

## CS 370 Spring 2017: Assignment 2

Due Thursday, June 8, 5:00 PM, in the Assignment Boxes, 4th Floor MC.

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Lectures: MWF 1:30 - 3:30 MC2035

Office Hours: Tues 11:00-12:00

1. **(15 marks)** Consider Newton's equations of motion for a two-body problem specified by:

$$x''(t) = -\frac{x(t)}{(x(t)^2 + y(t)^2)^{3/2}}; \quad x(0) = 0.4; \quad x'(0) = 0$$

$$y''(t) = -\frac{y(t)}{(x(t)^2 + y(t)^2)^{3/2}}; \quad y(0) = 0; \quad y'(0) = 2.$$

As  $t$  ranges from 0 to  $2\pi$ ,  $(x(t), y(t))$  defines an ellipse.

- (a) Put this initial value problem into the standard form:  $\mathbf{z}'(t) = f(t, \mathbf{z})$ .
  - (b) Write the corresponding dynamics function in Matlab.
  - (c) On the course site you will find a Matlab function, `y = ForwardEuler(f, t0, tf, N, z0)`, which takes as input the system dynamics function  $f$ , the initial and final  $t$  values,  $t_0$  and  $t_f$ , the number of steps,  $N$ , and the initial value,  $z_0$ , and implements the forward Euler method. Use this command to graph the resulting curves  $(x(t), y(t))$  from  $t = 0$  to  $t = 2\pi$  and verify that it indeed defines an ellipse.
2. **( 15 marks )** Consider the initial value problem

$$y'(t) = -2ty(t)^2, \quad y(0) = 1.$$

For this particular IVP it is easily checked that

$$y(t) = \frac{1}{1+t^2}$$

is the exact analytical solution. We will use this exact solution to determine the errors in various numerical methods applied to solve the IVP.

- (a) Write a Matlab function, `y = MidpointRule(f, t0, tf, N, y0)`, which takes as input a system dynamics function  $f$ , the initial and final  $t$  values,  $t_0$  and  $t_f$ , the number of steps,  $N$ , and the initial value,  $y_0$ , and implements the midpoint Runge-Kutta method (course notes on page 54). It should return the approximate solution as a vector  $y$  of length  $N+1$  such that

$$y_i \approx y(t_{i-1}) \quad \text{for } i = 1, 2, \dots, N+1$$

where  $t_i = t_0 + ih$ ,  $h = \frac{t_f - t_0}{N}$ , and  $y(t_{i-1})$  denotes the exact solution at  $t = t_{i-1}$ .

- (b) Write a Matlab script to apply the `ForwardEuler` function from the course web site to solve the IVP to value  $t_f = 4$  with  $N = 40$  time steps. Plot the analytical solution (using solid lines) and the numerical solution (using the `'.'` symbol for each point) on the same graph, for  $0 \leq t \leq 4$ .

- (c) Repeat part (c) using the `MidpointRule` function.
- (d) Calculate evidence illustrating the order of the forward Euler method as follows.  
 Let  $\text{err}(h)$  denote the error when computing with step size  $h$ . For a  $p$ -order method,  $\text{err}(h) \approx c \cdot h^p$  for some constant  $c$ , and so the effect of cutting the step size in half is

$$\frac{\text{err}(h/2)}{\text{err}(h)} \approx \frac{1}{2^p}$$

i.e. the error is reduced by a factor of  $2^p$ . Therefore, the order  $p$  of a given method can be illustrated by calculating the above ratios.

For the forward Euler method, calculate a vector of successive computed values for the solution at  $t = 4$  based on cutting the step size in half a number of times. Specifically, use  $N = 2^i \times 10$  for  $i = 0, 1, \dots, 10$ . Calculate the errors and the successive ratios  $\frac{\text{err}(h/2)}{\text{err}(h)}$ . From your experimental evidence, what is the order of the forward Euler method?

- (e) Repeat part (e) for the midpoint method. From your experimental evidence, what is the order of the midpoint rule method?

3. **(10 marks)** Suppose we are solving the ordinary differential equation

$$y'(t) = f(t, y(t)).$$

Determine the local truncation error of the method

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(t_{n-1}, y_{n-1})$$

where  $y_n$  is an approximation to  $y(t_n)$ , and  $t_n = nh$  with  $h$  the stepsize.

4. **(10 marks)** Consider the second-order Runge-Kutta method given by

$$y_{n+1} = y_n + \frac{h}{4} \left[ f(t_n, y_n) + 3f\left(t_n + \frac{2h}{3}, y_n + \frac{2}{3}hf(t_n, y_n)\right) \right].$$

Analyse the stability of this method by applying the test equation

$$y'(t) = -\lambda y(t), \quad \lambda > 0.$$

What is your conclusion about stability limitations on  $h$ ?

5. **(25 marks)** Consider a target moving in a two dimensional plane with location  $(x_t, y_t)$ . The target is moving with constant speed  $S_t$ , in direction given by the unit vector  $(\cos \theta_t, \sin \theta_t)$ , where  $\theta_t$  is the angle of the velocity vector with respect to the  $x$  axis.

The target is pursued by a pursuer, with location  $(x_p, y_p)$ . The pursuer is moving with constant speed  $S_p$ , in direction given by the unit vector  $(\cos \theta_p, \sin \theta_p)$ , where  $\theta_p$  is the angle of the velocity vector with respect to the  $x$  axis.

The pursuer attempts to hit the target, by turning as fast as possible towards the target. The minimum turning radius of the pursuer is  $R_p$ , and the minimum turning radius of the target is  $R_t$ .

