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')] =$$

$$\int \int d\lambda_m(p) d\lambda_m(p') \sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p) \sqrt{4w_{\mathbf{p}}^2} f(\mathbf{p})^* g(\mathbf{p}) 1 = \int d\lambda_m(p) 2w_{\mathbf{p}} f(\mathbf{p})^* g(\mathbf{p}) 1$.

Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}) 1$, where 1 is the identity operator on our Fock space.

Now we consider $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') 1$. Integrating once, we find this gives us $\int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}) 1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. \square

Question 3. Consider the theory for massive scalar bosons of mass m . Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation

$$\frac{\partial \varphi}{\partial t} = i[H_0, \varphi] \quad (2)$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f . Then, since $H_0 = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}'))$, we have

$$(H_0 \varphi)(f) = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}) \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}')),$$

$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}')) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}} A, \text{ where}$$

$$A =$$

$$a^\dagger(\mathbf{p}) a(\mathbf{p}) e^{-i(x,p)} a(\mathbf{p}') + a^\dagger(\mathbf{p}) a(\mathbf{p}) e^{i(x,p')} a^\dagger(\mathbf{p}') - e^{-i(x,p')} a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}) - e^{i(x,p')} a^\dagger(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a(\mathbf{p}')) - a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}) + e^{i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}') a(\mathbf{p}) - a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p})) + e^{i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}) a^\dagger(\mathbf{p}') a(\mathbf{p}))$$

$$= e^{-i(x,p')} [a^\dagger(\mathbf{p}), a(\mathbf{p}')] a(\mathbf{p}) + e^{i(x,p')} a^\dagger(\mathbf{p}) [a(\mathbf{p}), a^\dagger(\mathbf{p}')]]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. Also, notice that $[A, B] = AB - BA = (-1)(BA - AB) = -[B, A]$. Thus, A becomes

$$\begin{aligned} & e^{-i(x, \mathbf{p}')} (-1) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') a(\mathbf{p}) + e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') (e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p}')} a(\mathbf{p})) \end{aligned}$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\begin{aligned} & \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') (e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p}')} a(\mathbf{p})) \\ &= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}}}} (e^{i(x, \mathbf{p})} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p})} a(\mathbf{p})) \end{aligned}$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x, p) = tw_{\mathbf{p}} + \mathbf{x} \cdot \mathbf{p}$, so the time derivative of $e^{\pm i(x, p)} = \pm iw_{\mathbf{p}} e^{\pm i(x, p)}$. Thus, $\frac{\partial \varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}} (-e^{-i(x, p)} a(\mathbf{p}') + e^{i(x, p)} a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived from the commutator.

Thus, $\frac{\partial \varphi}{\partial t} = i[H_0, \varphi]$, up to a set of measure zero. \square

Question 4. In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude

$$\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle \quad (3)$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x : \varphi(x)^4 : + \mathcal{O}(g^2)$. We then have

$$\begin{aligned} \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle &= \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | : \varphi(x)^4 : | \mathbf{p}_1 \rangle + \mathcal{O}(g^2) \\ &= \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x \langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) : \varphi(x)^4 : a^\dagger(\mathbf{p}_1) | 0 \rangle + \mathcal{O}(g^2) \end{aligned}$$

For the first term, we notice that $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle = \langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) a^\dagger(\mathbf{p}_1) | 0 \rangle$. Applying the first two operators we get either ground state back if $\mathbf{p}_1 = \mathbf{p}_4$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0 | a(\mathbf{p}) \varphi(x) | 0 \rangle = \frac{e^{i(x, \mathbf{p})}}{\sqrt{2w_{\mathbf{p}}}}$, $\langle 0 | \varphi(x) a^\dagger(\mathbf{p}) | 0 \rangle = \frac{e^{-i(x, \mathbf{p})}}{\sqrt{2w_{\mathbf{p}}}}$.

$$\langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) : \varphi(x)^4 : a^\dagger(\mathbf{p}_1) | 0 \rangle = \langle 0 | a(\mathbf{p}_2) \varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p}_3) \varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p}_4) \varphi(x) | 0 \rangle \langle 0 | a^\dagger(\mathbf{p}_1) \varphi(x) | 0 \rangle.$$

