

# Algebraic Topology Refresher Problems

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Note: These are the only problems I had saved before I decided to make a page to preserve this stuff. Because there is so much machinery in algebraic topology, I have saved some of these problems as a refresher for whenever I return to it, as said machinery is very forgettable, beautiful though it may be. Also, these problems were done open-book and open-past-homework with Hatcher's "Algebraic Topology" textbook, so there may be some steps I skipped over.

Denotation:  $\mathbb{R}P^n$  is the space of all lines through the origin in  $\mathbb{R}^n$ ,  $S^n$  is the  $n$ -dimensional sphere, and  $D^n$  is the  $n$ -dimensional disc.

All problems were written by Professor Ralph Cohen, saxophone extraordinaire.

**Question 1.** *Is every covering space of  $\mathbb{R}P^2 \times \mathbb{R}P^3$  isomorphic to a product of covering spaces  $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^3$ , where  $p_1 : \tilde{X}_1 \rightarrow \mathbb{R}P^2$  and  $p_2 : \tilde{X}_2 \rightarrow \mathbb{R}P^3$ ? Why or why not?*

*Proof.* Let  $Y$  be the covering space of  $\mathbb{R}P^2 \times \mathbb{R}P^3$ . If there exists an isomorphism  $f : Y \rightarrow \tilde{X}_1 \times \tilde{X}_2$ , then from the relations  $p_1 = p_2 f, p_2 = p_1 f$ , it follows that  $(p_1 \times p_2)_* \pi_1(\tilde{X}_1 \times \tilde{X}_2) \cong p_*(Y)$  due to the induced isomorphisms. Since  $\mathbb{R}P^2$  and  $\mathbb{R}P^3$  are path-connected, we know from the product topology that a map  $f : Y \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^3$  is continuous if and only if the maps  $g : Y \rightarrow \mathbb{R}P^2, h : Y \rightarrow \mathbb{R}P^3$  defined by  $f = g \times h$  are continuous. Therefore a loop in  $\mathbb{R}P^2 \times \mathbb{R}P^3$  is equivalent to a pair of loops in  $\mathbb{R}P^2$  and  $\mathbb{R}P^3$ , and furthermore a homotopy on the product space is equivalent to a pair of homotopies on the corresponding components. Thus there exists a bijection

$$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3) \quad (1)$$

given by

$$[f] \mapsto ([g], [h]) \quad (2)$$

Next we compute the homology groups of  $\mathbb{R}P^n, n \in \{2, 3\}$ .

$\mathbb{R}P^n$  is topologized as the quotient space  $\mathbb{R}^{n+1} - \{0\}$  under the equivalence relation  $v \sim \lambda v$  for scalars  $\lambda \neq 0$ , so we can thus restrict to vectors of length 1, so  $\mathbb{R}P^n = S^n / (v \sim -v)$ . Thus  $\mathbb{R}P^n$  is the quotient space of a hemisphere  $D^n$  with antipodal points of  $\partial D^n$  identified. Since  $\partial D^n$  with antipodal points identified is just  $\mathbb{R}P^{n-1}$ , we see that  $\mathbb{R}P^n$  is obtained from  $\mathbb{R}P^{n-1}$  by attaching

an  $n$ -cell, with the quotient projection  $S^{n-1} \rightarrow \mathbb{R}P^{n-1}$  as the attaching map. By induction on  $n$ ,  $\mathbb{R}P^n$  has a CW structure  $e^0 \cup e^1 \cup \dots \cup e^n$  with one cell  $e^i$  in each dimension  $i \leq n$ . To compute the boundary map  $d_k$  we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1} \quad (3)$$

with  $q$  the quotient map. The map  $q\varphi$  is a homeomorphism when restricted to each component of  $S^{k-1} - S^{k-2}$ , and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of  $S^{k-1}$ , which has degree  $(-1)^k$ . Hence  $\deg q\varphi = \deg(1) + \deg(-1) = 1 + (-1)^k$ , and so the boundary maps  $d_k$  is either 0 or multiplication by 2, depending on whether  $k$  is odd or even. Thus the cellular chain complex for  $\mathbb{R}P^n$  is

