Physics?

Alec Lau

Here is a list of the highlights of physics, in my view, for reference (the physics I'd like to remember). These are chosen by their coolness factor or just their importance to physics.

Contents

1 Classical kinetic energy from relativity

1.1 Relativistic mass

We want to find momentum for an object that is reference-frame-invariant. We know from time dilation that $t = \gamma\tau$, where τ is the proper time in the object's rest frame.

$$p = m_0 \frac{dx}{d\tau} \tag{1}$$

$$= m_0 \frac{dx}{dt}\frac{dt}{d\tau} \tag{2}$$

$$= m_0 \frac{dx}{dt}\frac{d(\gamma\tau)}{d\tau} \tag{3}$$

$$p = m_0 \gamma v \tag{4}$$

where v is the velocity of the object that *we* see (since t is our time), not of a reference frame relative to another. Thus we have that relativistic mass m is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

1.2 Classical kinetic energy is a Taylor expansion from relativity and $E = mc^2$

In relativity, we have to introduce the spacetime, which is, here at least, just the normal 3 dimensions of space, plus time:

$$x = (t, x_1, x_2, x_3) \tag{6}$$

$$:= (x_0, x_1, x_2, x_3) \tag{7}$$

But the first dimension is in terms of time, and the rest in space. It's bad practice to have different units in the same vector. No problem, because then we'll set

$$x = (ct, x_1, x_2, x_3) \tag{8}$$

since c always a constant.

Suppose we're only moving in dimension x_1 with velocity v. For $\beta = \frac{v}{c}$, we have

$$x_1' = \frac{x_1 - \beta x_0}{\sqrt{1-\beta^2}}, x_0' = \frac{x_0 - \beta x_1}{\sqrt{1-\beta^2}} \tag{9}$$

Let's play around for a bit:

$$(x_1')^2 - (x_0')^2 = (x_0 - \beta x_1)^2\gamma - (x_1 - \beta x_0)^2\gamma \tag{10}$$

$$= \frac{x_0^2 - \beta^2 x_1^2 - x_1^2 - \beta^2 x_0^2}{1-\beta^2} \tag{11}$$

$$= x_0^2 - x_1^2 \tag{12}$$

Thus $x_0^2 - x_1^2$ is Lorentz invariant. In this spirit, let's define a momentum vector:

$$p = m(\frac{dx_0}{d\tau}, \frac{dx_1}{d\tau}, ...) \tag{13}$$

Since $x_0 = ct$, we have

$$p_0 = mc\frac{dt}{d\tau} = mc\gamma, \tag{14}$$

$$p_1 = mv\gamma \tag{15}$$

Assuming $v << c$, we have $\frac{v}{c} << 1$. We Taylor expand γ using $(1+x)^n = 1 + nx + ...$ to get

$$\gamma \approx 1 + \frac{v^2}{2c^2} + ... \tag{16}$$

Thus

$$p_1 \approx mv, \tag{17}$$

$$p_0 \approx mc + \frac{1}{2c}mv^2 \Rightarrow \tag{18}$$

$$cp_0 \approx mc^2 + \frac{1}{2}mv^2 \tag{19}$$

This is in terms of energy, since the last term is jsut kinetic energy. Thus, at rest, $E = mc^2$.

2 Maxwell's equations in matter

Let P be the electric dipole moment per unit volume. For a single dipole p we have

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot p}{r^2} \tag{20}$$

so, with $p = Pd\tau'$,

$$\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\hat{r}\cdot P(r')}{r^2}d\tau' && (21)\\ &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} P\cdot\nabla(\frac{1}{r})d\tau' && (22)\\ &= \frac{1}{4\pi\varepsilon_0}[\int_{\mathcal{V}} \nabla\cdot(\frac{P}{r})d\tau' - \int_{\mathcal{V}}\frac{1}{r}(\nabla\cdot P)d\tau'] && (23)\\ &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{S}}\frac{1}{r}P\cdot da' - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{1}{r}(\nabla\cdot P)d\tau' && (24) \end{aligned}$$

These terms look like the potentials for a surface charge $\sigma_b = P\cdot n$ and a volume charge $\rho_b = -\nabla\cdot P$, respectively.

For charge $\rho = \rho_b + \rho_f$, Gauss' Law reads

$$\varepsilon_0\nabla\cdot E = \rho = \rho_p + \rho_f = -\nabla\cdot P + \rho_f \tag{25}$$

so

$$\nabla\cdot(\varepsilon_0 E + P) = \rho_f \tag{26}$$

This D-field is the electric displacement field, and Gauss' Law in this term reads

$$\nabla\cdot D = \rho_f \tag{27}$$

For many materials, the polarization is proportional to the E-field, provided the E-field isn't too strong:

$$P = \varepsilon_0\chi_e E \tag{28}$$

where the constant of proportionality χ_e is the electric susceptibility. These materials are called linear dielectrics. Then we have

$$D = \varepsilon_0 E + P = \varepsilon_0(1 + \chi_e)E \Rightarrow \tag{29}$$

$$D = \varepsilon E, \tag{30}$$

$$\varepsilon := \varepsilon_0(1 + \chi_e) \tag{31}$$

This factor $\varepsilon_r := (1 + \chi_e)$ is called the relative permittivity or dielectric constant of the material.

