Notes on Dijkgraaf-Witten TQFTs

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1 Definitions

1.1 First Definition

This is the introduction of Dijkgraaf-Witten topological quantum field theory given by Dijkgraaf and Witten: For a group Γ, the classifying space $B\Gamma$ is the base space of the universal principal $\Gamma-$bundle $E\Gamma$. Any principal $\Gamma-$bundle E over a manifold M allows a bundle map into the universal bundle $E\Gamma$. This induces a bundle map $\gamma : M \to B\Gamma$. The topology of E is completely determined by the homotopy type of γ, so there is a bijective correspondence between $Map(M, B\Gamma)$ and principal $\Gamma-$bundles $E \to M$.

We care about principal $\Gamma-$bundles over M because the Γ action on the bundle encodes the (global) gauge transformation, and connections on said bundle (gauge fields) allow for derivatives that are invariant under Γ as well (because in general $\partial_\mu(g\phi) \neq g\partial_\mu(\phi)$, $g \in \Gamma$, but with a gauge field $D_\mu(g\phi) = gD_\mu(\phi)$).

The path integral is

$$Z(M) = \int \mathscr{D}Ae^{2\pi i S(A)}$$

with A a connection on a principal $\Gamma-$bundle. For this reason (and a couple more) a topological action $S(\rho)$ should be a value in $\mathbb{R}/\mathbb{Z}$. For Γ a finite group, every principal $\Gamma-$bundle has a unique flat connection corresponding to a homomorphism $\rho : \pi_1(M) \to \Gamma$. The actions should also be equivalent if they differ by a functional that only depends on the $\rho_{\partial M}$, because then the transition amplitudes (e^{iS}) could correspond to a change in definition of the external states. Also, if $\partial M = \varnothing$ and there exists a manifold B such that $\partial B = M$ such that ρ extends to a homomorphism $\pi_1(B) \to \Gamma$, then $S(\rho) = 0$, i.e. if $M = M_1 \# M_2$, $\partial B = M \sqcup -M_1 \sqcup M_2$, and $e^{iS(M)} = e^{iS(M_1)}e^{iS(M_2)}$. With these two requirements, the action functionals are then in bijective correspondence with $H^n(B\Gamma; \mathbb{R}/\mathbb{Z})$.

With the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

This induces an isomorphism

$$H^k(B\Gamma; \mathbb{Z}) \cong H^{k-1}(B\Gamma; \mathbb{R}/\mathbb{Z})$$

For a trivial principal $\Gamma-$bundle $E \to M^3$ with connection A and Γ a compact simple gauge group, the Chern-Simons action functional is

$$S(A) = \frac{k}{8\pi^2} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

where $k \in \mathbb{Z}$ ensures the integral is single valued, and Tr is an invariant quadratic form on the lie algebra of Γ. From cobordism theory we know that there exists a 4-manifold B^4 such that $\partial B = M$. Extending the trivial bundle E, and letting F be the curvature of any gauge field A' on B reducing to A on $\partial B = M$, this functional becomes

$$S(A) = \frac{k}{8\pi^2} \int_B Tr(F \wedge F) \mod 1$$

This is the integral of the differental form

$$\Omega(F) = \frac{k}{8\pi^2} Tr(F \wedge F) \in H^4(B\Gamma; \mathbb{R})$$

Since this differential form has integral periods, it is in the image of the natural map $H^4(B\Gamma; \mathbb{Z}) \to H^4(B\Gamma; \mathbb{R})$.

Choose any $\omega \in H^4(B\Gamma; \mathbb{Z})$ representing $\Omega(F) \in H^4(B\Gamma; \mathbb{R})$. Then the topological action becomes

$$S(A) = \frac{1}{n} [\int_B \Omega(F) - \langle \gamma^* \omega, B \rangle] \mod 1$$

If we transform ω into $\omega + \omega'$, where ω' is an $n-$torsion element, the action picks up a $\mathbb{Z}_n$ phase. If $\Omega(F) = 0$ as is the case for Γ finite, then ω is torsion and determines a cocycle $\alpha \in H^3(B\Gamma; \mathbb{R}, /\mathbb{Z})$ through the isomorphism $Tor(H^4(B\Gamma; \mathbb{Z})) \cong H^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. Then the action becomes

$$S = \langle \gamma^* \alpha, [M] \rangle$$

For gauge invariance, let A be a connection and A^g be its gauge transform. We can construct a connection A_t such that $A_0 = A, A_1 = A^g$ on the manifold $M \times I$. Thus we have

$$S(A) - S(A^g) = \int_{M \times I} \Omega(F)$$

1.2 Second Definition

A $(n+1)$-dimensional DW theory based on a finite group Γ is defined by the following. Choose a cocycle $w \in C^{n+1}(B\Gamma)$. For M a closed n-manifold, we define the vector space

$$\mathbb{A}(M) = \mathbb{C}[\text{maps}(M \to B\Gamma)]/\sim$$

where maps$(M \to B\Gamma)$ denotes the set of continuous maps from M to $B\Gamma$. The equivalence relation is the following: given a map $F : M \times I \to B\Gamma$, and defining $f_t : M \to B\Gamma$ as the restriction to $M \times \{t\}$, we have $f_1 \sim \langle w, [F(M \times I)]\rangle f_0$. For $w = 0$ this is just homotopy equivalence, if not this is a twisted version of homotopy. For the latter case we have to be careful to ensure everything is well-defined. $\mathbb{A}(M)$ is the predual to the Hilbert space: define

$$Z(M) = \mathbb{A}(M)^* = \{\text{linear maps}(\mathbb{A}(M) \to \mathbb{C})\}$$

More generally, as is traditional in TQFT, we consider a similar space for manifolds with boundary, such that the assignment to the boundary is fixed: For M a compact manifold, and fixed "crude" boundary condition map $c : \partial M \to B\Gamma$, define the vector space

$$\mathbb{A}(M; c) = \mathbb{C}[\text{maps}(M \to B\Gamma)^*_c]/\sim$$

where maps$(M \to B\Gamma)^*_c$ is the set of maps $M \to B\Gamma$ which restrict to c on ∂M, and the quotient is the same. In the same way, we have

$$Z(M; c) = \mathbb{A}(M; c)^*$$

Let Y be a closed $(n-1)$-manifold. In this theory, we associate a 1-category to such a manifold.

