

Lecture Notes for Modern Theoretical Condensed Matter

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1 Introduction

There are many rich frameworks used to describe various exotic regimes like the quantum hall regime, crystalline solids, topologically ordered phases of matter, and the phase transitions between them. But all of this progress has been built on two pillars of equilibrium statistical mechanics:

1. Thermalization

2. Phase structure

We assume, when studying phase transitions in the general sense, that the system is in thermal contact with some kind of bath/reservoir, and the system exchanges heat, particles, etc. with this bath, and eventually it reaches equilibrium at late times. This state is characterized by a few variables like temperature, chemical potential, etc. depending on the conservation laws in this system. Then the properties are developed by analysis of observables or order parameters:

$$\rho_{eq}(T, \mu, \dots) \tag{1}$$

$$\langle O \rangle = Tr[\rho_{eq} O] \tag{2}$$

The definition of a phase only makes sense in the thermodynamic limit, where we only care about the asymptotics of millions and millions of particles at late times. The central problem in statistical mechanics is how to keep track of all of this data. This is where equilibrium statistical mechanics comes in: a standard assumption of equilibrium statistical mechanics is the ergodic hypothesis, which states that, in time, all of phase space is explored, so we can replace all of these detailed equations of motion with macroscopic averages. These macrostates are characterized by a couple parameters like temperature, chemical potential... to find the equilibrium state, or, if we're at zero temperature, the ground state, and then characterize phases and phase transitions by evaluating correlation functions, order parameters, etc.

One question is what can we say about the dynamics of *isolated* many-body quantum systems, where there is no bath. How does such a system reach equilibrium? Usually we think about some sort of irreversible change from a tiny fraction of phase space, and, after a long time, having no way to know what that tiny fraction was. However, under unitary time evolution, we know where the initial state was. Thus the first question is what thermal equilibrium even means in this context.

Remark 1. *This is actually deeply connected to the black hole information paradox. According to Hawking's calculation for an evaporating black hole, the Hawking radiation should only be characterized by a small number of parameters such as the temperature or mass of the black hole, but a quantum black hole evolving unitarily doesn't forget any information.*

1.1 Thermalization & Localization

Here we will consider highly excited, strongly interacting many-body systems. A standard assumption of statistical mechanics is that this system goes to thermal equilibrium at late times. This makes sense, but must this always be true? What does “thermal equilibrium” mean in this

context, and how is it reached? How does classical hydrodynamics emerge from unitary quantum dynamics? Anderson, in 1958, gave the first example of a system which could be many-body “localized” and fail to go to thermal equilibrium. (*Absence of Diffusion in Certain Random Lattices*) The question he raised in 1958 is: Can an isolated, strongly interacting many-body system act as its own “bath” and bring its subsystems to thermal equilibrium? The two *generic* (not fine-tuned in any way) answers are

1. Yes: the system is thermalizing
2. No: the system is many-body localized. E.g. integrable models. These have extensively many conservation laws, so any thermalization must obey these conservation laws, so they don’t generically thermalize to a Gibbs ensemble where temperature is a parameter; instead they thermalize to generalized Gibbs ensembles.

There are some systems that are sort of intermediate to the above two possibilities. E.g. many-body scars, ie. systems where the vast majority of initial states do approach thermal equilibrium, but there are states that don’t.

If you have thermalizing on one hand and localization on the other hand, what is the whole chaos between the two? As of now, our current understanding is that there are only two *generic* possibilities, i.e. if we have one intermediate system and perturb it in parameter space, it goes to one or the other.

In thinking about these new systems, we think about different classes of models. We have

1. Time-independent Hamiltonians: $U(t) = e^{-iHt}$
2. Floquet systems: $H(t+T) = H(t), U(nT) = [U(T)]^n$
3. Unitary circuits (e.g. from quantum information theory)

Remark 2. *The Google experiment claiming quantum supremacy really just implemented a random unitary circuit and characterized the behavior of that circuit.*

A natural question is whether systems with more structure, e.g. have many-body eigenstates, is there more structure implied?

The **Eigenstate Thermalization Hypothesis** (ETH) states that if all “reasonable” initial states reach thermal equilibrium, then eigenstates of H must be thermal:

$$H |n\rangle = E_n |n\rangle \tag{3}$$

$$\text{Tr}_B |n\rangle \langle n| = \text{Tr}_B \frac{e^{-\beta_n H}}{Z} \tag{4}$$

Eigenstates are time-independent, so how would an eigenstate reach thermal equilibrium in time? ETH says that they must already be thermal. This means that, in systems that are thermalizing and obey ETH, every single eigenstate is thermal and forms a microcanonical ensemble.

The only “generic” (that we’re aware of) exception to thermalization is localization. This occurs in systems that are not translationally invariant. An actual experiment verifying this was done in a system of cold atoms (Choi et. al. Science (2016), Screiber et. al (2015)). A localized system retains local memory of its initial conditions forever i.e. is observable. Globally, unitary dynamics preserves all information, but locally we preserve local information.

So, when is ETH obeyed? Yes: (strong version) for all eigenstates at finite temperature. (weak version) for most eigenstates at finite temperature. This weak version occurs in many body quantum scars. For examples, the AKLT chain. No: In many body localized systems, integrable systems (generalized Gibbs ensemble). For some observable O at some temperature $\frac{1}{\beta}$,

$$\langle O \rangle_{\beta} = \frac{1}{Z} \text{Tr}[e^{-\beta H} O] \quad (5)$$

if $\tau = \infty, \beta = 0$, and

$$\langle O \rangle_{\beta=0} = \frac{\text{Tr}[O]}{Z} \quad (6)$$

The observable could be energy density. Most eigenstates entropically will live near this energy density. Each state has a global energy (with T some global temperature) and energy density E/V . For ρ for the whole system, we have

$$\text{Tr}_B[\rho] = \rho_A(t) \xrightarrow{t, V_{ol_B} \rightarrow \infty} \text{Tr}_B[\rho_{thermal}(t)] \quad (7)$$

so if we have a thermal density matrix ρ at the temperature set by the energy density of the state we’re considering (which, for an eigenstate, is just the energy density), the reduced density matrix of the subregion at late times approaches the reduced density matrix appropriate to a thermal density matrix.

1.2 Entanglement Dynamics

In systems that reach thermal equilibrium, the states approach a **volume law** which states that the rest of the region B can act as a bath for all spins in A . Then the Von Neumann entanglement entropy for region A agrees with the thermodynamic entropy at the temperature that’s appropriate to the eigenstate/initial state that we’re considering, so the entanglement entropy scales with the

volume. On the other hand, one of the best diagnostics of localization (because transport doesn't really work) is in the dynamics of entanglement in the eigenstates of an MBL system. In MBL systems, states have an **area law** for entanglement: only the spins right near the edge in A are entangled to the spins in B , and B is not able to act like a bath and bring all states in A to thermal equilibrium. So in MBL systems, entanglement entropy scales with the perimeter of the subregion, so the difference between thermalization and localization is really examining the entanglement properties of the system.

2 Thermalization & the Eigenstate Thermalization Hypothesis

In strongly interacting, highly excited systems, we have the natural question: Do these systems reach thermal equilibrium under their *own* internal unitary dynamics? As above, thermalizing systems answer yes, MBL systems, integrable systems, etc. answer no. For both cases, how and why to the dynamics establish equilibrium or not? Furthermore, how does unitary dynamics lead to apparently irreversible changes? The types of systems we'll consider are

1. Time-independent Hamiltonians $H(t) = H$, where energy is conserved, and we consider eigenstates of H
2. Floquet systems $H(t+T) = H(t)$. This means, in the way that momentum is not conserved in Bloch settings, energy is not conserved, although quasi-energy (energy mod ω) is conserved.
3. Random $H(t)$ with no symmetries.

For the first item, we have $U(t) = \tau e^{-i \int_0^t dt' H(t')}$, which, since $H(t) = H$, we have the energy eigenstates form a complete basis, so

$$|\psi_0\rangle = \sum_{\alpha} c_{\alpha} |\alpha\rangle \tag{8}$$

$$|\psi(t)\rangle = \sum_{\alpha} c_{\alpha} e^{-iE_{\alpha}t} |\alpha\rangle \tag{9}$$

All of the complexity is hidden in E_{α} and $|\alpha\rangle$. In traditional statmech, we have a system S within an environment E , with $|S| \ll |E|$. Then we specify the laws of conservation: we want to talk about the total energy or the total particle number, etc. These conservation laws define the appropriate equilibrium states as the maximum entropy states consistent all these different conservation laws. Usually the way this works is we take variables conjugate to the energy density or the particle number, assume the environment is at some fixed temperature or chemical potential,

and the system and the environment are exchanging energy, particles, etc. and the system is going to reach thermal equilibrium at late times as a result of this change. Thus we want

$$\rho_S(t) \rightarrow \rho_{eq}^S(T, \mu, \dots) \quad (10)$$

where ρ_{eq}^S is the maximum entropy state consistent with all of the conservation laws in this system.

Implicitly in this system we take advantage of classical chaos. A classically chaotic system ergodically explores all of phase space, so we can replace very detailed descriptions of all of these particles with macroscopic descriptions. What is the analogue in the many-body setting? We are still exploring how to correctly describe quantum chaos, and what the notion of ergodicity means in this many-body setting. What we do know how to define is the concept of thermalization.

Definition 1. *In a closed quantum system, **thermalization** is the property that the system successfully acts as a bath for its subsystems and brings them to equilibrium at late times and for large sizes.*

Definition 2. *A **subsystem** is a subset of the degrees of freedom of the whole system. For a subsystem A of system B (A 's complement, as a bath), we require (imprecisely) $\|A\| \ll \|B\|$, $\mathcal{H}_A = \{|a\rangle_A\}$, $\mathcal{H}_B = \{|b\rangle_B\}$, $\mathcal{H}_{total} = \mathcal{H}_A \otimes \mathcal{H}_B$ spanned by $\{|a\rangle_A \otimes |b\rangle_B\}$ with hamiltonian $H = H_A + H_B + H_{AB}$.*

We assume that there are some local degrees of freedom, e.g. spins on some site. The subsystem is some low-order constructions in terms of these local operators. The observables are also going to be low-order constructions in terms of local observables.

$$\sum_{i \in A} O_i \quad (11)$$

for O_i local operators. These are a vanishing subset of all the operators defined on our Hilbert space.

Definition 3. ***States** can be pure or mixed. A state is given by its density matrix ρ . This corresponds to the probability density in state space. We know $\text{Tr}[\rho] = 1$, and $\rho = \sum_n p_n |n\rangle \langle n|$. In general $\text{Tr}[\rho]^2 \leq 1$, with equality if and only if it is a pure state. The time evolution of this state is, again, given by the Schrödinger equation.*

So now we have a state defined in AB given by ρ_{AB} . If we want to define the state on the subsystem we take the reduced-density matrix $\text{Tr}_B[\rho_{AB}]$.

Definition 4. *What we mean by **thermal equilibrium** is*

$$\lim_{t, B \rightarrow \infty} \rho_A(t) = \text{Tr}_B(\rho_{AB}) \rightarrow \text{Tr}_B[\rho_{eq}] \quad (12)$$

The appropriate equilibrium state ρ_{eq} (which is not unique) respects all conservation laws in our system.

Remark 3. *There are classes known as integrable systems that have infinitely many local conserved quantities, taking the form*

$$N_q = \sum_i c_{q,i} O_{q,i} \quad (13)$$

$$q = \{0, 1, \dots\} \quad (14)$$

where the sum is over every site i .

Now we need to specify conserved quantities. We know energy density E/V is conjugate to temperature, number density is conjugate to chemical potential, and we need these quantities to be well-defined in the thermodynamic limit. We have the expectations $\langle E \rangle \propto V$, $\langle \Delta E \rangle \propto V^\alpha$, $\alpha < 1$. Only when we have

$$\lim_{V \rightarrow \infty} \frac{\langle \Delta E \rangle}{\langle E \rangle} \rightarrow 0 \quad (15)$$

do we have a well-defined energy density. Only then can we have a well-defined temperature for our system as a whole. So if we have initial states which are macroscopic superpositions of states that have very different (extensively different, to be precise) expectation values for energy or number, then the energy is so uncertain that we don't know what energy we're supposed to be thermalizing to, so we can't define an appropriate equilibrium ensemble because we don't know what the right temperature should be.

Definition 5. *If we start with an out-of-equilibrium initial state $|\psi_0\rangle$, which could be a product state, and we time-evolve it under a Hamiltonian, we see what the dynamics look like. We prepared $|\psi_0\rangle$ by preparing the ground state of some Hamiltonian and then, immediately after time $t = 0$ **quench** to a different Hamiltonian, which means we vary the experimental parameters as fast as we can to time evolve our initial state under a different Hamiltonian. In practice people start with product states. All such initial states behave well, i.e. have sub-extensive uncertainties in all*

conserved quantities.

