Bundle Theory Problems

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All problems written by Prof. Ralph Cohen. Referenced to Cohen's notes/textbook-in-progress, "Bundles, Homotopy, and Manifolds."

Question 1. *Let $\xi \to B$ be an n-dimensional vector bundle.*

1. *Define clutching functions of the $nk-$dimensional $k-$fold tensor product bundle $\otimes^k \xi \to B$ in terms of clutching functions of ξ.*

2. *Define clutching functions of the $k-$fold exterior product bundle $\wedge^k \xi \to B$ in terms of clutching functions of ξ.*

Proof. 1. Let $\xi \to B$ be an $n-$dimensional vector bundle with clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. We define clutching functions on the $k-$fold tensor bundle $\otimes^k \xi \to B$ by taking the product of our clutching functions on ξ:

$$\phi^{\otimes^k \xi}_{\alpha,\beta} : \qquad U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta} \times \ldots \times \phi_{\alpha,\beta}} GL_n(\mathbb{R}) \times \ldots \times GL_n(\mathbb{R}) \xrightarrow{\quad\otimes\quad} GL_{n^k}(\mathbb{R})$$

$$x \xmapsto{\phi_{\alpha,\beta} \times \ldots \times \phi_{\alpha,\beta}} A \times \ldots \times A \xmapsto{\quad\otimes\quad} A \otimes \ldots \otimes A$$

where $A \in GL_n(\mathbb{R})$ is the linear transformation on ξ is the image of the regular clutching function on ξ. Here the tensor product of two linear transformations $A_1 : \xi \to \xi, \ldots, A_k : \xi \to \xi$ is the induced linear transformation $A_1 \otimes \ldots \otimes A_k : \xi \otimes \ldots \otimes \xi \to \xi \otimes \ldots \otimes \xi$. This well-defined because we simply take the automorphism associated to the clutching function on ξ and tensor k copies of it; we get $\phi_{\beta,\alpha} = A^{-1} \otimes \ldots \otimes A^{-1}$, which, when applied before or after $\phi_{\alpha,\beta}$, we get $Id \otimes \ldots \otimes Id$. Thus $\phi_{\alpha,\beta} = \phi^{-1}_{\beta,\alpha}$.

2. We approach this problem by considering the vector space associated with $\otimes^k \xi$, for ξ and $n-$dimensional vector space. Consider an orthonormal basis $\{e_i\}_{0 \le i \le n}$ of ξ. We define an isomorphism from $\xi \times \ldots \times \xi$ to $\xi \otimes \ldots \otimes \xi$:

$$e_{i_1} \times \ldots \times e_{i_k} \mapsto e_{i_1} \otimes \ldots \otimes e_{i_k} \tag{1}$$

Thus we have nk generators of $\otimes^k \xi$, so $\otimes^k V$ is isomorphic to an $nk-$dimensional vector space. Since $\wedge^k \xi$ a quotient of this vector space, this should probably be isomorphic to a subspace.

We construct a basis of $\wedge^k\xi$. The symmetry quotient $a\otimes b+b\otimes a$ implies that $a\otimes b=-b\otimes a$, in particular $a\otimes a=-a\otimes a$, so $a\otimes a=0$, for $a,b\in\xi$. Thus there can be no repeated indices, and permutations of index combinations are linearly dependent. Thus, the allowable basis vectors are just the $\binom{n}{k}$ combinations of k entries spanning 1 to n. Thus $\wedge^k\xi$ is isomorphic to an $\binom{n}{k}$-dimensional vector space. If $k>n$, there must be repeated indices, and so the vector space is 0-dimensional. Thus we need our clutching functions to have image in $GL_{\binom{n}{k}}(\mathbb{R})$ from k elements in $GL_n(\mathbb{R})$. Antisymmetry, along with general objects one runs into with dealing with exterior algebras, are determinants. Suppose $[g_{ij}]\in GL_k(\mathbb{R})$ is the transition function associated with $U_i\cap U_j$, and associate the $k\times k$-submatrix of $[g_{ij}]$ denoted $[g_{ij}]_{h\in\binom{n}{k}}$ (henceforth we will say that $h\in S(\binom{n}{k})$ is an increasing sequence of k integers between 1 and n), with entries kept being the entries associated with the corresponding indices of the basis vectors of $\wedge^k\xi$. Define the element in $GL_{\binom{n}{k}}(\mathbb{R})$ as the matrix

$$(\mathrm{Det}[g_{ij}]_{h\in S(\binom{n}{k})})_{m,l\in\binom{n}{k}} \tag{2}$$

Example 1.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} Det\begin{pmatrix} a & b \\ d & e \end{pmatrix} & Det\begin{pmatrix} a & c \\ d & f \end{pmatrix} & Det\begin{pmatrix} b & b \\ e & f \end{pmatrix} \\ Det\begin{pmatrix} a & b \\ g & h \end{pmatrix} & Det\begin{pmatrix} a & c \\ g & i \end{pmatrix} & Det\begin{pmatrix} b & c \\ h & i \end{pmatrix} \\ Det\begin{pmatrix} d & e \\ g & h \end{pmatrix} & Det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & Det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix} \tag{3}$$

where the submatrix entries are determined by ranging across the elements of the $\binom{n}{k}$ permutations.