Let $\mathscr{A}(Y)$ be the 1-category with:

- Objects = all continuous maps $f : Y \to B\Gamma$ (NOT up to homotopy)
- Morphisms = the vector space $\mathbb{A}(Y \times I; x, y)$, where x, y are the boundary components of the "cylinder"
- Compositions are the stacking of cylinders

Said in a more calculation-friendly way, $\mathscr{A}(Y)$ is the 1-category with:

- Ob: $\{\alpha : \pi_1(Y) \to \Gamma\}$
- Mor $(\alpha \to \beta)$: $\mathbb{C}[\rho : \pi_{\leq 1}(Y \times I) | \rho|_{\pi_1(Y \times \{0\})} = \alpha, \rho|_{\pi_1(Y \times \{1\})} = \beta]$
- Stacking: Stack the Is

The minimal idempotents of this category are given by pairs $\{\alpha, \sigma\}$, where α is a homomorphism $\pi_1(Y) \to \Gamma$, and σ is a representation of $Z(\alpha)$, the centralizer of the image of α.

If we take any representation V or $\mathscr{A}(Y)$, we have

$$V \cong \oplus_{e=\text{min idem}} V_e$$

For disjoint unions, we have

$$\mathbb{A}(M^n \sqcup N^n) \cong \mathbb{A}(M^n) \otimes \mathbb{A}(N^n)$$
$$\mathscr{A}(X^k \sqcup Y^k) \cong \mathscr{A}(X^k) \times \mathscr{A}(Y^k), k < n$$

Example 1. For $n = 1, w = 0$, we consider $\mathbb{A}(S^1)$. These are unbased maps. Two circles are freely homotopic if their corresponding Γ elements are conjugate, so $\dim(\mathbb{A}(S^1)) = |\Gamma / \sim|$, where $\sim$ is conjugacy equivalence.

Example 2. For $n = 1, w = 0$, we consider $\mathbb{A}(I; x, y)$. This is a vector space of dimension the number of homotopy classes of paths from x to y. If we fix a path from y to x and compose this will all such paths, this becomes $|\pi_1(B\Gamma, x)| \cong \Gamma$.

Example 3. For $n = 1, w = 0$, $\mathscr{A}(pt) = \{pt \to B\Gamma\} \cong B\Gamma$. The morphisms in this 1-category are $\mathbb{A}(I; a, b)$ for two fixed points $a, b \in B\Gamma$. These satisfy the property that $\mathbb{A}(I; a, b) \otimes \mathbb{A}(I; b, c) \to \mathbb{A}(I; a, c)$

Example 4. For $n = 0, w = 0$, how many equivalence classes are there of objects in $\mathscr{A}(pt)$? In category theory, two abjects c, d are considered equivalent if there exist morphisms $u : c \to d, v : d \to c$ such that $uv = id_c, vu = id_d$. Since $\pi_0(B\Gamma) \cong 0$, there is one equivalence class between objects.

Example 5. For $n = 2, w = 0$, we'll describe the (equivalence classes) of objects in $\mathscr{A}(S^1)$. From above we know that the set of equivalence classes of objects has a bijection between conjugacy classes of Γ. This means that the only objects we can consider are basepoint-preserving maps $S^1 \to B\Gamma$. These correspond to elements $g \in \Gamma$. The morphisms are the vector space $\mathbb{A}(S^1 \times I; g, g')$. Thus we want to consider maps from $S^1 \times I$ into $B\Gamma$. If we trace the basepoints from g to g', we get another loop in $B\Gamma$ under the map $S^1 \times I$ into $B\Gamma$, represented by an element in Γ we'll call h. But how do we know which h works? If we cut the cylinder along h, we get a 2-cell, and we want the boundary of this 2-cell to be nullhomotopic, i.e. $hg'h^{-1}g^{-1} = 1 \in \Gamma$. Thus

$$mor([g] \to [g']) = \mathbb{C}[\{h \in \Gamma | g = hg'h^{-1}\}]$$
$$End([g]) = \mathbb{C}[N_g] := \mathbb{C}[\{h | g = hgh^{-1}\}]$$

or $gh = hg$.

1.3 $n = 2, \Gamma = S_3$ Example

First, let's see what happens when we consider S^1. Since $n = 2$, we have a category $\mathscr{A}(S^1)$. How many equivalence classes of objects are there in this category? From before we know that the set of equivalence classes are in bijective correspondence with the conjugacy classes of S_3. We give a presentation of S_3 as

$$S_3 = \langle r, a | a^3 = 1, r^2 = 1, rar = a^2 \rangle$$

a:

1	a
a	a
r	a^2
a^2	a
ar	a^2
a^2r	a^2

r:

1	r
a	a^2r
r	r
a^2	ar
ar	a^2r
a^2r	ar

a^2:

1	a^2
a	a^2
r	a
a^2	a^2
ar	a
a^2r	a

ar:

1	ar
a	r
r	a^2r
a^2	a^2r
ar	ar
a^2r	r

a^2r:

1	a^2r
a	ar
r	ar
a^2	r
ar	r
a^2r	a^2r

The conjugacy classes of S_3 are

$$\{Id\}, \{r, ar, a^2r\}, \{a, a^2\}$$

thus there are 3 equivalence classes of objects in $\mathscr{A}(S^1)$. Now we examine the endomorphism algebra of an object in each equivalence class. From above, the dimension is the number of elements of S_3 that commute with each group representing the object. The representative object of each equivalence is of order 1, 2, and 3, respectively. For $dim(End(1))$, we have 6 elements commuting with 1, for $dim(End(r, ar, a^2r))$, we have 2 elements commuting with each one, and $dim(End(a, a^2))$, we have 3 elements commuting with each one.

We then examine $\mathbb{A}(T^2)$. This is $\mathbb{A}(S^1 \times I; g, g), \forall g \in S_3$. For $g = Id$, all elements commute, and so there are 3 homotopy classes of loops (for the 3 conjugacy classes that all the elements fall under). For an element in the $\{r, ar, a^2r\}$ conjugacy class, there are 2 commuting elements. For an element in the $\{a, a^2\}$ conjugacy class, there are 3 commuting elements. Thus $dim(\mathbb{A}(T^2)) = 3 + 2 + 3 = 8$.

For objects $(g, h), g, h \in S_3$ up to conjugacy, we have the following list:

$$\langle 1, 1\rangle, \langle 1, a\rangle, \langle 1, r\rangle, \langle a, 1\rangle, \langle a, a\rangle, \langle a, r\rangle, \langle a, a^2\rangle, \langle r, 1\rangle, \langle r, a\rangle, \langle r, r\rangle, \langle r, ar\rangle$$

Now let M^3 be a 3-manifold such that $\partial M^3 \neq \varnothing$. Denote

$$R := \{\pi_1(M^3) \to \Gamma\}$$
$$R_\alpha := \{\pi_1(M^3) \to \Gamma, \text{ restr. to } \partial M^3 = \alpha\}$$

$Z(\alpha)$ acts on R_α, and $Z(\overline{\alpha})$ acts on $\mathbb{C}[R_\alpha]$. Our goal is to write $\mathbb{C}[R_\alpha]$ as a direct sum of "isotypical" components. The orbits are:

- $\{x\}$, so $\mathbb{C}[R_\alpha] = \mathbb{C}_{triv}$
- $\{x, y \neq x\}$

$$\begin{tikzcd} X \arrow[r, bend left, "r"] \arrow[loop below, "a"] & Y \arrow[l, bend left, "r"] \arrow[loop below, "a"] \end{tikzcd}$$

  so $\mathbb{C}[R_\alpha] = \mathbb{C}_{triv} \oplus \mathbb{C}_{sign}$

- $\{x, y, z\}$

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[loop below, "r"] & & Y \arrow[dl, "a"] \arrow[loop below, "r"] \\ & Z \arrow[ul, "a"] \arrow[loop below, "r"] & \end{tikzcd}$$

  so $\mathbb{C}[R_\alpha] = \mathbb{C}_{triv} \oplus \mathbb{C}^2_{std}$, since we know the dimension has to be 3.