This expression is equal to $(e^{i(x, \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1)}) / (\sqrt{16w_{\mathbf{p}_2} w_{\mathbf{p}_3} w_{\mathbf{p}_4} w_{\mathbf{p}_1}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want

to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8 - 1)!!$ diagrams, and $4!$ diagrams of this type. Thus we have $4! (e^{i(x, p_2 + p_3 + p_4 - p_1)})/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2 + p_3 + p_4 - p_1))/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}})$. Thus we have $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle = (-ig(2\pi)^4\delta^{(4)}(p_2 + p_3 + p_4 - p_1))/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}}) + \mathcal{O}(g^2)$. \square

Question 5. 1. Derive Maxwell's equations as the Euler-Lagrange equations of the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (4)$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory.

2. Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad (5)$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu, \quad (6)$$

leads to an energy-momentum tensor \hat{T} that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2); S = E \times B \quad (7)$$

Peskin & Schroeder 2.1.

Proof. 1. Let's first calculate $F^{\mu\nu}$. Given our identification with E^i and B^i , we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (8)$$

Treating A_ν as our dynamical variables, we take

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad (9)$$

$$\partial_\mu \frac{\partial}{\partial (\partial_\mu A_\nu)} \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial_\mu A_\nu - \partial_\nu A_\mu) \right] = 0 \quad (10)$$

$$\partial_\mu \frac{\partial}{\partial (\partial_\mu A_\nu)} \left[-\frac{1}{4} (2\partial_\mu A_\nu \partial_\mu A_\nu - 2\partial_\nu A_\mu \partial_\mu A_\nu) \right] = 0 \quad (11)$$

$$\partial_\mu \left[-\frac{1}{4} (4\partial_\mu A_\nu - 4\partial_\nu A_\mu) \right] = 0 \quad (12)$$

$$\partial_\mu F^{\mu\nu} = 0 \quad (13)$$

With the identification $F^{0i} = -E^i$, $F^{ij} = -\epsilon^{ijk} B^k$, we have $-\frac{\partial E}{\partial t} - \partial_i \epsilon^{ijk} B^k = 0$. Because I always forget the Levi-Civita symbols, we recall that

$$\epsilon^{ijk} \partial_j v_k = (\nabla \times v)^i \quad (14)$$

and thus $-\frac{\partial E}{\partial t} + \epsilon^{jik} \partial_i B^k = 0$, or

$$\nabla \times B = \frac{\partial E}{\partial t} \quad (15)$$

2. With this construction, we have

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad (16)$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\gamma)} \partial^\nu A_\gamma - \mathcal{L} \delta^{\mu\nu} + \partial_\lambda F^{\mu\lambda} A^\nu \quad (17)$$

$$= -F^{\mu\gamma} \partial^\nu A_\gamma + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} + \partial_\lambda (F^{\mu\lambda} A^\nu) \quad (18)$$

$$= F^{\mu\iota} (\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} - \partial_\lambda F^{\lambda\mu} A^\nu \quad (19)$$

$$= F^{\mu\iota} F_\iota^\nu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} - (0) A^\nu \quad (20)$$

This is now a viable energy-momentum tensor. We now $T^{\hat{0}0}$ and $T^{\hat{0}i}$:

$$T^{\hat{0}0} = F^{0\iota} F_\iota^0 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (21)$$

$$= E^\iota E_\iota + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (22)$$

We then have

$$\langle, \rangle = tr(\overline{F^{\mu\nu}} F^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (23)$$

$$= tr \left(\begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix} \right) \quad (24)$$

This is equal to $2(B^2 - E^2)$. Thus we have that $T^{\hat{\mu}\nu} = E^2 + \frac{1}{4} 2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $T^{\hat{0}i}$, we have

$$T^{\hat{0}i} = F^{0j} F_j^i + \frac{1}{2}(B^2 - E^2)g^{0i} \quad (25)$$

$$= E^j \epsilon_{jik} B^k g^{mi} + \frac{1}{2}(B^2 - E^2)g^{0i} \quad (26)$$

$$= \mathbf{E} \times \mathbf{B} = \mathbf{S} \quad (27)$$

□

Question 6. Consider the the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \quad (28)$$

(a) Find the conjugate momenta to $\phi(x)$, $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) \quad (29)$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m .

