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Useful Mathematical Preliminary Objects (that I have difficulty remembering)

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1 Algebra & Vector Spaces

A *group* G is a set closed under an operation $\star$ that is associative $(g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3)$, contains an identity e such that $e \star g = g \star e = g \forall g \in G$, and every element has an inverse such that $g \star g^{-1} = g^{-1} \star g = e$. A group is *abelian* if $g_1 \star g_2 = g_2 \star g_1$

A *ring* is a set closed under two operations $+, \times$ that is an abelian group under $+$, and contains an identity 1_R for the operation $\times$. $\times$ is distributive and associative.

A *field* is a ring where every element except maybe the $+$ identity has a multiplicative inverse sending it to the multiplicative identity. This forms a group structure for the elements except for maybe the additive identity. This group, called the multiplicative group, is also abelian.

A *module* M is an abelian group (operation denoted $+$) with a ring R such that, for all

$r, s \in R, x, y \in M$, we have

$$r(x+y) = rx + ry \tag{1}$$

$$(r+s)x = rx + sx \tag{2}$$

$$(rs)x = r(sx) \tag{3}$$

$$1_R x = x \tag{4}$$

This defines scalar multiplication.

A *vector space* is a module where R is a field.

An *algebra* A is a vector space with a binary operation $\cdot : A \times A \to A$ such that, for all $x, y, z \in K, r, s \in R$,

$$(x+y) \cdot z = (x \cdot z) + (y \cdot z) \tag{5}$$

$$x \cdot (y+z) = z \cdot y + x \cdot z \tag{6}$$

$$rx \cdot sy = (rs)x \cdot y \tag{7}$$

(These axioms define bilinearity)

2 Manifolds

A *topological space* is an ordered pair (X, τ) where X is a set an τ is a set of subsets of X such that:

The empty set and X belong to τ,

An arbitrary, finite or infinite union of elements of τ is in τ,

The intersection of any finite number of elements of τ is in τ.

τ is a topology on X, and defining a topology allows one to define continuity, connectedness, and convergence.

A *topological base (basis* B of a topological space X is a set of open subsets of X such that every open subset of X can be written as a union of elements in B. We say the base generates the topology, which makes sense, as the elements in τ are each a union of elements of B. For this to be well-used,

The base elements must cover X,

Let $B_1, B_2 \in B$ have $B_1 \cap B_2 := I$. For each $x \in I$, there is a $B' \in B$ such that $x \in B' \subseteq I$

Remark 1. *A second-countable space is a space with a countable base. A compact, metrizable space* *is necessarily second-countable. (Throwback to proving an uncountable collection of 1-simplices is not metrizable.)*

A homeomorphism is a map between topological spaces that is an injection, is continuous, and has a continuous inverse map.

A *manifold* is a topological space such that every point $p \in M$ has a neighborhood homeomorphic to Euclidean space of the same dimension.

A *tangent space* is a vector space at a point of a manifold that consists of vectors tangent to that point. The tangent space of a sphere is a cylinder with the same radius as the sphere.

A *chart* is such a homeomorphism.

An *atlas* is a collection of charts such that the preimage of every chart in the atlas covers the manifold.

A *transition map* is a map that transitions the image of the intersection of the preimage of multiple charts from the image of the one to the another.

A *Lie Group* is a group that is also a differentiable manifold. It provides a way to classify continuous symmetries (e.g. the rotation matrices in a dimension are a group and a differentiable manifold; one can smoothly rotate a sphere).

3 Differential Forms from a Field-Theoretic Perspective

3.1 Differential 1-forms on $\mathbb{R}^2$

A *vector field* is an association of each point of whatever space we're working in of a vector there. A *covector* field is an association of each point with a linear function from vectors there to $\mathbb{R}$.

For example, let us work in $\mathbb{R}^2$. Standard coordinates give us 2 "basic" covector fields on $\mathbb{R}^2$ called dx and dy. The first is a covector field which sends a vector to its x-coordinate, and the second is a covector field which sends a vector to its y-coordinate. Every covector ω on $\mathbb{R}^2$ can be written as $\omega = \alpha dx + \beta dy$ for unique functions α, β on $\mathbb{R}^2$.

The operator d is defined as follows:

Given a function f on $\mathbb{R}^2$, we define

$$df := \tfrac{\partial f}{\partial x} dx + \tfrac{\partial f}{\partial y} dy$$

The covectors in this covector field are *differential 1-forms* on $\mathbb{R}^2$.

