

Quantum Field Theory Problems

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Question 1. Consider the space \mathcal{D} of continuously differentiable complex-valued functions f on $[0, 1]$. Consider the operator A on $L^2([0, 1])$ with domain \mathcal{D} , defined by $A(f) = if'$. Is A symmetric? What happens if one considers instead the domain $\mathcal{D}_\alpha := \{f \in \mathcal{D} : f(1) = \alpha f(0)\}$, where α is a complex number with modulus 1? Written by Prof. Sourav Chatterjee.

Proof. We want to check if $\langle A\psi | \varphi \rangle = \langle \psi | A\varphi \rangle$. This gives us $\langle i\psi' | \varphi \rangle, \langle \psi | i\varphi' \rangle$. Rewriting our bra-kets into integrals, we have $\int_0^1 (i\psi')^* \varphi dx, \int_0^1 \psi^* i\varphi' dx$. Evaluating the former, we have $\int_0^1 (i\psi')^* \varphi dx = \int_0^1 (-i)\psi'^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx \neq \int_0^1 i\psi^* \varphi' dx$. Thus, on this general a domain, A is not symmetric.

If instead our domain is \mathcal{D}_α , then, evaluating the same integral, we have $\int_0^1 (i\psi')^* \varphi dx = [-i\psi^* \varphi]_0^1 - \int_0^1 (-i)\psi^* \varphi' dx = [-i\psi^*(1)\varphi(1) + i\psi^*(0)\varphi(0)] + \int_0^1 i\psi^* \varphi' dx$. Computing the first term, we have $[-i(\alpha\psi(0))^* \alpha\varphi(0) + i\psi^*(0)\varphi(0)] = [-i\alpha^* \alpha \psi^*(0)\varphi(0) + i\psi(0)\varphi(0)] = (1 - \alpha^* \alpha) i\psi^*(0)\varphi(0)$. Since α has modulus 1, $\alpha^* \alpha = 1$, and this term becomes zero and hence $\int_0^1 (A\psi)^* \varphi dx = \int_0^1 \psi^* A\varphi$, so A becomes symmetric on this domain. \square

Question 2. Recall the definition of the manifold X_m , the measure λ_m on X_m , and the Hilbert space $\mathcal{H} = L^2(X_m, d\lambda_m)$. Recall also the operator valued distributions $a(p)$ and $a^\dagger(p)$ on the bosonic Fock space of \mathcal{H} . Finally, recall the definitions of $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$. Assuming the commutation relations for $a(p)$ and $a^\dagger(p)$ as given, prove that

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \mathbb{K} \quad (1)$$

where \mathbb{K} is the identity operator on the Fock space. Written by Prof. Sourav Chatterjee.

Proof. Integrating this operator in Schwartz space, we have $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}') [a(\mathbf{p}), a^\dagger(\mathbf{p}')]$. Since $a(\mathbf{p}) = \frac{a(p)}{\sqrt{2w_{\mathbf{p}}}}, a^\dagger(\mathbf{p}') = \frac{a^\dagger(p')}{\sqrt{2w_{\mathbf{p}'}}}$, we can conclude $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} [a(p), a^\dagger(p')]$. The first expression then becomes $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')]$. We know from the notes that $[a(p), a^\dagger(p')] = \delta(p - p') 1$. We want to integrate this on our mass shell with respect to our probability measure in order to apply our useful distribution. Since $\int_{X_m} d\lambda_m(p) f(p) = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2w_{\mathbf{p}}} f(w_{\mathbf{p}}, \mathbf{p})$, we have the equality

$$\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')] =$$

$$\int \int d\lambda_m(p) d\lambda_m(p') \sqrt{4w_{\mathbf{p}} w_{\mathbf{p}'}}} f(\mathbf{p})^* g(\mathbf{p}') [a(p), a^\dagger(p')]$$

Integrating once, we find this is equal to $\int d\lambda_m(p) \sqrt{4w_{\mathbf{p}}^2} f(\mathbf{p})^* g(\mathbf{p}) 1 = \int d\lambda_m(p) 2w_{\mathbf{p}} f(\mathbf{p})^* g(\mathbf{p}) 1$.

