# Physics?

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Here is a list of the highlights of physics, in my view, for reference (the physics I'd like to remember). These are chosen by their coolness factor or just their importance to physics.

## Contents

# 1 Geometric Quantization

What's fascinating but somewhat frustrating is the fact that new physics theories arise nowadays from redefining previously understood concepts with increasing levels of abstraction. This note is to explain *intuitively* where the idea that a quantum state is a section of a principal bundle.

## 1.1 The right classical formulation to quantize, or how we get to symplectic manifolds

The point of mechanics is to describe the time development of the state of a physical system. In classical mechanics a state is given by a point $P$ on an $n-$manifold $M$, which is the configuration space.

**Example 1.** *For a rigid body constrained to move in a circle in three dimensions, $M = S^1 \times SO(3)$, which specifies the coordinates of the body's center of mass and its rotation.*

We'll label the position variables as $\{q_i\}$. The time development of the state is given by a curve $\gamma$ on $M$. To make sure all derivatives are well-defined everywhere, let's make things easy on ourselves and consider smooth manifolds.

Now we inject some physics in order to find the correct $\gamma$ curve. We assume the system has a Lagrange function

$$L = L(q, \dot{q}, t) = E_{kin} - E_{pot} \tag{1}$$

Our desired $\gamma$ curve minimizes the integral

$$\int_{t_0}^{t_1} L dt \tag{2}$$

From the Calculus of Variations, this means that $\gamma$ satisfies the Euler-Lagrange Equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, i = 1, ..., n \tag{3}$$

This is a system of ODEs in the $2n$-dimensional $TM$, since $\dot{q}$ is a tangent vector. The desired curve $\gamma$ on $M$ is the projection of the solution curve $\tilde{\gamma}$ of the E-L equations on $TM$.

Quantum mechanics is best expressed in Hamiltonian mechanics, so we'll want to consider Hamiltonian mechanics, defined in terms of $q$ and momentum $p$, which in Lagrangian mechanics, is defined by $p_i = \frac{\partial L}{\partial \dot{q}_i}$:

$$H(p, q, t) := p\dot{q} - L(q, \dot{q}, t), \tag{4}$$

where $\dot{q}$ is determined from $p = \frac{\partial L}{\partial \dot{q}}$. Thus we're less interested in $(q, \dot{q})$ coordinates and more interested in $(q, p)$ coordinates. But we've defined $q$ and know that $\dot{q}$ is a tangent vector in $T_qM$ - what is $p$ in this setup?

Since $L$ is real-valued and a function of a velocity vector, this leads us to $p$ being a covector:

**Example 2.** *If $L = \frac{1}{2}m|\dot{q}|^2 - V(q)$, then $p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}[\frac{1}{2}mg_{ij}q^iq^j] = mg_{ij}q^j$, which is a covector.*

The reasons for $p$ being a covector become more obvious as the framework becomes better-developed.

Now that we've established that $p$s are covectors, we know that $H(p, q, t)$ is a function on $T^*M$, and $H$ is actually the Legendre Transformation of $L$, going from $TM$ to $T^*M$.

Backing up a bit: we have a smooth manifold $M$, and energy function $H$ on its phase space $T^*M$. We want to find the path $\gamma(t)$ on $M$, and we know now that this will be the projection of a path $\tilde{\gamma}(t)$ on $T^*M$.

Fortunately, our function $H$ on $T^*M$ gives us such a curve; using the definition of the differential of $H$ and the definition of $H$ in terms of $L$, we have

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t}dt \tag{5}$$

$$dH = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{6}$$

therefore $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}, \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$.

So if we have a curve $\tilde{\gamma}(t)$ on $T^*M$, this is given by $\tilde{\gamma}(t) = (q(t), p(t))$. From our above-established Hamiltonian mechanics, we know that

$$\dot{\tilde{\gamma}}(t) = (\dot{q}(t)\frac{\partial}{\partial q(t)}, \dot{p}(t)\frac{\partial}{\partial p(t)}) \tag{7}$$

$$= (\frac{\partial H}{\partial p(t)}\frac{\partial}{\partial q(t)}, -\frac{\partial H}{\partial q(t)}\frac{\partial}{\partial p(t)}) \tag{8}$$

Thus the vector field $V := \frac{\partial H}{\partial p(t)}\frac{\partial}{\partial q(t)} - \frac{\partial H}{\partial q(t)}\frac{\partial}{\partial p(t)}$ describes the dynamics of our system.

From the theory of differential forms, there exists a 2-form $\omega = dq \wedge dp$ on $T^*M$ such that, for any vector $X$, the 2-form $\omega$ gives a 1-form $\omega(V, \cdot)$. Hamilton's equations are then equivalent to

$$\omega(V, X) = dH(X) \tag{9}$$

What do we need on $\omega$ ensure no funny business happens? First, we must make sure that we can always solve for $V$, i.e. a $V$ exists for all vector fields $X$. If $\omega$ is nondegenerate, then $V$ always exists, and through linear algebra is always unique.

Also, we want energy to be conserved. This entails that the Hamiltonian is constant along the flow due to the dynamics, i.e.

$$dH(V) = 0 \tag{10}$$

Therefore, $\omega(V, V) = 0$, so $\omega$ must be alternating. Now we have that $\omega$ must be a nondegenerate 2-form on $M$.