- If the size of the center is 6, Γ acts on itself by left multiplication, so we have the "regular representation" $\mathbb{C}_{triv} \oplus \mathbb{C}^2_{std} \oplus \mathbb{C}^2_{std} \oplus \mathbb{C}_{sign}$

1.4 Particle Types

A "crude" boundary condition on Y is a map $f : Y \to B\Gamma$ (NOT up to homotopy), or equivalently a homomorphism $\rho : \pi_1(Y) \to \Gamma$. An endomorphism of our "crude" boundary condition is a map $F : Y \times I \to B\Gamma$ such that $F(y,0) = F(y,1) = f(y)$, or $x \in Z(im(\rho))$, where Z is denoted as the centralizer of this image subgroup.

A non-"crude" boundary condition represents a particle type on Y (aka irreps or idempotents of Y), and is given by pairs

$$[\rho : \pi_1(Y) \to \Gamma, \alpha \in irrep(Z(im(\rho)))]/conj$$

where we mod out by conjugation via the observation in Example 1.

Let M be a connected $n-$manifold such that $\partial M = Y_1 \sqcup ... \sqcup Y_k$. Fix crude boundary conditions $\rho_i : \pi_1(Y_i) \to \Gamma$. The Hilbert space is given by

$$\mathbb{A}(M; \rho_1, ..., \rho_k) := \mathbb{C}[\{\alpha : \pi_1^m(M) \to \Gamma | \alpha_{\pi_1(Y_i)} = \rho_i, \forall 1 \leq i \leq k\}]$$

where $\pi_1^m(M)$ is the fundamental groupoid of M with fixed basepoints in each Y_i. Notice that the number of objects in a garden variety fundamental groupoid is uncountable, but with fixed basepoints in each boundary submanifold becomes finite, and so this Hilbert space is finite.

Example 6. Let $n = 2$, $M = S^1 \times I$, $\rho_1 = \rho_2 =$ the trivial homomorphism. Then $\mathbb{A}(S^1 \times I, triv, triv) \cong \mathbb{C}\Gamma$.

Example 7. Let $n = 3$, $M = S^3 \backslash [B^3 \sqcup B^3 \sqcup B^3]$, $\rho_i = triv$. Notice that $Y_i = S^2$, so $\pi_1(Y_i) = 1$. Then $\mathbb{A}(M; triv, triv, triv) \cong \mathbb{C}[\Gamma \times \Gamma]$. See 1.

If $\partial M = \emptyset$, then we mod out by conjugation: $\mathbb{A}(M) = \mathbb{C}[\{\pi_1(M) \to \Gamma\}/conj]$.

The group $Z(im(\rho_1)) \times ... \times Z(im(\rho_k))$ acts on $\mathbb{A}(M; \rho_1, ..., \rho_k)$ via conjugation (Note that we need to be careful about conjugation in the groupoid picture).

$$\mathbb{A}(M; \rho_1, ..., \rho_k) \cong \oplus_{\beta_1, ..., \beta_k} \mathbb{A}(M; (\rho_1, \beta_1), ..., (\rho_k, \beta_k))$$

where β_i is an irrep of $Z(im(\rho_i))$. The right side of the equation is the Hilbert space of particle types $(\rho_1, \beta_1), ..., (\rho_k, \beta_k)$.

To help with checking calculations, we can use this standard fact of TQFT:

$$dim(\mathbb{A}(Y \times S^1)) = \# \text{ irreps of } \mathscr{A}(Y)$$

Suppose $\partial M^3 = Y_0 \sqcup S^2$, then we fix points $b_0 \in Y_0$ and $b_1 \in Y_1$.

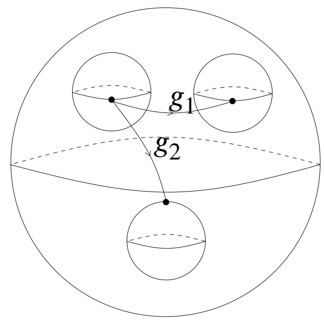


Figure 1: Three $(triv, triv)$ particles on the 3-sphere.

$$\begin{tikzcd} \cdot b_0 \arrow[loop left, "m"] & \cdot b_1 \arrow[l, "a"'] \end{tikzcd}$$

With this subcategory of the fundamental groupoid, we have an isomorphism of sets

$$\begin{aligned} \{\rho : \pi_{0,1} \to \Gamma\} &\cong_{set} \{\rho : \pi_1 \to \Gamma\} \times \Gamma \\ \rho &\mapsto \rho|_{End(b_0)} \times \rho(a) \end{aligned}$$

1.5 Extended TQFTs

When we have particles, we'd want to examine their statistics. In 2D TQFTs, the particles are points in the disc, and their motion groups are braid groups. The natural generalization for statistics of extended objects in higher dimensions are motions groups of links in S^3. Representations of these groups are used to model statistics of extended objects, e.g. closed strings.

Schwarz-type TQFTs are theories where the action functional is metric-independent, e.g. Chern-Simons theory.

Atiyah-type TQFTs are functors from the bordism category of manifolds to the category of finite-dimensional vector spaces with morphisms linear maps:

$$Z : \mathbf{Bord}_{n+1} \to \mathbf{Vec}$$

satisfying functoriality:

1. $Z(X) = Z(X') : V(Y_-) \to V(Y_+)$ if X, X' are equivalent cobordisms
2. $Z(Y \times I) = Id_{V(Y)}$
3. $Z(X_2 \cup X_1) = Z(X_2) \cdot Z(X_1)$

and monoidality:

1. $V(\emptyset) \cong \mathbb{C}$
2. $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$ with

$$\begin{tikzcd} V((Y_1 \sqcup Y_2) \sqcup Y_3) \arrow[r, "\cong"] \arrow[d] & (V(Y_1) \otimes V(Y_2)) \otimes V(Y_3) \arrow[d] \\ V(Y_1 \sqcup (Y_2 \sqcup Y_3)) \arrow[r, "\cong"] & V(Y_1) \otimes (V(Y_2) \otimes V(Y_3)) \end{tikzcd}$$

3. Unions with the empty set:

$$\begin{tikzcd} V(\emptyset \sqcup Y) \arrow[r, "\cong"] \arrow[d] & \mathbb{C} \otimes V(Y) \arrow[d] \\ V(Y) \arrow[r, "="] & V(Y) \end{tikzcd}$$

4. Symmetry: $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2) \to V(Y_2) \otimes V(Y_1) \cong V(Y_2 \sqcup Y_2)$

where the isomorphisms are canonical.

