Algebraic Topology Refresher Problems

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Note: These are the only problems I had saved before I decided to make a page to preserve this stuff. Because there is so much machinery in algebraic topology, I have saved some of these problems as a refresher for whenever I return to it, as said machinery is very forgettable, beautiful though it may be. Also, these problems were done open-book and open-past-homework with Hatcher's "Algebraic Topology" textbook, so there may be some steps I skipped over.

Denotation: $\mathbb{R}P^n$ is the space of all lines through the origin in $\mathbb{R}^n$, S^n is the $n-$dimensional sphere, and D^n is the $n-$dimensional disc.

All problems were written by Professor Ralph Cohen, saxophone extraordinaire.

Question 1. *Is every covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$ isomorphic to a product of covering spaces $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to \mathbb{R}P^2 \times \mathbb{R}P^3$, where $p_1 : \tilde{X}_1 \to \mathbb{R}P^2$ and $p_2 : \tilde{X}_2 \to \mathbb{R}P^3$? Why or why not?*

Proof. Let Y be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$. If there exists an isomorphism $f : Y \to \tilde{X}_1 \times \tilde{X}_2$, then from the relations $p_1 = p_2 f, p_2 = p_1 f$, it follows that $(p_1 \times p_2)_* \pi_1(\tilde{X}_1 \times \tilde{X}_2) \cong p_*(Y)$ due to the induced isomorphisms. Since $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, we know from the product topology that a map $f : Y \to \mathbb{R}P^2 \times \mathbb{R}P^3$ is continuous if and only if the maps $g : Y \to \mathbb{R}P^2, h : Y \to \mathbb{R}P^2$ defined by $f = g \times h$ are continuous. Therefore a loop in $\mathbb{R}P^2 \times \mathbb{R}P^3$ is equivalent to a pair of loops in $\mathbb{R}P^2$ and $\mathbb{R}P^3$, and furthermore a homotopy on the product space is equivalent to a pair of homotopies on the corresponding components. Thus there exists a bijection

$$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3) \tag{1}$$

given by

$$[f] \mapsto ([g], [h]) \tag{2}$$

Next we compute the homology groups of $\mathbb{R}P^n, n \in \{2, 3\}$.

$\mathbb{R}P^n$ is topologized as the quotient space $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$, so we can thus restrict to vectors of length 1, so $\mathbb{R}P^n = S^n/(v \sim -v)$. Thus $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an $n-$cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. By induction on n, $\mathbb{R}P^n$ has a CW structure $e^0 \cup e^1 \cup ... \cup e^n$ with one cell e^i in each dimension $i \leq n$. To compute the boundary map d_k we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1} \tag{3}$$

with q the quotient map. The map $q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence $\deg q\varphi = \deg(1) + \deg(-1) = 1 + (-1)^k$, and so the boundary maps d_k is either 0 or multiplication by 2, depending on whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is even} \tag{4}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is odd} \tag{5}$$

$$\tag{6}$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} \text{ if } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 \text{ if } k = \text{ odd}, 0 < k < n \\ 0 \text{ otherwise} \end{cases} \tag{7}$$

Thus we know that $H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. We know from homework 4 that H_1 is the abelianization of π_1 (since $\mathbb{R}P^n$ is path-connected and nonempty), but a group with cardinality 2 must be isomorphic to $\mathbb{Z}_2$ to have the group axioms still hold. Thus, $\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, and are manifolds, and so are locally path-connected semi-locally simply-connected as well. Thus we can apply the Galois Correspondence Theorem to say that, for every subgroup H of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there is an isomorphism class of covering spaces Y such that $p_*(Y) \cong H$, therefore all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ have their fundamental groups as subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the trivial subgroup (0,0), and groups with generator (1,0), (0,1), and (1,1). I claim that there is no pair of maps $p_1 : \tilde{X}_1 \to \mathbb{R}P^2, p_2 : \tilde{X}_2 \to \mathbb{R}P^3$ such that $(p_1 \times p_2)_*(\tilde{X}_2 \times \tilde{X}_3) \cong \{(0,0),(1,1)\}$. We see that $|\{(0,0),(1,1)\}| = 2$. By Lagrange's Theorem for any subgroup of this group must have cardinality that divides 2. Since 2 is prime, the only subgroup in it has cardinality 1 and is thus is the trivial subgroup. Thus, if we have $\pi_1(\tilde{X}_1)\times\pi_1(\tilde{X}_2) \cong \{(0,0),(1,1)\}$, a subgroup $\pi_1(\tilde{X}_1)\times 0 \cong 0$ and another subgroup $0\times\pi_1(\tilde{X}_2) \cong 0$. Thus both $\pi_1(\tilde{X}_1), \pi_1(\tilde{X}_2)$ are trivial subgroups, and it is impossible for $f((0,0)) \mapsto (1,1)$ if f is an isomorphism. If, without loss of generality $\pi_1(\tilde{X}_1) \times 0$ were the whole group, this has cardinality 2, but the group is $\{(0,0),(1,0)\}$, a different subgroup corresponding to a different covering space. Thus, $\{(0,0),(1,1)\} \ncong \pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2)$, so not all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ are isomorphic to a product of each summand's covering space. $\square$