Suppose we have a state ρ which is a candidate thermal state. ρ can be mixed or pure. From ρ we get $\rho_A = \text{Tr}_B(\rho)$. We say ρ is thermal if $\rho_A \rightarrow \rho_A^{eq}$. In the ETH, we consider eigenstates as our candidate states. These are pure states, but the eigenstate can individually be thermal if, in this limit, if (for $|n\rangle$ the n^{th} energy eigenstate)

$$\text{Tr}_B(|n\rangle\langle n|) \rightarrow \rho_A^{eq} \quad (16)$$

(One candidate for ρ^{eq} is $\frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle\langle n|$)

Remark 4. For a system with no eigenstates, it is considered thermalized if we reach the maximum entropy state. E.g. for Floquet systems or $H(t)$ with no conservation laws, $\rho_A^{eq} \propto 1_A$. The Gibbs ensemble at infinite temperature also approaches the identity.

If we have a thermal state, all local information is set by the conserved quantities, and has forgotten everything about the initial condition, i.e. if $\rho_0 \rightarrow \rho_{eq}$, all local information at late times is only set by the macroscopic conserved quantities, and all other information about the initial state is hidden in non-observable operators.

The ETH has two versions. The strong version states that every MB eigenstate at finite $t > 0$ is a thermal state (the weak version states most). Suppose $|\psi_0\rangle$ is a non-equilibrium initial pure state. Then $\langle E \rangle = \langle \psi_0 | H | \psi_0 \rangle$, $\langle \Delta E \rangle = \sqrt{\langle \psi_0 | H^2 | \psi_0 \rangle - \langle H \rangle^2}$. If $|\psi_0\rangle$ is a well-defined state, then $\frac{\Delta E}{E} \rightarrow 0$ as $V \rightarrow \infty$. Then $|\psi_0\rangle$ has a well-defined energy density. We can use the energy-entropy curve to define temperature via $\frac{dS}{dE} = \frac{1}{T}$. If we wait over time, $|\psi_0\rangle \xrightarrow{t} \text{Tr}_B[|\psi_0(t)\rangle\langle\psi_0(t)|] = \text{Tr}_B[\rho_0(t)] = \text{Tr}_B[\frac{e^{-\beta H}}{Z}]$. Notice that the state determines the temperature, because the system is isolated. For an energy eigenstate,

$$H |\alpha\rangle = E_\alpha |\alpha\rangle \quad (17)$$

$$|\psi_0\rangle = \sum_\alpha c_\alpha |\alpha\rangle \quad (18)$$

$$|\psi(t)\rangle = \sum_\alpha c_\alpha e^{-iE_\alpha t} |\alpha\rangle \quad (19)$$

$$\langle \psi(t) | O | \psi(t) \rangle = \sum_{\alpha, \beta} c_\beta^* c_\alpha e^{-i(E_\alpha - E_\beta)\tau} \langle \beta | O | \alpha \rangle \quad (20)$$

$$= \sum_\alpha |c_\alpha|^2 O_{\alpha\alpha} + \sum_{\alpha \neq \beta} c_\beta^* c_\alpha e^{-i(E_\alpha - E_\beta)\tau} O_{\beta\alpha} \quad (21)$$

so we have all of these random phases $e^{-i(E_\alpha - E_\beta)\tau} O$. In these hamiltonians, we assume there are no degeneracies. Degeneracies usually come from symmetries, but we consider that we are working

in one block of the hamiltonian, i.e. all symmetries have already been accounted for. If we have

$$E_{\beta_1} + \dots + E_{\beta_n} = E_{\alpha_1} + \dots + E_{\alpha_n} \quad (22)$$

Then we know that $|\beta_1 \dots \beta_n\rangle$ is some permutation of $|\alpha_1 \dots \alpha_n\rangle$ so there are no symmetries in the problem.

$\|\mathcal{H}_{AB}\| \propto \exp(V)$, e.g. 2^L for spin- $\frac{1}{2}$ in a 1-dimensional chain. The energies are extensive, so if we take the full spectrum from E_0 to E_{2^L-1} , we know since E_0, E_{2^L-1} scale with V , then $E_{2^L-1} - E_0 \propto V$. We have jammed in 2^L energy levels in this extensive spacing. Thus we have

$$\frac{E_0}{E_{2^L-1} - E_0} = \frac{\alpha V}{2^V} \propto e^{-V} \quad (23)$$

so we have exponentially decaying energy differences as the system grows. In the limit as $t \rightarrow \exp(V)$, we have that all of the differences become some random numbers and the $e^{-i(E_\alpha - E_\beta)\tau}$ term goes to zero, so we are left with the diagonal term $\sum_\alpha |c_\alpha|^2 O_{\alpha\alpha}$.

So thermal equilibrium is the diagonal piece $\sum_\alpha |c_\alpha|^2 O_{\alpha\alpha} \rightarrow O_{eq} = \text{Tr}[O \rho_{Gibbs}(t)]$. If this expression has to hold for absolutely every initial state, then we know that $O_{\alpha\alpha}$ must be thermal. This is the ETH. Srednicki's statement in particular is

$$\langle a | O | \beta \rangle = O(E) \delta_{\alpha\beta} + e^{-\frac{S(E)}{2}} f(E, \omega) R_{\alpha\beta} \quad (24)$$

where O is a smooth function of $E := \frac{E_\alpha + E_\beta}{2}$ ($\langle E \rangle = \langle O \rangle_{Th} + \mathcal{O}(V^{-1})$, $\langle O \rangle_{Th} = \frac{1}{Z} \text{Tr}[e^{-\beta H} O]$), and $\delta_{\alpha\beta}$ is the diagonal part, $f(E, \omega)$ is a smooth function of E and ω (usually decaying for large ω), and $R_{\alpha\beta}$ is a random number of mean 0 and standard deviation 1. This tells us that off-diagonal matrix elements are exponentially small.

3 Quantum Statistical Mechanics

The Transverse Field Ising Model (TFIM) is the system with Hamiltonian

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x \quad (25)$$

The ground state of H is ferromagnetically ordered when $J > h$, is a paramagnet for $h > J$, and the system undergoes a phase transition when $h = J$. For a system of L spins, let

$$\sigma_i^\alpha = 1 \otimes \dots \otimes 1 \otimes \sigma_i^\alpha \otimes 1 \otimes \dots \otimes 1, \alpha \in \{x, y, z\} \quad (26)$$

3.1 Block-diagonalization

The TFIM has a symmetry with the operator $P = \prod_i \sigma_i^z$, which flips all spins in the z basis. It is easy enough to create P and H using scipy's sparse matrix methods. Since P flips all spins, $P^2 = 1$, so P 's eigenvalues are ± 1 . Thus we can transform H into a block-diagonal matrix, corresponding to the $+1$ and -1 sectors. By finding the eigenvectors and eigenvalues of these matrices, we can greatly speed up the solving of the Hamiltonian: (the following code is in python using numpy and scipy sparse matrix methods)

```
def diagonalize_with_P_v1(H, P, L):
    '''assuming P is diagonal, i.e. using
    P = gen_op_prod(sz_list)'''
    diagP = P.diagonal()
    posP_inds = np.where(diagP==1)[0]
    posP_sector = np.ix_(posP_inds, posP_inds)
    negP_inds = np.where(diagP==-1)[0]
    negP_sector = np.ix_(negP_inds, negP_inds)
    Hfull = H.toarray()
    H_negP = Hfull[negP_sector]
    H_posP = Hfull[posP_sector]
    evals_negP = linalg.eigh(H_negP, eigvals_only=True)
    evals_posP = linalg.eigh(H_posP, eigvals_only=True)
    return [evals_negP, evals_posP]
```

Here is an alternative method for block-diagonalizing H : (using argsort)

```
def permute_in_place(M, p):
    '''reorder rows and columns of matrix M
    according to permutation vector p'''
    N = M.shape[0]
    for i in range(N):
        M[:, i] = M[p, i]
```

```

for i in range(N):
    M[i,:] = M[i,p]

def plot_matrix_structure(H):
    plt.imshow(np.where(np.abs(H)>1e-5,1,0),cmap='Greys')
    plt.xlabel(r'index_{$i$}')
    plt.ylabel(r'index_{$j$}')
    plt.title(r'$H_{ij}$')
    plt.show()

def diagonalize_with_P_v2(H, P, L, print_Hfull = False):
    '''assuming P is diagonal, i.e. using
    P = gen_op_prod(sz_list)'''

    def get_block_inds(conserved_op_eigs):
        _, inds = np.unique(conserved_op_eigs, return_index = True)
        return inds

    diagP = P.diagonal()
    perm = np.argsort(diagP)
    Hfull = H.toarray()
    permute_in_place(Hfull, perm)

    if print_Hfull:
        plot_matrix_structure(Hfull)

    _, block_inds = np.unique(diagP, return_index = True)
    block_inds = np.append(block_inds, [Hfull.shape[0]])

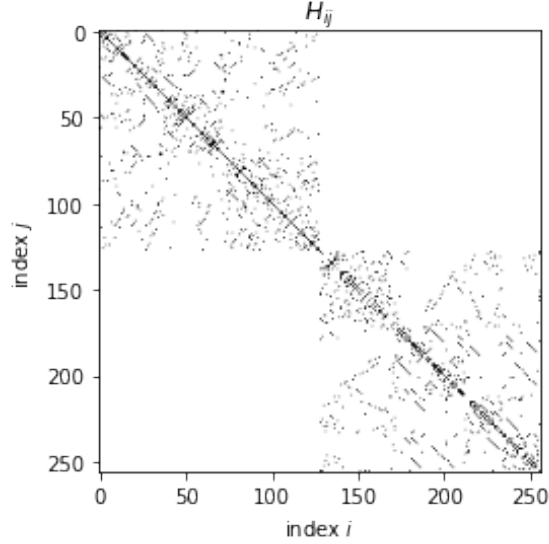
    evals = []
    for i in range(len(block_inds)-1):
        evals = np.append(evals, np.linalg.eigvalsh(Hfull[block_inds[i]:block_inds[i+1],
        block_inds[i+1]:block_inds[i+2]]))

    evals = np.sort(evals)

```

```
return evals
```

After this block-diagonalization, we get the following matrix representation of H with $L = 8$:



Thus we've partially split the Hamiltonian into two blocks of size $2^{L-1} \times 2^{L-1}$.

In general, to diagonalize H using parity symmetry, we need the eigenvectors of P . In the x -basis, they are of the form $|\pm_1, \pm_2, \dots, \pm_L\rangle$. This is done in the following code:

```
def gen_parity_eigenstate(n, L):
    '''generates the n'th parity eigenstate
    by taking binary representation of n and
    translating 0 -> |+> and 1 -> |-> '''
    b = np.binary_repr(n,width=L)
    state = [1,1] if b[0]=='1' else [1,-1]
    for j in range(1,L):
        if b[j]=='1':
            state = sparse.kron(state,[1,1], 'csc')
        else:
            state = sparse.kron(state,[1,-1], 'csc')
    state = state/ 2**(L/2) # properly normalize
    return state

def diagonalize_with_P_v3(H, P, L, print_Hfull = False):
    '''assuming P is in x-basis, i.e. using
    P = gen_op_prod(sx_list)'''
```

```

even_projector = np.zeros((2**(L-1),2**L))
odd_projector = np.zeros((2**(L-1),2**L))
even_counter, odd_counter = 0, 0

for n in range(2**L):
    state = gen_parity_eigenstate(n, L)

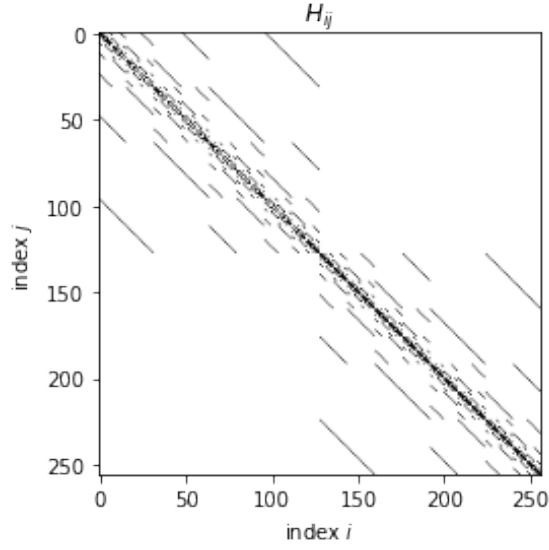
    if state[0,0]==state[0,2**L-1]:
        even_projector[even_counter, :] = state.toarray()
        even_counter += 1
    else:
        odd_projector[odd_counter, :] = state.toarray()
        odd_counter += 1

if print_Hfull:
    U = np.zeros((2**L,2**L))
    U[:2**(L-1), :] = even_projector
    U[2**(L-1):, :] = odd_projector
    H_transformed = U@H@U.T
    plot_matrix_structure(H_transformed)

# Get the parity sectors of the Hamiltonian with the projectors
H_even = even_projector@H@even_projector.T
H_odd = odd_projector@H@odd_projector.T
evals_even = linalg.eigh(H_even, eigvals_only=True)
evals_odd = linalg.eigh(H_odd, eigvals_only=True)
return np.sort(np.concatenate((evals_even, evals_odd)))

```

Resulting in the following matrix:



3.2 Open boundary conditions, parity and reflection

For open systems, we have spatial reflection symmetry, with reflection operator $R : i \rightarrow L - i$. For periodic systems, we have R , but we also have spatial translation invariance, with translation operator $T : i \rightarrow i + 1$.