Going back to integrating over momentum space, we find that this is equal to $\int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}) 1$, where 1 is the identity operator on our Fock space.

Now we consider $\int \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') 1$. Integrating once, we find this gives us $\int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(\mathbf{p})^* g(\mathbf{p}) 1$, the exact result (up to a set of measure zero) as our original commutator. Thus, $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. \square

Question 3. Consider the theory for massive scalar bosons of mass m . Let φ be the free field of this theory, and let H_0 be the Hamiltonian for free evolution. Give a formal proof of the relation

$$\frac{\partial \varphi}{\partial t} = i[H_0, \varphi] \quad (2)$$

Written by Prof. Sourav Chatterjee.

Proof. Suppose we have a Schwartz function f . Then, since $H_0 = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$ and $\varphi(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}'))$, we have

$$(H_0 \varphi)(f) = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}) \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}')),$$

$$(\varphi H_0)(f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2w_{\mathbf{p}'}}} (e^{-i(x,p)} a(\mathbf{p}') + e^{i(x,p)} a^\dagger(\mathbf{p}')) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} w_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Thus we have

$$[H_0, \varphi](f) = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}} A, \text{ where}$$

$$A =$$

$$a^\dagger(\mathbf{p}) a(\mathbf{p}) e^{-i(x,p)} a(\mathbf{p}') + a^\dagger(\mathbf{p}) a(\mathbf{p}) e^{i(x,p')} a^\dagger(\mathbf{p}') - e^{-i(x,p')} a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}) - e^{i(x,p')} a^\dagger(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p})$$

Factoring out scalars, we have

$$A = e^{-i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a(\mathbf{p}')) - a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}) + e^{i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p}))$$

Because $[a(\mathbf{p}), a(\mathbf{p}')] = 0$ and $[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0$, this is equal to

$$e^{-i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}') a(\mathbf{p}) - a(\mathbf{p}') a^\dagger(\mathbf{p}) a(\mathbf{p})) + e^{i(x,p')} (a^\dagger(\mathbf{p}) a(\mathbf{p}) a^\dagger(\mathbf{p}') - a^\dagger(\mathbf{p}) a^\dagger(\mathbf{p}') a(\mathbf{p}))$$

$$= e^{-i(x,p')} [a^\dagger(\mathbf{p}), a(\mathbf{p}')] a(\mathbf{p}) + e^{i(x,p')} a^\dagger(\mathbf{p}) [a(\mathbf{p}), a^\dagger(\mathbf{p}')]]$$

We know from the previous problem that $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. Also, notice that $[A, B] = AB - BA = (-1)(BA - AB) = -[B, A]$. Thus, A becomes

$$\begin{aligned} & e^{-i(x, \mathbf{p}')} (-1) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') a(\mathbf{p}) + e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') (e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p}')} a(\mathbf{p})) \end{aligned}$$

Now, with this helpful rearrangement, we have $[H_0, \varphi](f) =$

$$\begin{aligned} & \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}'}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') (e^{i(x, \mathbf{p}')} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p}')} a(\mathbf{p})) \\ &= \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{w_{\mathbf{p}}}{\sqrt{2w_{\mathbf{p}}}} (e^{i(x, \mathbf{p})} a^\dagger(\mathbf{p}) - e^{-i(x, \mathbf{p})} a(\mathbf{p})) \end{aligned}$$

Let's take the time derivative of $\varphi(f)$ and see what we get. Notice that $(x, p) = tw_{\mathbf{p}} + \mathbf{x} \cdot \mathbf{p}$, so the time derivative of $e^{\pm i(x, p)} = \pm iw_{\mathbf{p}} e^{\pm i(x, p)}$. Thus, $\frac{\partial \varphi}{\partial t} = \int_{\mathbb{R}^{1,3}} dx^4 f(x) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{iw_{\mathbf{p}'}}{\sqrt{2w_{\mathbf{p}'}}} (-e^{-i(x, p)} a(\mathbf{p}') + e^{i(x, p)} a^\dagger(\mathbf{p}'))$. This is simply i times the previous expression we derived from the commutator.