The current density $J = J_b + J_f + J_p$ has contributions from free charges J_f, bound charges J_b, and polarization effects J_p. Given the magnetization M of the medium, we have

$$J_b = \nabla \times M \tag{32}$$

derived in the same way as we derived $\rho_b = -\nabla \cdot P$. Furthermore,

$$J_p = \frac{\partial P}{\partial t} \tag{33}$$

Let

$$H := \frac{1}{\mu_0} B - M, \nabla \times H = J_f + \frac{\partial D}{\partial t} \tag{34}$$

Thus Maxwell's equations in matter are

$$\nabla \cdot D = \rho_f, \nabla \cdot B = 0 \tag{35}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \nabla \times H = J_f + \frac{\partial D}{\partial t} \tag{36}$$

3 Maxwell's equations are relativistic

First of all, *Maxwell's equations are not invariant to Lorentz transformations*! They are *covariant* under Lorentz transformations - observers in different reference frames will disagree on the values of E, B, e.tc. in their rest frames, but they will still obey Maxwell's Equations! Let's

reconsider the equations, simplified:

$$\nabla \cdot E = \rho_f, \tag{37}$$

$$\nabla \cdot B = 0, \tag{38}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{39}$$

$$\nabla \times B = J_f + \frac{\partial E}{\partial t} \tag{40}$$

Note that these are in natural units $4\pi = c = 1$. We get the same equations if we consider time as another coordinate (with an eye toward relativity), make our metric pseudo-Riemannian (+,-,-,-), and define the electromagnetic tensor (Faraday tensor) and corresponding current:

$$F^{\mu\nu} := \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, J^{\mu} = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \tag{41}$$

so Maxwell's Equations are neatly packaged into this single equation:

$$\partial_\mu F^{\mu\nu} = J^{\nu} \tag{42}$$

The Lorentz group describes operations corresponding to boosting an inertial reference frame by some velocity v. For simplicity, let's assume that we boost in the x-direction, and have the usual denotations $\beta := \frac{v}{c}, \gamma = \frac{1}{\sqrt{1-\beta^2}}$. A Lorentz transformation then gives

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} \gamma(ct - \beta x) \\ \gamma(x - \beta ct) \\ y \\ z \end{pmatrix} \tag{43}$$

A Lorentz transformation in this coordinate system (Minkowski space) is then given by the matrix

$$\Lambda^{\mu}_{\nu} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

So say we have a Faraday tensor in a 1^{st} coordinate system. If we want to transform our Faraday tensor according to a Lorentz transformation, we untransform a vector in these new coordinates to the old coordinates, feed it through the old Faraday tensor, then re-transform the resulting vector:

$$F^{'\mu\nu} = \Lambda F^{\mu\nu}\Lambda^T = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} \tag{45}$$

Next if we introduce an alternating 4-dimensional tensor $\epsilon^{\mu\nu\rho\sigma}$, it is not hard to check that

$$|\epsilon^{\mu\nu\rho\sigma} F^{'}_{\rho\sigma} F_{\mu\nu}| = E \cdot B \tag{46}$$

so $E \cdot B$ is Lorentz invariant. (Note: I've been pretty cavalier about coefficients, so there might be a factor of 2 or 4 or something in there)

4 Kinetic-theoretic derivation of temperature

We define temperature by saying that two systems in thermal equilibrium have the same temperature. The multiplicity of system 1 is g_1 and the multiplicity of system 2 is g_2. The total multiplicity of the system is $g = g_1 g_2$. Since the systems are in thermal equilibrium, we have $dU_1 = -dU_2$. To put the multiplicity in these terms, we take the total differential of the total multiplicity:

$$dg = dg_1 g_2 + g_1 dg_2 \tag{47}$$

$$= \frac{\partial g_1}{\partial U_1} g_2 dU_1 + g_1 \frac{\partial g_2}{\partial U_2} dU_2 \tag{48}$$

Since the multiplicity doesn't change in total, we have $dg = 0$. Thus we have

$$\frac{1}{g_1}\frac{\partial g_1}{\partial U_1} = \frac{1}{g_2}\frac{\partial g_2}{\partial U_2} \tag{49}$$

$$\frac{\partial \log g_1}{\partial U_1} = \frac{\partial \log g_2}{\partial U_2} \tag{50}$$

$$\frac{\partial \sigma_1}{\partial U_1} = \frac{\partial \sigma_2}{\partial U_2} \tag{51}$$

Thus we define $\frac{\partial \sigma_1}{\partial U_1} = \beta = \frac{1}{\tau}$. This is because, intuitively, at low temperature, the entropy depends greatly on the energy put into the system.

5 The partition function and why it's important

First we derive the Boltzmann factor. Suppose we have a generic system $\mathcal{A}$ in thermal contact with a reservoir $\mathcal{R}$, and we have two microstates of $\mathcal{A}$ with energy E_1, E_2. The ratio of the probabilites of these microstates is

$$\frac{P_1}{P_2} = \frac{g(E - E_1)}{g(E - E_2)} \tag{52}$$

Taking the log of both sides, we have

$$\log(P_1/P_2) = \sigma_R(E - E_1) - \sigma_R(E - E_2) \tag{53}$$

$$\approx \sigma_R(E) - \frac{d\sigma_R}{dE_R}E_1 - \sigma_R(E) + \frac{d\sigma_R}{dE_r}E_2 \tag{54}$$

$$= -\frac{E_1}{\tau} + \frac{E_2}{\tau} \Rightarrow \tag{55}$$

$$\frac{P_1}{P_2} = e^{-\beta(E_1 - E_2)} \tag{56}$$

Thus $P_i \propto e^{-\beta E_i}$. To get the actual probability, we divide by the sum over all microstates i. This is the partition function, Z. From this we can derive many properties of the system:

$$\langle E \rangle = -\frac{\partial Z}{\partial \beta}, \langle (\Delta E)^2 \rangle = \frac{\partial^2 Z}{\partial \beta^2} \tag{57}$$

Furthermore, Helmholtz free energy $F = E - \tau\sigma$ is given by

$$F = -k_B T \log Z = -\tau \log Z \tag{58}$$

and from there we get

$$\sigma = -(\frac{\partial F}{\partial \tau})_{V,N}, P = -(\frac{\partial F}{\partial V})_{T,N}, \mu = (\frac{\partial F}{\partial N})_{\tau,V} \tag{59}$$

6 Ideal gas law from quantum mechanics

How does one model a gas? Assume it's ideal, to start, so we don't have to deal with interactions and weird edge cases of particle size and such, and classical enough velocity. So picture an ideal gas as just a bunch of particles in a container. The particles are described by the wavefunction

that obeys the Schrödinger Equation:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)]\Psi(\mathbf{r},t), \tag{60}$$

$$[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)]\Psi(\mathbf{r},t) = E\Psi(\mathbf{r},t) \text{ (in the energy eigenbasis)} \tag{61}$$