Remark 5. Notice that $[H, R] = [H, T] = 0, [R, T] \neq 0$.

For open boundary conditions, we can simultaneously diagonalize H using R and P . $R^2 = 1$, so the eigenvalues of R are ± 1 . This simultaneous diagonalization should give blocks of size $2^L/4$.

Let $|\Psi_P\rangle$ be the eigenstates of P , i.e. states of the form $|\pm_1, \pm_2, \dots, \pm_L\rangle$. Then

$$|\Psi_{RP}\rangle := \frac{|\Psi_P\rangle \pm R|\Psi_P\rangle}{\sqrt{2}} \quad (27)$$

are eigenstates of both P and R . We implement this with the following code:

```
def bin2int(b):
    L = len(b)
    n = 0
    for i in range(L):
        if b[i] == '1':
            n += 2 ** (L - i - 1)
    return n

def lex_order(b1, b2):
    '''order bit strings based on the value of their
```

```

        integers in base 10. Smallest first.'''
n1, n2 = bin2int(b1), bin2int(b2)
if n1 == n2:
    return 0
elif n1 > n2:
    return 1
else:
    return -1

def gen_reflection_op(L):
    N = 2**L
    row, col = np.arange(N), np.zeros(N)
    data = np.ones(N)
    for i in range(N):
        col[i] = bin2int(np.binary_repr(i,width=L)[::-1])
    R = sparse.csr_matrix((data, (row, col)), shape=(N, N))
    return R

def ispalindrome(b):
    return (bin2int(b) == bin2int(b[::-1]))

def get_bitstring_parity(b):
    return 1 if b.count('0')%2 == 0 else -1

def diagonalize_with_PR(H, P, R, L, print_Hfull = False):
    P_even_R_even_states = []
    P_even_R_odd_states = []
    P_odd_R_even_states = []
    P_odd_R_odd_states = []

    for i in range(2**L):
        b = np.binary_repr(i,width=L)

        if ispalindrome(b):
            state = gen_parity_eigenstate(i, L)

```

```

    peig = get_bitstring_parity(b)
    if peig > 0:
        P_even_R_even_states.append(state.toarray()[0])
    else:
        P_odd_R_even_states.append(state.toarray()[0])

    if lex_order(b, b[::-1]) <= 0:
        continue

    state = gen_parity_eigenstate(i, L)
    reflected_state = gen_parity_eigenstate(bin2int(b[::-1]), L)

    R_even_state = (state + reflected_state)/np.sqrt(2)
    R_odd_state = (state - reflected_state)/np.sqrt(2)

    peig = (R_even_state*P*R_even_state.H).toarray()[0,0]
    if peig > 0:
        P_even_R_even_states.append(R_even_state.toarray()[0])
        P_even_R_odd_states.append(R_odd_state.toarray()[0])
    else:
        P_odd_R_even_states.append(R_even_state.toarray()[0])
        P_odd_R_odd_states.append(R_odd_state.toarray()[0])

    if print_Hfull:
        U_PR = sparse.csr_matrix(np.array(P_even_R_even_states+P_even_R_odd_states+P_odd_
H_PR = U_PR * H * U_PR.H
        plot_matrix_structure(H_PR.toarray())

    ee_projector = sparse.csr_matrix( np.array(P_even_R_even_states))
    eo_projector = sparse.csr_matrix( np.array(P_even_R_odd_states))
    oe_projector = sparse.csr_matrix( np.array(P_odd_R_even_states))
    oo_projector = sparse.csr_matrix( np.array(P_odd_R_odd_states))

    H_ee = ee_projector@H@ee_projector.H
    H_eo = eo_projector@H@eo_projector.H

```



```

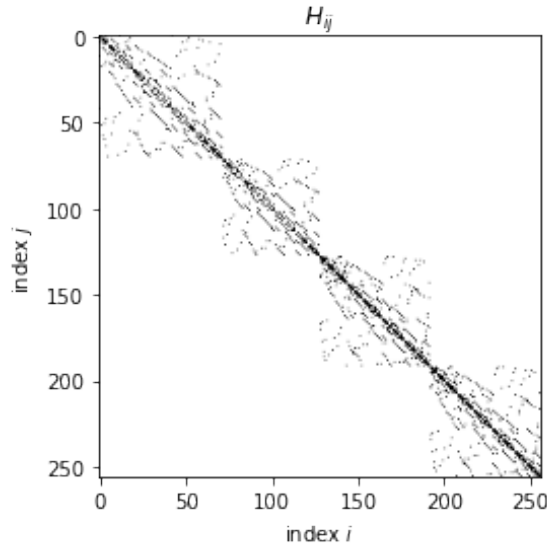
H_oe = oe_projector@H@oe_projector.H
H_oo = oo_projector@H@oo_projector.H

evals_ee = linalg.eigh(H_ee.toarray(), eigvals_only=True)
evals_eo = linalg.eigh(H_eo.toarray(), eigvals_only=True)
evals_oe = linalg.eigh(H_oe.toarray(), eigvals_only=True)
evals_oo = linalg.eigh(H_oo.toarray(), eigvals_only=True)

return np.sort(np.concatenate((evals_ee, evals_eo, evals_oe, evals_oo)))

```

This results in the diagonalization



3.3 Periodic boundary conditions, parity and translation

For periodic boundary conditions, we can use P and T . T splits the Hilbert space roughly into L blocks of length $2^L/L$. This is a much greater advantage than the factor of 2 from spatial reflection. P further splits each of these blocks into 2, giving a total of $2L$ blocks with size $\sim 2^L/(2L)$.

```

def translate_bitstring(b, d):
    '''translates the bitstring b by d positions'''
    b1 = b[0 : d]
    b2 = b[d :]
    return b2 + b1

def get_cycle_length(b):
    '''computes the number of translations
    allowed before bitstring b comes back

```

```

        to itself'''
n = bin2int(b)
L = len(b)
for i in range(1,L):
    if n == bin2int(translate_bitstring(b, i)):
        return i
return L

def get_T_representatives(L):
    '''get one representative from each set
    of basis states that are equivalent
    up to action of the translation op T'''
    reps = []
    for n in range(2**L):
        flag = False
        b = np.binary_repr(n, width=L)
        for d in range(1, L):
            if translate_bitstring(b, d) in reps:
                flag = True
        if not flag:
            reps.append(b)
    return reps

def get_PT_eigenstate(rep, k, L, cycle):
    state = gen_parity_eigenstate(bin2int(rep), L)
    for i in range(1, cycle):
        state += np.exp(2 * np.pi * 1j * k * i/L) *
            gen_parity_eigenstate(bin2int(translate_bitstring(rep, i)), L)
    state = state/np.sqrt(cycle)
    return state

def diagonalize_with_PT(H, P, L, print_Hfull = False):
    tol = 0.1/L
    P_even_states = []
    P_odd_states = []

```

```

reps = get_T_representatives(L)
cycle_lengths = {r:get_cycle_length(r) for r in reps}
n_up = {r:r.count('1') for r in reps}

countkp = 0
countkm = 0

indsplus = []
indsminus = []

for k in range(L):
    indsplus.append(countkp)
    indsminus.append(countkm)

    for r in reps:
        tmp = k*cycle_lengths[r]/L
        if np.abs(np.round(tmp) - tmp) < tol:
            state = get_PT_eigenstate(r, k, L, cycle_lengths[r])
            peig = np.real( (state*P*state.H).toarray()[0,0])
            if peig > 0:
                P_even_states.append(state.toarray()[0])
                countkp = countkp+1
            else:
                P_odd_states.append(state.toarray()[0])
                countkm = countkm+1
    indsplus.append(countkp)
    indsminus.append(countkm)

U_PT = sparse.csr_matrix(np.array(P_even_states+P_odd_states))
H_PT = U_PT * H * U_PT.H
Hfull = H_PT.toarray()

if print_Hfull:
    plot_matrix_structure(Hfull)

```

```

block_inds = np.append(np.array(indsplus), np.array(indsminus[1:]+indsplus[-1]))

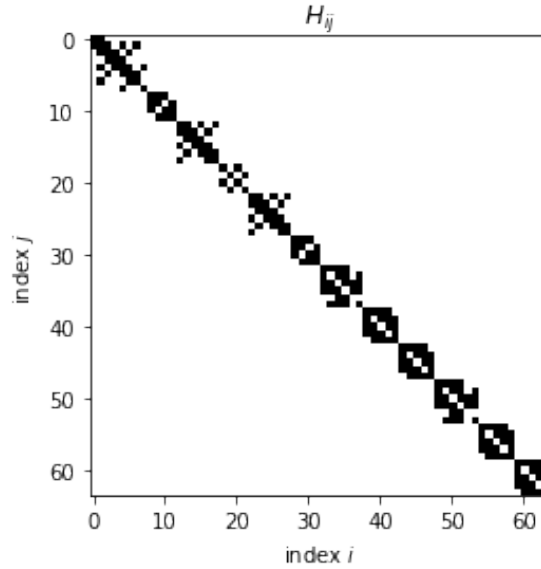
evals = []
for i in range(len(block_inds)-1):
    evals = np.append(evals, np.linalg.eigvalsh(Hfull[block_inds[i]:block_inds[i+1], block_inds[i]:block_inds[i+1]]))

evals = np.sort(evals)

return evals

```

This results in the diagonalization



While R and T do not commute, in the $k = 0$ and $k = \pm\pi$ sectors, reflection symmetry/anti-symmetry can still be utilized. This does not provide a large speed-up in general as compared to the gains seen by T .

4 Quantum chaos

We've seen that it is already pretty hard to extend what thermalization means from the classical to quantum context. It is similarly difficult to do this with chaos.

In classical chaos we usually use the description of a **Lyapunov exponent**

$$\lambda_L \sim \frac{dq(t)}{dq_0} \quad (28)$$

but having a Lyapunov exponent in the quantum case is harder because there's already a phase ambiguity, and the Schrödinger equation is linear; evolving using unitary time evolution doesn't change the amplitude. In quantum chaos we use “**OTOCs**,” or **Out-of-Time-Ordered Correlators**; and these are useful quantities for probing chaos in systems like black holes or a system with many degrees of freedom (large n systems). But for spin systems or very quantum systems we consider here it is difficult to make this work; they fail to give exponential dependence when considering strongly quantum spin- $\frac{1}{2}$ systems.

We also have to consider the quantum analogue of **ergodicity**, or the exploration of phase space. Traditionally this is described using canonical coordinates (q, p) , which are conjugate coordinates, and again it is hard to translate this to the quantum context because we can't even define position and momentum at the same time due to Heisenberg uncertainty.

The point is that, although we have some tools to describe these, like OTOCs and Random Matrix Theory (RMT), we need a whole new framework. We also can use the Riemann Zeta function; the eigenvalues of a random matrix are related to the zeros of the Riemann Zeta function via *Montgomery's Conjecture*. The relationship with RMT and quantum chaos is difficult via **Gutzwiller's Trace Formula**, which links the spectrum of random matrices in terms of the density of states with something classical which is a sum over periodic orbits:

$$\rho(E) \sim \sum_{\rho_0} A_{\rho_0} e^{iS\rho_0/\hbar} \quad (29)$$

There is a distinction between few-particle quantum chaos with a semiclassical analogue; we need to think about few particle quantum chaos in the semiclassical limit vs many-body quantum chaos for systems that are far from any kind of semiclassical limit (e.g. strongly spin- $\frac{1}{2}$ systems). There this notion of periodic orbits becomes far less well-defined because there isn't any natural classical analogue. Lyapunov exponents have to do with small perturbations in phase space leading to exponential differences in trajectories.

In classical chaos, we have a $2n$ -dimensional phase space

$$\mathcal{R} = (q_1, \dots, q_n, p_1, \dots, p_n) \quad (30)$$

evolving via Hamilton's equations

$$\frac{dq}{dt} = -\frac{\partial H}{\partial p} \quad (31)$$

$$\frac{dp}{dt} = \frac{\partial H}{\partial q} \quad (32)$$

Already in this system we assume that energy is conserved, and anything additionally conserved constrains our motion in phase space even more, which is why if we say in an integrable system, in the classical context we are constrained to the **invariant torus**. Classically, 2-body motion is pretty easy to do, but adding a third body classically makes the problem very difficult to solve.