The trajectories of the pursuer and target can be described by the following system of ODEs, where  $t$  is the time.

$$\begin{aligned} x'_p(t) &= S_p \cos \theta_p(t), & y'_p(t) &= S_p \sin \theta_p(t), & \theta'_p(t) &= S_p \frac{\theta_d(t) - \theta_p}{(|\theta_d(t) - \theta_p(t)| + D)} \\ x'_t(t) &= S_t \cos \theta_t(t), & y'_t(t) &= S_t \sin \theta_t(t), & \theta'_t(t) &= \frac{S_t}{R_t} \omega_t(t). \end{aligned}$$

where

$$\begin{aligned} (x_p, y_p) &= \text{x, y coordinates of pursuer} \\ (x_t, y_t) &= \text{x, y coordinates of target} \\ \theta_t &= \text{target moving in direction } \cos \theta_t \hat{x} + \sin \theta_t \hat{y} \\ \theta_p &= \text{pursuer moving in direction } \cos \theta_p \hat{x} + \sin \theta_p \hat{y} \\ S_p &= \text{speed of pursuer} \\ S_t &= \text{speed of target} \\ R_t &= \text{minimum turning radius, target} \\ \theta_d &= \text{direction from pursuer to target (see Figure 1)} \\ \omega_t &= \text{angular speed of target (fraction of maximum } S_t) \\ D &= \text{turning damping factor, pursuer} \end{aligned} \tag{1}$$

The damping factor  $D$  in equation (1) prevents the pursuer direction from changing too rapidly when  $\theta_d \simeq \theta_p$ .

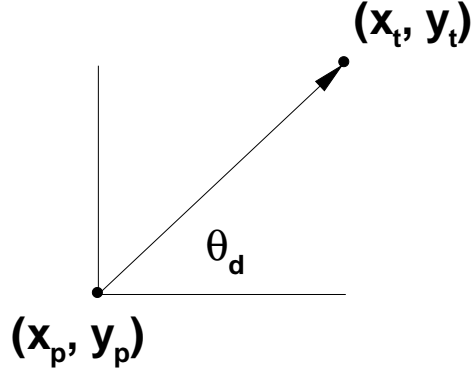


Figure 1: Direction from pursuer to target.

Parameter	Value
Initial Target Position	$(x_t, y_t) = (5, 5)$
Initial Pursuer Position	$(x_p, y_p) = (0, 0)$
Initial Target Direction	$\theta_t = 0.0$
Initial Pursuer Direction	$\theta_p = -\pi$
Integration Interval	$[0, 20]$
$R_t$	.25
$S_p$	2.0
$S_t$	1.0
Damping Factor	$D = 10^{-3}$
Hitting Distance	$d_{stop} = 10^{-2}$
Angular Speed of Target	$\omega_t = 0.1$
Error Tolerance	$tol = 10^{-6}$

Table 1: Data for pursuit problem.

### Objective

The objective of this assignment is to develop Matlab code which solves the pursuit-target problem, and determines the time (if it occurs) when the pursuer hits the target. The pursuer is considered to have hit the target if the distance between the target and the pursuer is less than  $d_{stop}$ , i.e.

$$\sqrt{(x_t - x_p)^2 + (y_t - y_p)^2} \leq d_{stop} . \quad (2)$$

The data for this problem is given in Table 1.

- (a) **(10 marks)** Write Matlab code to solve the ODE system (1). Use an event function (see course notes) to stop the simulation if the hitting criterion (2) is satisfied. In order to determine the direction  $\theta_d$  to the target (equation (1)), we have to be careful, since  $\theta_p, \theta_t$  can be less than or larger than  $2\pi$  (the pursuer/target could wind around a central point several times). As well, we want to make sure the pursuer turns towards the target in the shortest way possible. Use the supplied function *direction.m* (available on the course web site) to compute  $\theta_d$ .

Use the stiff solver *ode15s*. Use the following options to be passed to the ODE solver (see the course notes).

```
options = odeset('AbsTol', tol, 'RelTol', tol, 'MaxOrder', 5, 'Stats', 'on', ...
    'Events', @event, 'Refine', 4);
```

Submit a hard copy of your code, a plot of the trajectories of the pursuer and target, and the hitting time. The trajectory of the pursuer is given by the parametric curve  $(x_p(t), y_p(t))$ , and the trajectory of the target is given by the parametric curve  $(x_t(t), y_t(t))$  for  $t \in [0, T]$  where  $T$  is the stopping time or end of the integration interval (whichever is first).

- (b) **(5 marks)** Now, carry out some tests to see the effect of the error control  $tol$  on the solution accuracy. Fill in Table 2. Explain what you see.

$tol$	Hitting Time	Number of Function Evaluations
$10^{-3}$		
$10^{-4}$		
$10^{-5}$		
$10^{-6}$		
$10^{-7}$		
$10^{-8}$		
$10^{-9}$		

Table 2: Test of error control

- (c) **(5 marks)** Now, try solving the problem again, except use *ode45* as the solver. Use  $tol = 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$ . Compare with the solution using *ode15s*. Explain what you see.
- (d) **(5 marks)** Attempt to devise an evasion strategy for the target. The target is constrained so that  $S_t = 1.0$ , and the  $\omega_t \leq 1$ . Some ideas: when the distance between the target and the pursuer is less than a small multiple of  $R_t$ , try turning the target in a direction away from the pursuer. Increase the maximum integration interval to  $[0, 40]$ . Submit a pseudo-code description of your evasion strategy, a hard copy of your matlab code, and a plot of the target and pursuer trajectories.
- The five most interesting strategies (as determined by the TAs), will earn 3 bonus marks. In the case of disputes, the TAs decisions are final.