Bundle Theory Problems

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All problems written by Prof. Ralph Cohen. Referenced to Cohen's notes/textbook-in-progress, "Bundles, Homotopy, and Manifolds."

Question 1. *Let $\xi \to B$ be an n-dimensional vector bundle.*

1. *Define clutching functions of the $nk-$dimensional $k-$fold tensor product bundle $\otimes^k \xi \to B$ in terms of clutching functions of ξ.*

2. *Define clutching functions of the $k-$fold exterior product bundle $\wedge^k \xi \to B$ in terms of clutching functions of ξ.*

Proof. 1. Let $\xi \to B$ be an $n-$dimensional vector bundle with clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. We define clutching functions on the $k-$fold tensor bundle $\otimes^k \xi \to B$ by taking the product of our clutching functions on ξ:

$$\phi^{\otimes^k \xi}_{\alpha,\beta} : \qquad U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} GL_n(\mathbb{R}) \times \ldots \times GL_n(\mathbb{R}) \xrightarrow{\quad\otimes\quad} GL_{n^k}(\mathbb{R})$$

$$x \xmapsto{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} A \times \ldots \times A \xmapsto{\quad\otimes\quad} A \otimes \ldots \otimes A$$

where $A \in GL_n(\mathbb{R})$ is the linear transformation on ξ is the image of the regular clutching function on ξ. Here the tensor product of two linear transformations $A_1 : \xi \to \xi, \ldots, A_k : \xi \to \xi$ is the induced linear transformation $A_1 \otimes \ldots \otimes A_k : \xi \otimes \ldots \otimes \xi \to \xi \otimes \ldots \otimes \xi$. This well-defined because we simply take the automorphism associated to the clutching function on ξ and tensor k copies of it; we get $\phi_{\beta,\alpha} = A^{-1} \otimes \ldots \otimes A^{-1}$, which, when applied before or after $\phi_{\alpha,\beta}$, we get $Id \otimes \ldots \otimes Id$. Thus $\phi_{\alpha,\beta} = \phi^{-1}_{\beta,\alpha}$.

2. We approach this problem by considering the vector space associated with $\otimes^k \xi$, for ξ and $n-$dimensional vector space. Consider an orthonormal basis $\{e_i\}_{0 \le i \le n}$ of ξ. We define an isomorphism from $\xi \times \ldots \times \xi$ to $\xi \otimes \ldots \otimes \xi$:

$$e_{i_1} \times \ldots \times e_{i_k} \mapsto e_{i_1} \otimes \ldots \otimes e_{i_k} \tag{1}$$

Thus we have nk generators of $\otimes^k \xi$, so $\otimes^k V$ is isomorphic to an $nk-$dimensional vector space. Since $\wedge^k \xi$ a quotient of this vector space, this should probably be isomorphic to a subspace.

We construct a basis of $\wedge^k \xi$. The symmetry quotient $a \otimes b + b \otimes a$ implies that $a \otimes b = -b \otimes a$, in particular $a \otimes a = -a \otimes a$, so $a \otimes a = 0$, for $a, b \in \xi$. Thus there can be no repeated indices, and permutations of index combinations are linearly dependent. Thus, the allowable basis vectors are just the $\binom{n}{k}$ combinations of k entries spanning 1 to n. Thus $\wedge^k \xi$ is isomorphic to an $\binom{n}{k}$-dimensional vector space. If $k > n$, there must be repeated indices, and so the vector space is 0-dimensional. Thus we need our clutching functions to have image in $GL_{\binom{n}{k}}(\mathbb{R})$ from k elements in $GL_n(\mathbb{R})$. Antisymmetry, along with general objects one runs into with dealing with exterior algebras, are determinants. Suppose $[g_{ij}] \in GL_k(\mathbb{R})$ is the transition function associated with $U_i \cap U_j$, and associate the $k \times k$-submatrix of $[g_{ij}]$ denoted $[g_{ij}]_{h \in \binom{n}{k}}$ (henceforth we will say that $h \in S(\binom{n}{k})$ is an increasing sequence of k integers between 1 and n), with entries kept being the entries associated with the corresponding indices of the basis vectors of $\wedge^k \xi$. Define the element in $GL_{\binom{n}{k}}(\mathbb{R})$ as the matrix

$$(\mathrm{Det}[g_{ij}]_{h \in S(\binom{n}{k})})_{m,l \in \binom{n}{k}} \tag{2}$$