4.1 Random Matrix Theory

This was started in the 1950's by Wigner, trying to write down Hamiltonians for interactions within nuclei. We wrote a Hamiltonian as an $N \times N$ random matrix

$$H \rightarrow \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NN} \end{pmatrix} \quad (33)$$

where the entries were from a Gaussian distribution. The mean doesn't usually matter, but the variance is important.

Definition 6. *Entries taken from a Gaussian distribution independent from one another is denoted **i.i.d.** (This doesn't mean they are taken from a Gaussian but they are independently identically taken from this Gaussian distribution)*

Remark 6. *The universal properties of the spectra of a nuclei looks remarkably well like that of a random matrix. A random matrix has become one of the best known diagnostics of quantum chaos. When you have a system which is chaotic, ergodic, or whatever, the spectra of these systems end up sharing many universal properties with these random matrices.*

The whole problem with RMT is starting from a matrix such as the one above and computing the characteristic equation i.e. finding the eigenvalues $\lambda_1, \dots, \lambda_n$. We start with some probability distribution on our entries, $p(x_{11}, \dots, x_{NN})$. If they're all random we have N^2 , if symmetric then $\frac{N(N+1)}{2}$. Then we want to find a probability distribution on the eigenvalues $p(\lambda_1, \dots, \lambda_n)$ but most of the time on the spacings $p(s_1, \dots, s_{n-1})$, $s_i = \lambda_{i+1} - \lambda_i$.

Example 1. For a random real and symmetric 2×2 matrix, we have

$$H = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}; x_1, x_3 \in N(0, 1), x_2 \in H(0, \frac{1}{2}) \quad (34)$$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad (35)$$

$$-\lambda^2 + \text{Tr}[H]\lambda + \det(H) = 0 \quad (36)$$

$$\lambda_{1,2} = \frac{1}{2} \Gamma(x_1 - x_3)^2 \pm \sqrt{f(x_1, x_2, x_3)} \quad (37)$$

$$S = \lambda_2 - \lambda_1 = \sqrt{(x_1 - x_3)^2 + 4x_2^2} \quad (38)$$

$$p(s) = \int dx_1 \int dx_2 \int dx_3 p(x_1)p(x_2)p(x_3) \int (s - \sqrt{(x_1 - x_3)^2 + 4x_2^2}) \quad (39)$$

$$(40)$$

We then get the level spacings as

$$p(s) \propto s^\beta e^{-s^2} \quad (41)$$

where $\beta = 1$ for real entries and $\beta = 2$ for complex entries. This is known as **Wigner's Surmise**.

The ensemble that we draw our entries from depends on the symmetries of the Hamiltonian. If we have a real Hamiltonian, it is described by the **Gaussian Orthogonal Ensemble** (GOE), rather than the **Gaussian Unitary Ensemble** (GUE). The diagonals always have to be real to make the matrix Hermitian. We also have a symplectic ensemble (GSE), in which $\beta = 4$.

Remark 7. In the GOE ensemble we can diagonalize our Hamiltonian via an orthogonal matrix, and this means that the entries are real, which means we have a system which have time-reversal symmetry because energy is conserved. Thus if we have a system with time-reversal symmetry we're going to want to draw our matrices from the GOE, while if we have a Hamiltonian that breaks time-reversal symmetry we'll want to model it with the GUE. One representative of time-reversal symmetry is complex conjugation.

In one particle quantum chaos we can see the use of RMT. For something with a classical analogue, e.g. a billiard ball in an ellipsis-shaped area, the system is chaotic: it bounces all around the ellipsis. In the quantum case, we have $\hat{H} = \frac{\hat{p}^2}{2m} + V$ in the geometry in the problem and look at these quantum energy levels and look at the distribution of level spacings, we get that, for

quantum systems that have a classical analogue to a classically chaotic system, we end up getting BFS statistics. On the other hand, for quantum systems for which the classical limit is integrable, we end up with Poisson level statistics. This is known as the quantum chaos conjecture.

Now we are getting to many-body quantum chaos, where we consider systems like strong spin- $\frac{1}{2}$ systems, and there is no semi-classical analogue. There is no classical analogue to a spin- $\frac{1}{2}$ Ising chain or XXZ chain. For thermalizing systems, their spectra look like results from RMT. For integrable systems or MBL, there are many symmetries in our problem, our level statistics end up looking Poisson.

In large N , $p(s)$ for RMT and Hamiltonians match very well. For the density of states $p(E)$, the random matrix gives a semicircle, while near the edges, for most physical Hamiltonians, $p(E)$ looks much more like a Gaussian.

Definition 7. *We have been considering $\lambda_{i+1} - \lambda_i$. There is another diagnostic called the **Spectral Form Factor** that probes the correlations between all distances of spectrum as a function of time $K(t)$. If we go to early times we find large distances and if we look at later times we look at very nearby energies.*

Taking the staircase function $N(E) := \sum_n O(E - E_n)$ and going to the density of states $\rho(E) = \frac{dN}{dE} = \sum_n \delta(E - E_n)$, and then we go from the density of states to a two-point function $R(E) = \langle \rho(E_0 + \frac{E}{2}) \rho(E_0 - \frac{E}{2}) \rangle_{E_0}$ (in a small-energy window), and then going from the two-point function we get the spectral form factor by taking the Fourier transform of the two-point function:

$$N(E) \rightarrow \rho(E) \rightarrow R(E) \rightarrow K(t) \quad (42)$$

$$\sum_n O(E - E_n) \rightarrow \sum_n \delta(E - E_n) \rightarrow \langle \rho(E_0 + \frac{E}{2}) \rho(E_0 - \frac{E}{2}) \rangle_{E_0} \xrightarrow{F.T.} \int e^{iEt} R(E) dE \quad (43)$$

$$= \frac{1}{N} \sum_{n,m} e^{it(E_n - E_m)} \quad (44)$$

with some normalization. $K(t)$ decreases in time until we reach the **Thouless time**, where level repulsion starts kicking in, and then $K(t)$ increases with time. This manifests itself in what is known as a “ramp” in $K(t)$. See 1.

For large time, since the exponent is negative, we essentially get random phases that add to zero, so we reach a plateau in $K(t)$ starting at the **Heisenberg time**, which is around the dimension of the Hilbert space. The height of this plateau depends on how we normalize the spectral form factor ($\frac{1}{N}$) and if we have degeneracies, because if we have degeneracies we don’t need $n = m$

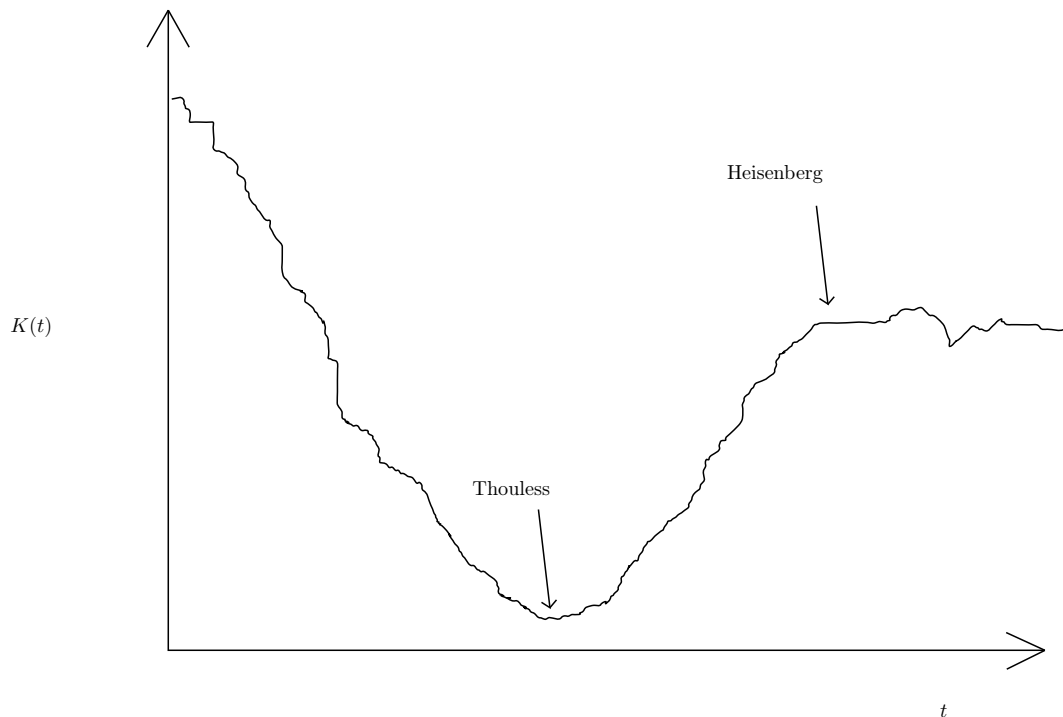


Figure 1: $K(t)$ graph

for $E_n = E_m$; therefore if we have degeneracies the plateau is essentially higher. The benefits of studying $K(t)$ are:

1. It is not sensitive to being in the correct symmetry sector (not proven but checked numerically for many systems)
2. We can calculate $K(t)$ without having to know our full spectrum: $K(t) \sim \sum_{p_0} A_{p_0} e^{i \frac{S_{p_0}}{\hbar}}$ in the semi-classical limit.

5 Breakdown of Thermalization

Integrable systems don't thermalize, and there is an interesting experiment by David Weiss's group in 2006 that created a "quantum newton's cradle," which took a large amount of strongly interacting atoms (a bose gas), tuned their parameters to attain integrability, and they observed oscillation for long periods of time.

The only known generic, stable exceptions to thermalization localized systems. There are generally three versions of localization:

1. Many-body localization (MBL). These are interacting, isolated, highly excited systems. This started with Anderson in 1958, but most literature is recent.
2. Anderson localization. These are isolated, highly excited non-interacting particles.

3. Ground state phase transitions in the presence of disordered interactions. These are not necessarily isolated. Some examples of transitions are metal-to-insulator, superconductor-to-insulator, superfluid-to-Bose-glass transitions.

The first report of a system where thermalization may fail was by Anderson in 1958. He considered a system of disordered (randomly placed) spins in a semiconductor with no mobile charges (so the spins were the only degrees of freedom that were relevant). He was considering the system in the presence of random interactions; spin-spin exchange interactions (e.g. $xx + yy$), density-density interactions (e.g. coulomb interactions), etc. These spins were in an applied Zeeman field. No spin-phonon interactions were present, so it was an isolated system at high temperature. The setting that was novel was the idea that Anderson localization occurs in systems that are not translationally invariant (in the presence of “quenched disorder”). All realistic solids will have disorder in the form of impurities, defects, etc. This was revolutionary because the starting point of the study of crystalline solids uses Bloch’s theorem, where we assume discrete translational symmetry and eigenstates are delocalized Bloch waves. Anderson discovered that disorder can *fundamentally* change the behavior of materials, leading to brand new phenomenology: Bloch’s theorem tells us that our wavefunctions are delocalized for all of real space. Anderson showed that disorder can *exponentially* localize wavefunctions in real space. Thus if we solve a problem with disorder, we’ll find that our eigenstates, instead of being spread across all space, remain exponentially stuck near different regions in real space. This is counterintuitive from usual quantum mechanics; yes we have a disordered potential, we still might be able to tunnel, so we can get delocalized wavefunctions.

“Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.” - Philip W. Anderson

In general there are very few exact results about localized.

5.1 Single-Particle Anderson Localization

We have a system with Hamiltonian

$$H = \sum_i h_i c_i^\dagger c_i + J(c_i^\dagger c_{i+1} + h.c.), \quad h_i \in [-W, W] \quad (45)$$

where c_i are fermionic operators $\{c_i^\dagger, c_j\} = \delta_{ij}$. This localization comes from detuning (for now this means that the potentials on adjacent sites are very different). The source of this detuning can be disorder, deterministic potentials (quasi-periodicity, e.g. $V(x), W \cos(kx)$). This gives a

localization

$$|\phi(r)|^2 \sim e^{-r/\xi} \quad (46)$$

of single-particle wavefunctions, in the locator expansion $J \ll W$ (strong localization). For example, consider a system with 2 sites, with site energy ϵ_1, ϵ_2 , and a hopping term. Then

$$H = \begin{pmatrix} \epsilon_1 & J \\ J & \epsilon_2 \end{pmatrix} \quad (47)$$

when $|\epsilon_1 - \epsilon_2| \ll J$, then

$$H \approx \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \quad (48)$$

Then we have

$$|\Psi_{1,2}\rangle \approx \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle) \quad (49)$$

so our wavefunctions are delocalized.