Atiyah-type TQFTs don't necessarily lead to representations of motion groups. For this we need **extended TQFTs**.

A **k-extended (n+1)-TQFT** (also denoted **(n+1,-k)-TQFT**) is a TQFT extended from $(n+1)-$manifolds all the way back to $(n-k)-$manifolds.

Our definition of a Dikgraaf-Witten TQFT is a 1-extended TQFT. A **1-extended (n+1)−TQFT** is a projectively symmetric monoidal functor from the category $Bord_n^{n+1}$ to Vec, the category of finite-dimensional complex vector spaces. This is so far a projective Atiyah-type TQFT, but we assign a semisimple category $\mathscr{C}(\Sigma)$ to each oriented closed $(n-1)-$manifold Σ and a finite-dimensional vector space $V(Y;\{X_l\})$ to each oriented $n-$manifold Y with parametrized and labeled boundary components by $X_l \in \prod_{\mathscr{C}}(\partial Y)$ (where $\prod_{\mathscr{C}}$ is a complete set of simple representatives of the category $\mathscr{C}$) subject to the usual empty set axiom, disc axiom, tube axiom, disjoint union axiom, duality axioms, and gluing axioms:

1. (Empty manifold axiom) $V(\emptyset) = 1, \mathbb{C}$, or Vec if $\emptyset$ is a manifold of dimension $n+1, n$, or $n-1$, respectively.

2. (Disc axiom) $V(D^n; X) \cong \mathbb{C}$ if $X = 1$ (the tensor unit), and 0 otherwise.

3. (Tube axiom) $V(S^{n-1} \times I; X_i, X_j) \cong \mathbb{C}$ if $X_i \cong X_j^*$, and 0 otherwise, where the isomorphisms are isomorphisms as vector spaces and functorial isomorphisms. (See Frobenius-Schur indicators of labels)

4. (Disjoint union axiom) $V(Y_1 \sqcup Y_2; X_{l_1} \sqcup X_{l_2}) \cong V(Y_1; X_{l_1}) \otimes V(Y_2; X_{l_2})$, associatively and compatible with mapping class group projective actions.

5. (Duality axiom 1) $V(-Y; X_l) \cong V(Y; X_l)^*$

6. (Duality axiom 2) The isomorphisms $V(Y) \to V(-Y)^*, V(-Y) \to V(Y)^*$ are mutually adjoint.

7. (Duality axiom 3) Given $f : (Y_1; X_{l_1}) \to (Y_2; X_{l_2})$, let $\overline{f} : (-Y_1; X_{l_1}^*) \to (-Y_2; X_{l_2}^*)$ be the induced map. Then $\langle x, y\rangle = \langle V(f)x, V(\overline{f})y\rangle$, with $x \in V(Y_1; X_{l_1}), y \in V(Y_2; X_{l_2})$.

8. (Duality axiom 4) $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle\langle x_2, y_2\rangle$, for $x_1 \otimes x_2 \in V(Y_1 \sqcup Y_2), y_1 \otimes y_2 \in V(-Y_1 \sqcup -Y_2)$

9. (Gluing axiom) If Y_{gl} is the manifold resulting from gluing two boundary components Σ of a manifold Y, then $V(Y_{gl}) \cong \oplus_{X_i \in \prod_{\mathscr{C}}(\Sigma)} V(Y; (X_i, X_i^*))$. This isomorphism is associative and compatible with mapping class group actions. Moreover, there exist nonzero numbers $s_j, j \in \prod_{\mathscr{C}}$ such that $\langle \oplus_{j\in\prod_{\mathscr{C}}} x_j, \oplus_{j\in\prod_{\mathscr{C}}} y_j\rangle = \sum_{j\in\prod_{\mathscr{C}}} s_j \langle x_j, y_j\rangle$

2 Cutting and Gluing

For DW theory to be a self-respecting topological quantum field theory, it has to be local. For $n = 0, w = 0$, let Y be such that $Y = Y_1 \# Y_2$, where $\partial Y_{1,2} = S^1$. What we want is to be able to write $\mathbb{A}(Y)$ in terms of $\mathbb{A}(Y_1)$ and $\mathbb{A}(Y_2)$. These new manifolds have boundary S^1 that must agree when mapped to $B\Gamma$. Where this S^1 must map to on $B\Gamma$ yields different elements of $\mathbb{A}(Y)$. Thus we have

$$\begin{aligned} \mathbb{A}(Y) := \oplus_c \mathbb{A}(Y_1; c) \otimes \mathbb{A}(Y_2; c) / \sim \\ \alpha e \otimes \beta \sim \alpha \otimes e\beta \end{aligned}$$

for all $\alpha \in \mathbb{A}(Y_1; c), \beta \in \mathbb{A}(Y_2; d)$, and e a morphism from c to d. This means topologically that, if we have Y_1 with boundary c and Y_2 with boundary d, it doesn't matter whether we glue a cylinder from c to d to Y_1 or Y_2. Y_1 glued with the cylinder and Y_2 mapped to $B\Gamma$ is homotopically the same as Y_1 and Y_2 glued with the cylinder, so they ought to be the same algebraically. This is in fact the only quotient relationship we need.

More generally, let Y be a compact manifold with boundary given by $\partial Y = S_+ \sqcup S_- \sqcup S_0$, where $S_+ = -S_-$. Let Y_{gl} denote the manifold obtained by gluing S_+ to S_- in Y. We want a gluing map from $\mathbb{A}(Y; a, b, c) \xrightarrow{gl} \mathbb{A}(Y_{gl}; c)$ for various a and b. It is easy to see topologically that the gluing map

$$gl : \oplus_{x \in \mathbb{A}(S_\pm)} \mathbb{A}(Y; x, x, c) \to \mathbb{A}(Y_{gl}; c)$$

is surjective via isotopy. See figure 2.