Example 1.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} Det\begin{pmatrix} a & b \\ d & e \end{pmatrix} & Det\begin{pmatrix} a & c \\ d & f \end{pmatrix} & Det\begin{pmatrix} b & b \\ e & f \end{pmatrix} \\ Det\begin{pmatrix} a & b \\ g & h \end{pmatrix} & Det\begin{pmatrix} a & c \\ g & i \end{pmatrix} & Det\begin{pmatrix} b & c \\ h & i \end{pmatrix} \\ Det\begin{pmatrix} d & e \\ g & h \end{pmatrix} & Det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & Det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix} \tag{3}$$

where the submatrix entries are determined by ranging across the elements of the $\binom{n}{k}$ *permutations.*

This is clearly an injection, as scaling any entry in the preimage scales a unique combination of entries in the image. Thus, by changing one entry in the image, one cannot counteract this change by changing another entry. Furthermore, Id_n trivially maps to $Id_{\binom{n}{k}}$, since all off-diagonal elements have the determinants of matrices with only one nonzero element, and the diagonal elements have determinants of identity matrices. Thus the image is in a subgroup of $GL_{\binom{n}{k}}(\mathbb{R})$, and this map has a well-defined inverse. To check that this is a well-defined clutching function, we consider the clutching function associated with $[g_{ij}^{-1}]$. We have

$$A_{ij}^{-1} = \frac{n!\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b} A_{i_1}^{j_1} \dots A_{i_b}^{j_b}}{k!\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b} A_{i_2}^{j_2} \dots A_{i_b}^{j_b}}, \text{ for } b := \frac{n!}{k!} \tag{4}$$

$$= \frac{n\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b} \mathrm{Det}[[g_{ij}]_{h_1}] \dots \mathrm{Det}[[g_{ij}]_{h_b}]}{\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b} \mathrm{Det}[[g_{ij}]_{h_2}] \dots \mathrm{Det}[[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}). \tag{5}$$

where h_α is the corresponding permutation with the i_α, j_α entry in our matrix.

$$(g_{ij})^{-1} = \frac{n\epsilon^{i_1 \dots i_n}\epsilon_{j_1 \dots j_n} g_{i_1 j_1} \dots g_{i_n j_n}}{\epsilon^{i i_2 \dots i_n}\epsilon_{j j_2 \dots j_n} g_{i_2 j_2} \dots g_{i_n j_n}} \tag{6}$$

$$(\mathrm{Det}(g_{ij}^{-1}))_{h \in S(\binom{n}{k})})_{m,l \in \binom{n}{k}} = \frac{n\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b}\epsilon^{s_1 \dots t_k}\epsilon_{s_1 \dots t_k} g_{s_1 t_1} \dots g_{s_k t_k}\epsilon^{i''_1 \dots i''_k}\epsilon_{j''_1 \dots j''_k} g_{i''_1 j''_1} \dots g_{i''_k j''_k}}{\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b}\epsilon^{i'_1 \dots i'_k}\epsilon_{j'_1 \dots j'_k} g_{i'_1 j'_1} \dots g_{i'_k j'_k}\epsilon^{i''_1 \dots i''_k}\epsilon_{j''_1 \dots j''_k} g_{i''_1 j''_1} \dots g_{i''_k j''_k}} \tag{7}$$

$$= \frac{n\epsilon^{i_1 \dots i_b}\epsilon_{j_1 \dots j_b} \mathrm{Det}[[g_{ij}]_{h_1}] \dots \mathrm{Det}[[g_{ij}]_{h_b}]}{\epsilon^{i i_2 \dots i_b}\epsilon_{j j_2 \dots j_b} \mathrm{Det}[[g_{ij}]_{h_2}] \dots \mathrm{Det}[[g_{ij}]_{h_b}]}, h_\alpha \in S(\binom{n}{k}) \tag{8}$$

where in the last step we rewrite our scare labels for elements of $S(\binom{n}{k})$, and we find this is exactly equal to A_{ij}^{-1}. Thus A_{ij}^{-1} is the image of g_{ij}^{-1}, and so our clutching functions are well-defined.

