Quantum Field Theory Problems

Alec Lau

Question 1. *Consider the space $\mathcal{D}$ of continuously differentiable complex-valued functions f on [0,1]. Consider the operator A on $L^2([0,1])$ with domain $\mathcal{D}$, defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.*

Proof. We want to check if $\langle A\psi|\varphi\rangle = \langle\psi|A\varphi\rangle$. This gives us $\langle i\psi'|\varphi\rangle, \langle\psi|i\varphi'\rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^*\varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^*\varphi dx = \int_0^1 (-i)\psi'^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx) \neq \int_0^1 i\psi^*\varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is D_α, then, evaluating the same integral, we have $\int_0^1 (i\psi')^*\varphi dx = [-i\psi^*\varphi]_0^1 - \int_0^1 (-i)\psi^*\varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^*\varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^*\alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^*\alpha\psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1-\alpha^*\alpha)i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^*\alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^*\varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. □

Question 2. *Recall the definition of the manifold X_m, the measure λ_m on X_m, and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of $\mathcal{H}$. Finally, recall the definitions of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that*

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\mathbb{1} \tag{1}$$

where $\mathbb{1}$ is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}')[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_\mathbf{p}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}}[a(p), a^\dagger(p')]$. The first expression then becomes $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{4w_\mathbf{p}w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p - p')1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2w_\mathbf{p}} f(w_\mathbf{p}, \mathbf{p})$, we have the equality

$$\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}}}f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')] =$$
$$\int\int d\lambda_m(p)d\lambda_m(p')\sqrt{4w_{\mathbf{p}}w_{\mathbf{p}'}}f(\mathbf{p})^* g(\mathbf{p}')[a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p)\sqrt{4w_{\mathbf{p}}^2}f(\mathbf{p})^* g(\mathbf{p})1 = \int d\lambda_m(p)2w_{\mathbf{p}}f(\mathbf{p})^* g(\mathbf{p})1$. Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3\mathbf{p}}{(2\pi)^3}f(\mathbf{p})^* g(\mathbf{p})1$, where 1 is the identity operator on our Fock space.

Now we consider $\int\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}f(\mathbf{p})^* g(\mathbf{p}')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')1$. Integrating once, we find this gives us $\int \frac{d^3\mathbf{p}}{(2\pi)^3}f(\mathbf{p})^* g(\mathbf{p})1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. □

Question 3. *Consider the theory for massive scalar bosons of mass m. Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation*

$$\frac{\partial\varphi}{\partial t} = i[H_0, \varphi] \tag{2}$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f. Then, since $H_0 = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}}dx^4 f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$, we have

$$(H_0\varphi)(f) = \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})\int_{\mathbb{R}^{1,3}}dx^4 f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}'),$$
$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}}dx^4 f(x)\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2w_{\mathbf{p}'}}}(e^{-i(x,p)}a(\mathbf{p}' + e^{i(x,p)}a^\dagger(\mathbf{p}')\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}w_{\mathbf{p}}a^\dagger(\mathbf{p})a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}}dx^4 f(x)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}}A, \text{ where}$$
$$A =$$
$$a^\dagger(\mathbf{p})a(\mathbf{p})e^{-i(x,p)}a(\mathbf{p}') + a^\dagger(\mathbf{p})a(\mathbf{p})e^{i(x,p')}a^\dagger(\mathbf{p}') - e^{-i(x,p')}a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) - e^{i(x,p')}a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a(\mathbf{p}')) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p}')a(\mathbf{p}) - a(\mathbf{p}')a^\dagger(\mathbf{p})a(\mathbf{p})) + e^{i(x,p')}(a^\dagger(\mathbf{p})a(\mathbf{p})a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')a(\mathbf{p}))$$
$$= e^{-i(x,p')}[a^\dagger(\mathbf{p}), a(\mathbf{p}')]a(\mathbf{p}) + e^{i(x,p')}a^\dagger(\mathbf{p})[a(\mathbf{p}), a^\dagger(\mathbf{p}')]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. Also, notice that $[A, B] = AB - BA = (-1)(BA - AB) = -[B, A]$. Thus, A becomes