Lastly, we want to make sure that $V$ doesn't depend on $t$, i.e. Newton's laws don't change with time. This also gets rid of that pesky $\frac{\partial H}{\partial t}$ term in the $dH$ differential. The way to determine how a tensor field changes along a vector field $V$ is with the *Lie derivative* $\mathcal{L}_V$, so we check the Lie derivative of $\omega$ along $V$:

$$\begin{aligned}
\mathcal{L}_V \omega &= d\omega(V, \cdot, \cdot) + d(\omega(V, \cdot)) &(11)\\
&= d\omega(V, \cdot, \cdot) + ddH &(12)\\
&= d\omega(V, \cdot, \cdot) &(13)
\end{aligned}$$

since we want this change to be 0, we require that $d\omega = 0$. Thus, $\omega$ must be a closed, nondegenerate 2-form on $M$, i.e. $(M, \omega)$ must be a symplectic manifold. This works out, because $\omega$ was ready for the taking in order to define Hamilton's equations as $\omega(V, \cdot) = dH$.

## 1.2 Prequantization

On any pair of differentiable functions $f, g$ on phase space $T^*M$, the **Poisson bracket** is defined by

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{14}$$

Hamilton's equations can then be written as

$$\dot{q} = \{q, H\}, \dot{p} = \{p, H\} \tag{15}$$

Poisson-bracketing any function with $H$ actually gives its time derivative (go figure). Recall that this is the condition of the quantum Hamiltonian.

To quantize the system, we need to promote the state from a point in phase space $T^*M$ to a vector in a Hilbert space $\mathcal{H}$, and promote the Hamiltonian function $H$ and any classical observable function $f$ to operators on $\mathcal{H}$.

This is done (will expand on this later) by looking for a Poisson subalgebra of differentiable functions on $T^*M$ that is isomorphic to the Lie algebra of a Lie group $G$, then taking an irreducible unitary representation of $G$, and its associated infinitesimal representation of $G$'s Lie algebra.

# 2 Classical kinetic energy from relativity

## 2.1 Relativistic mass

We want to find momentum for an object that is reference-frame-invariant. We know from time dilation that $t = \gamma\tau$, where $\tau$ is the proper time in the object's rest frame.

$$\begin{aligned} p &= m_0 \frac{dx}{d\tau} && (16)\\ &= m_0 \frac{dx}{dt}\frac{dt}{d\tau} && (17)\\ &= m_0 \frac{dx}{dt}\frac{d(\gamma\tau)}{d\tau} && (18)\\ p &= m_0 \gamma v && (19) \end{aligned}$$

where $v$ is the velocity of the object that *we* see (since $t$ is our time), not of a reference frame relative to another. Thus we have that relativistic mass $m$ is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{20}$$

## 2.2 Classical kinetic energy is a Taylor expansion from relativity and $E = mc^2$

In relativity, we have to introduce the spacetime, which is, here at least, just the normal 3 dimensions of space, plus time:

$$x = (t, x_1, x_2, x_3) \tag{21}$$

$$:= (x_0, x_1, x_2, x_3) \tag{22}$$

But the first dimension is in terms of time, and the rest in space. It's bad practice to have different units in the same vector. No problem, because then we'll set

$$x = (ct, x_1, x_2, x_3) \tag{23}$$

since $c$ always a constant.

Suppose we're only moving in dimension $x_1$ with velocity $v$. For $\beta = \frac{v}{c}$, we have

$$x_1' = \frac{x_1 - \beta x_0}{\sqrt{1-\beta^2}}, x_0' = \frac{x_0 - \beta x_1}{\sqrt{1-\beta^2}} \tag{24}$$

Let's play around for a bit:

$$\begin{aligned}
(x_1')^2 - (x_0')^2 &= (x_0 - \beta x_1)^2\gamma - (x_1 - \beta x_0)^2\gamma & (25)\\
&= \frac{x_0^2 - \beta^2 x_1^2 - x_1^2 - \beta^2 x_0^2}{1-\beta^2} & (26)\\
&= x_0^2 - x_1^2 & (27)
\end{aligned}$$

Thus $x_0^2 - x_1^2$ is Lorentz invariant. In this spirit, let's define a momentum vector:

$$p = m(\frac{dx_0}{d\tau}, \frac{dx_1}{d\tau}, ...) \tag{28}$$

Since $x_0 = ct$, we have

$$\begin{aligned}
p_0 &= mc\frac{dt}{d\tau} = mc\gamma, & (29)\\
p_1 &= mv\gamma & (30)
\end{aligned}$$

Assuming $v << c$, we have $\frac{v}{c} << 1$. We Taylor expand $\gamma$ using $(1+x)^n = 1 + nx + ...$ to get

$$\gamma \approx 1 + \frac{v^2}{2c^2} + ... \tag{31}$$

Thus

$$\begin{aligned}
p_1 &\approx mv, & (32)\\
p_0 &\approx mc + \frac{1}{2c}mv^2 \Rightarrow & (33)\\
cp_0 &\approx mc^2 + \frac{1}{2}mv^2 & (34)
\end{aligned}$$

This is in terms of energy, since the last term is jsut kinetic energy. Thus, at rest, $E = mc^2$.