If, however, $|\epsilon_1 - \epsilon_2| \gg J$, then the wavefunctions look like that in ??.

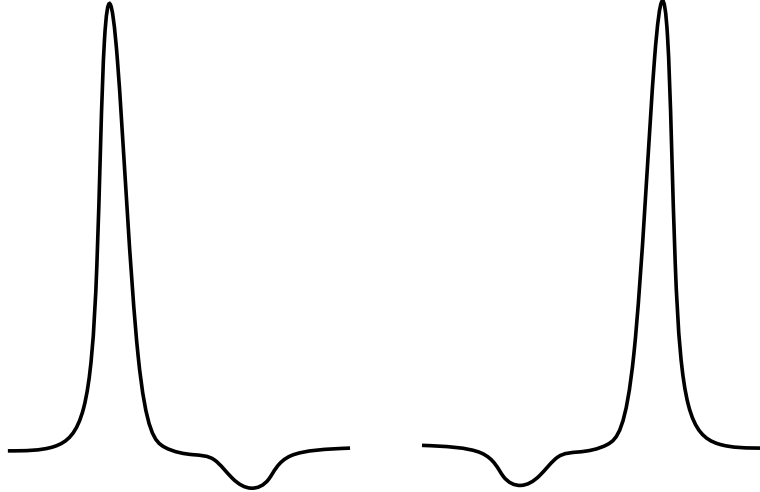


Figure 2: Wavefunction when $|\epsilon_1 - \epsilon_2| \gg J$.

so just for the 2 by 2 example, the degree of the hybridization of the wavefunctions is set by this competition between the differences in energies between the two sites, and the mixing between the

sites induced by the hopping.

So when we have a system with many different sites, and $W \gg J$, the off-resonant hopping is not able to hybridize our single-particle wavefunctions. What Anderson asked is if we had some particle localized in space, how to that particle spread out in time. Well, at first-order, we only look at nearest neighbor hoppings:

$$\frac{V_{i,i+1}}{\epsilon_i - \epsilon_{i+1}} \quad (50)$$

At next-order, this site can couple to its next nearest neighbor.

$$\frac{V_{i,i+1}}{\epsilon_i - \epsilon_{i+1}} \frac{V}{\Delta\epsilon} \quad (51)$$

So at higher and higher orders, we get multiple factors of V divided by ΔE :

$$\frac{V_{i,i+1}}{\epsilon_i - \epsilon_{i+1}} \frac{V}{\Delta\epsilon} \dots \frac{V}{\Delta\epsilon} \quad (52)$$

so how likely are we to get a resonance, i.e. when $V \sim \Delta E$. The key point is that, as we go out to a distance r , our effective hopping t_{eff} , for some site and its distant partner, we have

$$t_{eff} \sim t^r \sim \exp\left(\frac{r}{\xi}\right) \quad (53)$$

But notice that $\Delta E(r) \sim \text{poly}(\frac{1}{r})$, because, when we examine how far we have to go before we find an energy difference that is small, we sample r different spacings from our distribution, and we want to know the chance of hitting something small. So the detunings are going down by a polynomial of r , but the effective couplings are going down exponentially in r , so we always get that this hopping doesn't really hybridize. Thus our eigenstates have a localization center, and end up looking like

$$|\Psi_\alpha(r_\alpha)|^2 \sim e^{-|r-r_\alpha|/\xi} \quad (54)$$

where $\xi = f(\frac{W}{J})$. When $J \geq W$ (weak localization), in 1 and 2 dimensions, no matter how weak a disorder strength is, it is still enough to exponentially localize all our single-particle wavefunctions.

3-dimensions, we can get a “**mobility edge**,” where we can have the edge seen in 3.

In 1D we can analytically prove localization (strong and weak).

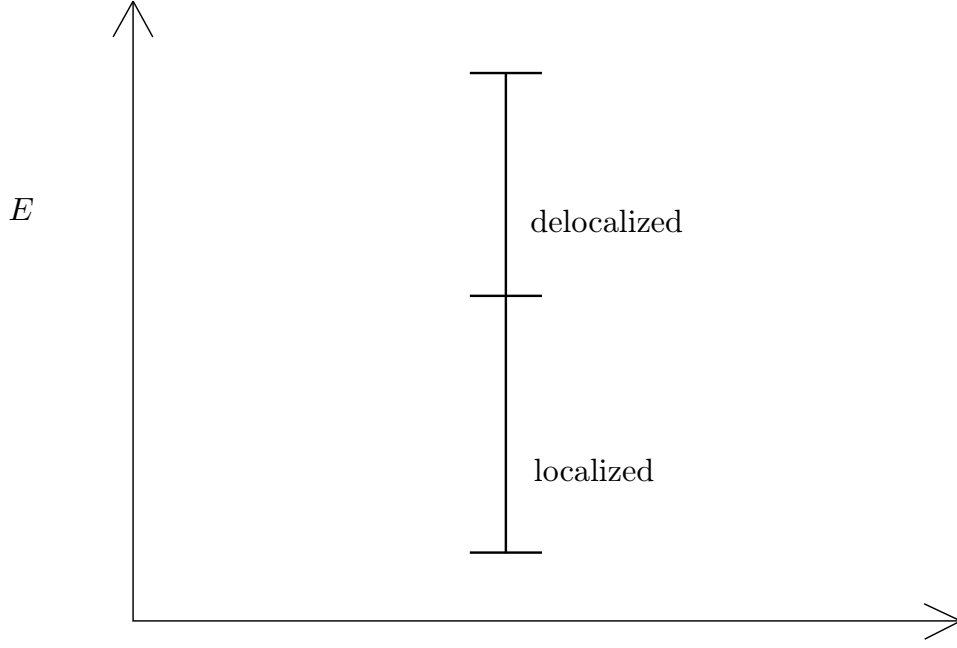


Figure 3: Mobility edge.

5.2 Many-body localization

For MBL systems, we have

$$H = \sum_{\alpha} e_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_{\alpha}^{\dagger} a_{\beta}^{\dagger} a_{\gamma} a_{\delta} \quad (55)$$

where $a_{\alpha}^{\dagger} = \sum_r d_{\alpha}(r) c_r^{\dagger}$ are single particle Anderson eigenorbitals. See [4](#).

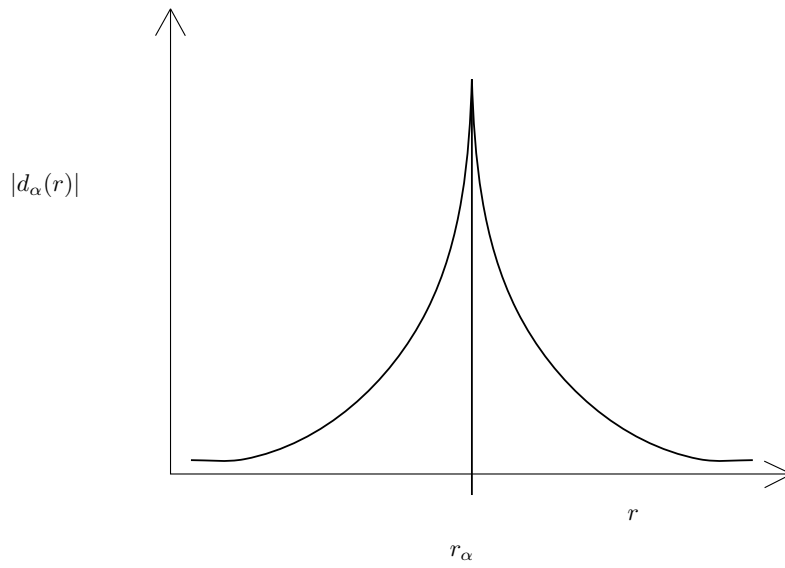


Figure 4: Eigenorbitals.

and we've added in density-density interactions $V = \sum_i c_i^\dagger c_i c_{i+1}^\dagger c_{i+1}$. Suppose we have a strong Anderson localized problem (so there's no single-particle mobility edge). So we have strong localized single-particle eigenorbitals, and now we want to add in weak interactions, and ask if that localization is stable in the addition of interactions. So now we do perturbation theory in the interaction term V .

Historically, MBL was defined via transport properties, e.g. by examining σ_{DC} (conductivity). Einstein's relation gives

$$\sigma = e^2 D \nu \quad (56)$$

where D is the diffusion constant and ν is the density of states, so in the absence of diffusion, $\sigma = 0$. Since $\rho(T) = \frac{1}{\sigma(T)}$, for

1. Metal: $\rho(T) \rightarrow \rho_0$ as $T \rightarrow 0$
2. Insulator: $\rho(T) \rightarrow \infty$ as $T \rightarrow 0$
3. MBL insulator $\rho(T) = \infty$ for some range of $T > 0$

These historical definitions don't tell the full story; transport can be subdiffusive $\sigma_{DC} = 0$ even in thermalizing systems because of Griffiths effects.

In a single-particle system with a mobility edge, conductivity has Arrhenius form

$$\sigma(T) \propto e^{-(E_{mc} - E_F)/T} \quad (57)$$

with E_F the Fermi energy. In MBL, we want to consider systems with no single-particle mobility edge. Here, in the presence of phonons, we get phonon-assisted hopping and variable-range conductivity

$$\sigma(T) \propto T^\alpha e^{(\delta/T)^{\frac{1}{d}+1}} \text{ (Mott)} \quad (58)$$

This will be true in the presence of any bath with a continuous spectrum down to $\omega \rightarrow 0$.

The general framework we'll take is the following. Choose a local basis. In this basis, H is as close to diagonal as possible. Then we can rewrite our H as

$$H = H_D + H_{OD} \quad (59)$$

we examine H_{OD} to find out whether or not we can have resonances. When we have many-body dynamics, we can treat H_D as $\sum_{\alpha} E_{\alpha}^{sp} a_{\alpha}^{\dagger} a_{\alpha}$ and H_{OD} as $\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} a_{\alpha}^{\dagger} a_{\beta}^{\dagger} a_{\gamma} a_{\delta}$. When we write H like this, we view our many-body dynamics as single-particle dynamics on this lattice of basis states. The basis states are then characterized by the occupations of all of the single-particle orbitals (Anderson orbitals)

$$|n\rangle = |n_1 = \{0, 1\}, n_2 = \{0, 1\}, \dots\rangle = \prod (a_{\alpha}^{\dagger})^{n_{\alpha}} |0\rangle \quad (60)$$

Then we want to connect this to a different state $|m\rangle$ in which the occupations of four of these orbitals have been changed:

$$|m\rangle = a_{\alpha}^{\dagger} a_{\beta}^{\dagger} a_{\gamma} a_{\delta} |n\rangle \quad (61)$$

See 5.

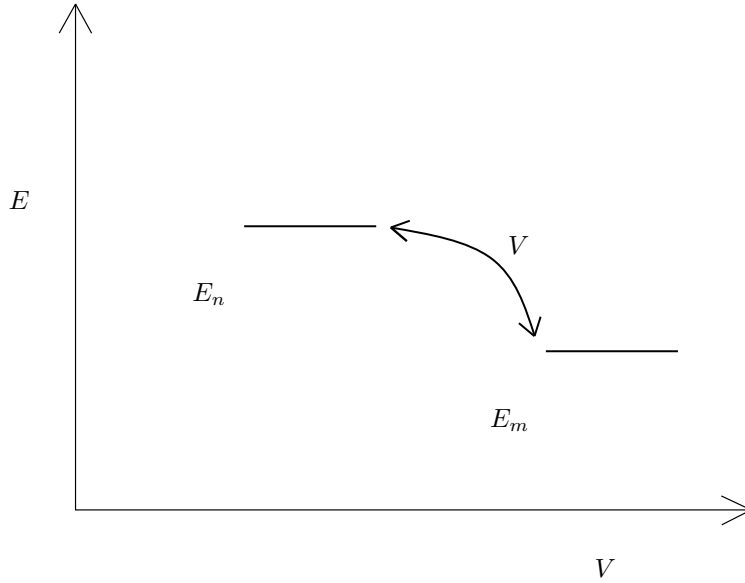


Figure 5: Energy jumping via V .

Is $V \ll E_n - E_m$? In this problem we say that localization is stable if we find $|V| \ll E_{\gamma} + E_{\delta} - E_{\alpha} - E_{\beta}$. Why? This comes back to the (non-index) δ in the expression

$$\sigma(T) \propto T^{\alpha} e^{-(\delta/T)^{1/d+1}} \quad (62)$$

Definition 8. δ is a very important scale in localization. It is the average energy difference between

states that live within a localization volume.