Thus, $\frac{\partial \varphi}{\partial t} = i[H_0, \varphi]$, up to a set of measure zero. \square

Question 4. In φ^4 field theory, compute the first order term in the perturbative expansion of the scattering amplitude

$$\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle \quad (3)$$

Written by Prof. Sourav Chatterjee.

Proof. In a first order Dyson series expansion of S gives us $1 - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x : \varphi(x)^4 : + \mathcal{O}(g^2)$. We then have

$$\begin{aligned} \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle &= \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 : \varphi(x)^4 : | \mathbf{p}_1 \rangle + \mathcal{O}(g^2) \\ &= \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle - \frac{ig}{4!} \int_{\mathbb{R}} d^4 x \langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) : \varphi(x)^4 : a^\dagger(\mathbf{p}_1) | 0 \rangle + \mathcal{O}(g^2) \end{aligned}$$

For the first term, we notice that $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle = \langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) a^\dagger(\mathbf{p}_1) | 0 \rangle$. Applying the first two operators we get either ground state back if $\mathbf{p}_1 = \mathbf{p}_4$ or 0 if not. Annihilating the ground state with the third operator, we get 0, so in both cases $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1 \rangle = 0$. Focusing on the integrand, we recall the following useful rules: $\langle 0 | a(\mathbf{p}) \varphi(x) | 0 \rangle = \frac{e^{i(x, \mathbf{p})}}{\sqrt{2w_{\mathbf{p}}}}$, $\langle 0 | \varphi(x) a^\dagger(\mathbf{p}) | 0 \rangle = \frac{e^{-i(x, \mathbf{p})}}{\sqrt{2w_{\mathbf{p}}}}$.

$$\langle 0 | a(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) : \varphi(x)^4 : a^\dagger(\mathbf{p}_1) | 0 \rangle = \langle 0 | a(\mathbf{p}_2) \varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p}_3) \varphi(x) | 0 \rangle \langle 0 | a(\mathbf{p}_4) \varphi(x) | 0 \rangle \langle 0 | a^\dagger(\mathbf{p}_1) \varphi(x) | 0 \rangle.$$

This expression is equal to $(e^{i(x, \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1)}) / (\sqrt{16w_{\mathbf{p}_2} w_{\mathbf{p}_3} w_{\mathbf{p}_4} w_{\mathbf{p}_1}})$ for each suitable contraction diagram. Since the scattering involves 1 incoming particle and three outgoing particles, we want

to consider all contraction diagrams of the "four all connected to the center $\varphi(x)$ operator"-shape. The $\varphi(x)^4$ operator has 4 tails, to which the incoming and outgoing particles get connected. Since there are 8 operators, there are $(8 - 1)!!$ diagrams, and $4!$ diagrams of this type. Thus we have $4! (e^{i(x, p_2 + p_3 + p_4 - p_1)})/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}})$ terms. Sticking these back into our integral and integrating, we get $(-\frac{ig}{4!}(4!)(2\pi)^4\delta^{(4)}(p_2 + p_3 + p_4 - p_1))/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}})$. Thus we have $\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1 \rangle = (-ig(2\pi)^4\delta^{(4)}(p_2 + p_3 + p_4 - p_1))/(\sqrt{16w_{\mathbf{p}_2}w_{\mathbf{p}_3}w_{\mathbf{p}_4}w_{\mathbf{p}_1}}) + \mathcal{O}(g^2)$. \square

Question 5. *Derive Maxwell's equations as the Euler-Lagrange equations of the action*

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (4)$$

treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$. Construct the energy-momentum tensor for this theory. Peskin & Schroeder 2.1.