□

Question 2. *1. Notice that the tensor product of two one-dimensional vector bundles ("line bundles") over a space B is still a one dimensional vector bundle . Show that the set of isomorphism classes of one-dimensional (real) vector bundles over B is an abelian monoid with respect to tensor product. In particular, what is the unit of this monoid?*

2. Show that in fact this abelian monoid is an abelian group.

Proof. 1. First we note that the clutching functions of a vector bundle uniquely determine the isomorphism class of said bundle. In the spirit of the first problem, we define our clutching functions for the $k-$fold tensor product of line bundles by multiplying the clutching functions of a single line bundle:

$$\phi^{\otimes^k \mathbb{R}}_{\alpha,\beta} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\ldots\times\phi_{\alpha,\beta}} \mathbb{R}^* \times \ldots \times \mathbb{R}^* \xrightarrow{\cong} \mathbb{R} \tag{9}$$

$$\to \mathbb{R} \otimes \ldots \otimes \mathbb{R} \tag{10}$$

where that last isomorphism is due to having the basis vector, comprised of the e_i basis vector for the i^{th} tensor factor, as $e_1 \otimes \ldots \otimes e_k^*$. This is an associative operation, as $s_{\alpha,\beta}(\tilde{s}_{\alpha,\beta} s'_{\alpha,\beta}) \to (e_1 \otimes e_2) \otimes e_3 \cong e_1 \otimes (e_2 \otimes e_3) \leftarrow (s_{\alpha,\beta}\tilde{s}_{\alpha,\beta})s'_{\alpha,\beta}$. This map is abelian, since we can simply map $e_i \times e_j \to e_j \otimes e_i$, as an isomorphism, and as $e_i \times e_j \to e_i \otimes e_j$ is an isomorphism, $\otimes\mathbb{R}' \cong \mathbb{R}' \otimes \mathbb{R}$. Thus these are the same isomorphism class. It remains to show that there is an identity element. From the properties of the tensor product, $e_1 \otimes e_2 = e_1 e_2 \otimes 1 = 1 \otimes e_1 e_2$. Furthermore, $s_{\alpha,\beta}s_{\beta,\alpha} = 1$. Thus we have a natural isomorphism from principal bundles $(B \times \{1\} = B)$ $\mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ given by $e_1 \to e_1 \otimes 1 \cong 1 \otimes e_1$. Using the inverse map (that map was an isomorphism) Thus the identity element is given by $1 \in \mathbb{R}^*$. Since we have associativity, commutativity, and an identity element, the isomorphism classes of 1-dimensional vector bundles forms an abelian monoid.

2. It remains to prove that every element has a unique inverse, as an abelian group is an abelian monoid where every element has a unique inverse. Consider the map

$$\phi^{\otimes^2 \mathbb{R}}_{\alpha,\alpha} : U_\alpha \cap U_\beta \xrightarrow{\phi_{\alpha,\beta}\times\phi_{\beta,\alpha}} \mathbb{R}^* \times \mathbb{R}^* \xrightarrow{\otimes} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R} \tag{11}$$

$$x \to a \times a^{-1} \to a \otimes a^{-1} \xrightarrow{b} aa^{-1} \otimes 1 \to 1 \otimes 1 \xrightarrow{c} 1 \tag{12}$$

where the b map is due to the linearity of the tensor product and the c map is due to the natural isomorphism defined above. Since $\phi_{\beta,\alpha}$ is the unique inverse clutching function to $\phi_{\alpha,\beta}$, each element in this monoid has a unique inverse, and thus the isomorphism classes of 1-dimensional vector bundles is an abelian group with the tensor product operation.

$\square$

Question 3. *Let X be a space with a basepoint $x_0 \in X$. Recall that the (reduced) suspension of*

X, ΣX, is the space

$$\Sigma X = X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\} \tag{13}$$

Here I am thinking of S^1 as the unit complex numbers. Let $(Y, y0)$ be another space with basepoint. Consider the (based) "loop space"

$$\Omega Y = Map((S^1, \{1\}), (Y, y_0)) \tag{14}$$

This is the space of maps from S^1 to Y that take $1 \in S^1$ to the basepoint $y_0 \in Y$, endowed with the compact - open topology.