This is clearly an injection, as scaling any entry in the preimage scales a unique combination of entries in the image. Thus, by changing one entry in the image, one cannot counteract this change by changing another entry. Furthermore, Id_n trivially maps to $Id_{\binom{n}{k}}$, since all off-diagonal elements have the determinants of matrices with only one nonzero element, and the diagonal elements have determinants of identity matrices. Thus the image is in a subgroup of $GL_{\binom{n}{k}}(\mathbb{R})$, and this map has a well-defined inverse. To check that this is a well-defined clutching function, we consider the clutching function associated with $[g_{ij}^{-1}]$. We have

$$A_{ij}^{-1} = \frac{n!\epsilon^{i_1...i_b}\epsilon_{j_1...j_b}A_{i_1}^{j_1}...A_{i_b}^{j_b}}{k!\epsilon^{ii_2...i_b}\epsilon_{jj_2...j_b}A_{i_2}^{j_2}...A_{i_b}^{j_b}}, \text{ for } b := \frac{n!}{k!} \tag{4}$$

$$= \frac{n\epsilon^{i_1...i_b}\epsilon_{j_1...j_b}\text{Det}[[g_{ij}]_{h_1}]...\text{Det}[[g_{ij}]_{h_b}]}{\epsilon^{ii_2...i_b}\epsilon_{jj_2...j_b}\text{Det}[[g_{ij}]_{h_2}]...\text{Det}[[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}). \tag{5}$$

where h_α is the corresponding permutation with the i_α, j_α entry in our matrix.

$$(g_{ij})^{-1} = \frac{n\epsilon^{i_1...i_n}\epsilon_{j_1...j_n}g_{i_1j_1}...g_{i_nj_n}}{\epsilon^{ii_2...i_n}\epsilon_{jj_2...j_n}g_{i_2j_2}...g_{i_nj_n}} \tag{6}$$

$$(\text{Det}(g_{ij}^{-1}))_{h\in S(\binom{n}{k})})_{m,l\in\binom{n}{k}} = \frac{n\epsilon^{i_1...i_b}\epsilon_{j_1...j_b}\epsilon^{s_1...t_k}\epsilon_{s_1...t_k}g_{s_1t_1}...g_{s_kt_k}\epsilon^{i''_1...i''_k}\epsilon_{j''_1...j''_k}g_{i''_1j''_1}...g_{i''_kj''_k}}{\epsilon^{ii_2...i_b}\epsilon_{jj_2...j_b}\epsilon^{i'_1...i'_k}\epsilon_{j'_1...j'_k}g_{i'_1j'_1}...g_{i'_kj'_k}\epsilon^{i''_1...i''_k}\epsilon_{j''_1...j''_k}g_{i''_1j''_1}...g_{i''_kj''_k}} \tag{7}$$

$$= \frac{n\epsilon^{i_1...i_b}\epsilon_{j_1...j_b}\text{Det}[[g_{ij}]_{h_1}]...\text{Det}[[g_{ij}]_{h_b}]}{\epsilon^{ii_2...i_b}\epsilon_{jj_2...j_b}\text{Det}[[g_{ij}]_{h_2}]...\text{Det}[[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}) \tag{8}$$

where in the last step we rewrite our scare labels for elements of $S(\binom{n}{k})$, and we find this is exactly equal to A_{ij}^{-1}. Thus A_{ij}^{-1} is the image of g_{ij}^{-1}, and so our clutching functions are well-defined.

□

Question 2. *1. Notice that the tensor product of two one-dimensional vector bundles ("line bundles") over a space B is still a one dimensional vector bundle . Show that the set of isomorphism classes of one-dimensional (real) vector bundles over B is an abelian monoid with respect to tensor product. In particular, what is the unit of this monoid?*