Proof. First we calculate $F^{\mu\nu}$. Given our identification with E^i and B^i , we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (5)$$

We notice that

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} F^{\alpha\beta} = (\delta_\alpha^\gamma \delta_\beta^\lambda - \delta_\alpha^\lambda \delta_\beta^\gamma) F^{\alpha\beta} = F^{\gamma\lambda} - F^{\lambda\gamma}, \quad (6)$$

with

$$F^{\lambda\gamma} = (\partial_\lambda A_\gamma - \partial_\gamma A_\lambda) = -(\partial_\gamma A_\lambda - \partial_\lambda A_\gamma) = -F^{\gamma\lambda} \quad (7)$$

Similarly, we have

$$F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} = F_{\alpha\beta} (\delta^{\gamma\alpha} \delta^{\lambda\beta} - \delta^{\lambda\alpha} \delta^{\gamma\beta}) = F^{\gamma\lambda} - F^{\lambda\gamma} = 2F^{\gamma\lambda} \Rightarrow \quad (8)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\gamma A_\lambda)} = \frac{\partial F_{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} F^{\alpha\beta} + F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial(\partial_\gamma A_\lambda)} = (-\frac{1}{4}) 4F^{\gamma\lambda} \quad (9)$$

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} = 0 \quad (10)$$

Thus, by the Euler-Lagrange Equations, we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{1}{4} \partial_\mu F^{\mu\nu} = 0 \quad (11)$$

Writing out the sum over μ and substituting $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$, we get

$$-\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} \quad (12)$$

$$= -\frac{\partial \mathbf{E}}{\partial t} - \partial_1 \epsilon^{1jk} B^k - \partial_2 \epsilon^{2jk} B^k - \partial_3 \epsilon^{3jk} B^k \quad (13)$$

$$= -\frac{\partial \mathbf{E}}{\partial t} + \partial_1 \epsilon^{j1k} B^k + \partial_2 \epsilon^{j2k} B^k + \partial_3 \epsilon^{j3k} B^k \quad (14)$$

$$= -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = 0 \quad (15)$$

Maxwell's Equations are

$$\mu_0 \mathbf{J} = -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} \quad (16)$$

For a system with no sources, $\mathbf{J} = 0$, so this comes down to the Euler-Lagrange equations found above.

In constructing the energy-momentum tensor, we examine the usual formula in Peskin & Schroeder:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\gamma)} \partial^\nu A_\gamma - \mathcal{L} \delta^{\mu\nu} \quad (17)$$

$$= -F^{\mu\gamma} \partial^\nu A_\gamma + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\nu} \quad (18)$$

where $g^{\alpha\beta}$ is our Minkowski metric. We see that, from the above assertion that $F^{\mu\nu} = -F^{\nu\mu}$, so

this is not a symmetric tensor. Thus we add $\partial_\lambda(F^{\mu\lambda}A^\nu)$ to $T^{\mu\nu}$. This term is antisymmetric in its first two indices, and thus is divergenceless, so adding this gives us the same globally conserved momentum and energy. Writing this out, we have

$$\widehat{T^{\mu\nu}} := T^{\mu\nu} + \frac{1}{4}\partial_\lambda(F^{\mu\lambda}A^\nu) \quad (19)$$

$$= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (\partial_\lambda F^{\mu\lambda})A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \quad (20)$$

$$= F^{\mu\iota}(\partial_\iota A^\nu - \partial^\nu A_\iota) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (-\partial_\lambda F^{\lambda\mu})A^\nu \quad (21)$$

$$= F^{\mu\iota}F_\iota^\nu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\mu\nu} + (-0)A^\nu \quad (22)$$