1. *Prove that there is a bijection*

$$[\Sigma X, Y] \cong [X, \Omega Y] \tag{15}$$

   Here the notation [−, −] denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

2. *Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the proof of Corollary 4.10 in the text. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".*

Proof. 1. We being by considering elements of $[X, \Omega Y]$. Define a basepoint-preserving map $f \in [X, \Omega Y]$ for some $x \in X$ by $f(x)$. This is thus a basepoint-preserving map from $(S^1, \{1\})$ to (Y, y_0), denoted $f(x)(t)$. Since $f(x)(t)$ is basepoint-preserving, $f(x)(1) = y_0$. Furthermore, since f is basepoint-preserving, $f(x_0)(t) = y_0$. We now examine elements of $[\Sigma X, Y]$. Suppose we have a basepoint-preserving map $g \in [\Sigma X, Y]$. If we consider $(x, t) \in X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\}$, we have that $(x, 1) = (x_0, t), \forall x \in X, t \in S^1$. Thus we have $g(x, 1) = g(x_0, t) = y_0$. We have our correspondence as $[g(x, t)] \mapsto [f(x)(t)]$, given by the corresponding that, for $g : \Sigma X \to Y$, we associate the family of loops $f(x)(t)$ by restricting g to the images of the loops $\{x\} \times S^1 \subset \Sigma X$. We first prove surjectivity. Given a map $j : (S^1, \{1\}) \to (Y, y_0)$, we have the preimage of this map in $[X, \Omega Y]$ to be the $x \in X$ such that $g(x,t) \subset Y$ is of the same homotopy type in Y (we fix x and let t span S^1 to get the same loop). Thus we associate the homotopy type of $g(x,t)$ with the homotopy type of $[f(x)(t)]$, so the map is surjective. Suppose $[g_1(x_1,t_1)] \neq [g_2(x_2,t_2)]$. Then we associate the maps $[f_1(x_1)(t_1)], [f_2(x_2)(t_2)] \in [X, \Omega Y]$, respectively. Since $x_1 \neq x_2, [g_1] \neq [g_2]$, the maps $f_1(x_1) : S^1 \to Y, f_2(x_2) : S^1 \to Y$ are not homotopically equivalent, and thus the loops are not homotopically equivalent. Thus $[f_1(x_1)(t_1)] \neq [f_2(x_2)(t_2)]$. We check that this works for the basepoint map: $[g(x_0,t)] \mapsto [f(x_0)(t)] = [\text{the constant map}]$.

Notice that $\pi_n(Y, y_0) = [S^n, Y]$, so it suffices to prove that $\Sigma S^{n-1} = S^n$. We do this using $CW-$complexes:

$$\frac{[e^0 \sqcup e^n/\sim] \times [e^0 \sqcup e^1]/\sim}{[e^0 \sqcup e^n/\sim] \times e^0 \cup [e^0 \sqcup e^1/\sim] \times e^0} = \frac{(e^0 \sqcup e^0) \times (e^0 \times e^1) \times (e^0 \times e^1) \times (e^1 \times e^n)/\sim}{[e^0 \times e^0] \cup [e^0 \times e^1] \cup [e^0 \times e^n]/\sim} \tag{16}$$

$$= e^0 \sqcup (e^n \times e^1)/\sim \tag{17}$$

$$= S^{n+1} \tag{18}$$

where we quotient out the usual way, i.e. the terms in the quotient "cancel" the equal terms in the space and become a single point e^0. Thus we have

$$\pi_{n-1}(\Omega Y, \epsilon_0) \cong [S^{n-1}, \Omega Y] \cong [\Sigma S^{n-1}, Y] \cong [S^n, Y] \cong \pi_n(Y, y_0) \tag{19}$$