2. Show that in fact this abelian monoid is an abelian group.

Proof. 1. First we note that the clutching functions of a vector bundle uniquely determine the isomorphism class of said bundle. In the spirit of the first problem, we define our clutching functions for the $k-$fold tensor product of line bundles by multiplying the clutching functions of a single line bundle:

$$\phi^{\otimes^k \mathbb{R}}_{\alpha,\beta} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} \mathbb{R}^* \times \ldots \times \mathbb{R}^* \xrightarrow{\cong} \mathbb{R} \tag{9}$$

$$\to \mathbb{R} \otimes \ldots \otimes \mathbb{R} \tag{10}$$

where that last isomorphism is due to having the basis vector, comprised of the e_i basis vector for the i^{th} tensor factor, as $e_1 \otimes \ldots \otimes e^*_k$. This is an associative operation, as $s_{\alpha,\beta}(\tilde{s}_{\alpha,\beta} s'_{\alpha,\beta}) \to (e_1 \otimes e_2) \otimes e_3 \cong e_1 \otimes (e_2 \otimes e_3) \leftarrow (s_{\alpha,\beta}\tilde{s}_{\alpha,\beta})s'_{\alpha,\beta}$. This map is abelian, since we can simply map $e_i \times e_j \to e_j \otimes e_i$, as an isomorphism, and as $e_i \times e_j \to e_i \otimes e_j$ is an isomorphism, $\otimes \mathbb{R}' \cong \mathbb{R}' \otimes \mathbb{R}$. Thus these are the same isomorphism class. It remains to show that there is an identity element. From the properties of the tensor product, $e_1 \otimes e_2 = e_1 e_2 \otimes 1 = 1 \otimes e_1 e_2$. Furthermore, $s_{\alpha,\beta} s_{\beta,\alpha} = 1$. Thus we have a natural isomorphism from principal bundles $(B \times \{1\} = B)$ $\mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ given by $e_1 \to e_1 \otimes 1 \cong 1 \otimes e_1$. Using the inverse map (that map was an isomorphism) Thus the identity element is given by $1 \in \mathbb{R}^*$. Since we have associativity, commutativity, and an identity element, the isomorphism classes of 1-dimensional vector bundles forms an abelian monoid.

2. It remains to prove that every element has a unique inverse, as an abelian group is an abelian monoid where every element has a unique inverse. Consider the map

$$\phi^{\otimes^2 \mathbb{R}}_{\alpha,\alpha} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\phi_{\beta,\alpha}} \mathbb{R}^* \times \mathbb{R}^* \xrightarrow{\otimes} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R} \tag{11}$$

$$x \to a \times a^{-1} \to a \otimes a^{-1} \xrightarrow{b} aa^{-1} \otimes 1 \to 1 \otimes 1 \xrightarrow{c} 1 \tag{12}$$

where the b map is due to the linearity of the tensor product and the c map is due to the natural isomorphism defined above. Since $\phi_{\beta,\alpha}$ is the unique inverse clutching function to $\phi_{\alpha,\beta}$, each element in this monoid has a unique inverse, and thus the isomorphism classes of 1-dimensional vector bundles is an abelian group with the tensor product operation.

□

Question 3. *Let X be a space with a basepoint $x_0 \in X$. Recall that the (reduced) suspension of*

X, ΣX, is the space

$$\Sigma X = X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\} \tag{13}$$

Here I am thinking of S^1 as the unit complex numbers. Let $(Y, y0)$ be another space with basepoint. Consider the (based) "loop space"

$$\Omega Y = Map((S^1, \{1\}), (Y, y_0)) \tag{14}$$

This is the space of maps from S^1 to Y that take $1 \in S^1$ to the basepoint $y_0 \in Y$, endowed with the compact - open topology.

1. *Prove that there is a bijection*

$$[\Sigma X, Y] \cong [X, \Omega Y] \tag{15}$$

Here the notation $[-, -]$ denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

2. *Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the proof of Corollary 4.10 in the text. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".*

Proof. 1. We being by considering elements of $[X, \Omega Y]$. Define a basepoint-preserving map $f \in [X, \Omega Y]$ for some $x \in X$ by $f(x)$. This is thus a basepoint-preserving map from $(S^1, \{1\})$ to (Y, y_0), denoted $f(x)(t)$. Since $f(x)(t)$ is basepoint-preserving, $f(x)(1) = y_0$. Furthermore, since f is basepoint-preserving, $f(x_0)(t) = y_0$. We now examine elements of $[\Sigma X, Y]$. Suppose we have a basepoint-preserving map $g \in [\Sigma X, Y]$. If we consider $(x, t) \in X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\}$, we have that $(x, 1) = (x_0, t), \forall x \in X, t \in S^1$. Thus we have $g(x, 1) = g(x_0, t) = y_0$. We have our correspondence as $[g(x, t)] \mapsto [f(x)(t)]$, given by the corresponding that, for $g : \Sigma X \to Y$, we associate the family of loops $f(x)(t)$ by restricting g to the images of the loops $\{x\} \times S^1 \subset \Sigma X$. We first prove surjectivity. Given a map $j : (S^1, \{1\}) \to (Y, y_0)$, we have the preimage of this map in $[X, \Omega Y]$ to be the $x \in X$ such that $g(x,t) \subset Y$ is of the same homotopy type in Y (we fix x and let t span S^1 to get the same loop). Thus we associate the homotopy type of $g(x,t)$ with the homotopy type of $[f(x)(t)]$, so the map is surjective. Suppose $[g_1(x_1,t_1)] \neq [g_2(x_2,t_2)]$. Then we associate the maps $[f_1(x_1)(t_1)], [f_2(x_2)(t_2)] \in [X, \Omega Y]$, respectively. Since $x_1 \neq x_2, [g_1] \neq [g_2]$, the maps $f_1(x_1) : S^1 \to Y, f_2(x_2) : S^1 \to Y$ are not homotopically equivalent, and thus the loops are not homotopically equivalent. Thus $[f_1(x_1)(t_1)] \neq [f_2(x_2)(t_2)]$. We check that this works for the basepoint map: $[g(x_0,t)] \mapsto [f(x_0)(t)] = [\text{the constant map}]$.