This is obviously symmetric in indices, and so is now a viable energy-momentum tensor. We now check $\widehat{T^{00}}$ and $\widehat{T^{0i}}$:

$$\widehat{T^{00}} = F^{0\iota}F_\iota^0 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (23)$$

$$= E^\iota E_\iota + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (24)$$

Where, I assume, the second product is the Frobenius inner product of this tensor. In examining this inner product, we have

$$\langle , \rangle = \text{tr}(\overline{F^{\mu\nu}}F^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (25)$$

$$= \text{tr} \begin{pmatrix} -E^2 & & & \\ & -E_x^2 + B_z^2 + B_y^2 & & \\ & & -E_y^2 + B_z^2 + B_x^2 & \\ & & & -E_z^2 + B_x^2 + B_y^2 \end{pmatrix} \quad (26)$$

This is equal to $2(B^2 - E^2)$. Thus, we have that $\widehat{T^{\mu\nu}} = E^2 + \frac{1}{4}2(B^2 - E^2) = \frac{1}{2}(E^2 + B^2)$.

For $\widehat{T^{0i}}$, we have

$$\widehat{T^{0i}} = F^{0j} F_j^i + \frac{1}{2} (B^2 - E^2) g^{0i} \quad (27)$$

$$= E^j \epsilon_{jik} B^k g^{mi} + \frac{1}{2} (B^2 - E^2) g^{0i} \quad (28)$$

$$= \mathbf{E} \times \mathbf{B} = \mathbf{S} \quad (29)$$

□

Question 6. Consider the the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \quad (30)$$

(a) Find the conjugate momenta to $\phi(x)$, $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) \quad (31)$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m .

(c) Rewrite the conserved charge

$$Q = \int d^3x \frac{i}{2} (\phi^* \pi^* - \pi \phi) \quad (32)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

Peskin & Schroeder, 2.2.

Proof. (a) We have that $p(\mathbf{x}) = \frac{\partial L}{\partial \dot{\phi}(\mathbf{x})} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int d^4x (\partial_\mu \phi^* g_\nu^\mu \partial^\nu \phi -$

$$m^2\phi^*\phi = \frac{\partial}{\partial\dot{\phi}(\mathbf{x})} \int d^4x \left(\frac{\partial\phi^*}{\partial t} \frac{\partial\phi}{\partial t} - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi \right) = \frac{\partial}{\partial\dot{\phi}(\mathbf{x})} \int d^4x (\dot{\phi}^*\dot{\phi} - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi).$$

Thus, $\pi = \dot{\phi}^*$. Similarly, $\pi^* = \dot{\phi}$. Since ϕ, ϕ^* are the dynamical variables, the canonical commutation relations are

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (33)$$

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0 \quad (34)$$

from quantization of the Klein-Gordon field given in the textbook. Given the equation for the Hamiltonian, we have

$$H = \int d^3x \left[\sum_{a,b} \pi_a(\mathbf{x}) \dot{\phi}_b(\mathbf{x}) - \mathcal{L} \right] \quad (35)$$

$$= \int d^3x [\pi^* \dot{\phi}^* + \pi \dot{\phi} - \mathcal{L}] \quad (36)$$

$$= \int d^3x [\dot{\phi} \dot{\phi}^* + \dot{\phi}^* \dot{\phi} - \mathcal{L}] \quad (37)$$

$$= \int d^3x [2\dot{\phi} \dot{\phi}^* - \dot{\phi} \dot{\phi}^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \quad (38)$$

$$= \int d^3x [\dot{\phi} \dot{\phi}^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \quad (39)$$

$$= \int d^3x [\pi^* \pi + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi] \quad (40)$$

We want to compute $i\frac{\partial\phi}{\partial t}$ via the Heisenberg Equation of Motion, so we calculate $[\phi, H]$.