2. First we prove injectivity. Since $\overline{f}(g)(t) = f(g,t)$, we notice the basepoint-preserving-ness of $\overline{f}$. For basepoint $g_0 \in G$, we have $\overline{f}(g_0)(t) = f(g_0,t) = f(g_0,t')$, for any $t' \in S^1$, equal to the constant map. Consider a class of a nullhomotopic loop in $\pi_n(G)$. We have $\overline{f}(g_0)(t)$ is the constant map, mapping the point g_0 to ϵ_0, the constant map $S^1 \to BG$. This is because $(g_0,t) = g_0$, so g_0 gets mapped to $\{1\} \in S^1$ which gets constantly mapped to the basepoint of BG, as it is constant for all $t \in S^1$. Thus the identity of $\pi_n(G)$ maps to the identity of $\pi_n(\Omega BG)$. Now we prove surjectivity. Suppose we have a homotopy class of an $n-$dimensional loop $[n] \in \pi_n(\Omega BG)$. This corresponds to an $(n+1)-$dimensional homotopy

class of BG through the isomorphism proved above. We seek to create a principal G-bundle over ΣG that is trivial on both cones. Define this bundle as

$$C_+ := G \times [1, -1]/\sim, c_- := G \times [-1, 1] \tag{20}$$

$$E := C_+ \times G \cup_{Id} C_- \times G \tag{21}$$

By theorem 4.8 in the text, there is a bijective correspondence given by

$$\psi : [\Sigma G, BG] \to Prin_G(\Sigma G) \tag{22}$$

$$f \mapsto f^*(E) \tag{23}$$

such that $f^*(EG) = E$. Thus let $[g] \in \pi_{n+1}(\Sigma G)$ be the homotopy class of an $(n+1)-$loop that maps its loop in ΣG to said $(n+1)-$loop in BG induced by f, well-defined because f is a bijection. Thus let g be the $\overline{f}(g)(t)$. We know this must exist due to part a). Therefore, $\overline{f}$ is an isomorphism in π_n. Since this did not depend on n, all such homotopy groups are isomorphic.

□

Question 4. *For any space X let $Vect^d(X)$ denote the set of isomorphism classes of $d-$dimensional vector bundles over X.*

1. *Compute $Vect^d(S^1)$. Justify your answer.*
2. *Compute the fundamental group of the Grassmannian, $\pi_1(Gr_d(\mathbb{R}^\infty))$.*
3. *Let X be a simply-connected space. Prove that any one-dimensional vector bundle over X is trivial.*

Proof.

Lemma 1. *There is a bijective correspondence between principal bundles and homotopy groups $Prin_G(S^n) \cong \pi_{n-1}(G)$ where as a set $\pi_{n-1}(G) = [S^{n-1}, x_0; G, \{1\}]$, which refers to (based) homotopy classes of basepoint preserving maps from the sphere S^{n-1} with basepoint $x_0 \in S^{n-1}$, to the group G with basepoint the identity 1$\in G$.*

Proof. Let $p : E \to S^n$ be a principal $G-$bundle. Write S^n as the union of its upper and lower hemispheres

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_- \tag{24}$$

Since $D^n_\pm$ are contractible, the restiction of E to each of the hemispheres is trivial, so if we fix a trivialization of the fiber of E above $x_0 \in S^{n-1} \subset S^n$, we can extend this trivialization to the upper and lower hemispheres. For θ a clutching function on the equator $\theta : S^{n-1} \to G$, we can then write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{25}$$

that is, for $(x, g) \in (D^n_+ \times G)$, we have $(x, g) \sim (x, \theta(x)g) \in (D^n_- \times G)$. Since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}(G) \tag{26}$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G) \tag{27}$$

By restricting to the upper and lower hemispheres, Ψ defines maps

$$\Psi_+ : D^n_+ \to G \tag{28}$$

$$\Psi_- : D^n_- \to G \tag{29}$$

which both map $x_0 \in S^{n-1}$ to the identity $1 \in G$, and have the property

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x) \tag{30}$$

or $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. By considering the linear homotopy $\Psi_+(tx)\theta_1(tx)\Psi_-(tx)^{-1}$ for $t \in [0,1]$, we can see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, for 0 the origin in $D^n_\pm$, i.e. the north and south poles of the sphere. Since $\Psi_\pm$ are defined on connected spaces, their images lie on a connected component of G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi + (0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\theta : Prin_G(S^n) \to \pi_{n-1}(G)$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G) \tag{31}$$

