Math 283 Problem Set 2

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Question 1. *Let P be a presentation of a group with l generators and m relations. Let X_P be the four-manifold obtained from $\#^l(S^1 \times S^3)$ by doing surgery along loops that represent the relations. Calculate the Euler characteristic of X_P in terms of l and m.*

Proof. The Euler characteristic is given, for finite CW-complexes such as these, as the alternating sum

$$\chi = k_0 - k_1 + k_2 - ... \tag{1}$$

where k_n is the number of cells in the complex. A cellular decomposition of S^n is a single 0-cell and a single n-cell. Thus $\chi(S^1) = \chi(S^3) = 0$. The Euler characteristic of $(S^1 \times S^3)$ is then 0×0. For an l-fold connected sum $N := (S^1 \times S^3)\#...\#(S^1 \times S^3)$, we remove a 4-ball from each manifold and glue along the boundary, S^3. Thus for each connected sum we remove 2 4-cells, add the resulting Euler characteristics, and subtract that of their intersection, S^3. Thus, since we have $(l-1)$ sums, we have

$$\begin{aligned}
\chi(N) &= (l-1)\chi((S^1 \times S^3) - B^4) - (l-1)\chi(S^3) && (2)\\
&= (l-1)[\chi(S^1 \times S^3) - 2\chi(B^4)] - (l-1) \cdot 0 && (3)\\
&= (l-1)[0-2] && (4)\\
&= 2 - 2l && (5)
\end{aligned}$$

Thus $\chi(N) = 2 - 2l$.

For each relation, we perform a surgery:

$$(\#^l(S^1 \times S^3) - (S^1 \times B^3)) \cup_{S^1 \times S^2} (B^2 \times S^2) \tag{6}$$

Thus for each relation we excise out $S^1 \times B^3$ and glue in $B^2 \times S^2$, so we subtract $m \cdot \chi(S^1 \times B^3)$ and add $m \cdot \chi(B^2 \times S^2)$. We have $\chi(S^1) = 0$, so $\chi(S^1 \times B^3) = \chi(S^1 \times S^2) = 0$. We have $\chi(S^2) = 2$ and $\chi(B^2) = +k_2 = 1$, so $\chi(S^2 \times B^2) = 2$, so we add $2m$ to the Euler characteristic of X_P. Therefore

$\chi(X_P) = 2 - 2l + 2m$. $\square$

Question 2. *Let X be a compact topological 4-manifold with possibly nonempty boundary ∂X. Suppose that X is oriented, and therefore it has a fundamental class $[X] \in H_4(X, \partial X)$. Consider the symmetric bilinear form*

$$Q : H^2(X, \partial X; \mathbb{Z}) \otimes H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z},\ Q(a, b) = \langle a \smile b, [X] \rangle \tag{7}$$

Prove that Q is nondegenerate (det(Q) $\neq$0) if and only if ∂X is a rational homology sphere, i.e. $H(\partial X; \mathbb{Q}) \cong H(S^3; \mathbb{Q})$.

Proof. Notice that the cup product is Poincaré dual to the intersection product. If $i_* : H_2(\partial X; \mathbb{Q}) \to H_2(X; \mathbb{Q})$ is the induced homomorphism given by the inclusion, notice that $[i_*(\alpha)]^* \smile [\beta] = 0$, for $\alpha \in H_2(\partial X; \mathbb{Q}), \beta \in H_2(X; \mathbb{Q})$. This is because the cup product is Poincaré Dual to intersections, so either the singular chain β does not intersect with that of $i_*(\alpha)$, or $\beta \in H_2(\partial X)$, in which case $[i_*(\alpha)]^* \smile [\beta] = 0 \in H^4(\partial X; \mathbb{Q}) = 0$. Since the map i_* must always be the trivial map, the only way the pairing $[\alpha]^* \smile [\beta]^*$ can be nondegenerate is if $H_2(\partial X; \mathbb{Q}) = 0$. Since ∂X is a boundary itself, and since X is compact orientable, ∂X is closed and orientable, and so we can apply Poincaré Duality. Thus $H_0(\partial X; \mathbb{Q}) \cong H_3(\partial X; \mathbb{Q}) \cong \mathbb{Q}, H_1(\partial Z; \mathbb{Q}) \cong H_2(\partial X; \mathbb{Q}) \cong 0$, so ∂X is a rational homology sphere.

If ∂X is a rational homology sphere, then we have the exact sequence

$$... \to H^1(\partial X; \mathbb{Q}) \to H^2(X, \partial X; \mathbb{Q}) \to H^2(X; \mathbb{Q}) \to H^2(\partial X; \mathbb{Q}) \to ... \tag{8}$$

$$... \to 0 \to H^2(X, \partial; \mathbb{Q}) \to H^2(X; \mathbb{Q}) \to 0 \to ... \tag{9}$$

And therefore $H^2(X, \partial X; \mathbb{Q}) \cong H^2(X; \mathbb{Q})$, for any coefficients. Thus we have a nondegenerate pairing due to Poincaré Duality given by $\langle a \smile b, \alpha \rangle \to \mathbb{Q}$, for $\alpha \in H^4(X; \mathbb{Q}) \cong \mathbb{Q}$. Thus Q is nondegenerate. $\square$

Question 3. *An n-knot is the image of a smooth embedding $f : S^n \hookrightarrow S^{n+2}$. Given a 1-knot $K \subset S^3$, we can construct a 2-knot $\Sigma_K \subset S^4$, called the spun knot of K, as follows. Around a point $p \in K$, choose a small ball $U = B^3(p, \varepsilon) \subset S^3$ such that K intersects U in a standardly* *embedded interval $I \subset U$. Let $B = S^3 \backslash U$ and $\beta = K \backslash I \subset B$. (Thus, β is a "knotted arc" in the three-dimensional ball B.) Identify S^4 with the result of gluing*

$$(\partial B \times D^2) \cup_{\partial B \times S^1} (B^1 \times S^1) \tag{10}$$

and let

$$\Sigma_K = (\partial \beta \times D^2) \cup_{\partial \beta \times S^1} (\beta \times S^1) \subset S^4 \tag{11}$$

1. *Prove that $\pi_1(S^4 \backslash \Sigma_K)$ is isomorphic to $\pi_1(S^3 \backslash K)$*

2. *Consider the torus knot $T_{p,q} \subset S^3$ obtained by taking a standard embedding of T^2 in S^3 and pre-composing it with the map $\psi : S^1 \to T^2$, $\psi(x) = (px, qx)$, where we identified $S^1 \cong \mathbb{R}\mathbb{Z}$ and $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Calculate $\pi_1(S^3 \backslash T_{p,q})$.*

3. *Conclude that there is no diffeomorphism $F : S^4 \to S^4$ taking the standard sphere $S^2 \subset S^4$ into the spun trefoil $\Sigma_{T_{3,2}}$.*

Proof. 1. We have $S^4 \backslash \Sigma_K$ as

$$((\partial B \backslash \partial \beta) \times D^2) \cup_{\partial B \backslash \partial \beta \times S^1} ((B \backslash \beta) \times S^1) \tag{12}$$

Notice that $\partial B = S^2, \partial \beta = S^0$. Thus $\partial B \backslash \partial \beta = S^1$, so we get

$$S^4 \backslash \Sigma_K = (S^1 \times D^2) \cup_{S^1 \times S^1} (B \backslash \beta \times S^1) \tag{13}$$

