Contents

Useful Mathematical Preliminary Objects (that I have difficulty remembering)

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1 Algebra & Vector Spaces

A *group* G is a set closed under an operation $\star$ that is associative $(g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3)$, contains an identity e such that $e \star g = g \star e = g \forall g \in G$, and every element has an inverse such that $g \star g^{-1} = g^{-1} \star g = e$. A group is *abelian* if $g_1 \star g_2 = g_2 \star g_1$

A *ring* is a set closed under two operations $+, \times$ that is an abelian group under $+$, and contains an identity 1_R for the operation $\times$. $\times$ is distributive and associative.

A *field* is a ring where every element except maybe the $+$ identity has a multiplicative inverse sending it to the multiplicative identity. This forms a group structure for the elements except for maybe the additive identity. This group, called the multiplicative group, is also abelian.

A *module* M is an abelian group (operation denoted $+$) with a ring R such that, for all $r, s \in R, x, y \in M$, we have

$$r(x + y) = rx + ry \tag{1}$$

$$(r + s)x = rx + sx \tag{2}$$

$$(rs)x = r(sx) \tag{3}$$

$$1_R x = x \tag{4}$$

This defines scalar multiplication.

A *vector space* is a module where R is a field.

An *algebra* A is a vector space with a binary operation $\cdot : A \times A \to A$ such that, for all

$x, y, z \in K, r, s \in R,$

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z) \tag{5}$$

$$x \cdot (y + z) = z \cdot y + x \cdot z \tag{6}$$

$$rx \cdot sy = (rs)x \cdot y \tag{7}$$

(These axioms define bilinearity)

2 Manifolds

A *topological space* is an ordered pair (X, τ) where X is a set an τ is a set of subsets of X such that:

The empty set and X belong to τ,

An arbitrary, finite or infinite union of elements of τ is in τ,

The intersection of any finite number of elements of τ is in τ.

τ is a topology on X, and defining a topology allows one to define continuity, connectedness, and convergence.

A *topological base (basis* B of a topological space X is a set of open subsets of X such that every open subset of X can be written as a union of elements in B. We say the base generates the topology, which makes sense, as the elements in τ are each a union of elements of B. For this to be well-used,

The base elements must cover X,

Let $B_1, B_2 \in B$ have $B_1 \cap B_2 := I$. For each $x \in I$, there is a $B' \in B$ such that $x \in B' \subseteq I$

Remark 1. *A second-countable space is a space with a countable base. A compact, metrizable space is necessarily second-countable. (Throwback to proving an uncountable collection of 1-simplices is not metrizable.)*

A homeomorphism is a map between topological spaces that is an injection, is continuous, and has a continuous inverse map.

A *manifold* is a topological space such that every point $p \in M$ has a neighborhood homeomorphic to Euclidean space of the same dimension.

A *tangent space* is a vector space at a point of a manifold that consists of vectors tangent to that point. The tangent space of a sphere is a cylinder with the same radius as the sphere.

A *chart* is such a homeomorphism.

An *atlas* is a collection of charts such that the preimage of every chart in the atlas covers the manifold.

A *transition map* is a map that transitions the image of the intersection of the preimage of multiple charts from the image of the one to the another.

A *Lie Group* is a group that is also a differentiable manifold. It provides a way to classify continuous symmetries (e.g. the rotation matrices in a dimension are a group and a differentiable manifold; one can smoothly rotate a sphere).

3 Differential Forms from a Field-Theoretic Perspective

3.1 Differential 1-forms on $\mathbb{R}^2$

A *vector field* is an association of each point of whatever space we're working in of a vector there. A *covector* field is an association of each point with a linear function from vectors there to $\mathbb{R}$.

For example, let us work in $\mathbb{R}^2$. Standard coordinates give us 2 "basic" covector fields on $\mathbb{R}^2$ called dx and dy. The first is a covector field which sends a vector to its x-coordinate, and the second is a covector field which sends a vector to its y-coordinate. Every covector ω on $\mathbb{R}^2$ can be written as $\omega = \alpha dx + \beta dy$ for unique functions α, β on $\mathbb{R}^2$.

The operator d is defined as follows:

Given a function f on $\mathbb{R}^2$, we define

$$df := \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The covectors in this covector field are *differential 1-forms* on $\mathbb{R}^2$.

