# Lecture XVI: Spherically symmetric stars

(Dated: October 25, 2019)

## I. OVERVIEW

We now turn our attention to relativistic stars. We will focus in this lecture on the derivation of the metric for a static spherically symmetric star and the relations between metric quantities, density, and pressure.

This lecture covers roughly Chapter 10 of the book.

## II. SYMMETRIES

Let us first define carefully what we mean by a "spherically symmetric" system. Recall that a sphere of radius $r$ has a metric

$$\mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1}$$

where $\theta$ is colatitude and $\phi$ is longitude. Note that "$r$" is not necessarily a physically measurable radius with a ruler (we have not said anything yet about the "interior" to the sphere), but $2\pi r$ is a physically measurable circumference and $4\pi r^2$ is a physically measurable area. This establishes $g_{\theta\theta} = r^2$, $g_{\theta\phi} = 0$, and $g_{\phi\phi} = r^2 \sin^2\theta$.

In a 4D spacetime, we need two more coordinates. We will use $r$ itself as one of the coordinates, so we need one more which we will call $t$ (so far, "$t$" is just a name). The coordinates $r$ and $t$ will tell us which 2-sphere we are on, and $\theta$ and $\phi$ tell us where on the 2-sphere. In this case, the vector $\boldsymbol{e}_r$ that points to larger radius but without changing colatitude or longitude must be orthogonal to $\boldsymbol{e}_\theta$ that points south, since otherwise the dot product $\boldsymbol{e}_r \cdot \boldsymbol{e}_\theta$ could be used to distinguish south from north. Thus spherical symmetry requires $g_{r\theta} = \boldsymbol{e}_r \cdot \boldsymbol{e}_\theta = 0$. Similar rules eliminate $g_{r\phi}$, $g_{t\theta}$, and $g_{t\phi}$. Then:

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + 2g_{rt}\,\mathrm{d}r\,\mathrm{d}t + g_{rr}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{2}$$

Moreover, $g_{tt}$, $g_{rt}$, and $g_{rr}$ are functions of only $r$ and $t$.

A further simplification can be obtained if the system is *stationary* or "time-translation invariant": this means that we can choose the coordinates such that the metric coefficients don't depend on $t$. In this case:

$$\mathrm{d}s^2 = g_{tt}(r)\,\mathrm{d}t^2 + 2g_{rt}(r)\,\mathrm{d}r\,\mathrm{d}t + g_{rr}(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{3}$$

We can eliminate $g_{rt}$ if we define a new time variable $\bar t$ by:

$$\bar t = t + \int \frac{g_{rt}(r)}{g_{tt}(r)}\,\mathrm{d}r \quad \rightarrow \quad \mathrm{d}t = \mathrm{d}\bar t - \frac{g_{rt}(r)}{g_{tt}(r)}\mathrm{d}r. \tag{4}$$

This change of coordinate system gets rid of the cross-term $g_{\bar r\bar t}$, and in what follows once we have done this we will omit the bar. It is furthermore conventional to write $g_{tt}$ and $g_{rr}$ as exponentials:

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \mathrm{e}^{2\Lambda(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{5}$$

This is the usual form for the metric of a spherically symmetric star. We write the coordinates in the order $tr\theta\phi$. We expect by looking at this that $\Phi(r)$ will be identified with the usual Newtonian gravitational potential in the weak-field limit.

In fact this doesn't completely describe the coordinate system, since we could rescale $t$ by $t = \mathrm{e}^{\Delta}\bar t$ with constant $\Delta$ and then $\Phi(r) \rightarrow \Phi(r) + \Delta$. Thus we expect an arbitrary additive offset to $\Phi$. For isolated stars, we will fix this by setting $\Phi = 0$ at $r = \infty$.

## III. THE STRESS-ENERGY TENSOR

For a stationary spherically symmetric star, we will go to a particular location in the star. We suppose the matter to be stationary in the sense that a fluid parcel has $\mathrm{d}r/\mathrm{d}\tau = 0$ (no radial motion) and $\mathrm{d}\theta/\mathrm{d}\tau = \mathrm{d}\phi/\mathrm{d}\tau = 0$ (no angular motion). Thus this parcel must have 4-velocity

$$u^\alpha \rightarrow (\mathrm{e}^{-\Phi}, 0, 0, 0), \quad u_\alpha \rightarrow (-\mathrm{e}^{\Phi}, 0, 0, 0). \tag{6}$$

If the fluid in the star is a perfect fluid, described by a density and a pressure, then

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) \to \begin{pmatrix} e^{2\Phi}\rho & 0 & 0 & 0 \\ 0 & e^{2\Lambda}p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 p \sin^2\theta \end{pmatrix}. \tag{7}$$

Again, in a stationary spherically symmetric star, we take $\rho$ and $p$ to be functions of $r$ only.