Question 2. *Let $X = \mathbb{R}P^2 \vee S^3$ and $Y = \mathbb{R}P^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.*

Proof. From the above, we know that the chain complex for $\mathbb{R}P^3$ is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \tag{8}$$

With any G coefficients, this becomes

$$0 \to G \xrightarrow{0} G \xrightarrow{2} G \xrightarrow{0} G \to 0 \tag{9}$$

so we have

$$H_0(\mathbb{R}P^3; G) \cong G, H_1(\mathbb{R}P^3; G) \cong G/2G, H_2(\mathbb{R}P^3; G) \cong 0, H_3(\mathbb{R}P^3; G) \cong G, \tag{10}$$

$$H_0(\mathbb{R}P^2; G) \cong G, H_1(\mathbb{R}P^2; G) \cong G/2G, H_2(\mathbb{R}P^2; G) \cong 0 \tag{11}$$

For $n > 0$ take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\tilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\tilde{H}_0(S^n) = 0$ By induction on n, starting with the case of S^0, we see that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$. Thus $H_1(S^3; G) \cong H_2(S^3; G) \cong 0, H_3(S^3) \cong G)$, due to the equivalence of $\tilde{H}_n$ and H_n for $n > 0$. Since $S^3, \mathbb{R}P^2$ are both path-connected and nonempty, $S^3 \vee \mathbb{R}P^2$ is path-connected and nonempty. By definition, $H_0(S^3 \vee \mathbb{R}P^2) = C_0(S^3 \vee \mathbb{R}P^2)/\text{Im } \partial_1$ since $\partial_0 = 0$. Define a homomorphism $\epsilon : C_0(S^3 \vee \mathbb{R}P^2) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. This is obviously surjective since $S^3 \vee \mathbb{R}P^2$ is nonempty. Ker $\epsilon =$ Im ∂_1 since $S^3 \vee \mathbb{R}P^2$ is path-connected, and thus ϵ induces an isomorphism.

We conclude that $H_0(S^3 \vee \mathbb{R}P^2; G) \cong G$. Since reduced homology is the same as homology relative to a basepoint, we know that, for $n > 0$,

$$\tilde{H}_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3) \oplus H_n(\mathbb{R}P^2) \tag{12}$$

Thus we have $H_1(S^3 \vee \mathbb{R}P^2; G) \cong G/2G, H_2(S^3 \vee \mathbb{R}P^2; G) \cong 0, H_3(S^3 \vee \mathbb{R}P^2; G) \cong G$. These are the same (up to isomorphism) homology groups as $\mathbb{R}P^3$.

In calculating cohomology for any G coefficients, we notice that $H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(X; \mathbb{Z}), G)$.

Lemma 1. *Hom*$(\mathbb{Z}, G) \cong G$, *Hom*$(G/2G, G) \cong 0$

Proof. By mapping 1 to each element of G, we get a cardinality of G. Since this is a homomorphism, the structure of the image is preserved, and $fg(n) = f(n) \star g(n)$, where $\star$ is the group operation of G. Since every element of the group is hit in the image, and the composition of these homomorphisms is mapped to the group operation, we have all elements of G following the same structure of G, and thus is isomorphic to G.

In order for there to be a nontrivial homomorphism, orders of elements must match from $G/2G$ to G. However, this is not the case, as we mod out by $2G$, so no generator of $G/2G$ has the same order as an element in G. Thus, the only homomorphism possible is the trivial homomorphism. □

Using this, we calculate cohomology for $\mathbb{R}P^3$, and, using the rules of Ext on page 195 of Hatcher, we find

$$H^0(\mathbb{R}P^3;G) \cong G, H^1(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z},G) \cong 0 \oplus 0 \cong 0, \tag{13}$$

$$H^2(\mathbb{R}P^3;G) \cong 0 \oplus \mathrm{Ext}(\mathbb{Z}_2,G) \cong 0 \oplus G/2G \cong G/2G, \tag{14}$$

$$H^3(\mathbb{R}P^3;G) \cong G \oplus 0 \tag{15}$$

We also have

$$H^n(S^3 \vee \mathbb{R}P^2;G) \cong \mathrm{Hom}(H_n(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \oplus \mathrm{Ext}(H_{n-1}(S^3 \vee \mathbb{R}P^2;\mathbb{Z}),G) \tag{16}$$

Since the homology groups are isomorphic, the cohomology groups are isomorphic as well.