From the Seifert-Van-Kampen Theorem we compute the fundamental group as

$$\begin{aligned}
\pi_1(S^4 \backslash \Sigma_K) &\cong (\pi_1(S^1 \times D^2) * \pi_1(B \backslash \beta \times S^1))/\pi_1(S^1 \times S^1) &(14)\\
&\cong (\pi_1(S^1) \times \pi_1(D^2)) * (\pi_1(B/\beta) \times \pi_1(S^1))/(\pi_1(S^1) \times \pi_1(S^1)) &(15)\\
&\cong (\mathbb{Z} \times 1) * (\pi_1(B \backslash \beta) \times \mathbb{Z})/\mathbb{Z} \times \mathbb{Z} &(16)\\
&\cong \pi_1(B \backslash \beta) &(17)
\end{aligned}$$

$B\backslash\beta$ can be thought of $\mathbb{R}^3$ with β glued to two holes in S^2 removed, because β is compact and its boundary is that of the removed twice-punctured sphere. Compressing this S^2, we find that $B\backslash\beta$ is homotopy equivalent to $\mathbb{R}^3\backslash K$, as the sphere simply connected the loop of the gap left behind by β. Since they are homotopy equivalent, $\pi_1(S^4\backslash\Sigma_K) \cong \pi_1(S^3\backslash K)$

2. Since T^2 is compact, and we can consider $\mathbb{R}^3\backslash T_{p,q}$ under the identification $S^3 = \mathbb{R}^3 \cup \{\}$. The graph $\psi(x) = (px, qx) \subset \mathbb{R}^2$ before quotienting by $\mathbb{Z}^2$ can be thought of as a line segment from (0,0) to (p, q). If this segment intersects $\mathbb{Z}^2$ n times before it ends (before $x < 1$), then we get the wedge sum of $(n + 1)$ torus knots $T_{\overline{p},\overline{q}}$. We henceforth assume p, q are coprime. Under the quotient $(px, qx) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}, x \in [0, 1]$, we have q loops (to avoid confusion, we'll call these loopty-loops). Each loopty-loop corresponds to an element in $\pi_1(S^3\backslash T_{p,q}) \cong \pi_1(\mathbb{R}^3\backslash T_{p,q})$, when a loop is made from a path from the basepoint through this loopty-loop. Since each such loop can be homotoped around the loopty-loop to another loopty-loop, this subgroup of loopty-loops under the quotient of this factor is singly-generated. After q of these loops, we arrive back at the origin. We can do the same by quotienting out the first factor: $(px, qx) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. After p of these loops, we arrive back at the origin as well. Thus $\pi_1(\mathbb{R}^3\backslash T_{p,q})$ has two generators a, b such that $a^p = b^q$, and in general (for p, q not necessarily coprime) $\pi_1(S^3\backslash T^{p,q)=\langle a_1,b_1,\ldots,a_k,b_k|a_i^p=b_i^q\forall 1\leq i\leq k\rangle}$ where k is the common factor of p and q.

3. The standard sphere is the spun knot of the unknot, which has $\pi_1(S^3\backslash K) \cong \pi_1(\mathbb{R}^3\backslash S^1) \cong \mathbb{Z}$. This is not isomorphic to $\pi_1(S^4\backslash\Sigma_{T_{3,2}}) \cong \pi_1(S^3\backslash T_{3,2}) \cong \langle a, b|a^3 = b^2\rangle$, because, for instance, the latter is not abelian.

□

Question 4. *Let $\Sigma \subset S^4$ be a 2-knot. For any $\theta \in [0, 2\pi]$, let $r_\theta : S^2 \to S^2$ be the rotation by angle θ about a fixed axis. Take a tubular neighborhood of Σ of the form $V \cong= S^2 \times D^2$ and re-glue it by the diffeomorphism*

$$\varphi : S^2 \times S^1 \to S^2 \times S^1, \varphi(x, \theta) = (r_\theta(x), \theta), \tag{18}$$

where we identified $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. We obtain a four-manifold

$$G_\Sigma = (S^4 \backslash V) \cup_{\partial V} (S^2 \times D^2), \tag{19}$$

where the boundaries are identified via the map φ. This is called the Gluck twist of S^4 along Σ. Prove that if Σ is the standard sphere $S^2 \subset S^4$, then G_Σ is diffeomorphic to S^4.

Proof. We have

$$G_\Sigma = (S^4 \backslash V) \cup_{\partial V} (S^2 \times D^2) \tag{20}$$

Identify $S^4 = \partial D^5 = \partial(D^3 \times D^2) = \partial D^3 \times D^2 \cup_{S^2 \times S^1} D^3 \times \partial D^2 = S^2 \times D^2 \cup_{S^2 \times S^1} D^3 \times S^1$. We give charts $(\psi_i(x_1', x_2', x_3'), \varphi_i(x_4', x_5'))$, and our diffeomorphism is given by $f(x_1, x_2, x_3, x_4, x_5) = (r_{(x_1^2+x_2^2+x_3^2)\theta}, \cos((x_4^2+x_5^2)\theta), \sin((x_4^2+x_5^2)\theta))$. This results in G_Σ, and is a diffeomorphism, because f is composed of trigonometric functions with polynomial arguments. These are both smooth, and r_θ and θ are both bijective. Df_x is a block-diagonal matrix with the first block the differential of a 3d rotation matrix and 2nd block that of a 2d rotation matrix. Thus G_Σ is diffeomorphic to S^4. □

Question 5. *For $n \geq 1$, prove that the positive definite forms $Q_1 = E_8 \oplus n\langle 1 \rangle$ and $Q_2 = (8+n)\langle 1 \rangle$ are not equivalent over $\mathbb{Z}$, even though they have equal rank, signature and parity. (Hint: Count the number of vectors of length 1.)*

Proof. We can easily see that Q_2 has $8+n$ vectors of length 1, the $\mathbf{e}_i, 1 \leq i \leq 8+n$. Now we count vectors of length 1 in Q_1. We already have n by the same logic, so we need to count the number of vectors of length 1 for E_8. Splitting up E_8 into two index-shifting matrices, $2 * Id$, and one more extra matrix, we get the length from Q_1 of a vector with entries $a_i \in \mathbb{Z}$ such that

$$\sum_{i=1}^{8} 2a_i^2 + \sum_{i=1}^{6} 2a_i a_{i+1} + 2a_5^2 = 1 \tag{21}$$

This, however, is impossible for $a_i \in \mathbb{Z}$, because we can simply divide each side by 2, and we get a sum of multiples of integers supposed to equal $\frac{1}{2}$. Thus, no matter how we change bases, we cannot get $8+n$ vectors of length 1 from Q_1, so $Q_1 \neq Q_2$. □

Question 6. *Let $P(d)$ denote the vector space of all homogeneous polynomials in $\mathbb{C}[z_0, z_1, z_2, z_3]$ of degree $d \geq 1$. For $f = f(z_0, ..., z_3) \in P(d)$, let*

$$Z(f) = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 | f(z_0, z_1, z_2, z_3) = 0\} \tag{22}$$

be the zero set of f.

1. *Calculate the dimension of $P(d)$ as a vector space.*
2. *Consider the set*

$$Z = \{(z, f) \in \mathbb{C}P^3 \times P(d) | f(z) = 0\}. \tag{23}$$

   Show that Z is a submanifold of $\mathbb{C}P^3 \times P(d)$.

3. *Consider the map $p : Z \to P(d)$ given by composing the inclusion of Z in $\mathbb{C}P^3 \times P(d)$ with projection to the second factor. Show that $f \in P(d)$ is a regular value of p if and only if the complex valued functions*

$$f, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \tag{24}$$

   have no common zero in $\mathbb{C}^4 \backslash \{0\}$.

