Useful Mathematical Preliminary Objects (that I have difficulty remembering)

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1 Algebra

A *group* G is a set closed under an operation $\star$ that is associative ($g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3$), contains an identity e such that $e \star g = g \star e = g \forall g \in G$, and every element has an inverse such that $g \star g^{-1} = g^{-1} \star g = e$. A group is *abelian* if $g_1 \star g_2 = g_2 \star g_1$

A *ring* is a set closed under two operations $+, \times$ that is an abelian group under $+$, and contains an identity 1_R for the operation $\times$. $\times$ is distributive and associative.

A *field* is a ring where every element except maybe the $+$ identity has a multiplicative inverse sending it to the multiplicative identity. This forms a group structure for the elements except for maybe the additive identity. This group, called the multiplicative group, is also abelian.

A *module* M is an abelian group (operation denoted $+$) with a ring R such that, for all

$r, s \in R, x, y \in M$, we have

$$r(x+y) = rx + ry \tag{1}$$

$$(r+s)x = rx + sx \tag{2}$$

$$(rs)x = r(sx) \tag{3}$$

$$1_R x = x \tag{4}$$

This defines scalar multiplication.

A *vector space* is a module where R is a field.

An *algebra* A is a vector space with a binary operation $\cdot : A \times A \to A$ such that, for all $x, y, z \in K, r, s \in R$,

$$(x+y) \cdot z = (x \cdot z) + (y \cdot z) \tag{5}$$

$$x \cdot (y+z) = z \cdot y + x \cdot z \tag{6}$$

$$rx \cdot sy = (rs)x \cdot y \tag{7}$$

(These axioms define bilinearity)

2 Topology

A *topological space* is an ordered pair (X, τ) where X is a set an τ is a set of subsets of X such that:

The empty set and X belong to τ,

An arbitrary, finite or infinite union of elements of τ is in τ,

The intersection of any finite number of elements of τ is in τ.

τ is a topology on X, and defining a topology allows one to define continuity, connectedness, and convergence.

A *topological base (basis* B of a topological space X is a set of open subsets of X such that every open subset of X can be written as a union of elements in B. We say the base generates the topology, which makes sense, as the elements in τ are each a union of elements of B. For this to be well-used,

The base elements must cover X,

Let $B_1, B_2 \in B$ have $B_1 \cap B_2 := I$. For each $x \in I$, there is a $B' \in B$ such that $x \in B' \subseteq I$

Remark 1. *A second-countable space is a space with a countable base.*

A homeomorphism is a map between topological spaces that is an injection, is continuous, and has a continuous inverse map.

2.1 Notions in Symplectic Geometry

A *symplectic manifold* is a manifold with a closed, nondegenerate 2-form ω called the symplectic form. These show up in cotangent bundles of manifold. For a system modeled as a manifold, the cotangent bundle describes the phase space (space of all possible configurations of the system, e.g. Hilbert space) of the system.

Any real-valued differentiable function H on a symplectic manifold can be an energy function i.e Hamiltonian. Associated to any Hamiltonian is a Hamiltonian vector field, the integral curves of which (curves sketched along the vector field from the differential equation) is a solution to Hamilton's equations.

A *Hamiltonian flow* or *symplectomorphism* is the flow of this field on the symplectic manifold.

A Field in terms of Differential Forms

The Electromagnetic Field F is given by the 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dx + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt \tag{8}$$

Computing dF gives us

$$dF = (\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z})[dx \wedge dy \wedge dz] + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t})[dx \wedge dy \wedge dt] + ... \tag{9}$$

Setting $dF = 0$, we find the first two Maxwell's Equations $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. For the other two Maxwell's Equations, we use $d * F = 4\pi\rho$:

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt \tag{10}$$

with

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt \tag{11}$$

where the metric used in the hodge star is the Lorentz metric.

2.2 Curvature (an actually intuitive approach)

Most textbooks introduce the notion of curvature and then admit that there's no intuition behind this definition, said definition being $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X\nabla_Y - \nabla_Y\nabla_Y - \nabla_{[X,Y]}$. This is grossly unintuitive and only pertains to the Levi-Civita connection. However, Hori et. al. introduce this concept in clear simplicity, which I'll repeat here.

