Analytic Number Theory Problems

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All mistakes are to be emailed to aszlau@gmail.com. All problems written by Prof. Kannan Soundararajan and Prof. Andrew Glanville's "Multiplicative Number Theory" textbook, unless otherwise marked.

Contents

1 The Prime Number Theorem

1.1 Partial Summation

1.1.1 Different Forms of the Prime Number Theorem

Question 1. *Given the conjecture*

$$\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x \tag{1}$$

where

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m \text{ for } p \text{ prime and } m \geq 1 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

and the conjecture

$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x} \tag{3}$$

Use partial summation to prove that (1) and (3) are equivalent and both are equivalent to the conjecture

$$\theta(x) := \sum_{p \le x} \log(p) = x + o(x) \tag{4}$$

Definition 1. ***Partial Summation:*** *Given a sequence $a_n \in \mathbb{C}$ and a function $f : \mathbb{R} \to \mathbb{C}$, set $S(t) = \sum_{k \le t} a_k$, it is easy to conclude that*

$$\sum_{n=A+1}^{B} a_n f(n) = S(B)f(B) - S(A)f(A) - \sum_{n=A}^{B-1} S(n)(f(n+1) - f(n)) \tag{5}$$

and, if f is continuously differentiable on $[A, B]$, then

$$\sum_{A<n\le B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt \tag{6}$$

Proof. We start with (3). Notice that, if we choose

$$a_n = \begin{cases} 1 \text{ if } n = p \text{ for p prime} \\ 0 \text{ otherwise} \end{cases} \tag{7}$$

and $f(x) = \log x$, then

$$\theta(x) = \sum_{n \le x} a_n f(n) \tag{8}$$
$$= (\sum_{n\le x} a_n) \log x - \int_2^x (\sum_{n\le t} a_n)(\log t)' dt \tag{9}$$
$$= (\sum_{p\le x} 1) \log x - \int_2^x (\sum_{p\le t} 1)\frac{1}{t} dt \tag{10}$$
$$= \pi(x) \log x - \int_2^x \pi(t)\frac{1}{t} dt \tag{11}$$
$$\sim \frac{x}{\log x} \log x - \int_2^x \frac{t}{t \log t} dt \tag{12}$$
$$\sim x - \int_2^x \frac{1}{\log t} dt \tag{13}$$
$$\sim x + (-li(x)) \tag{14}$$

It remains to prove that $(-li(x)) \in o(x)$. Thus we examine the asymptotic behavior of $-li(x)/x$. By L'Hospital's Rule, we have

$$\lim_{x\to\infty} \frac{-li(x)}{x} = \lim_{x\to\infty} -\frac{1}{\log x} \tag{15}$$

Since $\log x$ diverges, we have this limit equal to 0, so (4) is true if and only if $\pi(x) \sim \frac{x}{\log x}$.

Now we move on to the conjecture in (1). We can easily see that

$$\sum_{p \le x} \log p \le \sum_{n \le x} \log n \Rightarrow \tag{16}$$

$$\theta(x) \le \psi(x) \sim \tag{17}$$

$$x + o(x) \le \psi(x) \tag{18}$$

Next we observe that

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p \le x} \log p \sum_{k \le \log_p x} 1 = \sum_{p \le x} \log p [\frac{\log x}{\log p}] \quad \le \sum_{p \le x} \log p \frac{\log x}{\log p} = \sum_{p \le x} \log x \Rightarrow \tag{19}$$

$$\psi(x) \le \pi(x) \log x \tag{20}$$

which, assuming conjecture (3), we then deduce

$$\psi(x) \le\sim \frac{x}{\log x} \log x = x \tag{21}$$

Thus we have

$$\theta \le \psi(x) \le\sim x \tag{22}$$

$$x - li(x) \le \psi(x) \le\sim x \tag{23}$$

using Conjecture (4). Thus we have, assuming Conjecture (3) and Conjecture (4), that

$$\psi(x) = \sum_{n \le x} \Lambda(n) \sim x \tag{24}$$

□

1.1.2 Adding reciprocals

Note: my version of the paper has $\sum_{n \le x}^{N} \frac{1}{N}$. I'm pretty sure the denominator should be n, as that sum is just 1.