4. *Prove that the set $U(d) \subset P(d)$ of regular values of p is connected.*
5. *Deduce that for $f \in U(d)$, the zero set $Z(f)$ is a smooth four-dimensional manifold whose diffeomorphism type is independent of f. We let Z_d denote any manifold of the form $Z(f)$ for $f \in U(d)$.*

Proof. 1. For our vector space, our ground field is $\mathbb{C}$, with basis vectors all possible words of $z_0, ..., z_3$ of d characters (because our polynomials are homogeneous), independent of order (this is because, in adding a single degree, we have 4 choices for which variable we multiply this "basis vector" by). We allow repeats of variables, and we need to choose 4 variables from a certain number of variables, so we have 4 choose d with repetition, i.e. $h := \dim(P(d)) = \binom{4+d-1}{d}$.

2. Notice that, for $f \in P(d)$, if $f(z) = 0$, then $\lambda f(z) = 0, \forall \lambda \in \mathbb{C}$, so the $f \in P(d)$ such that $f(z) = 0$ form a subspace of the vector space $P(d)$. Since $P(d)$ is a $4d$-dimensional complex vector space, it is isomorphic to $\mathbb{C}^{4n}$, and similarly such $f(z) \in P(d)$ are isomorphic to a subspace of $\mathbb{C}^{4n}$, and thus a submanifold of $\mathbb{C}^{4n}$. Now choose some $f \in P(d)$. Since f is a homogeneous polynomial, the Jacobian matrix is

$$[\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}] \tag{25}$$

has full rank at every point except $(0, ..., 0)$, which is not in $\mathbb{C}P^3$. This is because each $\frac{\partial f}{\partial z_i}$ consists of terms with z_0 or $d-1$ factors of other z_is. By the Preimage theorem the zero locus of f is then a submanifold of $\mathbb{C}P^3$. With the product topology, Z is then a submanifold of $\mathbb{C}P^3 \times P(d)$.

3. We examine the Jacobian of the projection, since the Jacobian of the inclusion map is the identity:

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_0} & \cdots & \frac{\partial f_1}{\partial z_3} & \frac{\partial f_1}{\partial v_1} & \cdots & \frac{\partial f_1}{\partial v_h} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_h}{\partial z_0} & \cdots & \frac{\partial f_h}{\partial z_3} & \frac{\partial f_h}{\partial v_1} & \cdots & \frac{\partial f_h}{\partial v_h} \end{pmatrix} \tag{26}$$

where v_i denotes an ordered basis of $P(d)$, if you want via the natural isomorphism from $P(d)$ to $\mathbb{C}^h$ mapping monomials to basis vectors, where we recall $h = \binom{4+d-1}{d}$, and f_i is f's monomial term corresponding to the monomial v_i. Notice that the last h columns are diagonal with entries the coefficients of our monomials. If f is zero at a particular point, then since it is in Z, we showed above that the first four columns must have full rank. Furthermore, if $\frac{\partial f}{\partial z_i} = 0$ at a particular point, then the other columns in the first four columns are not all zero if and only if not all columns are multiples of each other. Therefore this matrix has full rank if and only if f and all its partial derivatives do not have a shared zero, and therefore f is a regular value with this same condition.

4. If the above conditions fail, the premise of the Implicit function theorem is violated, and all $g \notin U(d)$ give points $g^{-1}(0)$ of codimension 1, as they are described by equations. Therefore,

for $f \in U(d)$, the hypersurfaces described by $f^{-1}(0)$s form a connected subset, because p is continuous.

5. Because $U(d)$ is connected, the hypersurface $f^{-1}(0)$ describes points that can be smoothly deformed into any other smooth hypersurface, via. Since this has codimension 1 in $\mathbb{C}$, it has real dimension 4. *NOTE* According to Ciprian this uses Ehresmann's Fibration Theorem.

□

Question 7. *Given homogeneous polynomials p_i of degree d_i in $n+1$ variables, for $i = 1, ..., n-2$, let*

$$S = S(d_1, ..., d_{n-2}) = \{[z_0 : z_1 : ... : z_n] \in \mathbb{C}P^n | p_i(z_0, ..., z_n) = 0, \forall i\} \tag{27}$$

The subset S is a smooth, simply connected, four-dimensional submanifold of $\mathbb{C}P^n$ whose diffeomorphism type only depends on d_i. The manifold S is called the complete intersection surface of multidegree $(d_1, ..., d_{n-2})$.

1. *Compute the characteristic numbers $c_2(TS)[S]$ and $c_1(TS)^2[S]$.*
2. *Compute the Euler characteristic and signature of S.*
3. *Show that S is spin if and only if $d_i - (n+1)$ is even.*

Proof. 1. For the inclusion $i : S \to \mathbb{C}P^n$, denote x by the pullback $i^*(\omega) \in H^2(S; \mathbb{Z})$ of $\omega = PD([\mathbb{C}P^{n-1}]) \in H^2(\mathbb{C}P^n; \mathbb{Z})$. We proceed as in class. We have

$$T\mathbb{C}P^n|_S = TS \oplus \nu S \tag{28}$$

By the Whitney product formula we have

$$c(T\mathbb{C}P^n|_S) = c(TS)c(\nu S) \tag{29}$$

$$(1+x)^{n+1} = (1 + c_1(TS) + c_2(TS))(1 + c_1(\nu S) + c_2(\nu S)) \tag{30}$$

since $rank(TS) = 2, dim(S) = 4$. We can easily expand this, because $x, c_1 \in H^2(S; \mathbb{Z}), c_2 \in$

$H^4(S;\mathbb{Z})$. Thus we get

$$[\sum_{k=0}^{n+1} \binom{n+1}{k} x^k](1 + c_1(\nu S) + c_2(\nu S))^{-1} = 1 + c_1(TS) + c_2(TS) \quad (31)$$

$$[1 + (n+1)x + \frac{n(n+1)}{2}x^2](1 - c_1(\nu S) + c_1^2(\nu S) - c_2(\nu S)) = 1 + c_1(TS) + c_2(TS) \quad (32)$$

To find $c_i(TS)$, we check which terms are in $H^{2i}(S;\mathbb{Z})$. Thus we have

$$\frac{n(n+1)}{2}x^2 - (n+1)xc_1(\nu S) - c_2(\nu S) + c_1^2(\nu S) = c_2(TS) \quad (33)$$

$$(n+1)x - c_1(\nu S) = c_1(TS) \quad (34)$$

We now need to find $c_1(\nu S)$. Since S is a manifold, all S_{d_i} intersect transversely. Thus the $S_{d_i} \cap S$ is transverse, so we can think of each S_{d_i} being a section of νS. Thus $S_{d_i} \cap S$ is a manifold of the zero section of a line bundle L_i. We then have $c_1(L_i) = PD([S_{d_i} \cap S])$. This is equal to $i^*(PD([S_{d_i}]))$ Since $[S_{d_i}] = d_i[S_1] \in H_{n-2}(\mathbb{C}P^n;\mathbb{Z})$ (we showed in the last homework that the diffeomorphism type only depends on d), we have $c_1(L_i) = d_i i^*(PD([S_1])) = d_i i^*(\omega) = d_i x$.

