Beautiful Problems

Alec Lau

Contents

1 Bounding Steps of the Euclidean Algorithm

Written by Prof. Cheng-Chiang Tsai, bless his soul.

Question 1. *Prove that if in Euclid's algorithm we begin with two integers $0 < x, y < 2^{32}$, then we need no more than 45 divisions to find out $gcd(x, y)$.*

2 Ramanujan and String Theory

Written by Brandon Rayhaun, heartthrob of the entire undergraduate physics student body at Stanford University, including me. Paraphrased here.

Question 2. *String theory is a candidate theory for quantum gravity. In addition to having point-like (0-dimensional) excitations, it features 1-dimensional strings and higher-dimensional objects called "branes" on which the end points of strings can be anchored. In this problem, you will observe a connection between 1) the energy spectrum of a quantum string stretched between two D-branes and 2) a famous function from number theory studied by Ramanujan.*

a) Let's start by doing a bit of classical string theory. To make it sound less scary, let's agree that we are studying a piano string stretched between the ends of a Steinway. It has length L, mass M, and tension T and we will denote its displacement at position x and time t as $y(x, t)$. Such a piano string satisfies a wave equation whose general solution (once one *imposes the boundary conditions $y(0, t) = y(L, t) = 0$ is of the form*

$$y(x,t) = \sqrt{2}\sum_{n=1}^{\infty} \xi_n(t) \sin \frac{n\pi x}{L} \tag{1}$$

Compute the kinetic and potential energies of this solution

$$K = \frac{1}{2}\frac{M}{L}\int_0^L dx\dot{y}^2, U = \frac{1}{2}T\int_0^L dx(\frac{\partial y}{\partial x})^2 \tag{2}$$

in terms of ξ_n and $\dot{\xi}_n$. You should find that the total energy $E = K + U$ is that of infinitely many harmonic oscillators, one for each mode ξ_n. What is the "mass" of each oscillator? What is the frequency ω_n of each oscillator?

b) *The simplest quantum string theories arise essentially by promoting classical systems (like the piano string considered in the previous part) to quantum ones. As you observed, the classical piano string is more or less equivalent to infinitely many decoupled harmonic oscillators (a different oscillator governing each harmonic), so let's work towards developing a quantum version of this. First consider the quantum Hamiltonian which describes just two oscillators with arbitrary frequencies ω_1, ω_2:*

$$\hat{H} = \frac{\hat{p}_1^2}{2M} + \frac{1}{2}M\omega_1^2\hat{\xi}_1^2 + \frac{\hat{p}_2^2}{2M} + \frac{1}{2}M\omega_2^2\hat{\xi}_2^2 \tag{3}$$

where $\hat{p}_n = -i\hbar\frac{\partial}{\partial_n}$ for $n = 1, 2$. Using a separation of variables argument, derive the energy eigenfunctions and energy eigenvalues for this Hamiltonian.

c) *Write down the energy spectrum (i.e. just the energy eigenvalues, don't worry about the eigenfunctions) of a Hamiltonian describing N decoupled harmonic oscillators each with arbitrary frequency,*

$$\hat{H} = \sum_{n=1}^{N}(\frac{\hat{p}_n^2}{2m} + \frac{1}{2}m\omega_n^2\hat{\xi}_n^2) \tag{4}$$

by generalizing the expression found in the previous part.

d) *Now, consider an infinite number of decoupled quantum harmonic oscillators ($N \to \infty$), and this time specialize your frequencies so that the frequency of the nth oscillator is the frequency ω_n you found in part a). Write down the energy spectrum for this particular configuration of oscillators by using your general expression from the previous part and taking $N \to \infty$. Simplify your answer using the fact that $\sum_{i=1}^{\infty} i = -\frac{1}{12}$. This is the energy spectrum of a quantum string stretched between two D-branes.*

e) *A partition of n is an unordered set of positive integers whose sum is n. We will define $p(n)$ to be the number of partitions of a number n. Consider the function*

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{5}$$

where $q \in \mathbb{C}$ with modulus ≤ 1. The series expansion of its inverse

$$\frac{1}{\eta(q)} = \sum_{n=0}^{\infty} p(n) q^{n-1/24} \tag{6}$$

famously serves as the generating function for $p(n)$. What is the relationship between $\frac{1}{\eta(q)}$ and the energy spectrum of the quantum string you derived in the previous part?

3 The Wave Equation on a Riemannian Manifold

Written by Jonathan Luk.

Question 3. *Let M be an n-dimensional manifold and g be a metric. Given a $\binom{k}{l}$-tensor field F, define a $\binom{k}{l+1}$-tensor field ∇F by*

$$(\nabla F)(Z, X_1, ..., X_k, \omega_1, ..., \omega_l) := (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) \tag{7}$$

where we recall

$$\nabla_Z[F(X_1, ..., X_k, \omega_1, ..., \omega_l)] = (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) + \sum_{i=1}^{k} F(X_1, ..., \nabla_Z X_i, ..., X_k, \omega_1, ..., \omega_l) \tag{8}$$

$$+ \sum_{j=1}^{l} F(X_1, ..., X_k, \omega_1, ..., \nabla_Z \omega_i, ..., \omega_l) \tag{9}$$

where for a function f, $\nabla_Z f = Zf$.

a) *Let $f : M \to \mathbb{R}$ be a smooth function. Prove that*

$$(\nabla^2 f)(X, Y) = X(Yf) - (\nabla_X Y) f \tag{10}$$

b) *Prove that, in local coordinates,*

$$\Delta f := \sum_{i,j=1}^{n} (g^{-1})^{ij} (\nabla^2 f)(\partial_i, \partial_j) = \frac{1}{\sqrt{\det g}} \partial_i ((g^{-1})^{ij} \sqrt{\det g} \partial_j f) \tag{11}$$

where one may use the following facts from linear algebra without proof:

$$\partial_i (g^{-1})^{jk} = -(g^{-1})^{jl}(g^{-1})^{mk}\partial_i g_{lm}, \partial_i \log(\det g) = (g^{-1})^{jk}\partial_i g_{jk} \tag{12}$$

4 Algebraic Topology with Statistics of Particles

Written by me. To be fair, I really like algebraic topology and this argument, formalized(?) by me.

Question 4. *Use the fundamental group of the configuration space of two identical particles to prove that, in $\mathbb{R}^3$, there can exist only bosons and fermions, but in $\mathbb{R}^2$ there can exist any particle statistics.*

5 A Surjection between Groups

Written by the a-MAZING Prof. Tom Church.

Question 5. *Let $G := APB(\mathbb{Z}^2)$ be the group of adjacency-preserving bijections $f : \mathbb{Z}^2 \to \mathbb{Z}^2$. You can use without proof that this is a group. Suppose that $G \twoheadrightarrow H$ is a surjective homomorphism, and H is an abelian group. Prove that $|H|$ is finite. (The original version of this problem states also to prove that $|H| = 8$, but I lost points on that part)*

6 Infinite Integers

Written again by Prof. Tom Church.

Question 6. *Let R denote the set of infinite integers. Two examples are*

$$a = ...000000001 \tag{13}$$

$$b = ...562951413 \tag{14}$$

(b is π backwards)

Prove that there exists at least one solution $z \in R$ to the equation $z^3 = 7$.