If we can prove $H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) \not\cong H^*(\mathbb{R}P^3;\mathbb{Z}_2)$, then this means the spaces are not homotopy equivalent. Plug in $G = \mathbb{Z}_2$. From Example 3.8 in Hatcher, we have $H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$, and $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^4)$, where $|\alpha| = |\beta| = 1$. Suppose we have $\beta \in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$. Then $\beta \smile \beta = \beta^2 \in H^2(\mathbb{R}P^3;\mathbb{Z}_2)$, and $\beta^2 \smile \beta = \beta^3 \in H^3(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and $\beta^3 \neq 0$. Noticing that $H^1(\mathbb{R}P^2 \vee S^3) \cong H^1(\mathbb{R}P^2) \oplus H^1(S^3)$, we have $(\alpha, 0) \in H^1(\mathbb{R}P^2 \vee S^3)$. Suppose there is an isomorphism $f : \mathbb{R}P^2 \vee S^3 \to \mathbb{R}P^3$. Then for $\alpha' = f(\beta) \in H^1(\mathbb{R}P^2;\mathbb{Z}_2)$, and for $a = f(\beta^2) \in H^2(\mathbb{R}P^2;\mathbb{Z}_2)$. We now have $(\alpha', 0) \smile (a, 0) = (\alpha' a, 0) \in H^3(\mathbb{R}P^2 \vee S^3)$. But $H^3(\mathbb{R}P^2;\mathbb{Z}_2) \cong 0$, so $\alpha' a = 0$. But since f is a ring isomorphism, then $f(\beta^3 \neq 0) = \alpha' a = 0$. Since f maps a nonzero element to 0, it cannot be an isomorphism, so the cup product structures of $\mathbb{R}P^2 \vee S^3$ and $\mathbb{R}P^3$ are not homotopically equivalent. □

Question 3. *Let M^n be a closed, path connected, orientable manifold. Let $x \in U \subset M$ where U is an open neighborhood homeomorphic to $\mathbb{R}^n$. Consider the "pinch map," $p : M^n \to S^n$ defined as the composition*

$$p : M^n \xrightarrow{quotient} M^n/(M^n - U) \xrightarrow{homeo} S^n \tag{17}$$

Show that

$$p_* : H_n(M^n;\mathbb{Z}) \to H_n(S^n;\mathbb{Z}) \tag{18}$$

is an isomorphism.

Proof. Since M is closed and path-connected, we can apply Theorem 3.26 in Hatcher to conclude that $H_n(M) \cong H_n(M, M-x)$. Hatcher states that the isomorphism between the two factors though $H_n(M, M-U)$ for U any neighborhood in M containing x. This is because the homomorphism $i_* : H_n(M, M-U) \to H_n(M, M-x)$ induced by inclusion is bijective, since $X-U$ is a deformation retract of $X-x$. By excision, $H_n(X, X-U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x)$. Since $\mathbb{R}^n$ is contractible, this is isomorphic to $H_{n-1}(\mathbb{R}^n - U)$. The second map is isomorphic for any $x \in U$, because $\mathbb{R}^n - U$ and $\mathbb{R}^n - x$ deformation retract onto a sphere centered at x. Thus $H_n(M) \cong H_n(M, M-U)$.

We notice that $(M, M-U)$ is a good pair; Simply take an open cover ϵ-thick covering the boundary of U, and add this to $M-U$. Because $U \cong \mathbb{R}^n$, the overlap of our covering with U can easily be deformation retracted until we are left with $M-U$. Since $(M, M-U)$ is a good pair, we can have a neighborhood V be a neighborhood of U in M that deformation retracts onto U. We have a commutative diagram

$$\begin{tikzcd}
H_n(M,U) \arrow[r] \arrow[d, "q_*"] & H_n(M,V) \arrow[d, "q_*"] & H_n(M-U, V-U) \arrow[l] \arrow[d, "q_*"] \\
H_n(M/U, U/U) \arrow[r] & H_n(M/U, V, U) & H_n(M/U - U/U, V/U - U/U) \arrow[l]
\end{tikzcd}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (M, V, U) the groups $H_n(V, U)$ are zero for all n, because a deformation retraction of V onto U gives a homotopy equivalence of pairs $(V, U) \cong (U, U)$, and $H_n(U, U) = 0$. The deformation retraction of V onto U induces a deformation retraction of V/U onto U/U so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of U. From the commutativity of the diagram it follows that the left q_* map is an isomorphism. We see then that $H_n(M, M-U) \cong$

$H_n(M/(M-U),(M-U)/(M-U)) \cong H_n(M/(M-U))$. Thus $H_n(M) \cong H_n(M/(M-U))$. Since $M/(M-U)$ is homeomorphic to S^n, they better have isomorphic homology groups. Call the isomorphism induced by the homeomorphism between $H_n(M/(M-U))$ and $H_n(S^n)$ g, and the isomorphism between $H_n(M)$ and $H_n(M/(M-U))$, shown via the diagram. Now we have that $p_* := f \circ g$, a composition of isomorphisms, so $p_* : H_n(M) \to H_n(S^n)$ is an isomorphism. □

Question 4. *Prove that if M^3 is a closed, simply connected manifold, then there is a map $g : M^3 \to S^3$ that induces an isomorphism in homology groups in all dimensions. This is a weaker statement of the Poincaré Conjecture, proved in 2003 by G. Perelman.*

Proof. Define $\tilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$. The map $\mu_x \mapsto x$ defines a two-to-one surjection, and, due to everything being nice and manifold-y, we see that $\tilde{M}$ is a two-sheeted covering space of M^3.