Thus we have $\nu S = L_1 \oplus ... \oplus L_{n-2}$, and, by the Whitney product formula, we have

$$c(\nu S) = \prod_{i=1}^{n-2}(1 + d_i x) \quad (35)$$

Looking at the terms of the above product that contain only one x gives us $c_1(\nu S)$, and the terms with x^2 give us $c_2(\nu S)$. Expanding the sum gives us

$$c_1(\nu S) = x\sum_{i}^{n-2} d_i \quad (36)$$

$$c_2(\nu S) = x^2 \sum_{i<j} d_i d_j \quad (37)$$

Therefore we have

$$c_1(TS) = (n+1)x - x\sum_i^{n-2} d_i \tag{38}$$

$$c_2(TS) = \frac{n(n+1)}{2}x^2 - (n+1)x^2\sum_i^{n-2} d_i - x^2\sum_{i<j} d_i d_j + (x\sum_i^{n-2} d_i)^2 \tag{39}$$

$$= (\frac{n(n+1)}{2} - (n+1)\sum_i^{n-2} d_i - \sum_{i<j} d_i d_j + (\sum_i^{n-2} d_i)^2)x^2 \tag{40}$$

$$c_1^2(TS) = ((n+1) - \sum_i^{n-2} d_i)^2 x^2 \tag{41}$$

we now compute x^2. We have

$$\langle x^2, [S]\rangle = \langle (i_1^*(...i_{n-2}^*(g)...))^2, [S]\rangle \tag{42}$$

$$= \langle g^2, i_{n-2,*}(...i_{1,*}([S])...)\rangle \tag{43}$$

$$= \langle g^2 \cup PD(i_{n-2,*}(...i_{1,*}([S])...)), [\mathbb{C}P^3]\rangle \tag{44}$$

$$= \langle g^2 \cup \prod d_i g, [\mathbb{C}P^3]\rangle \tag{45}$$

$$= \prod d_i \rangle g^3, [\mathbb{C}P^3]\rangle \tag{46}$$

$$= \prod d_i \tag{47}$$

Thus we have $c_1^2(TS) = ((n+1) - \sum_i^{n-2} d_i)^2 \prod d_i, c_2[S] = (\frac{n(n+1)}{2} - (n+1)\sum_i^{n-2} d_i - \sum_{i<j} d_i d_j + (\sum_i^{n-2} d_i)^2)\prod d_i$.

2. Since $e(TS) = c_{r/2}(TS) = c_2(TS)$, we have

$$e(TS) = (\frac{n(n+1)}{2} - (n+1)\sum_i^{n-2} d_i - \sum_{i<j} d_i d_j + (\sum_i^{n-2} d_i)^2)x^2 \tag{48}$$

Since $p_1 = c_1^2 - 2c_2$, $\sigma(S) = \frac{1}{3}p_1$, and $\chi(S) = e(TS)$, we have

$$
\begin{aligned}
\sigma(S) &= \frac{1}{3}c_1^2 - \frac{2}{3}c_2 && (49)\\
&= \frac{1}{3}((n+1) - \sum_i^{n-2} d_i)^2 x^2 - \frac{2}{3}(\frac{n(n+1)}{2} - (n+1)\sum_i^{n-2} d_i - \sum_{i<j} d_i d_j + (\sum_i^{n-2} d_i)^2)x^2 && (50)\\
&= \frac{x^2}{3}((n+1)^2 - 2(n+1)\sum_i^{n-2} d_i + (\sum_i^{n-2} d_i)^2) && (51)\\
&- n(n+1) - 2(n+1)\sum_i^{n-2} d_i - \sum_{i<j} d_i d_j + (\sum_i^{n-2} d_i)^2) && (52)\\
&= \frac{x^2}{3}((n+1) - (\sum_i^{n-2} d_i)^2 + 2\sum_{i<j} d_i d_j) && (53)\\
&= \frac{x^2}{3}((n+1) - \sum_i^{n-2} d_i^2) && (54)\\
&= \prod d_i \frac{1}{3}((n+1) - \sum_i^{n-2} d_i^2) && (55)
\end{aligned}
$$

3. $w_2(TS) = 0$ if and only if S is spin, so since $\pi_1(S) = 1$, $c_1(TS) \equiv w_2(TS) \bmod 2$. Thus $c_1(TS) \equiv 0 \bmod 2$ if and only if S is spin, so S is spin if and only if $(n+1) - \sum_i d_i$ is even.

□

Question 8. *An Enriques surface X is the quotient of a $K3$ surface by a free involution. Note that $\pi_1(X) = \mathbb{Z}/2$. Calculate the Euler characteristic and the signature of X, and from here find its intersection form.*

Proof. A $K3$ surface is the double-cover of an Enriques surface, so we divide $\sigma(K3), \chi(K3)$ by 2. From class, we have the following for hypersurfaces:

$$\sigma = \frac{d(4-d^2)}{3}, \chi = d(d^2 - 4d + 6) \tag{56}$$

For a $K3$ surface this yields $\sigma(K3) = -16, \chi(K3) = 24$. Thus for an Enriques surface we have $\sigma = -8, \chi = 12$. Since $2 + rk(Q) = \chi = 12$, we have $rk(Q) = 10$. Since $rk(Q) \neq \sigma$, Q is indefinite, and thus $Q = -E_8 \oplus H$. □

Question 9. *Let X be a four-manifold with a (g, k)-trisection. Calculate the Euler characteristic* *of X in terms of g and k.*

Proof. The Euler characteristic is the alternating sum of the number of n-cells $k_0 - k_1 + k_2 - \ldots$. A trisection of X is given by $X_1 \cup X_2 \cup X_3 = X$, where $X_i = \natural^k(S^1 \times B^3), X_i \cap X_j = \natural^g(S^1 \times B^2), X_1 \cap X_2 \cap X_3 = \Sigma_g$, a surface of genus g. Notice that Σ_g can be built by taking the $g-$fold connected sum of tori $S^1 \times S^1$. When we take the g-fold connected sum, we subtract 1 pair of 2-cells from the middle summands and 1 from each end summand, so we have $\chi(\Sigma_g) = g\chi(S^1 \times S^1) - 2(g-1) = 2 - 2g$. $\natural^\alpha(S^1 \times B^\beta)$ is a genus $(\alpha\ \beta + 1)-$handlebody. Attaching a β-handle is homotopy equivalent to attaching a $\beta-$cell, so a genus g n-handlebody has Euler characteristic $k_n + gk_{n-1} = \pm 1 \mp g$. Thus $\chi(X_i) = (1-k)$ and $\chi(X_i \cap X_j) = (-1+g)$. When we take the union $X_1 \cup X_2 \cup X_3$, we over-count $X_1 \cap X_2, X_2 \cap X_3$, and $X_1 \cap X_3$ each by 1, and over-count $X_1 \cap X_2 \cap X_3$ by 2. Thus we have

$$\chi(X) = 3\chi(\natural^k(S^1 \times B^3)) - 3\chi(\natural^6(S^1 \times B^2)) - 2\chi(\Sigma_g) \tag{57}$$

$$= 3(1-k) - 3(-1+g) - 2(2-2g) \tag{58}$$

$$= 2 + g - 3k \tag{59}$$

□

Question 10. *Prove using Kirby calculus that the manifolds $X_1 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ and $X_2 = (S^2 \times S^2) \# \overline{\mathbb{C}P^2}$ are diffeomorphic.*

Proof. See Figure 1 □

Question 11. *Prove that the boundary of the E_8 plumbing is diffeomorphic to the result of +1 surgery on the trefoil, where all components are labeled +1.*

Proof. Since $\mathbb{C}P^2$ is closed, we blow up E_8, since $\partial E_8 \# \overline{\mathbb{C}P^2} = \partial E_8$. This allows us to blow down. See Figure 2 Then we simplify the final diagram in Figure 2 as the blow up of the +1 trefoil knot, as in Figure 3. □

Question 12.

Proof. See Figure 4. □

Question 13.