## IV. THE CHRISTOFFEL SYMBOLS AND HYDROSTATIC EQUILIBRIUM

A stationary star must be in *hydrostatic equilibrium*: this means that the pull of gravity downward balances the pressure gradient pushing upward. To see how this works in GR, let's recall the conservation of stress-energy in the form

$$T_\mu{}^\nu{}_{;\nu} = T_\mu{}^\nu{}_{,\nu} - \Gamma^\alpha{}_{\mu\nu} T_\alpha{}^\nu + \Gamma^\nu{}_{\alpha\nu} T_\mu{}^\alpha = 0. \tag{8}$$

In GR, there is not any gravitational "force" on the right-hand side of the equation that acts as a source of energy or momentum; rather the coordinate system (e.g., given value of $r$) is accelerating upward relative to a freely-falling frame. This acceleration of the coordinate system is described by the Christoffel symbols.

The down-up-index version of the stress-energy tensor – which we are using here – is

$$T_\mu{}^\nu \to \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{9}$$

The Christoffel symbols can be computed in the usual way. They are as follows (using the $'$ symbol to indicate an $r$-derivative):

$$\begin{aligned} \Gamma^r{}_{tt} &= e^{2\Phi-2\Lambda}\Phi', \\ \Gamma^t{}_{rt} = \Gamma^t{}_{tr} &= \Phi', \\ \Gamma^r{}_{rr} &= \Lambda', \\ \Gamma^r{}_{\theta\theta} &= -re^{-2\Lambda}, \\ \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{r}, \\ \Gamma^r{}_{\phi\phi} &= -re^{-2\Lambda}\sin^2\theta, \\ \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} &= \frac{1}{r}, \\ \Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} &= \cot\theta, \end{aligned} \tag{10}$$

and the others are zero.

In Eq. (8), the $t$, $\theta$, and $\phi$ components are all trivially satisfied. The $r$ component of the equation reads:

$$(p') - \left(-\Phi'\rho + \Lambda' p + \frac{1}{r}p + \frac{1}{r}p\right) + \left(\Phi' p + \Lambda' p + \frac{1}{r}p + \frac{1}{r}p\right) = 0 \tag{11}$$

(the parentheses correspond to terms in Eq. 8) or

$$p' = -\Phi'(\rho + p). \tag{12}$$

This is equivalent to the Newtonian equation $p' = \Phi'\rho$, but with the added pressure term on the right-hand side: in GR, it takes pressure to hold up pressure!

## V. THE EINSTEIN EQUATIONS

We now turn to the Einstein equations, which we will need to relate $\Lambda$ and $\Phi$ to the properties of the star. We need the Ricci tensor first; recall that

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\delta\alpha}\Gamma^\delta{}_{\mu\nu} - \Gamma^\alpha{}_{\delta\nu}\Gamma^\delta{}_{\mu\alpha}. \tag{13}$$