We discuss injectivity. Suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2, where

$$E_i = (D^n_+ \times G) \cup_{\theta_i} (D^n_- \times G) \tag{32}$$

Let $H : S^{n-1} \times [-1,1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator $S^{n-1} \subset S^n$ with $S^{n-1} \times [-1,1]$. Write $\mathcal{D}_+ = D^2_+ \cup \overline{\mathcal{N}}$ and

$\mathcal{D}_- = D^2_- \cup \overline{\mathcal{N}}$. Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \overline{\mathcal{N}} \cong S^1 \times [-1,1] \tag{33}$$

Thus we may form the principal $G-$bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_+ \times G \tag{34}$$

where, by abuse of notation, H is the composition $\overline{\mathcal{N}} \cong S^{n-1} \times [-1,1] \xrightarrow{H} G$. If we deformation retract $\overline{\mathcal{N}}$ to S^{n-1} and contract $D^2_\pm$ to D_-, we get that E is isomorphic to E_1 and E_2. □

Lemma 2. *There are bijective correspondences*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{C}P^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z}) \tag{35}$$

Similarly, there are bijective correspondences

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2,1)] \cong H^1(X;\mathbb{Z}_2) \tag{36}$$

Proof. The last correspondence takes a map $f : X \to \mathbb{C}P^\infty$ to the class

$$c_1 = f^*(c) \in H^2(X;\mathbb{Z}) \tag{37}$$

where $c \in H^2(\mathbb{C}P^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)-$bundle). These other correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X)\mathbb{C}[X, BGL(1,\mathbb{C})]$, and that $\mathbb{C}P^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because we can express $\mathbb{C}P^\infty$ in its homogeneous form as $\mathbb{C}P^\infty = \lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C})$, and that $\lim_{n\to\infty}(\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$.

For the other case, we have the last correspondence taking a map $f : X \to \mathbb{R}P^\infty$ to the class $\omega_1 = f^*(\omega) \in H^1(X;\mathbb{Z}_2$, where $\omega \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $\omega_1 \in H^1(X;\mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel-Whitney class of ζ (or of the corresponding principal $O(1)-$ bundle). $\square$

1. Let $V_d(\mathbb{R}^n)$ be the Stiefel manifold as in the text. We claim that the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ to the first n coordinates induces a nullhomotopic inclusion of $V_d(\mathbb{R}^n)$ into $V_d(\mathbb{R}^{2n})$. Let $\iota : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho \in V_d(\mathbb{R}^n) \subset V_d(\mathbb{R}^{2n})$, we have a homotopy $t\iota + (1-t)\rho$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V_d(\mathbb{R}^n)$ onto the element ι. Hence the limiting space $V_d(\mathbb{R}^\infty)$ is aspherical with a free $GL(d,\mathbb{R})-$action. Therefore the projection

$$V_d(\mathbb{R}^\infty) \to V_d(\mathbb{R}^\infty)/GL(d,\mathbb{R}) = Gr_d(\mathbb{R}^\infty) \tag{38}$$

is a universal $GL(d,\mathbb{R})-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BGL(d,\mathbb{R})$, so we have a classification

$$Vect^d(S^1) \cong Prin_{GL(d,\mathbb{R})}(S^1) \cong [S^1, BGL(d,\mathbb{R})] \cong [S^1, Gr_d(\mathbb{R}^\infty)] = \pi_1(Gr_d(\mathbb{R}^\infty)) \tag{39}$$

Thus it remains to compute $\pi_1(Gr_d(\mathbb{R}^\infty))$. Let $V_d^O(\mathbb{R}^n)$ be the Stiefel manifold of orthonormal $d-$frames in $\mathbb{R}^n$. Let $\iota' : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a linear embedding with image an orthonormal frame in the last n coordinates in $\mathbb{R}^{2n}$. For any $\rho' \in V_d^O(\mathbb{R}^n) \subset V_d^O(\mathbb{R}^{2n})$, we have a homotopy $t\iota' + (1-t)\rho'$ that defines a one-parameter family of linear embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{2n}$, and hence a contraction of the image in $V_d^O(\mathbb{R}^n)$ onto the element ι'. Hence the limiting space $V_d^O(\mathbb{R}^\infty)$ is aspherical with a free $O(d)-$action. Therefore the projection