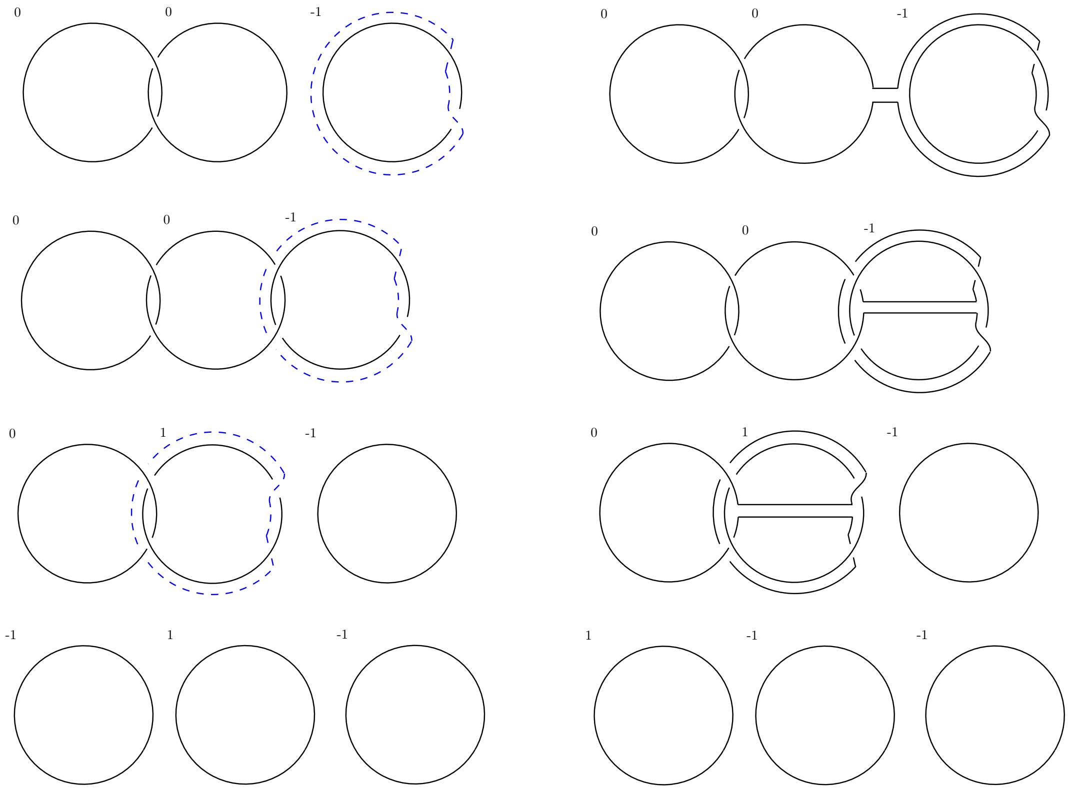

Figure 1: Kirby calculus from $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}$.

Proof. See Figure 5 $\square$

Question 14. *See Figure 6*

Question 15. *Given a finite presentation of a group G, explain how to construct a Kirby diagram for a 4-manifold X with $\pi_1(X) = G$.*

Proof. We follow the hint and what was done in class. For l generators, we start with $\#^l(S^1 \times S^3)$. We attach l 1-handles with $2l$ 2-spheres in our Kirby diagram for each connected summand. For r relations, we embed circles by connecting the spheres representing the generators in the characters in the relation. Since we attach a 2-handle along these circles, we kill the word represented by this relation. We surger these out to obtain a 4-manifold with boundary (a manifold with 0-handles, 1-handles, and 2-handles). By attaching the 2-handles to this manifold and doubling in order to connect the circles, we get a Kirby diagram for a manifold X such that $\pi_1(X)$. $\square$

Question 16. *Let X be a smooth, closed, connected (but not necessarily simply connected) 4-manifold.*

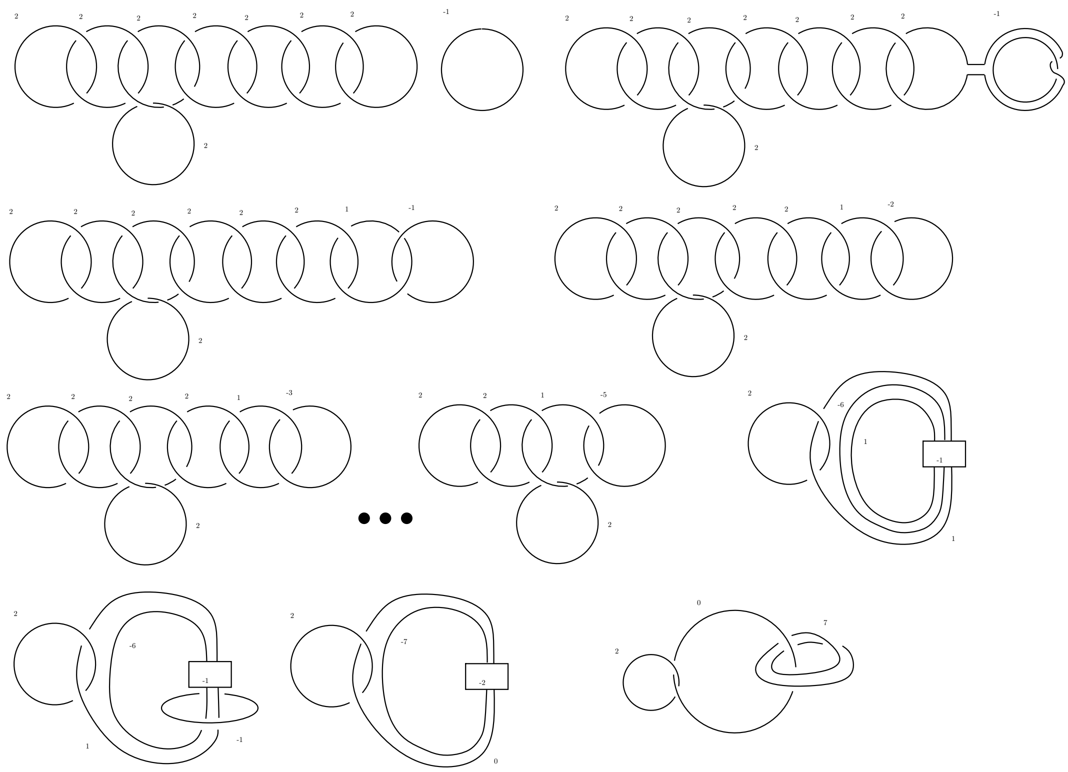

Figure 2: Kirby calculus on $E_8\#\overline{\mathbb{C}P^2}$. (Sorry the font is so small; I couldn't figure out how to fix it in time.)

1. *Prove that $\omega_2(TX)$ is the lift of an integral class, i.e. it lives in the image of the natural map* $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}/2)$.