Question 2. *Prove that for any integer $N \ge 1$,*

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt \tag{25}$$

Deduce that, for any real $x \geq 1$,

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{26}$$

where γ *is the Euler-Mascheroni constant*

$$\gamma = \lim_{N \to \infty} (\sum_{n=1}^{N} \frac{1}{n} - \log N) = 1 - \int_1^{\infty} \frac{\{t\}}{t^2} dt \tag{27}$$

Note that, for $t \in \mathbb{R}$, $[t]$ is the integral part of t, and $\{t\}$ is the rest of t.

Proof. We use partial summation again. Let $f(x) = \frac{1}{x}$ and $a_n = 1$. Thus, by partial summation, we have

$$\sum_{n \leq x} \frac{1}{n} = [N]\frac{1}{N} + \log 1 + \int_1^N t \frac{1}{t^2} dt \tag{28}$$

$$= [N]\frac{1}{N} + \int_1^N \frac{1}{t^2}(t - \{t\}) dt \tag{29}$$

$$= 1 + \int_1^N \frac{t}{t^2} dt - \int_1^N \frac{\{t\}}{t^2} dt \tag{30}$$

$$= 1 + \log N - \log 1 - \int_1^N \frac{\{t\}}{t^2} dt \tag{31}$$

$$= \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt \tag{32}$$

For any *real* x, we have, through partial summation,

$$\sum_{n \leq x} \frac{1}{n} = [N]\frac{1}{N} + \int_1^N t \frac{1}{t^2} dt \tag{33}$$

$$= \frac{x - \{x\}}{x} + \log N - \int_1^N \frac{\{t\}}{t^2} dt \tag{34}$$

$$= \log N + 1 - \frac{\{x\}}{x} + \int_N^{\infty} \frac{\{t\}}{t^2} dt - \int_1^{\infty} \frac{\{t\}}{t^2} dt \tag{35}$$

$$= \log N + \gamma - \frac{\{x\}}{x} + \int_N^{\infty} \frac{\{t\}}{t^2} dt \tag{36}$$

It remains to prove that $\frac{\{x\}}{x}$ and $\int_N^{\infty} \frac{\{t\}}{t^2} dt$ are in $O(\frac{1}{x})$. Starting with the former, we see that since $\{x\} < 1$, we have that $|\frac{\{x\}}{x}| < \frac{1}{x}$, so $\frac{\{x\}}{x} \in O(\frac{1}{x})$. Similarly, we have

$$|\int_N^{\infty} \frac{\{t\}}{t^2} dt| \leq \int_N^{\infty} |\{t\}||\frac{1}{t^2}| dt \leq \int_N^{\infty} \frac{1}{t^2} dt \in O(\frac{1}{x}) \tag{37}$$

Thus we conclude

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O(\frac{1}{x}) \tag{38}$$

□

1.1.3 log$N$!

Question 3. *For an integer $N\geq 1$, show that*

$$\log N!=N\log N-N+1+\int_1^N\frac{\{t\}}{t}dt \tag{39}$$

Using that $\int_1^x(\{t\}-1/2)dt=(\{x\}^2-\{x\})/2$, show that

$$\int_1^N\frac{\{t\}}{t}dt=\frac{1}{2}\log N-\frac{1}{2}\int_1^N\frac{\{t\}-\{t\}^2}{t^2}dt \tag{40}$$

Conclude that $N!\sim C\sqrt{N}(N/e)^N$, where you can take as fact that

$$C=\exp(1-\frac{1}{2}\int_1^\infty\frac{\{t\}-\{t\}^2}{t^2}dt)=\sqrt{2\pi} \tag{41}$$

