

Analytic Number Theory Problems

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All mistakes are to be emailed to aszlau@gmail.com. All problems written by Prof. Kannan Soundararajan and Prof. Andrew Glanville’s “Multiplicative Number Theory” textbook, unless otherwise marked.

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1 The Prime Number Theorem

1.1 Partial Summation

1.1.1 Different Forms of the Prime Number Theorem

Question 1. *Given the conjecture*

$$\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x \quad (1)$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for } p \text{ prime and } m \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and the conjecture

$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x} \quad (3)$$

Use partial summation to prove that (1) and (3) are equivalent and both are equivalent to the conjecture

$$\theta(x) := \sum_{p \leq x} \log(p) = x + o(x) \quad (4)$$

Definition 1. Partial Summation: Given a sequence $a_n \in \mathbb{C}$ and a function $f : \mathbb{R} \rightarrow \mathbb{C}$, set $S(t) = \sum_{k \leq t} a_k$, it is easy to conclude that

$$\sum_{n=A+1}^B a_n f(n) = S(B)f(B) - S(A)f(A) - \sum_{n=A}^{B-1} S(n)(f(n+1) - f(n)) \quad (5)$$

and, if f is continuously differentiable on $[A, B]$, then

$$\sum_{A < n \leq B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt \quad (6)$$

Proof. We start with (3). Notice that, if we choose

$$a_n = \begin{cases} 1 & \text{if } n = p \text{ for } p \text{ prime} \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

and $f(x) = \log x$, then

$$\theta(x) = \sum_{n \leq x} a_n f(n) \quad (8)$$

$$= \left(\sum_{n \leq x} a_n \right) \log x - \int_2^x \left(\sum_{n \leq t} a_n \right) (\log t)' dt \quad (9)$$

$$= \left(\sum_{p \leq x} 1 \right) \log x - \int_2^x \left(\sum_{p \leq t} 1 \right) \frac{1}{t} dt \quad (10)$$

$$= \pi(x) \log x - \int_2^x \pi(t) \frac{1}{t} dt \quad (11)$$

$$\sim \frac{x}{\log x} \log x - \int_2^x \frac{t}{t \log t} dt \quad (12)$$

$$\sim x - \int_2^x \frac{1}{\log t} dt \quad (13)$$

$$\sim x + (-li(x)) \quad (14)$$

It remains to prove that $(-li(x)) \in o(x)$. Thus we examine the asymptotic behavior of $-li(x)/x$.

By L'Hospital's Rule, we have

$$\lim_{x \rightarrow \infty} \frac{-li(x)}{x} = \lim_{x \rightarrow \infty} -\frac{1}{\log x} \quad (15)$$

Since $\log x$ diverges, we have this limit equal to 0, so (4) is true if and only if $\pi(x) \sim \frac{x}{\log x}$.

Now we move on to the conjecture in (1). We can easily see that

$$\sum_{p \leq x} \log p \leq \sum_{n \leq x} \log n \Rightarrow \quad (16)$$

$$\theta(x) \leq \psi(x) \sim \quad (17)$$

$$x + o(x) \leq \psi(x) \quad (18)$$

Next we observe that

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \log p \sum_{k \leq \log_p x} 1 = \sum_{p \leq x} \log p \left[\frac{\log x}{\log p} \right] \leq \sum_{p \leq x} \log p \frac{\log x}{\log p} = \sum_{p \leq x} \log x \Rightarrow \quad (19)$$

$$\psi(x) \leq \pi(x) \log x \quad (20)$$

which, assuming conjecture (3), we then deduce

$$\psi(x) \leq \sim \frac{x}{\log x} \log x = x \quad (21)$$

Thus we have

$$\theta \leq \psi(x) \leq \sim x \quad (22)$$

$$x - li(x) \leq \psi(x) \leq \sim x \quad (23)$$

using Conjecture (4). Thus we have, assuming Conjecture (3) and Conjecture (4), that

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \sim x \quad (24)$$

□

1.1.2 Adding reciprocals

Note: my version of the paper has $\sum_{n \leq x}^N \frac{1}{N}$. I'm pretty sure the denominator should be n , as that sum is just 1.

Question 2. *Prove that for any integer $N \geq 1$,*

$$\sum_{n=1}^N \frac{1}{n} = \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt \quad (25)$$

Deduce that, for any real $x \geq 1$,

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \quad (26)$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log N \right) = 1 - \int_1^{\infty} \frac{\{t\}}{t^2} dt \quad (27)$$

Note that, for $t \in \mathbb{R}$, $[t]$ is the integral part of t , and $\{t\}$ is the rest of t .