Since M^3 is simply connected, $\tilde{M}$ has either one or two components since it is a two-sheeted covering space of M^3. If it has two components, they are each mapped nomeomorphically to M^3 by the covering projection, so M^3 is orientable, being homreomorphic to a component of the orientable manifold $\tilde{M}$. Thus M^3 is orientable, and $H_0(M^3) \cong \mathbb{Z}$ because simply connected implies nonempty and path-connected. Now since M^3 is a closed and orientable manifold, we can use Poincaré duality. Also from Theorem 3.26 (part c), $H_i(M^3) \cong 0, i > 3$. Because M^3 is simply-connected, $\pi_1(M^3) \cong 0 \cong H_1(M^3)$. $H^1(M^3) \cong \text{Hom}(0,\mathbb{Z}) \oplus \text{ Ext}(\mathbb{Z},\mathbb{Z}) \cong 0$. By Poincare duality, $H_2(M^3) \cong H^1(M^3) \cong 0$. Also by Poincaré Duality, $H_3(M^3) \cong H^0(M^3) \cong \text{Hom}(H_0(M^3),\mathbb{Z}) \oplus \text{Ext}(0,\mathbb{Z}) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$. To sum this up, we have $H_0(M^3) \cong H^3(M^3) \cong \mathbb{Z}, H_1(M^3) \cong H_2(M^3) \cong 0 \cong H_i(M^3), i > 3$. These are the exact homology groups of S^3, so let g be the isomorphism between their homology groups. □

Question 5. *Is $(S^2 \times S^4) \vee S^8$ homotopy equivalent to a compact closed manifold? Explain.*

Proof. Let $a_i \in H^i(S^2;\mathbb{Z}), b_i \in H^i(S^4;\mathbb{Z})$ be generators of their cohomology groups. From the definition of the external cup product we have $p_1^*(a) \smile p_2^*(b) \in H^*(X \times Y; R)$, for p_1, p_2 projection maps. For $H^0(S^2 \times S^4) \cong \mathbb{Z}$ because this is space and path-connected. Let p_1^*, p_2^* be induced homomorphisms from the projection $S^2 \times S^4 \to S^2, S^2 \times S^4 \to S^4$, respectively. We have $p_1^*(a_1) \smile p_2^*(b_1) = 0 \smile 0 = p_1^*(a_1) \smile 0 = 0 \smile p_2^*(b_1) = 0 \in H^1(S^2 \times S^4;\mathbb{Z})$. We also have $p_1^*(a_2) \smile$

$p_2^*(b_2) = p_1^*(a_2) \smile p_2^*(0) \in H^2(S^2 \times S^4; \mathbb{Z})$ as a generator for H^2. The other nonzero generator b_4, when cupped with another generator $a_i, i \neq 2$ is $0 \smile p_2^*(b_4) \in H^4(S^2 \times S^4; \mathbb{Z})$, which is the generator of H^4. For all other combinations when $a_i \neq a_2, b_j \neq 4$ we have trivial H^{i+j}. With $0 \neq p_1^*(a_2) \smile p_2^*(b_4) \in H^6(S^2 \times S^4; \mathbb{Z})$, we conclude that $H^i(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ (has a single generator infinite with $\mathbb{Z}$ coefficients) when $i = 0, 2, 4, 6$ and trivial otherwise. We might think we would run into trouble with $p_1^*(a_2) \smile p_1^*(a_2)$, but because this is the pullback of the generator $H^2(S^2)$ under p_1, and in $H^2(S^2), a_2 \smile a_2 = 0$, this still holds in $H^2(S^2 \times S^4)$. By the Kunneth Formula, we have $H^*(S^2 \times S^4; R) \cong \mathbb{Z}[a_2]/(a_2^2) \otimes_R \mathbb{Z}[a_4]/(a_4^2), |a_2| = 2, |a_4| = 4$.

For $\tilde{H}^*((S^2 \times S^4) \vee S^8)$ (we need not worry about H^0 since the space is nonempty and path-connected), we use the fact from Hatcher that $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong \tilde{H}^*(S^2 \times S^4) \oplus \tilde{H}^*(S^8)$. For $\tilde{H}^*(S^8)$, we know that $H_i \cong H^i \cong \mathbb{Z}$ for $i = 0, 8$, and $\cong 0$ if else. Thus our cohomology ring is $\tilde{H}^*(S^8; \mathbb{Z}) \cong \mathbb{Z}[b]/(b^2), |b| = 8$. Thus we have $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong [\mathbb{Z}[a_2]/(a_2^2) \otimes \mathbb{Z}[a_4]/(a_4^2)] \oplus \mathbb{Z}[b]/(b^2), |a_2| = 2, |a_4| = 4, |b| = 8$.