$$V_d^O(\mathbb{R}^\infty) \to V_d^O(\mathbb{R}^\infty)/O(d) = Gr_d(\mathbb{R}^\infty) \tag{40}$$

is a universal $O(d)-$bundle, so the infinite Grassmannian is the classifying space $Gr_d(\mathbb{R}^\infty) = BO(d)$. Thus we have

$$\pi_1(Gr_d(\mathbb{R}^\infty)) \cong \pi_1(BO(d)) \cong [S^1, BO(d)] \cong Prin_{O(d)}(S^1) \xrightarrow{x} \pi_0(O(n)) \tag{41}$$

where the last x map is a bijection due to Lemma 1. $O(d)$ has two connected components: the Det map is a polynomial and thus continuous. This maps $O(d)$ to either 1 or -1. Therefore, since $\pi_0(O(d))$ is the set of connected components $O(d)$, it is a two-element group, and is thus $\mathbb{Z}/2\mathbb{Z}$. Thus $Vect^d(S^1) \cong \pi_1(Gr_d(\mathbb{R}^\infty)) \cong \mathbb{Z}/2\mathbb{Z}$.

2. See part a).

3. From Theorem 4.14, we have

$$Vect^1(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{R}P^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2) \tag{42}$$

Since X is simply connected, $H^1(X; \mathbb{Z}_2) = 0$, and so there is only one element in $Vect^1(X)$, i.e. there is only one isomorphism class of 1-dimensional vector bundles, which must be the trivial bundle.

□

Question 5. *Let T^2 be a closed, connected, orientable surface (two-dimensional manifold). Show that there are infinitely many nonisomorphic complex line bundles over T^2.*

Proof. From Theorem 4.13 in the book, we have

$$Vect^1(T) \cong Prin_{U(1)}(T) \cong [T, BU(1)] = [T, \mathbb{C}P^\infty] = [T, K(\mathbb{Z}, 2)] \cong H^2(T, \mathbb{Z}) \tag{43}$$

Since T is closed and orientable, we can apply Poincaré Duality. Due to Poincaré Duality, we have $H_2(T, \mathbb{Z}) \cong H_0(T)$. Since T is connected, we want to show that T is path-connected because it's a manifold. Let $x \in T$ be any point in T. Let U denote the open neighborhood of x with is locally path-connected. This is assured because there exists an open neighborhood of x that is homeomorphic to $\mathbb{R}^2$, which is everywhere path-connected. Let $y \in T \backslash U$. There exists an open neighborhood of y that is path-connected by the same argument. Thus $U, T \backslash U$ are open, and $U \cup T \backslash U = T$. Since U is nonempty, T must be path-connected, and the boundary of any image of a singular chain is homotopic to the boundary of a point (since every point can be homotoped to any other point), and is thus zero. Thus we have

$$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0 \Rightarrow \tag{44}$$

$$H_0(T) = \frac{C_0}{Im(\partial_1)} \tag{45}$$

$$\varepsilon : C_0 \to \mathbb{Z} \tag{46}$$

$$\varepsilon(\sum_i n_i\sigma_i) \mapsto \sum_i n_i \tag{47}$$

This is obviously a surjective map since T is nonempty. We want to show that Ker(ε) = Im(∂). For a 1-simplex $\sigma : \Delta^1 \to X$, we have $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|[v_1] - \sigma|[v_0]) = 1 - 1 = 0$, so Im$(\partial)$ $\subset$Ker(ε). Now suppose $\varepsilon(\sum_i n_i\sigma_1) = 0$. The σ_i's are singular 0-simplices i.e. points of T. Choose a path $f_t : [0,1] \to T$ from a basepoint x_0 to $\sigma_i(v_0)$, with σ_0 the singular 0-simplex with image x_0. f_t is a singular 1-simplex, and $\partial f_t = \sigma_i - \sigma_0$. Thus $\partial(\sum_i n_i f_t) = \sum n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i - 0$. Therefore $\sum_i n_i\sigma_i$ is a boundary. Thus Im(∂) $\subset$Ker(ε), so Ker(ε) = Im(∂), and thus $H_0(T) \cong \mathbb{Z}$. This has infinitely many elements, so $H^2(T,\mathbb{Z}) \cong \mathbb{Z}$ has infinitely many elements, so $Vect^1(T)$ has infinitely many elements i.e. isomorphism classes of complex line bundles. □