2. *Prove that there is a short exact sequence of groups*

$$1 \to \mathbb{Z}/2 \to Spin^c(4) \to S^1 \times SO(4) \to 1. \tag{60}$$

3. *The above sequence induces a long exact sequence on cohomology. Using this, prove that X admits a $Spin^c$ structure.*

4. *Let (S,γ) be a $Spin^c$ structure on X, with determinant line bundle L. Prove that $c_1(L) \equiv \omega_2(TX) \mod 2$, and therefore $c_1(L)$ is a characteristic element.*

Proof. 1. If TX has an almost-complex structure, $Vect(X) \xrightarrow{\omega_2} H^2(X;\mathbb{Z}_2)$ factors through the map $Vect(X) \xrightarrow{c_1} H^2(X;\mathbb{Z})$ giving a total map

$$TX \xrightarrow{c_1} H^2(X;\mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(X;\mathbb{Z}_2) \tag{61}$$

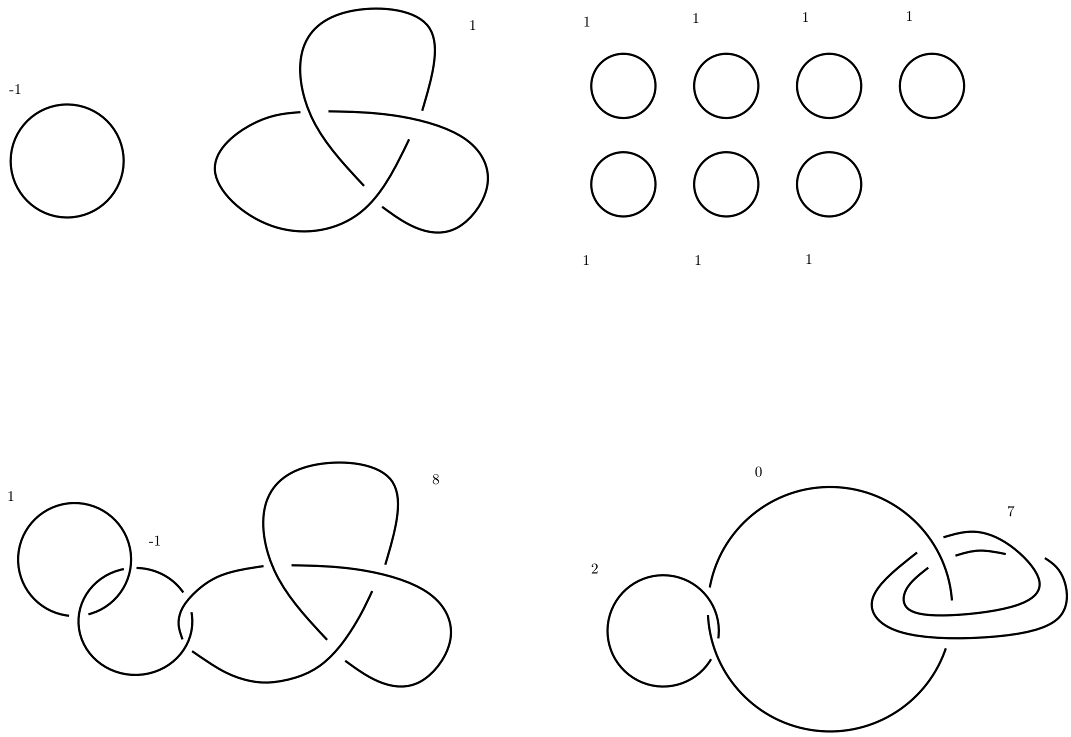

Figure 3: Kirby calculus from $E_8\#\overline{\mathbb{C}P^2}$ to $T_H\#\mathbb{C}P^2$

As for the real case: For Q a unimodular symmetric bilinear form defined over a free $\mathbb{Z}$-module Z, we reduce it by quotienting the ideal $2Z$ to define $Q' :=$ mod 2. If we take some integers a, b, we have $(a+b)(a+b) = aa + bb \mod 2$, and is thus $\mathbb{Z}_2$-linear. For every $\mathbb{Z}_2$-linear function f, there exists some a such that $f(\cdot) = Q'(a, \cdot)$. Then there must exist a characteristic $b \in Z$ such that $Q'(a,a) = Q'(a,b)$. Therefore, there must exist an element $b' \in Z$ such that $b' \equiv b \mod 2$, so there is always a characteristic element for all $a \in Z$. Thus we can always find an integral class $w' \in H^2(M;\mathbb{Z})$ such that $w' \equiv w \in H^2(M;\mathbb{Z}_2)$, and w must be $w_2(TM)$.

2. In dimension 4, $spin^c(4) = \{(A,B) \in U(2) \times U(2) | \det(A) = \det B\}$. This acts on $\mathbb{C}^2$ via $(h_1, h_2) \mapsto h_1 x h_2^{-1}, x \in \mathbb{C}^2$. We can set A, B such that $\det(A) = \det(B) = 1$, and multiply each by a scaling factor of modulus 1, i.e. $e^{i\theta}$ to keep unitarity: $\det(\lambda A) = \lambda^2 \det(A) = \lambda^2$. This mods out the sign of λ. Thus $spin^c(4) = (SU(2) \times SU(2)) \times_{\mathbb{Z}/2\mathbb{Z}} S^1$, where we identify

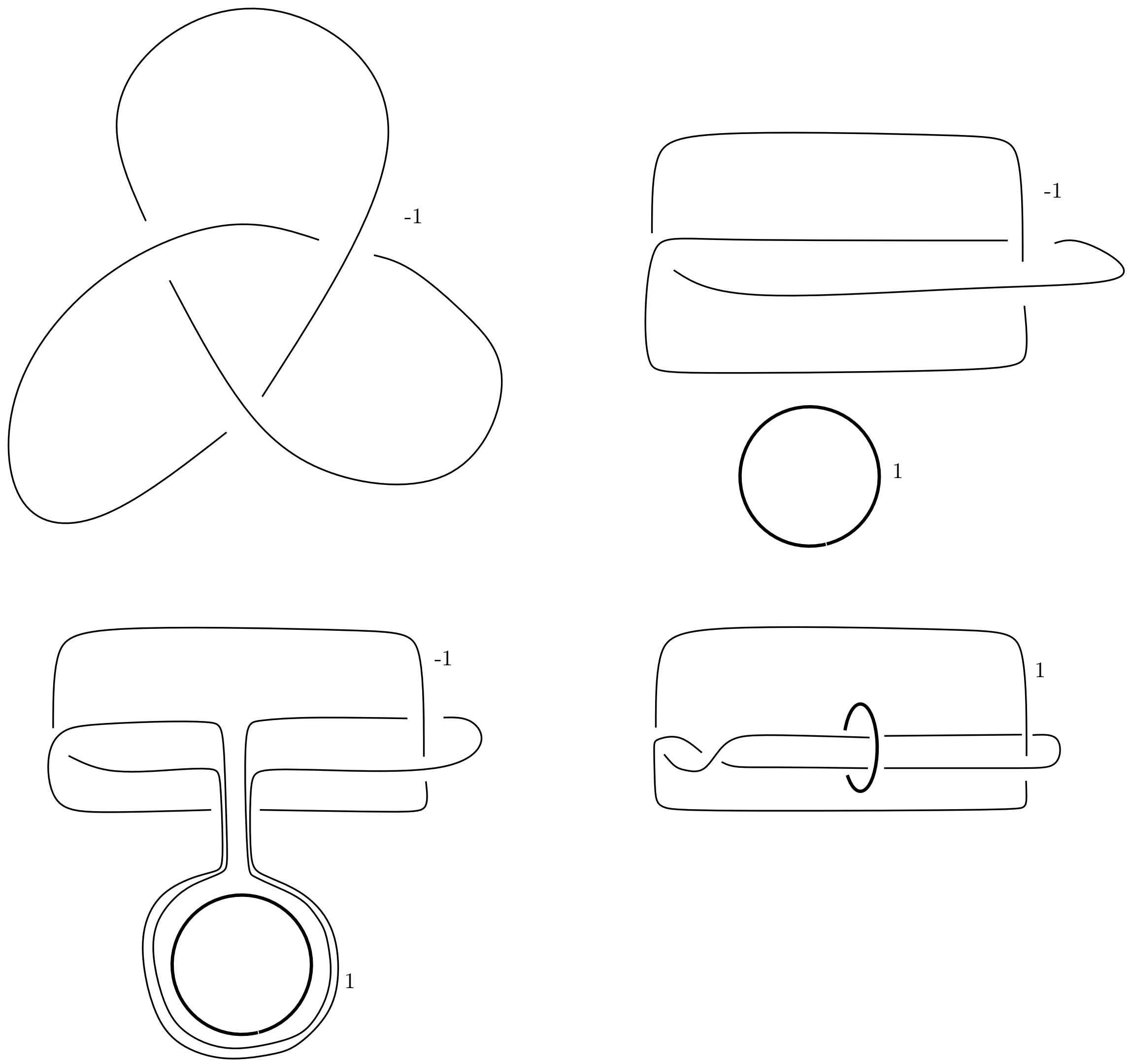

Figure 4: Kirby calculus from the -1 surgery on a trefoil to +1 surgery on the figure 8 knot.