# 3 Maxwell's equations in matter

Let $P$ be the electric dipole moment per unit volume. For a single dipole $p$ we have

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot p}{r^2} \tag{35}$$

so, with $p = Pd\tau'$,

$$\begin{align}
V &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{\hat{r}\cdot P(r')}{r^2}d\tau' \tag{36}\\
&= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} P\cdot\nabla(\frac{1}{r})d\tau' \tag{37}\\
&= \frac{1}{4\pi\varepsilon_0}[\int_{\mathcal{V}}\nabla\cdot(\frac{P}{r})d\tau' - \int_{\mathcal{V}}\frac{1}{r}(\nabla\cdot P)d\tau'] \tag{38}\\
&= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{S}}\frac{1}{r}P\cdot da' - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{1}{r}(\nabla\cdot P)d\tau' \tag{39}
\end{align}$$

These terms look like the potentials for a surface charge $\sigma_b = P\cdot n$ and a volume charge $\rho_b = -\nabla\cdot P$, respectively.

For charge $\rho = \rho_b + \rho_f$, Gauss' Law reads

$$\varepsilon_0\nabla\cdot E = \rho = \rho_p + \rho_f = -\nabla\cdot P + \rho_f \tag{40}$$

so

$$\nabla\cdot(\varepsilon_0 E + P) = \rho_f \tag{41}$$

This $D$-field is the electric displacement field, and Gauss' Law in this term reads

$$\nabla\cdot D = \rho_f \tag{42}$$

For many materials, the polarization is proportional to the $E$-field, provided the $E$-field isn't too strong:

$$P = \varepsilon_0\chi_e E \tag{43}$$

where the constant of proportionality $\chi_e$ is the electric susceptibility. These materials are called linear dielectrics. Then we have

$$D = \varepsilon_0 E + P = \varepsilon_0(1+\chi_e)E \Rightarrow \tag{44}$$

$$D = \varepsilon E, \tag{45}$$

$$\varepsilon := \varepsilon_0(1+\chi_e) \tag{46}$$

This factor $\varepsilon_r := (1+\chi_e)$ is called the relative permittivity or dielectric constant of the material.

The current density $J = J_b + J_f + J_p$ has contributions from free charges $J_f$, bound charges $J_b$, and polarization effects $J_p$. Given the magnetization $M$ of the medium, we have

$$J_b = \nabla \times M \tag{47}$$

derived in the same way as we derived $\rho_b = -\nabla \cdot P$. Furthermore,

$$J_p = \frac{\partial P}{\partial t} \tag{48}$$

Let

$$H := \frac{1}{\mu_0}B - M, \nabla \times H = J_f + \frac{\partial D}{\partial t} \tag{49}$$

Thus Maxwell's equations in matter are

$$\nabla \cdot D = \rho_f, \nabla \cdot B = 0 \tag{50}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \nabla \times H = J_f + \frac{\partial D}{\partial t} \tag{51}$$

# 4 Maxwell's equations are relativistic

First of all, *Maxwell's equations are not invariant to Lorentz transformations*! They are *co-variant* under Lorentz transformations - observers in different reference frames will disagree on the values of $E, B$, e.tc. in their rest frames, but they will still obey Maxwell's Equations! Let's reconsider the equations, simplified:

$$\nabla \cdot E = \rho_f, \tag{52}$$

$$\nabla \cdot B = 0, \tag{53}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{54}$$

$$\nabla \times B = J_f + \frac{\partial E}{\partial t} \tag{55}$$

Note that these are in natural units $4\pi = c = 1$. We get the same equations if we consider time as another coordinate (with an eye toward relativity), make our metric pseudo-Riemannian (+,-,-,-), and define the electromagnetic tensor (Faraday tensor) and corresponding current:

$$F^{\mu\nu} := \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, J^\mu = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \tag{56}$$

so Maxwell's Equations are neatly packaged into this single equation:

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{57}$$

The Lorentz group describes operations corresponding to boosting an inertial reference frame by some velocity $v$. For simplicity, let's assume that we boost in the $x$-direction, and have the usual denotations $\beta := \frac{v}{c}, \gamma = \frac{1}{\sqrt{1-\beta^2}}$. A Lorentz transformation then gives

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \gamma(ct - \beta x) \\ \gamma(x - \beta ct) \\ y \\ z \end{pmatrix} \tag{58}$$

A Lorentz transformation in this coordinate system (Minkowski space) is then given by the matrix

$$\Lambda^\mu_\nu \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{59}$$

So say we have a Faraday tensor in a $1^{st}$ coordinate system. If we want to transform our Faraday tensor according to a Lorentz transformation, we untransform a vector in these new coordinates to the old coordinates, feed it through the old Faraday tensor, then re-transform the resulting vector:

$$F^{'\mu\nu} = \Lambda F^{\mu\nu}\Lambda^T = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} \tag{60}$$

Next if we introduce an alternating 4-dimensional tensor $\epsilon^{\mu\nu\rho\sigma}$, it is not hard to check that

$$|\epsilon^{\mu\nu\rho\sigma} F^{'}_{\rho\sigma} F_{\mu\nu}| = E \cdot B \tag{61}$$

so $E \cdot B$ is Lorentz invariant. (Note: I've been pretty cavalier about coefficients, so there might be a factor of 2 or 4 or something in there)

# 5 Kinetic-theoretic derivation of temperature

We define temperature by saying that two systems in thermal equilibrium have the same temperature. The multiplicity of system 1 is $g_1$ and the multiplicity of system 2 is $g_2$. The total multiplicity of the system is $g = g_1 g_2$. Since the systems are in thermal equilibrium, we have $dU_1 = -dU_2$. To put the multiplicity in these terms, we take the total differential of the total multiplicity:

$$dg = dg_1 g_2 + g_1 dg_2 \tag{62}$$

$$= \frac{\partial g_1}{\partial U_1} g_2 dU_1 + g_1 \frac{\partial g_2}{\partial U_2} dU_2 \tag{63}$$