Notice that $\pi_n(Y, y_0) = [S^n, Y]$, so it suffices to prove that $\Sigma S^{n-1} = S^n$. We do this using $CW-$complexes:

$$\frac{[e^0 \sqcup e^n/\sim] \times [e^0 \sqcup e^1]/\sim}{[e^0 \sqcup e^n/\sim] \times e^0 \cup [e^0 \sqcup e^1/\sim] \times e^0} = \frac{(e^0 \sqcup e^0) \times (e^0 \times e^1) \times (e^0 \times e^1) \times (e^1 \times e^n)/\sim}{[e^0 \times e^0] \cup [e^0 \times e^1] \cup [e^0 \times e^n]/\sim} \tag{16}$$

$$= e^0 \sqcup (e^n \times e^1)/\sim \tag{17}$$

$$= S^{n+1} \tag{18}$$

where we quotient out the usual way, i.e. the terms in the quotient "cancel" the equal terms in the space and become a single point e^0. Thus we have

$$\pi_{n-1}(\Omega Y, \epsilon_0) \cong [S^{n-1}, \Omega Y] \cong [\Sigma S^{n-1}, Y] \cong [S^n, Y] \cong \pi_n(Y, y_0) \tag{19}$$

2. First we prove injectivity. Since $\overline{f}(g)(t) = f(g,t)$, we notice the basepoint-preserving-ness of $\overline{f}$. For basepoint $g_0 \in G$, we have $\overline{f}(g_0)(t) = f(g_0,t) = f(g_0,t')$, for any $t' \in S^1$, equal to the constant map. Consider a class of a nullhomotopic loop in $\pi_n(G)$. We have $\overline{f}(g_0)(t)$ is the constant map, mapping the point g_0 to ϵ_0, the constant map $S^1 \to BG$. This is because $(g_0,t) = g_0$, so g_0 gets mapped to $\{1\} \in S^1$ which gets constantly mapped to the basepoint of BG, as it is constant for all $t \in S^1$. Thus the identity of $\pi_n(G)$ maps to the identity of $\pi_n(\Omega BG)$. Now we prove surjectivity. Suppose we have a homotopy class of an $n-$dimensional loop $[n] \in \pi_n(\Omega BG)$. This corresponds to an $(n+1)-$dimensional homotopy

class of BG through the isomorphism proved above. We seek to create a principal G-bundle over ΣG that is trivial on both cones. Define this bundle as

$$C_+ := G \times [1,-1]/\sim, c_- := G \times [-1,1] \tag{20}$$

$$E := C_+ \times G \cup_{Id} C_- \times G \tag{21}$$

By theorem 4.8 in the text, there is a bijective correspondence given by

$$\psi : [\Sigma G, BG] \to Prin_G(\Sigma G) \tag{22}$$

$$f \mapsto f^*(E) \tag{23}$$

such that $f^*(EG) = E$. Thus let $[g] \in \pi_{n+1}(\Sigma G)$ be the homotopy class of an $(n+1)-$loop that maps its loop in ΣG to said $(n+1)-$loop in BG induced by f, well-defined because f is a bijection. Thus let g be the $\overline{f}(g)(t)$. We know this must exist due to part a). Therefore, $\overline{f}$ is an isomorphism in π_n. Since this did not depend on n, all such homotopy groups are isomorphic.

□

Question 4. *For any space X let $Vect^d(X)$ denote the set of isomorphism classes of $d-$dimensional vector bundles over X.*

1. *Compute $Vect^d(S^1)$. Justify your answer.*
2. *Compute the fundamental group of the Grassmannian, $\pi_1(Gr_d(\mathbb{R}^\infty)$).*
3. *Let X be a simply-connected space. Prove that any one-dimensional vector bundle over X is trivial.*

Proof.