3.2 Differential 2-forms on $\mathbb{R}^2$

It is easy to see that, given a 1-form $\omega = \alpha dx + \beta dy$ on $\mathbb{R}^2$, that if $\omega = df$ for some f on $\mathbb{R}^2$, then

$$\frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y} \tag{8}$$

Let V be a vector space, and V^* be its dual space (sends elements of V to the field of coefficients). If V is 2-dimensional, then V^* is 2-dimensional, and so 2-covectors $\wedge^2 V^*$ is $\binom{2}{2}$=1-dimensional. If dx, dy is a basis for V^*, then $dx \wedge dy$ is *the* basis for $\wedge^2 V^*$. A *2-covector field* is an association for every point a 2-covector. We define the map d as

$$d(\alpha dx + \beta dy) = (\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}) dx \wedge dy \tag{9}$$

Example 1. $\int_{C_1} \omega \stackrel{?}{=} \int_{C_1} df = \int_a^b \frac{df}{dt} dt = f(b) - f(a)$

If our 1-form in this example is in fact a df of some f (?), with f nice and continuous, then df is *exact.*

If $dw = 0$ for some $k-$form, then w is *closed.* As you can see, exactness implies closedness due to $d^2 = 0$, but not necessarily the converse. Thus a exact form is the image of d, and a closed form is the image of d, further hinting at d begin the chain map in De Rham cohomology.

Closedness implies exactness on a contractible domain via the Poincare Lemma.

For those with experience in differential topology, a differential 2-form ω on a manifold M gives, for each $p \in M$, a bilinear form

$$\omega_p : T_pM \times T_pM \to \mathbb{R} \tag{10}$$

A bilinear form on a vector space V is nondegenerate if, for all $v \in V$, $\langle w, v \rangle = 0$ implies $w = 0$. A two form is nondegenerate if w_p is nondegenerate for all $p \in M$.

3.3 Wedge Product

Recall the rules for wedge product for one forms:

$$\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \tag{skew-symmetric}$$

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \tag{associativity}$$

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \wedge \mathbf{v} = c_1\mathbf{u}_1 \wedge \mathbf{v} + c_2\mathbf{u}_2 \wedge \mathbf{v} \tag{bilinearity}$$

$$\mathbf{u} \wedge (c_1\mathbf{v}_1 \wedge c_2\mathbf{v}_2) = c_1\mathbf{u} \wedge \mathbf{v}_1 + c_2\mathbf{u} \wedge \mathbf{v}_2 \tag{bilinearity}$$

$$\mathbf{u} \wedge \mathbf{u} = 0 \tag{*only for 1-forms}$$

We can construct k-forms by wedging a $(k-n)$-form with a n-form. The d operation sends k-forms to $(k+1)$-forms.

Example 2. $d(Fdy \wedge dz + Gdz \wedge dx + Hdx \wedge dy) =$
$(F_x dx + F_y dy + F_z dz) \wedge dy \wedge dz + (G_x dx + G_y dy + G_z dz) \wedge dz \wedge dx + (H_x dx + H_y dy + H_z dz) \wedge dx \wedge dy$
$= (F_x + G_y + H_z) dx \wedge dy \wedge dz$

It is easy to show $d^2 = 0$ (Hinting that k-forms may have a cohomological structure, perhaps named after De Rham)

A k-form is meant to be integrated over a k-manifold.

3.4 Symplectic Manifold

A *symplectic manifold* is a manifold with a closed, nondegenerate 2-form ω called the symplectic form. These show up in cotangent bundles of manifold. For a system modeled as a manifold, the cotangent bundle describes the phase space (space of all possible configurations of the system, e.g. Hilbert space) of the system.

Any real-valued differentiable function H on a symplectic manifold can be an energy function i.e Hamiltonian. Associated to any Hamiltonian is a Hamiltonian vector field, the integral curves of which (curves sketched along the vector field from the differential equation) is a solution to Hamilton's equations.

A *Hamiltonian flow* or *symplectomorphism* is the flow of this field on the symplectic manifold.

3.5 Hodge Star

The *Hodge Star* sends k-forms to $(n-k)$-forms in an n-dimensional manifold. (It maps k-dimensional vectors to $(n-k)$-dimensional vectors in an n-dimensional vector space.)