Question 6. *A vector bundle η is said to be stably trivial if for some $k \in \mathbb{Z}$, the Whitney sum $\eta \oplus \epsilon^k$ is a trivial vector bundle, where ϵ^k denotes the standard trivial bundle of dimension k. Let M be an $n-$dimensional smooth, closed manifold, and suppose that there exists an immersion*

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{48}$$

1. *Prove that the tangent bundle TM is stably trivial.*
2. *Show that the sphere S^n has stably trivial tangent bundle for every n. (A manifold with stably trivial tangent bundle is called "stably parallelizable".)*
3. *Show that the tangent bundle $TS^2 \to S^2$ is not trivial, but $TS^2 \oplus \epsilon^1$ is trivial.*

Proof. 1. Since we have an immersion

$$f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k} \tag{49}$$

we have a monomorphism

$$\begin{tikzcd} T(M \times \mathbb{R}^k) \arrow[r, "Df"] \arrow[d] & T\mathbb{R}^{n+k} \arrow[d] \\ M \times \mathbb{R}^k \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{R}^{n+k} \\ M & \end{tikzcd} \qquad \begin{tikzcd} \zeta \arrow[r, "\overline{\gamma}"] \arrow[d] & \xi \arrow[d] \\ X \arrow[r, "\gamma"] & Y \end{tikzcd}$$

so that $\gamma_x : \zeta_x \to \xi_{\gamma(x)}$ is a monomorphism of fibers. Since M is $n-$dimensional, since γ_x is injective, it must be an isomorphism as well. Thus we have an isomorphism of vector bundles $T(M \times \mathbb{R}^k) \cong f^*(T\mathbb{R}^{n+k})$. We have isomorphisms

$$T(M \times \mathbb{R}^k) \cong \pi^*(TM) \oplus \epsilon^k,\ T\mathbb{R}^{n+k} \cong \epsilon^{n+k} \tag{50}$$

$$(v, e) \mapsto v \oplus e,\ (w) \mapsto w \tag{51}$$

Thus we have an isomorphism $\pi^*(TM) \otimes \epsilon^k \cong \epsilon^{n+k}$ of vector bundles over $M \times \mathbb{R}^k$. The pullback of this along a section of π yields $TM \oplus \epsilon^k \cong \epsilon^{n+k}$.

2. We consider the standard embedding $f : S^n \to \mathbb{R}^{n+1}, f(x_1, ..., x_{n+1}) = (x_1, ..., x_{n+1})$. This is obviously an embedding and thus an immersion. The unit normal vector in this embedding is $\frac{\mathbf{x}}{|\mathbf{x}|}$ with respect to the usual euclidean metric. This is nowhere-vanishing on S^n, so the normal bundle is given by $t\frac{\mathbf{x}}{|\mathbf{x}|}, t \in \mathbb{R}$. We have the isomorphism from the trivial line bundle ϵ^1 to the normal bundle by $v \mapsto v\frac{\mathbf{x}}{|\mathbf{x}|}$. Thus we have $TS^n \oplus \nu(S^n) \cong TS^n \oplus \epsilon^1 = \mathbb{R}^{n+1} \cong \epsilon^{n+1}$. Thus TS^n is a stably trivial bundle for all n.

3. From part b) we know that $TS^2 \oplus \epsilon^1 \cong \epsilon^{2+1}$, so $TS^2 \oplus \epsilon^1$ is trivial, so it remains to show that $TS^2 \to S^2$ is not trivial. If TS^2 was nontrivial, there would exist nowhere vanishing sections on S^2, i.e. a nowhere-vanishing vector field on S^2. However, by the Hairy Ball Theorem, such a vector field cannot exist on S^2. Therefore there is no global section on S^2, and TS^2 is nontrivial.

$\square$