Since the multiplicity doesn't change in total, we have $dg = 0$. Thus we have

$$\frac{1}{g_1}\frac{\partial g_1}{\partial U_1} = \frac{1}{g_2}\frac{\partial g_2}{\partial U_2} \tag{64}$$

$$\frac{\partial \log g_1}{\partial U_1} = \frac{\partial \log g_2}{\partial U_2} \tag{65}$$

$$\frac{\partial \sigma_1}{\partial U_1} = \frac{\partial \sigma_2}{\partial U_2} \tag{66}$$

Thus we define $\frac{\partial \sigma_1}{\partial U_1} = \beta = \frac{1}{\tau}$. This is because, intuitively, at low temperature, the entropy depends greatly on the energy put into the system.

# 6 The partition function and why it's important

First we derive the Boltzmann factor. Suppose we have a generic system $\mathcal{A}$ in thermal contact with a reservoir $\mathcal{R}$, and we have two microstates of $\mathcal{A}$ with energy $E_1, E_2$. The ratio of the probabilites of these microstates is

$$\frac{P_1}{P_2} = \frac{g(E-E_1)}{g(E-E_2)} \tag{67}$$

Taking the log of both sides, we have

$$\begin{aligned}
\log(P_1/P_2) &= \sigma_R(E-E_1) - \sigma_R(E-E_2) && (68)\\
&\approx \sigma_R(E) - \frac{d\sigma_R}{dE_R}E_1 - \sigma_R(E) + \frac{d\sigma_R}{dE_r}E_2 && (69)\\
&= -\frac{E_1}{\tau} + \frac{E_2}{\tau} \Rightarrow && (70)\\
\frac{P_1}{P_2} &= e^{-\beta(E_1-E_2)} && (71)
\end{aligned}$$

Thus $P_i \propto e^{-\beta E_i}$. To get the actual probability, we divide by the sum over all microstates $i$. This is the partition function, $Z$. From this we can derive many properties of the system:

$$\langle E\rangle = -\frac{\partial Z}{\partial \beta}, \langle(\Delta E)^2\rangle = \frac{\partial^2 Z}{\partial \beta^2} \tag{72}$$

Furthermore, Helmholtz free energy $F = E - \tau\sigma$ is given by

$$F = -k_B T \log Z = -\tau \log Z \tag{73}$$

and from there we get

$$\sigma = -(\frac{\partial F}{\partial \tau})_{V,N}, P = -(\frac{\partial F}{\partial V})_{T,N}, \mu = (\frac{\partial F}{\partial N})_{\tau,V} \tag{74}$$

# 7 Ideal gas law from quantum mechanics

How does one model a gas? Assume it's ideal, to start, so we don't have to deal with interactions and weird edge cases of particle size and such, and classical enough velocity. So picture an ideal gas as just a bunch of particles in a container. The particles are described by the wavefunction that obeys the Schrödinger Equation:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)]\Psi(\mathbf{r},t), \tag{75}$$

$$[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)]\Psi(\mathbf{r},t) = E\Psi(\mathbf{r},t) \text{ (in the energy eigenbasis)} \tag{76}$$

Suppose we consider $N$ such particles confined to a box of side lengths $L, W, H$. Then we impose an infinite potential outside of this box to ensure there are indeed $N$ particles inside, with no potential inside the box because in general there wouldn't be. This is a 3D particle-in-a-box scenario. Solving the Schrödinger Equation yields

$$\Psi = \sqrt{\frac{2}{L}}\sin(\frac{n_1\pi}{L})\sqrt{\frac{2}{W}})\sin(\frac{n_2\pi}{W})\sqrt{\frac{2}{H}})\sin(\frac{n_3\pi}{H}) \tag{77}$$

with energy

$$E = \frac{\hbar^2\pi^2}{2m}[(\frac{n_1^2}{L^2}) + (\frac{n_2^2}{W^2}) + (\frac{n_3^2}{H^2})] \tag{78}$$

From statistical mechanics, we know that the partition function is given by

$$Z = \sum_i e^{-E_i/\tau} \tag{79}$$

so in our case the partition function for one particle is

$$Z_1 = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2m}[(\frac{n_1^2}{L^2})+(\frac{n_2^2}{W^2})+(\frac{n_3^2}{H^2})]} \tag{80}$$

$$= (\sum_{n_1=1}^{\infty} e^{-\hbar^2\pi^2 n_1^2/2m\tau L^2})^3 \text{ without loss of generality} \tag{81}$$

$$\approx (\int_0^{\infty} e^{n_1^2[-\hbar^2\pi^2/2m\tau L^2]})^3 \tag{82}$$

$$:= (\int_0^{\infty} e^{-\gamma n_1^2})^3 \tag{83}$$

$$:= (\gamma^{-\frac{1}{2}}\int_0^{\infty} e^{-x^2}dx)^3 \tag{84}$$

$$= (\frac{1}{2}(\frac{\pi}{\gamma})^{\frac{1}{2}})^3 \tag{85}$$

$$= LWH(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}} = V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}} \tag{86}$$