In a single-particle context, if we have a density of states $\nu = \frac{\# \text{states}}{\text{energy} \cdot \text{volume}}$, then

$$\delta = \frac{1}{\nu \xi^t} \quad (63)$$

$V(r)$ is short ranged, i.e. $V(r) = a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta$. Unless these orbitals $\alpha, \beta, \gamma, \delta$ with localization centers $r_\alpha, r_\beta, r_\gamma, r_\delta$ are not within some localization volume (or length), this matrix element is time. But when the centers are within some localization length, states have single-particle energies that are different by order W , so we get some discrete set of energy levels. Then we get a locator expansion within this interaction strength V . In the literature they have

$$\langle \Phi_m^{MB} | C_i^\dagger | \Psi_n^{MB} \rangle = \sum_\alpha V_\alpha O_\alpha \quad (64)$$

For localization with interactions, we consider the following Hamiltonian

$$H = \sum_i h_i \sigma_i^z + J \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z) \quad (65)$$

where the $h_i \sigma_i^z$ are the “detuned” onsite fields (disordered/quasi-periodic), $\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y$ represents hopping, and $\sigma_i^z \sigma_{i+1}^z$ represents interactions. We can do a Jordan-Wigner transformation and map to spinless fermions, where up \mapsto occupied, down \mapsto empty:

$$H \mapsto \sum_i h_i \sigma_i^z + J \sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_i, c_{i+1}^\dagger c_{i+1}) \quad (66)$$

This is the “**Standard Model**” of MBL, an interacting version of Anderson’s model. The parameters are J, W and, as a function of W/J , the system can be localized or thermalizing.

The simplest example is $J = 0$. Here $|n\rangle = |\uparrow \downarrow \uparrow \uparrow \dots \downarrow\rangle$. This is not thermal; it violates ETH. There is no transport of spin/energy. The dynamics is just Larmor precession. There are extensively many constants of motion, $\{\sigma_i^z\}$ are constant.

$$[H, \sigma_i^z] = 0, [\sigma_i^z, \sigma_j^z] = 0 \quad (67)$$

(coming from l -bits, to be discussed later) Here we have 2^L energy states. If we take a state’s single spin and flip it, the energy changes by $2|h_i|$ which is of order W , which is far away in energy. If we want to take that state and find a state with energy arbitrarily close to it, we’ll need to flip order L states. This definitely violates ETH: the energies don’t vary smoothly with local operators. See

6.

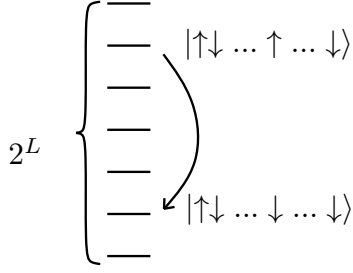


Figure 6: Energy difference of flipping a single spin

This is an integrable model (all free-fermion models are integrable, but this is the simplest), because we have all $\{\sigma_i^z\}$ are conserved quantities, and many-body eigenstates specified by giving the expectation value under every one of these states. There is also no level repulsion (Poisson statistics, because adjacent states are characterized by different values).

5.3 Emergent Integrability

The only kind of MBL we really understand and can prove things about is the notion of emergent integrability. In this picture, the idea is that we continue to have extensively many local integrals of motion, but they get dressed. So we start with

$$H = \sum_i h_i \sigma_i^z + J \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z) \quad (68)$$

where $H = \sum_i h_i \sigma_i^z$ are called **l-bits** (bits because they are spin $\frac{1}{2}$; they could be qubits more generally, and “l” because they are localized). In the locator limit $J \ll W$. Then the statement is that the Hamiltonian can be rewritten in the following form:

$$H = \sum_i \tilde{h}_i \tau_i^z + \sum_{ij} \tilde{J}_{ij} \tau_i^z \tau_j^z + \sum_{ijk} \tilde{K}_{ijk} \tau_i^z \tau_j^z \tau_k^z + \dots \quad (69)$$

This Hamiltonian is local, and \tilde{J}_{ij} is exponentially decaying with $i-j$. The new variables τ_i^z satisfy

$$\tau_i^z = U^\dagger \sigma_i^z U, \quad (70)$$

$$[H, \tau_i^z] = 0, [\tau_i^z, \tau_j^z] = 0 \quad (71)$$

and are considered as “dressed operators.” If we have a matrix V that diagonalizes our Hamiltonian, we have

$$\tau_i = V^\dagger \sigma_i V \quad (72)$$

These τ_i s are going to be local themselves. τ_i^z admit an expansion of strings of different σ

$$\tau_i^z = \sigma_i^z + c_{ij} \sigma_i^x \sigma_j^y + d_{ijk} \sigma_i^x \sigma_j^y \sigma_k^x + \dots \quad (73)$$

Most of the weight of this expansion is focused on these Pauli strings, with is localized near the site. The existence of such a V is one of the operational definitions of MBL. The existence of l -bits is known as “emergent integrability.” This is “emergent” because these local operators τ_i^z that all commute with our Hamiltonian, but they are not conservation laws that we “put in” at the start. The dressing of the τ_i^z s depends on V which intrinsically depends on every one of the disordered potential terms and the parameters of our Hamiltonian. Local conservation laws come out of these parameters. This is responsible for the stability of MBL. Unlike integrable models where we can perturb it arbitrarily and destroy all conservation laws, with these models we can perturb our model weakly and our matrix V gets weakly perturbed, and all of our l -bits change a little bit, but their existence doesn’t disappear. Thus we can weakly perturb our system and get MBL.

The brute force way to find V is extremely computationally costly (go figure, V transforms H to the eigenbasis). But the locality of V is determined by the permutation of its columns. For our physical $\{\sigma\}$ degrees of freedom, called “**p-bits**,” our physical basis states look like products of σ^z :

$$|\uparrow\downarrow \dots \downarrow\rangle \quad (74)$$

and our eigenstates look like product states of $\{\tau^z\}$. if we want to go from p-bits to l-bits via V , notice that V gives back an eigenstate when given a physical basis state. For V to be local, we want the p-bit to look similar to the l-bit with some minor corrections. If we have an eigenstate with most of its weight on the physical basis state that we care about, we want V to map this

basis state to the similar l-bit:

$$V |\uparrow\downarrow \dots \downarrow\rangle := \begin{pmatrix} \dots & 0 & \dots \\ \dots & 0 & \dots \\ \dots & \vdots & \dots \\ \dots & 0.8 & \dots \\ \dots & 0.9 & \dots \\ \dots & 0.8 & \dots \\ \dots & \vdots & \dots \\ \dots & 0.0001 & \dots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (75)$$

$$(76)$$

We have a massive freedom in how to choose V , because we have $2^L!$ eigenvector permutations. Further, for every eigenvector, we can choose a phase factor. So V is not unique, and finding the most local V is computationally impossible (it scales exponentially). So we have to be clever about finding V . If we are deep in the disordered phase so that every eigenstate looks like a product state with weak dressing, then pick the V that looks the most diagonal and we're fine. The problem is that anything interesting is harder. Thus we have to worry about those rare resonances. So when we have some basis state that is almost in resonance with some other, and they start mixing, this proliferation of resonances drives the transition out from the MBL phase toward thermalization. Thus how to treat those resonances and how to localize the effect of those resonances to keep V to be as local as possible is the challenge.

Remark 8. *When a unitary operator is called “local,” it means that it maps local operators to local operators.*

Remark 9. *If the Hamiltonian was just*

$$H = \sum_i h_i \sigma_i^z + J(X_i X_{i+1} + Y_i Y_{i+1}) \quad (77)$$

then

$$H = \sum_{\alpha} \epsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} \quad (78)$$

where we note that $a_{\alpha}^{\dagger} a_{\alpha}$ define our τ_i^z . So V is just the linear transformation that takes the

physical basis to the single-particle eigenorbital basis.

This gives a bunch of properties of MBL eigenstates. One of these is entanglement dynamics. In thermalization and MBL, we don't need to think about transport because we don't even need conserved quantities. When we talk about whether or not something thermalizes, we want to ask if the reduced density matrix is approximately the reduced thermal density matrix, i.e. the maximal-entropy state consistent with all of the conservation laws in the system. So if we have no conservation laws in our system, this is proportional to the identity. We want to ask if our subsystem is getting as entangled as it can with the rest of our system. If it is then the system is thermalizing, and if not it is localized. In terms of the ETH, what we're saying is that the different many-body eigenstates are not going to be thermal. This means that if we take some eigenstate, it looks like a product state in this l-basis, and the l-bits are just dressed versions of the physical bits. If we make an entanglement cut across some level of energies, we ask what the entanglement shared across this cut in physical real space. Because the l-bits have these tails, all the entanglement is only coming from some number of l-bits that live within some localization length of the entanglement cut. So the entanglement entropy for MBL eigenstates is going to obey some kind of area law.

What do we know about the MBL phase transition? The existence of the MBL phase (existence of l-bits) was proven to all orders in perturbation theory in a local interaction V , including higher dimensions, and almost proven including non-perturbative effects in 1D lattice models with exponentially decaying interactions. There are lots of open questions; possible non-perturbative instabilities in higher dimensions, with longer ranged interactions, etc. There is lots of numerical evidence for existence of the thermal phase (but no proof). This is given by a dynamical phase transition to a thermalizing phase as a function of disorder strength/interaction strength...

5.4 Properties of the MBL phase

If we have these τ bits that commute with H , if we were to prepare some initial state with some definite value on the τ_i , that value cannot change. This is why MBL systems are said to have "quantum memory." The approach to equilibrium is a slow power law because of slow "dephasing" dynamics.

MBL eigenstates have an area law entanglement. Product states have zero entanglement, "dressed" l-bits only have local correlations. For A a subsystem in a bath B , only the boundary spins are entangled, whereas for a thermal state, B is a bath for all spins in A . Naturally, low entanglement gives efficient representations. One order parameter for MBL, since transport isn't a good one, is entanglement entropy.

MBL exists as its own well-defined concept, independent of transport and conserved quantities.

The dynamics of entanglement offer a good description of the MBL system. In thermalizing systems, entanglement grows as a power law in time:

$$S(t) \propto t \quad (79)$$

In localization, we get a logarithmic growth of entanglement in time, coming from dephasing. If $H = J_{ij}z_i z_j$, if we have:

$$\uparrow\uparrow, S(t) = 0 \quad (80)$$

$$\uparrow\nearrow, S(t) = 0 \quad (81)$$

$$\nwarrow\nearrow = (c_1 \uparrow + c_2 \downarrow), (c'_1 \uparrow + c'_2 \downarrow) \quad (82)$$

where \nearrow represents a precession. If we have a precession on spin j , there is an effective field acting on spin j which cares about spin i . If spin i is up, spin j precesses a different way than if spin i were down. So the neutral precession cares about the other spin, which is dephasing. In summation, we have

Thermalizing	MBL
ETH obeyed	Not obeyed
Volume law entanglement	Area law
Dynamics of entanglement is $Poly(t)$	$S(t) \sim \log(t)$
DC transport could be 0 if there are Griffiths effects	No DC transport
All local information is lost	Forms local “quantum” memory

The transition between the two is a major research problem.

6 1-bit Picture of MBL

The standard approach to many-body physics is to study the ground state properties ($\tau = 0$ properties). Our local Hamiltonian is given by

$$H = \sum_i H_i \quad (83)$$

In plotting the energy spectrum, we have a gap Δ from the ground state to the next state. This many-body bandwidth is extensive in system size $\sim O(V)$, but the number of states is exponential $N \sim e^V$. Only the ground state is protected by this gap $\Delta \sim O(1)$. The ground state contains the most pronounced quantum effects. In quantum entanglement, if we have a gapped ground state,

the entanglement region of subsystem A S_A scales with $|\partial A|$, the “Area law.” The spins on ∂A are coupled to spins in the bath only up to a finite correlation length $\xi(\Delta)$. If we have a gapless ground state (we find these at critical points) $S_A \sim \log(|A|)$. If we have thermal eigenstates at some finite temperature (finite energy density E/L), then $S_A \sim |A|$ (obeys a volume law). This looks classical, and A is completely decohered.

We study phases and phase transitions by studying (classifying) gapped ground states. From here we can get topological order, symmetry breaking, etc, classify gapped ground states with or without symmetries present. With this framework we can create a landscape of phases; regions of phases in parameterspace. In deforming parameters, we ask how we can deform while keeping a gap. We say these hamiltonians are considered to be in the same phase. To pass between phases we must cross over a gapless region. See 7.