Proof. From rules of logarithms, we have $\log N!=\log(N(N-1)...(2)(1))=\log N+\log(N-1)+...+\log 2+\log 1$. We use partial summation once again. Let $a_n=1$, and $f(x)=\log x$. From the partial summation formula given by (6), we have

$$\begin{aligned}\log N! &= N\log N-0-\int_1^N(\sum_{n\leq t}1)\frac{dt}{t} && (42)\\ &= N\log N-\int_1^N\frac{[t]}{t}dt && (43)\\ &= N\log N-\int_1^N\frac{t-\{t\}}{t}dt && (44)\\ &= N\log N-\int_1^N dt+\int_1^N\frac{\{t\}}{t}dt && (45)\\ &= N\log N-N+1+\int_1^N\frac{\{t\}}{t}dt && (46)\end{aligned}$$

As for the next part, we notice (38):

$$\begin{aligned}
\int_1^N \frac{\{t\}}{t}dt &= \int_1^N \frac{\{t\}-\frac{1}{2}+\frac{1}{2}}{t}dt && (47)\\
&= \int_1^N \frac{1}{t}(\{t\}-\frac{1}{2})dt + \int_1^N \frac{1}{2t}dt && (48)\\
&= \frac{1}{t}\frac{\{t\}^2-\{t\}}{2}|_1^N - \int_1^N \frac{1}{2}\frac{\{t\}^2-\{t\}}{-t^2}dt + \frac{1}{2}\log N + \frac{1}{2}\log 1 && (49)\\
&= 0 + \int_1^N \frac{1}{2}\frac{\{t\}^2-\{t\}}{t^2}dt + \frac{1}{2}\log N && (50)\\
&= \frac{1}{2}\log N - \int_1^N \frac{1}{2}\frac{\{t\}-\{t\}^2}{t^2}dt && (51)
\end{aligned}$$

Thus we have

$$\begin{aligned}
\log N! &= N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^N \frac{\{t\}-\{t\}^2}{t^2}dt && (52)\\
&= N\log N - N + 1 + \frac{1}{2}\log N - \frac{1}{2}\int_1^\infty \frac{\{t\}-\{t\}^2}{t^2}dt + \frac{1}{2}\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt && (53)\\
&= N\log N - N + \frac{1}{2}\log N + \log C + \frac{1}{2}\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt && (54)
\end{aligned}$$

Taking the exponent of both sides, we get

$$N! = N^N \cdot \frac{1}{e^N}\sqrt{N}\cdot C\sqrt{e^{\int_N^\infty \frac{\{t\}-\{t\}^2}{t^2}dt}} \qquad (55)$$

Now we examine the integral in the exponent. First we split it up.

$$\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2}dt - \frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2}dt \le |\frac{1}{2}\int_N^\infty \frac{\{t\}}{t^2}dt| - |\frac{1}{2}\int_N^\infty \frac{\{t\}^2}{t^2}dt| \le \frac{1}{2}\int_N^\infty |\{t\}||\frac{1}{t^2}|dt - \frac{1}{2}\int_N^\infty |\{t\}^2|\frac{1}{t^2}dt \qquad (56)$$

It is easy to see that the limit as N approaches infinity the value of this integral converges to 0, so

$$\begin{aligned}
N! \sim N^N \cdot \frac{1}{e^N}\sqrt{N}\cdot C\sqrt{e}^0 \Rightarrow && (57)\\
N! \sim C\sqrt{N}(N/e)^N && (58)
\end{aligned}$$

□

Definition 2. *The **Riemann Zeta Function** is given by*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p (1-\frac{1}{p^s})^{-1} \qquad (59)$$

1.1.4 The Riemann Zeta Function

Note: My copy of the problem has a later part. In another copy of the book, this part is labeled as a problem which requires complex analysis, something I haven't learned yet, so I'm skipping that for now.