Therefore, since the particles are not interacting, the partition function for $N$ particles is

$$Z = [V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}]^N \tag{87}$$

Since we now have the partition function for our system, we can calculate $F = -\tau \log Z$ and $P = -(\frac{\partial F}{\partial V})_\tau$. Let's do this:

$$F = -\tau \log([V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}]^N) \tag{88}$$

$$= -N\tau \log(V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}) \tag{89}$$

$$P = -\frac{\partial}{\partial V} - N\tau \log(V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}) \tag{90}$$

$$= N\tau \frac{1}{V} \Rightarrow \tag{91}$$

$$PV = N\tau \tag{92}$$

# 8 Solving the quantum harmonic oscillator using only operators and commutation relations

Our operators are $H, a$, and $a^\dagger$, where these are the Hamiltonian, annihilation, and creation operators, respectively. Their commutation relations are:

$$[H, a] = -\hbar\omega a \tag{93}$$

$$[H, a^\dagger] = \hbar\omega a^\dagger \tag{94}$$

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{95}$$

Let $u_n$ be an eigenfunction.

$$[H, a]u_n = -\hbar\omega a u_n \tag{96}$$

$$Hau_n - aHu_n = -\hbar\omega a u_n \tag{97}$$

$$H(au_n) - E_n(au_n) = -\hbar\omega a u_n \tag{98}$$

$$H(au_n) = (E_n - \hbar\omega)(au_n) \tag{99}$$

Thus $a$ lowers the energy by $\hbar\omega$. In a similar calculation, we find that $a^\dagger$ raises the energy by $\hbar\omega$.

Since the harmonic oscillator energy cannot be negative, we have to have a ground state $u_0$

such that $au_0 = 0$. Thus

$$Hu_0 = \hbar\omega(a^\dagger a + \frac{1}{2})u_0 = \frac{1}{2}\hbar\omega u_0 \tag{100}$$

Thus the ground state is $\frac{1}{2}\hbar\omega$, and in general

$$E_n = (n + \frac{1}{2})\hbar\omega \tag{101}$$

# 9 Aharonov-Bohm effect

Maxwell's Equations are:

$$\begin{gathered}\nabla \cdot E = \frac{\rho}{\epsilon_0}, \nabla \cdot B = 0, \\ \nabla \times E = -\frac{\partial B}{\partial t}, \nabla \times B = \mu_0[j + \epsilon_0 \frac{\partial E}{\partial t}]\end{gathered}$$

Since $B$ is divergenceless, we can write $B = \nabla \times A$, the curl of the vector potential $A$.

Consider a long solenoid to model a constant field $B$ inside the solenoid and $B = 0$ outside the solenoid. Thus outside the solenoid, the vector potential $A$ must satisfy

$$\begin{aligned} B &= \nabla \times A, \\ \oint_C A \cdot dr = \int_S (\nabla \times A) \cdot dS &= \int_S B \cdot dS = \Phi \end{aligned}$$

for $C$ a path around the solenoid, with $\Phi$ the magnetic flux through the solenoid. Thus we choose

$$A = \frac{\Phi}{2\pi r}\hat{\phi}$$

Thus although the $B$-field outside the solenoid is 0, the $A-$field is nonzero.

Now we investigate what happens to a particle with charge $q$ in this field. The lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + q\dot{x}A - q\phi$$

Solving the Euler-Lagrange equations gives us the Lorentz force law:

$$m\ddot{x} = qE + q\dot{x} \times B$$

Take the Legendre transformation of this to get the Hamiltonian, which is

$$H = \dot{x}p - \mathcal{L} = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{(p - qA)^2}{2m} + q\phi$$

where we not that canonical momentum, $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$, is not gauge-invariant, but kinetic momentum, $p - qA$, is gauge-invariant and physically measurable. We how quantize this Hamiltonian:

$$H = \frac{1}{2m}(p - qA(r))^2 + q\phi + V(r),$$

where $V(r)$ is some other potential. Writing $p = -i\hbar\nabla$, we get the Schrödinger equation as

$$[\frac{1}{2m}(-i\hbar\nabla - qA(r))^2 + q\phi + V(r)]\psi = i\hbar\frac{\partial \psi}{\partial t}$$

The solution to this equation is

$$\psi(r,t) = e^{\frac{iq}{\hbar}\int_0^r A(r)dr}\psi'(r,t) \tag{102}$$

where the initial integration point is 0, chosen arbitrarily due to gauge invariance. Taking the gradient of this function, we get

$$-\frac{\hbar^2}{2m}\nabla^2\psi' - V\psi' = i\hbar\frac{\partial \psi'}{\partial t} \tag{103}$$

Thus if we turn on a magnetic field, the wavefunction gains a phase, called the **Berry phase**. Thus if we wind the particle around the solenoid, we get

$$\psi(r,t) = e^{\frac{iq}{\hbar}\oint_C A(r)dr}\psi'(r,t) \tag{104}$$

$$= e^{\frac{iq}{\hbar}\Phi}\psi'(r,t) \tag{105}$$

so we gain a phase.

