Lie Algebras & their Relation to Conformal Field Theory

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Root System

A *hyperplane* is a plane is a plane with dimension 1 less than its ambient space, e.g. sheets in $\mathbb{R}^3$, a line in a sheet, etc.

Let E be a finite-dimensional Euclidean vector space with the Euclidean inner product $(\cdot, \cdot)$. A *root system* Φ is a set of nonzero vectors called *roots* in E such that:

The roots span E.

The only scalar multiples of a root $\alpha \in \Phi$ are α and $-\alpha$.

$\forall \alpha \in \Phi$, Φ is closed under reflection through the hyperplane perpendicular to α.

For any roots $\alpha, \beta \in \Phi$, the projection of β through the span of α is either an integer or half-integer multiple of α.

Weyl Group

The *Isometry Group* on a metric space is the set of distance-preserving bijective maps from the space to itself, with composition the group operation.

Example 1. *The isometry group of S^2 is $O(3)$. (group of $n \times n$ matrices where $M^T M = MM^T = Id$)*

The *Weyl Group* is a subgroup of the isometry group of the root system, generated by reflections of the hyperplanes orthogonal to the roots.

Lie Algebras

A *Lie Algebra* is an algebra $\mathfrak{g}$ where the binary operation is bilinear,

$[ax + by, z] = a[x, z] + b[y, z], [x, ay + bz] = a[x, y] + b[x, z]$

alternative,

$[x, x] = 0, \forall x \in \mathfrak{g}$

and satisfies the *Jacobi Identity*

$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

Remark 1. *We can show that $[x, y] = -[y, z]$ by considering $[x + y, x + y]$.*

Example 2. $\mathfrak{g} = \{X \in Mat(n, \mathbb{C}) | \forall t \in \mathbb{R}, e^{tX} \in G\}$*, for G a lie group.*

Example 3. *$SL(n, \mathbb{R})$, the matrix group with real entries and determinant 1, has lie algebra $n \times n$ matrices with real values and trace 0.*

Physics Flash

The angular momentum operators have the same commutation relations as $\mathfrak{so}(3)$ of $SO(3) < O(3)$, where $SO(3)$ is the group of othorgonal matrices of determinant 1.

Representations of Lie Algebras

Given a vector space V, $\mathfrak{gl}(V)$ is the lie algebra of linear automorphisms on V, with the bracket given by $[X, Y] = XY - YX$.

A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V) \tag{1}$$

A representation is called faithful if its kernel is is 0. We can construct a representation for any Lie algebra $\mathfrak{g}$:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{2}$$

$$ad(x)(y) \mapsto [x, y] \tag{3}$$

This representation is called the *adjoint representation*.

Semi-Simple Lie Algebras

A lie algebra is *simple* if it is not abelian and has no non-trivial ideals. (recall a left ideal is a subset of our algebraic object such that the subset is closed under addition, scalar multiplication, and LEFT multiplication from another vector)

A lie algebra is *semi-simple* if it can be expressed as a (is isomorphic to a) direct sum of simple lie algebras.

In this way one can think of a simple lie algebra as being one-dimensional.

The *Cartan Subalgebra* is the maximal abelian ideal of our lie algebra (generated by all $H^i, i = 1, ..., r$ such that $[H^i, H^j] = 0$) denoted $\mathfrak{h}$.

Root System of a Semi-Simple Lie Algebra

The remaining generators of $\mathfrak{g}$ are chosen such that $[H^i, E^\alpha] = \alpha^i E^\alpha$, (recall angular momentum operators/spin operators in quantum mechanics) and α^i is a component of the vector $\alpha = (\alpha^1, ..., \alpha^r)$. α is called a root and E^α is the corresponding ladder operator. α is therefore a nonzero element of $\mathfrak{h}^*$. The roots are the nonzero weights for the adjoint representation. These roots form, if you can believe it, a root system.

Remark 2. *Because $\mathfrak{h}$ is the maximal subalgebra, of $\mathfrak{g}$, the roots are non-degenerate.*

In the adjoint representation we have

$$ad(H^i)E^\alpha = \alpha^i E^\alpha \mapsto H^i |\alpha\rangle = \alpha^i |\alpha\rangle$$

The one-to-one correspondence with between the states $|\alpha\rangle$ and E^α reflect the nondegenerate character of roots. In this representation, the zero eigenvalue has degeneracy r for each state $|H^i\rangle$. The adjoint representation is a representation using the Lie algebra itself as the vector space. We define an inner product using the Killing Form

$$\tilde{K}(X, Y) = Tr(adXadY) \tag{4}$$

$$K(X, Y) = \frac{1}{2g} Tr(adXadY) \tag{5}$$

where g is the dual coxeter number of the algebra.