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \text{ if } n \text{ is even} \quad (4)$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \text{ if } n \text{ is odd} \quad (5)$$

$$(6)$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{if } k = \text{odd}, 0 < k < n \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Thus we know that  $H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$ . We know from homework 4 that  $H_1$  is the abelianization of  $\pi_1$  (since  $\mathbb{R}P^n$  is path-connected and nonempty), but a group with cardinality 2 must be isomorphic to  $\mathbb{Z}_2$  to have the group axioms still hold. Thus,  $\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$ , and  $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .  $\mathbb{R}P^2$  and  $\mathbb{R}P^3$  are path-connected, and are manifolds, and so are locally path-connected semi-locally simply-connected as well. Thus we can apply the Galois Correspondence Theorem to say that, for every subgroup  $H$  of  $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , there is an isomorphism class of covering spaces  $Y$  such that  $p_*(Y) \cong H$ , therefore all covering spaces of  $\mathbb{R}P^2 \times \mathbb{R}P^3$  have their fundamental groups as subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

The subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  are the trivial subgroup  $(0,0)$ , and groups with generator  $(1,0)$ ,  $(0,1)$ , and  $(1,1)$ . I claim that there is no pair of maps  $p_1 : \tilde{X}_1 \rightarrow \mathbb{R}P^2, p_2 : \tilde{X}_2 \rightarrow \mathbb{R}P^3$  such that  $(p_1 \times p_2)_*(\tilde{X}_2 \times \tilde{X}_3) \cong \{(0,0), (1,1)\}$ . We see that  $|\{(0,0), (1,1)\}| = 2$ . By Lagrange's Theorem for any subgroup of this group must have cardinality that divides 2. Since 2 is prime, the only subgroup in it has cardinality 1 and is thus is the trivial subgroup. Thus, if we have  $\pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2) \cong \{(0,0), (1,1)\}$ , a subgroup  $\pi_1(\tilde{X}_1) \times 0 \cong 0$  and another subgroup  $0 \times \pi_1(\tilde{X}_2) \cong 0$ . Thus both  $\pi_1(\tilde{X}_1), \pi_1(\tilde{X}_2)$  are trivial subgroups, and it is impossible for  $f((0,0)) \mapsto (1,1)$  if  $f$  is an isomorphism. If, without loss of generality  $\pi_1(\tilde{X}_1) \times 0$  were the whole group, this has cardinality 2, but the group is  $\{(0,0), (1,0)\}$ , a different subgroup corresponding to a different covering space. Thus,  $\{(0,0), (1,1)\} \not\cong \pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2)$ , so not all covering spaces of  $\mathbb{R}P^2 \times \mathbb{R}P^3$  are isomorphic to a product of each summand's covering space.  $\square$

**Question 2.** Let  $X = \mathbb{R}P^2 \vee S^3$  and  $Y = \mathbb{R}P^3$ . Prove that the homology and cohomology groups of  $X$  and  $Y$  are isomorphic with any coefficients, but that  $X$  and  $Y$  do not have the same homotopy type.

*Proof.* From the above, we know that the chain complex for  $\mathbb{R}P^3$  is

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \quad (8)$$

With any  $G$  coefficients, this becomes

$$0 \rightarrow G \xrightarrow{0} G \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0 \quad (9)$$

so we have

$$H_0(\mathbb{R}P^3; G) \cong G, H_1(\mathbb{R}P^3; G) \cong G/2G, H_2(\mathbb{R}P^3; G) \cong 0, H_3(\mathbb{R}P^3; G) \cong G, \quad (10)$$

$$H_0(\mathbb{R}P^2; G) \cong G, H_1(\mathbb{R}P^2; G) \cong G/2G, H_2(\mathbb{R}P^2; G) \cong 0 \quad (11)$$