Lemma 1. *There is a bijective correspondence between principal bundles and homotopy groups $Prin_G(S^n) \cong \pi_{n-1}(G)$ where as a set $\pi_{n-1}(G) = [S^{n-1}, x_0; G, \{1\}]$, which refers to (based) homotopy classes of basepoint preserving maps from the sphere S^{n-1} with basepoint $x_0 \in S^{n-1}$, to the group G with basepoint the identity 1∈ G.*

Proof. Let $p : E \to S^n$ be a principal $G-$bundle. Write S^n as the union of its upper and lower hemispheres

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_- \tag{24}$$

Since $D^n_\pm$ are contractible, the restiction of E to each of the hemispheres is trivial, so if we fix a trivialization of the fiber of E above $x_0 \in S^{n-1} \subset S^n$, we can extend this trivialization to the upper and lower hemispheres. For θ a clutching function on the equator $\theta : S^{n-1} \to G$, we can then write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{25}$$

that is, for $(x, g) \in (D^n_+ \times G)$, we have $(x, g) \sim (x, \theta(x)g) \in (D^n_- \times G)$. Since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}(G) \tag{26}$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G) \tag{27}$$

By restricting to the upper and lower hemispheres, Ψ defines maps

$$\Psi_+ : D^n_+ \to G \tag{28}$$

$$\Psi_- : D^n_- \to G \tag{29}$$

which both map $x_0 \in S^{n-1}$ to the identity $1 \in G$, and have the property

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x) \tag{30}$$

or $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. By considering the linear homotopy $\Psi_+(tx)\theta_1(tx)\Psi_-(tx)^{-1}$ for $t \in [0,1]$, we can see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, for 0 the origin in $D^n_\pm$, i.e. the north and south poles of the sphere. Since $\Psi_\pm$ are defined on connected spaces, their images lie on a connected component of G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t=0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t=1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi + (0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\theta : Prin_G(S^n) \to \pi_{n-1}(G)$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{31}$$

We discuss injectivity. Suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2, where

$$E_i = (D^n_+ \times G) \cup_{\theta_i} (D^n_- \times G) \tag{32}$$

Let $H : S^{n-1} \times [-1,1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator $S^{n-1} \subset S^n$ with $S^{n-1} \times [-1,1]$. Write $\mathcal{D}_+ = D^2_+ \cup \overline{\mathcal{N}}$ and

$\mathcal{D}_- = D^2_- \cup \overline{\mathcal{N}}$. Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \overline{\mathcal{N}} \cong S^1 \times [-1,1] \tag{33}$$

Thus we may form the principal $G-$bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_+ \times G \tag{34}$$

where, by abuse of notation, H is the composition $\overline{\mathcal{N}} \cong S^{n-1} \times [-1,1] \xrightarrow{H} G$. If we deformation retract $\overline{\mathcal{N}}$ to S^{n-1} and contract $D^2_\pm$ to D_-, we get that E is isomorphic to E_1 and E_2. □

Lemma 2. *There are bijective correspondences*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{C}P^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z}) \tag{35}$$

Similarly, there are bijective correspondences

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2,1)] \cong H^1(X;\mathbb{Z}_2) \tag{36}$$

Proof. The last correspondence takes a map $f : X \to \mathbb{C}P^\infty$ to the class

$$c_1 = f^*(c) \in H^2(X;\mathbb{Z}) \tag{37}$$

where $c \in H^2(\mathbb{C}P^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)-$bundle). These other correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X)\mathbb{C}[X, BGL(1,\mathbb{C})]$, and that $\mathbb{C}P^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because we can express $\mathbb{C}P^\infty$ in its homogeneous form as $\mathbb{C}P^\infty = \lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C})$, and that $\lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$.

For the other case, we have the last correspondence taking a map $f : X \to \mathbb{R}P^\infty$ to the class $\omega_1 = f^*(\omega) \in H^1(X;\mathbb{Z}_2$, where $\omega \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $\omega_1 \in H^1(X;\mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel-Whitney class of ζ (or of the corresponding principal $O(1)-$ bundle). $\square$

