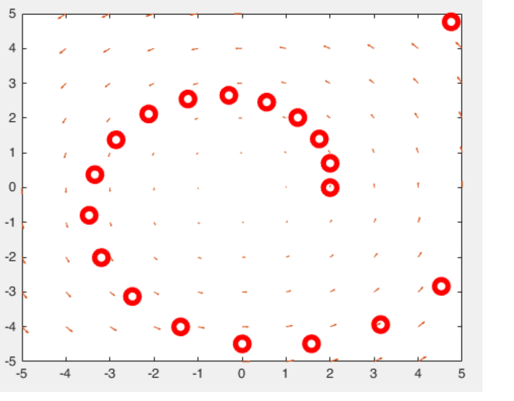
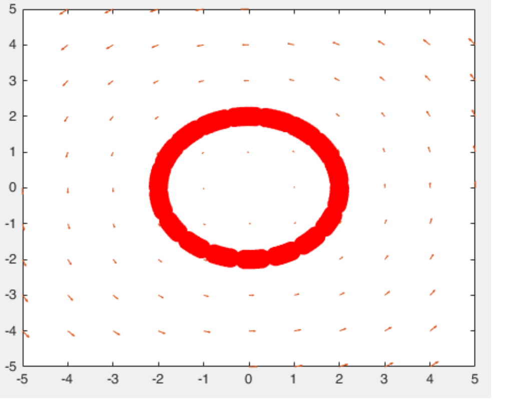
Time Integration Methods So Far



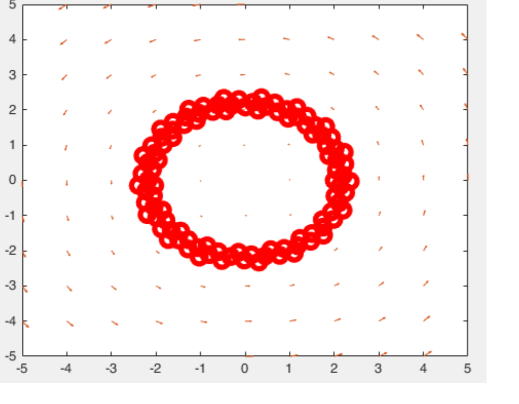
Forward Euler: Use slope at starting point.

- Local Truncation Error (LTE): $O(h^2)$. Explicit. Single-Step.



Trapezoidal: Use average of slope at start and end of step.

- LTE: $O(h^3)$. **Implicit**. Single-Step.



Improved Euler: Use average of slope at start and (approximate) end.

- LTE: $O(h^3)$. Explicit. Single-Step.

Improved Euler / Trapezoidal Example

We previously applied F.E. to the problem:

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

with initial conditions $x(t_0) = 2, y(t_0) = 0, t_0 = 0.$

1) Apply *improved Euler* with time step size $h = 2$ to find x, y at $t = 4.$

2) What equations do we need to solve if we apply the trapezoidal method?

Even More Schemes!

There are many more time integration schemes, each with its own particular properties. (Impossible to cover them all.)

We will see a few more, and (continue to) focus on:

1. Truncation error
2. Explicit v.s. implicit
3. Single-step v.s. multistep
4. Stability (Still to come!)

Backwards (Implicit) Euler method

Similar to forward Euler, but implicit.

Forward Euler:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Start of Step Slope,
i.e. time t_n.

Backwards Euler uses the slope from **only** the end of the step:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

End of Step Slope,
i.e. time t_{n+1}.

Its local truncation error is $O(h^2)$, like F.E.

Explicit “Runge Kutta” schemes

Improved Euler can (equivalently) be written as:

$k_1 = h \cdot f(t_n, y_n),$

$k_2 = h \cdot f(t_n + h, y_n + k_1),$

$y_{n+1} = y_n + \frac{k_1}{2} + \frac{k_2}{2}\,.$

There is an entire family of similar schemes: *Runge Kutta* methods.

Often written in this form.