Example 3. *In a 3-dimensional Euclidean space, we can associate to every vector a plane orthogonal to that vector, and every plane an oriented normal vector.*

$$\mathbf{u} \wedge *\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}, \dim \mathbf{u} = \dim \mathbf{v} = k < n, \dim \mathbf{w} = n \tag{11}$$

where n is the dimension of our vector space.

A Field in terms of Differential Forms

The Electromagnetic Field F is given by the 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dx + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt \tag{12}$$

Computing dF gives us

$$dF = (\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z})[dx \wedge dy \wedge dz] + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t})[dx \wedge dy \wedge dt] + ... \tag{13}$$

Setting $dF = 0$, we find the first two Maxwell's Equations $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. For the other two Maxwell's Equations, we use $d * F = 4\pi\rho$:

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt \tag{14}$$

with

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt \tag{15}$$

where the metric used in the hodge star is the Lorentz metric.

3.6 Connections and Curvature (an actually intuitive approach)

Most textbooks introduce the notion of curvature and then admit that there's no intuition behind this definition, said definition being $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_Y - \nabla_{[X,Y]}$. This is grossly unintuitive and only pertains to the Levi-Civita connection. However, Hori et. al. introduce this concept in clear simplicity, which I'll repeat here.

We want to be able to differentiate vectors. It's tempting to write

$$\lim_{\epsilon \to 0} \frac{v(x+\epsilon) - v(x)}{\epsilon} \tag{16}$$

but this definition makes no sense. "$+\epsilon$" isn't defined on a manifold, and we can't subtract vectors living in different vector spaces. (this is why we need a way to *connect* these vector space fibers). In resolving the first issue, we'll choose to differentiate in the i^{th} direction: denote the point whose i^{th} coordinate has been advanced by ϵ by $x + \epsilon\partial_i$. In resolving the second issue, we'll need an i-dependent automorphism. Since

ϵ is small, we need our automorphism to be close to the identity; write it as $\mathbf{1} + \epsilon A_i$, for A_i an arbitrary endomorphism. Thus we have

$$D_i v = \frac{(\mathbf{1} + \epsilon A_i)(v(x + \epsilon \partial_i)) - v(x)}{\epsilon} \tag{17}$$

For $v := v^a \partial_a$, we get

$$v^a(x + \epsilon \partial_i) = v^a(x) + \epsilon \partial_i v^a(x) \Rightarrow \tag{18}$$

$$(D_i v)^a = \partial_i v^a + (A_i)^a_b v^b \tag{19}$$

Thus we have D an operator that sends vectors to vectors $v \mapsto D_i v$, and the vector w sens $v \mapsto D_w v = w^i D_i v = \langle Dv, w \rangle$, where we define the vector-valued one-form $Dv = (D_i v) dx^i$, so $D = d + A$ is our **connection**. This is why the covariant derivative along a vector field V is not $\frac{dV}{dt}(t)$, but $\frac{D}{dt}V(t)$, because the former vector doesn't belong to the tangent plane of the curve, i.e. the first issue with our first guess.

The curvature is intuitively the acceleration of a curve, or the concavity, etc. Either way, the data is encoded in a second derivative of sorts. We recast R as D^2, where $D = d + A$, for A our one-form. Thus $R = dA + A \wedge A$.

3.7 Morse Theory

Consider M a smooth manifold and $f : M \to \mathbb{R}$ a smooth function with nondegenerate critical points (the Hessian of f at these points is nonsingular). If no critical values of f occur between the numbers a and b, for $a < b$, then the subspace on which f takes values less than a is a deformation retract of the subspace where f is less than b; simply define a metric and flow the manifold via the vector field $-\nabla f / |\nabla f|^2$ for time $b - a$.

3.8 Chern Classes

4 Algebraic Geometry Preliminaries

4.1 Projective Spaces

In solving roots of polynomials, one wants to make a couple changes: work over $\mathbb{C}$, and work in projective space, i.e. work in $\mathbb{CP}^n$. This is because $\mathbb{C}$ is algebraically closed; If we have a family of polynomial equations, the number of solutions will remain constant as we change the coefficients. Furthermore, in projective space, one adds a 'line at infinity.' Two parallel lines don't intersect on the real plane, but they intersect at a point on the line at infinity.