We may expand these and find that the non-zero components are

$$\begin{aligned}
R_{tt} &= ([\mathrm{e}^{2\Phi-2\Lambda}\Phi']') - (0) + \left(\left[\Phi' + \Lambda' + \frac{2}{r}\right]\mathrm{e}^{2\Phi-2\Lambda}\Phi'\right) - (2\mathrm{e}^{2\Phi-2\Lambda}\Phi'^2) \\
&= \mathrm{e}^{2\Phi-2\Lambda}\left[\Phi'' + \Phi'^2 - \Lambda'\Phi' + \frac{2}{r}\Phi'\right], \\
R_{rr} &= (\Lambda'') - \left(\left[\Phi' + \Lambda' + \frac{2}{r}\right]'\right) + \left(\left[\Phi' + \Lambda' + \frac{2}{r}\right]\Lambda'\right) - \left(\Phi'^2 + \Lambda'^2 + \frac{1}{r^2} + \frac{1}{r^2}\right) \\
&= -\Phi'' + \Phi'\Lambda' + \frac{2}{r}\Lambda' - \Phi'^2, \\
R_{\theta\theta} &= ([-r\mathrm{e}^{-2\Lambda}]') - (-\csc^2\theta) + \left(\left[\Phi' + \Lambda' + \frac{2}{r}\right][-r\mathrm{e}^{-2\Lambda}]\right) - \left(-2\mathrm{e}^{-2\Lambda} + \cot^2\theta\right) \\
&= (-1 + r\Lambda' - r\Phi')\mathrm{e}^{-2\Lambda} + 1, \quad \text{and} \\
R_{\phi\phi} &= ([-r\mathrm{e}^{-2\Lambda}\sin^2\theta]' + \partial_\theta[-\sin\theta\cos\theta]) - (0) + \left(\left[\Phi' + \Lambda' + \frac{2}{r}\right][-r\mathrm{e}^{-2\Lambda}\sin^2\theta] - \cos^2\theta\right) \\
&\quad - \left(-2\mathrm{e}^{-2\Lambda}\sin^2\theta - 2\cos^2\theta\right) \\
&= [(-1 + r\Lambda' - r\Phi')\mathrm{e}^{-2\Lambda} + 1]\sin^2\theta.
\end{aligned} \tag{14}$$

(the parentheses correspond to terms in Eq. 13, and we used $\Gamma^\beta{}_{t\beta} = 0$, $\Gamma^\beta{}_{r\beta} = \Phi' + \Lambda' + \frac{2}{r}$, $\Gamma^\beta{}_{\theta\beta} = \cot\theta$, and $\Gamma^\beta{}_{\phi\beta} = 0$). Adding these up we find that

$$\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= -\mathrm{e}^{-2\Phi}R_{tt} + \mathrm{e}^{-2\Lambda}R_{rr} + \frac{R_{\theta\theta} + \csc^2\theta\, R_{\phi\phi}}{r^2} \\
&= \mathrm{e}^{-2\Lambda}\left[-2\Phi'' - 2\Phi'^2 + 2\Phi'\Lambda' - \frac{4}{r}\Phi' + \frac{4}{r}\Lambda' - \frac{2}{r^2}\right] + \frac{2}{r^2}.
\end{aligned} \tag{15}$$

This leads us to the Einstein tensor components:

$$\begin{aligned}
G_{tt} &= R_{tt} + \frac{1}{2}\mathrm{e}^{2\Phi}R = \mathrm{e}^{2\Phi-2\Lambda}\left[\frac{2}{r}\Lambda' - \frac{1}{r^2}\right] + \frac{1}{r^2}\mathrm{e}^{2\Phi}, \\
G_{rr} &= R_{rr} - \frac{1}{2}\mathrm{e}^{2\Lambda}R = \frac{2}{r}\Phi' + \frac{1}{r^2} - \frac{1}{r^2}\mathrm{e}^{2\Lambda}, \\
G_{\theta\theta} &= R_{\theta\theta} - \frac{1}{2}r^2R = \mathrm{e}^{-2\Lambda}\left[r^2\Phi'' + r^2\Phi'^2 - r^2\Phi'\Lambda' + r\Phi' - r\Lambda'\right], \quad \text{and} \\
G_{\theta\theta} &= R_{\phi\phi} - \frac{1}{2}r^2\sin^2\theta\, R = \mathrm{e}^{-2\Lambda}\sin^2\theta\left[r^2\Phi'' + r^2\Phi'^2 - r^2\Phi'\Lambda' + r\Phi' - r\Lambda'\right].
\end{aligned} \tag{16}$$

Equating these with the stress-energy components gives

$$\begin{aligned}
\mathrm{e}^{-2\Lambda}\left[\frac{2}{r}\Lambda' - \frac{1}{r^2}\right] + \frac{1}{r^2} &= 8\pi\rho, \\
\left[\frac{2}{r}\Phi' + \frac{1}{r^2}\right]\mathrm{e}^{-2\Lambda} - \frac{1}{r^2} &= 8\pi p, \quad \text{and} \\
\mathrm{e}^{-2\Lambda}\left[\Phi'' + \Phi'^2 - \Phi'\Lambda' + \frac{1}{r}\Phi' - \frac{1}{r}\Lambda'\right] &= 8\pi p.
\end{aligned} \tag{17}$$

Because of spherical symmetry, the $\theta\theta$ and $\phi\phi$ components give the same result. In fact, because of automatic conservation of the source, out of the 3 equations of Eq. (17) and the 1 equation of Eq. (12), only 3 are independent. In fact, if we took the $r$ derivative of the second equation in Eq. (17), and used Eq. (12) on the right hand side to replace $p'$ with $p$ and $\rho$, and then substituted the first equation of Eq. (17) for $\rho$, we would get the last equation in Eq. (17). Therefore, our basic set of equations will be the first two equations in Eq. (17) and Eq. (12).