Weights

For an arbitrary representation, a basis $\{|\lambda\rangle\}$ can always be found such that $H^i |\lambda\rangle = \lambda^i |\lambda\rangle$ eigenvalues exist. These eigenvalues build the vector $\lambda = (\lambda^1, ..., \lambda^r)$ called a *weight*. The *weight space* with weight λ is a subspace of V:

$$V_\lambda = \{v \in V : \forall H^i \in \mathfrak{h}, H \cdot v = \lambda(H^i)v\} \tag{6}$$

where $\lambda(H^i)$ is the weight associated to V_λ. A weight of the representation is a linear functional λ such that the corresponding V_λ is nonzero. The nonzero elements of the weight space are *weight vectors.*

(This is to say the weight vectors are the simulataneous eigenvectors for the action of the elements of $\mathfrak{h}$ with weight $\lambda = (\lambda^1, ..., \lambda^r)$)

A simple root can be written as a linear combination of a set of fundamental weights (weights that satisfy $\omega_i(H^{\alpha_j}) = \delta_{ij}$)

Examples

Let $E_{jk} \in \mathbb{C}^{3\times 3}$ be the 3×3 matrix whose j, k^{th} element is equal to 1 and the rest are equal to 0, for $j, k \in \{1, 2, 3\}$.

For the lie group $SL_3(\mathbb{C})$ we define for the lie algebra $H_{jk} = E_{jj} - E_{kk}$

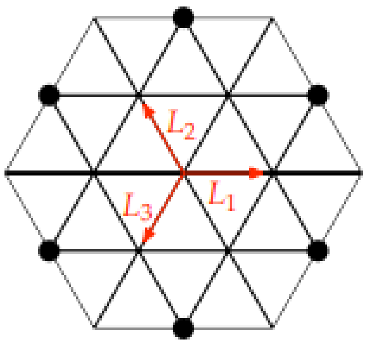
$$\mathfrak{h} = \mathbb{C}H_{12} + \mathbb{C}H_{13} + \mathbb{C}H_{23} \tag{7}$$

is a Cartan subalgebra. Define L_i be the following:

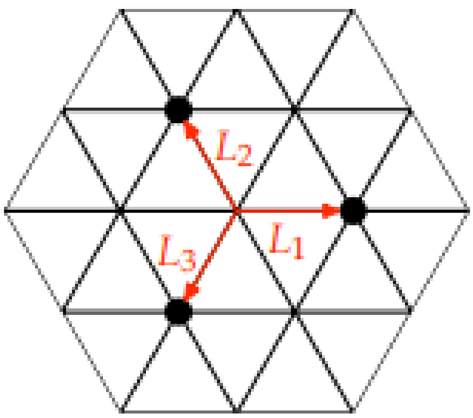
$$L_i : \mathfrak{h} \to \mathbb{C} \tag{8}$$

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \mapsto x_i \tag{9}$$

With this notation we can see that $[H_{jk}, E_{rs}] = ((L_r - L_s)(H_{jk}))E_{rs}$, so $L_r - L_s$ are the roots of the representation, drawn with a dot.



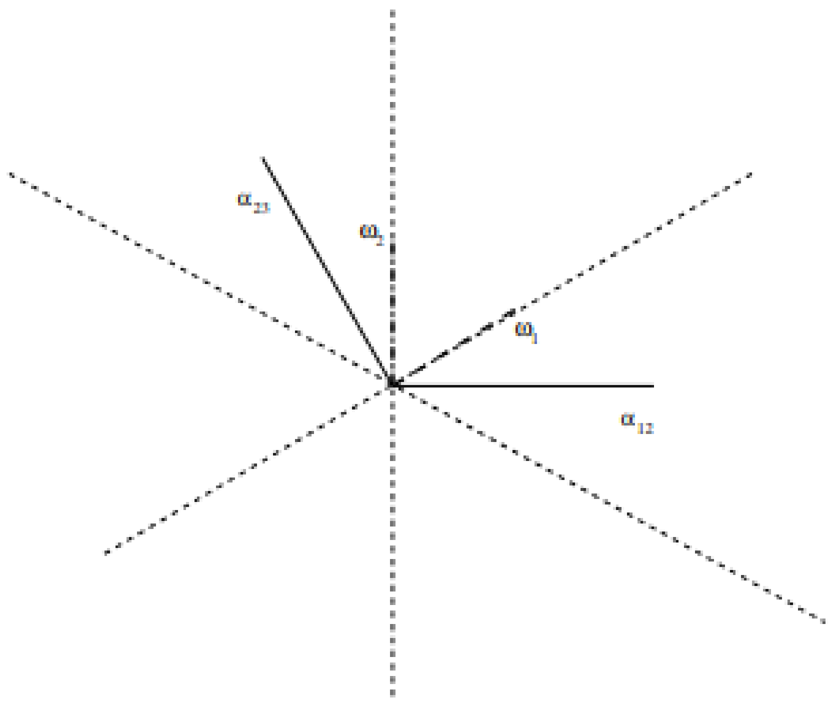
Letting e_i be the i^{th} unit vector, we see that $H_{jk}e_i = \delta_{ji}e_i - \delta_{ki}e_i = L_i(H_{jk})e_i$, so the weights are L_i, shown in the diagram below with a dot:



(images from Clara Loh's notes: "Representation theory of Lie algebras")

Here's another example explaining these diagrams:

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2 \\ \alpha_2 &= -\omega_1 + 2\omega_1 \end{aligned}$$



Simple Roots and Fundamental Weights for $SU(3)$

(image from notes of Peter Woit)