(Explicit) Midpoint method

Another explicit Runge Kutta scheme (with LTE $O(h^3)$) is the *explicit midpoint method* (course notes p54):

$$k_1 = h \cdot f(t_n, y_n),$$
$$k_2 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$y_{n+1} = y_n + k_2.$$

Intuition?

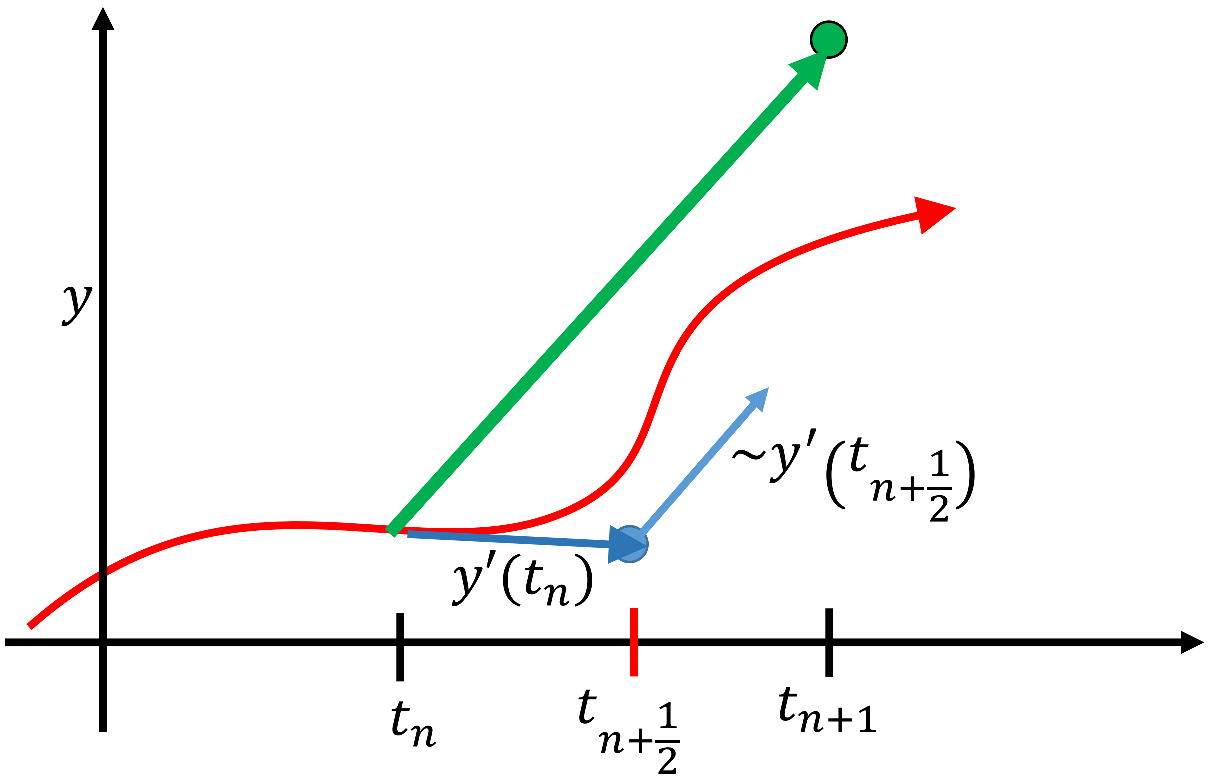
(Explicit) Midpoint method - Intuition

1. Take a FE step to the "halfway" point in time.
2. Evaluate the slope there.
3. Use *that* slope to take a full step from the start.

