Useful Mathematical Preliminary Objects (that I have difficulty remembering)

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Algebra & Vector Spaces

A *group* G is a set closed under an operation $\star$ that is associative $(g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3)$, contains an identity e such that $e \star g = g \star e = g \forall g \in G$, and every element has an inverse such that $g \star g^{-1} = g^{-1} \star g = e$. A group is *abelian* if $g_1 \star g_2 = g_2 \star g_1$

A *ring* is a set closed under two operations $+, \times$ that is an abelian group under $+$, and contains an identity 1_R for the operation $\times$. $\times$ is distributive and associative.

A *field* is a ring where every element except maybe the $+$ identity has a multiplicative inverse sending it to the multiplicative identity. This forms a group structure for the elements except for maybe the additive identity. This group, called the multiplicative group, is also abelian.

A *module* M is an abelian group (operation denoted $+$) with a ring R such that, for all $r, s \in R, x, y \in M$, we have

$$r(x+y) = rx + ry \tag{1}$$

$$(r+s)x = rx + sx \tag{2}$$

$$(rs)x = r(sx) \tag{3}$$

$$1_R x = x \tag{4}$$

This defines scalar multiplication.

A *vector space* is a module where R is a field.

An *algebra* A is a vector space with a binary operation $\cdot : A \times A \to A$ such that, for all $x, y, z \in K, r, s \in R$,

$$(x+y) \cdot z = (x \cdot z) + (y \cdot z) \tag{5}$$

$$x \cdot (y+z) = z \cdot y + x \cdot z \tag{6}$$

$$rx \cdot sy = (rs)x \cdot y \tag{7}$$

(These axioms define bilinearity)

Manifolds

A *topological space* is an ordered pair (X, τ) where X is a set an τ is a set of subsets of X such that:

$$\text{The empty set and } X \text{ belong to } \tau,$$

$$\text{An arbitrary, finite or infinite union of elements of } \tau \text{ is in } \tau,$$

$$\text{The intersection of any finite number of elements of } \tau \text{ is in } \tau.$$

τ is a topology on X, and defining a topology allows one to define continuity, connectedness, and convergence.

A *topological base (basis* B of a topological space X is a set of open subsets of X such that every open subset of X can be written as a union of elements in B. We say the base generates the topology, which makes sense, as the elements in τ are each a union of elements of B. For this to be well-used,

$$\text{The base elements must cover } X,$$

$$\text{Let } B_1, B_2 \in B \text{ have } B_1 \cap B_2 := I. \text{ For each } x \in I, \text{ there is a } B' \in B \text{ such that } x \in B' \subseteq I$$

Remark 1. *A second-countable space is a space with a countable base. A compact, metrizable space is necessarily second-countable. (Throwback to proving an uncountable collection of 1-simplices is not metrizable.)*

A homeomorphism is a map between topological spaces that is an injection, is continuous, and has a continuous inverse map.

A *manifold* is a topological space such that every point $p \in M$ has a neighborhood homeomorphic to Euclidean space of the same dimension.

A *tangent space* is a vector space at a point of a manifold that consists of vectors tangent to that point. The tangent space of a sphere is a cylinder with the same radius as the sphere.

A *chart* is such a homeomorphism.

An *atlas* is a collection of charts such that the preimage of every chart in the atlas covers the manifold.

A *transition map* is a map that transitions the image of the intersection of the preimage of multiple charts from the image of the one to the another.

A *Lie Group* is a group that is also a differentiable manifold. It provides a way to classify continuous symmetries (e.g. the rotation matrices in a dimension are a group and a differentiable manifold; one can smoothly rotate a sphere).

Differential Forms from a Field-Theoretic Perspective

Example 1. $\int_{C_1} \omega \stackrel{?}{=} \int_{C_1} df = \int_a^b \frac{df}{dt} dt = f(b) - f(a)$

If our 1-form in this example is in fact a df of some f (?), with f nice and continuous, then df is *exact.* If $dw = 0$ for some $k-$form, then w is *closed.* As you can see, exactness implies closedness due to

$d^2 = 0$, but not necessarily the converse. Thus a exact form is the image of d, and a closed form is the image of d, further hinting at d begin the chain map in De Rham cohomology.