We want to be able to differentiate vectors. It's tempting to write

$$\lim_{\epsilon\to 0} \frac{v(x+\epsilon) - v(x)}{\epsilon} \tag{12}$$

but this definition makes no sense. "$+\epsilon$" isn't defined on a manifold, and we can't subtract vectors living in different vector spaces. (this is why we need a way to *connect* these vector space fibers). In resolving the first issue, we'll choose to differentiate in the i^{th} direction: denote the point whose i^{th} coordinate has been advanced by ϵ by $x + \epsilon\partial_i$. In resolving the second issue, we'll need an i-dependent automorphism. Since ϵ is small, we need our automorphism to be close to the identity; write it as $\mathbf{1} + \epsilon A_i$, for A_i an arbitrary endomorphism. Thus we have

$$D_i v = \frac{(\mathbf{1} + \epsilon A_i)(v(x+\epsilon\partial_i)) - v(x)}{\epsilon} \tag{13}$$

For $v := v^a\partial_a$, we get

$$v^a(x+\epsilon\partial_i) = v^a(x) + \epsilon\partial_i v^a(x) \Rightarrow \tag{14}$$

$$(D_i v)^a = \partial_i v^a + (A_i)^a_b v^b \tag{15}$$

Thus we have D an operator that sends vectors to vectors $v \mapsto D_i v$, and the vector w sens $v \mapsto D_w v = w^i D_i v = \langle Dv, w\rangle$, where we define the vector-valued one-form $Dv = (D_i v)dx^i$, so $D = d + A$ is our **connection**. This is why the covariant derivative along a vector field V is not $\frac{dV}{dt}(t)$, but $\frac{D}{dt}V(t)$, because the former vector doesn't belong to the tangent plane of the curve, i.e. the first issue with our first guess.

The curvature is intuitively the acceleration of a curve, or the concavity, etc. Either way, the data is encoded in a second derivative of sorts. We recast R as D^2, where $D = d + A$, for A our one-form. Thus $R = dA + A \wedge A$.

2.3 Morse Theory

Consider M a smooth manifold and $f : M \to \mathbb{R}$ a smooth function with nondegenerate critical points (the Hessian of f at these points is nonsingular). If no critical values of f occur between the numbers a and b, for $a < b$, then the subspace on which f takes values less than a is a deformation retract of the subspace where f is less than b; simply define a metric and flow the manifold via the vector field $-\nabla f/|\nabla f|^2$ for time $b-a$.

2.4 Chern Classes

2.5 Sheaves

2.5.1 Motivating Example

Suppose we have a topological manifold X. We wish to think about differentiable functions on X. In order to be well-defined, we need to consider all differentiable functions on all open subsets on X. On each open set $U \subset X$ we have a ring of differentiable functions, denoted $\mathscr{O}(U)$. Well, what about open sets within this open set? We can restrict a differentiable function on an open set to a smaller open set, and therefore get another differentiable function. I.e., if $U \subset V$ is an inclusion of open sets, we have a restriction map

$$res_{V,U} : \mathscr{O}(V) \to \mathscr{O}(U) \tag{16}$$

What about a third open set $W \subset V$? The restriction should commute from $U \subset V$ and $W \subset U$ to $W \subset V$:

$$\begin{tikzcd}
\mathscr{O}(V) \arrow[rr, "res_{W,V}"] \arrow[dr, "res_{V,U}"'] & & \mathscr{O}(W) \\
& \mathscr{O}(U) \arrow[ur, "res_{U,W}"'] &
\end{tikzcd}$$

One can also get an open set from a collection of smaller open sets. So suppose we take two differentiable functions f_1, f_2 on an open set U, and let $\{U_i\}$ be an open cover of U. If our two functions agree on the open cover, they better agree on U. Thus, if $f_1, f_2 \in \mathscr{O}(U)$ and $res_{U,U_i} f_1 = res_{U,U_i} f_2$, then $f_1 = f_2$.

Furthermore, what about the opposite direction, i.e. $\{U_i\}$ to U? We need to keep track of overlaps. Thus, given $f_i \in \mathscr{O}(U_i)$, for all i, such that $res_{U_i, U_i \cap U_j} f_i = res_{U_j, U_i \cap U_j} f_j$, for all i, j, given there is some $f \in \mathscr{O}(U)$ such that $res_{U,U_i} f = f_i$, for all i. We didn't use differentiability here, so hence we generalize to sheaves.

2.5.2 Presheaves and Sheaves

A **presheaf** $\mathscr{F}$ on a topological space X with

1. To each open set $U \subset X$, we associate an object $\mathscr{F}(U)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over** U, often called sections of $\mathscr{F}$, called **global sections**.

2. For each inclusion $U \hookrightarrow V$, we have a restriction morphism $res_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$

3. $res_{U,U} = id_{\mathscr{F}(U)}$

4. If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then restriction maps commute:

$$\begin{tikzcd} \mathscr{F}(W) \arrow[rr, "res_{W,U}"] \arrow[dr, "res_{W,V}"'] & & \mathscr{F}(U) \\ & \mathscr{F}(V) \arrow[ur, "res_{V,U}"'] & \end{tikzcd}$$

A presheaf is a **sheaf** if it satisfies two more axioms, corresponding to the open cover requirements used the example:

1. (**Identity axiom**) If $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$ with $res_{U,U_i} f_1 = res_{U,U_j} f_2$ for all i, then $f_1 = f_2$.