Example 4. *Three conics; Over the reals, the parabola's infinitely-extending sides become more and more parallel to each other, and thus intersect the line at infinity at one point. Since the asymptotics approach being parallel, the parabola is tangent to the line at infinity; it intersects the line at infinity at two points* *infinitesimally close together. The hyperbola intersects the line at infinity twice; these two points are determined by the slope of the asymptotics. The ellipse doesn't intersect the line at infinity.*

One can think about $\mathbb{CP}^n$ as a quotient space, or the space of $\mathbb{C}^\times$ orbits. It is also a smooth $2n-$manifold, as can be easily proven via the charts $[z_0, ..., z_n] \mapsto (z_0/z_i, ..., z_n/z_i)$ with domains $[z_0, ..., z_n]|z_i \neq 0$.

Example 5. *For a degree 3 polynomial in $\mathbb{C}p^n$ given by $a_1X^3+a_2Y^3+a_3Z^3+a_4XYZ+...+a_{10}YZ^2$. There are 10 parameters determining the roots of this polynomial. 8 of these can be eliminated by a homogeneous, linear change of variables (something like $X \mapsto XZ + Y^2$ or whatever). Then we can remove one of the remaining 2 variables because scaling doesn't matter. Thus there's one complex parameter that can't be removed, and determines the structure of the complex structure of this curve. This is an elliptic curve, which is a Riemann surface of genus 1.*

One can describe **weighted projective spaces** via different torus actions. Before we defined $\mathbb{CP}^n$ by the quotient relation $[z_0, ..., z_n] \sim [\omega z_0, ..., \omega z_n]$, but we could define a different torus action $[z_0, ..., z_n] \mapsto [\lambda^{\omega_0} z_0, ..., \lambda^{\omega_n} z_n]$. Here the $\mathbb{C}^\times$ action isn't free (there can exist fixed points).

4.2 Toric Varieties

4.2.1 Cones

For a finite set of vectors $\{v_i\}$ in $\mathbb{R}^n$, the set

$$\sigma = \{x \in \mathbb{R}^n | x = \lambda_1 v_1 + ... + \lambda_n v_r, 0 \leq \lambda_i \in \mathbb{R}, \} \tag{20}$$

is a polyhedral cone with generators $\{v_i\}$.

Example 6. *In $\mathbb{R}^2$ with canonical basis vectors e_1, e_2 as the generators, the first quadrant is a cone.*

4.2.2 Dual Cone

The **dual cone** is defined by the set

$$\overline{\sigma} = \{u \in (\mathbb{R}^n)^* | u(v) \geq 0, \forall v \in \sigma\} \tag{21}$$

Example 7. *For $\sigma = e_2$, the dual cone is the upper half-plane.*

To construct a **toric variety**, we start with $\mathbb{C}^N$ and an action by an algebraic torus $(\mathbb{C}^*)^m, m < N$. We identify and then subtract a subset U that is fixed by a continuous subgroup of $(\mathbb{C}^*)^m$. The variety is then

$$\mathbb{P} = (\mathbb{C}^N \backslash U)/(\mathbb{C}^*)^m \tag{22}$$

This still has an algebraic torus action by the group $(\mathbb{C}^*)^{N-m}$ descending from the natural $(\mathbb{C}^*)^N$ action on $\mathbb{C}^N$.

4.3 Sheaves

4.3.1 Motivating Example

Suppose we have a topological manifold X. We wish to think about differentiable functions on X. In order to be well-defined, we need to consider all differentiable functions on all open subsets on X. On each open set $U \subset X$ we have a ring of differentiable functions, denoted $\mathscr{O}(U)$. Well, what about open sets within this open set? We can restrict a differentiable function on an open set to a smaller open set, and therefore get another differentiable function. I.e., if $U \subset V$ is an inclusion of open sets, we have a restriction map

$$res_{V,U} : \mathscr{O}(V) \to \mathscr{O}(U) \tag{23}$$

What about a third open set $W \subset V$? The restriction should commute from $U \subset V$ and $W \subset U$ to $W \subset V$:

$$\begin{tikzcd} \mathscr{O}(V) \arrow[rr, "res_{W,V}"] \arrow[dr, "res_{V,U}"'] & & \mathscr{O}(W) \\ & \mathscr{O}(U) \arrow[ur, "res_{U,W}"'] & \end{tikzcd}$$

One can also get an open set from a collection of smaller open sets. So suppose we take two differentiable functions f_1, f_2 on an open set U, and let $\{U_i\}$ be an open cover of U. If our two functions agree on the open cover, they better agree on U. Thus, if $f_1, f_2 \in \mathscr{O}(U)$ and $res_{U,U_i} f_1 = res_{U,U_i} f_2$, then $f_1 = f_2$.