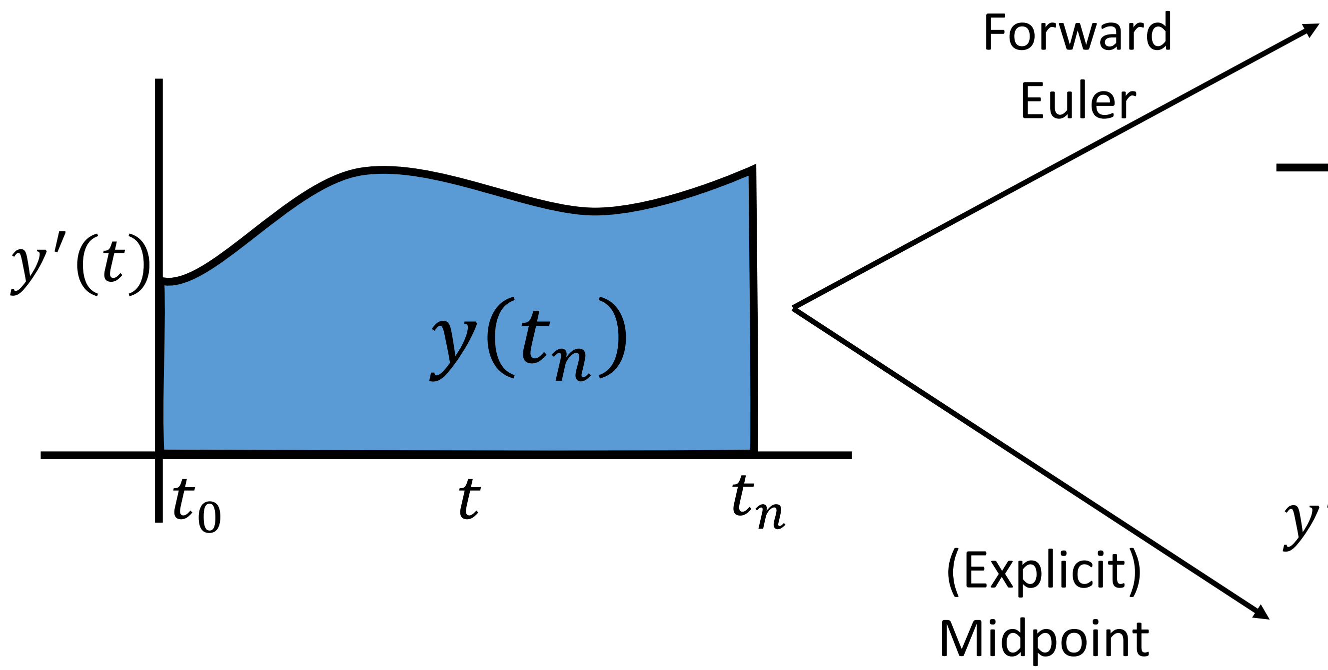
Equivalent expression:

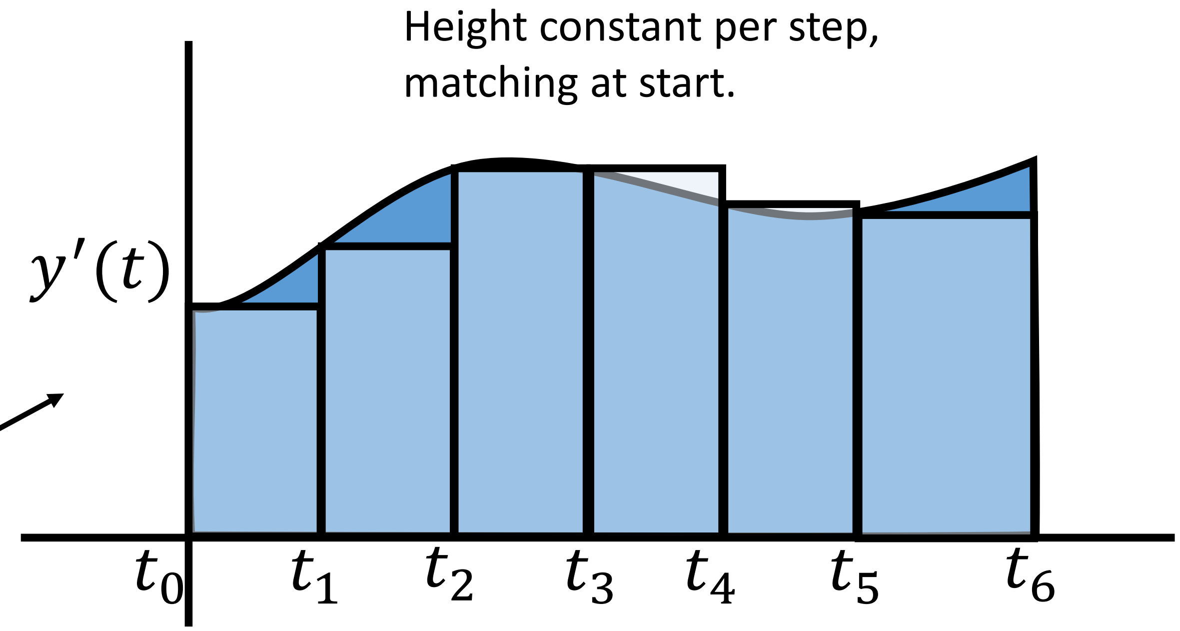
$$y^*_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(t_n, y_n),$$