Suppose we consider N such particles confined to a box of side lengths L, W, H. Then we impose an infinite potential outside of this box to ensure there are indeed N particles inside, with no potential inside the box because in general there wouldn't be. This is a 3D particle-in-a-box scenario. Solving the Schrödinger Equation yields

$$\Psi = \sqrt{\frac{2}{L}}\sin(\frac{n_1\pi}{L})\sqrt{\frac{2}{W}})\sin(\frac{n_2\pi}{W})\sqrt{\frac{2}{H}})\sin(\frac{n_3\pi}{H}) \tag{62}$$

with energy

$$E = \frac{\hbar^2\pi^2}{2m}[(\frac{n_1^2}{L^2}) + (\frac{n_2^2}{W^2}) + (\frac{n_3^2}{H^2})] \tag{63}$$

From statistical mechanics, we know that the partition function is given by

$$Z = \sum_i e^{-E_i/\tau} \tag{64}$$

so in our case the partition function for one particle is

$$Z_1 = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2m}[(\frac{n_1^2}{L^2})+(\frac{n_2^2}{W^2})+(\frac{n_3^2}{H^2})]} \tag{65}$$

$$= (\sum_{n_1=1}^{\infty} e^{-\hbar^2\pi^2 n_1^2/2m\tau L^2})^3 \text{ without loss of generality} \tag{66}$$

$$\approx (\int_0^{\infty} e^{n_1^2[-\hbar^2\pi^2/2m\tau L^2]})^3 \tag{67}$$

$$:= (\int_0^{\infty} e^{-\gamma n_1^2})^3 \tag{68}$$

$$:= (\gamma^{-\frac{1}{2}}\int_0^{\infty} e^{-x^2}dx)^3 \tag{69}$$

$$= (\frac{1}{2}(\frac{\pi}{\gamma})^{\frac{1}{2}})^3 \tag{70}$$

$$= LWH(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}} = V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}} \tag{71}$$

Therefore, since the particles are not interacting, the partition function for N particles is

$$Z = [V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}]^N \tag{72}$$

Since we now have the partition function for our system, we can calculate $F = -\tau \log Z$ and $P = -(\frac{\partial F}{\partial V})_\tau$. Let's do this:

$$F = -\tau \log([V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}]^N) \tag{73}$$

$$= -N\tau \log(V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}) \tag{74}$$

$$P = -\frac{\partial}{\partial V} - N\tau \log(V(\frac{m\tau}{2\pi\hbar^2})^{\frac{3}{2}}) \tag{75}$$

$$= N\tau \frac{1}{V} \Rightarrow \tag{76}$$

$$PV = N\tau \tag{77}$$

7 Solving the quantum harmonic oscillator using only operators and commutation relations

Our operators are H, a, and $a^\dagger$, where these are the Hamiltonian, annihilation, and creation operators, respectively. Their commutation relations are:

$$[H, a] = -\hbar\omega a \tag{78}$$

$$[H, a^\dagger] = \hbar\omega a^\dagger \tag{79}$$

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{80}$$

Let u_n be an eigenfunction.

$$[H, a]u_n = -\hbar\omega a u_n \tag{81}$$

$$Hau_n - aHu_n = -\hbar\omega a u_n \tag{82}$$

$$H(au_n) - E_n(au_n) = -\hbar\omega a u_n \tag{83}$$

$$H(au_n) = (E_n - \hbar\omega)(au_n) \tag{84}$$

Thus a lowers the energy by $\hbar\omega$. In a similar calculation, we find that $a^\dagger$ raises the energy by $\hbar\omega$.

Since the harmonic oscillator energy cannot be negative, we have to have a ground state u_0

such that $au_0 = 0$. Thus

$$Hu_0 = \hbar\omega(a^\dagger a + \frac{1}{2})u_0 = \frac{1}{2}\hbar\omega u_0 \tag{85}$$

Thus the ground state is $\frac{1}{2}\hbar\omega$, and in general

$$E_n = (n + \frac{1}{2})\hbar\omega \tag{86}$$

8 Aharonov-Bohm effect

Maxwell's Equations are:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \nabla \cdot B = 0,$$
$$\nabla \times E = -\frac{\partial B}{\partial t}, \nabla \times B = \mu_0[j + \epsilon_0 \frac{\partial E}{\partial t}]$$

Since B is divergenceless, we can write $B = \nabla \times A$, the curl of the vector potential A.

Consider a long solenoid to model a constant field B inside the solenoid and $B = 0$ outside the solenoid. Thus outside the solenoid, the vector potential A must satisfy

$$B = \nabla \times A,$$
$$\oint_C A \cdot dr = \int_S (\nabla \times A) \cdot dS = \int_S B \cdot dS = \Phi$$

for C a path around the solenoid, with Φ the magnetic flux through the solenoid. Thus we choose

$$A = \frac{\Phi}{2\pi r}\hat{\phi}$$

Thus although the B-field outside the solenoid is 0, the $A-$field is nonzero.

Now we investigate what happens to a particle with charge q in this field. The lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + q\dot{x}A - q\phi$$

Solving the Euler-Lagrange equations gives us the Lorentz force law:

$$m\ddot{x} = qE + q\dot{x} \times B$$

Take the Legendre transformation of this to get the Hamiltonian, which is

$$H = \dot{x}p - \mathcal{L} = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{(p - qA)^2}{2m} + q\phi$$

where we not that canonical momentum, $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$, is not gauge-invariant, but kinetic momentum, $p - qA$, is gauge-invariant and physically measurable. We how quantize this Hamiltonian:

$$H = \frac{1}{2m}(p - qA(r))^2 + q\phi + V(r),$$

where $V(r)$ is some other potential. Writing $p = -i\hbar\nabla$, we get the Schrödinger equation as

$$[\frac{1}{2m}(-i\hbar\nabla - qA(r))^2 + q\phi + V(r)]\psi = i\hbar\frac{\partial \psi}{\partial t}$$

The solution to this equation is

$$\psi(r,t) = e^{\frac{iq}{\hbar}\int_0^r A(r)dr}\psi'(r,t) \tag{87}$$

where the initial integration point is 0, chosen arbitrarily due to gauge invariance. Taking the gradient of this function, we get

$$-\frac{\hbar^2}{2m}\nabla^2\psi' - V\psi' = i\hbar\frac{\partial \psi'}{\partial t} \tag{88}$$

Thus if we turn on a magnetic field, the wavefunction gains a phase, called the **Berry phase**. Thus if we wind the particle around the solenoid, we get

$$\psi(r,t) = e^{\frac{iq}{\hbar}\oint_C A(r)dr}\psi'(r,t) \tag{89}$$

$$= e^{\frac{iq}{\hbar}\Phi}\psi'(r,t) \tag{90}$$

so we gain a phase.