(c) Rewrite the conserved charge

$$Q = \int d^3x \frac{i}{2} (\phi^* \pi^* - \pi \phi) \quad (30)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the

generalization of part (c), and the other three given by

$$Q^i = \int d^3x \frac{i}{2} (\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b) \quad (31)$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Peskin & Schroeder, 2.2.

Proof. (a) We have that $p(\mathbf{x}) = \frac{\partial L}{\partial \dot{\phi}(\mathbf{x})} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* g_\nu^\mu \partial^\nu \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\dot{\phi}^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi)$. Thus, $\pi = \dot{\phi}^*$. Similarly, $\pi^* = \dot{\phi}$. Since ϕ, ϕ^* are the dynamical variables, the canonical commutation relations are

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (32)$$

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0 \quad (33)$$

from quantization of the Klein-Gordon field given in the textbook. Given the equation for the Hamiltonian, we have

$$H = \int d^3x [\sum_{a,b} \pi_i(\mathbf{x}) \dot{\phi}_i(\mathbf{x}) - \mathcal{L}] \quad (34)$$

$$= \int d^3x [\pi^* \dot{\phi}^* + \pi \dot{\phi} - \mathcal{L}] \quad (35)$$

$$= \int d^3x [\dot{\phi} \dot{\phi}^* + \dot{\phi}^* \dot{\phi} - \mathcal{L}] \quad (36)$$

$$= \int d^3x [2\dot{\phi} \dot{\phi}^* - \dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \quad (37)$$

$$= \int d^3x [\dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \quad (38)$$

$$= \int d^3x [\pi^* \pi + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi] \quad (39)$$

We want to compute $i\frac{\partial\phi}{\partial t}$ via the Heisenberg Equation of Motion, so we calculate $[\phi, H]$.

$$i\frac{\partial\phi}{\partial t} = [\phi, H] \quad (40)$$

$$= [\phi(x'), \int d^3x (\pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi)] \quad (41)$$

$$= \int d^3x [\phi(x'), \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi] \quad (42)$$

$$= \int d^3x ([\phi(x'), \pi^* \pi] + [\phi, \nabla\phi^* \cdot \nabla\phi] + m^2 [\phi, \phi^* \phi]) \quad (43)$$

$$= \int d^3x \delta^{(3)}(x' - x) i\pi^*(x) \quad (44)$$

$$= i\pi^*(x) \quad (45)$$

$$i\frac{\partial\pi^*}{\partial t} = [\pi^*, H] \quad (46)$$

$$= [\pi^*(x'), \int d^3x (\pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi)] \quad (47)$$

$$= \int d^3x [\pi^*(x'), \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi] \quad (48)$$

$$(\text{integrating by parts}) = \int d^3x ([\pi^*(x'), \pi^* \pi] + [\pi^*(x'), \phi^* (-\nabla^2 + m^2) \phi]) \quad (49)$$

$$= \int d^3x \delta^{(3)}(x' - x) (-i) (-\nabla^2 + m^2) \phi(x) \quad (50)$$

$$= i(\nabla^2 - m^2) \phi \quad (51)$$

Since $i\frac{\partial\phi}{\partial t} = i\pi^*$ and $i\frac{\partial\pi^*}{\partial t} = i(\nabla^2 - m^2)\phi$, so $\frac{\partial^2\phi}{\partial t^2} = (\nabla^2 - m^2)\phi$, which is the Klein-Gordon equation.

(b) Since ϕ satisfies the Klein-Gordon equation, and, in the same way, so does ϕ^* , we take the Fourier transform to gain more insight into $\nabla^2\phi$:

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}) \Rightarrow \quad (52)$$

$$[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi(\mathbf{p}) = 0, \quad [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi^*(\mathbf{p}) = 0 \quad (53)$$

We write ϕ in terms of two real valued scalar free fields ψ_1, ψ_2 , of which we already know the theory:

$$\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \quad \phi^* = \frac{\psi_1 - i\psi_2}{\sqrt{2}} \quad (54)$$