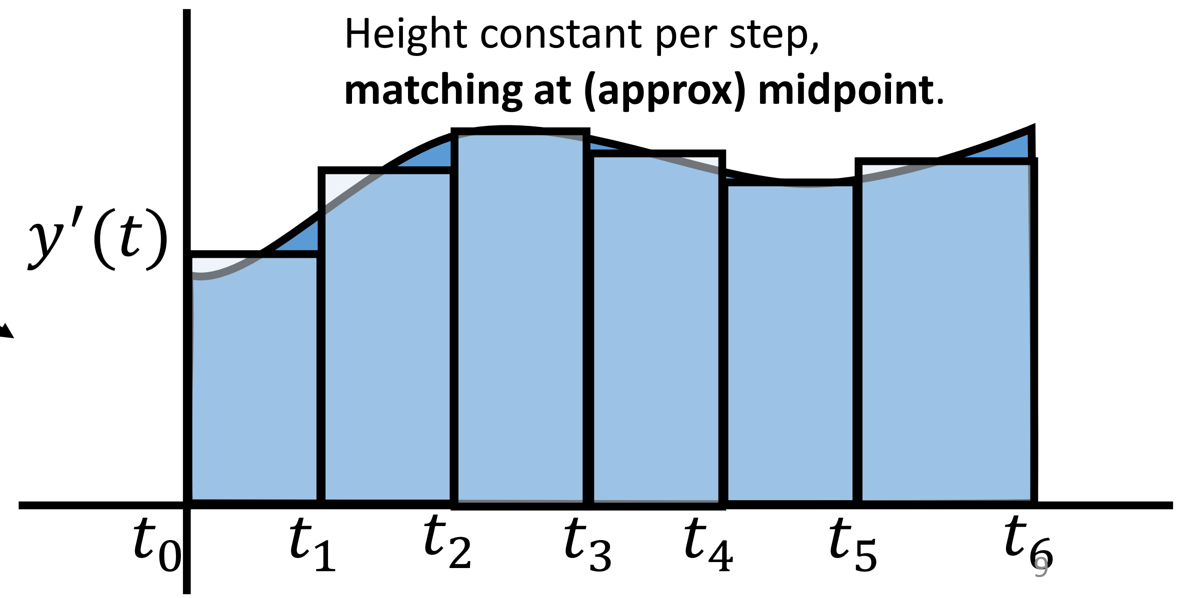
$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y^*_{n+1/2}\right)$$



Area Integration View







4^{th} Order Runge Kutta

Similar schemes exist for higher orders, $O(h^{\alpha})$ for $\alpha = 4{,}5{,}6 \ldots$
"Classical" Runge-Kutta, or "RK4", with LTE of $O(h^5)$:

$$k_1 = h \cdot f(t_n, y_n),\ k_2 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),\ k_4 = f(t_n + h, y_n + k_3),$$
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Again, evaluate $y'(t) = f(t, y)$ at various intermediate positions, and take a specific linear combination to find y_{n+1}.

RK4 – Area integration (& interpolation)

RK4 also approximates area under the derivative curve! (But not using rectangle or trapezoids.)

Fit a ***quadratic*** to the start, middle, and end points, and exactly integrate area beneath.

(AKA "Simpson's rule".)