Furthermore, what about the opposite direction, i.e. $\{U_i\}$ to U? We need to keep track of overlaps. Thus, given $f_i \in \mathscr{O}(U_i)$, for all i, such that $res_{U_i,U_i \cap U_j} f_i = res_{U_j,U_i \cap U_j} f_j$, for all i, j, given there is some $f \in \mathscr{O}(U)$ such that $res_{U,U_i} f = f_i$, for all i. We didn't use differentiability here, so hence we generalize to sheaves.

4.3.2 Presheaves and Sheaves

A **presheaf** $\mathscr{F}$ on a topological space X with

1. To each open set $U \subset X$, we associate an object $\mathscr{F}(U)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over U**, often called sections of $\mathscr{F}$, called **global sections**.
2. For each inclusion $U \hookrightarrow V$, we have a restriction morphism $res_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$
3. $res_{U,U} = id_{\mathscr{F}(U)}$
4. If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then restriction maps commute:

$$\begin{tikzcd} \mathscr{F}(W) \arrow[rr, "res_{W,U}"] \arrow[dr, "res_{W,V}"'] & & \mathscr{F}(U) \\ & \mathscr{F}(V) \arrow[ur, "res_{V,U}"'] & \end{tikzcd}$$

A presheaf is a **sheaf** if it satisfies two more axioms, corresponding to the open cover requirements used the example:

1. (**Identity axiom**) If $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$ with $res_{U,U_i} f_1 = res_{U,U_j} f_2$ for all i, then $f_1 = f_2$.

2. (**Gluability axioim**) If $\{U_i\}_{i\in I}$ is an open cover of U, then, given $f_i \in \mathscr{F}(U_i)$ for all i such that $res_{U_i,U_i\cap U_j} f_i = res_{U_j,U_j\cap U_i} f_j$ for all i, j, then there is some $f \in \mathscr{F}(U)$ such that $res_{u,u_i} f = f_i$, for all i.

Example 8. *We let $\mathbb{Z}$ denote the sheaf of integer-values functions, with $\mathbb{Z}(U)$ the locally constant integer-valued functions on U, and $\mathbb{Z}(X)$ is the group of globally-defined integer-valued functions. This is a vector space of dimension the number of connected components of X.*

Example 9. *$\mathbb{R}$ and $\mathbb{C}$ are sheaves of real and complex constant functions.*

Example 10. *$\mathcal{O}$ is the sheaf of holomorphic functions, with $\mathcal{O}(U)$ the set of holomorphic functions, with dimension equal to the number of connected components of U's topological space. This only works if X is compact, since the only global holomorphic functions on a compact connected space are constants.*

Example 11. *$\mathcal{O}^*$ is the sheaf of nowhere-zero holomorphic functions.*

If $\mathscr{F}$ is the category of vector spaces, sheaves inherit many properties from linear algebra. If $\mathscr{F}$ is the category of abelian groups, shaves inhereit many properties from homological algebra. A map between sheaves defined maps on the corresponding abelian groups, and its kernel defined the kernel sheaf.

Example 12. *We can have exact sequences of sheaves:*

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{times\ 2\pi i} \mathcal{O}^* \to 0 \tag{24}$$

This sequence isn't exact on every open set, e.g. $\mathbb{C} - \{0\}$, but is exact for open sets small enough, e.g. with trivial cohomology.

4.3.3 Cěch Cohomology

Let $\mathscr{F}$ be the category of abelian groups. For the sheaf relative to a cover $\{U_\alpha\}$ of X, we define our cochain complexes in the following way:

$$C^0(\mathscr{F}) = \prod_\alpha \mathscr{F}(U_\alpha) \tag{25}$$

$$C^1(\mathscr{F}) = \prod_{(\alpha,\beta)} \mathscr{F}(U_\alpha \cap U_\beta) \tag{26}$$

$$\vdots \tag{27}$$

where we require total anti-symmetry with higher cochains ($\sigma_{U_\alpha,U_\beta} = -\sigma_{U_\beta,U_\alpha}$). The chain maps are given by

$$(\delta_0\sigma)_{U,V} = \sigma_V - \sigma_U \tag{28}$$

$$(\delta_1\rho)_{U,V,W} = \rho_{V,W} - \rho_{U,W} + \rho_{U,V} \tag{29}$$

$$\vdots \tag{30}$$

The cohomology groups are thus defined by

$$H^i(\mathscr{F}) = Ker\delta_i / Im\delta_{i-1} \tag{31}$$

Something special about Cěch cohomology is that an exact sequnence of sheaves

$$0 \to A \to B \to C \to 0 \tag{32}$$

induces a long exact sequence in cohomology:

$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to ... \tag{33}$$

4.4 Stalks and Germs

4.4.1 Motivating Example

The germ of a differentiable function at a point $p \in X$ is an object of the form

$$\{(f, \text{open } U) : p \in U, f \in \mathscr{O}(U)\} \tag{34}$$

modulo the relation $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ containing p where $f|_W = g|_W$. In other words, two functions that are the same in an open neighbrohood of p have the same germ, even though they may be different elsewhere. The stalk in this example is the set of germs at p, and denote it $\mathscr{O}_p$. The stalk here is a ring: a germ can be the sum of two germs, defined on the intersection of those two germs' sets.

4.4.2 Definitions

The **stalk** of a presheaf $\mathscr{F}$ at a point p is the set of **germs** of $\mathscr{F}$ at p, denoted $\mathscr{F}_p$. The germ is the same definition as above, just for any category:

$$\{(f, \text{open } U) : p \in U, f \in \mathscr{F}(U)\} \tag{35}$$

Germs correspond to sections over some open set containing p, with two sections considered the same if they agree on some smaller open set. Equivalently, a stalk is the colimit of all $\mathscr{F}(U)$ over all open sets U containing p:

$$\mathscr{F}_p = \lim_{\to} \mathscr{F}(U) \tag{36}$$

The same definition holds for sheaves as well as presheaves.

4.5 Stacks and Schemes

4.5.1 Fibered categories

A **fibered category** is a category c with a functor to a category $\mathcal{C}$ such that, for any morphism $f : X \to Y$ in $\mathcal{C}$, and any object $y \in c$ with image Y, we have a pullback $f : x \to y$ in $\mathcal{C}$ by F:

This element is called F^*y

$$\begin{tikzcd}
y \arrow[d] & x \arrow[l, dashed] & \in c \\
Y & X \arrow[l] & \in \mathcal{C}
\end{tikzcd}$$

4.5.2 Sieves and Grothendieck Topology

We want a way to systematically select an open subset of an open set that is stable under inclusion. Thus we put all open subsets under a sieve to "select" open subsets. For an object $c \in \mathcal{C}$, a **sieve** S is a subfunctor of $\mathrm{Hom}(-, c)$. Suppose $\mathcal{C}(X)$ is the category of open sets of X, with morphisms the inclusions maps of open sets. In this example, for any open subset $V \subset U$, $S(V)$ is a subset of $\mathrm{Hom}(V, U)$, which as singular element the open immersion $V \to U$. Then V is "selected" by S if $S(V)$ is nonempty. If $W \subset V$, there is a morphism $S(W) \to S(V)$ given by composition with the inclusion $W \hookrightarrow V$. If $S(V)$ is nonempty, then $S(W)$ is nonempty.

If S is a sieve on X, and there is a morphism $f : X \to Y$, there is a sieve on Y given by left composition with f called the **pullback sieve** of S along f, denoted f^*S.

The **Grothendieck topology** J on a category $\mathcal{C}$ is a collection of distinguished sieves on each object $c \in \mathcal{C}$ called **covering sheaves** of c, subject to the following axioms:

1. (Identity) $\mathrm{Hom}(-, X)$ is a covering sieve on X for any object $X \in \mathcal{C}$.
2. (Base Change) If S is a covering sieve on X, and $f : X \to Y$ is a morphism, then the pullback f^*S is a covering sieve on Y.
3. (Local Character) Let S be a covering sieve on X, and T any sieve on X. Suppose that, for each object $Y \in \mathcal{C}$ and each morphism $f : Y \to X$ in $S(Y)$, the pullback sieve f^*T is a covering sieve on Y. Then T is a covering sieve on X.