2. (**Gluability axioim**) If $\{U_i\}_{i \in I}$ is an open cover of U, then, given $f_i \in \mathscr{F}(U_i)$ for all i such that $res_{U_i, U_i \cap U_j} f_i = res_{U_j, U_j \cap U_i} f_j$ for all i, j, then there is some $f \in \mathscr{F}(U)$ such that $res_{u,u_i} f = f_i$, for all i.

Example 1. *We let $\mathbb{Z}$ denote the sheaf of integer-values functions, with $\mathbb{Z}(U)$ the locally constant integer-valued functions on U, and $\mathbb{Z}(X)$ is the group of globally-defined integer-valued functions. This is a vector space of dimension the number of connected components of X.*

Example 2. *$\mathbb{R}$ and $\mathbb{C}$ are sheaves of real and complex constant functions.*

Example 3. *$\mathcal{O}$ is the sheaf of holomorphic functions, with $\mathcal{O}(U)$ the set of holomorphic functions, with dimension equal to the number of connected components of U's topological space. This only works if X is compact, since the only global holomorphic functions on a compact connected space are constants.*

Example 4. *$\mathcal{O}^*$ is the sheaf of nowhere-zero holomorphic functions.*

If $\mathscr{F}$ is the category of vector spaces, sheaves inherit many properties from linear algebra. If $\mathscr{F}$ is the category of abelian groups, shaves inhereit many properties from homological algebra. A map between sheaves defined maps on the corresponding abelian groups, and its kernel defined the kernel sheaf.

Example 5. *We can have exact sequences of sheaves:*

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{times\ 2\pi i} \mathcal{O}^* \to 0 \tag{17}$$

This sequence isn't exact on every open set, e.g. $\mathbb{C} - \{0\}$, but is exact for open sets small enough, e.g. with trivial cohomology.

2.5.3 Cěch Cohomology

Let $\mathscr{F}$ be the category of abelian groups. For the sheaf relative to a cover $\{U_\alpha\}$ of X, we define our cochain complexes in the following way:

$$C^0(\mathscr{F}) = \prod_\alpha \mathscr{F}(U_\alpha) \tag{18}$$

$$C^1(\mathscr{F}) = \prod_{(\alpha,\beta)} \mathscr{F}(U_\alpha \cap U_\beta) \tag{19}$$

$$\vdots \tag{20}$$

where we require total anti-symmetry with higher cochains ($\sigma_{U_\alpha,U_\beta} = -\sigma_{U_\beta,U_\alpha}$). The chain maps are given by

$$(\delta_0\sigma)_{U,V} = \sigma_V - \sigma_U \tag{21}$$

$$(\delta_1\rho)_{U,V,W} = \rho_{V,W} - \rho_{U,W} + \rho_{U,V} \tag{22}$$

$$\vdots \tag{23}$$

The cohomology groups are thus defined by

$$H^i(\mathscr{F}) = Ker\delta_i/Im\delta_{i-1} \tag{24}$$

Something special about Cěch cohomology is that an exact sequnence of sheaves

$$0 \to A \to B \to C \to 0 \tag{25}$$

induces a long exact sequence in cohomology:

$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to ... \tag{26}$$

2.6 Stalks and Germs

2.6.1 Motivating Example

The germ of a differentiable function at a point $p \in X$ is an object of the form

$$\{(f, \text{open } U) : p \in U, f \in \mathscr{O}(U)\} \tag{27}$$

modulo the relation $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ containing p where $f|_W = g|_W$. In other words, two functions that are the same in an open neighbrohood of p have the same germ, even though they may be different elsewhere. The stalk in this example is the set of germs at p, and denote it $\mathscr{O}_p$. The stalk here is a ring: a germ can be the sum of two germs, defined on the intersection of those two germs' sets.

2.6.2 Definitions

The **stalk** of a presheaf $\mathscr{F}$ at a point p is the set of **germs** of $\mathscr{F}$ at p, denoted $\mathscr{F}_p$. The germ is the same definition as above, just for any category:

$$\{(f, \text{open } U) : p \in U, f \in \mathscr{F}(U)\} \tag{28}$$

Germs correspond to sections over some open set containing p, with two sections considered the same if they agree on some smaller open set. Equivalently, a stalk is the colimit of all $\mathscr{F}(U)$ over all open sets U containing p:

$$\mathscr{F}_p = \lim_{\rightarrow} \mathscr{F}(U) \tag{29}$$

The same definition holds for sheaves as well as presheaves.