Question 4. *Prove that for Re(s)>1,*

$$\zeta(s) = s\int_1^\infty \frac{[y]}{y^{s+1}}dy = \frac{s}{s-1} - s\int_1^\infty \frac{\{y\}}{y^{s+1}}dy \tag{60}$$

Proof. We use partial summation again. We see that, for $a_n = 1, f(x) = \frac{1}{x^s}$, we have $\zeta(s) = \sum_1^\infty a_n f(n)$, so, using the usual partial summation formula,

$$\begin{aligned}
\zeta(s) &= \sum_1^\infty a_n f(n) = \lim_{N\to\infty} \sum_1^N a_n f(n) && (61)\\
&= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - \int_1^N [y]\frac{1}{y^s}dy] && (62)\\
&= \lim_{N\to\infty} [[N]\frac{1}{N^s} - [1]\frac{1}{1^s} - [N]\frac{1}{N^s} + [1]\frac{1}{1^s} + s\int_1^N [y]\frac{1}{y^{s+1}}dy] && (63)\\
&= \lim_{N\to\infty} [s\int_1^N [y]\frac{1}{y^s}dy] && (64)\\
&= s\int_1^\infty [y]\frac{1}{y^s}dy && (65)
\end{aligned}$$

We write this final integral in a different way:

$$\begin{aligned}
s\int_1^\infty [y]\frac{1}{y^s}dy &= \lim_{N\to\infty} s\int_1^N \frac{y-\{y\}}{y^{s+1}}dy && (66)\\
&= s\int_1^N \frac{y}{y^{s+1}}dy - s\int_1^N \frac{\{y\}}{y^{s+1}}dy && (67)\\
&= \lim_{N\to\infty} [s\int_1^N \frac{1}{y^s}dt - s\int_1^N \frac{\{y\}}{y^{s+1}}dy] && (68)\\
&= \lim_{N\to\infty} [-s\frac{1}{s-1}(\frac{1}{t^{s-1}}|_1^N) - s\int_1^N \frac{\{y\}}{y^{s+1}}dy] && (69)\\
&= \lim_{N\to\infty} [-\frac{s}{s-1}(\frac{1}{N^{s-1}} - \frac{1}{1^{s-1}}) - s\int_1^N \frac{\{y\}}{y^{s+1}}dt] && (70)
\end{aligned}$$

Since $Re(s) > 1$, we have $Re(s) - 1 > 0$, so, evaluating the limit, we get that this expression is equivalent to

$$\frac{-s}{s-1}(0-1) - s\int_1^N \frac{\{y\}}{y^{s+1}}dt = \frac{s}{s-1} - s\int_1^N \frac{\{y\}}{y^{s+1}}dt \tag{71}$$

□

1.2 Chebyshev's Elementary Estimates

1.2.1 $\lim \frac{\psi(x)}{x}$

Question 5. *Prove that*

$$\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \geq 1 \geq \lim_{x\to\infty} \inf \frac{\psi(x)}{x} \tag{72}$$

so that if $\lim_{x\to\infty} \frac{\psi(x)}{x}$ *exists, it must be equal to 1.*

Note that $\log n = \sum_{d|n} \Lambda(d)$, so

$$\sum_{n<x} \log n = \sum_{n\leq x} \sum_{n=dk} \Lambda(d) = \sum_{k=1}^{\infty} \psi(\frac{x}{k}) \tag{73}$$

and, by Stirling's Formula, we have

$$\sum_{k=1}^{\infty} \psi(\frac{x}{k}) = x\log x - x + O(\log x) \tag{74}$$

Proof. We start with $\lim\sup \frac{\psi(x)}{x} \geq 1$. Suppose not. Then there exists $\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, $\frac{\psi(x)}{x} \leq (1-\epsilon)$. Then we have

$$\sum_{k=1}^{\infty} \psi(x/k) \leq \sum_{k=1}^{x/x_0} \psi(x/k) + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{75}$$

$$x\log x - x + O(\log x) \leq (1-\epsilon)x \sum_{k=1}^{x/x_0} \frac{1}{k} + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{76}$$

$$x\log x - x + O(\log x) \leq (1-\epsilon)x(\log x - \log x_0 + \gamma + O(\frac{1}{x})) + \sum_{x/x_0<k}^{\infty} \psi(x/k) \tag{77}$$

$$\epsilon x\log x + O(x) \leq \sum_{x/x_0<k}^{\infty} \psi(x/k) \leq \psi(x_0)(x-x_0) \tag{78}$$

where γ is the Euler-Mascheroni constant. Since the LHS is $O(xlogx)$ and the RHS is $O(x)$, this cannot be true for all $x > x_0$. Thus $\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \geq 1$.