Now we want to describe the kernel of this map. Let $e \in \mathbb{A}(S_\pm \times I; a, b)$. Since $gl_b(Y \cup_{S_+} e) \sim gl_a(Y \cup_{S_-} e)$ in Y_{gl}, we have

$$(Y \cup_{S_+} e) - (Y \cup_{S_-} e) \in \ker(gl) \subset \oplus_{x \in S_\pm} \mathbb{A}(Y; x, x, c)$$

We claim that $(Y \cup_{S_+} e) - (Y \cup_{S_-} e)$ generates all of $\ker(gl)$. This is known as the **Gluing theorem**:

Theorem 1. *Let Y be a manifold such that $\partial Y = S_+ \sqcup S_- \sqcup S_0$ with $S_+ = -S_-$, and let Y_{gl} be the manifold obtained from gluing S_+ to S_-. Note that $\partial Y_{gl} = S_0$. Let $C \in Maps(Y \to B\Gamma; a, b, c)$ and $e \in Maps(S_\pm \times I \to B\Gamma; a, b)$ for some $a, b \in Maps(S_\pm \to B\Gamma)$. Let $L \subset \oplus_x \mathbb{A}(Y; x, x, c)$ be the subspace generated by all elements of the form $Ce - eC$. Then there is a natural isomorphism*

$$\mathbb{A}(Y_{gl}; c) \cong \oplus_x \mathbb{A}(Y; x, x, c) / L$$

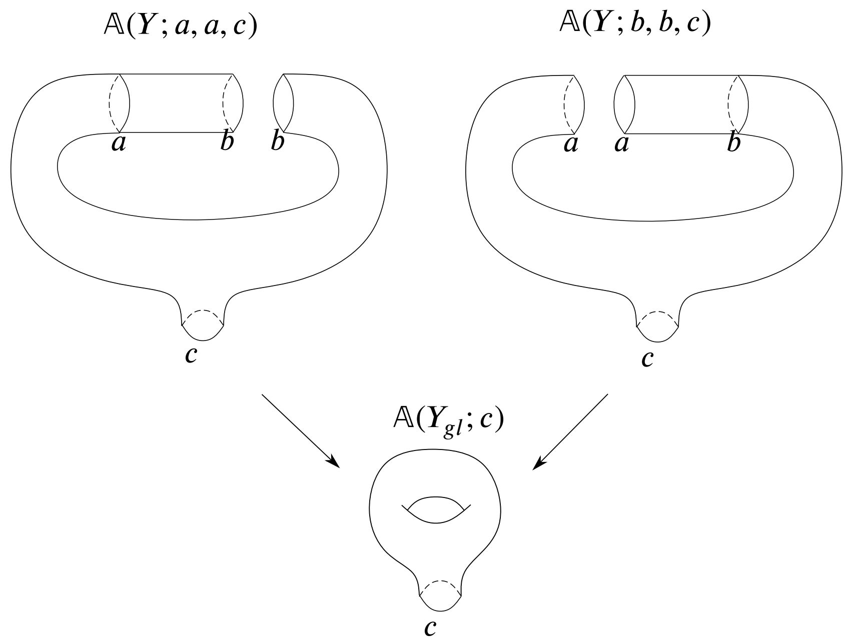


Figure 2: Gluing annuli

2.1 Tube Category

The gluing theorem can be usefully restated in terms of actions on a **tube category** (also called **cylinder category**). For a manifold S, this category's objects are maps $S \to B\Gamma$ and its morphisms are $\mathbb{A}(S \times I; a, b)$. Composition is given by gluing tubes. We restate the gluing theorem in these terms:

Theorem 2. *Let W be a vector space and linear maps $f_a : \mathbb{A}(Y; a, a, c)$ for all $x \in \mathbb{A}(Y; a, a, c)$, such that for all $e : x \to y$, the following diagram commutes:*

$$\begin{tikzcd}
& \mathbb{A}(Y; a, a, c) \arrow[dr, "f_a"] & \\
\mathbb{A}(Y; a, b, c) \arrow[ur, "Id_a \times e"] \arrow[dr, "e \times Id_b"'] & & W \\
& \mathbb{A}(Y; b, b, c) \arrow[ur, "f_b"'] &
\end{tikzcd}$$

then there exists a map $g : \mathbb{A}(Y_{gl}; c) \to W$ such that $f_a = g \cdot gl_a$ for all x.

This rephrasing makes it easy to generalize to different target categories.

3 Algebra prerequisites

3.1 Modules and algebras

Here we recall the definition of an algebra. For R a ring with multiplicative identity 1_R, a **left R-module** is an abelian group $(M, +)$ with an operation $\cdot : R \times M \to M$, such that, for all $r, s \in R$ and $x, y \in M$,

$$\begin{aligned} r \cdot (x + y) &= r \cdot x + r \cdot y \\ (r + s) \cdot x &= r \cdot x + s \cdot y \\ (rs) \cdot x &= r \cdot (s \cdot x) \\ 1_R \cdot x &= x \end{aligned}$$

For a **right R-module**, flip the actions above, using the map $\cdot : M \times R \to M$. When R is a field then M is a **vector space**.

An **algebra** over a field is a vector space A with a map $\times : A \times A \to A$ that is right and left distributive, and scalar compatible ($ax \times by = (ab)[x \times y]$).

3.2 Idempotents

An **idempotent** of A is an element e such that

$$e^2 = e$$

and by induction $e^n = e, n \geq 1$. Two idempotents e_1, e_2 are orthogonal if $e_1 e_2 = 0 = e_2 e_1$. It is quite easy to see that:

1. If e_1, e_2 are commuting idempotents, then e_1, e_2 is also an idempotent.
2. If e is an idempotent, then $Id - e$ is an idempotent.
3. If e_1, e_2 are orthogonal idempotents, then $e_1 + e_2$ is an idempotent.
4. If e is an idempotent, e and $Id - e$ are orthogonal.

An idempotent e is **minimal** if and only if eAe is 1-dimensional. Also, e is minimal if and only if e cannot be written as a sum of two nonzero idempotents $e_1 + e_2$.

Example 1. Let $A = \mathbb{C}[\mathbb{Z}_k]$, where t is the generator. Then the minimal idempotents are

$$e_j = \frac{1}{k} \sum_{n=1}^{k} e^{\frac{i2\pi jn}{k}} t^n$$

Example 2. Let $A = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$. Then the minimal idempotents are

$$\frac{1}{4}[(0,0) + (1,0) + (0,1) + (1,1)]$$
$$\frac{1}{4}[(0,0) + (1,0) - (0,1) - (1,1)]$$
$$\frac{1}{4}[(0,0) - (1,0) - (0,1) + (1,1)]$$
$$\frac{1}{4}[(0,0) - (1,0) + (0,1) - (1,1)]$$

3.3 Morita Equivalence

If we consider a module over an algebra, we get a way for an algebra to act on a vector space. Whereas representations provide a way for groups to act on vector spaces, modules provide a way for algebras to act on vector spaces. Here we introduce **Morita equivalence**. Two rings are **Morita equivalent** if the categories of modules over these rings are equivalent.

