

Algebraic Topology Refresher Problems

Alec Lau

Note: These are the only problems I had saved before I decided to make a page to preserve this stuff. Because there is so much machinery in algebraic topology, I have saved some of these problems as a refresher for whenever I return to it, as said machinery is very forgettable, beautiful though it may be. Also, these problems were done open-book and open-past-homework with Hatcher's "Algebraic Topology" textbook, so there may be some steps I skipped over.

All problems were written by Professor Ralph Cohen, saxophone extraordinaire.

Question 1. *Is every covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$ isomorphic to a product of covering spaces $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^3$, where $p_1 : \tilde{X}_1 \rightarrow \mathbb{R}P^2$ and $p_2 : \tilde{X}_2 \rightarrow \mathbb{R}P^3$? Why or why not?*

Proof. Let Y be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^3$. If there exists an isomorphism $f : Y \rightarrow \tilde{X}_1 \times \tilde{X}_2$, then from the relations $p_1 = p_2 f, p_2 = p_1 f$, it follows that $(p_1 \times p_2)_* \pi_1(\tilde{X}_1 \times \tilde{X}_2) \cong p_*(Y)$ due to the induced isomorphisms. Since $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, we know from the product topology that a map $f : Y \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^3$ is continuous if and only if the maps $g : Y \rightarrow \mathbb{R}P^2, h : Y \rightarrow \mathbb{R}P^3$ defined by $f = g \times h$ are continuous. Therefore a loop in $\mathbb{R}P^2 \times \mathbb{R}P^3$ is equivalent to a pair of loops in $\mathbb{R}P^2$ and $\mathbb{R}P^3$, and furthermore a homotopy on the product space is equivalent to a pair of homotopies on the corresponding components. Thus there exists a bijection

$$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^3) \quad (1)$$

given by

$$[f] \mapsto ([g], [h]) \quad (2)$$

Next we compute the homology groups of $\mathbb{R}P^n, n \in \{2, 3\}$.

$\mathbb{R}P^n$ is topologized as the quotient space $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$, so we can thus restrict to vectors of length 1, so $\mathbb{R}P^n = S^n / (v \sim -v)$. Thus $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n -cell, with the quotient projection $S^{n-1} \rightarrow \mathbb{R}P^{n-1}$ as the attaching map. By induction on n ,

$\mathbb{R}P^n$ has a CW structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one cell e^i in each dimension $i \leq n$. To compute the boundary map d_k we compute the degree of the composition

$$S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1} \quad (3)$$

with q the quotient map. The map $q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1} , which has degree $(-1)^k$. Hence $\deg q\varphi = \deg(1) + \deg(-1) = 1 + (-1)^k$, and so the boundary maps d_k is either 0 or multiplication by 2, depending on whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \text{ if } n \text{ is even} \quad (4)$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \text{ if } n \text{ is odd} \quad (5)$$

$$(6)$$

It follows that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{if } k = \text{odd}, 0 < k < n \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Thus we know that $H_1(\mathbb{R}P^2) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. We know from homework 4 that H_1 is the abelianization of π_1 (since $\mathbb{R}P^n$ is path-connected and nonempty), but a group with cardinality 2 must be isomorphic to \mathbb{Z}_2 to have the group axioms still hold. Thus, $\pi_1(\mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are path-connected, and are manifolds, and so are locally path-connected semi-locally simply-connected as well. Thus we can apply the Galois Correspondence Theorem to say that, for every subgroup H of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there is an isomorphism class of covering spaces Y such that $p_*(Y) \cong H$, therefore all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ have their fundamental groups as subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the trivial subgroup $(0,0)$, and groups with generator $(1,0)$,

$(0,1)$, and $(1,1)$. I claim that there is no pair of maps $p_1 : \tilde{X}_1 \rightarrow \mathbb{R}P^2, p_2 : \tilde{X}_2 \rightarrow \mathbb{R}P^3$ such that $(p_1 \times p_2)_*(\tilde{X}_2 \times \tilde{X}_3) \cong \{(0,0), (1,1)\}$. We see that $|\{(0,0), (1,1)\}| = 2$. By Lagrange's Theorem for any subgroup of this group must have cardinality that divides 2. Since 2 is prime, the only subgroup in it has cardinality 1 and is thus is the trivial subgroup. Thus, if we have $\pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2) \cong \{(0,0), (1,1)\}$, a subgroup $\pi_1(\tilde{X}_1) \times 0 \cong 0$ and another subgroup $0 \times \pi_1(\tilde{X}_2) \cong 0$. Thus both $\pi_1(\tilde{X}_1), \pi_1(\tilde{X}_2)$ are trivial subgroups, and it is impossible for $f((0,0)) \mapsto (1,1)$ if f is an isomorphism. If, without loss of generality $\pi_1(\tilde{X}_1) \times 0$ were the whole group, this has cardinality 2, but the group is $\{(0,0), (1,0)\}$, a different subgroup corresponding to a different covering space. Thus, $\{(0,0), (1,1)\} \not\cong \pi_1(\tilde{X}_1) \times \pi_1(\tilde{X}_2)$, so not all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^3$ are isomorphic to a product of each summand's covering space. \square