For  $n > 0$  take  $(X, A) = (D^n, S^{n-1})$  so  $X/A = S^n$ . The terms  $\tilde{H}_i(D^n)$  in the long exact sequence for this pair are zero since  $D^n$  is contractible. Exactness of the sequence then implies that the maps  $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$  are isomorphisms for  $i > 0$  and that  $\tilde{H}_0(S^n) = 0$ . By induction on  $n$ , starting with the case of  $S^0$ , we see that  $\tilde{H}_n(S^n) \cong \mathbb{Z}$  and  $\tilde{H}_i(S^n) = 0$  for  $i \neq n$ . Thus  $H_1(S^3; G) \cong H_2(S^3; G) \cong 0, H_3(S^3) \cong G$ , due to the equivalence of  $\tilde{H}_n$  and  $H_n$  for  $n > 0$ . Since  $S^3, \mathbb{R}P^2$  are both path-connected and nonempty,  $S^3 \vee \mathbb{R}P^2$  is path-connected and nonempty. By definition,  $H_0(S^3 \vee \mathbb{R}P^2) = C_0(S^3 \vee \mathbb{R}P^2)/\text{Im } \partial_1$  since  $\partial_0 = 0$ . Define a homomorphism  $\epsilon : C_0(S^3 \vee \mathbb{R}P^2) \rightarrow \mathbb{Z}$  by  $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$ . This is obviously surjective since  $S^3 \vee \mathbb{R}P^2$  is nonempty.  $\text{Ker } \epsilon = \text{Im } \partial_1$  since  $S^3 \vee \mathbb{R}P^2$  is path-connected, and thus  $\epsilon$  induces an isomorphism.

We conclude that  $H_0(S^3 \vee \mathbb{R}P^2; G) \cong G$ . Since reduced homology is the same as homology relative to a basepoint, we know that, for  $n > 0$ ,

$$\tilde{H}_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3) \oplus H_n(\mathbb{R}P^2) \quad (12)$$

Thus we have  $H_1(S^3 \vee \mathbb{R}P^2; G) \cong G/2G, H_2(S^3 \vee \mathbb{R}P^2; G) \cong 0, H_3(S^3 \vee \mathbb{R}P^2; G) \cong G$ . These are the same (up to isomorphism) homology groups as  $\mathbb{R}P^3$ .

In calculating cohomology for any  $G$  coefficients, we notice that  $H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(X; \mathbb{Z}), G)$ .

**Lemma 1.**  $\text{Hom}(\mathbb{Z}, G) \cong G, \text{Hom}(G/2G, G) \cong 0$

*Proof.* By mapping 1 to each element of  $G$ , we get a cardinality of  $G$ . Since this is a homomorphism, the structure of the image is preserved, and  $fg(n) = f(n) \star g(n)$ , where  $\star$  is the group operation of  $G$ . Since every element of the group is hit in the image, and the composition of these homomorphisms is mapped to the group operation, we have all elements of  $G$  following the same structure of  $G$ , and thus is isomorphic to  $G$ .

In order for there to be a nontrivial homomorphism, orders of elements must match from  $G/2G$  to  $G$ . However, this is not the case, as we mod out by  $2G$ , so no generator of  $G/2G$  has the same order as an element in  $G$ . Thus, the only homomorphism possible is the trivial homomorphism.  $\square$

Using this, we calculate cohomology for  $\mathbb{R}P^3$ , and, using the rules of Ext on page 195 of Hatcher,

we find

$$H^0(\mathbb{R}P^3; G) \cong G, H^1(\mathbb{R}P^3; G) \cong 0 \oplus \text{Ext}(\mathbb{Z}, G) \cong 0 \oplus 0 \cong 0, \quad (13)$$

$$H^2(\mathbb{R}P^3; G) \cong 0 \oplus \text{Ext}(\mathbb{Z}_2, G) \cong 0 \oplus G/2G \cong G/2G, \quad (14)$$

$$H^3(\mathbb{R}P^3; G) \cong G \oplus 0 \quad (15)$$

We also have

$$H^n(S^3 \vee \mathbb{R}P^2; G) \cong \text{Hom}(H_n(S^3 \vee \mathbb{R}P^2; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(S^3 \vee \mathbb{R}P^2; \mathbb{Z}), G) \quad (16)$$

Since the homology groups are isomorphic, the cohomology groups are isomorphic as well.