In other words, if we split a beam around a solenoid, the two paths, once recombined, yield a loop around the solenoid, so the interference pattern of the recombined beam will shift by a phase $\frac{e\Phi}{\hbar}$

## 9.1 Path integral formulation of Aharonov-Bohm effect

We have that

$$\langle x_1, t_1 | x_0, t_0 \rangle = \int_{x_0}^{x_1} \mathcal{D}[x(t)] \exp[i\frac{S(t_1,t_0)}{\hbar}] = \int_{x_0}^{x_1} \mathcal{D}[x(t)] \exp[\frac{i}{\hbar}\int_{t_0}^{t_1} \mathcal{L}(\dot{x}(t), x(t)) dt]$$

where the Lagrangian $\mathcal{L}(\dot{x}(t), x(t))$ is the same as above. If we turn on the magnetic field, the action becomes $S_0 + e\int A \cdot dl$.

If we have a double-slit experiment, this time with a solenoid between the slits, the probability amplitude that we detect the particle at some point is proportional to

$$\int_{\text{all paths through slit 1}} \exp(\frac{i}{\hbar}S_0 + \frac{ie}{\hbar}\int A \cdot dl) + \int_{\text{all paths through slit 2}} \exp(\frac{i}{\hbar}S_0 + \frac{ie}{\hbar}\int A \cdot dl) \Rightarrow$$
$$\exp(\frac{ie}{\hbar}\int_{slit1} A \cdot dl - \frac{ie}{\hbar}\int_{slit2} A \cdot dl) = \exp(\frac{ie}{\hbar}\oint_C A \cdot dl)$$

This time, we can use Stokes' theorem:

$$\frac{ie}{\hbar}\oint_C A \cdot dl = \frac{ie}{\hbar}\oint_{enclosed} B \cdot dS = \frac{ie}{\hbar}\Phi$$

# 10 The Google paper

When Google claimed a demonstration of quantum supremacy in 2019, here is what they did in a nutshell: Send a bunch of qubits through a quantum circuit and measure the output as a bitstring $x_i$ (e.g. $x_i = \{1001011010...\}$). Sampling many of these outputs gives us a probability distribution of the bitstrings. If quantum interference is at play, certain bitstrings are more likely than others, whereas in a classical random circuit, we should obviously get a uniform distribution of bitstrings. As the number of qubits grows, the ability to classically compute this distribution grows exponentially with qubit number and circuit depth. To measure how "quantum" a random circuit on $n$ qubits is, we use the **linear cross-entropy benchmarking fidelity** value:

$$\mathcal{F}_{XEB} = 2^n \langle P(x_i) \rangle_i - 1 \tag{106}$$

when the qubits do not interact, the probability of any fixed $x_i$ should be $\frac{1}{2^n}$, and so $\mathcal{F}_{XEB}$ should tend to 0. On the other hand, if interactions are occurring, $\mathcal{F}_{XEB}$ should tend to 1.

The Google experiment was performed using the Google Sycamore processor. Th Sycamore processor contains 142 transmon qubits. A **transmon qubit** is a superconducting island connected via a Josephson junction (in Google's case a DC SQUID) to a larger superconductor that acts as a Cooper pair reservoir in series with a capacitor (which we'll call a gate), with both island and reservoir connected to a shunt with an additional capacitor.

The state of the qubit is the number of Cooper pairs that have tunneled across the junction. The Hamiltonian of this system is

$$H = \sum_n [E_C(n - \frac{C_g V_g}{2e}) |n\rangle \langle n| - \frac{1}{2} E_J(|n\rangle \langle n+1| + |n+1\rangle \langle n|)], \tag{107}$$

$$E_C = \frac{(2e)^2}{2(C_g + C_J)} \tag{108}$$

where $C_g$ is the shunt capacitance, $C_J$ is the capacitance across the Josephson junction, $V_g$ is the voltage across the shunt capacitor, and $E_J$ is the energy across the Josephson junction. The presence of the shunted capacitor increases the ratio of $E_J/E_C$, making the Josephson junction less susceptible to charge noise. The flux through the DC SQUID is controlled by an on-chip bias line.

## 11 How quantum field theory respects causality

The propagator $D(x-y) := \langle 0| \phi(x)\phi(y) |0\rangle$ need not be zero for $x, y$ spacelike - what matters is whether a measurement performed at one point can affect the measurement at another point separated spacelike from the first. We will use the Klein-Gordon field as an illustrative example.

In regular quantum mechanics, the probability for a free particle to go from $x$ to $y$ in time $t$ is

$$U(t) = \langle y| e^{-iHt} |x\rangle \tag{109}$$

If we use energy $E = \frac{p^2}{2m}$, or even relativistic energy $E = \sqrt{p^2 + m^2}$, we get a nonzero probability for all $t$:

$$U(t) = \langle y| e^{-it\sqrt{(p^2+m^2}} |x\rangle \tag{110}$$

$$= \langle y| e^{-it\sqrt{p^2+m^2}} |p\rangle \langle p| |x\rangle \tag{111}$$

$$= \frac{1}{(2\pi)^3} \int e^{-it\sqrt{(pc)^2+(cm^2)^2}} e^{ip\cdot(y-x)} d^3p \tag{112}$$

$$\text{spherical coordinates} = \frac{1}{2\pi^2|y-x|} \int_0^\infty p \sin(p|y-x|) e^{-it\sqrt{p^2+m^2}} dp \tag{113}$$

Assume $y >> t$. Using the method of stationary phase, we have a stationary point for the phase function $px - t\sqrt{p^2+m^2}$ at the point $p = \frac{imy}{\sqrt{y^2-t^2}}$. When considering this $p$ value, we have

$$U(t) \sim e^{-m\sqrt{y^2-t^2}} \tag{114}$$

which still gives a nonzero probability far outside the light-cone.