Question 2. Let $X = \mathbb{R}P^2 \vee S^3$ and $Y = \mathbb{R}P^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.

Proof. From the above, we know that the chain complex for $\mathbb{R}P^3$ is

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \quad (8)$$

With any G coefficients, this becomes

$$0 \rightarrow G \xrightarrow{0} G \xrightarrow{2} G \xrightarrow{0} G \rightarrow 0 \quad (9)$$

so we have

$$H_0(\mathbb{R}P^3; G) \cong G, H_1(\mathbb{R}P^3; G) \cong G/2G, H_2(\mathbb{R}P^3; G) \cong 0, H_3(\mathbb{R}P^3; G) \cong G, \quad (10)$$

$$H_0(\mathbb{R}P^2; G) \cong G, H_1(\mathbb{R}P^2; G) \cong G/2G, H_2(\mathbb{R}P^2; G) \cong 0 \quad (11)$$

For $n > 0$ take $(X, A) = (D^n, S^{n-1})$ so $X/A = S^n$. The terms $\tilde{H}_i(D^n)$ in the long exact

sequence for this pair are zero since D^n is contractible. Exactness of the sequence then implies that the maps $\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for $i > 0$ and that $\tilde{H}_0(S^n) = 0$. By induction on n , starting with the case of S^0 , we see that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$. Thus $H_1(S^3; G) \cong H_2(S^3; G) \cong 0, H_3(S^3) \cong G$, due to the equivalence of \tilde{H}_n and H_n for $n > 0$. Since $S^3, \mathbb{R}P^2$ are both path-connected and nonempty, $S^3 \vee \mathbb{R}P^2$ is path-connected and nonempty. By definition, $H_0(S^3 \vee \mathbb{R}P^2) = C_0(S^3 \vee \mathbb{R}P^2)/\text{Im } \partial_1$ since $\partial_0 = 0$. Define a homomorphism $\epsilon : C_0(S^3 \vee \mathbb{R}P^2) \rightarrow \mathbb{Z}$ by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. This is obviously surjective since $S^3 \vee \mathbb{R}P^2$ is nonempty. $\text{Ker } \epsilon = \text{Im } \partial_1$ since $S^3 \vee \mathbb{R}P^2$ is path-connected, and thus ϵ induces an isomorphism.

We conclude that $H_0(S^3 \vee \mathbb{R}P^2; G) \cong G$. Since reduced homology is the same as homology relative to a basepoint, we know that, for $n > 0$,

$$\tilde{H}_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3 \vee \mathbb{R}P^2) \cong H_n(S^3) \oplus H_n(\mathbb{R}P^2) \quad (12)$$

Thus we have $H_1(S^3 \vee \mathbb{R}P^2; G) \cong G/2G, H_2(S^3 \vee \mathbb{R}P^2; G) \cong 0, H_3(S^3 \vee \mathbb{R}P^2; G) \cong G$. These are the same (up to isomorphism) homology groups as $\mathbb{R}P^3$.

In calculating cohomology for any G coefficients, we notice that $H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(X; \mathbb{Z}), G)$.

Lemma 1. $\text{Hom}(\mathbb{Z}, G) \cong G, \text{Hom}(G/2G, G) \cong 0$

Proof. By mapping 1 to each element of G , we get a cardinality of G . Since this is a homomorphism, the structure of the image is preserved, and $f \star g(n) = f(n) \star g(n)$, where \star is the group operation of G . Since every element of the group is hit in the image, and the composition of these homomorphisms is mapped to the group operation, we have all elements of G following the same structure of G , and thus is isomorphic to G .