$$e^{-i(x,p')}(-1)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')a(\mathbf{p}) + e^{i(x,p')}a^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{w_\mathbf{p}}{\sqrt{2w_{\mathbf{p}'}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')(e^{i(x,p')}a^\dagger(\mathbf{p}) - e^{-i(x,p')}a(\mathbf{p}))$$
$$= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{w_\mathbf{p}}{\sqrt{2w_\mathbf{p}}}(e^{i(x,p)}a^\dagger(\mathbf{p}) - e^{-i(x,p)}a(\mathbf{p}))$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x,p) = tw_\mathbf{p} + \mathbf{x}\cdot\mathbf{p}$, so the time derivative of $e^{\pm i(x,p)} = \pm iw_\mathbf{p}e^{\pm i(x,p)}$. Thus, $\frac{\partial\varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}}(-e^{-i(x,p)}a(\mathbf{p}') + e^{i(x,p)}a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived form the commutator. Thus, $\frac{\partial\varphi}{\partial t} = i[H_0, \varphi]$, up to a set of measure zero. □

Question 4. *In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude*

$$\langle \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4 | S | \boldsymbol{p}_1\rangle \tag{3}$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!}\int_\mathbb{R} d^4x : \varphi(x)^4 : +\mathcal{O}(g^2)$. We then have

$$\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|S|\mathbf{p_1}\rangle = \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_\mathbb{R} d^4x\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}| : \varphi(x)^4 : |\mathbf{p_1}\rangle + \mathcal{O}(g^2)$$
$$= \langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle - \frac{ig}{4!}\int_\mathbb{R} d^4x\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle + \mathcal{O}(g^2)$$

For the first term, we notice that $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = \langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4})a^\dagger(\mathbf{p_1})|0\rangle$. Applying the first two operators we get either ground state back if $\mathbf{p_1} = \mathbf{p_4}$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}|\mathbf{p_1}\rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0|a(\mathbf{p})\varphi(x)|0\rangle = \frac{e^{i(x,p)}}{\sqrt{2w_\mathbf{p}}}$, $\langle 0|\varphi(x)a^\dagger(\mathbf{p})|0\rangle = \frac{e^{-i(x,p)}}{\sqrt{2w_\mathbf{p}}}$. $\langle 0|a(\mathbf{p_2})a(\mathbf{p_3})a(\mathbf{p_4}) : \varphi(x)^4 : a^\dagger(\mathbf{p_1})|0\rangle = \langle 0|a(\mathbf{p_2})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_3})\varphi(x)|0\rangle\langle 0|a(\mathbf{p_4})\varphi(x)|0\rangle\langle 0|a^\dagger(\mathbf{p_1})\varphi(x)|0\rangle$. This expression is equal to $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8-1)!!$ diagrams, and 4! diagrams of this type. Thus we have 4! $(e^{i(x,p_2+p_3+p_4-p_1)})/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}})$. Thus we have $\langle \mathbf{p_2},\mathbf{p_3},\mathbf{p_4}|S|\mathbf{p_1}\rangle = (-ig(2\pi)^4\delta^{(4)}(p_2+p_3+p_4-p_1))/(\sqrt{16w_{\mathbf{p_2}}w_{\mathbf{p_3}}w_{\mathbf{p_4}}w_{\mathbf{p_1}}}) + \mathcal{O}(g^2)$. □

Question 5. *1. Derive Maxwell's equations as the Euler-Lagrange equations of the action*

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4}$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard from by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory.

2. Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{5}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{6}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2+B^2); S = E \times B \tag{7}$$

Peskin & Schroeder 2.1.

Proof. 1. Let's first calculate $F^{\mu\nu}$. Given our identification with E^i and B^i, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8}$$

Treating A_ν as our dynamical variables, we take

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{9}$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)] = 0 \tag{10}$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}[-\frac{1}{4}(2\partial_\mu A_\nu \partial_\mu A_\nu - 2\partial_\nu A_\mu \partial_\mu A_\nu)] = 0 \tag{11}$$

$$\partial_\mu[-\frac{1}{4}(4\partial_\mu A_\nu - 4\partial_\nu A_\mu)] = 0 \tag{12}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{13}$$