Two categories $\mathscr{C}, \mathscr{D}$ are equivalent if there exists functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that there exist natural isomorphisms

$$\epsilon : F \circ G \to Id_D$$
$$\eta : Id_C \to G \circ F$$

Equivalently, a functor $F : \mathscr{C} \to \mathscr{D}$ yields an equivalence if:

1. For any two objects $c_1, c_2 \in \mathscr{C}$, the map

   $$\hom_{\mathscr{C}}(c_1, c_2) \to \hom_{\mathscr{D}}(F(c_1), F(c_2))$$

   is bijective (**fully faithful**). When this map is surjective, F is called **full**, and when it's injective, F is called **faithful**.

2. Each object $d \in \mathscr{D}$ is isomorphic to an object of the form $F(c)$ for $c \in \mathscr{C}$. (**Essentially surjective**, or **dense**)

The classic example of Morita equivalent rings is a ring R and the ring S of $n \times n$ matrices with entries in R, for any n.

Theorem 3. *Let $(R, 1)$ be a ring and $S = M_n(R)$ be the ring of $n \times n$ matrices with entries in R. Then $R \cong_M S$.*

Proof. Let M be a (right) $R-$module. Let $F(M) = \{(m_1, ..., m_n) | m_i \in M\}$. $F(M)$ becomes a module over $M_n(R)$, where all "vectors" $aF(M)$ arise from matrix-vector multiplication for $a \in M_n(R)$.

If $f : M_1 \to M_2$ is a module homomorphism (morphism in the category of $R-$modules), we have $F(f) : F(M_1) \to F(M_2)$ given by $F(f)(m_1, ..., m_2) = (f(m_1), ..., f(m_n))$, so F is a covariant functor.

We have functors F from $R-$modules to $S-$modules. Now we need a functor going the opposite way. Let N be an $S-$module. Let $e(r)$ be the $n \times n$ matrix where the $(0,0)^{th}$ entry is $r \in R$, and 0 everywhere else. Note that $e(1)$ is an idempotent, and $e(1)e(r) = e(r)e(1)$.

Let $G(N) = \{se(1) | s \in N\}$. Define the scalar multiplication with $r \in R$ by $se(1) \cdot r := se(1)e(r) = se(r)e(1)$. Since $se(r) \in N$, $G(N)$ is an $R-$module. If $g : N_1 \to N_2$ is an $S-$module homomorphism, let $G(g)(se(1)) = g(s)e(1)$. One can then easily check that G is a covariant functor.

Now we compute that, for M an $R-$module,

$$G \circ F(M) = \{(m_1, ..., m_n)e | m_i \in M\} = \{\begin{pmatrix} m_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} | m \in M\} \cong M$$

and, for N an $S-$module,

$$F \circ G(N) = \{s_1 e(1), ..., s_n e(1) | s_i \in N\}$$

Denote the matrix with the $(i,i)^{th}$ component equal to 1 and all other entries equal to 0 by e_{ii}. Note that e_{ii} is idempotent, $e_{ii}e_{jj} = 0$, and $\sum_i e_{ii} = 1$. Then $N = Ne_{11} \oplus ... \oplus Ne_{nn}$. Note also that $Ne_{ii} \cong Ne_{jj}$ as M_n modules. Let $\pi_i : N \to Ne_{ii}$ be the projection map and $\psi_i : Ne_{ii} \to N$ be the embedding, and $\phi_{ij} : Ne_{ii} \to Ne_{jj}$ be the isomorphism from Ne_{ii} to Ne_{jj}. Note that these maps are M_n-module homomorphisms since $e_{ii}A = Ae_{ii}$ for an $R-$module A.

Take any $s \in N$. Then we have a homomorphism

$$\begin{aligned} \alpha : N &\to F \circ G(N) \\ \alpha : s &\mapsto (\pi_1(s), ..., \pi_n(s)) \\ &\mapsto (\phi_{11}\pi_1(s), ..., \phi_{n1}\pi_n(s)) \in F \circ G(N) \end{aligned}$$

and a homomorphism

$$\begin{aligned} \beta : F \circ G(N) &\to N \\ \beta : (s_1 e_1, ..., s_n e_1) &\mapsto (\phi_1 1(s_1 e_1), ..., \phi_{1n}(s_n e_1)) \\ &\mapsto \psi_1(\phi_{11}(s_1 e_1)) + ... + \psi_n(\phi_{1n}(s_n e_1)) \end{aligned}$$

By inspection $\beta = \alpha^{-1}$ and vise versa, so $F \circ G(N) \cong N$. □

3.4 Fusion Categories

Let R be a ring. An $R-$**linear category** $\mathscr{C}$ is a category such that, for all $A, B \in Obj(\mathscr{C})$, the set of morphisms $Hom(A, B)$ in $\mathscr{C}$ has the structure of an $R-$module, and composition of morphisms is $R-$bilinear. If all hom sets $Hom(A, B)$ are abelian groups and composition of morphisms is bilinear, then $\mathscr{C}$ is **preadditive**.

A **monoidal category** (**tensor category**) $\mathscr{C}$ is a category equipped with:

1. A functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

called a **tensor product**,

2. an object $1 \in \mathscr{C}$ with natural isomorphisms

$$\begin{aligned} \lambda_x &: 1 \otimes X \to X \\ \rho_x &: X \otimes 1 \to X \end{aligned}$$

for all $X \in Obj(\mathscr{C})$,

3. Natural isomorphisms, for all $A, B, C \in Obj(\mathscr{C})$, such that

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

such that the **triangle identity** is satisfied (the following diagram commutes):

$$\begin{tikzcd} A \otimes (1 \otimes B) \arrow[rr, "a_{A,1,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & (A \otimes 1) \otimes B \arrow[dl, "1_A \otimes \lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

and the **pentagon identity** is satisfied (the following diagram commutes) for all $A, B, C, D \in Obj(\mathscr{C})$:

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "a_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes Id_D"] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "Id_A \otimes a_{B,C,D}"']
\end{tikzcd}$$

Think about the objects of the category being a monoid under the operation $\otimes$ A linear category $\mathscr{C}$ is **additive** if every finite set of objects has a biproduct $\oplus$ (think about direct sums).

An additive category is **preabelian** if every morphism $f : X \to Y$ has a kernel and cokernel $(Y/Im(f))$.

A preabelian category is **abelian** if every "injective" morphism (monomorphism) is the kernel of some morphism, and every "surjective" morphism (epimorphism) is the cokernel of some morphism. The quotes around *injective* and *surjective* note that they are the generalizations of injective/surjective maps. Thus an abelian category is a generalization of the category of abelian groups, that allows for things like exact sequences to arise naturally.