In order for there to be a nontrivial homomorphism, orders of elements must match from $G/2G$ to G . However, this is not the case, as we mod out by $2G$, so no generator of $G/2G$ has the same order as an element in G . Thus, the only homomorphism possible is the trivial homomorphism. \square

Using this, we calculate cohomology for $\mathbb{R}P^3$, and, using the rules of Ext on page 195 of Hatcher, we find

$$H^0(\mathbb{R}P^3; G) \cong G, H^1(\mathbb{R}P^3; G) \cong 0 \oplus \text{Ext}(\mathbb{Z}, G) \cong 0 \oplus 0 \cong 0, \quad (13)$$

$$H^2(\mathbb{R}P^3; G) \cong 0 \oplus \text{Ext}(\mathbb{Z}_2, G) \cong 0 \oplus G/2G \cong G/2G, \quad (14)$$

$$H^3(\mathbb{R}P^3; G) \cong G \oplus 0 \quad (15)$$

We also have

$$H^n(S^3 \vee \mathbb{R}P^2; G) \cong \text{Hom}(H_n(S^3 \vee \mathbb{R}P^2; \mathbb{Z}), G) \oplus \text{Ext}(H_{n-1}(S^3 \vee \mathbb{R}P^2; \mathbb{Z}), G) \quad (16)$$

Since the homology groups are isomorphic, the cohomology groups are isomorphic as well.

If we can prove $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \not\cong H^*(\mathbb{R}P^3; \mathbb{Z}_2)$, then this means the spaces are not homotopy equivalent. Plug in $G = \mathbb{Z}_2$. From Example 3.8 in Hatcher, we have $H^*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$, and $H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^4)$, where $|\alpha| = |\beta| = 1$. Suppose we have $\beta \in H^1(\mathbb{R}P^3; \mathbb{Z}_2)$. Then $\beta \smile \beta = \beta^2 \in H^2(\mathbb{R}P^3; \mathbb{Z}_2)$, and $\beta^2 \smile \beta = \beta^3 \in H^3(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and $\beta^3 \neq 0$. Noticing that $H^1(\mathbb{R}P^2 \vee S^3) \cong H^1(\mathbb{R}P^2) \oplus H^1(S^3)$, we have $(\alpha, 0) \in H^1(\mathbb{R}P^2 \vee S^3)$. Suppose there is an isomorphism $f : \mathbb{R}P^2 \vee S^3 \rightarrow \mathbb{R}P^3$. Then for $\alpha' = f(\beta) \in H^1(\mathbb{R}P^2; \mathbb{Z}_2)$, and for $a = f(\beta^2) \in H^2(\mathbb{R}P^2; \mathbb{Z}_2)$. We now have $(\alpha', 0) \smile (a, 0) = (\alpha'a, 0) \in H^3(\mathbb{R}P^2 \vee S^3)$. But $H^3(\mathbb{R}P^2; \mathbb{Z}_2) \cong 0$, so $\alpha'a = 0$. But since f is a ring isomorphism, then $f(\beta^3 \neq 0) = \alpha'a = 0$. Since f maps a nonzero element to 0, it cannot be an isomorphism, so the cup product structures of $\mathbb{R}P^2 \vee S^3$ and $\mathbb{R}P^3$ are not homotopically equivalent. \square

Question 3. Let M^n be a closed, path connected, orientable manifold. Let $x \in U \subset M$ where U is an open neighborhood homeomorphic to \mathbb{R}^n . Consider the “pinch map,” $p : M^n \rightarrow S^n$ defined as the composition

$$p : M^n \xrightarrow{\text{quotient}} M^n/(M^n - U) \xrightarrow{\text{homeo}} S^n \quad (17)$$

Show that

$$p_* : H_n(M^n; \mathbb{Z}) \rightarrow H_n(S^n; \mathbb{Z}) \quad (18)$$

is an isomorphism.

Proof. Since M is closed and path-connected, we can apply Theorem 3.26 in Hatcher to conclude that $H_n(M) \cong H_n(M, M - x)$. Hatcher states that the isomorphism between the two factors though $H_n(M, M - U)$ for U any neighborhood in M containing x . This is because the homomorphism $i_* : H_n(M, M - U) \rightarrow H_n(M, M - x)$ induced by inclusion is bijective, since $X - U$ is a deformation retract of $X - x$. By excision, $H_n(X, X - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x)$. Since \mathbb{R}^n is contractible, this is isomorphic to $H_{n-1}(\mathbb{R}^n - U)$. The second map is isomorphic for any $x \in U$, because $\mathbb{R}^n - U$ and $\mathbb{R}^n - x$ deformation retract onto a sphere centered at x . Thus $H_n(M) \cong H_n(M, M - U)$.