3.2 Differential 2-forms on $\mathbb{R}^2$

It is easy to see that, given a 1-form $\omega = \alpha dx + \beta dy$ on $\mathbb{R}^2$, that if $\omega = df$ for some f on $\mathbb{R}^2$, then

$$\frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y} \tag{8}$$

Let V be a vector space, and V^* be its dual space (sends elements of V to the field of coefficients). If V is 2-dimensional, then V^* is 2-dimensional, and so 2-covectors $\wedge^2 V^*$ is $\binom{2}{2}$=1-dimensional. If dx, dy is a basis for V^*, then $dx \wedge dy$ is *the* basis for $\wedge^2 V^*$. A *2-covector field* is an association for every point a 2-covector. We define the map d as

$$d(\alpha dx + \beta dy) = (\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y})dx \wedge dy \tag{9}$$

Example 1. $\int_{C_1} \omega \overset{?}{=} \int_{C_1} df = \int_a^b \frac{df}{dt} dt = f(b) - f(a)$

If our 1-form in this example is in fact a df of some f (?), with f nice and continuous, then df is *exact.*

If $dw = 0$ for some $k-$form, then w is *closed.* As you can see, exactness implies closedness due to $d^2 = 0$, but not necessarily the converse. Thus a exact form is the image of d, and a closed form is the image of d, further hinting at d begin the chain map in De Rham cohomology.

Closedness implies exactness on a contractible domain via the Poincare Lemma.

For those with experience in differential topology, a differential 2-form ω on a manifold M gives, for each $p \in M$, a bilinear form

$$\omega_p : T_pM \times T_pM \to \mathbb{R} \tag{10}$$

A bilinear form on a vector space V is nondegenerate if, for all $v \in V$, $\langle w, v\rangle = 0$ implies $w = 0$. A two form is nondegenerate if w_p is nondegenerate for all $p \in M$.

3.3 Wedge Product

Recall the rules for wedge product for one forms:

$$\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \tag{skew-symmetric}$$

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \tag{associativity}$$

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \wedge \mathbf{v} = c_1\mathbf{u}_1 \wedge \mathbf{v} + c_2\mathbf{u}_2 \wedge \mathbf{v} \tag{bilinearity}$$

$$\mathbf{u} \wedge (c_1\mathbf{v}_1 \wedge c_2\mathbf{v}_2) = c_1\mathbf{u} \wedge \mathbf{v}_1 + c_2\mathbf{u} \wedge \mathbf{v}_2 \tag{bilinearity}$$

$$\mathbf{u} \wedge \mathbf{u} = 0 \tag{*only for 1-forms}$$

We can construct k-forms by wedging a $(k-n)$-form with a n-form. The d operation sends k-forms to $(k+1)$-forms.

Example 2. $d(Fdy \wedge dz + Gdz \wedge dx + Hdx \wedge dy) =$
$(F_x dx + F_y dy + F_z dz) \wedge dy \wedge dz + (G_x dx + G_y dy + G_z dz) \wedge dz \wedge dx + (H_x dx + H_y dy + H_z dz) \wedge dx \wedge dy$
$= (F_x + G_y + H_z) dx \wedge dy \wedge dz$

It is easy to show $d^2 = 0$ (Hinting that k-forms may have a cohomological structure, perhaps named after De Rham)

A k-form is meant to be integrated over a k-manifold.

3.4 Symplectic Manifold

A *symplectic manifold* is a manifold with a closed, nondegenerate 2-form ω called the symplectic form. These show up in cotangent bundles of manifold. For a system modeled as a manifold, the cotangent bundle describes the phase space (space of all possible configurations of the system, e.g. Hilbert space) of the system.

Any real-valued differentiable function H on a symplectic manifold can be an energy function i.e Hamiltonian. Associated to any Hamiltonian is a Hamiltonian vector field, the integral curves of which (curves sketched along the vector field from the differential equation) is a solution to Hamilton's equations.

A *Hamiltonian flow* or *symplectomorphism* is the flow of this field on the symplectic manifold.

3.5 Hodge Star

The *Hodge Star* sends k-forms to $(n-k)$-forms in an n-dimensional manifold. (It maps k-dimensional vectors to $(n-k)$-dimensional vectors in an n-dimensional vector space.)

Example 3. *In a 3-dimensional Euclidean space, we can associate to every vector a plane orthogonal to that vector, and every plane an oriented normal vector.*

$$\mathbf{u} \wedge *\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}, \dim \mathbf{u} = \dim \mathbf{v} = k < n, \dim \mathbf{w} = n \tag{11}$$

where n is the dimension of our vector space.