4.5.3 Prestacks and Stacks

Take a category c and a functor to a category $\mathcal{C}$. Take any $U \in \mathcal{C}$ and objects $x, y \in c$ with image U under this functor. Consider the functor G from objects over U to sets taking $F : V \to U$ to $\mathrm{Hom}(F^*x, F^*y)$. Obviously c must be a fibered category over $\mathcal{C}$. If this G functor is a sheaf, then c is a **prestack**.

Take any object $V \in \mathcal{C}$. Consider a "covering" of V given by $\{V_i\}$ with elements x_i in the fiber over V_i, and morphisms f_{ji} between the restrictions of x_i and x_j to $V_{ij} = V_i \times_V V_j$ satisfying the compatibility condition $f_{ki} = f_{kj} f_{ji}$. If each x_i are the "pullbacks" of an element x with image V, then c is a **stack**.

4.5.4 Locally Ringed Spaces

A **ringed space** is a sheaf with category the category of commutative rings, often called the **structure sheaf**. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings.

4.5.5 Affine Schemes

An **affine scheme** is a locally ringed space isomorphic to the spectrum of a commutative ring R, i.e. the set of all prime ideals of R.

4.5.6 Schemes

A **scheme** is a locally ringed space X with an open covering $\{U_i\}$ such that each U_i, as a locally ringed space, is an affine scheme. One can think of a scheme as being covered by "coordinate charts" which are affine schemes.

5 Physics

5.1 The Calculus of Variations

An *action* is given by the integral of a *Lagrangian*, and, for fields, the Lagrangian is the integral of a *Lagrangian Density.*

Let $\mathcal{V}$ be the space of admissible functions that the Lagrangian can take as input, and let $T\mathcal{V}$ be the space of admissible variations of those functions, such that, for every $y \in \mathcal{V}, y + \alpha\delta y \in \mathcal{V}$ for all $\alpha \in \mathbb{R}, \delta y \in T\mathcal{V}$. Usually these spaces vary due to boundary conditions.

One of the secrets of the universe is that nature (classically, at least) takes the path in $\mathcal{V}$ that minimizes the action.

$$I[y(\cdot)] = \int_a^b L(t, y(t), y'(t))dt \tag{37}$$

In minimizing this functional, we get a $\overline{y}(t) \in \mathcal{V}$ such that $\overline{y}(t) + \alpha\delta y(t)$ increases the value of I, for all $\alpha \in \mathbb{R}, \delta y(t) \in T\mathcal{V}$. We define the variation of I in the direction of δy by $\langle \delta I[\overline{y}], \delta y\rangle$. Setting this equal to zero, we get the *Euler-Lagrange equations*:

$$\begin{aligned} 0 = \langle \delta I[\overline{y}], \delta y\rangle &= \frac{d}{d\alpha}[\int_a^b f(t, y(\alpha, t), \frac{\partial y(\alpha, t)}{\partial t})dt]|_{\alpha=0} & (38)\\ &= [\int_a^b (\frac{\partial f}{\partial t}\frac{\partial t}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha})]_{\alpha=0} & (39)\end{aligned}$$

for $y = \overline{y} + \alpha\delta y$. Often times, to get rid of derivatives of δy, one will integrate by parts to obtain whatever Lagrangian's Euler-Lagrange equations. In obtaining these Euler-Lagrange equations, you plug $\overline{y} + \alpha\delta y$ into the Lagrangian, differentiate the Lagrangian with respect to α, and set $\alpha = 0$.

5.2 Variations

The variation follows similar rules of the total differential, except the differentials (1-forms) are replaced by the virtual displacements and their time derivatives, i.e.

$$dy(x_1, ..., x_n) = \frac{\partial y}{\partial x_1}dx_1 + ... + \frac{\partial y}{\partial x_n}dx_n \rightarrow \tag{40}$$

$$\delta L(q, \dot{q}) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{41}$$

Lemma 1. *For $d\delta A, dA \in \wedge^k$, $d\delta A \wedge *dA = dA \wedge *d\delta A$.*

Proof. $d\delta A \wedge dA = (-1)^q \langle d\delta A, dA\rangle \wedge_i^n e_i = (-1)^q \langle dA, d\delta A\rangle \wedge_i^n e_i = dA \wedge d\delta A$, where q is associated with the inner product of our vector space. $\square$