1. Let $V_d(\mathbb{R}^n)$ be the Stiefel manifold as in the text. We claim that the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ to the first n coordinates induces a nullhomotopic inclusion of $V_d(\mathbb{R}^n)$ into $V_d(\mathbb{R}^{2n})$. Let $\iota : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho \in V_d(\mathbb{R}^n) \subset V_d(\mathbb{R}^{2n})$, we have a homotopy $t\iota + (1-t)\rho$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V_d(\mathbb{R}^n)$ onto the element ι. Hence the limiting space $V_d(\mathbb{R}^\infty)$ is aspherical with a free $GL(d,\mathbb{R})-$action. Therefore the projection

$$V_d(\mathbb{R}^\infty) \to V_d(\mathbb{R}^\infty)/GL(d,\mathbb{R}) = Gr_d(\mathbb{R}^\infty) \tag{38}$$

is a universal $GL(d,\mathbb{R})-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BGL(d,\mathbb{R})$, so we have a classification

$$Vect^d(S^1) \cong Prin_{GL(d,\mathbb{R})}(S^1) \cong [S^1, BGL(d,\mathbb{R})] \cong [S^1, Gr_d(\mathbb{R}^\infty)] = \pi_1(Gr_d(\mathbb{R}^\infty)) \tag{39}$$

Thus it remains to compute $\pi_1(Gr_d(\mathbb{R}^\infty))$. Let $V^O_d(\mathbb{R}^n)$ be the Stiefel manifold of orthonormal $d-$frames in $\mathbb{R}^n$. Let $\iota' : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image an orthonormal frame in the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho' \in V^O_d(\mathbb{R}^n) \subset V^O_d(\mathbb{R}^{2n})$, we have a homotopy $t\iota' + (1-t)\rho'$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V^O_d(\mathbb{R}^n)$ onto the element ι'. Hence the limiting space $V^O_d(\mathbb{R}^\infty)$ is aspherical with a free $O(d)-$action. Therefore the projection

$$V^O_d(\mathbb{R}^\infty) \to V^O_d(\mathbb{R}^\infty)/O(d) = Gr_d(\mathbb{R}^\infty) \tag{40}$$

is a universal $O(d)-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BO(d)$. Thus we have

$$\pi_1(Gr_d(\mathbb{R}^\infty)) \cong \pi_1(BO(d)) \cong [S^1, BO(d)] \cong Prin_{O(d)}(S^1) \xrightarrow{x} \pi_0(O(n)) \tag{41}$$

where the last x map is a bijection due to Lemma 1. $O(d)$ has two connected components: the Det map is a polynomial and thus continuous. This maps $O(d)$ to either 1 or -1. Therefore, since $\pi_0(O(d))$ is the set of connected components $O(d)$, it is a two-element group, and is thus $\mathbb{Z}/2\mathbb{Z}$. Thus $Vect^d(S^1) \cong \pi_1(Gr_d(\mathbb{R}^\infty)) \cong \mathbb{Z}/2\mathbb{Z}$.

2. See part a).

3. From Theorem 4.14, we have

$$Vect^1(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2) \tag{42}$$

Since X is simply connected, $H^1(X;\mathbb{Z}_2) = 0$, and so there is only one element in $Vect^1(X)$, i.e. there is only one isomorphism class of 1-dimensional vector bundles, which must be the trivial bundle.

$\square$

Question 5. *Let T^2 be a closed, connected, orientable surface (two-dimensional manifold). Show that there are infinitely many nonisomorphic complex line bundles over T^2.*

Proof. From Theorem 4.13 in the book, we have

$$Vect^1(T) \cong Prin_{U(1)}(T) \cong [T, BU(1)] = [T, \mathbb{C}P^\infty] = [T, K(\mathbb{Z}, 2)] \cong H^2(T, \mathbb{Z}) \tag{43}$$

Since T is closed and orientable, we can apply Poincaré Duality. Due to Poincaré Duality, we have $H_2(T,\mathbb{Z}) \cong H_0(T)$. Since T is connected, we want to show that T is path-connected because it's a manifold. Let $x \in T$ be any point in T. Let U denote the open neighborhood of x with is locally path-connected. This is assured because there exists an open neighborhood of x that is homeomorphic to $\mathbb{R}^2$, which is everywhere path-connected. Let $y \in T \backslash U$. There exists an open neighborhood of y that is path-connected by the same argument. Thus $U, T\backslash U$ are open, and $U \cup T\backslash U = T$. Since U is nonempty, T must be path-connected, and the boundary of any image of a singular chain is homotopic to the boundary of a point (since every point can be homotoped to any other point), and is thus zero. Thus we have