If we can prove  $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \not\cong H^*(\mathbb{R}P^3; \mathbb{Z}_2)$ , then this means the spaces are not homotopy equivalent. Plug in  $G = \mathbb{Z}_2$ . From Example 3.8 in Hatcher, we have  $H^*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$ , and  $H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^4)$ , where  $|\alpha| = |\beta| = 1$ . Suppose we have  $\beta \in H^1(\mathbb{R}P^3; \mathbb{Z}_2)$ . Then  $\beta \smile \beta = \beta^2 \in H^2(\mathbb{R}P^3; \mathbb{Z}_2)$ , and  $\beta^2 \smile \beta = \beta^3 \in H^3(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2$ , and  $\beta^3 \neq 0$ . Noticing that  $H^1(\mathbb{R}P^2 \vee S^3) \cong H^1(\mathbb{R}P^2) \oplus H^1(S^3)$ , we have  $(\alpha, 0) \in H^1(\mathbb{R}P^2 \vee S^3)$ . Suppose there is an isomorphism  $f : \mathbb{R}P^2 \vee S^3 \rightarrow \mathbb{R}P^3$ . Then for  $\alpha' = f(\beta) \in H^1(\mathbb{R}P^2; \mathbb{Z}_2)$ , and for  $a = f(\beta^2) \in H^2(\mathbb{R}P^2; \mathbb{Z}_2)$ . We now have  $(\alpha', 0) \smile (a, 0) = (\alpha'a, 0) \in H^3(\mathbb{R}P^2 \vee S^3)$ . But  $H^3(\mathbb{R}P^2; \mathbb{Z}_2) \cong 0$ , so  $\alpha'a = 0$ . But since  $f$  is a ring isomorphism, then  $f(\beta^3 \neq 0) = \alpha'a = 0$ . Since  $f$  maps a nonzero element to 0, it cannot be an isomorphism, so the cup product structures of  $\mathbb{R}P^2 \vee S^3$  and  $\mathbb{R}P^3$  are not homotopically equivalent.  $\square$

**Question 3.** Let  $M^n$  be a closed, path connected, orientable manifold. Let  $x \in U \subset M$  where  $U$  is an open neighborhood homeomorphic to  $\mathbb{R}^n$ . Consider the “pinch map,”  $p : M^n \rightarrow S^n$  defined as the composition

$$p : M^n \xrightarrow{\text{quotient}} M^n / (M^n - U) \xrightarrow{\text{homeo}} S^n \quad (17)$$

Show that

$$p_* : H_n(M^n; \mathbb{Z}) \rightarrow H_n(S^n; \mathbb{Z}) \quad (18)$$

is an isomorphism.

*Proof.* Since  $M$  is closed and path-connected, we can apply Theorem 3.26 in Hatcher to conclude that  $H_n(M) \cong H_n(M, M - x)$ . Hatcher states that the isomorphism between the two factors though  $H_n(M, M - U)$  for  $U$  any neighborhood in  $M$  containing  $x$ . This is because the homomorphism  $i_* : H_n(M, M - U) \rightarrow H_n(M, M - x)$  induced by inclusion is bijective, since  $X - U$  is a deformation retract of  $X - x$ . By excision,  $H_n(X, X - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x)$ . Since  $\mathbb{R}^n$  is contractible, this is isomorphic to  $H_{n-1}(\mathbb{R}^n - U)$ . The second map is isomorphic for any  $x \in U$ , because  $\mathbb{R}^n - U$  and  $\mathbb{R}^n - x$  deformation retract onto a sphere centered at  $x$ . Thus  $H_n(M) \cong H_n(M, M - U)$ .

We notice that  $(M, M - U)$  is a good pair; Simply take an open cover  $\epsilon$ -thick covering the boundary of  $U$ , and add this to  $M - U$ . Because  $U \cong \mathbb{R}^n$ , the overlap of our covering with  $U$  can easily be deformation retracted until we are left with  $M - U$ . Since  $(M, M - U)$  is a good pair, we can have a neighborhood  $V$  be a neighborhood of  $U$  in  $M$  that deformation retracts onto  $U$ . We have a commutative diagram

$$\begin{array}{ccccc} H_n(M, U) & \xrightarrow{\quad\quad\quad} & H_n(M, V) & \xleftarrow{\quad\quad\quad} & H_n(M - U, V - U) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(M/U, U/U) & \xrightarrow{\quad\quad\quad} & H_n(M/U, V/U) & \xleftarrow{\quad\quad\quad} & H_n(M/U - U/U, V/U - U/U) \end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple  $(M, V, U)$  the groups  $H_n(V, U)$  are zero for all  $n$ , because a deformation retraction of  $V$  onto  $U$  gives a homotopy equivalence of pairs  $(V, U) \cong (U, U)$ , and  $H_n(U, U) = 0$ . The deformation retraction of  $V$  onto  $U$  induces a deformation retraction of  $V/U$  onto  $U/U$  so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map  $q_*$  is an isomorphism since  $q$  restricts to a homeomorphism on the complement of  $U$ . From the commutativity of the diagram it follows that the left  $q_*$  map is an isomorphism. We see then that  $H_n(M, M - U) \cong$