In other words, if we split a beam around a solenoid, the two paths, once recombined, yield a loop around the solenoid, so the interference pattern of the recombined beam will shift by a phase $\frac{e\Phi}{\hbar}$

8.1 Path integral formulation of Aharonov-Bohm effect

We have that

$$\langle x_1, t_1 | x_0, t_0 \rangle = \int_{x_0}^{x_1} \mathcal{D}[x(t)] \exp[i \frac{S(t_1, t_0)}{\hbar}] = \int_{x_0}^{x_1} \mathcal{D}[x(t)] \exp[\frac{i}{\hbar} \int_{t_0}^{t_1} \mathcal{L}(\dot{x}(t), x(t)) dt]$$

where the Lagrangian $\mathcal{L}(\dot{x}(t), x(t))$ is the same as above. If we turn on the magnetic field, the action becomes $S_0 + e \int A \cdot dl$.

If we have a double-slit experiment, this time with a solenoid between the slits, the probability amplitude that we detect the particle at some point is proportional to

$$\int_{\text{all paths through slit 1}} \exp(\frac{i}{\hbar} S_0 + \frac{ie}{\hbar} \int A \cdot dl) + \int_{\text{all paths through slit 2}} \exp(\frac{i}{\hbar} S_0 + \frac{ie}{\hbar} \int A \cdot dl) \Rightarrow$$
$$\exp(\frac{ie}{\hbar} \int_{slit1} A \cdot dl - \frac{ie}{\hbar} \int_{slit2} A \cdot dl) = \exp(\frac{ie}{\hbar} \oint_C A \cdot dl)$$

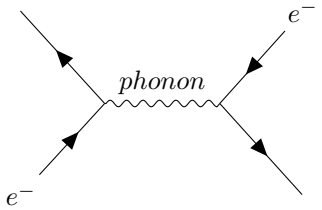
This time, we can use Stokes' theorem:

$$\frac{ie}{\hbar} \oint_C A \cdot dl = \frac{ie}{\hbar} \oint_{enclosed} B \cdot dS = \frac{ie}{\hbar} \Phi$$

9 SQUIDS as tunable Josephson junctions

9.1 Superconductivity via cooper pairs

SQUIDs are created using superconducting wires, typically pure (99%) niobum or lead with ~10% gold or indium, since lead doesn't respond well to temperature changes. Superconductivity here works because of **cooper pairs**, which we can model as two electrons moving through a crystal lattice of positively-charged ions. As an electron moves, the positively charged ions are attracted to the electron, resulting in a region of higher positive charge density. As the electron moves, this region follows, attracting another electron (with opposite spin from the first). This binds two electrons via a phonon:



The binding energy of a cooper pair is easily overcome at temperatures higher than several Kelvin, so the quantum processor must be kept at extremely cold temperatures.

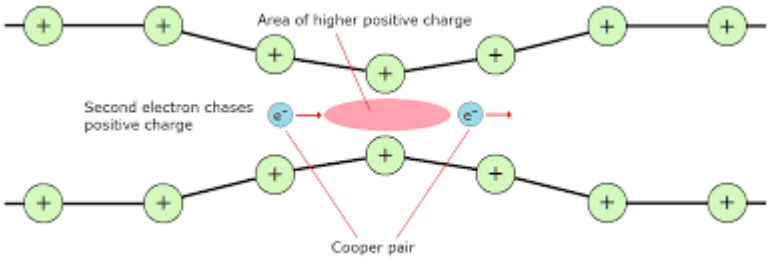


Figure 1: Cooper pairs.

9.2 The Josephson Effect

We consider a Josephson junction, i.e. an insulator with a potential difference of V sandwiched between two superconductors A and B. According to Ginzburg and Landau, the free energy of a superconductor can be expressed in terms of a complex order parameter field

$$\psi_A := \sqrt{\rho_A} e^{i\phi_A}, \psi_B := \sqrt{\rho_B} e^{i\phi_B} \tag{91}$$

We interpret these as wavefunctions of the cooper pairs in the superconductors. The Schrodinger Equation is then

$$i\hbar \frac{\partial}{\partial t}(\sqrt{\rho_A} e^{i\phi_A})\sqrt{\rho_B} e^{i\phi_B} = \begin{pmatrix} eV & K \\ K & -eV \end{pmatrix} \begin{pmatrix} \sqrt{\rho_A} e^{i\phi_A} \\ \sqrt{\rho_B} e^{i\phi_B} \end{pmatrix} \tag{92}$$

where K is a constant intrinsic to the to the Junction. Simplifying this, we get the following Schrodinger Equation and its complex conjugate:

$$\varphi := \phi_A - \phi_B \tag{93}$$

$$\dot{\sqrt{\rho_A}} \pm i\sqrt{\rho_A}\dot{\phi}_A = \frac{1}{\pm i\hbar}(eV\sqrt{\rho_A} + K\sqrt{\rho_B} e^{\pm i(\varphi)}) \tag{94}$$