With the identification $F^{0i} = -E^i, F^{ij} = -\epsilon^{ijk}B^k$, we have $-\frac{\partial E}{\partial t} - \partial_i \epsilon^{ijk} B^k = 0$. Because I always forget the Levi-Civita symbols, we recall that

$$\epsilon^{ijk}\partial_j v_k = (\nabla \times v)^i \tag{14}$$

and thus $-\frac{\partial E}{\partial t} + \epsilon^{jik}\partial_i B^k = 0$, or

$$\nabla \times B = \frac{\partial E}{\partial t} \tag{15}$$

2. With this construction, we have

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{16}$$
$$= \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\gamma)}\partial^\nu A_\gamma - \mathcal{L}\delta^{\mu\nu} + \partial_\lambda F^{\mu\lambda}A^\nu \tag{17}$$
$$= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) \tag{18}$$
$$= F^{\mu\iota}(\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} - \partial_\lambda F^{\lambda\mu}A^\nu \tag{19}$$
$$= F^{\mu\iota}F_\iota^\nu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} - (0)A^\nu \tag{20}$$

This is now a viable energy-momentum tensor. We now $T^{\hat{0}0}$ and $T^{\hat{0}i}$:

$$T^{\hat{0}0} = F^{0\iota}F_\iota^0 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{21}$$
$$= E^\iota E_\iota + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{22}$$

We then have

$$\langle,\rangle = tr(\overline{F^{\mu\nu}}F^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_Z & -B_y & B_x & 0 \end{pmatrix} \tag{23}$$

$$= tr(\begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix} \tag{24}$$

This is equal to $2(B^2 - E^2)$. Thus we have that $T^{\hat{\mu\nu}} = E^2 + \frac{1}{4}2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $\hat{T^{0i}}$, we have

$$\hat{T^{0i}} = F^{0j}F^i_j + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{25}$$

$$= E^j \epsilon_{jik} B^k g^{mi} + \frac{1}{2}(B^2 - E^2)g^{0i} \tag{26}$$

$$= \mathbf{E} \times \mathbf{B} = \mathbf{S} \tag{27}$$

□

Question 6. *Consider the the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is*

$$S = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \tag{28}$$

(a) *Find the conjugate momenta to $\phi(x), \phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is*

$$H = \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \tag{29}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) *Diagonlize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.*

(c) *Rewrite the conserved charge*

$$Q = \int d^3x \frac{i}{2}(\phi^*\pi^* - \pi\phi) \tag{30}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) *Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where a = 1,2. Show that there are now four conserved charges, one given by the* *generalization of part (c), and the other three given by*

$$Q^i = \int d^3x \frac{i}{2}(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b) \tag{31}$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Peskin & Schroeder, 2.2.

Proof. (a) We have that $p(\mathbf{x}) = \frac{\partial L}{\partial \dot{\phi}(\mathbf{x})} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2\phi^*\phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x(\partial_\mu \phi^* g^\mu_\nu \partial^\nu \phi - m^2\phi^*\phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x(\frac{\partial \phi^*}{\partial t}\frac{\partial \phi}{\partial t} - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x(\dot{\phi}^*\dot{\phi} - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi)$. Thus, $\pi = \dot{\phi}^*$. Similarly, $\pi^* = \dot{\phi}$. Since ϕ, ϕ^* are the dynamical variables, the canonical commutation relations are

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{32}$$

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0 \tag{33}$$

from quantization of the Klein-Gordon field given in the textbook. Given the equation for the Hamiltonian, we have

$$H = \int d^3x[\sum_{a,b} \pi_i(\mathbf{x})\dot{\phi}_i(\mathbf{x}) - \mathcal{L}] \tag{34}$$

$$= \int d^3x[\pi^*\dot{\phi}^* + \pi\dot{\phi} - \mathcal{L}] \tag{35}$$

$$= \int d^3x[\dot{\phi}\dot{\phi}^* + \dot{\phi}^*\dot{\phi} - \mathcal{L}] \tag{36}$$

$$= \int d^3x[2\dot{\phi}\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \tag{37}$$

$$= \int d^3x[\dot{\phi}\dot{\phi}^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \tag{38}$$

$$= \int d^3x[\pi^*\pi + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \tag{39}$$

We want to compute $i\frac{\partial\phi}{\partial t}$ via the Heisenberg Equation of Motion, so we calculate $[\phi, H]$.