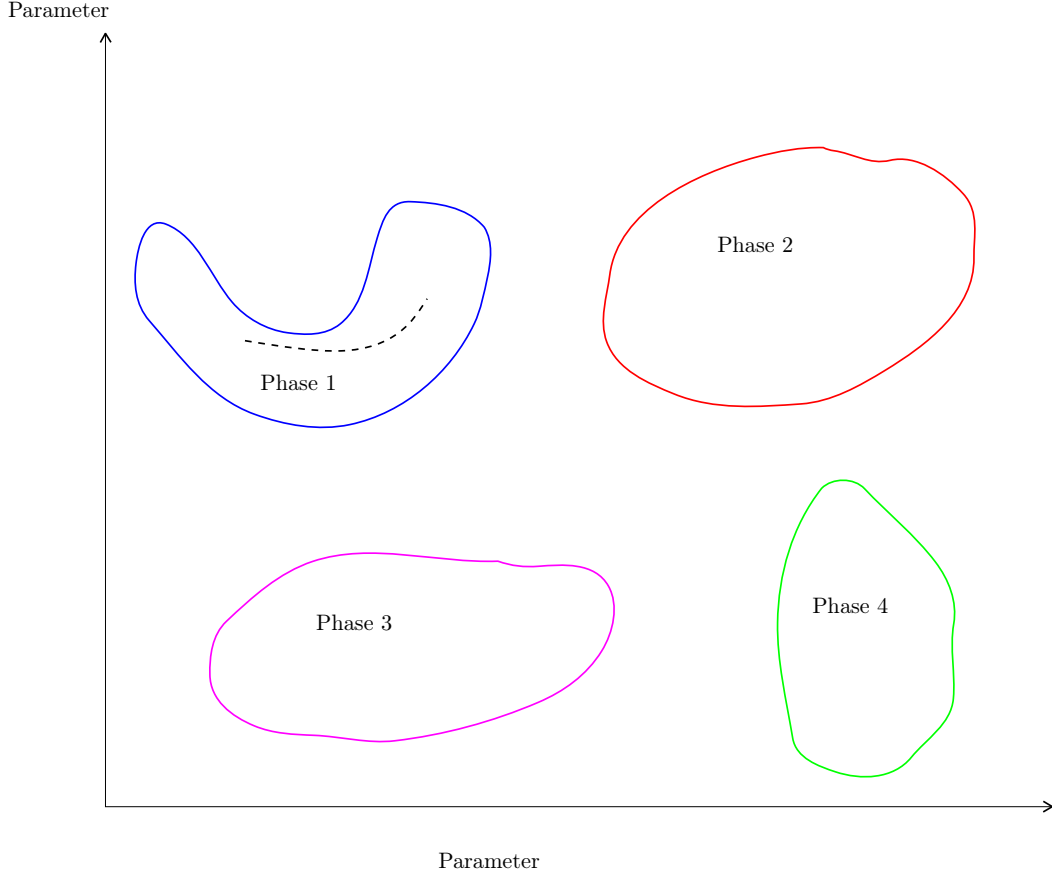


Figure 7: Phase landscape.

Definition 9. Quasi-adiabatic continuity (*Hastings-Wen, 2005*) As long as the gap doesn't close when mapping between sets of parameters λ_0, λ_1 , there exists a local unitary transformation

which relates the ground states

$$|\psi_{\lambda_1}\rangle \approx V_\lambda |\psi_{\lambda_0}\rangle \quad (84)$$

with finite error in expectation values of local observables.

This unitary V “dresses,” or “smears,” local observables into quasi-local observables

$$\tilde{O}_\lambda^i = V^\dagger O_i V \quad (85)$$

The gap Δ comes hand-in-hand with a finite correlation length ξ , i.e.

$$\langle \psi_0 | O_x O_y | \psi_0 \rangle \sim e^{-|x-y|/\xi} \quad (86)$$

so as long as the scale in which our operators are getting smeared is comparable to ξ , we’ll still capture the correlations accurately.

Definition 10. *Local V is often called **finite depth unitary**. See ??.*

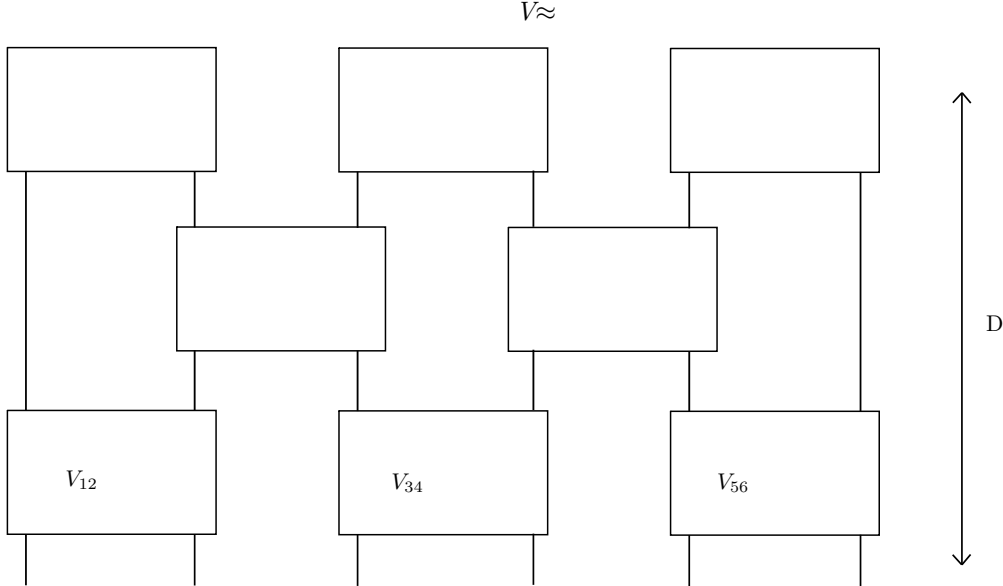


Figure 8: Quantum circuit.

Definition 11. *In terms of D in the diagram, we have a bound on how fast our correlations can spread in space, given by the **Lieb-Robinson speed** v_{LR} .*

Suppose we start with some operator on site 3, and act on it with the first layer of unitaries. Everything outside of this $V_{34} O_3 V_{34}^\dagger$ operation hasn’t been touched by this operator O_3 , so it is

just the identity. Then we act on it by the next layer of unitaries, which have preimage affected by the V^\dagger, V , so with the next layer the operator has spread to sites 5 and 6. So we get a light-cone, seen in ??.

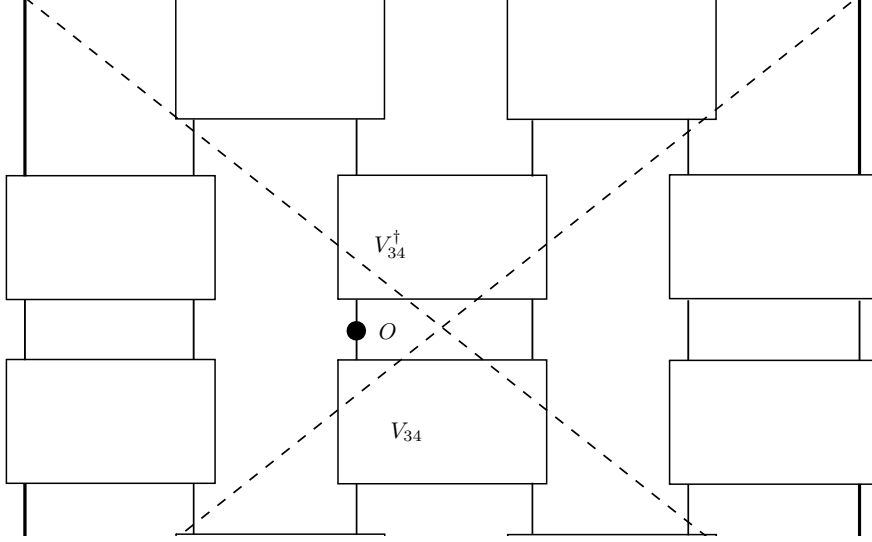


Figure 9: Light cone in a quantum circuit.

In particular, if our circuit has a finite depth D , our operator cannot grow in size any more than D from where it started.

Remark 10. *When acting with this unitary transformation, while the ground state is protected by a gap, the states in the middle of the spectrum will in general mix “like crazy” with one another:*

$$\Delta\lambda - O(1) \gg e^{-V} \quad (87)$$

so there is no notion of adiabatic continuity.

Remark 11. *This has only been proven in 1D, but is expected to hold in 2D.*

Example 2. *For the Transverse Field Ising Model (TFIM), we have*

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x \quad (88)$$

we have a phase transition at $h/J = 1$, where we have a paramagnet for $h/J > 1$ and a ferromagnet

otherwise, where the gap goes to 0 at this critical point. When $h = 0$, we have eigenstates

$$|\psi_{\pm}^0\rangle = |\uparrow \dots \uparrow\rangle \pm |\downarrow \dots \downarrow\rangle \quad (89)$$

The power of Hasting and Wen's theorem is that, when $h \ll 1$, there exists a V_h with

$$|\psi_{\pm}^h\rangle = V |\psi_{\pm}^0\rangle \quad (90)$$

$$\tilde{\sigma}_j^z \tilde{\sigma}_h^{i,z} = V_h \sigma_i^z \sigma_j^z V_h^\dagger \quad (91)$$

since we need Ising-even operators to act on these states, and we have

$$\sigma_h^{i,z} \tilde{\sigma}_h^{j,z} |\psi_{\pm}^h\rangle = 1 \quad (92)$$

This formalism says that we have an operator that dresses our operators and eigenstates.

The question of localization is: Is there any way in which highly excited states can resemble gapped ground states (and not thermalize)? Quantum coherence at high energies corresponds to new kinds of order (localization-protected quantum order). We can evade thermalization at high energy densities. One way to do this is to have extensively many symmetries.

Example 3. $H_0 = h \sum_i \sigma_i^z$. This model is certainly not thermal: $[H_0, \sigma_i^z] = 0$, $|n\rangle = |\uparrow \downarrow \dots \downarrow\rangle$ are product states in the z basis. $S = 0$ for any cuts. We can take all 2^L states. This isn't very interesting, because it's extremely unstable to the addition of small perturbations. $H_\lambda := H_0 + H_1$, where

$$H_1 = J \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \dots \quad (93)$$

we can write this in terms of creation and annihilation operators

$$H = J \sum_i \sigma_i^\dagger \sigma_{i+1} + \sigma_i \sigma_{i+1}^\dagger + \dots \quad (94)$$

Under spin flips our states mix, get delocalized eigenstates, and high entanglement (with interactions). See ??.

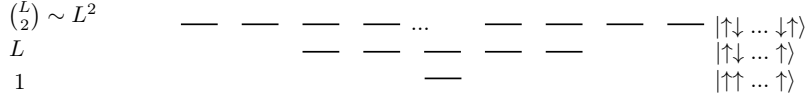


Figure 10: Energy for $H = \sum_i h_i \sigma_i^z$.

Now let's add disorder:

Example 4. $H_0 = \sum_i h_i \sigma_i^z + \sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z + \dots$, disordered with disorder strength W . We perturb with H_1 . H_1 allows flip flops between nearest neighbors. For a flip

$$|n\rangle \rightarrow \sigma_i^+ \sigma_{i+1}^- |n\rangle \quad (95)$$

$$\Delta H_0 \sim O(W) \quad (96)$$

There is no resonance if $J \ll W$. This is a detuning between states in Hilbert space.

When we have a thermalizing system, and a dense many-body spectrum, we can take two states and create an avoided level-crossing. The local properties of neighboring eigenstates are similar. We have a matrix element between these neighboring eigenstates

$$M.E. = \langle E_{n+1} | H_1 | E_n \rangle \sim J e^{-S/2} \quad (97)$$

$$\Delta E \sim e^{-S}; S \propto V \Rightarrow \quad (98)$$

$$M.E. \gg \Delta E \quad (99)$$

For the localized system, the local properties of neighboring eigenstates is very different. For $H_0 = \sum h_i \sigma_i^z$,

$$|E_n\rangle = |\uparrow \downarrow \dots \uparrow \uparrow\rangle \quad (100)$$

$$|E_{n+1}\rangle = |\downarrow \downarrow \dots \uparrow \downarrow\rangle \quad (101)$$

If we flip we get an energy difference of order W .

Definition 12. The **mobility gap** is the typical energy difference between states that differ only in a local subregion.

So, what even are l-bits, and how do we construct them? If we know for a diagonalizable H we have

$$V^\dagger H V = H_{diag} \quad (102)$$

$$|E_i\rangle = V |e_i\rangle \quad (103)$$

where $|e_i\rangle$ are the basis states in the z basis $|\uparrow\downarrow \dots \uparrow\rangle$. Let

$$V = (v_1, v_2, \dots, v_{2^L}) \quad (104)$$

$$\tau_i^\alpha = V \sigma_i^\alpha V^\dagger \quad (105)$$

where τ_i^α is the dressed operator (l-bit) and σ_i^α is the bare (physical) spins (p-bit).

$$\tau_i^z |E_i\rangle = \pm |E_i\rangle \quad (106)$$

$$= V \sigma_i^z V^\dagger V |e_i\rangle \quad (107)$$

$$= \pm V |e_i\rangle \quad (108)$$

$$= \pm |E_i\rangle \quad (109)$$

This means that $[H, \tau_i^z] = 0, \forall i$, and H can be written in terms of these τ :

$$H = \tilde{h}_i \tau_i^z + \tilde{J}_{ij} \tau_i^z \tau_j^z + \tilde{J}_{ijk} \tau_i^z \tau_j^z \tau_k^z + \dots \quad (110)$$

at no point have we assumed anything about thermalization, many-body localization, etc. All we started with was a V that diagonalizes the Hamiltonian.