Adding and subtracting these equations together isolates $\dot{\sqrt{\rho_A}}$ and $\dot{\phi_A}$, respectively. With ρ_A proportional to the charge density of the cooper pairs, we get the two Josephson relations:

$$I = I_c \cos(\varphi) \tag{95}$$

$$\frac{\partial(\varphi)}{\partial t} = \frac{2e}{\hbar} V \tag{96}$$

9.3 Magnetic flux quantum

The probability current in quantum mechanics is

$$J = \frac{1}{2m}[\Psi^*(-i\hbar\nabla)\Psi - \Psi(-i\hbar\nabla)\Psi^*] \tag{97}$$

and, for a spin-0 particle in an electromagnetic field (see Aharonov-Bohm effect), the probability current is

$$J = \frac{1}{2m}[\Psi^*(-i\hbar\nabla)\Psi - \Psi(-i\hbar\nabla)\Psi^* - 2qA|\Psi|^2], \tag{98}$$

$$\Psi = \sqrt{\rho(r)}e^{i\theta(r)} \text{ (Ginzburg-Landau order parameter)} \tag{99}$$

Plugging in wavefunction and realizing that J must be zero in the loop, we get

$$J = \frac{\hbar}{m}[\nabla\theta - \frac{q}{\hbar}A]\rho = 0 \tag{100}$$

$$\Rightarrow \nabla\theta = \frac{q}{\hbar}A \tag{101}$$

Therefore the magnetic flux threading the loop must be

$$\Phi := \int\int Bda = \int\int \nabla \times Ada \xrightarrow{Stokes} \oint A \cdot dl = \frac{\hbar}{q}\oint \nabla\theta \cdot dl = 2\pi\frac{\hbar}{q}z \tag{102}$$

$$= 2\pi\frac{\hbar}{2e}z \tag{103}$$

$$:= \Phi_0 z \tag{104}$$

since the integral starts and ends at the same point (and $\theta = \theta_0 + 2\pi z$). Thus the magnetic flux must be an integer multiple of $2\pi\frac{\hbar}{2e} =: \Phi_0$, i.e. quantized. The K constant intrinsic to the Josephson junction in the previous subsection is defined as the inverse of this when $z = 1$:

$$K = \frac{2e}{2\pi\hbar} \tag{105}$$

$$\Rightarrow \frac{\partial}{\partial t}(\phi_B - \phi_A) = \frac{2e}{\hbar}V(t) \tag{106}$$

Note that this voltage exists whether or not there is a magnetic field present, as it is due to the kinetic energy of the cooper pairs.

9.4 SQUID design

A SQUID (Superconducting QUantum Interference Device) consists of a superconducting wire that splits into two wires each with a Josephson junction (insulator with potential V across it) on it, then rejoining into one wire, forming a loop. The current through the wire is then

$$I = I_1 \sin\varphi_1 + I_2 \sin\varphi_2 \tag{107}$$

An external magnetic field B_{ext} is applied to to the SQUID, contributing a flux Φ_{ext} threading the loop. The total phase difference around the loop must be an integer multiple of the magnetic flux quantum:

$$2\pi n = \phi_2 - \phi_1 - \phi_{ext} \tag{108}$$

Thus the current of the SQUID is

$$I_{SQUID}(\Phi_{ext}) = \sqrt{I_1^2 + I_2^2 + 2I_1 I_2 \cos(\frac{2\pi\Phi_{ext}}{\Phi_0})} \tag{109}$$

Thus a SQUID can be used as a tunable Josephson junction.

10 The Google paper

When Google claimed a demonstration of quantum supremacy in 2019, here is what they did in a nutshell: Send a bunch of qubits through a quantum circuit and measure the output as a bitstring x_i (e.g. $x_i = \{1001011010...\}$). Sampling many of these outputs gives us a probability distribution of the bitstrings. If quantum interference is at play, certain bitstrings are more likely than others, whereas in a classical random circuit, we should obviously get a uniform distribution of bitstrings. As the number of qubits grows, the ability to classically compute this distribution grows exponentially with qubit number and circuit depth. To measure how "quantum" a random circuit on n qubits is, we use the **linear cross-entropy benchmarking fidelity** value:

$$\mathcal{F}_{XEB} = 2^n \langle P(x_i) \rangle_i - 1 \tag{110}$$

when the qubits do not interact, the probability of any fixed x_i should be $\frac{1}{2^n}$, and so $\mathcal{F}_{XEB}$ should tend to 0. On the other hand, if interactions are occurring, $\mathcal{F}_{XEB}$ should tend to 1.

The Google experiment was performed using the Google Sycamore processor. Th Sycamore

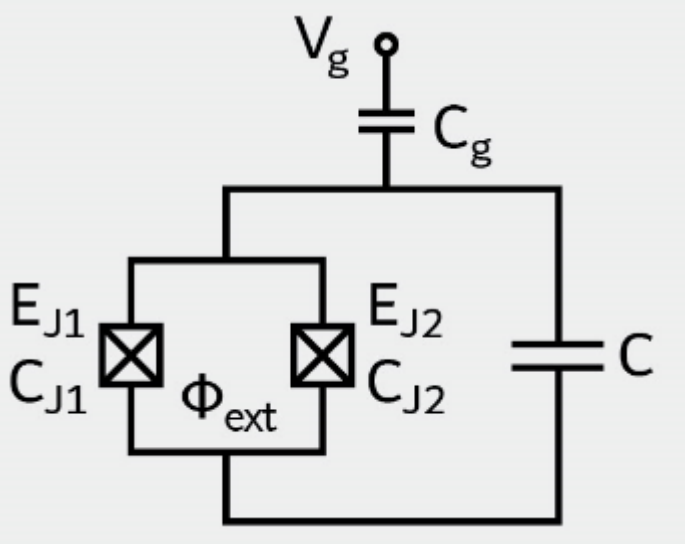


Figure 2: Transmon qubit with DC SQUID as tunable Josephson junction.

processor contains 142 transmon qubits. A **transmon qubit** is a superconducting island connected via a Josephson junction (in Google's case a DC SQUID) to a larger superconductor that acts as a Cooper pair reservoir in series with a capacitor (which we'll call a gate), with both island and reservoir connected to a shunt with an additional capacitor. See the diagram. The state of the qubit is the number of Cooper pairs that have tunneled across the junction. The Hamiltonian of this system is

$$H = \sum_n [E_C(n - \frac{C_g V_g}{2e}) \left|n\right\rangle \left\langle n\right| - \frac{1}{2} E_J(\left|n\right\rangle \left\langle n+1\right| + \left|n+1\right\rangle \left\langle n\right|)], \tag{111}$$

$$E_C = \frac{(2e)^2}{2(C_g + C_J)} \tag{112}$$

where C_g is the shunt capacitance, C_J is the capacitance across the Josephson junction, V_g is the voltage across the shunt capacitor, and E_J is the energy across the Josephson junction. The presence of the shunted capacitor increases the ratio of E_J/E_C, making the Josephson junction less susceptible to charge noise. The flux through the DC SQUID is controlled by an on-chip bias line.