$$i\frac{\partial\phi}{\partial t} = [\phi, H] \tag{40}$$
$$= [\phi(x'), \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \tag{41}$$
$$= \int d^3x[\phi(x'), \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{42}$$
$$= \int d^3x([\phi(x'), \pi^*\pi] + [\phi, \nabla\phi^* \cdot \nabla\phi] + m^2[\phi, \phi^*\phi]) \tag{43}$$
$$= \int d^3x\delta^{(3)}(x' - x)i\pi^*(x) \tag{44}$$
$$= i\pi^*(x) \tag{45}$$
$$i\frac{\partial\pi^*}{\partial t} = [\pi^*, H] \tag{46}$$
$$= [\pi^*(x'), \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \tag{47}$$
$$= \int d^3x[\pi^*(x'), \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{48}$$
$$\text{(integrating by parts)} = \int d^3x([\pi^*(x'), \pi^*\pi] + [\pi^*(x'), \phi^*(-\nabla^2 + m^2)\phi]) \tag{49}$$
$$= \int d^3x\delta^{(3)}(x' - x)(-i)(-\nabla^2 + m^2)\phi(x) \tag{50}$$
$$= i(\nabla^2 - m^2)\phi \tag{51}$$

Since $i\frac{\partial\phi}{\partial t} = i\pi^*$ and $i\frac{\partial\pi^*}{\partial t} = i(\nabla^2 - m^2)\phi$, so $\frac{\partial^2\phi}{\partial t^2} = (\nabla^2 - m^2)\phi$, which is the Klein-Gordon equation.

(b) Since ϕ satisfies the Klein-Gordon equation, and, in the same way, so does ϕ^*, we take the Fourier transform to gain more insight into $\nabla^2\phi$:

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}) \Rightarrow \tag{52}$$
$$[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi(\mathbf{p}) = 0,\ [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi^*(\mathbf{p}) = 0 \tag{53}$$

We write ϕ in terms of two real valued scalar free fields ψ_1, ψ_2, of which we already know the theory:

$$\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}},\ \phi^* = \frac{\psi_1 - i\psi_2}{\sqrt{2}} \tag{54}$$

Since ψ_1, ψ_2 are independent free fields, both must satisfy the harmonic oscillator equation:

$$\tfrac{1}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2+m^2)]\psi_1 = 0,\ \tfrac{\pm i}{\sqrt{2}}[\tfrac{\partial^2}{\partial t^2} + (p^2+m^2)]\psi_2 = 0 \Rightarrow$$
$$[\tfrac{\partial^2}{\partial t^2} + (p^2+m^2)]\psi_1 = 0,\ [\tfrac{\partial^2}{\partial t^2} + (p^2+m^2)]\psi_2 = 0 \Rightarrow$$
$$\omega_1 = \sqrt{p_1^2+m^2},\ \omega_2 = \sqrt{p_2^2+m^2}$$

Since the frequencies of the oscillators have independent momentums and ϕ is not hermitian, we create two different creation and annihilation operators:

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \frac{i}{\sqrt{2\omega_i}}p_i,\ a_i^\dagger = \sqrt{\frac{\omega_i}{2}}q_i - \frac{i}{\sqrt{2\omega_i}}p_i,\ i \in \{1,2\} \tag{55}$$

where $q_1 = \phi, q_2 = \phi^*, p_1 = \pi, p_2 = \pi^*$, with the notation in the spirit of Peskin and Schroeder. These creation operators, given their frequencies, represent creating two different particles with mass m. From the theory of a real-valued scalar free field, we know that

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \tag{56}$$
$$\phi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \tag{57}$$

The two different operators ensure that ϕ is not hermitian. From above we know that $\pi = \dot{\phi}^*, \pi^* = \dot{\phi}$, and, using our real-valued scalar free field as reference, we have

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \tag{58}$$
$$\pi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}(a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \tag{59}$$

Finally, we rewrite our Hamiltonian in terms of our operators:

$$H = \int d^3\mathbf{x}(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \tag{60}$$

$$= \int d^3\mathbf{x}(\int\int \frac{d^3\mathbf{p}d^3\mathbf{p'}}{(2\pi)^6}\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{p'}}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p'})e^{i(\mathbf{p}-\mathbf{p'})\cdot\mathbf{x}} - a_1(\mathbf{p})a_2(\mathbf{p'})e^{i(\mathbf{p}+\mathbf{p'})\cdot\mathbf{x}} \tag{61}$$

$$- a_2^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p'})e^{-i(\mathbf{p}+\mathbf{p'})\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})a_2(\mathbf{p'})e^{i(\mathbf{p'}-\mathbf{p})\cdot\mathbf{x}}\} \tag{62}$$

$$+ \int\int \frac{d^3\mathbf{p}d^3\mathbf{p'}}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{p'}}}[-i\mathbf{p}a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + i\mathbf{p}a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}] \tag{63}$$