We notice that $(M, M - U)$ is a good pair; Simply take an open cover ϵ -thick covering the boundary of U , and add this to $M - U$. Because $U \cong \mathbb{R}^n$, the overlap of our covering with U can easily be deformation retracted until we are left with $M - U$. Since $(M, M - U)$ is a good pair, we can have a neighborhood V be a neighborhood of U in M that deformation retracts onto U . We have a commutative diagram

$$\begin{array}{ccccc} H_n(M, U) & \xrightarrow{\quad\quad\quad} & H_n(M, V) & \xleftarrow{\quad\quad\quad} & H_n(M - U, V - U) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(M/U, U/U) & \xrightarrow{\quad\quad\quad} & H_n(M/U, V/U) & \xleftarrow{\quad\quad\quad} & H_n(M/U - U/U, V/U - U/U) \end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (M, V, U) the groups $H_n(V, U)$ are zero for all n , because a deformation retraction of V onto U gives a homotopy equivalence of pairs $(V, U) \cong (U, U)$, and $H_n(U, U) = 0$. The deformation retraction of V onto U induces a deformation retraction of V/U onto U/U so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of U . From the commutativity of the diagram it follows that the left q_* map is an isomorphism. We see then that $H_n(M, M - U) \cong$

$H_n(M/(M-U), (M-U)/(M-U)) \cong H_n(M/(M-U))$. Thus $H_n(M) \cong H_n(M/(M-U))$. Since $M/(M-U)$ is homeomorphic to S^n , they better have isomorphic homology groups. Call the isomorphism induced by the homeomorphism between $H_n(M/(M-U))$ and $H_n(S^n)$ g , and the isomorphism between $H_n(M)$ and $H_n(M/(M-U))$, shown via the diagram. Now we have that $p_* := f \circ g$, a composition of isomorphisms, so $p_* : H_n(M) \rightarrow H_n(S^n)$ is an isomorphism. \square

Question 4. *Prove that if M^3 is a closed, simply connected manifold, then there is a map $g : M^3 \rightarrow S^3$ that induces an isomorphism in homology groups in all dimensions. This is a weaker statement of the Poincaré Conjecture, proved in 2003 by G. Perelman.*

Proof. Define $\tilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$. The map $\mu_x \mapsto x$ defines a two-to-one surjection, and, due to everything being nice and manifold-y, we see that \tilde{M} is a two-sheeted covering space of M^3 .

Since M^3 is simply connected, \tilde{M} has either one or two components since it is a two-sheeted covering space of M^3 . If it has two components, they are each mapped homeomorphically to M^3 by the covering projection, so M^3 is orientable, being homeomorphic to a component of the orientable manifold \tilde{M} . Thus M^3 is orientable, and $H_0(M^3) \cong \mathbb{Z}$ because simply connected implies nonempty and path-connected. Now since M^3 is a closed and orientable manifold, we can use Poincaré duality. Also from Theorem 3.26 (part c), $H_i(M^3) \cong 0, i > 3$. Because M^3 is simply-connected, $\pi_1(M^3) \cong 0 \cong H_1(M^3)$. $H^1(M^3) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) \cong 0$. By Poincaré duality, $H_2(M^3) \cong H^1(M^3) \cong 0$. Also by Poincaré Duality, $H_3(M^3) \cong H^0(M^3) \cong \text{Hom}(H_0(M^3), \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$. To sum this up, we have $H_0(M^3) \cong H^3(M^3) \cong \mathbb{Z}, H_1(M^3) \cong H_2(M^3) \cong 0 \cong H_i(M^3), i > 3$. These are the exact homology groups of S^3 , so let g be the isomorphism between their homology groups. \square

Question 5. *Is $(S^2 \times S^4) \vee S^8$ homotopy equivalent to a compact closed manifold? Explain.*

Proof. Let $a_i \in H^i(S^2; \mathbb{Z}), b_i \in H^i(S^4; \mathbb{Z})$ be generators of their cohomology groups. From the definition of the external cup product we have $p_1^*(a) \smile p_2^*(b) \in H^*(X \times Y; \mathbb{R})$, for p_1, p_2 projection maps. For $H^0(S^2 \times S^4) \cong \mathbb{Z}$ because this is space and path-connected. Let p_1^*, p_2^* be induced homomorphisms from the projection $S^2 \times S^4 \rightarrow S^2, S^2 \times S^4 \rightarrow S^4$, respectively. We have $p_1^*(a_1) \smile p_2^*(b_1) = 0 \smile 0 = p_1^*(a_1) \smile 0 = 0 \smile p_2^*(b_1) = 0 \in H^1(S^2 \times S^4; \mathbb{Z})$. We also have $p_1^*(a_2) \smile$