11 How quantum field theory respects causality

The propagator $D(x-y) := \left\langle 0\right| \phi(x)\phi(y) \left|0\right\rangle$ need not be zero for x, y spacelike - what matters is whether a measurement performed at one point can affect the measurement at another point separated spacelike from the first. We will use the Klein-Gordon field as an illustrative example.

In regular quantum mechanics, the probability for a free particle to go from x to y in time t is

$$U(t) = \langle y| \, e^{-iHt} \, |x\rangle \tag{113}$$

If we use energy $E = \frac{p^2}{2m}$, or even relativistic energy $E = \sqrt{p^2 + m^2}$, we get a nonzero probability for all t:

$$\begin{aligned} U(t) &= \langle y| \, e^{-it\sqrt{(p^2+m^2}} \, |x\rangle & (114)\\ &= \langle y| \, e^{-it\sqrt{p^2+m^2}} \, |p\rangle \, \langle p| \, |x\rangle & (115)\\ &= \frac{1}{(2\pi)^3} \int e^{-it\sqrt{(pc)^2+(cm^2)^2}} e^{ip\cdot(y-x)} d^3p & (116)\\ \text{spherical coordinates} &= \frac{1}{2\pi^2|y-x|} \int_0^\infty p \sin(p|y-x|) e^{-it\sqrt{p^2+m^2}} dp & (117) \end{aligned}$$

Assume $y >> t$Using the method of stationary phase, we have a stationary point for the phase function $px - t\sqrt{p^2+m^2}$ at the point $p = \frac{imy}{\sqrt{y^2-t^2}}$. When considering this p value, we have

$$U(t) \sim e^{-m\sqrt{y^2-t^2}} \tag{118}$$

which still gives a nonzero probability far outside the light-cone.

The fields $\phi(x)$ and $\phi(y)$ act on vacuum to create excitations at x and y (in spacetime), respectively. We will see how creating one with the other already there compares for the two points via $[\phi(x), \phi(y)]$. In the Klein-Gordon field, we have

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} [(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_q e^{-iq\cdot y} + a_q^\dagger e^{iq\cdot y})] & (119)\\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) & (120) \end{aligned}$$

If $(x-y)^2 < 0$, we can boost by $\beta = \frac{t}{|x|^2} < 1$ to make the coordinates purely spacelike:

$$\begin{aligned} (t, \cdot, \cdot, \cdot) &\mapsto (\gamma(t - \beta x), \cdot, \cdot, \cdot) & (121)\\ &= (\gamma(t - \frac{t}{|x|^2} x), \cdot, \cdot, \cdot) & (122) \end{aligned}$$

(there's always a Lorentz boost where $t = 0$.) Using this, we can have

$$(t, x, y, z) \xrightarrow{R_1} (t, \sqrt{x^2 + y^2}, 0, z) \tag{123}$$

$$\xrightarrow{R_2} (t, \sqrt{x^2 + y^2 + z^2}, 0, 0) \tag{124}$$

$$\xrightarrow{\beta = \frac{t}{|x|^2}} (0, \sqrt{x^2 + y^2 + z^2 - t^2}, 0, 0) \tag{125}$$

$$\xrightarrow{R_\pi} -(0, \sqrt{x^2 + y^2 + z^2 - t^2}, 0, 0) \tag{126}$$

$$\xrightarrow{(\beta R_2 R_1)^{-1}} -(t, x, y, z) \tag{127}$$

$$\tag{128}$$

This is a continuous transformation that can only be achieved if $(x-y)^2 < 0$. Time-like four-vectors are essentially purely time-like four-vectors (i.e. (t,0,0,0)), and there is only one dimension to rotate in, so we can't rotate like we can in 3-dimensions. Thus, if x and y are separated space-like, we can send $(x - y) \mapsto -(x - y)$:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) \mapsto \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{-ip\cdot(x-y)}) = 0 \tag{129}$$

and causality is preserved.

12 Dirac Theory

Dirac wanted to make the Schrödinger equation relativistic. Do to this, we try to make the Hamiltonian in the Schrödinger equation

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{130}$$

$$\hat{H}\Psi = E\Psi \tag{131}$$

a relativistic description of energy. From relativity, we have the relativistic energy relationship:

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{132}$$

We want to find a self-adjoint operator that, when squared, gives $\hat{E}^2 - (\hat{p}c)^2$. The Schrödinger Equation gives us the $\hat{E}$ operator as $i\hbar\frac{\partial}{\partial t}$ and the $\hat{p}$ operator as $-i\hbar\nabla$, so we want a self-adjoint operator that, when squared, gives $\hbar^2(-\frac{\partial^2}{\partial t^2} + c^2\nabla^2)$. It turns out that the **Clifford Algebra**

allows us to do this:

$$\hbar(i\gamma_0\frac{\partial}{\partial t}+c\sum_{n=1}^{3}\gamma_n\frac{\partial}{\partial x_n})\Psi=m_0c^2\Psi \tag{133}$$

where the γ_i satisfy the algebra

$$\gamma_i^2=Id, \tag{134}$$

$$\gamma_i\gamma_j+\gamma_j\gamma_i=0, i\neq j \tag{135}$$

We can write this much more succinctly. Recall that four-gradient ∂^μ is given by $(\frac{1}{c}\frac{\partial}{\partial t},-\nabla)$. We can write the above in terms of the four-gradient by pulling out an i:

$$(i\hbar\gamma_\mu\partial^\mu-m_0c^2)\Psi=0 \tag{136}$$

$$:=(i\partial\!\!\!/-m)\Psi=0 \tag{137}$$

in natural units and with Feynman's notation. This is the **Dirac Equation**. To make this concrete, and to make this operator self-adjoint, we consider skew-Hermitian representations of this Clifford Algebra. That is, we demand representations such that $rep(\gamma_i)^*=-rep(\gamma_i),\forall i$. An easy such representation of this is the **Weyl** or **chiral** representation (sign conventions and γ_0 representation may vary):

$$\gamma_0=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \gamma_i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix} \tag{138}$$

where the σ_i are the usual Pauli matrices. In this represenation, the Ψ "wavefunction" must be a 4-entry vector, i.e. a four component field called a **spinor**.

13 Using $\sum_{\mathbb{Z}}=-\frac{1}{12}$ in string theory

String theory is a candidate theory for quantum gravity. In addition to having point-like (0-dimensional) excitations, it features 1-dimensional strings and higher-dimensional objects called branes on which the end points of strings can be anchored. The energy spectrum of a quantum string stretched between two D-branes is related to a famous function from number theory studied

by Ramanujan.

We start by classical string theory. Consider a string stretched between two ends. It has length L, mass M, and tension T and we will denote its displacement at position x and time t as $y(x,t)$. Such a string satisfies a wave equation whose general solution (once one imposes the boundary conditions $y(0,t) = y(L,t) = 0$ is of the form

$$y(x,t) = \sqrt{2}\sum_{n=1}^{\infty} \xi_n(t) \sin\frac{n\pi x}{L} \tag{139}$$

We compute the kinetic and potential energies of this solution

$$K = \frac{1}{2}\frac{M}{L}\int_0^L dx\dot{y}^2, U = \frac{1}{2}T\int_0^L dx(\frac{\partial y}{\partial x})^2 \tag{140}$$

in terms of ξ_n and $\dot{\xi}_n$: The modes of y(x, t) are identical (up to normalization) to the eigenfunctions of the infinite square well. The key property well exploit is that these functions are orthonormal. Then

$$K = \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)\int_0^L dx 2\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L}) \tag{141}$$

$$= \frac{1}{2}\frac{M}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dot{\xi}_n(t)\dot{\xi}_m(t)2L\delta(nm) \tag{142}$$

$$= \frac{1}{2}M\sum_{n=1}^{\infty}\dot{\xi}_n(t)^2 \tag{143}$$

Using a very similar calculation, we have

$$U = \frac{1}{2}T\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\xi_n(t)\xi_m(t)\frac{n\pi}{L}\frac{m\pi}{L}\int dx 2\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} \tag{144}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{Tn^2\pi^2}{L}\xi_n(t)^2 \tag{145}$$

Thus we see that the total energy looks like that of infinitely many harmonic oscillators, each with mass M and with the n^{th} oscillator having frequency $\omega_n = \sqrt{\frac{T\pi^2}{M:}}n = \omega_1 n$. The total energy $E = K + U$ is that of infinitely many harmonic oscillators, one for each mode ξ_n.

The simplest quantum string theories arise essentially by promoting classical systems (like the piano string considered in the previous part) to quantum ones. The classical string is more or less equivalent to infinitely many decoupled harmonic oscillators (a different oscillator governing each harmonic), so we work towards developing a quantum version of this. First consider the quantum

Hamiltonian which describes just two oscillators with arbitrary frequencies ω_1, ω_2:

$$\hat{H} = \frac{\hat{p}_1^2}{2M} + \frac{1}{2}M\omega_1^2\hat{\xi}_1^2 + \frac{\hat{p}_2^2}{2M} + \frac{1}{2}M\omega_2^2\hat{\xi}_2^2 \tag{146}$$

where $\hat{p}_n = -i\hbar\frac{\partial}{\partial_n}$ for $n = 1, 2$. Using a separation of variables argument, we assume for now that the solution to the time-independent solution Ψ has the form $\Psi(\xi_1, \xi_2) = X(\xi_1)Y(\xi_2)$. Plugging this into the TISE and dividing by Ψ, we get

$$-\frac{1}{X}\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial\xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 = E - (-\frac{1}{Y}\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial\xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2) \tag{147}$$

Following the standard separation of arguments process, we conclude that this is equal to a constant E, and calling $E = E_1 + E_2$, with

$$-\frac{\hbar^2}{2M}\frac{\partial^2 X}{\partial\xi_1^2} + \frac{1}{2}M\omega_1^2\xi_1^2 X = E_1 X \tag{148}$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2 Y}{\partial\xi_2^2} + \frac{1}{2}M\omega_2^2\xi_2^2 Y = E_2 Y \tag{149}$$

Thus $X(\xi_1$ and $Y(\xi_2)$ are harmonic oscillator eigenstates with energies that add up to E. Thus we can have the eigenstates be

$$\Psi_{n_1,n_2}(\xi_1, \xi_2) \propto H_{n_1}(\sqrt{\frac{M\omega_1}{\hbar}}\xi_1)H_{n_2}(\sqrt{\frac{M\omega_2}{\hbar}}\xi_2)\exp(-\frac{M}{2\hbar}(\xi_1^2\omega_1 + \xi_2^2\omega_2)) \tag{150}$$

with energy

$$E_{n_1,n_2} = \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2}) \tag{151}$$

For N decoupled harmonic oscillators, this result generalizes:

$$E_{n_1,\ldots,n_N} = \sum_{i=1}^{N}\hbar\omega_i(n_i + \frac{1}{2}) \text{ for } n_i \in \mathbb{Z}\forall i \tag{152}$$

Now, we consider an infinite number of decoupled quantum harmonic oscillators ($N \to \infty$). This time we specialize our frequencies so that the frequency of the n^{th} oscillator is the frequency ω_n found before:

Taking the limit as $N \to \infty$, from the previous part we have the energy spectrum as

$$E_{n_1,n_2,\ldots} = \sum_{k=1}^{\infty} \hbar\omega_k (n_k + \frac{1}{2}) \tag{153}$$

$$= \sum_{k=1}^{\infty} \hbar k\omega_1 (n_k + \frac{1}{2}) \tag{154}$$

$$= \hbar\omega_1 \sum_{k=1}^{\infty} k n_k + \frac{\hbar\omega_1}{2} \sum_{k=1}^{\infty} k \tag{155}$$

$$= \hbar\omega_1 (-\frac{1}{24} + \sum_{k=1}^{\infty} k n_k) \tag{156}$$

This is the energy spectrum of a quantum string stretched between two D-branes.