Any manifold homotopically equivalent to $(S^2 \times S^4) \vee S^8$ must be an 8-manifold. From Theorem 3.26 in Hatcher, if a manifold is not oriented, then $H_8(M; \mathbb{Z}) \cong 0 \Rightarrow H^8(M; \mathbb{Z}) \cong 0$, which cannot be possible, as $H^8((S^2 \times S^4) \vee S^8)$ is nontrivial. Thus a manifold that is homotopy equivalent must be oriented, since in that case $H^8(M; \mathbb{Z}) \cong H^8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. Oriented closed manifolds satisfy Poincaré Duality. If a closed manifold were to be homotopy equivalent to $(S^2 \times S^4) \vee S^8$, since the latter is path-connected the former better be path-connected. Suppose that $(S^2 \times S^4) \vee S^8$ satisfies Poincaré Duality. Consider the fundamental homology class $[M] \in H_8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. From Poincaré Duality, we know that, for $\alpha \in H^2(M)$, where α is a generator, $[M] \frown \alpha$ generates $H_6(M)$, since $D(\alpha) = [M] \frown \alpha$ is an isomorphism.

Examining the cap product, we have

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|[v_0, ..., v_k])\sigma|[v_k, ..., v_{k+l}]) \tag{19}$$

$$= \psi(\sigma|[v_0, ..., v_k])\psi(\sigma|[v_k, ..., v_{k+l}]) = (\varphi \smile \psi)(\sigma) \tag{20}$$

Thus, $\psi([M] \frown \alpha) = (\alpha \smile \beta)([M])$, where $\psi \in H^6(M)$ is the generator. Since $[M] \frown \alpha$ is a generator, and ψ is a generator homomorphism, $\psi([M] \frown \alpha)$ is a generator for the ring we are in (here we are using $\mathbb{Z}$). Thus, for $\beta \in H^4((S^2 \times s^4) \vee S^8)$ the generator, $1 = \psi([M] \frown \alpha) =$

$(\alpha \smile \psi)([M]) = (\alpha \smile (\alpha \smile \beta))([M]) = 0([M]) = 0$, a contradiction. This is because $\alpha \smile \alpha = 0$ from the ring structure we derived earlier. Because of this, Poincare Duality is not satisfied, so any orientable or otherwise closed manifold has a different ring structure and therefore is not homotopy equivalent. □

Question 6. *Prove that the Poincaré Duality theorem implies that if F is a field and M^n is a closed $F-$oriented manifold with its fundamental class $[M^n] \in H_n(M^n; F)$, then the pairing*

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \to F \tag{21}$$

$$\phi \times \psi \mapsto \langle \phi \cup \psi, [M^n] \rangle \tag{22}$$

is nonsingular for every $k = 0, ..., n$.

Proof. For F a field, M^n a closed $F-$oriented manifold with fundamental class $[M^n] \in H_n(M^n; F)$, the pairing

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \to F \tag{23}$$

$$\phi \times \psi \mapsto (\phi \smile \psi)([M^n]) \tag{24}$$

is nonsingular if $H^k(M^n; F) \cong \text{Hom}(H^{n-k}(M^n; F), F)$ and $H^{n-k}(M^n; F) \cong \text{Hom}(H^k(M^n; F), F)$.

For the first isomorphism, we want to relate $H^k(M^n; F)$ with $\text{Hom}(H^{n-k}(M^n; F), F)$. Using Poincaré Duality, we have

$$\begin{tikzcd} H^{n-k}(M^n; F) \arrow[rr, "\cong"] \arrow[d] & & H_k(M^n; F) \arrow[d] \\ \text{Hom}(H^{n-k}(M^n; F), F) \arrow[rr, phantom, "\cong"] & & \text{Hom}(H_k(M^n; F), F) \end{tikzcd}$$

$\text{Hom}(H^{n-k}(M^n; F), F) \cong \text{Hom}(H_k(M^n; F), F)$ via the hom-dual of Poincaré Duality. We can now relate $H^k(M^n; F)$ with $\text{Hom}(H_k(M^n; F), F)$ through the Universal Coefficient Theorem. The homology groups $H_k(M^n; F)$ are the homology groups of the chain complex of free $F-$modules with basis the singular $n-$simplices in M^n. From the Universal Coefficient Theorem, we have the exact sequence

$$0 \to \text{Ext}(H_{k-1}(M^n; F), F) \to H^k(M^n; F) \to \text{Hom}(H_k(M^n; F), F) \to 0 \tag{25}$$

Since M^n is closed, we have that $H_{k-1}(M^n; F), F)$ is finitely generated. Now we examine $\mathrm{Ext} H_{k-1}(M^n; F), F)$.