$$i\frac{\partial\phi}{\partial t} = [\phi, H] \quad (41)$$

$$= [\phi(x'), \int d^3x (\pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \quad (42)$$

$$= \int d^3x [\phi(x'), \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \quad (43)$$

$$= \int d^3x ([\phi(x'), \pi^* \pi] + [\phi, \nabla\phi^* \cdot \nabla\phi] + m^2[\phi, \phi^*\phi]) \quad (44)$$

$$= \int d^3x \delta^{(3)}(x' - x) i\pi^*(x) \quad (45)$$

$$= i\pi^*(x) \quad (46)$$

$$i\frac{\partial\pi^*}{\partial t} = [\pi^*, H] \quad (47)$$

$$= [\pi^*(x'), \int d^3x (\pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)] \quad (48)$$

$$= \int d^3x [\pi^*(x'), \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \quad (49)$$

$$(\text{integrating by parts}) = \int d^3x ([\pi^*(x'), \pi^* \pi] + [\pi^*(x'), \phi^*(-\nabla^2 + m^2)\phi]) \quad (50)$$

$$= \int d^3x \delta^{(3)}(x' - x) (-i)(-\nabla^2 + m^2)\phi(x) \quad (51)$$

$$= i(\nabla^2 - m^2)\phi \quad (52)$$

Since $i\frac{\partial\phi}{\partial t} = i\pi^*$ and $i\frac{\partial\pi^*}{\partial t} = i(\nabla^2 - m^2)\phi$, so $\frac{\partial^2\phi}{\partial t^2} = (\nabla^2 - m^2)\phi$, which is the Klein-Gordon equation.

(b) Since ϕ satisfies the Klein-Gordon equation, and, in the same way, so does ϕ^* , we take the Fourier transform to gain more insight into $\nabla^2\phi$:

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}) \Rightarrow \quad (53)$$

$$[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi(\mathbf{p}) = 0, \quad [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\phi^*(\mathbf{p}) = 0 \quad (54)$$

We write ϕ in terms of two real valued scalar free fields ψ_1, ψ_2 , of which we already know the theory:

$$\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \quad \phi^* = \frac{\psi_1 - i\psi_2}{\sqrt{2}} \quad (55)$$

Since ψ_1, ψ_2 are independent free fields, both must satisfy the harmonic oscillator equation:

$$\begin{aligned} \frac{1}{\sqrt{2}}[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 &= 0, \quad \frac{\pm i}{\sqrt{2}}[\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow \\ [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_1 &= 0, \quad [\frac{\partial^2}{\partial t^2} + (p^2 + m^2)]\psi_2 = 0 \Rightarrow \\ \omega_1 &= \sqrt{p_1^2 + m^2}, \quad \omega_2 = \sqrt{p_2^2 + m^2} \end{aligned}$$

Since the frequencies of the oscillators have independent momentums and ϕ is not hermitian, we create two different creation and annihilation operators:

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \frac{i}{\sqrt{2\omega_i}}p_i, \quad a_i^\dagger = \sqrt{\frac{\omega_i}{2}}q_i - \frac{i}{\sqrt{2\omega_i}}p_i, \quad i \in \{1, 2\} \quad (56)$$

where $q_1 = \phi, q_2 = \phi^*, p_1 = \pi, p_2 = \pi^*$, with the notation in the spirit of Peskin and Schroeder. These creation operators, given their frequencies, represent creating two different particles with mass m . From the theory of a real-valued scalar free field, we know that

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (57)$$

$$\phi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \quad (58)$$

The two different operators ensure that ϕ is not hermitian. From above we know that $\pi = \dot{\phi}^*, \pi^* = \dot{\phi}$, and, using our real-valued scalar free field as reference, we have

$$\pi = \int \frac{d^3\mathbf{p}}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}) \quad (59)$$

$$\pi^* = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_1(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - a_2^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (60)$$