A partition of n is an unordered set of positive integers whose sum is n. We will define $p(n)$ to be the number of partitions of a number n. Consider the function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{157}$$

where $q \in \mathbb{C}$ with modulus ≤ 1. The series expansion of its inverse

$$\frac{1}{\eta(q)} = \sum_{n=0}^{\infty} p(n) q^{n-1/24} \tag{158}$$

famously serves as the generating function for $p(n)$. For an arbitrary positive integer m, we can get degeneracy depending on our choices for n_k. Thus $\sum_{k=1}^{\infty} k n_k = p(m)$. Thus

$$\frac{1}{\eta(q)} \sim \sum_{\text{energies } E} (\text{number of states of energy } E) q e^{\frac{E}{\hbar\omega_1}} \tag{159}$$

14 Geometric Quantization

14.1 Prequantization

What's fascinating but somewhat frustrating is the fact that new physics theories arise nowadays from redefining previously understood concepts with increasing levels of abstraction. This note is to explain *intuitively* where the idea that a quantum state is a section of a principal bundle.

When we want to consider a physical system, we start with a phase space. This is a space with coordinates describing possible spatial coordinates and momentum coordinates. In regular classical mechanics, we usually consider $\mathbb{R}^6$, where the first three coordinates describe position, and the last three coordinates describe position in momentum space (momentum in the three directions). But

why limit ourselves to this boring a space? We know that, for each position coordinate (i.e. direction) we can have a direction for momentum, so our phase space must be even-dimensional. But supposing we want to constrain our system, like if we can to consider particles on a curve or plane, we'd want an even-dimensional submanifold M of $\mathbb{R}^{2n}$. To make sure all derivatives are well-defined everywhere, let's make things easy on ourselves and consider smooth manifolds.

To define the dynamics on our manifold, we want some sort of energy function defined on every state (point) of our system (manifold). Thus we want an energy function, H for Hamiltonian on our manifold:

$$H : M \to \mathbb{R} \tag{160}$$

giving us an energy value for every state we're in. To make things realistic, we'd better have H be a *smooth* real-valued function on M - we want states sufficiently close together to have sufficiently close energies. It turns out that just having H be real-valued is all we need to determine the physics of our system (e.g. Newton's laws). How does one state change into another state? The geometric setting that describes how all points change into other points on a manifold is a *flow* along a (smooth) vector field: each point has a vector attached to it (and sufficiently close points have sufficiently "close" vectors - more on how two vectors can be "close" later), and the vector points in the direction that the state will evolve to.

Inventory: we have a smooth manifold M, and energy function H on M, and we want a smooth vector field, which we'll call V, describing the dynamics. Dynamics are determined by energy, so we have a hint that we can determine the vector field describing these dynamics with H. The dynamics should vary locally, as dynamics depend on how the energy changes locally, so we want the differential dH. In multidimensional spaces, how H changes depends on which direction you're doing. So if you were to give a direction (ahem tangent vector) into dH, you'd have a value on how H is changing. Thus dH is a covector field on M.

So now we want a vector field on M (the dynamics on M) and we physically know we should get it from dH. Thus we want a way to go from dH to a V. Furthermore, this should be a *linear* way to go from dH to a V, since Newton's equations are linear differential equations. Thus we should look for a field attaching each point on M to a value in

$$Hom(T^*M, TM) \tag{161}$$

$$(Hom \text{ is the set of linear ways to go from } T^*M \text{ to } TM) \tag{162}$$

i.e. we want a section $M \to Hom(T^*M, TM)$. It's more convenient to dualize this section into

$M \to Hom(TM, T^*M)$. This is the same as a section $M \to T^*M \otimes T^*M$, which we'll label by ω. We want our ω such that

$$\omega : TM \to T^*M \tag{163}$$

$$V \to \omega(V, \cdot) \tag{164}$$

$$\omega(V, \cdot) = dH(\cdot) \tag{165}$$

This way, if we have a vector field X (if we know in which direction we'll start out), we can solve for V (the dynamics of our system) by taking

$$\omega(V, X) = dH(X) \tag{166}$$

What can we impose on ω to ensure no funny business happens? First, we must make sure that we can always solve for V, i.e. a V exists for all vector fields X. If ω is nondegenerate, then V always exists, and through linear algebra is always unique.

Also, we want energy to be conserved. This entails that the Hamiltonian is constant along the flow due to the dynamics, i.e.

$$dH(V) = 0 \tag{167}$$

Therefore, $\omega(V, V) = 0$, so ω must be alternating. Now we have that ω must be a nondegenerate 2-form on M.

Lastly, we want to make sure that V doesn't depend on t, i.e. Newton's laws don't change with time. The way to determine how a tensor field changes along a vector field V is with the *Lie derivative* $\mathcal{L}_V$, so we check the Lie derivative of ω along V:

$$\mathcal{L}_V \omega = d\omega(V, \cdot, \cdot) + d(\omega(V, \cdot)) \tag{168}$$

$$= d\omega(V, \cdot, \cdot) + ddH \tag{169}$$

$$= d\omega(V, \cdot, \cdot) \tag{170}$$

since we want this change to be 0, we require that $d\omega = 0$. Thus, ω must be a closed, nondegenerate 2-form on M, i.e. (M, ω) must be a symplectic manifold.