An abelian category $\mathscr{C}$ is **semisimple** if there is a collection of simple objects $A_i \in Obj(\mathscr{C})$ (an object is **strongly simple** in an abelian $\Bbbk-$linear category if $End(A_i) \cong \Bbbk$, and if $\Bbbk$ is algebraically closed every simple object is strongly simple) such that any $A \in Obj(\mathscr{C})$ is the direct sum of finitely many simple objects. Alternatively, a linear monoidal category with ground field $\Bbbk$ is **semisimple** if:

1. It has finite biproducts $\oplus$,

2. There is a morphism $e : A \to A$ with an object B and morphisms $r : A \to B, s : B \to A$ such that $s \circ r = e, r \circ s = Id_B$,

3. There exist objects X_i labeled by an index set I such that $Hom(X_i, X_j) \cong \delta_{ij}\Bbbk$ such that, for any $A, B \in \mathscr{C}$, there is a natural isomorphism

$$\bigoplus_{i\in I} Hom(A, X_i) \otimes Hom(X_i, B) \cong Hom(A, B)$$

A monoidal category $(\mathscr{C}, \otimes, 1)$ is (left, right) **rigid** if, for every object X, there is a (resp. left, right) inverse X^* such that there are natural isomorphisms

$$X^* \otimes X \cong 1$$
$$(X \otimes X^* \cong 1)$$

If the category is left and right rigid the category is said to be rigid. The operation of taking duals is a contravariant functor on a rigid category.

Kuperberg proved that finite, connected, semisimple, rigid monoidal (tensor) categories are linear.

A **fusion** category is a linear, finite, strongly semisimple rigid monoidal category.

4 Condensed Matter

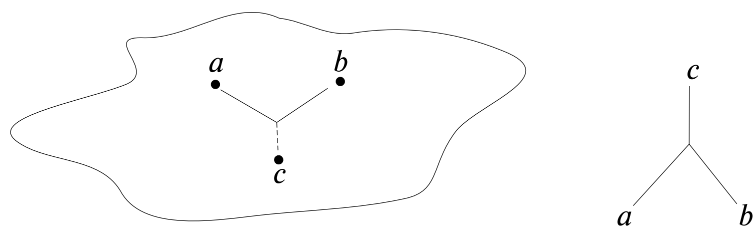


Figure 3: Fusion of anyons.

Suppose we have a (2D) sample with two anyons a and b fusing to get c. See 3. The direct sum $\oplus_c V_{ab}^c$ is a decomposition of $\mathscr{H}_a \otimes \mathscr{H}_b$ This corresponds to a quantum state in an N_{ab}^c−dimensional Hilbert space V_{ab}^c. In topological quantum field theory, V_{ab}^c is the vector space corresponding to the 3-punctured 2-sphere. More complicated Hilbert spaces (and 2-manifolds) can be constructed from such V_{ab}^c (3-punctured spheres). The decomposition of these Hilbert spaces is modeled using fusion categories - this is because fusion can be much more complicated, as in 4.

We can build any type of particle fusion in a TQFT by a 3-punctured sphere. By gluing 3-punctured spheres together we can get a fusion of any number of particles, but we have to be specific about how we build the associated Hilbert spaces. For instance, we need associators:

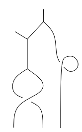


Figure 4: More complicated fusion of anyons.

$$
\begin{tikzcd}
& D & \\
& \cdot \arrow[u] & \\
A \arrow[ur] & B \arrow[u] & C \arrow[ul]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& D & \\
& \cdot \arrow[u] & \\
\cdot \arrow[ur] & & C \arrow[ul] \\
A \arrow[u] & B \arrow[ul] &
\end{tikzcd}
=
\begin{tikzcd}
& D & \\
& \cdot \arrow[u] & \\
A \arrow[ur] & & \cdot \arrow[ul] \\
& B \arrow[ur] & C \arrow[u]
\end{tikzcd}
$$

that give $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, pentagon equations, and $6j-$ symbols (see 5).

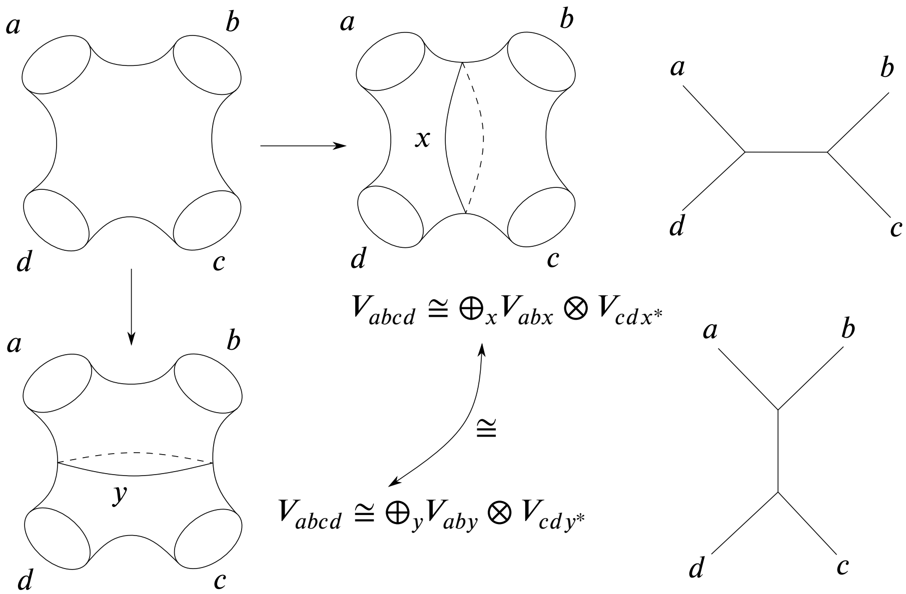


Figure 5: $6j$-symbols in TQFT.

4.1 Kitaev's Lattice Model

4.1.1 Toric Code

Consider a square lattice with toric boundary conditions with spins associated to each edge. For each vertex v, define the operator

$$A = \prod_{i \in v} \sigma_i^x$$

and for each plaquette (square bounded by four edges) define the operator

$$B = \prod_{i \in p} \sigma_i^z$$

where the product in A is given over the four edges touching v and the product in B is given over the four edges bordering the plaquette p. The stabilizer space for this code is $|\psi\rangle$ such that

$$\begin{aligned} A_v |\psi\rangle &= |\psi\rangle, \forall v, \\ B_p |\psi\rangle &= |\psi\rangle, \forall p \end{aligned}$$

The stabilizer space of this code is 4-dimensional, and thus can encode 2 qubits. Violations of the stabilizer space is the syndrome of the code, and their positions represent quasiparticles. The hamiltonian is given by

$$H = -J\sum_v A_v - J\sum_p B_p, J > 0$$

4.1.2 Kitaev's Lattic Model

Let R be a finite-dimensional semisimple Hopf algebra. For a compact oriented surface Σ with a cell decomposition Δ with orientation o, we define the Hilbert space

$$\mathscr{H}_K(\Sigma, \Delta, o) = \otimes_{\text{edges}} R$$

We need not specify an orientation of Δ: If o' differs by o by the reversal of orientation of a single edge e, we have the isomorphism

$$\begin{aligned} \mathscr{H}_k(\Sigma, \Delta, o) &\to \mathscr{H}_k(\Sigma, \Delta, o') \\ x_e &\mapsto S(x_e) \end{aligned}$$