We wish to define a free resolution of the free $F-$module that is $H_{k-1}(M^n; F), F)$:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H_{k-1}(M^n; F) \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow H_{k-1}(M^n; F) \longrightarrow H_{k-1}(M^n; F) \longrightarrow 0$$

Dualizing the last line, we have the exact sequence

$$0 \xleftarrow{f_1} H_{k-1}(M^n; F) \xleftarrow{f_0} H_{k-1}(M^n; F) \longleftarrow 0$$

The definition of Ext is $\mathrm{Ker}(f_1)/\mathrm{Im}(f_0) = H_{k-1}(M^n; F), F)/H_{k-1}(M^n; F), F) = 0$. Thus our exact sequence from the Universal Coefficient Theorem becomes

$$0 \to 0 \to H^k(M^n; F) \to \mathrm{Hom}(H_k(M^n; F), F)) \to 0 \tag{26}$$

Therefore, $H^k(M^n; F) \cong \mathrm{Hom}(H_k(M^n; F), F) \cong \mathrm{Hom}(H^{n-k}(M^n; F), F)$.

That F needs to be a field comes from the first isomorphism above. Denote the free $F-$module $C_k(M^n; F)$ with basis the singular $k-$simplices in M^n. Suppose there are j $k-$simplices in M^n. Then by the Structure Theorem for Principal Ideal Domains, $C_k(M^n; F) \cong F^j$ if F is a field. Thus $\mathrm{Hom}(C_k(M^n; F), F) \cong \mathrm{Hom}(F^j, F) \cong \mathrm{Hom}(C_k(M^n), F)$. Treating $\mathrm{Hom}(C_k(M^n; F), F)$ as a dual complex, the homology groups are the cohomology groups $H^k(M^n; F)$.

We can get the second requirement $H^{n-k}(M^n; F) \cong \mathrm{Hom}(H^k(M^n; F), F)$ the same way, provided that we can take

$$H^{n-k}(M^n; F) \times H^k(M^n; F) \to F \tag{27}$$

$$\psi \times \phi \mapsto (\psi \smile \phi)([M^n]) = (\phi \smile \psi)([M^n]) \tag{28}$$

or, in terms, the cup product commutes. We check that this is true. For a singular $n-$simplex $\sigma : \Delta^n \to M^n$, we have

$$(\phi \smile \psi)([M^n]) = \phi(\sigma|[v_0, ..., v_k]) \cdot \psi(\sigma|[v_k, ..., v_n]) \tag{29}$$

$$= \psi(\sigma|[v_k, ..., v_n]) \cdot \phi(\sigma|[v_0, ..., v_k]) = (\psi \smile \phi)([M^n]) \tag{30}$$

since the product in F given by $\cdot$ commutes in a field, and relabeling the vertices of our $n-$simplex. Therefore $H^{n-k}(M^n;F) \cong \mathrm{Hom}(H^k(M^n;F),F)$, and the pairing given by (2) is nonsingular. $\square$

Question 7. *Show that*

$$H_c^*(\mathbb{R}^n;G) \cong \tilde{H}^*(S^n;G) \tag{31}$$

and more generally that if X is a topological space so that in its one-point compactification $X \cup \infty$, the point ∞ has a contractible neighborhood, then

$$H_c^*(X;G) \cong \tilde{H}^*(X \cup \infty;G) \tag{32}$$

where H_c is the cohomology with compact supports.

Proof. In computing $H_c^*(\mathbb{R}^n;G)$, we compute the limit group $\xrightarrow{lim} H^i(\mathbb{R}^n,\mathbb{R}^n-K;G)$, for K compact subsets $K \subset \mathbb{R}^n$. We let each compact subset K be the ball B_k of integer radius k. This is a compatible choice because the integers are a directed set, and any compact subset of $\mathbb{R}^n$ can be contained in a ball of some integer radius. We then use the exact sequence that comes with relative cohomology:

$$H^0(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^0(\mathbb{R}^n;G) \to H^0(\mathbb{R}^n-B_k;G) \to \tag{33}$$

$$H^1(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^1(\mathbb{R}^n;G) \to H^1(\mathbb{R}^n-B_k;G) \to ... \tag{34}$$

$$H^i(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to H^i(\mathbb{R}^n;G) \to H^i(\mathbb{R}^n-B_k;G) \to ... \tag{35}$$

Since $\mathbb{R}^n$ is simply connected, $H^0(\mathbb{R}^n;G) \cong G$, and $H^i(\mathbb{R}^n;G) \cong 0, i>0$. Examining $H^0(\mathbb{R}^n,\mathbb{R}^n-B_k;G)$, we see that this is given by $\mathrm{Hom}(C_0(\mathbb{R}^n)/C_0(\mathbb{R}^n-B_k),G)$. Since both $\mathbb{R}^n$ and $\mathbb{R}^n-B_k$ are connected, this group is trivial. Therefore, our exact sequence becomes

$$0 \to G \to H^0(\mathbb{R}^n-B_k;G) \to \tag{36}$$

$$H^1(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to 0 \to H^1(\mathbb{R}^n-B_k;G) \to ... \tag{37}$$

$$0 \to H^i(\mathbb{R}^n-B_k;G) \to H^{i+1}(\mathbb{R}^n,\mathbb{R}^n-B_k;G) \to 0 \to ... \tag{38}$$