$$\cdot [i\mathbf{p'}a_1(\mathbf{p'})e^{i\mathbf{p'}\cdot\mathbf{x}} - i\mathbf{p'}a_2^\dagger(\mathbf{p'})e^{-i\mathbf{p'}\cdot\mathbf{x}}] \tag{64}$$

$$+ m^2\int\int \frac{d^3\mathbf{p}d^3\mathbf{p'}}{(2\pi)^6}\frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{p'}}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p'})e^{i(\mathbf{p'}-\mathbf{p})\cdot\mathbf{x}} + a_1^\dagger(\mathbf{p})a_2^\dagger(\mathbf{p'})e^{-i(\mathbf{p}+\mathbf{p'})\cdot\mathbf{x}} \tag{65}$$

$$+ a_2(\mathbf{p})a_1(\mathbf{p'})e^{i(\mathbf{p}+\mathbf{p'})\cdot\mathbf{x}} + a_2(\mathbf{p})a_2^\dagger(\mathbf{p'})e^{i(\mathbf{p}-\mathbf{p'})\cdot\mathbf{x}}\}) \tag{66}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3}(\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \tag{67}$$

$$+ \frac{p^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\} \tag{68}$$

$$+ \frac{m^2}{2\omega_\mathbf{p}}\{a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2(\mathbf{p})a_2^\dagger(\mathbf{p})\}) \tag{69}$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3}[\frac{\omega_\mathbf{p}}{2}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p})a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p})a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\} \tag{70}$$

$$+ \frac{p^2+m^2}{2\omega_\mathbf{p}}\{a_1(\mathbf{p})a_1^\dagger(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) + a_2^\dagger(\mathbf{p})a_2(\mathbf{p})\}] \tag{71}$$

where in (70) the middle two terms have a positive sign because we subtract $\mathbf{p}\cdot\mathbf{p'} = \mathbf{p}\cdot(-\mathbf{p})$. Furthermore, since $\mathbf{p} \neq -\mathbf{p}$, we can commute our operators. Thus we have the Hamiltonian as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\{a_1a_1^\dagger + a_2a_2^\dagger\} \tag{72}$$

Since this Hamiltonian is constructed purely out of constants and operators whose eigenvectors are momentum eigenstates, our Hamiltonian is now diagonalized. The indices 1 and 2 represent the two particles of mass m.

(c) This is just plugging in our values for momentum and position and integrating, like the previous problem. We have

$$Q = \int d^3\mathbf{x}\frac{i}{2}(\phi^*\pi^* - \pi\phi) \tag{73}$$

$$= \int d^3\mathbf{x}\frac{i}{2}\int d^3\mathbf{p}\int d^3\mathbf{p'}\frac{1}{(2\pi)^6}\frac{1}{2}[(-i)(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}) \tag{74}$$

$$- i(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}})] \tag{75}$$

$$= \int d^3\mathbf{p}\frac{1}{(2\pi)^3}\frac{1}{4}([a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \tag{76}$$

$$+ [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) - a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]) \tag{77}$$

$$= \frac{1}{2}\int d^3\mathbf{p}\frac{1}{(2\pi)^3}[a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \tag{78}$$

This means that this theory has two particle types: one created by $a_1^\dagger(\mathbf{p})$ and one created by $a_2^\dagger(\mathbf{p})$. In examining $[Q, a_i^\dagger]\,|n\rangle$ for some state n-particle state $|n\rangle$, we can deduce the charge. It is easy to see that $[a_1, a_2] = 0, [a_i, a_i^\dagger] = 1$ since ψ_1, ψ_2 are independent fields. Thus

$$[Q, a_1^\dagger] = a_1^\dagger, [Q, a_2^\dagger] = -a_2^\dagger \tag{79}$$

This means that the charges are valued at 1 unit for particles created by $a_1^\dagger$ and -1 for particles created by $a_2^\dagger$].