$e^{i\theta}$ with S^1. We then have a map

$$\mathbb{R}^4 \to \mathbb{R}^4 \tag{62}$$

$$x \mapsto \lambda A x (\lambda B)^{-1} \tag{63}$$

$$spin^c \to SO(4) \times S^1 \tag{64}$$

The kernel of this map, by inspection, is $A = B = Id, A = B = -Id, \lambda = \pm 1$. Thus we have an exact sequence

$$1 \to \mathbb{Z}/2 \to spin^c \to SO(4) \times S^1 \to 1 \tag{65}$$

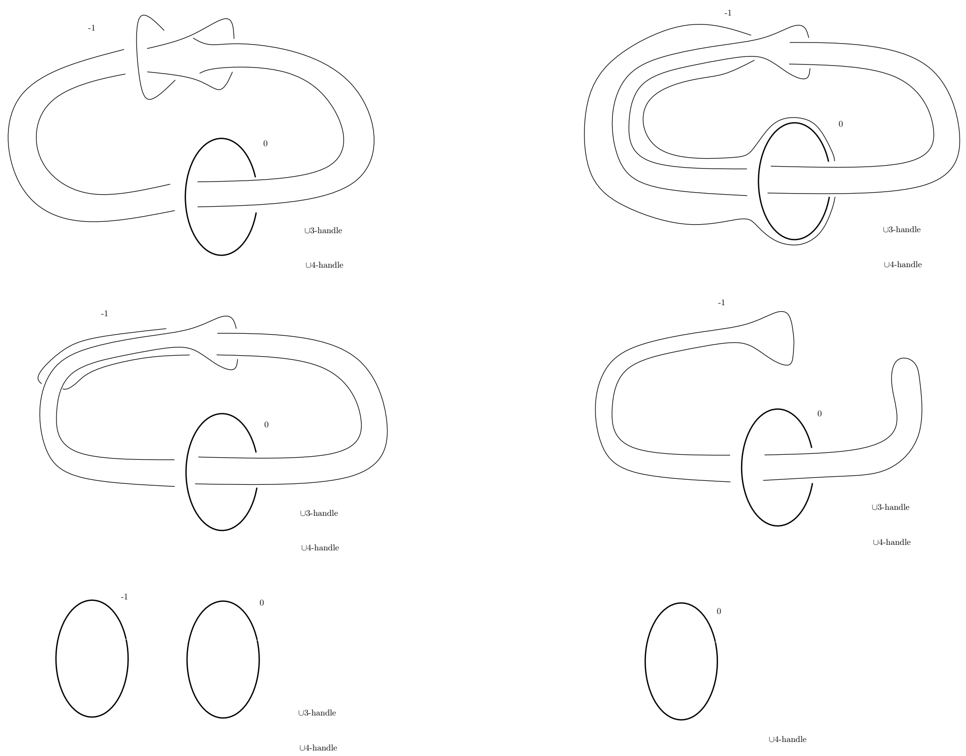

Figure 5: Kirby calculus to $\mathbb{C}P^2$. The text was a lot smaller than I wanted, but we use handle cancellation to get rid of the 0-circle.

3. The above short exact sequence induces a long exact sequence in cohomology:

$$0 \to H^1(X;\mathbb{Z}_2) \to H^1(X;Spin^c) \to H^1(X;SO(4)\times S^1) \xrightarrow{\delta} \tag{66}$$

$$H^2(X;\mathbb{Z}_2) \to H^2(X;Spin^c) \to H^2(X;SO(4)\times S^1) \to ... \tag{67}$$

This gives the exact sequence

$$H^1(X;Spin^c) \to H^1(X;SO(4)) \oplus H^1(X;S^1) \xrightarrow{\omega_2 \oplus c_1'} H^2(X;\mathbb{Z}_2) \tag{68}$$

where ω_2 is the Stiefel-Whitney class of the principal-$SO(4)$ bundle and c_1' is the mod 2 chern class of the principal-$U(1)$ bundle. We have that

$$Prin_{U(1)}X \cong [X, BU(1)] \cong Prin_{GL(1,\mathbb{C})} = Vect^1(X) \cong H^2(X;\mathbb{Z}) \tag{69}$$

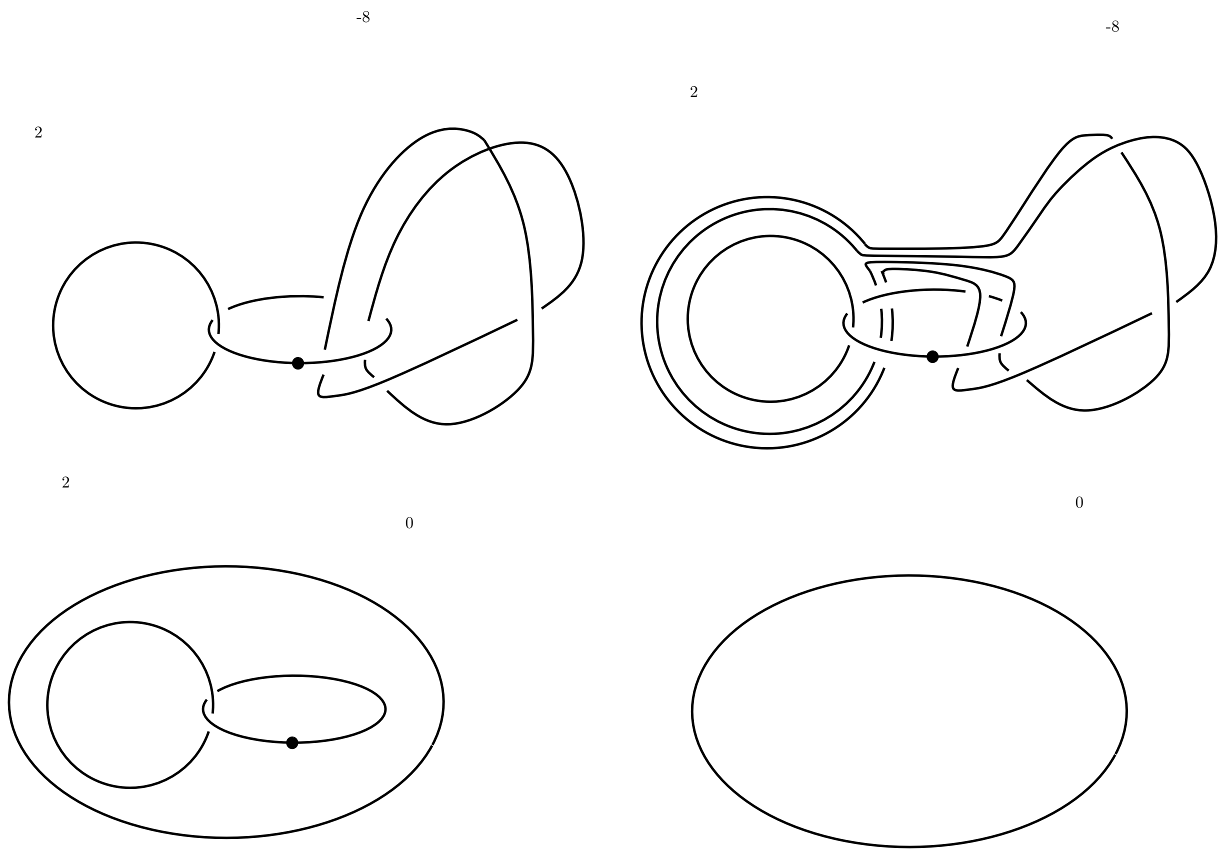

Figure 6: Unlink the -8 framed 2-handle, and cancel the 1-handle with this 2 framed 2-handle. After that, cancel the 3-handle with the 0-framed unknot.