Notice that $\mathbb{R}^n - B_k$ is homotopically equivalent to S^n. Therefore we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & \mathrm{Hom}(C_0(S^n),G) & \longrightarrow \; ... \; \longrightarrow & \mathrm{Hom}(C_i(S^n),G) & \longrightarrow & ... \\
 & & & & \downarrow & & \downarrow & & \\
0 & & \longrightarrow & & \tilde{H}^0(S^n;G) & \longrightarrow \; ... \; \longrightarrow & \tilde{H}^i(S^n;G) & \longrightarrow & ... \\
 & & & & \downarrow \cong & & \downarrow \cong & & \\
0 & & \longrightarrow & & H^0(\mathbb{R}^n,\mathbb{R}^n - B_k;G) & \longrightarrow \; ... \; \longrightarrow & H^i(\mathbb{R}^n,\mathbb{R}^n - B_k;G) & \longrightarrow & ...
\end{array}$$

Therefore $H^i(\mathbb{R}^n,\mathbb{R}^n - B_k;G) \cong \tilde{H}^i(S^n;G)$. Since $\mathbb{R}^n - B_k \simeq S^n \simeq \mathbb{R}^n - B_{k+1}$, we have that $H^i(\mathbb{R}^n,\mathbb{R}^n - B_k;G) \cong H^i(\mathbb{R}^n,\mathbb{R}^n - B_{k+1};G)$. Thus, $\xrightarrow{lim} H^i(\mathbb{R}^n,\mathbb{R}^n - B_k;G) \cong H^i_c(\mathbb{R}^n;G) \cong \tilde{H}^i(S^n;G)$. Because of this homotopy equivalence, these cohomology groups must have isomorphic ring structure, so $H^*_c(\mathbb{R}^n;G) \cong \tilde{H}^*(S^n;G)$.

More generally, for a topological space X such that the one-point compactification $X \cup \infty$ has a neighborhood of $\{\infty\}$ that is contractible, we examine the compactly supported cohomology of X. Therefore, there exists a contractible open set $U \subset X \cup \infty$ containing ∞. The compact subsets $K \subset X$ form a directed set under inclusion, since the union of two compact sets is compact. We have $\lim_{\rightarrow} H^i(X, X - K;G) = H^i_c(X;G)$. Let H be the complement of U in $X \cup \infty$. Since U is open, H is closed. Since $H \subset X \cup \infty$, and $X \cup \infty$ is compact, H is bounded. Therefore H is compact. Due to excision, since $\infty \in U \subset X \cup \infty$ has closure in U, we have

$$H^i(X, X - H) \cong H^i(X \cup \infty, X - H \cup \infty) = H^i(X \cup \infty, U \cup \infty) = H^i(X \cup \infty, U) \qquad (39)$$

for all i. Since U is contractible, $\lim_{\rightarrow} H^i(X, X - K) = H^i(X, X - H)$. This is because U can contract to a smaller open neighborhood, encompassing any larger compact subset of $X \cup \infty$. We now examine $H^*(X \cup \infty, U;G)$. We have

$$\begin{array}{cccccccc}
 & 0 & \longrightarrow & C^0(X\cup\infty;G)/C^0(U;G) & \longrightarrow & C^1(X\cup\infty;G)/C^1(U;G) & \xrightarrow{...} & C^i(X\cup\infty;G)/C^i(U;G)... \\
 & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & 0 & \longrightarrow & C^0(X\cup\infty;G)/\mathbb{Z} & \longrightarrow & C^1(X\cup\infty;G) & \xrightarrow{...} & C^i(X\cup\infty;G)... \\
 & & & & & \downarrow & & \\
0 \longrightarrow & \mathbb{Z} & \longrightarrow & C^0(X\cup\infty;G) & \longrightarrow & C^1(X\cup\infty;G) & \xrightarrow{...} & C^i(X\cup\infty;G)... \\
 & & & & & \downarrow & & \\
 & 0 & \longrightarrow & \tilde{H}^0(X\cup\infty;G) & \longrightarrow & \tilde{H}^1(X\cup\infty;G) & \xrightarrow{...} & \tilde{H}^i(X\cup\infty;G)...
\end{array}$$

since $C^i(U;G) \cong C^i(\infty;G)$, as U is homotopically equivalent to a point. Thus $H^*(X \cup \infty, U;G) \cong \tilde{H}^*(X \cup \infty;G)$. Since X has the same singular structure of $X \cup \infty$ in $0 <$ dimensions, the ring structure is the same. Thus $H_c^*(X;G) \cong \tilde{H}^*(X \cup \infty; G)$. $\square$