Closedness implies exactness on a contractible domain via the Poincare Lemma.

For those with experience in differential topology, a differential 2-form ω on a manifold M gives, for each $p \in M$, a bilinear form

$$\omega_p : T_pM \times T_pM \to \mathbb{R} \tag{8}$$

A bilinear form on a vector space V is nondegenerate if, for all $v \in V$, $\langle w, v\rangle = 0$ implies $w = 0$. A two form is nondegenerate if w_p is nondegenerate for all $p \in M$.

Wedge Product

Recall the rules for wedge product for one forms:

$$\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \tag{skew-symmetric}$$

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \tag{associativity}$$

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \wedge \mathbf{v} = c_1\mathbf{u}_1 \wedge \mathbf{v} + c_2\mathbf{u}_2 \wedge \mathbf{v} \tag{bilinearity}$$

$$\mathbf{u} \wedge (c_1\mathbf{v}_1 \wedge c_2\mathbf{v}_2) = c_1\mathbf{u} \wedge \mathbf{v}_1 + c_2\mathbf{u} \wedge \mathbf{v}_2 \tag{bilinearity}$$

$$\mathbf{u} \wedge \mathbf{u} = 0 \tag{*only for 1-forms}$$

We can construct k-forms by wedging a $(k-n)$-form with a n-form. The d operation sends k-forms to $(k+1)$-forms.

Example 2. $d(Fdy \wedge dz + Gdz \wedge dx + Hdx \wedge dy) =$
$(F_xdx + F_ydy + F_zdz) \wedge dy \wedge dz + (G_xdx + G_ydy + G_zdz) \wedge dz \wedge dx + (H_xdx + H_ydy + H_zdz) \wedge dx \wedge dy$
$= (F_x + G_y + H_z)dx \wedge dy \wedge dz$

It is easy to show $d^2 = 0$ (Hinting that k-forms may have a cohomological structure, perhaps named after De Rham)

A k-form is meant to be integrated over a k-manifold.

Symplectic Manifold

A *symplectic manifold* is a manifold with a closed, nondegenerate 2-form ω called the symplectic form. These show up in cotangent bundles of manifold. For a system modeled as a manifold, the cotangent bundle describes the phase space (space of all possible configurations of the system, e.g. Hilbert space) of the system.

Any real-valued differentiable function H on a symplectic manifold can be an energy function i.e Hamiltonian. Associated to any Hamiltonian is a Hamiltonian vector field, the integral curves of which (curves sketched along the vector field from the differential equation) is a solution to Hamilton's equations.

A *Hamiltonian flow* or *symplectomorphism* is the flow of this field on the symplectic manifold.

Hodge Star

The *Hodge Star* sends k-forms to $(n-k)$-forms in an n-dimensional manifold. (It maps k-dimensional vectors to $(n-k)$-dimensional vectors in an n-dimensional vector space.)

Example 3. *In a 3-dimensional Euclidean space, we can associate to every vector a plane orthogonal to that vector, and every plane an oriented normal vector.*

$$\mathbf{u} \wedge *\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}, \dim \mathbf{u} = \dim \mathbf{v} = k < n, \dim \mathbf{w} = n \tag{9}$$

where n is the dimension of our vector space.

A Field in terms of Differential Forms

The Electromagnetic Field F is given by the 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dx + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt \tag{10}$$

Computing dF gives us

$$dF = (\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} dx \wedge dy \wedge dz) + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}) dx \wedge dy \wedge dt + ... \tag{11}$$

Setting $dF = 0$, we find the first two Maxwell's Equations $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. For the other two Maxwell's Equations, we use $d * F = 4\pi\rho$:

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt \tag{12}$$

with

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt \tag{13}$$

where the metric used in the hodge star is the Lorentz metric.