(d) For two complex scalar fields, the lagrangian is then

$$\mathcal{L} = \partial_\mu\phi_1^*\partial^\mu\phi_1 - m^2\phi_1^*\phi_1 + \partial_\mu\phi_2^*\partial^\mu\phi_2 - m^2\phi_2^*\phi_2 \tag{80}$$

We then have

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\Delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\Delta\phi_2 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1^*)}\Delta\phi_1^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2^*)}\Delta\phi_2^* - \mathcal{J}^\mu \tag{81}$$

$$= \partial^\mu\phi_1^*\Delta\phi_1 + \partial^\mu\phi_2^*\Delta\phi_2 + \partial^\mu\phi_1\Delta\phi_1^* + \partial^\mu\phi_2\Delta\phi_2^* - \mathcal{J}^\mu \tag{82}$$

$$\to Q = \int d^3x(\pi_1^*\Delta\phi_1 + \pi_2^*\Delta\phi_2 + \pi_1\Delta\phi_1^* + \pi_2\Delta\phi_2^*) \tag{83}$$

If we set

$$\Phi := \begin{pmatrix} \phi_1 \\ \\ \phi_2 \end{pmatrix} \tag{84}$$

we rewrite our theory as

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi \tag{85}$$

$$Q = \int d^3x (\dot{\Phi}^\dagger \Delta\Phi + (\Delta\Phi)^\dagger \dot{\Phi}) \tag{86}$$

The symmetry of this lagrangian is

$$\Phi \mapsto M\Phi \tag{87}$$

for $M \in U(2)$. We know that this system should have $U(1)$ symmetry from the above problem. Using the $\det : U(n) \to U(1)$ map, we have a short exact sequence

$$SU(2) \to U(2) \to U(1) \tag{88}$$

giving us $U(2) = SU(2) \times U(1)$. For this reason, since $U(1)$ is just a complex number, each conserved charge from this symmetry has the commutation relations of $SU(2)$. In order to put this into a continuous symmetry picture, we exponentiate an element $\sigma \in SU(2)$ is a factor $i(\alpha_1, \alpha_2)$ and take $\alpha_1, \alpha_2 \to 1$:

$$\Phi \mapsto e^{i(\alpha_1, \alpha_2)\sigma} \Phi \tag{89}$$

$$\Delta\Phi \mapsto i\sigma\Phi \tag{90}$$

$$\Delta\Phi^* \mapsto -i\sigma\Phi \tag{91}$$

$SU(2)$ is generated by the Pauli matrices, so we have conserved charges

$$Q^i = i \int d^3x(\dot{\Phi}^\dagger \sigma^i \Phi - \Phi^\dagger \sigma^i \dot{\Phi}) \tag{92}$$

$$= i \int d^3x(\phi_a^* \sigma_{ab}^i \pi_b^* - \pi_a \sigma_{ab}^i \phi_b) \tag{93}$$

Generalizing to n independent identical complex scalar fields, we let $\Phi = (\phi_1, ..., \phi_n)^T$, and our symmetry becomes $U(n) = SU(n) \times U(1)$, meaning the charges we get are of the same form as what we got, but replacing the σ^i with $n-$dimensional skew-hermitian matrices.

□

Question 7. *Evaluate the function*

$$\langle 0| \phi(x)\phi(y) |0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)} \tag{94}$$

for $(x-y)$ *spacelike so that* $(x-y)^2 = -r^2$*, explicitly in terms of Bessel functions.*

Proof. We have

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \tag{95}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} \frac{dp}{(2\pi)^3} \frac{p^2}{\sqrt{p^2+m^2}} e^{ipr\cos\theta} \tag{96}$$

θ is the angle between p and $(x-y)$, which also works for the conversion to spherical coordinates. We then have

$$D(x-y) = \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} \int_0^{\pi} d\theta \sin\theta (\sum_{n=-\infty}^{\infty} J_n(pr) e^{in\theta}) \tag{97}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} \int_0^{\pi} d\theta \sin\theta (J_0(pr) + 2\sum_{n=1}^{\infty} i^n J_n(pr)\cos(n\theta)) \tag{98}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} [2J_0(pr) + 2\sum_{n=1}^{\infty} i^n J_n(pr) \frac{\cos n\pi + 1}{1-n^2}] \tag{99}$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} \frac{dpp^2}{\sqrt{p^2+m^2}} [J_0(pr) + \sum_{n=1}^{\infty} J_{2n}(pr) \frac{2}{1-4n^2}] \tag{100}$$

As in the book, the integrand has branch cuts on the imaginary axis starting at $p = \pm im$, so we push the contour up to wrap around the upper branch cut. With $\rho = -ip$, we get

$$D(x-y) = \frac{-i}{2\pi^2}\int_m^\infty d\rho \frac{-\rho^2}{\rho^2 - m^2}[J_0(i\rho r) + \sum_{n=1}^{\infty} J_{2n}(i\rho r)\frac{2}{1-4n^2}] \tag{101}$$

□