Question 8. *1. A theorem of Hopf states that if X is a path connected space of the homotopy type of a CW-complex, and it is endowed with a basepoint, then there is an isomorphism,*

$$[X, S^1]_\cdot \xrightarrow{\cong} H^1(X, \mathbb{Z}) \tag{40}$$

$$f \to f^*(\sigma) \tag{41}$$

where $\sigma \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator, and $[X, S^1]_\cdot$ denotes the based homotopy classes of basepoint preserving maps from X to S^1. Let M^n be a closed, oriented, connected n-dimensional manifold with basepoint $x_0 \in M^n$. Suppose $\alpha \in H^1(M;\mathbb{Z})$. Let $f_\alpha : M \to S^1$ represent α via Hopf's theorem. Let $N = f_\alpha^{-1}(t)$ where $t \in S^1$ is a regular value of f_α. Show that the homology class $[N] \in H_{n-1}(M)$ is Poincaré dual to $\alpha \in H^1(M)$.

2. Prove, using Hopf's theorem, the following theorem of Thom: If M^n is a closed, orientable manifold, then any homology class in $H_{n-1}(M^n)$ is represented by the fundamental class of a smooth codimension one, closed, oriented submanifold.

Proof. 1. Treat $t \in S^1$ as an embedded 0-dimensional submanifold. Since t is a regular value of f_α, $N = f_\alpha^{-1}(t)$ is a submanifold of M of dimension $n-1$ by the Regular Value Theorem (for proof that any α can correspond to a *smooth* map, see part b)). Furthermore, $f \pitchfork t$ since t is 0-dimensional; since t is a regular value, f is submersive at t, and thus has image the entire tangent space T_tS^1. We then can invoke the previous problem:

$$[f_\alpha^{-1}(t)] = [N] \in H_{n-1}(M;\mathbb{Z}) \tag{42}$$

$$= f_\alpha^*(D_{S^1}([t])) \cap [M] \tag{43}$$

We now examine $f_\alpha^*(D_{S^1}([t]))$. From the above theorem, we have $[t] \in H_0(S^1;\mathbb{Z})$ the fundamental class of t, up to a sign difference depending on our orientation. Therefore, we have $D_{S^1}([t]) \in H^1(S^1;\mathbb{Z})$ is a generator. Thus $f_\alpha^*(D_{S^1}([t])) = \alpha \in H^1(M;\mathbb{Z})$, up to a sign. Therefore $\alpha \cap [M] = [N]$, i.e. α is Poincaré Dual to $[N]$. (This works out because every manifold is a $CW-$complex, and M is closed, oriented, and connected and thus path-connected)

2. We use the previous part for inspiration. Suppose we have a class $\beta \in H_{n-1}(M)$. Take its Poincaré Dual $D_M(\beta) \in H^1(M)$. In the same way as in part a), let f_β be a smooth map from M to S^1. First we check that f_β is in the same homotopy class as a smooth map. From the Whitney Embedding theorem, we call the smooth embedding $g : S^1 \to \mathbb{R}^2$. We want $g^{-1} \circ g \circ f : M \to S^1$ to be homotopic to a smooth map, so $g \circ f : M \to \mathbb{R}^2$ has to be homotopic to a smooth map. This is a standard fact in analysis: for $\epsilon > 0$, there exists a differentiable function h such that, when we divide up our map $(g \circ f) := \sum_i^n (g \circ f)_i : M \to \mathbb{R}$, we have $|(g \circ f)_i - h| < \epsilon$; the graph of $(g \circ f)_i$ is a continuous section of the trivial bundle $M \times \mathbb{R}$. In any $\epsilon-$neighborhood of $(g \circ f)_i$ there is a differentiable section h. This is the h we want. Since the $\epsilon-$neighborhood is continuously mapped to $(g \circ f)_i$, h homotopic to $(g \circ f)_i$. Thus $(g \circ f)$ (the whole map $M \to \mathbb{R}^2$) maps to a tubular neighborhood of S^1. Since ηS^1 can be smoothly deformed to S^1 (call this map π), we have a differentiable map $\pi \circ (g \circ f) : M \to S^1 \subset \mathbb{R}^2$ that is homotopic to f. From Corollary 8.5 in the notes, the set of regular values of f_β is residual, so there exists a regular value $t' \in S^1$ of this map. Through the Regular Value Theorem, we have $f_\beta^{-1}(t')$ is a submanifold of M. This is oriented because, for $U \subset S^1$ an open subset containing $t \in S^1$, $f_\beta^{-1}(U)$ is an open subset of M and therefore orientable via f_β^{-1}. By similar reasoning, $f_\beta^{-1}(t)$ is closed. From the previous part, $D_M([f_\beta^{-1}(t')]) = D_M(\beta) \in H^1(M)$, i.e. $[f_\beta^{-1}(t')] = \beta \in H^1(M)$. Thus every homology class in $H_{n-1}(M)$ is represented by the fundamental class of a codimension 1, closed, orientable submanifold.

$\square$