Since ψ_1, ψ_2 are independent free fields, both must satisfy the harmonic oscillator equation:

$$\begin{aligned} \frac{1}{\sqrt{2}}[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 &= 0, \quad \frac{\pm i}{\sqrt{2}}[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow \\ [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 &= 0, \quad [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow \\ \omega_1 &= \sqrt{p_1^2 + m^2}, \quad \omega_2 = \sqrt{p_2^2 + m^2} \end{aligned}$$

Since the frequencies of the oscillators have independent momentums and ϕ is not hermitian, we create two different creation and annihilation operators:

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \frac{i}{\sqrt{2\omega_i}}p_i, \quad a_i^\dagger = \sqrt{\frac{\omega_i}{2}}q_i - \frac{i}{\sqrt{2\omega_i}}p_i, \quad i \in \{1, 2\} \quad (55)$$

where $q_1 = \phi, q_2 = \phi^*, p_1 = \pi, p_2 = \pi^*$, with the notation in the spirit of Peskin and Schroeder. These creation operators, given their frequencies, represent creating two different particles with mass m . From the theory of a real-valued scalar free field, we know that

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (56)$$

$$\phi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \quad (57)$$

The two different operators ensure that ϕ is not hermitian. From above we know that $\pi = \dot{\phi}^*, \pi^* = \dot{\phi}$, and, using our real-valued scalar free field as reference, we have

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \quad (58)$$

$$\pi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (59)$$

Finally, we rewrite our Hamiltonian in terms of our operators:

$$H = \int d^3\mathbf{x} (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) \quad (60)$$

$$= \int d^3\mathbf{x} \left(\int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}') e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} - a_1(\mathbf{p}) a_2(\mathbf{p}') e^{i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} \right. \quad (61)$$

$$\left. - a_2^\dagger(\mathbf{p}) a_1^\dagger(\mathbf{p}') e^{-i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}') e^{i(\mathbf{p}'-\mathbf{p}) \cdot \mathbf{x}} \right\} \quad (62)$$

$$+ \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} [-i\mathbf{p} a_1^\dagger(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}} + i\mathbf{p} a_2(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}}] \quad (63)$$

$$\cdot [i\mathbf{p}' a_1(\mathbf{p}') e^{i\mathbf{p}' \cdot \mathbf{x}} - i\mathbf{p}' a_2^\dagger(\mathbf{p}') e^{-i\mathbf{p}' \cdot \mathbf{x}}] \quad (64)$$

$$+ m^2 \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}') e^{i(\mathbf{p}'-\mathbf{p}) \cdot \mathbf{x}} + a_1^\dagger(\mathbf{p}) a_2^\dagger(\mathbf{p}') e^{-i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} \quad (65)$$

$$+ a_2(\mathbf{p}) a_1(\mathbf{p}') e^{i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}') e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} \} \quad (66)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p}) a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p}) a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right. \quad (67)$$

$$\left. + \frac{p^2}{2\omega_{\mathbf{p}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}) \} \right) \quad (68)$$

$$+ \frac{m^2}{2\omega_{\mathbf{p}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}) \} \quad (69)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{\omega_{\mathbf{p}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p}) a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p}) a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right. \quad (70)$$

$$\left. + \frac{p^2 + m^2}{2\omega_{\mathbf{p}}} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right] \quad (71)$$

where in (70) the middle two terms have a positive sign because we subtract $\mathbf{p} \cdot \mathbf{p}' = \mathbf{p} \cdot (-\mathbf{p})$.

Furthermore, since $\mathbf{p} \neq -\mathbf{p}$, we can commute our operators. Thus we have the Hamiltonian as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \omega_{\mathbf{p}} \{ a_1 a_1^\dagger + a_2 a_2^\dagger \} \quad (72)$$

Since this Hamiltonian is constructed purely out of constants and operators whose eigenvectors are momentum eigenstates, our Hamiltonian is now diagonalized. The indices 1 and 2 represent the two particles of mass m .