The fields $\phi(x)$ and $\phi(y)$ act on vacuum to create excitations at $x$ and $y$ (in spacetime), respectively. We will see how creating one with the other already there compares for the two points via $[\phi(x), \phi(y)]$. In the Klein-Gordon field, we have

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} [(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_q e^{-iq\cdot y} + a_q^\dagger e^{iq\cdot y})] \tag{115}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) \tag{116}$$

If $(x-y)^2 < 0$, we can boost by $\beta = \frac{t}{|x|^2} < 1$ to make the coordinates purely spacelike:

$$(t, \cdot, \cdot, \cdot) \mapsto (\gamma(t - \beta x), \cdot, \cdot, \cdot) \tag{117}$$

$$= (\gamma(t - \frac{t}{|x|^2} x), \cdot, \cdot, \cdot) \tag{118}$$

(there's always a Lorentz boost where $t = 0$.) Using this, we can have

$$(t, x, y, z) \xrightarrow{R_1} (t, \sqrt{x^2+y^2}, 0, z) \tag{119}$$

$$\xrightarrow{R_2} (t, \sqrt{x^2+y^2+z^2}, 0, 0) \tag{120}$$

$$\xrightarrow{\beta = \frac{t}{|x|^2}} (0, \sqrt{x^2+y^2+z^2-t^2}, 0, 0) \tag{121}$$

$$\xrightarrow{R_\pi} -(0, \sqrt{x^2+y^2+z^2-t^2}, 0, 0) \tag{122}$$

$$\xrightarrow{(\beta R_2 R_1)^{-1}} -(t, x, y, z) \tag{123}$$

$$\tag{124}$$

This is a continuous transformation that can only be achieved if $(x-y)^2 < 0$. Time-like four-vectors are essentially purely time-like four-vectors (i.e. (t,0,0,0)), and there is only one dimension to rotate in, so we can't rotate like we can in 3-dimensions. Thus, if $x$ and $y$ are separated space-like, we can send $(x-y) \mapsto -(x-y)$:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) \mapsto \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{-ip\cdot(x-y)}) = 0 \tag{125}$$

and causality is preserved.

## 12 Dirac Theory

Dirac wanted to make the Schrödinger equation relativistic. Do to this, we try to make the Hamiltonian in the Schrödinger equation

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{126}$$

$$\hat{H}\Psi = E\Psi \tag{127}$$

a relativistic description of energy. From relativity, we have the relativistic energy relationship:

$$E^2 = (pc)^2 + (m_0c^2)^2 \tag{128}$$

We want to find a self-adjoint operator that, when squared, gives $\hat{E}^2 - (\hat{p}c)^2$. The Schrödinger Equation gives us the $\hat{E}$ operator as $i\hbar\frac{\partial}{\partial t}$ and the $\hat{p}$ operator as $-i\hbar\nabla$, so we want a self-adjoint operator that, when squared, gives $\hbar^2(-\frac{\partial^2}{\partial t^2} + c^2\nabla^2)$. It turns out that the **Clifford Algebra** allows us to do this:

$$\hbar(i\gamma_0\frac{\partial}{\partial t} + c\sum_{n=1}^{3}\gamma_n\frac{\partial}{\partial x_n})\Psi = m_0c^2\Psi \tag{129}$$

where the $\gamma_i$ satisfy the algebra

$$\gamma_i^2 = Id, \tag{130}$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0, i \neq j \tag{131}$$

We can write this much more succinctly. Recall that four-gradient $\partial^\mu$ is given by $(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$. We can write the above in terms of the four-gradient by pulling out an $i$:

$$(i\hbar\gamma_\mu\partial^\mu - m_0c^2)\Psi = 0 \tag{132}$$

$$:= (i\partial\!\!\!/ - m)\Psi = 0 \tag{133}$$

in natural units and with Feynman's notation. This is the **Dirac Equation**. To make this concrete, and to make this operator self-adjoint, we consider skew-Hermitian representations of this Clifford Algebra. That is, we demand representations such that $rep(\gamma_i)^* = -rep(\gamma_i), \forall i$. An easy such representation of this is the **Weyl** or **chiral** representation (sign conventions and $\gamma_0$ representation may vary):

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{134}$$

where the $\sigma_i$ are the usual Pauli matrices. In this represenation, the $\Psi$ "wavefunction" must be a 4-entry vector, i.e. a four component field called a **spinor**.

# 13 Using $\sum_{\mathbb{Z}} = -\frac{1}{12}$ in string theory

String theory is a candidate theory for quantum gravity. In addition to having point-like (0-dimensional) excitations, it features 1-dimensional strings and higher-dimensional objects called branes on which the end points of strings can be anchored. The energy spectrum of a quantum string stretched between two D-branes is related to a famous function from number theory studied by Ramanujan.

We start by classical string theory. Consider a string stretched between two ends. It has length $L$, mass $M$, and tension $T$ and we will denote its displacement at position $x$ and time $t$ as $y(x,t)$. Such a string satisfies a wave equation whose general solution (once one imposes the boundary conditions $y(0,t) = y(L,t) = 0$ is of the form

$$y(x,t) = \sqrt{2} \sum_{n=1}^{\infty} \xi_n(t) \sin \frac{n\pi x}{L} \tag{135}$$

We compute the kinetic and potential energies of this solution

$$K = \frac{1}{2} \frac{M}{L} \int_0^L dx \dot{y}^2, U = \frac{1}{2} T \int_0^L dx (\frac{\partial y}{\partial x})^2 \tag{136}$$

in terms of $\xi_n$ and $\dot{\xi}_n$: The modes of y(x, t) are identical (up to normalization) to the eigenfunctions of the infinite square well. The key property well exploit is that these functions are orthonormal.