Definition 13. *For MBL, V can be made local with MBL! This is a precise definition of MBL.*

When τ is local, the fact $[H, \tau_i^z] = 0$ is emergent integrability.

Remark 12. *The basis in which H is diagonal need not be the local naive computational basis.*

Traditionally we had

$$H = H_{Anderson} + H_{Interaction} \quad (111)$$

$$= h_i \sigma_i^z + XX + YY + H_{Interaction} \quad (112)$$

$$= \sum_{\alpha} e_{\alpha} d_{\alpha}^{\dagger} d_{\alpha} + V \sum_{\alpha\beta\gamma\delta} d_{\alpha}^{\dagger} d_{\beta}^{\dagger} d_{\gamma} d_{\delta} \quad (113)$$

where $d_\alpha^\dagger d_\alpha$ are Anderson eigenstates, and we transform \uparrow to an occupied Anderson orbital and \downarrow is unoccupied, and $\tau_i^z = \sigma_i^z$. Thus the basis in which we diagonalize depends of the problem.

6.1 Implications

The Hamiltonian is given by

$$H = \tilde{h}_i \tilde{\tau}_i + \tilde{J}_{ij} \tau_i^z \tau_j^z + \tilde{J}_{ijk} \tau_i^z \tau_j^z \tau_k^z + \dots \quad (114)$$

and \tilde{J}_{ij} are exponentially decaying.

Remark 13. *We are considering very local interactions. A large part of the motivation to consider interactions is because a real solid has Coulomb interactions which are not local.*

The implications are:

1. The eigenstates are product states of τ_i^z : $\tau_i^z |E_n\rangle = \pm 1 |E_n\rangle$. If we make an entanglement cut $|\uparrow\downarrow\uparrow|\downarrow\ldots\uparrow\rangle_{\tau^z}$, the states overlap in their tails as in Figure ??.

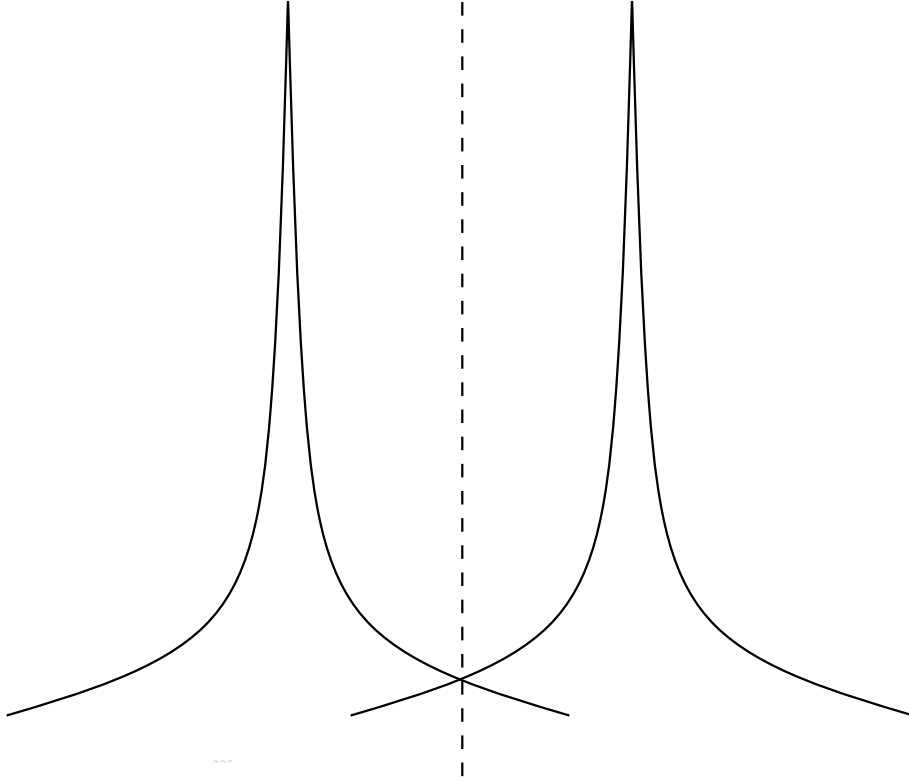


Figure 11: Overlap of state $|\uparrow\downarrow\uparrow|\downarrow\ldots\uparrow\rangle$ over our entanglement cut.

and we get an area law. This is one of the best known diagnostics of MBL: highly excited eigenstates in the middle of the spectrum have area-law entanglement similar to gapped ground states.

2. Because $[\tau_i^z, H]$, $\frac{1}{D} \text{Tr}[\tau_i^z \sigma_i^\alpha] \sim O(1)$. Dressed operators have finite overlap with physical operators. If we plot $O_i(t)$ vs. time as in , we find that it approaches some value $C_0 > 1$ and it stays there forever. See [12](#).

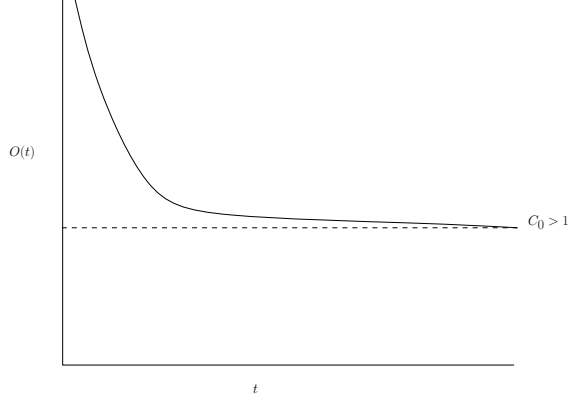


Figure 12: $O(t)$: Quantum memory of an operator.

This is quantum memory, i.e. lack of thermalization. Even if we wait infinitely long in an infinitely large system, this observable will remember its initial condition. Experimentally, one can always object to the inevitable finiteness in system size and time. This is a valid objection, and we're not sure if eventually this phenomenon breaks down.

3. ETH is not obeyed.
4. There is no level repulsion. Eigenstates can just pass through each other and obey Poissonian statistics.
5. Slow dephasing dynamics. Log-growth of entanglement entropy.

In the “Standard Model” of thermodynamics, we have

$$H = \sum_i h_i \sigma_i^z + J \sum (\sigma_i^z \sigma_{i+1}^z + \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \quad (115)$$

with $h_i \in [-W, W]$. The phase transition at a value in W/J is a huge, exciting area of research for a bunch of reasons:

1. It is based on dynamics, not thermodynamics. Dynamics are tied to the properties of individual eigenstates.
2. It entails a restoration of statistical mechanics. The transition goes from where statmech works to where it doesn't.
3. It is a transition which looks like a quantum to classical phase transition.
4. It is driven by a singular change in the entanglement properties in individual MB eigenstates.

6.2 Log-growth of entanglement

For a system with a subregion A in a region B , we have a Hilbert space

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \quad (116)$$

$$|\psi\rangle = \sum_{i=1}^{D_A} \sum_{j=1}^{D_B} c_{ij} |i\rangle_A |j\rangle_B \quad (117)$$

where $D_A = \dim(\mathcal{H}_A)$. We do a singular value decomposition to get

$$\sum_{n=1}^R \sqrt{p_n} |m\rangle_A |n\rangle_B \quad (118)$$

where R is the rank of this c_{ij} matrix, and $\sum_n p_n = 1$, and $S_{VN}(A) = S_{VN}(B) = -\text{Tr}[\rho_A \log \rho_A]$ (with $\rho_A = \text{Tr}_B[|\psi\rangle\langle\psi|]$) and therefore $S_{VN}(A) = -\sum_n p_n \log p_n$. More generally we have **Renyi entropy**:

$$S_\alpha = \frac{1}{1-\alpha} \log\left(\sum_n p_n^\alpha\right) \quad (119)$$

For a generic local chaotic system, we expect

$$S_{VN}(t) \propto t \quad (120)$$

In a thermalizing disordered system, transport can be subdiffusive due to Griffiths effects. If our transport is subdiffusive, we know $\sigma_{DC}(t) = 0$. With these effects,

$$S(t) \sim t^\alpha, \alpha < 1 \quad (121)$$

(growth of entropy is sublinear) This is growing by entanglement velocity, bounded by the Lieb-Robinson speed (see the lightcone section).

$$H_{l-bit} = \sum_i J_{ij} \sigma_i^z \sigma_j^z + h_i \sigma_i^z \quad (122)$$

where $J_{ij} \sim e^{|i-j|/\xi}$. For 2 sites, we have

$$H = J \sigma_1^z \sigma_2^z \quad (123)$$

We consider families of initial states. Suppose

1. $|\psi_0\rangle = |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$. Here there is no entanglement.
2. $|\psi_0\rangle = |\uparrow\rangle \cdot (\alpha |\uparrow\rangle + \beta |\downarrow\rangle)$. The second spin feels an effective field. So effectively we have $|\uparrow\rangle \cdot (J(1) + \sigma_2^z)$, which isn't entangled.
3. $|\psi_0\rangle = (\alpha |\uparrow\rangle + \beta |\downarrow\rangle) \otimes (\alpha |\uparrow\rangle + \beta |\downarrow\rangle)$. Now, depending on what one spin is doing, the other spin wants to do different things. This generates entanglement between spins 1 and 2. Explicitly,

$$|\psi_0\rangle = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}_1 \otimes \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}_2 \quad (124)$$

$$|\psi(t)\rangle = \begin{pmatrix} \cos^2 \alpha e^{iJt} \\ \cos \alpha \sin \alpha e^{-iJt} \\ \cos \alpha \sin \alpha e^{-iJt} \\ \sin^2 \alpha e^{iJt} \end{pmatrix} \quad (125)$$

$$\rho_A(t) = \begin{pmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha (e^{2iJt} \cos^2 \alpha + e^{-2iJt} \sin^2 \alpha) \\ \cos \alpha \sin \alpha (e^{2iJt} \cos^2 \alpha + e^{-2iJt} \sin^2 \alpha) & \sin^2 \alpha \end{pmatrix} \quad (126)$$

For $\alpha = \frac{\pi}{4}$, we have

$$|\psi_0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (127)$$

$$\rho_A(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \cos(2Jt) \\ \frac{1}{2} \cos(2Jt) & \frac{1}{2} \end{pmatrix} \quad (128)$$

so $S(t)$ is maximized if $Jt = \frac{\pi}{4}$; the off-diagonal entries go to zero. More generally, when $Jt \sim O(1)$.

Going to the MBL problem, we have a whole bunch of spins with some entropy cut. There is an effective interaction between two spins across this cut that are r apart.

$$H = J_0 e^{-r/\xi} \sigma_i^z \sigma_{i+r}^z \quad (129)$$

$$= J_{eff} \sigma_i^z \sigma_{i+1}^z \quad (130)$$

when $J_{eff}t \sim 1$, the entanglement is maximized, i.e. when $\log(J_0 t) \sim r$. In an actual MBL system we have a whole lot of couplings, so the off-diagonal entries in H are $f(t)$, which contain many different incommensurate frequencies. Graphing the entropies, we get the figure 13.

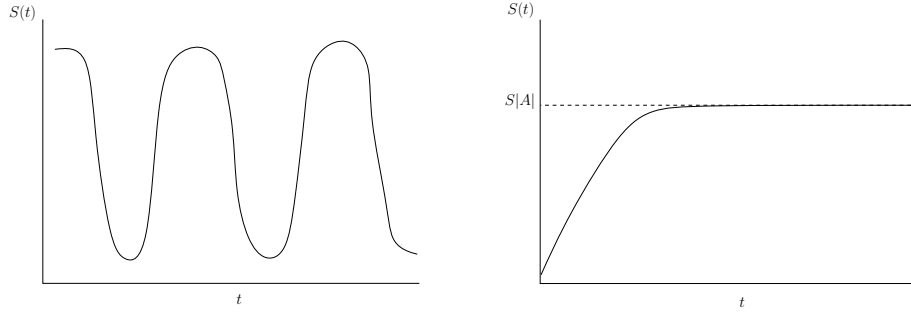


Figure 13: Entropies of different-sized systems.

where we can see that the entropy saturates at some value $s|A|$, and thus obeys a volume law, and $0 < s < \log(2)$, and s depends on the initial state.