$$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0 \Rightarrow \tag{44}$$

$$H_0(T) = \frac{C_0}{Im(\partial_1)} \tag{45}$$

$$\varepsilon : C_0 \to \mathbb{Z} \tag{46}$$

$$\varepsilon(\sum_i n_i \sigma_i) \mapsto \sum_i n_i \tag{47}$$

This is obviously a surjective map since T is nonempty. We want to show that $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(\partial)$. For a 1-simplex $\sigma : \Delta^1 \to X$, we have $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$, so $\mathrm{Im}(\partial) \subset \mathrm{Ker}(\varepsilon)$. Now suppose $\varepsilon(\sum_i n_i \sigma_1) = 0$. The σ_i's are singular 0-simplices i.e. points of T. Choose a path $f_t : [0,1] \to T$ from a basepoint x_0 to $\sigma_i(v_0)$, with σ_0 the singular 0-simplex with image x_0. f_t is a singular 1-simplex, and $\partial f_t = \sigma_i - \sigma_0$. Thus $\partial(\sum_i n_i f_t) = \sum n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i - 0$. Therefore $\sum_i n_i \sigma_i$ is a boundary. Thus $\mathrm{Im}(\partial) \subset \mathrm{Ker}(\varepsilon)$, so $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(\partial)$, and thus $H_0(T) \cong \mathbb{Z}$. This has infinitely many elements, so $H^2(T, \mathbb{Z}) \cong \mathbb{Z}$ has infinitely many elements, so $Vect^1(T)$ has infinitely many elements i.e. isomorphism classes of complex line bundles. □

Question 6. *A vector bundle η is said to be stably trivial if for some $k \in \mathbb{Z}$, the Whitney sum $\eta \oplus \epsilon^k$ is a trivial vector bundle, where ϵ^k denotes the standard trivial bundle of dimension k. Let M be an $n-$dimensional smooth, closed manifold, and suppose that there exists an immersion*

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{48}$$

1. *Prove that the tangent bundle TM is stably trivial.*
2. *Show that the sphere S^n has stably trivial tangent bundle for every n. (A manifold with stably trivial tangent bundle is called "stably parallelizable".)*
3. *Show that the tangent bundle $TS^2 \to S^2$ is not trivial, but $TS^2 \oplus \epsilon^1$ is trivial.*

Proof. 1. Since we have an immersion

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{49}$$

we have a monomorphism

$$\begin{tikzcd} T(M\times\mathbb{R}^k) \arrow[r, "Df"] \arrow[d] & T\mathbb{R}^{n+k} \arrow[d] \\ M\times\mathbb{R}^k \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{R}^{n+k} \\ M & \end{tikzcd} \qquad \begin{tikzcd} \zeta \arrow[r, "\overline{\gamma}"] \arrow[d] & \xi \arrow[d] \\ X \arrow[r, "\gamma"] & Y \end{tikzcd}$$

so that $\gamma_x : \zeta_x \to \xi_{\gamma(x)}$ is a monomorphism of fibers. Since M is $n-$dimensional, since γ_x is injective, it must be an isomorphism as well. Thus we have an isomorphism of vector bundles $T(M\times\mathbb{R}^k) \cong f^*(T\mathbb{R}^{n+k})$. We have isomorphisms

$$T(M\times\mathbb{R}^k) \cong \pi^*(TM)\oplus\epsilon^k,\, T\mathbb{R}^{n+k} \cong \epsilon^{n+k} \tag{50}$$

$$(v,e)\mapsto v\oplus e,\, (w)\mapsto w \tag{51}$$

Thus we have an isomorphism $\pi^*(TM)\otimes\epsilon^k \cong \epsilon^{n+k}$ of vector bundles over $M\times\mathbb{R}^k$. The pullback of this along a section of π yields $TM\oplus\epsilon^k \cong \epsilon^{n+k}$.

2. We consider the standard embedding $f : S^n \to \mathbb{R}^{n+1}, f(x_1,...,x_{n+1}) = (x_1,...,x_{n+1})$. This is obviously an embedding and thus an immersion. The unit normal vector in this embedding is $\frac{\mathbf{x}}{|\mathbf{x}|}$ with respect to the usual euclidean metric. This is nowhere-vanishing on S^n, so the normal bundle is given by $t\frac{\mathbf{x}}{|\mathbf{x}|}, t\in\mathbb{R}$. We have the isomorphism from the trivial line bundle ϵ^1 to the normal bundle by $v\mapsto v\frac{\mathbf{x}}{|\mathbf{x}|}$. Thus we have $TS^n\oplus\nu(S^n)\cong TS^n\oplus\epsilon^1 = \mathbb{R}^{n+1}\cong\epsilon^{n+1}$. Thus TS^n is a stably trivial bundle for all n.

3. From part b) we know that $TS^2\oplus\epsilon^1\cong\epsilon^{2+1}$, so $TS^2\oplus\epsilon^1$ is trivial, so it remains to show that $TS^2\to S^2$ is not trivial. If TS^2 was nontrivial, there would exist nowhere vanishing sections on S^2, i.e. a nowhere-vanishing vector field on S^2. However, by the Hairy Ball Theorem, such a vector field cannot exist on S^2. Therefore there is no global section on S^2, and TS^2 is nontrivial.

$\square$

Question 7. *Let M^n be a smooth, closed, oriented manifold of dimension n. Consider the diagonal* *embedding,*

$$\Delta_M : M \hookrightarrow M \times M \tag{52}$$

$$x \to (x, x) \tag{53}$$

Let ν_{Δ_M} be the normal bundle of this embedding.