5.2.1 Lagrangian in terms of differential forms

Here we find the Euler-Lagrange equations for a Lagrangian written in differential forms, because Lagrangians written in differential forms can be daunting on first (through fourth) viewing. For $L = \int \mathcal{L}$, where

$$\mathcal{L} = -\frac{1}{2}dA \wedge *dA - A \wedge *J \tag{42}$$

We start with the usual way:

$$\begin{aligned}
0 = \frac{d}{d\alpha}L(A + \alpha\delta A)|_{\alpha=0} &= \frac{d}{d\alpha}\int [-\frac{1}{2}d(A + \alpha\delta A) \wedge *d(A + \alpha\delta A) - (A + \alpha\delta A) \wedge *J]|_{\alpha=0} && (43)\\
&= \frac{d}{d\alpha}\int [-\frac{1}{2}(dA \wedge *dA + dA \wedge *d\alpha\delta A + d\alpha\delta A \wedge *dA + d\alpha\delta A \wedge *d\alpha\delta A) && (44)\\
&\quad - A \wedge *J - \alpha\delta A \wedge *J]|_{\alpha=0} && (45)\\
&= \frac{d}{d\alpha}[L(A) + \int [-\frac{1}{2}(dA \wedge *d\alpha\delta A + d\alpha\delta A \wedge *dA + \alpha^2 d\delta A \wedge *d\alpha\delta A) && (46)\\
&\quad - \alpha\delta A \wedge *J]]|_{\alpha=0} && (47)\\
&= \int [-\frac{1}{2}(dA \wedge *d\delta A + d\delta A \wedge *dA + 2\alpha d\delta A \wedge *d\alpha\delta A) - \delta A \wedge *J]|_{\alpha=0} && (48)\\
&= \int [-\frac{1}{2}(dA \wedge *d\delta A + d\delta A \wedge *dA) - \delta A \wedge *J] && (49)\\
\text{(from Lemma 1)} \quad &= -\int [d\delta A \wedge *dA + \delta A \wedge *J] && (50)\\
\text{(integrating by parts)} \quad &= \int [-\delta A \wedge d * dA - \delta A \wedge *J] && (51)\\
&= \int [\delta A \wedge (-d * dA - *J)] && (52)
\end{aligned}$$

In order for this to be 0 for all admissible variations δA, we need

$$d * dA = - * J \tag{53}$$

The Laplacian on 1-forms is given by $\Delta = *d * dA$. Thus, taking the Hodge star operation on both sides, we have

$$\Delta A = -J \tag{54}$$

5.3 Hamiltonian Mechanics

For a given Lagrangian L the coordinates y, y' are replaced by the coordinates position and momentum (q, p) by the transformation

$$p_i = \frac{\partial L}{\partial y_i'} \tag{55}$$

This is based on the *Legendre transformation* between tangent and cotangent bundles

$$TQ \longrightarrow T^*Q$$

$$(y, y') \longrightarrow (q, p)$$

between tangent and cotangent bundles.

The *Hamiltonian function H on phase space T^*Q* is given by

$$H(p, q, t) := py' - L(y, y', t), p = \frac{\partial L}{\partial y'} \tag{56}$$

$$y' = \frac{\partial H}{\partial p}, p' = -\frac{\partial H}{\partial y} \tag{57}$$

From the symplectic viewpoint, we can say that there exists a 2-form and inner product i such that, for any vector field X, the 2-form ω yields a 1-form $i(X)\omega$. Hamilton's equations are then equivalent to

$$i(X_H)\omega = dH \tag{58}$$

5.4 Poisson Bracket

The *Poisson Bracket* is given by

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \text{ for } f, g \in \mathcal{F}(T^*Q) \tag{59}$$

Here Hamilton's equations can be written as

$$q' = \{q, H\}, p' = \{p, H\} \tag{60}$$

Generally, for the time development of an observable given by f, the above system must satisfy the condition

$$\dot{f} = \{f, H\} \tag{61}$$

5.4.1 Quantization

The transition should give, for the Hamilton function H and the classical observables f, promote these to self-adjoint operators $\hat{H}, \hat{f}$ in a complex Hilbert space $\mathcal{H}$. The time course should shift to the quantum case

$$\dot{\hat{f}} = c[\hat{f}, \hat{H}], c = -\frac{ih}{2\pi} \tag{62}$$

where [,] is the natural *Lie bracket* given by $[A, B] = AB - BA$.