A Field in terms of Differential Forms

The Electromagnetic Field F is given by the 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dx + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt \tag{12}$$

Computing dF gives us

$$dF = (\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} dx \wedge dy \wedge dz) + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}) dx \wedge dy \wedge dt + ... \tag{13}$$

Setting $dF = 0$, we find the first two Maxwell's Equations $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. For the other two Maxwell's Equations, we use $d * F = 4\pi\rho$:

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt \tag{14}$$

with

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt \tag{15}$$

where the metric used in the hodge star is the Lorentz metric.

4 Physics

4.1 The Calculus of Variations

An *action* is given by the integral of a *Lagrangian*, and, for fields, the Lagrangian is the integral of a *Lagrangian Density.*

Let $\mathcal{V}$ be the space of admissible functions that the Lagrangian can take as input, and let $T\mathcal{V}$ be the space of admissible variations of those functions, such that, for every $y \in \mathcal{V}, y + \alpha\delta y \in \mathcal{V}$ for all $\alpha \in \mathbb{R}, \delta y \in T\mathcal{V}$. Usually these spaces vary due to boundary conditions.

One of the secrets of the universe is that nature (classically, at least) takes the path in $\mathcal{V}$ that minimizes the action.

$$I[y(\cdot)] = \int_a^b L(t, y(t), y'(t))dt \tag{16}$$

In minimizing this functional, we get a $\overline{y}(t) \in \mathcal{V}$ such that $\overline{y}(t) + \alpha\delta y(t)$ increases the value of I, for all $\alpha \in \mathbb{R}, \delta y(t) \in T\mathcal{V}$. We define the variation of I in the direction of δy by $\langle \delta I[\overline{y}], \delta y\rangle$. Setting this equal to zero, we get the *Euler-Lagrange equations*:

$$0 = \langle \delta I[\overline{y}], \delta y\rangle = \frac{d}{d\alpha}[\int_a^b f(t, y(\alpha,t), \frac{\partial y(\alpha,t)}{\partial t})dt]|_{\alpha=0} \tag{17}$$

$$= [\int_a^b (\frac{\partial f}{\partial t}\frac{\partial t}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha})]_{\alpha=0} \tag{18}$$

for $y = \overline{y} + \alpha\delta y$. Often times, to get rid of derivatives of δy, one will integrate by parts to obtain whatever Lagrangian's Euler-Lagrange equations. In obtaining these Euler-Lagrange equations, you plug $\overline{y} + \alpha\delta y$ into the Lagrangian, differentiate the Lagrangian with respect to α, and set $\alpha = 0$.

4.2 Hamiltonian Mechanics

For a given Lagrangian L the coordinates y, y' are replaced by the coordinates position and momentum (q, p) by the transformation

$$p_i = \frac{\partial L}{\partial y'_i} \tag{19}$$

This is based on the *Legendre transformation* between tangent and cotangent bundles

$$TQ \longrightarrow T^*Q$$

$$(y, y') \longrightarrow (q, p)$$

between tangent and cotangent bundles.

The *Hamiltonian function H on phase space* T^*Q is given by

$$H(p, q, t) := py' - L(y, y', t), p = \frac{\partial L}{\partial y'} \tag{20}$$

$$y' = \frac{\partial H}{\partial p}, p' = -\frac{\partial H}{\partial y} \tag{21}$$

From the symplectic viewpoint, we can say that there exists a 2-form and inner product i such that, for any vector field X, the 2-form ω yields a 1-form $i(X)\omega$. Hamilton's equations are then equivalent to

$$i(X_H)\omega = dH \tag{22}$$

4.3 Poisson Bracket

The *Poisson Bracket* is given by

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \text{ for } f, g \in \mathcal{F}(T^*Q) \tag{23}$$

Here Hamilton's equations can be written as

$$q' = \{q, H\}, p' = \{p, H\} \tag{24}$$

Generally, for the time development of an observable given by f, the above system must satisfy the condition

$$\dot{f} = \{f, H\} \tag{25}$$

4.3.1 Quantization

The transition should give, for the Hamilton function H and the classical observables f, promote these to self-adjoint operators $\hat{H}, \hat{f}$ in a complex Hilbert space $\mathcal{H}$. The time course should shift to the quantum case

$$\dot{\hat{f}} = c[\hat{f}, \hat{H}], c = -\frac{ih}{2\pi} \tag{26}$$

where [,] is the natural *Lie bracket* given by $[A, B] = AB - BA$.