Proof. We use partial summation again. Let $f(x) = \frac{1}{x}$ and $a_n = 1$. Thus, by partial summation, we have

$$\sum_{n \leq x} \frac{1}{n} = [N] \frac{1}{N} + \log 1 + \int_1^N t \frac{1}{t^2} dt \quad (28)$$

$$= [N] \frac{1}{N} + \int_1^N \frac{1}{t^2} (t - \{t\}) dt \quad (29)$$

$$= 1 + \int_1^N \frac{t}{t^2} dt - \int_1^N \frac{\{t\}}{t^2} dt \quad (30)$$

$$= 1 + \log N - \log 1 - \int_1^N \frac{\{t\}}{t^2} dt \quad (31)$$

$$= \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt \quad (32)$$

For any real x , we have, through partial summation,

$$\sum_{n \leq x} \frac{1}{n} = [N] \frac{1}{N} + \int_1^N t \frac{1}{t^2} dt \quad (33)$$

$$= \frac{x - \{x\}}{x} + \log N - \int_1^N \frac{\{t\}}{t^2} dt \quad (34)$$

$$= \log N + 1 - \frac{\{x\}}{x} + \int_N^{\infty} \frac{\{t\}}{t^2} dt - \int_1^{\infty} \frac{\{t\}}{t^2} dt \quad (35)$$

$$= \log N + \gamma - \frac{\{x\}}{x} + \int_N^{\infty} \frac{\{t\}}{t^2} dt \quad (36)$$

It remains to prove that $\frac{\{x\}}{x}$ and $\int_N^{\infty} \frac{\{t\}}{t^2} dt$ are in $O(\frac{1}{x})$. Starting with the former, we see that since $\{x\} < 1$, we have that $|\frac{\{x\}}{x}| < \frac{1}{x}$, so $\frac{\{x\}}{x} \in O(\frac{1}{x})$. Similarly, we have

$$\left| \int_N^{\infty} \frac{\{t\}}{t^2} dt \right| \leq \int_N^{\infty} |\{t\}| \frac{1}{t^2} dt \leq \int_N^{\infty} \frac{1}{t^2} dt \in O\left(\frac{1}{x}\right) \quad (37)$$

Thus we conclude

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \quad (38)$$

□

1.1.3 $\log N!$

Question 3. For an integer $N \geq 1$, show that

$$\log N! = N \log N - N + 1 + \int_1^N \frac{\{t\}}{t} dt \quad (39)$$

Using that $\int_1^x (\{t\} - 1/2) dt = (\{x\}^2 - \{x\})/2$, show that

$$\int_1^N \frac{\{t\}}{t} dt = \frac{1}{2} \log N - \frac{1}{2} \int_1^N \frac{\{t\} - \{t\}^2}{t^2} dt \quad (40)$$

Conclude that $N! \sim C\sqrt{N}(N/e)^N$, where you can take as fact that

$$C = \exp\left(1 - \frac{1}{2} \int_1^\infty \frac{\{t\} - \{t\}^2}{t^2} dt\right) = \sqrt{2\pi} \quad (41)$$

Proof. From rules of logarithms, we have $\log N! = \log(N(N-1)\dots(2)(1)) = \log N + \log(N-1) + \dots + \log 2 + \log 1$. We use partial summation once again. Let $a_n = 1$, and $f(x) = \log x$. From the partial summation formula given by (6), we have

$$\log N! = N \log N - 0 - \int_1^N \left(\sum_{n \leq t} 1\right) \frac{dt}{t} \quad (42)$$

$$= N \log N - \int_1^N \frac{[t]}{t} dt \quad (43)$$

$$= N \log N - \int_1^N \frac{t - \{t\}}{t} dt \quad (44)$$

$$= N \log N - \int_1^N dt + \int_1^N \frac{\{t\}}{t} dt \quad (45)$$

$$= N \log N - N + 1 + \int_1^N \frac{\{t\}}{t} dt \quad (46)$$

As for the next part, we notice (38):

$$\int_1^N \frac{\{t\}}{t} dt = \int_1^N \frac{\{t\} - \frac{1}{2} + \frac{1}{2}}{t} dt \quad (47)$$

$$= \int_1^N \frac{1}{t} (\{t\} - \frac{1}{2}) dt + \int_1^N \frac{1}{2t} dt \quad (48)$$

$$= \frac{1}{t} \frac{\{t\}^2 - \{t\}}{2} \Big|_1^N - \int_1^N \frac{1}{2} \frac{\{t\}^2 - \{t\}}{-t^2} dt + \frac{1}{2} \log N + \frac{1}{2} \log 1 \quad (49)$$

$$= 0 + \int_1^N \frac{1}{2} \frac{\{t\}^2 - \{t\}}{t^2} dt + \frac{1}{2} \log N \quad (50)$$

$$= \frac{1}{2} \log N - \int_1^N \frac{1}{2} \frac{\{t\} - \{t\}^2}{t^2} dt \quad (51)$$