## VI. SOLVING FOR THE RADIAL STRUCTURE

At this point, it is customary to make another substitution. We write the new function $m(r)$ (so far just a name!) defined by

$$m = \frac{1}{2}r(1 - \mathrm{e}^{-2\Lambda}) \quad \leftrightarrow \quad \mathrm{e}^{-2\Lambda} = 1 - \frac{2m}{r} \quad \leftrightarrow \quad \Lambda = -\frac{1}{2}\ln\left(1 - \frac{2m}{r}\right). \tag{18}$$

This is convenient when working with the Einstein equations because

$$m' = \frac{1}{2}(1 - \mathrm{e}^{-2\Lambda}) + r\Lambda'\mathrm{e}^{-2\Lambda} = 4\pi r^2\rho \tag{19}$$

if we use the $tt$ Einstein equation and do the appropriate algebraic manipulation. The $rr$ equation can be re-written with the new variable as

$$\left[\frac{2}{r}\Phi' + \frac{1}{r^2}\right]\left(1 - \frac{2m}{r}\right) - \frac{1}{r^2} = 8\pi p. \tag{20}$$

We can solve this for $\Phi'$:

$$\Phi' = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \tag{21}$$

We finally recall Eq. (12):

$$p' = -\Phi'(\rho + p). \tag{22}$$

Equations (19), (21), and (22) are the basic equations of relativistic stellar structure. They are known as the *Tolman-Oppenheimer-Volkoff equations* or TOV equations. To solve for the 4 functions – $m(r)$, $\Phi(r)$, $\rho(r)$, and $p(r)$ – we need these three equations as well as an equation relating $\rho$ to $p$ (an *equation of state*, which depends on the type of matter; for example, for a neutron star, the equation of state contains the nuclear physics needed to solve the structure of the star).

The metric now reads

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2m(r)/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{23}$$

We also need boundary conditions.

### A. Exterior solution

We can see that if the star has a boundary at $r = R$, then exterior to that boundary we have $\rho = p = 0$ and thence $m' = 0$. Thus we have $m(r)$ constant for $r > R$; we call this constant $M$ (again, so far just a name). Then Eq. (21) says

$$\Phi' = \frac{M}{r(r - 2M)} \quad \to \quad \Phi = \int \frac{M}{r(r - 2M)}\,\mathrm{d}r = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right). \tag{24}$$

(I set the integration constant to give $\Phi = 0$ at $r = \infty$.) This lead us to the metric outside the star:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{25}$$

This is the *Schwarzschild metric*. At large radii ($r \gg M$), we have a small perturbation around flat spacetime with $h_{tt} = 2M/r$ and thus gravitational potential $\phi = -M/r$; thus we infer that $M$ is the gravitational mass of the star. It is the mass appearing in Kepler's 3rd law if the star itself is relativistic ($R/M \sim$ a few) but the particle orbiting it is far away ($r/M \gg 1$).

### B. Inner condition

We need a solution at $r = 0$; for a star, the condition would be that $m(0) = 0$. If the central density of the star is $\rho_\mathrm{c}$, then the mass equation tells us that

$$m = \int_0^r 4\pi r^2 \rho \, \mathrm{d}r = \frac{4}{3}\pi \rho_\mathrm{c} r^3 + \ldots , \tag{26}$$

where the ... represents higher order terms in $r$. Thus we have

$$\Phi' = \frac{\frac{4}{3}\pi r^3 \rho_\mathrm{c} + 4\pi r^3 p_\mathrm{c}}{r^2} + \ldots = \frac{4\pi(\rho_\mathrm{c} + 3p_\mathrm{c})}{3} r + \ldots \quad \to \quad \Phi = \Phi_\mathrm{c} + \frac{2\pi(\rho_\mathrm{c} + 3p_\mathrm{c})}{3} r^2 + \ldots . \tag{27}$$

Thus the TOV equations allow a solution with a smooth gravitational potential at $r = 0$.