(c) This is just plugging in our values for momentum and position and integrating, like the

previous problem. We have

$$Q = \int d^3\mathbf{x} \frac{i}{2} (\phi^* \pi^* - \pi \phi) \quad (73)$$

$$= \int d^3\mathbf{x} \frac{i}{2} \int d^3\mathbf{p} \int d^3\mathbf{p}' \frac{1}{(2\pi)^6} \frac{1}{2} [(-i)(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}) \quad (74)$$

$$- i(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}})] \quad (75)$$

$$= \int d^3\mathbf{p} \frac{1}{(2\pi)^3} \frac{1}{4} ([a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \quad (76)$$

$$+ [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) - a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]) \quad (77)$$

$$= \frac{1}{2} \int d^3\mathbf{p} \frac{1}{(2\pi)^3} [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \quad (78)$$

This means that this theory has two particle types: one created by $a_1^\dagger(\mathbf{p})$ and one created by $a_2^\dagger(\mathbf{p})$. In examining $[Q, a_i^\dagger] |n\rangle$ for some state n -particle state $|n\rangle$, we can deduce the charge.

It is easy to see that $[a_1, a_2] = 0, [a_i, a_i^\dagger] = 1$ since ψ_1, ψ_2 are independent fields. Thus

$$[Q, a_1^\dagger] = a_1^\dagger, [Q, a_2^\dagger] = -a_2^\dagger \quad (79)$$

This means that the charges are valued at 1 unit for particles created by a_1^\dagger and -1 for particles created by a_2^\dagger .

(d) For two complex scalar fields, the lagrangian is then

$$\mathcal{L} = \partial_\mu \phi_1^* \partial^\mu \phi_1 - m^2 \phi_1^* \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 - m^2 \phi_2^* \phi_2 \quad (80)$$

We then have

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \Delta \phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Delta \phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1^*)} \Delta \phi_1^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2^*)} \Delta \phi_2^* - \mathcal{J}^\mu \quad (81)$$

$$= \partial^\mu \phi_1^* \Delta \phi_1 + \partial^\mu \phi_2^* \Delta \phi_2 + \partial^\mu \phi_1 \Delta \phi_1^* + \partial^\mu \phi_2 \Delta \phi_2^* - \mathcal{J}^\mu \quad (82)$$

$$\rightarrow Q = \int d^3x (\pi_1^* \Delta \phi_1 + \pi_2^* \Delta \phi_2 + \pi_1 \Delta \phi_1^* + \pi_2 \Delta \phi_2^*) \quad (83)$$

If we set

$$\Phi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (84)$$

we rewrite our theory as

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi \quad (85)$$

$$Q = \int d^3x (\dot{\Phi}^\dagger \Delta \Phi + (\Delta \Phi)^\dagger \dot{\Phi}) \quad (86)$$

The symmetry of this lagrangian is

$$\Phi \mapsto M\Phi \quad (87)$$

for $M \in U(2)$. We know that this system should have $U(1)$ symmetry from the above problem. Using the $\det : U(n) \rightarrow U(1)$ map, we have a short exact sequence

$$SU(2) \rightarrow U(2) \rightarrow U(1) \quad (88)$$

giving us $U(2) = SU(2) \times U(1)$. For this reason, since $U(1)$ is just a complex number, each conserved charge from this symmetry has the commutation relations of $SU(2)$. In order to put this into a continuous symmetry picture, we exponentiate an element $\sigma \in SU(2)$ is a factor $i(\alpha_1, \alpha_2)$ and take $\alpha_1, \alpha_2 \rightarrow 1$:

$$\Phi \mapsto e^{i(\alpha_1, \alpha_2)\sigma} \Phi \quad (89)$$

$$\Delta \Phi \mapsto i\sigma \Phi \quad (90)$$

$$\Delta \Phi^* \mapsto -i\sigma \Phi \quad (91)$$

$SU(2)$ is generated by the Pauli matrices, so we have conserved charges

$$Q^i = i \int d^3x (\dot{\Phi}^\dagger \sigma^i \Phi - \Phi^\dagger \sigma^i \dot{\Phi}) \quad (92)$$

$$= i \int d^3x (\phi_a^* \sigma_{ab}^i \pi_b^* - \pi_a \sigma_{ab}^i \phi_b) \quad (93)$$

Generalizing to n independent identical complex scalar fields, we let $\Phi = (\phi_1, \dots, \phi_n)^T$, and our symmetry becomes $U(n) = SU(n) \times U(1)$, meaning the charges we get are of the same form as what we got, but replacing the σ^i with n -dimensional skew-hermitian matrices.

□