$H_n(M/(M-U), (M-U)/(M-U)) \cong H_n(M/(M-U))$ . Thus  $H_n(M) \cong H_n(M/(M-U))$ . Since  $M/(M-U)$  is homeomorphic to  $S^n$ , they better have isomorphic homology groups. Call the isomorphism induced by the homeomorphism between  $H_n(M/(M-U))$  and  $H_n(S^n)$   $g$ , and the isomorphism between  $H_n(M)$  and  $H_n(M/(M-U))$ , shown via the diagram. Now we have that  $p_* := f \circ g$ , a composition of isomorphisms, so  $p_* : H_n(M) \rightarrow H_n(S^n)$  is an isomorphism.  $\square$

**Question 4.** *Prove that if  $M^3$  is a closed, simply connected manifold, then there is a map  $g : M^3 \rightarrow S^3$  that induces an isomorphism in homology groups in all dimensions. This is a weaker statement of the Poincaré Conjecture, proved in 2003 by G. Perelman.*

*Proof.* Define  $\tilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$ . The map  $\mu_x \mapsto x$  defines a two-to-one surjection, and, due to everything being nice and manifold-y, we see that  $\tilde{M}$  is a two-sheeted covering space of  $M^3$ .

Since  $M^3$  is simply connected,  $\tilde{M}$  has either one or two components since it is a two-sheeted covering space of  $M^3$ . If it has two components, they are each mapped homeomorphically to  $M^3$  by the covering projection, so  $M^3$  is orientable, being homeomorphic to a component of the orientable manifold  $\tilde{M}$ . Thus  $M^3$  is orientable, and  $H_0(M^3) \cong \mathbb{Z}$  because simply connected implies nonempty and path-connected. Now since  $M^3$  is a closed and orientable manifold, we can use Poincaré duality. Also from Theorem 3.26 (part c),  $H_i(M^3) \cong 0, i > 3$ . Because  $M^3$  is simply-connected,  $\pi_1(M^3) \cong 0 \cong H_1(M^3)$ .  $H^1(M^3) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) \cong 0$ . By Poincaré duality,  $H_2(M^3) \cong H^1(M^3) \cong 0$ . Also by Poincaré Duality,  $H_3(M^3) \cong H^0(M^3) \cong \text{Hom}(H_0(M^3), \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$ . To sum this up, we have  $H_0(M^3) \cong H^3(M^3) \cong \mathbb{Z}, H_1(M^3) \cong H_2(M^3) \cong 0 \cong H_i(M^3), i > 3$ . These are the exact homology groups of  $S^3$ , so let  $g$  be the isomorphism between their homology groups.  $\square$

**Question 5.** *Is  $(S^2 \times S^4) \vee S^8$  homotopy equivalent to a compact closed manifold? Explain.*

*Proof.* Let  $a_i \in H^i(S^2; \mathbb{Z}), b_i \in H^i(S^4; \mathbb{Z})$  be generators of their cohomology groups. From the definition of the external cup product we have  $p_1^*(a) \smile p_2^*(b) \in H^*(X \times Y; \mathbb{R})$ , for  $p_1, p_2$  projection maps. For  $H^0(S^2 \times S^4) \cong \mathbb{Z}$  because this is space and path-connected. Let  $p_1^*, p_2^*$  be induced homomorphisms from the projection  $S^2 \times S^4 \rightarrow S^2, S^2 \times S^4 \rightarrow S^4$ , respectively. We have  $p_1^*(a_1) \smile p_2^*(b_1) = 0 \smile 0 = p_1^*(a_1) \smile 0 = 0 \smile p_2^*(b_1) = 0 \in H^1(S^2 \times S^4; \mathbb{Z})$ . We also have  $p_1^*(a_2) \smile$