### C. Overall solution

Normally one solves the TOV equations by taking an equation of state $p(\rho)$, and a central density $\rho_\mathrm{c}$. One can then write $\rho(p)$ as an algebraic relation (no derivatives), and integrate the TOV equations for $m'$, $\Phi'$, and $p'$ outward, solving for $m(r)$, $\Phi(r) - \Phi_\mathrm{c}$, and $p(r)$. One cannot solve for the constant $\Phi_\mathrm{c}$ this way, but the TOV equations don't require the zero-point of potential. When (if?) the pressure drops to zero at some radius, we call that radius $R$ the surface of the star, set $M = m(R)$, and then use the matching to the Schwarzschild solution to determine $\Phi_\mathrm{c}$.

### D. Total mass and "gravitational energy"

At this point, we come back to the question of how the total mass of the star relates to the sum of all of the mass elements. If we take a 3D spatial slice through the star − a surface of $t =$constant − we can see that the volume element on that surface is

$$\mathrm{d}V = (\mathrm{e}^\Lambda \, \mathrm{d}r)(r \, \mathrm{d}\theta)(r \sin\theta \, \mathrm{d}\phi) = \mathrm{e}^\Lambda r^2 \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{28}$$

There is then a mass obtained by adding up the energy (which for a star made of an ideal gas includes the rest mass energy as well as kinetic energy of the particles) of all of the elements:

$$\int \rho \, \mathrm{d}V = \int_0^R \mathrm{e}^\Lambda 4\pi r^2 \rho \, \mathrm{d}r. \tag{29}$$

The total mass of the system is different from this, with a deficit of

$$\delta M = M - \int_0^R \rho \, \mathrm{d}V = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r - \int_0^R \mathrm{e}^\Lambda 4\pi r^2 \rho \, \mathrm{d}r = \int_0^R 4\pi r^2 (1 - \mathrm{e}^\Lambda) \rho \, \mathrm{d}r. \tag{30}$$

Since $m > 0$ and hence $\Lambda > 0$, we will have $\delta M < 0$: the total mass is less than the sum of the energy of the constituent parts.

In the weak-field limit, where $m/r \ll 1$, we can approximate

$$1 - \mathrm{e}^\Lambda = 1 - \frac{1}{\sqrt{1 - 2m/r}} \approx -\frac{m}{r}, \tag{31}$$

so

$$\delta M \approx \int_0^R 4\pi r^2 \left(-\frac{m}{r}\right) \rho \, \mathrm{d}r = \int_0^M -\frac{m}{r} \, \mathrm{d}m. \tag{32}$$

Now what is interesting is that for a shell at radius $r$, $-m/r$ is the Newtonian gravitational potential contributed by mass interior to the shell, i.e. at $\tilde{r}$ with $0 \le \tilde{r} < r$. (The total gravitational potential $\phi$ is more negative than $-m/r$, because it contains contributions from shells outside $r$.) Thus $\delta M$ as written is the Newtonian gravitational potential energy: $-\delta M$ is the work done by gravity if the star is assembled from the inside out, one shell at a time. We conclude that in general relativity, "gravitational potential energy" is part of the gravitational mass of a star.

### E. Gravitational redshift

We note that for a stationary observer ($r$, $\theta$, and $\phi$ unchanging), the relation of proper to coordinate time is

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \mathrm{e}^{\Phi}. \tag{33}$$

This means that a stationary source of radiation emitting at proper frequency $\omega_0$ will have observed frequency $\omega_{\mathrm{obs}} = \mathrm{e}^{\Phi}\omega_0$ if seen at $\infty$. This factor is often described using the *gravitational redshift* $z_{\mathrm{grav}}$:

$$1 + z_{\mathrm{grav}} = \frac{\omega_0}{\omega} = \mathrm{e}^{-\Phi}. \tag{34}$$

The gravitational redshift at the surface of a star is

$$1 + z_{\mathrm{grav}} = \frac{1}{\sqrt{1 - 2M/R}}. \tag{35}$$

This is an "exact" result to all orders for a spherical star – no first-order approximations!