Thus we have

$$\log N! = N \log N - N + 1 + \frac{1}{2} \log N - \frac{1}{2} \int_1^N \frac{\{t\} - \{t\}^2}{t^2} dt \quad (52)$$

$$= N \log N - N + 1 + \frac{1}{2} \log N - \frac{1}{2} \int_1^\infty \frac{\{t\} - \{t\}^2}{t^2} dt + \frac{1}{2} \int_N^\infty \frac{\{t\} - \{t\}^2}{t^2} dt \quad (53)$$

$$= N \log N - N + \frac{1}{2} \log N + \log C + \frac{1}{2} \int_N^\infty \frac{\{t\} - \{t\}^2}{t^2} dt \quad (54)$$

Taking the exponent of both sides, we get

$$N! = N^N \cdot \frac{1}{e^N} \sqrt{N} \cdot C \sqrt{e^{\int_N^\infty \frac{\{t\} - \{t\}^2}{t^2} dt}} \quad (55)$$

Now we examine the integral in the exponent. First we split it up.

$$\frac{1}{2} \int_N^\infty \frac{\{t\}}{t^2} dt - \frac{1}{2} \int_N^\infty \frac{\{t\}^2}{t^2} dt \leq \left| \frac{1}{2} \int_N^\infty \frac{\{t\}}{t^2} dt \right| - \left| \frac{1}{2} \int_N^\infty \frac{\{t\}^2}{t^2} dt \right| \leq \frac{1}{2} \int_N^\infty |\{t\}| \frac{1}{t^2} dt - \frac{1}{2} \int_N^\infty |\{t\}^2| \frac{1}{t^2} dt \quad (56)$$

It is easy to see that the limit as N approaches infinity the value of this integral converges to 0, so

$$N! \sim N^N \cdot \frac{1}{e^N} \sqrt{N} \cdot C \sqrt{e^0} \Rightarrow \quad (57)$$

$$N! \sim C \sqrt{N} (N/e)^N \quad (58)$$

□

Definition 2. The **Riemann Zeta Function** is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (59)$$

1.1.4 The Riemann Zeta Function

Note: My copy of the problem has a later part. In another copy of the book, this part is labeled as a problem which requires complex analysis, something I haven't learned yet, so I'm skipping that for now.

Question 4. *Prove that for $\operatorname{Re}(s) > 1$,*

$$\zeta(s) = s \int_1^\infty \frac{[y]}{y^{s+1}} dy = \frac{s}{s-1} - s \int_1^\infty \frac{\{y\}}{y^{s+1}} dy \quad (60)$$

Proof. We use partial summation again. We see that, for $a_n = 1$, $f(x) = \frac{1}{x^s}$, we have $\zeta(s) = \sum_1^\infty a_n f(n)$, so, using the usual partial summation formula,

$$\zeta(s) = \sum_1^\infty a_n f(n) = \lim_{N \rightarrow \infty} \sum_1^N a_n f(n) \quad (61)$$

$$= \lim_{N \rightarrow \infty} \left[[N] \frac{1}{N^s} - [1] \frac{1}{1^s} - \int_1^N [y] \frac{1}{y^s} dy \right] \quad (62)$$

$$= \lim_{N \rightarrow \infty} \left[[N] \frac{1}{N^s} - [1] \frac{1}{1^s} - [N] \frac{1}{N^s} + [1] \frac{1}{1^s} + s \int_1^N [y] \frac{1}{y^{s+1}} dy \right] \quad (63)$$

$$= \lim_{N \rightarrow \infty} \left[s \int_1^N [y] \frac{1}{y^s} dy \right] \quad (64)$$

$$= s \int_1^\infty [y] \frac{1}{y^s} dy \quad (65)$$

We write this final integral in a different way:

$$s \int_1^\infty [y] \frac{1}{y^s} dy = \lim_{N \rightarrow \infty} s \int_1^N \frac{y - \{y\}}{y^{s+1}} dy \quad (66)$$

$$= s \int_1^N \frac{y}{y^{s+1}} dy - s \int_1^N \frac{\{y\}}{y^{s+1}} dy \quad (67)$$