1. *Show that there is an isomorphism of vector bundles over M,*

$$\nu_{\Delta_M} \cong TM \tag{54}$$

   where TM is the tangent bundle of M.

2. *Let $\tau : M \times M \to T(\nu_{\Delta_M})$ be the Thom collapse map. Here $T(\nu_{\Delta_M})$ is the Thom space of the normal bundle. Consider the composition map in homology,*

$$\phi : H_p(M^n; \mathbb{Z}) \times H_q(M^n; \mathbb{Z}) \xrightarrow{x} H_{p+q}(M \times M; \mathbb{Z}) \xrightarrow{*} H_{p+q}(T(\nu_{\Delta_M}); \mathbb{Z}) \xrightarrow{\cap u} H_{p+q-n}(M^n; \mathbb{Z}). \tag{55}$$

   The first map in this sequence is the cross product, and the last map in this sequence is the Thom isomorphism in homology, given by capping with the Thom class. Show that this composition map ϕ is equal, up to sign, to the intersection product:

$$\phi(\alpha, \beta) = \pm\alpha \cdot \beta. \tag{56}$$

Proof. 1. We have, for $\Delta(x) = (x, x)$, that $T\Delta(x) \cong \{(x, x, v, v) | x \in M, v \in T_xM\}$. We then calculate the normal bundle ν_{Δ_M}: $\{(v_1, v_2) \in TM \times TM | v \cdot v_1 + v \cdot v_2 = 0\}$. In examining the orthogonality condition, we get that

$$v \cdot v_1 + v \cdot v_2 = 0 \Rightarrow \tag{57}$$

$$v \cdot v_1 = -v \cdot v_2 \tag{58}$$

$$v_1 = -v_2 \tag{59}$$

Therefore the normal bundle is $\{(x, x, v, -v) | x \in M, v \in T_x M\}$. This is isomorphic to TM via the isomorphism

$$\Psi(x, x, v, -v) \mapsto (x, v) \tag{60}$$

$$\Psi^{-1}(x, v) \mapsto (x, x, v, -v) \tag{61}$$

This is injective because, if $(x, x, v, -v) \neq (x', x', v', -v')$ as a set, we get their image under Ψ as (x, v) vs. (x', v'), which are not equivalent. This is surjective because $(x, v) \in TM$ has preimage $(x, x, v, -v)$, which is indeed in ν_{Δ_M}. This is well-defined, because the quotient relations hold under Ψ, as the normal bundle is subject to the same quotient relations as that of TM, and scalar multiplication by -1 doesn't affect anything.

2. For $\alpha \in H_p(M; \mathbb{Z}), \beta \in H_q(M; \mathbb{Z})$, we have their cross product as $\pm[\alpha \times \beta] \in H_{p+q}(M \times M; \mathbb{Z})$. If we take the induced homomorphism in homology of the Thom collapse map of the embedding $MM \times M$, we have that $\pm\tau_*(\alpha \times \beta) \in H_{p+q}(T(\nu_{\Delta_M}); \mathbb{Z})$. We know from part a) that $\nu_{\Delta_M} \cong TM$, so $H_{p+q}(T(\nu_{\Delta_M}); \mathbb{Z}) \cong H_{p+q}(T(TM); \mathbb{Z})$, so the Thom isomorphism can be applied from $H_{p+q}(T(\nu_{\Delta_M}); \mathbb{Z})$ to $H_{p+q-n}(\Delta(M); \mathbb{Z})$, since M is oriented. From the reasoning in Theorem 9.4 in the notes, we have

$$[\pm\tau_*(\alpha \times \beta) \in H_{p+q}(T(\nu_{\Delta_M}); \mathbb{Z}) \cong H_{p_q}(T(TM); \mathbb{Z})] \frown [u] \in H^n(T(TM); \mathbb{Z}) \tag{62}$$

$$= \pm[\alpha \times \beta \cap \Delta(M)] \in H_{p+q-n}(\Delta(M); \mathbb{Z}) \cong H_{p+q-n}(M; \mathbb{Z}) \tag{63}$$

Notice that the intersection of $\alpha \times \beta$ with $\Delta(M) \subset M \times M$ are the exact points when α and β intersect: if $(a, b) \in \alpha \times \beta$ is equal to $(x, x) \in \Delta(M) \subset M \times M$, then $x = a \in \alpha, x = b \in \beta$. Thus the class $\alpha \times \beta \cap \Delta(M) \subset M \times M$ is equal to $\pm[\alpha \cap \beta] \in H_{p+q-n}(M; \mathbb{Z})$, which is the intersection product $\pm\alpha \cdot \beta$.

□