$p_2^*(b_2) = p_1^*(a_2) \smile p_2^*(0) \in H^2(S^2 \times S^4; \mathbb{Z})$ as a generator for H^2 . The other nonzero generator b_4 , when cupped with another generator $a_i, i \neq 2$ is $0 \smile p_2^*(b_4) \in H^4(S^2 \times S^4; \mathbb{Z})$, which is the generator of H^4 . For all other combinations when $a_i \neq a_2, b_j \neq 4$ we have trivial H^{i+j} . With $0 \neq p_1^*(a_2) \smile p_2^*(b_4) \in H^6(S^2 \times S^4; \mathbb{Z})$, we conclude that $H^i(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ (has a single generator infinite with \mathbb{Z} coefficients) when $i = 0, 2, 4, 6$ and trivial otherwise. We might think we would run into trouble with $p_1^*(a_2) \smile p_1^*(a_2)$, but because this is the pullback of the generator $H^2(S^2)$ under p_1 , and in $H^2(S^2), a_2 \smile a_2 = 0$, this still holds in $H^2(S^2 \times S^4)$. By the Kunneth Formula, we have $H^*(S^2 \times S^4; \mathbb{R}) \cong \mathbb{Z}[a_2]/(a_2^2) \otimes_{\mathbb{R}} \mathbb{Z}[a_4]/(a_4^2), |a_2| = 2, |a_4| = 4$.

For $\tilde{H}^*((S^2 \times S^4) \vee S^8)$ (we need not worry about H^0 since the space is nonempty and path-connected), we use the fact from Hatcher that $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong \tilde{H}^*(S^2 \times S^4) \oplus \tilde{H}^*(S^8)$. For $\tilde{H}^*(S^8)$, we know that $H_i \cong H^i \cong \mathbb{Z}$ for $i = 0, 8$, and $\cong 0$ if else. Thus our cohomology ring is $\tilde{H}^*(S^8; \mathbb{Z}) \cong \mathbb{Z}[b]/(b^2), |b| = 8$. Thus we have $\tilde{H}^*((S^2 \times S^4) \vee S^8) \cong [\mathbb{Z}[a_2]/(a_2^2) \otimes \mathbb{Z}[a_4]/(a_4^2)] \oplus \mathbb{Z}[b]/(b^2), |a_2| = 2, |a_4| = 4, |b| = 8$.

Any manifold homotopically equivalent to $(S^2 \times S^4) \vee S^8$ must be an 8-manifold. From Theorem 3.26 in Hatcher, if a manifold is not oriented, then $H_8(M; \mathbb{Z}) \cong 0 \Rightarrow H^8(M; \mathbb{Z}) \cong 0$, which cannot be possible, as $H^8((S^2 \times S^4) \vee S^8)$ is nontrivial. Thus a manifold that is homotopy equivalent must be oriented, since in that case $H^8(M; \mathbb{Z}) \cong H^8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. Oriented closed manifolds satisfy Poincaré Duality. If a closed manifold were to be homotopy equivalent to $(S^2 \times S^4) \vee S^8$, since the latter is path-connected the former better be path-connected. Suppose that $(S^2 \times S^4) \vee S^8$ satisfies Poincaré Duality. Consider the fundamental homology class $[M] \in H_8((S^2 \times S^4) \vee S^8) \cong \mathbb{Z}$. From Poincaré Duality, we know that, for $\alpha \in H^2(M)$, where α is a generator, $[M] \frown \alpha$ generates $H_6(M)$, since $D(\alpha) = [M] \frown \alpha$ is an isomorphism.

Examining the cap product, we have

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|[v_0, \dots, v_k])\sigma|[v_k, \dots, v_{k+l}]) \quad (19)$$

$$= \psi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}]) = (\varphi \smile \psi)(\sigma) \quad (20)$$

Thus, $\psi([M] \frown \alpha) = (\alpha \smile \beta)([M])$, where $\psi \in H^6(M)$ is the generator. Since $[M] \frown \alpha$ is a generator, and ψ is a generator homomorphism, $\psi([M] \frown \alpha)$ is a generator for the ring we are in (here we are using \mathbb{Z}). Thus, for $\beta \in H^4((S^2 \times S^4) \vee S^8)$ the generator, $1 = \psi([M] \frown \alpha) =$

$(\alpha \smile \psi)([M]) = (\alpha \smile (\alpha \smile \beta))([M]) = 0([M]) = 0$, a contradiction. This is because $\alpha \smile \alpha = 0$ from the ring structure we derived earlier. Because of this, Poincare Duality is not satisfied, so any orientable or otherwise closed manifold has a different ring structure and therefore is not homotopy equivalent. □