Then

$$K = \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)\int_0^L dx 2\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L}) \tag{137}$$

$$= \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)2L\delta(nm) \tag{138}$$

$$= \frac{1}{2}M\sum_{n=1}^{\infty}\dot{\xi}_n(t)^2 \tag{139}$$

Using a very similar calculation, we have

$$U = \frac{1}{2}T\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\xi_n(t)\xi_m(t)\frac{n\pi}{L}\frac{m\pi}{L}\int dx 2\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} \tag{140}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{Tn^2\pi^2}{L}\xi_n(t)^2 \tag{141}$$

Thus we see that the total energy looks like that of infinitely many harmonic oscillators, each with mass $M$ and with the $n^{th}$ oscillator having frequency $\omega_n = \sqrt{\frac{T\pi^2}{M:}}n = \omega_1 n$. The total energy $E = K + U$ is that of infinitely many harmonic oscillators, one for each mode $\xi_n$.

The simplest quantum string theories arise essentially by promoting classical systems (like the piano string considered in the previous part) to quantum ones. The classical string is more or less equivalent to infinitely many decoupled harmonic oscillators (a different oscillator governing each harmonic), so we work towards developing a quantum version of this. First consider the quantum Hamiltonian which describes just two oscillators with arbitrary frequencies $\omega_1, \omega_2$:

$$\hat{H} = \frac{\hat{p}_1^2}{2M} + \frac{1}{2}M\omega_1^2\hat{\xi}_1^2 + \frac{\hat{p}_2^2}{2M} + \frac{1}{2}M\omega_2^2\hat{\xi}_2^2 \tag{142}$$

where $\hat{p}_n = -i\hbar\frac{\partial}{\partial_n}$ for $n = 1, 2$. Using a separation of variables argument, we assume for now that the solution to the time-independent solution $\Psi$ has the form $\Psi(\xi_1, \xi_2) = X(\xi_1)Y(\xi_2)$. Plugging this into the TISE and dividing by $\Psi$, we get

$$-\frac{1}{X}\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial\xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 = E - (-\frac{1}{Y}\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial\xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2) \tag{143}$$

Following the standard separation of arguments process, we conclude that this is equal to a constant $E$, and calling $E = E_1 + E_2$, with

$$-\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial \xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 X = E_1 X \tag{144}$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial \xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2 Y = E_2 Y \tag{145}$$

Thus $X(\xi_1$ and $Y(\xi_2)$ are harmonic oscillator eigenstates with energies that add up to $E$. Thus we can have the eigenstates be

$$\Psi_{n_1,n_2}(\xi_1,\xi_2) \propto H_{n_1}(\sqrt{\frac{M\omega_1}{\hbar}}\xi_1)H_{n_2}(\sqrt{\frac{M\omega_2}{\hbar}}\xi_2)\exp(-\frac{M}{2\hbar}(\xi_1^2\omega_1 + \xi_2^2\omega_2)) \tag{146}$$

with energy

$$E_{n_1,n_2} = \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2}) \tag{147}$$

For $N$ decoupled harmonic oscillators, this result generalizes:

$$E_{n_1,\ldots,n_N} = \sum_{i=1}^{N} \hbar\omega_i(n_i + \frac{1}{2}) \text{ for } n_i \in \mathbb{Z}\forall i \tag{148}$$

Now, we consider an infinite number of decoupled quantum harmonic oscillators ($N \to \infty$). This time we specialize our frequencies so that the frequency of the $n^{th}$ oscillator is the frequency $\omega_n$ found before:

Taking the limit as $N \to \infty$, from the previous part we have the energy spectrum as

$$E_{n_1,n_2,\ldots} = \sum_{k=1}^{\infty} \hbar\omega_k(n_k + \frac{1}{2}) \tag{149}$$

$$= \sum_{k=1}^{\infty} \hbar k\omega_1(n_k + \frac{1}{2}) \tag{150}$$

$$= \hbar\omega_1 \sum_{k=1}^{\infty} kn_k + \frac{\hbar\omega_1}{2}\sum_{k=1}^{\infty} k \tag{151}$$

$$= \hbar\omega_1(-\frac{1}{24} + \sum_{k=1}^{\infty} kn_k) \tag{152}$$

This is the energy spectrum of a quantum string stretched between two D-branes.

A partition of $n$ is an unordered set of positive integers whose sum is $n$. We will define $p(n)$ to be the number of partitions of a number $n$. Consider the function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{153}$$

where $q \in \mathbb{C}$ with modulus $\leq 1$. The series expansion of its inverse

$$\frac{1}{\eta(q)} = \sum_{n=0}^{\infty} p(n) q^{n-1/24} \tag{154}$$

famously serves as the generating function for $p(n)$. For an arbitrary positive integer $m$, we can get degeneracy depending on our choices for $n_k$. Thus $\sum_{k=1}^{\infty} k n_k = p(m)$. Thus

$$\frac{1}{\eta(q)} \sim \sum_{\text{energies } E} (\text{number of states of energy } E) q e^{\frac{E}{\hbar\omega_1}} \tag{155}$$