$$= \lim_{N \rightarrow \infty} \left[s \int_1^N \frac{1}{y^s} dt - s \int_1^N \frac{\{y\}}{y^{s+1}} dy \right] \quad (68)$$

$$= \lim_{N \rightarrow \infty} \left[-s \frac{1}{s-1} \left(\frac{1}{t^{s-1}} \Big|_1^N \right) - s \int_1^N \frac{\{y\}}{y^{s+1}} dy \right] \quad (69)$$

$$= \lim_{N \rightarrow \infty} \left[-\frac{s}{s-1} \left(\frac{1}{N^{s-1}} - \frac{1}{1^{s-1}} \right) - s \int_1^N \frac{\{y\}}{y^{s+1}} dt \right] \quad (70)$$

Since $\operatorname{Re}(s) > 1$, we have $\operatorname{Re}(s) - 1 > 0$, so, evaluating the limit, we get that this expression is equivalent to

$$\frac{-s}{s-1} (0 - 1) - s \int_1^N \frac{\{y\}}{y^{s+1}} dt = \frac{s}{s-1} - s \int_1^N \frac{\{y\}}{y^{s+1}} dt \quad (71)$$

□

1.2 Chebyshev's Elementary Estimates

1.2.1 $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x}$

Question 5. *Prove that*

$$\limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq 1 \geq \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} \quad (72)$$

so that if $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x}$ exists, it must be equal to 1.

Note that $\log n = \sum_{d|n} \Lambda(d)$, so

$$\sum_{n < x} \log n = \sum_{n \leq x} \sum_{n=dk} \Lambda(d) = \sum_{k=1}^{\infty} \psi\left(\frac{x}{k}\right) \quad (73)$$

and, by Stirling's Formula, we have

$$\sum_{k=1}^{\infty} \psi\left(\frac{x}{k}\right) = x \log x - x + O(\log x) \quad (74)$$

Proof. We start with $\limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq 1$. Suppose not. Then there exists $\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, $\frac{\psi(x)}{x} \leq (1 - \epsilon)$. Notice that

$$\psi\left(\frac{x}{k}\right) \leq \frac{\psi(x)}{k}, \forall k \in \mathbb{Z} \quad (75)$$

Then, for $x > x_0$,

$$\sum_{k=1}^{\infty} \psi(x/k) = \sum_{k=1}^x \psi(x/k) \leq \sum_{k=1}^x \frac{\psi(x)}{k} \leq (1 - \epsilon)x \sum_{k=1}^x \frac{1}{k} = (1 - \epsilon)x[\log x + \gamma + O(\frac{1}{x})] \quad (76)$$

Using the above noted relation, we then have

$$x \log x - x + O(\log x) \leq (1 - \epsilon)x[\log x + \gamma + O(\frac{1}{x})] \quad (77)$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{1}{x} [x \log x - x + O(\log x)] \leq \frac{1}{x} [(1 - \epsilon)x[\log x + \gamma + O(\frac{1}{x})]] \quad (78)$$

$$\Rightarrow \log x - 1 \leq (1 - \epsilon)[\log x + \gamma] \quad (79)$$

so $-\epsilon \log x + (1 - \epsilon)\gamma + 1$ must be positive, where γ is the Euler-Mascheroni constant. However, since $\log x$ diverges, this cannot be true for all $x > x_0$.

We follow the same approach as with \limsup . We suppose by contradiction that there exists

$\epsilon > 0$ such that, for all $x > x_0$ for some $x_0 \geq 2$, such that $\frac{\psi(x)}{x} \geq (1 + \epsilon)$. We then have

$$(1 + \epsilon)x \sum_{k=1}^{x/x_0} \frac{1}{k} + \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq \sum_{k=1}^{\infty} \psi(x/k) \quad (80)$$

$$(1 + \epsilon)x \left[\log \frac{x}{x_0} + \gamma + O\left(\frac{1}{x}\right) \right] + \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq x \log x - x + O(\log x) \quad (81)$$

$$\frac{1}{x} \sum_{\lfloor x/x_0 \rfloor < k}^{\infty} \psi(x/k) \leq -\epsilon \log x - 1 + (1 + \epsilon) \log x_0 - (1 + \epsilon)\gamma \quad (82)$$

Since $\psi(x)$ is strictly nonnegative for all $x \in \mathbb{Z}^+$, this cannot be true. Thus

$$\lim_{x \rightarrow \infty} \sup \frac{\psi(x)}{x} \geq 1 \geq \lim_{x \rightarrow \infty} \inf \frac{\psi(x)}{x} \quad (83)$$

□