Finally, we rewrite our Hamiltonian in terms of our operators:

$$H = \int d^3\mathbf{x} (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) \quad (61)$$

$$= \int d^3\mathbf{x} \left(\int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}') e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} - a_1(\mathbf{p}) a_2(\mathbf{p}') e^{i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} \right. \quad (62)$$

$$\left. - a_2^\dagger(\mathbf{p}) a_1^\dagger(\mathbf{p}') e^{-i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}') e^{i(\mathbf{p}'-\mathbf{p}) \cdot \mathbf{x}} \right\} \quad (63)$$

$$+ \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} [-i\mathbf{p} a_1^\dagger(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}} + i\mathbf{p} a_2(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}}] \quad (64)$$

$$\cdot [i\mathbf{p}' a_1(\mathbf{p}') e^{i\mathbf{p}' \cdot \mathbf{x}} - i\mathbf{p}' a_2^\dagger(\mathbf{p}') e^{-i\mathbf{p}' \cdot \mathbf{x}}] \quad (65)$$

$$+ m^2 \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{p}'}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}') e^{i(\mathbf{p}'-\mathbf{p}) \cdot \mathbf{x}} + a_1^\dagger(\mathbf{p}) a_2^\dagger(\mathbf{p}') e^{-i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} \quad (66)$$

$$+ a_2(\mathbf{p}) a_1(\mathbf{p}') e^{i(\mathbf{p}+\mathbf{p}') \cdot \mathbf{x}} + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}') e^{i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}} \} \quad (67)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p}) a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p}) a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right. \quad (68)$$

$$\left. + \frac{p^2}{2\omega_{\mathbf{p}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}) \} \right) \quad (69)$$

$$+ \frac{m^2}{2\omega_{\mathbf{p}}} \{ a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2(\mathbf{p}) a_2^\dagger(\mathbf{p}) \} \quad (70)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{\omega_{\mathbf{p}}}{2} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) - a_1(\mathbf{p}) a_2(-\mathbf{p}) - a_2^\dagger(\mathbf{p}) a_1^\dagger(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right. \quad (71)$$

$$\left. + \frac{p^2 + m^2}{2\omega_{\mathbf{p}}} \{ a_1(\mathbf{p}) a_1^\dagger(\mathbf{p}) + a_1^\dagger(\mathbf{p}) a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p}) a_1(-\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \} \right] \quad (72)$$

where in (70) the middle two terms have a positive sign because we subtract $\mathbf{p} \cdot \mathbf{p}' = \mathbf{p} \cdot (-\mathbf{p})$.

Furthermore, since $\mathbf{p} \neq -\mathbf{p}$, we can commute our operators. Thus we have the Hamiltonian as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \omega_{\mathbf{p}} \{ a_1 a_1^\dagger + a_2 a_2^\dagger \} \quad (73)$$

Since this Hamiltonian is constructed purely out of constants and operators whose eigenvectors are momentum eigenstates, our Hamiltonian is now diagonalized. The indices 1 and 2 represent the two particles of mass m .

(c) This is just plugging in our values for momentum and position and integrating, like the

previous problem. We have

$$Q = \int d^3\mathbf{x} \frac{i}{2} (\phi^* \pi^* - \pi \phi) \quad (74)$$

$$= \int d^3\mathbf{x} \frac{i}{2} \int d^3\mathbf{p} \int d^3\mathbf{p}' \frac{1}{(2\pi)^6} \frac{1}{2} [(-i)(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} - a_2(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}}) \quad (75)$$

$$- i(a_1^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} - a_2(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}})(a_1(\mathbf{p}')e^{i\mathbf{p}'\cdot\mathbf{x}} + a_2(\mathbf{p}')e^{-i\mathbf{p}'\cdot\mathbf{x}})] \quad (76)$$

$$= \int d^3\mathbf{p} \frac{1}{(2\pi)^3} \frac{1}{4} ([a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) + a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \quad (77)$$

$$+ [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) + a_1^\dagger(\mathbf{p})a_2^\dagger(-\mathbf{p}) - a_2(\mathbf{p})a_1(-\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})]) \quad (78)$$

$$= \frac{1}{2} \int d^3\mathbf{p} \frac{1}{(2\pi)^3} [a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2(\mathbf{p})a_2^\dagger(\mathbf{p})] \quad (79)$$

□