$p_2^*(b_2) = p_1^*(a_2) \smile p_2^*(0) \in H^2(S^2 \times S^4; \mathbb{Z})$  as a generator for  $H^2$ . The other nonzero generator  $b_4$ , when cupped with another generator  $a_i, i \neq 2$  is  $0 \smile p_2^*(b_4) \in H^4(S^2 \times S^4; \mathbb{Z})$ , which is the generator of  $H^4$ . For all other combinations when  $a_i \neq a_2, b_j \neq 4$  we have trivial  $H^{i+j}$ . With  $0 \neq p_1^*(a_2) \smile p_2^*(b_4) \in H^6(S^2 \times S^4; \mathbb{Z})$ , we conclude that  $H^i(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$  (has a single generator infinite with  $\mathbb{Z}$  coefficients) when  $i = 0, 2, 4, 6$  and trivial otherwise. We might think we would run into trouble with  $p_1^*(a_2) \smile p_1^*(a_2)$ , but because this is the pullback of the generator  $H^2(S^2)$  under  $p_1$ , and in  $H^2(S^2), a_2 \smile a_2 = 0$ , this still holds in  $H^2(S^2 \times S^4)$ . By the Kunneth Formula, we have  $H^*(S^2 \times S^4; \mathbb{R}) \cong \mathbb{Z}[a_2]/(a_2^2) \otimes_{\mathbb{R}} \mathbb{Z}[a_4]/(a_4^2), |a_2| = 2, |a_4| = 4$ .

For  $\tilde{H}^*((S^2 \times S^4) \vee S^8)$  (we need not worry about  $H^0$  since the space is nonempty and path-connected), we use the fact from Hatcher that  $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong \tilde{H}^*(S^2 \times S^4) \oplus \tilde{H}^*(S^8)$ . For  $\tilde{H}^*(S^8)$ , we know that  $H_i \cong H^i \cong \mathbb{Z}$  for  $i = 0, 8$ , and  $\cong 0$  if else. Thus our cohomology ring is  $\tilde{H}^*(S^8; \mathbb{Z}) \cong \mathbb{Z}[b]/(b^2), |b| = 8$ . Thus we have  $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong [\mathbb{Z}[a_2]/(a_2^2) \otimes \mathbb{Z}[a_4]/(a_4^2)] \oplus \mathbb{Z}[b]/(b^2), |a_2| = 2, |a_4| = 4, |b| = 8$ .

Any manifold homotopically equivalent to  $(S^2 \times S^4) \vee S^8$  must be an 8-manifold. From Theorem 3.26 in Hatcher, if a manifold is not oriented, then  $H_8(M; \mathbb{Z}) \cong 0 \Rightarrow H^8(M; \mathbb{Z}) \cong 0$ , which cannot be possible, as  $H^8((S^2 \times S^4) \vee S^8)$  is nontrivial. Thus a manifold that is homotopy equivalent must be oriented, since in that case  $H^8(M; \mathbb{Z}) \cong H^8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$ . Oriented closed manifolds satisfy Poincaré Duality. If a closed manifold were to be homotopy equivalent to  $(S^2 \times S^4) \vee S^8$ , since the latter is path-connected the former better be path-connected. Suppose that  $(S^2 \times S^4) \vee S^8$  satisfies Poincaré Duality. Consider the fundamental homology class  $[M] \in H_8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$ . From Poincaré Duality, we know that, for  $\alpha \in H^2(M)$ , where  $\alpha$  is a generator,  $[M] \frown \alpha$  generates  $H_6(M)$ , since  $D(\alpha) = [M] \frown \alpha$  is an isomorphism.

Examining the cap product, we have

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|[v_0, \dots, v_k])\sigma|[v_k, \dots, v_{k+l}]) \quad (19)$$

$$= \psi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]) = (\varphi \smile \psi)(\sigma) \quad (20)$$

Thus,  $\psi([M] \frown \alpha) = (\alpha \smile \beta)([M])$ , where  $\psi \in H^6(M)$  is the generator. Since  $[M] \frown \alpha$  is a generator, and  $\psi$  is a generator homomorphism,  $\psi([M] \frown \alpha)$  is a generator for the ring we are in (here we are using  $\mathbb{Z}$ ). Thus, for  $\beta \in H^4((S^2 \times S^4) \vee S^8)$  the generator,  $1 = \psi([M] \frown \alpha) =$



$(\alpha \smile \psi)([M]) = (\alpha \smile (\alpha \smile \beta))( [M]) = 0([M]) = 0$ , a contradiction. This is because  $\alpha \smile \alpha = 0$  from the ring structure we derived earlier. Because of this, Poincare Duality is not satisfied, so any orientable or otherwise closed manifold has a different ring structure and therefore is not homotopy equivalent. □