Thus we have

$$H^1(X;Spin^c) \to H^1(X;SO(4)) \oplus H^2(X;\mathbb{Z}) \xrightarrow{\omega_2 \oplus f} H^2(X;\mathbb{Z}_2) \tag{70}$$

Since we have $\omega_2(P_{SO(4)}) = f(U(1))$ due to part a), we have a $Spin^c$ structure

$$P_{Spin^c} \to P_{SO(4)} \times P(S^1) \to X \tag{71}$$

4. Since X is a 4-manifold, TX is a principal-$SO(4)$ bundle, and so we have the spin structure

$$P_{Spin^c} \to P_{SO(4)} \times_{\det} P_{U(1)} \to X \tag{72}$$

so, from above, we have $c_1(L) = \omega_2(TX) \in H^2(X;\mathbb{Z}_2)$, i.e. $c_1(L) \equiv \omega_2(TX) \mod 2$, and so $c_1(L)$ is a characteristic element.

□

Question 17. *Show that every closed, oriented 3-manifold admits a $Spin^c$ structure.*

Proof. We study the tangent bundle of a closed, oriented 3-manifold. Suppose $\omega_2 \in H^2(X;\mathbb{Z}_2)$ is nonzero. Choose a loop γ in X that is Poincaré dual to ω_2. Then there is a spin structure on $X-\gamma$ that doesn't extend over γ. Let Y be a dual surface transverse to γ at a point x. The normal bundle $Y\times D$ to Y is equal to the total normal bundle of an immersion of Y in $\mathbb{R}^3$, so the $Y\times D$ normal bundle has a spin structure, classified by $H^1(Y;\mathbb{Z}_2)\cong H^1(Y-x;\mathbb{Z}_2)$. $H^1(Y-x;\mathbb{Z}_2)$ classifies spin structures on $((Y-x)\times D)$, which must agree with $(Y\times D)$, so the spin structure extends across γ, a contradiction. Thus $\omega_2(X)=0$, so X is spin. Since we have

$$H^2(X;\mathbb{Z}_2)\to H^2(PSO(3);\mathbb{Z}_2)\to H^2(SO(3);\mathbb{Z}_2) \tag{73}$$

and $\pi_2(SO(3))=0$ and X is orientable, TX is trivial, so X admits a $Spin^c$ structure. □

Question 18. *Let X be a closed, connected, smooth 4-manifold that admits an almost complex structure.*

1. *Prove that $p_1(TX)=c_1^2(TX)-2c_2(TX)$.*
2. *Prove that there exists $h\in H^2(X;\mathbb{Z})$ such that $h\equiv\omega_2(TX)\mod 2$ and $h^2[X]=3\sigma(X)+2\chi(X)$, where σ denotes the signature and χ denotes the Euler characteristic.*
3. *If X is simply connected, prove that that $b_2^+(X)$ is odd.*

Proof. 1. The k^{th} Pontryagin class for a real vector bundle E is given by

$$p_k(E)=(-1)^k c_{2k}(E\otimes_{\mathbb{R}}\mathbb{C})\in H^4(X;\mathbb{Z}) \tag{74}$$

Our almost-complex structure has two eigenvalues: $\pm i$. This gives a decomposition of $TX\otimes\mathbb{C}=T^+\oplus T^-$ into the two corresponding eigenspaces once we have i from $\otimes\mathbb{C}$. Then we have

$$p_1(TX)=-c_2(T^+\oplus T^-) \tag{75}$$

To find this, we take the formula

$$c(T^+ \oplus T^-) = c(T^+)c(T^-) \tag{76}$$

Since $T^\pm$ are complexified, T^+ is isomorphic to TX, and T^- is the conjugate bundle TX^*. Thus we find $p_1(TX)$ by grouping the elements of $-c(TX)^2$ in the second homology class. Since we have $c_i(TX*) = (-1)^i c_i(TX)$, we get

$$c(TX)^2 = (1 + c_1(TX) + c_2(TX))(1 - c_1(TX) + c_2(TX)) \tag{77}$$

$$= 1 + c_1 + c_2 - c_1 - c_1^2 - c_1c_2 + c_2 + c_1c_2 + c_2^2 \Rightarrow \tag{78}$$

$$c_2(TX \otimes \mathbb{C}) = -c_1^2(TX) + 2c_2(TX) \tag{79}$$

Adding in the -1 factor from the Pontryagin formula, we get

$$p_1(TX) = c_1^2(TX) - 2c_2(TX) \tag{80}$$

2. c_1 exists as a class $h \in H^2(X;\mathbb{Z})$ such that $h \equiv \omega_2(TX) \mod 2$, since $\omega_{2k} \equiv c_k \mod 2$. Also, $c_2(TX) = \chi(X)$, so we have

$$p_1(TX) = h^2 - 2\chi(X) \tag{81}$$

By the Hirzebruch Signature theorem, we have

$$p_1(TX)([X]) = 3\sigma(X) \tag{82}$$

so we have

$$3\sigma(X) = h^2([X]) - 2\chi(X)([X]) \Rightarrow \tag{83}$$

$$h^2([X]) = 3\sigma(X) + 2\chi(X) \tag{84}$$

3. If X is simply-connected, then $\chi = 2 + b_2^+ + b_2^-$. $h = c_1$ is characteristic. If Q is definite, then

$b_2^- = 0$. If Q is indefinite, then $Q = Q \oplus \langle 1 \rangle \oplus \langle -1 \rangle$, since these have the same rank, signature, and parity. Then $Q = (b_2^+ + 1)\langle 1 \rangle \oplus (b_2^+ 1)\langle -1 \rangle$, with h having odd components in this basis, because it is a characteristic class. Each component can thus be written as $(2k+1), k \in \mathbb{Z}$, and $(2k+1)^2 = 4k(k+1)+1$. Since either k or $(k+1)$ is even, each component is congruent to 1 mod 8. Thus we have $Q(h,h) \equiv (b_2^+ + 1) - (b_2^- + 1) = \sigma \mod 8$. Therefore $h^2 \equiv \sigma \mod 8$.

$$h^2 = 2(\sigma + \chi) + \sigma \Rightarrow \tag{85}$$

$$2(\sigma + \chi) + \sigma - h^2 = 0 \tag{86}$$

$$2(\sigma + \chi) \equiv 0 \mod 8 \tag{87}$$

$$\sigma + \chi \equiv 0 \mod 4 \tag{88}$$

$$b_2^+ - b_2^- + 2 + b_2^+ b_2^- \equiv 0 \mod 4 \tag{89}$$

$$2 + 2b_2^+ \equiv 0 \mod 4 \tag{90}$$

$$2b_2^+ \equiv 2 \mod 4 \tag{91}$$

Therefore b_2^+ must be $(2k+1), k \in \mathbb{Z}$, i.e. b_2^+ must be odd.

□