Beautiful Problems

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Contents

1 Bounding Steps of the Euclidean Algorithm

Written by Prof. Cheng-Chiang Tsai.

Question 1. *Prove that if in Euclid's algorithm we begin with two integers $0 < x, y < 2^{32}$, then we need no more than 45 divisions to find out $gcd(x, y)$.*

2 2023 Numbers to a Perfect Square

IMO 2023 Problem 4.

Question 2. *Let $x_1, x_2, ..., x_{2023}$ be pairwise different positive real numbers such that*

$$a_n = \sqrt{(x_1 + x_2 + ... + x_n)(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n})} \tag{1}$$

is an integer for every $n = 1, 2, ..., 2023$. Prove $a_n \geq 3034$.

3 Rational degree 2 curves are plane curves of degree 2

Written by Sheldon Katz.

Question 3. *Show that any rational curve of degree 2 in $\mathbb{P}^2$ is a plane curve of degree 2.*

4 The Wave Equation on a Riemannian Manifold

Written by Jonathan Luk.

Question 4. *Let M be an n-dimensional manifold and g be a metric. Given a $\binom{k}{l}$-tensor field F, define a $\binom{k}{l+1}$-tensor field ∇F by*

$$(\nabla F)(Z, X_1, ..., X_k, \omega_1, ..., \omega_l) := (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) \tag{2}$$

where we recall

$$\nabla_Z[F(X_1, ..., X_k, \omega_1, ..., \omega_l)] = (\nabla_Z F)(X_1, ..., X_k, \omega_1, ..., \omega_l) + \sum_{i=1}^{k} F(X_1, ..., \nabla_Z X_i, ..., X_k, \omega_1, ..., \omega_l) \tag{3}$$

$$+ \sum_{j=1}^{l} F(X_1, ..., X_k, \omega_1, ..., \nabla_Z \omega_i, ..., \omega_l) \tag{4}$$

where for a function f, $\nabla_Z f = Zf$.

a) Let $f : M \to \mathbb{R}$ be a smooth function. Prove that

$$(\nabla^2 f)(X, Y) = X(Yf) - (\nabla_X Y)f \tag{5}$$

b) Prove that, in local coordinates,

$$\Delta f := \sum_{i,j=1}^{n} (g^{-1})^{ij} (\nabla^2 f)(\partial_i, \partial_j) = \frac{1}{\sqrt{\det g}} \partial_i((g^{-1})^{ij} \sqrt{\det g} \partial_j f) \tag{6}$$

where one may use the following facts from linear algebra without proof:

$$\partial_i (g^{-1})^{jk} = -(g^{-1})^{jl} (g^{-1})^{mk} \partial_i g_{lm}, \partial_i \log(\det g) = (g^{-1})^{jk} \partial_i g_{jk} \tag{7}$$

5 Algebraic Topology with Statistics of Particles

Question 5. *Use the fundamental group of the configuration space of two identical particles to prove that, in* $\mathbb{R}^3$*, there can exist only bosons and fermions, but in* $\mathbb{R}^2$ *there can exist any particle statistics.*

6 A Surjection between Groups

Written by the a-MAZING Prof. Tom Church.

Question 6. *Let* $G := APB(\mathbb{Z}^2)$ *be the group of adjacency-preserving bijections* $f : \mathbb{Z}^2 \to \mathbb{Z}^2$*. You can use without proof that this is a group. Suppose that* $G \twoheadrightarrow H$ *is a surjective homomorphism, and* H *is an abelian group. Prove that* $|H|$ *is finite. (The original version of this problem states also to prove that* $|H| = 8$*, but I lost points on that part)*

7 Infinite Integers

Written again by Prof. Tom Church.

Question 7. *Let* R *denote the set of infinite integers. Two examples are*

$$a = ...000000001 \tag{8}$$

$$b = ...562951413 \tag{9}$$

(b *is* π *backwards)*

Prove that there exists at least one solution $z \in R$ *to the equation* $z^3 = 7$*.*

8 A Quantum Error-Correcting Code

Written by Prof. Douglas Stanford.

Question 8. *There is no four-qubit code that can protect an encoded qubit against single-qubit errors on any site. However, suppose that we only care about protecting against arbitrary errors on the first qubit, and against bit-flip errors on the other three.*

a) How many possible errors are there? (Include "no error" as well in your count.)

b) In the stabilizer formalism, how many generators are you allowed to use, in order that the code subspace should be two-dimensional?

c) *Using the stabilizer formalism, devise a four-qubit error correcting code that will protect against all ofthe errors described above.*

d) *Does your code "accidentally" also protect against some other error(s)?*

9 A Generalization of Shor

Written again by Prof. Douglas Stanford.

Question 9. *Consider the code with code subspace spanned by*

$$|0\rangle_L = \frac{1}{4}(|0000\rangle + |1111\rangle)^{\otimes 4} \tag{10}$$
$$|1\rangle_L = \frac{1}{4}(|0000\rangle - |1111\rangle)^{\otimes 4} \tag{11}$$

1. *Find a set of stabilizer generators for this code.*

2. *Find logical operators X_L and Z_L such that*

$$X_L\,|0\rangle_L = |1\rangle_L \tag{12}$$

   and so on. There is no unique answer, but try to find a set of logical operators that use as few physical Pauli operators as possible.

3. *Is it possible to find an X_L and a Z_L that commute with each other?*

10 Real Projective Space

Written by Prof. Ralph Cohen.

Question 10. 1. *Let $x \in S^n$, and $[x] \in \mathbb{R}P^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{R}P^n \Rightarrow \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{R}P^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.*

2. *Use the above to show that $\mathbb{R}P^n$ is compact.*