We call a **site** of a cell decomposition a pair (v, p) of a vertex and plaquette. To each site we associate a vertex operator $A^a_{v,p} : \mathscr{H}_k(\Sigma, \Delta) \to \mathscr{H}_k(\Sigma, \Delta)$ by acting by $a^{(n)}$ on each edge x_n touching the vertex. We also associate a plaquette operator $B^\alpha_{v,p} : \mathscr{H}_k(\Sigma, \Delta) \to \mathscr{H}_k(\Sigma, \Delta)$ by acting by $\alpha^{(n)}$ on each edge x_n bordering the plaquette. If we define $h \in R, \overline{h} \in \overline{R}$ as the Haar integrals of $R, \overline{R}$, respectively, then the Hamiltonian is given by

$$H = \sum_v (1 - A^h_v) + \sum_p (1 - B^{\overline{h}}_v)$$

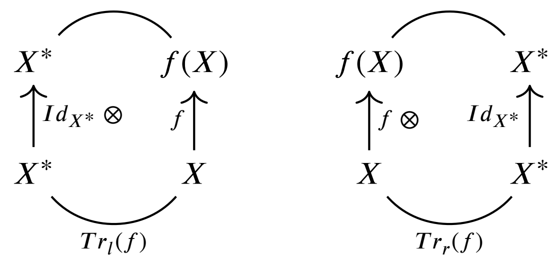
Note that since, $h^2 = h$ and h is central, all $A^h_v, B^{\overline{h}}_p$ commute with each other, and each is idempotent.

4.2 Turaev-Viro Model

The **Turaev-Viro** (-**Barrett-Westbury**) **Model** is a 3D TQFT based on a fusion category $\mathscr{C}$. If Γ is a finite group and $\mathscr{C} = Vect_\Gamma$, the category of $\Gamma-$graded vector spaces, this becomes Dijkgraaf-Witten theory.

A **pivotal category** is a rigid monoidal category equipped with a monoidal natural isomorphism $A \to (A^*)^*$.

For a pivotal category, the left trace $Tr_l(f)$ and right trace $Tr_r(f)$ of a morphism f are given by



The **left (right) dimension** of $X \in Obj(\mathscr{C})$ is given by $dim_l(X) = Tr_l(Id_X)$ ($dim_r(X) = Tr_r(Id_X)$).

A **spherical category** is a pivotal category where the left and right trace operations coincide on all objects.

Given a spherical fusion category $\mathscr{A}$, the **Turaev-Viro Model** associates a vector space $H^{\mathscr{A}}_{TV}(\Sigma, \Delta)$ given a closed oriented surface Σ with a cell decomposition Δ defined as follows: for every oriented edge $e \in \Delta$, associate a simple object l_e such that $l_{\overline{e}} = l_e^*$, and build our vector space via

$$H^{\mathscr{A}}_{TV}(\Sigma, \Delta) = \oplus_l [\otimes_{C|\partial C = e_1 \cup \ldots \cup e_n} Hom_{\mathscr{A}}(1, l_{e_1} \otimes \ldots \otimes l_{e_n})]$$

where the e_i are taken in the counterclockwise order on ∂C. Given a cobordism M between (Σ, Δ) and (Σ', Δ') we define an operator $Z(M) : H^{\mathscr{A}}_{TV}(\Sigma, \Delta) \to H^{\mathscr{A}}_{TV}(\Sigma', \Delta')$ based on a cell decomposition of M, and actually independently of any specific cell decomposition: If $\partial M = \overline{\Sigma} \sqcup \Sigma'$, then

$$H^{\mathscr{A}}_{TV}(\partial M) = H^{\mathscr{A}}_{TV}(\Sigma)^* \otimes H^{\mathscr{A}}_{TV}(\Sigma') = \hom(H^{\mathscr{A}}_{TV}(\Sigma), H^{\mathscr{A}}_{TV}(\Sigma'))$$

given by

$$Z_{TV}(M) = dim(\mathscr{A})^{-2v(M)} \sum_l (ev(\otimes_C Z(C, l)) \prod_e dim(l(e))^{n_e})$$

where e runs over all unoriented edges in M, $v(M)$ is the number of internal vertices in M plus half of the internal vertices in ∂M, and n_e is 1 if the edge e is internal and $\frac{1}{2}$ if $e \in \partial M$.

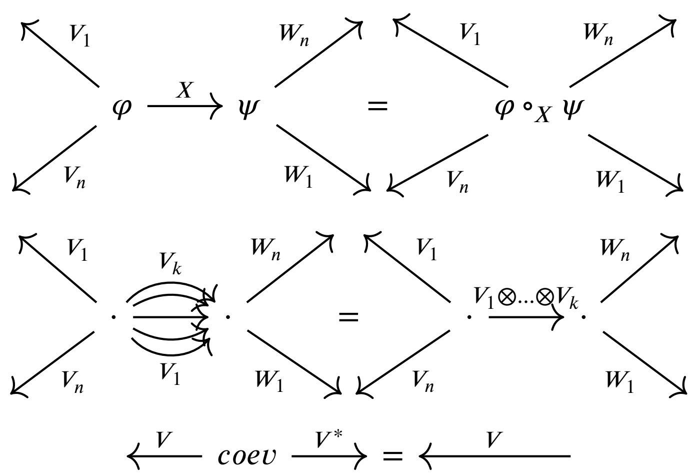
4.3 Levin-Wen Model

(These are also called stringnet models)
Here we once again begin with a spherical fusion category $\mathscr{A}$ and consider colored graphs Γ on Σ. Edges of Γ are colored by an object of $\mathscr{A}$ and vertices are colored by $\hom(1, V_1 \otimes ... \otimes V_n)$, where each V_i is the object associated to the edge (ordered counterclockwise) intersecting the vertex in question. The orientation of the edge is outward, so if the object associated to the inward edge is V_i, the object with the outward orientation should be V_i^*.

We define the stringnet space

$$H^{str}(\Sigma) = \{\text{Formal linear combinations of colored graphs on } \Sigma\}/\text{Local relations}$$

The local relations are, for graphs equivalent outside of a disc and inside the disc differ by the following 3 relations, the graphs are equivalent.



The is a well-known theorem that $H_{TV}^{\mathscr{A}}(\Sigma)$ is canonically isomorphic to $H^{str}(\Sigma)$.

It is not hard to see that, if the category is a group G, the coloring of Γ with elements in G encode a map into BG. This is why Levin-Wen models are a generalization of Dijkgraaf-Witten theories.