We follow the same approach as with lim sup. We suppose by contradiction that there exists

$\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, such that $\frac{\psi(x)}{x} \geq (1+\epsilon)$. We then have

$$(1+\epsilon)x \sum_{k=1}^{x/x_0} \frac{1}{k} + \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq \sum_{k=1}^{\infty} \psi(x/k) \tag{79}$$

$$(1+\epsilon)x[\log \frac{x}{x_0} + \gamma + O(\frac{1}{x})] + \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq x \log x - x + O(\log x) \tag{80}$$

$$\frac{1}{x} \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq -\epsilon \log x - 1 + (1+\epsilon) \log x_0 - (1+\epsilon)\gamma \tag{81}$$

Since $\psi(x)$ is strictly nonnegative for all $x \in \mathbb{Z}^+$, this cannot be true for all $x > x_0$. Thus

$$\lim_{x \to \infty} \sup \frac{\psi(x)}{x} \geq 1 \geq \lim_{x \to \infty} \inf \frac{\psi(x)}{x} \tag{82}$$

□

1.2.2 Proof of Bertrand's postulate

Question 6. *Given that*

$$\psi(2x) - \psi(x) + \psi(2x/3) \geq x \log 4 + O(\log x)$$

Proof that there exists a prime between N and $2N$ for large N.

It is given to us that $\psi(x) \leq x \log 4 + O((\log x)^2)$. (To see this, just subtract $\psi(2x) - \psi(x)$ using the approximation given in the previous problem). Therefore, we get that $\pi(x) \leq (\log 4 + o(1)) \frac{x}{\log x}$

Proof. First we rearrange terms and take the given bound, resulting in

$$\psi(2x) - \psi(x) \geq x \log 4 - \psi(2x/3) + O(\log x) \Rightarrow \tag{83}$$

$$\psi(2x) - \psi(x) \geq \frac{1}{3} x \log 4 + O(\log x) \tag{84}$$

Notice that $\psi(x) = \sum_{p \leq x} \log p \lfloor \frac{\log x}{\log p} \rfloor$. Then this inequality becomes

$$\sum_{p \leq 2x} \log p \lfloor \frac{\log 2x}{\log p} \rfloor - \sum_{p \leq x} \log p \lfloor \frac{\log x}{\log p} \rfloor \geq \frac{1}{3} x \log 4 + O(\log x) \tag{85}$$

The LHS is less than or equal to

$$\begin{aligned}
\sum_{p \leq 2x} \log 2x - \sum_{p \leq x} \log x &\geq \frac{1}{3} x \log 4 + O(\log x) \\
[\pi(2x) - \pi(x)] \log x + \pi(2x) \log 2 &\geq \frac{1}{3} x \log 4 + O(\log x) \\
\pi(2x) - \pi(x) &\geq \frac{x}{3 \log x} \log 4 - \pi(2x) \frac{\log 2}{\log x} + O(1) \\
\pi(2x) - \pi(x) &\geq \frac{2x}{3} \frac{\log 2}{\log x} - \pi(2x) \frac{\log 2}{\log x} + O(1)
\end{aligned}$$

Thus we have that there exists a prime number between $2x$ and x if $\pi(2x) < \frac{2x}{3}$ for large x. As we bounded $\pi(x) \leq (\log 4 + o(1)) \frac{x}{\log x} < \frac{x}{3}$, this is the case for large enough x. $\square$