## VII. EXAMPLE: CONSTANT-DENSITY STAR

For a general equation of state, the TOV equations must be solved numerically. However, a few cases can be done analytically and are instructive. We will take here the case of a constant-density equation of state, $\rho = \rho_\star$, because it is simple and illustrates some important points about relativistic stars (even though it is not very realistic). The central condition is specified by a central pressure $p_{\mathrm{c}}$.

We first see that the mass integral is simple:

$$m' = 4\pi r^2 \rho_\star \quad \rightarrow \quad m = \frac{4}{3}\pi r^3 \rho_\star. \tag{36}$$

Now Eqs. (21) and (22) can be combined to give

$$p' = -\frac{m + 4\pi r^3 p}{r(r-2m)}(\rho + p) = -\frac{\frac{4}{3}\pi r^3(\rho_\star + p)(\rho_\star + 3p)}{r(r - \frac{8}{3}\pi r^3 \rho_\star)} = -\frac{4\pi r(\rho_\star + p)(\rho_\star + 3p)}{3(1 - \frac{8}{3}\pi r^2 \rho_\star)}. \tag{37}$$

We can move the two terms with pressure to the left-hand side to get

$$\frac{\mathrm{d}p}{(\rho_\star + p)(\rho_\star + 3p)} = -\frac{4\pi r}{3(1 - \frac{8}{3}\pi r^2 \rho_\star)}\,\mathrm{d}r. \tag{38}$$

This can be integrated to give

$$\frac{1}{2}\ln\frac{\rho_\star + 3p}{\rho_\star + p} = \frac{1}{4}\ln\left(1 - \frac{8}{3}\pi r^2 \rho_\star\right) + \text{const}. \tag{39}$$

By looking at the center of the star, we see that the constant is $\frac{1}{2}\ln\frac{\rho_\star + 3p_{\mathrm{c}}}{\rho_\star + p_{\mathrm{c}}}$. If we define

$$\xi = \frac{\rho_\star + 3p}{\rho_\star + p} \quad \rightarrow \quad p = \rho_\star \frac{\xi - 1}{3 - \xi}, \tag{40}$$

then we find that

$$\xi = \xi_{\mathrm{c}}\sqrt{1 - \frac{8}{3}\pi r^2 \rho_\star}. \tag{41}$$

Note that $\xi$ is an increasing function of $p$, ranging from $\xi = 1$ when $p = 0$ to $\xi \rightarrow 3$ as $p \rightarrow \infty$.

The radius $R$ of the star is found when $p$ drops to zero or when $\xi = 1$: this is at

$$R = \sqrt{\frac{3}{8\pi\rho_\star}(1 - \xi_{\mathrm{c}}^{-2})}. \tag{42}$$

The mass of the star is

$$M = \frac{4}{3}\pi R^3 \rho_\star \tag{43}$$

and the radius-to-mass ratio is

$$\frac{R}{M} = \frac{3}{4\pi\rho_\star R^2} = \frac{2}{1-\xi_c^{-2}}. \tag{44}$$

As we take this star and add mass (again keeping the constant density equation of state), $R$ and $\xi_c$ will increase. However, there is a limit to this process: the central pressure approaches $\infty$ as $\xi_c \to 3$. In this limit, we have:

$$R \to \sqrt{\frac{27}{64\pi\rho_\star}}, \quad M \to \sqrt{\frac{1}{12\pi\rho_\star}}, \quad \frac{R}{M} \to \frac{9}{4}, \quad z_{\rm grav} \to 2. \tag{45}$$

For nuclear density, $\rho_\star = 2.3 \times 10^{14}\,\mathrm{g/cm^3} = 2.3 \times 10^{17}\,\mathrm{kg/m^3}$, we would have

$$M_{\rm max} = \frac{c^3}{G\sqrt{12\pi G\rho_\star}} = 1.7 \times 10^{31}\,\mathrm{kg} = 8.4M_\odot. \tag{46}$$

The actual upper limit to $M$ depends on the equation of state of the matter; in the case of neutron stars, it depends on as-yet-uncertain properties of nuclear/QCD physics at very high densities, and this is an active area of research in both astronomy and theoretical and experimental nuclear physics. In any case, any realistic equation of state has some maximum mass $M_{\rm max}$, and because nuclear matter is compressible the true maximum mass is less than $8.4M_\odot$ (it is probably close to $2M_\odot$). Trying to add more mass to such an object leads to a circumstance with no equilibrium solution, and the object undergoes gravitational collapse and creates a black hole.