Beautiful Problems

Alec Lau

Contents

1   Bounding Steps of the Euclidean Algorithm

Written by Prof. Cheng-Chiang Tsai.

Question 1. *Prove that if in Euclid's algorithm we begin with two integers $0 < x, y < 2^{32}$, then we need no more than 45 divisions to find out $gcd(x, y)$.*

2   The Wave Equation on a Riemannian Manifold

Written by Jonathan Luk.

Question 2. *Let M be an n-dimensional manifold and g be a metric. Given a $\binom{k}{l}$-tensor field F, define a $\binom{k}{l+1}$-tensor field ∇F by*

$$(\nabla F)(Z, X_1, ..., X_k, \omega_1, ..., \omega_l) := (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) \tag{1}$$

where we recall

$$\nabla_Z[F(X_1, ..., X_k, \omega_1, ..., \omega_l)] = (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) + \sum_{i=1}^{k} F(X_1, ..., \nabla_Z X_i, ..., X_k, \omega_1, ..., \omega_l)$$

(2)

$$+ \sum_{j=1}^{l} F(X_1, ..., X_k, \omega_1, ..., \nabla_Z \omega_i, ..., \omega_l)$$

(3)

where for a function f, $\nabla_Z f = Zf$.

a) *Let $f : M \to \mathbb{R}$ be a smooth function. Prove that*

$$(\nabla^2 f)(X, Y) = X(Yf) - (\nabla_X Y)f$$

(4)

b) *Prove that, in local coordinates,*

$$\Delta f := \sum_{i,j=1}^{n} (g^{-1})^{ij} (\nabla^2 f)(\partial_i, \partial_j) = \frac{1}{\sqrt{\det g}} \partial_i ((g^{-1})^{ij} \sqrt{\det g} \partial_j f)$$

(5)

where one may use the following facts from linear algebra without proof:

$$\partial_i (g^{-1})^{jk} = -(g^{-1})^{jl} (g^{-1})^{mk} \partial_i g_{lm}, \partial_i \log(\det g) = (g^{-1})^{jk} \partial_i g_{jk}$$

(6)

3 Algebraic Topology with Statistics of Particles

Written by me. To be fair, I really like algebraic topology and this argument, formalized(?) by me.

Question 3. *Use the fundamental group of the configuration space of two identical particles to prove that, in $\mathbb{R}^3$, there can exist only bosons and fermions, but in $\mathbb{R}^2$ there can exist any particle statistics.*

4 A Surjection between Groups

Written by the a-MAZING Prof. Tom Church.

Question 4. *Let $G := APB(\mathbb{Z}^2)$ be the group of adjacency-preserving bijections $f : \mathbb{Z}^2 \to \mathbb{Z}^2$. You can use without proof that this is a group. Suppose that $G \twoheadrightarrow H$ is a surjective homomorphism, and H is an abelian group. Prove that $|H|$ is finite. (The original version of this problem states also to prove that $|H| = 8$, but I lost points on that part)*

5 Infinite Integers

Written again by Prof. Tom Church.

Question 5. *Let R denote the set of infinite integers. Two examples are*

$$a = ...000000001 \tag{7}$$

$$b = ...562951413 \tag{8}$$

(b is π backwards)

Prove that there exists at least one solution $z \in R$ to the equation $z^3 = 7$.

6 A Quantum Error-Correcting Code

Written by Prof. Douglas Stanford.

Question 6. *There is no four-qubit code that can protect an encoded qubit against single-qubit errors on any site. However, suppose that we only care about protecting against arbitrary errors on the first qubit, and against bit-flip errors on the other three.*

a) *How many possible errors are there? (Include "no error" as well in your count.)*

b) *In the stabilizer formalism, how many generators are you allowed to use, in order that the code subspace should be two-dimensional?*

c) *Using the stabilizer formalism, devise a four-qubit error correcting code that will protect against all ofthe errors described above.*

d) *Does your code "accidentally" also protect against some other error(s)?*

7 A Generalization of Shor

Written again by Prof. Douglas Stanford.

Question 7. *Consider the code with code subspace spanned by*

$$|0\rangle_L = \frac{1}{4}(|0000\rangle + |1111\rangle)^{\otimes 4} \tag{9}$$

$$|1\rangle_L = \frac{1}{4}(|0000\rangle - |1111\rangle)^{\otimes 4} \tag{10}$$

1. *Find a set of stabilizer generators for this code.*

2. Find logical operators X_L and Z_L such that

$$X_L \left|0\right\rangle_L = \left|1\right\rangle_L \tag{11}$$

and so on. There is no unique answer, but try to find a set of logical operators that use as few physical Pauli operators as possible.

3. Is it possible to find an X_L and a Z_L that commute with each other?

8   Real Projective Space

Written by Prof. Ralph Cohen.

Question 8.   *1. Let $x \in S^n$, and $[x] \in \mathbb{R}P^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{R}P^n \Rightarrow \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{R}P^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.*

2. Use the above to show that $\mathbb{R}P^n$ is compact.