Homework #2 – Solutions

1. Horizon in a matter-filled universe. [15 points]
Consider a flat universe filled with only non-relativistic matter and no cosmological constant. Take the Hubble constant today to be H_0. We showed in class that for this model universe, $a(t) \propto t^{2/3}$.

(a) [6 points] *Solve for the expansion history of the Universe $a(t)$, and show that the present age of the Universe is $t_0 = 2/(3H_0)$. What is this in years if $H_0 = 70$ km/s/Mpc?*

The proportionality says

$$\frac{1}{a}\frac{da}{dt} \propto a^{-3/2} \tag{1}$$

so

$$dt \propto a^{1/2}\,da \quad \rightarrow \quad t \propto a^{3/2} + \text{constant}. \tag{2}$$

Setting $t = 0$ at the Big Bang, we get that the constant is zero and $a \propto t^{2/3}$. Then

$$H = \frac{da/dt}{a} = \frac{(2/3)t^{-1/3}}{t^{2/3}} = \frac{2}{3t}, \tag{3}$$

regardless of the proportionality constant. This must hold today so $H_0 = 2/(3t_0)$ and $t_0 = 2/(3H_0)$.
Now we see that if $H_0 = 70$ km/s/Mpc, then

$$t_0 = \frac{2}{3(70\,\text{km/s/Mpc})} \cdot \frac{3.086 \times 10^{19}\,\text{km}}{1\,\text{Mpc}} \cdot \frac{1\,\text{yr}}{3.16 \times 10^7\,\text{s}} = 9.3 \times 10^9\,\text{yr}. \tag{4}$$

(b) [6 points] *Evaluate the conformal age of the universe today, η_0. Consider the most distant visible objects in the universe (i.e. those for which the light they emitted just after the Big Bang is reaching our telescopes today). What is the present-day distance to those objects, in light-years?*

The conformal time today is

$$\eta_0 = \int_0^{t_0} \frac{dt}{a} = \int_0^{t_0} \frac{dt}{(t/t_0)^{2/3}} = 3t_0^{2/3}t^{1/3}\Big|_0^{t_0} = 3t_0 = 28\,\text{Gyr}. \tag{5}$$

This means that the present-day distance to the cosmological horizon is 28 billion light-years.

(c) [3 points] *Explain, in words or equations that would make sense to a first-year undergraduate, why the answer to (b) in light-years is greater than the answer to (a) in years.*

It may seem odd that in a universe that is 9.3 billion years old, we can see objects farther away than 9.3 billion light-years. However, this means that light can travel a physical distance of 9.3 billion years. Since each region of space continued expanding after the light has crossed it, the present-day length of the path the light has taken can be larger than 9.3 billion light-years.

2. Distance measures. [20 points]
Write a computer program or script in the language of your choice to compute $D_A(z)$ and $D_L(z)$ for a given choice of Ω_m and Ω_K. Attach your code.
Make a plot of $D_A(z)$ and $D_L(z)$ over the range $0 < z < 1.5$ for the cases of (i) an $\Omega_\mathrm{m} = 1$, $\Omega_K = 0$ cosmology (Einstein-de Sitter); (ii) an $\Omega_\mathrm{m} = 0.3$, $\Omega_K = 0$ cosmology (Λ+dark matter); and (iii) an $\Omega_\mathrm{m} = 0.3$, $\Omega_K = 0.7$ (open, low-density, no Λ) cosmology.
You might want to numerically check some of your values against Ned Wright's online calculator.

Here is a computer program (in C) to write a table of the distances, and for $H_0 = 70$ km/s/Mpc. There are of course many correct ways to do this!

```
#include <stdlib.h>
#include <math.h>
```

```
#include <stdio.h>

/* Distance functions (in Mpc) -- output to *DA and *DL; H0 in km/s/Mpc */
void get_D(double z, double H0, double Omega_M, double Omega_K, double *DA, double *DL) {

  long i,N=100000;
  double dz, chi, r, K, z1;

  /* Get radial distance */
  chi = 0;
  dz = z/N;
  for(i=0;i<N;i++) {
    z1 = 1. + dz*(i+0.5);
    chi += dz*299792.458/H0/sqrt(Omega_M*z1*z1*z1 + Omega_K*z1*z1 + 1.-Omega_M-Omega_K);
  }

  /* Get comoving angular diameter distance */
  K = -Omega_K/(299792.458/H0*299792.458/H0);
  if (Omega_K>1e-7) {
    r = 1./sqrt(-K)*sinh(sqrt(-K)*chi);
  } else if (Omega_K<-1e-7) {
    r = 1./sqrt(K)*sin(sqrt(K)*chi);
  } else {
    r = chi - K*chi*chi*chi/6.;
  }

  /* Return DA and DL */
  *DA = r/(1.+z);
  *DL = r*(1.+z);
}

int main(void) {
  double z, DA, DL, H0, Omega_M, Omega_K;

  for(z=0.02;z<1.51;z+=.02) {
    printf("%6.4lf", z);

    H0 = 70.; Omega_M = 1.; Omega_K = 0.;
    get_D(z,H0,Omega_M,Omega_K,&DA,&DL);
    printf("       %8.2f %8.2f", DA, DL);

    H0 = 70.; Omega_M = 0.3; Omega_K = 0.;
    get_D(z,H0,Omega_M,Omega_K,&DA,&DL);
    printf("       %8.2f %8.2f", DA, DL);

    H0 = 70.; Omega_M = 0.3; Omega_K = 0.7;
    get_D(z,H0,Omega_M,Omega_K,&DA,&DL);
    printf("       %8.2f %8.2f", DA, DL);

    printf("\n");
  }

  return(0);
}
```

A plot of the resulting output is shown in Figure 1.

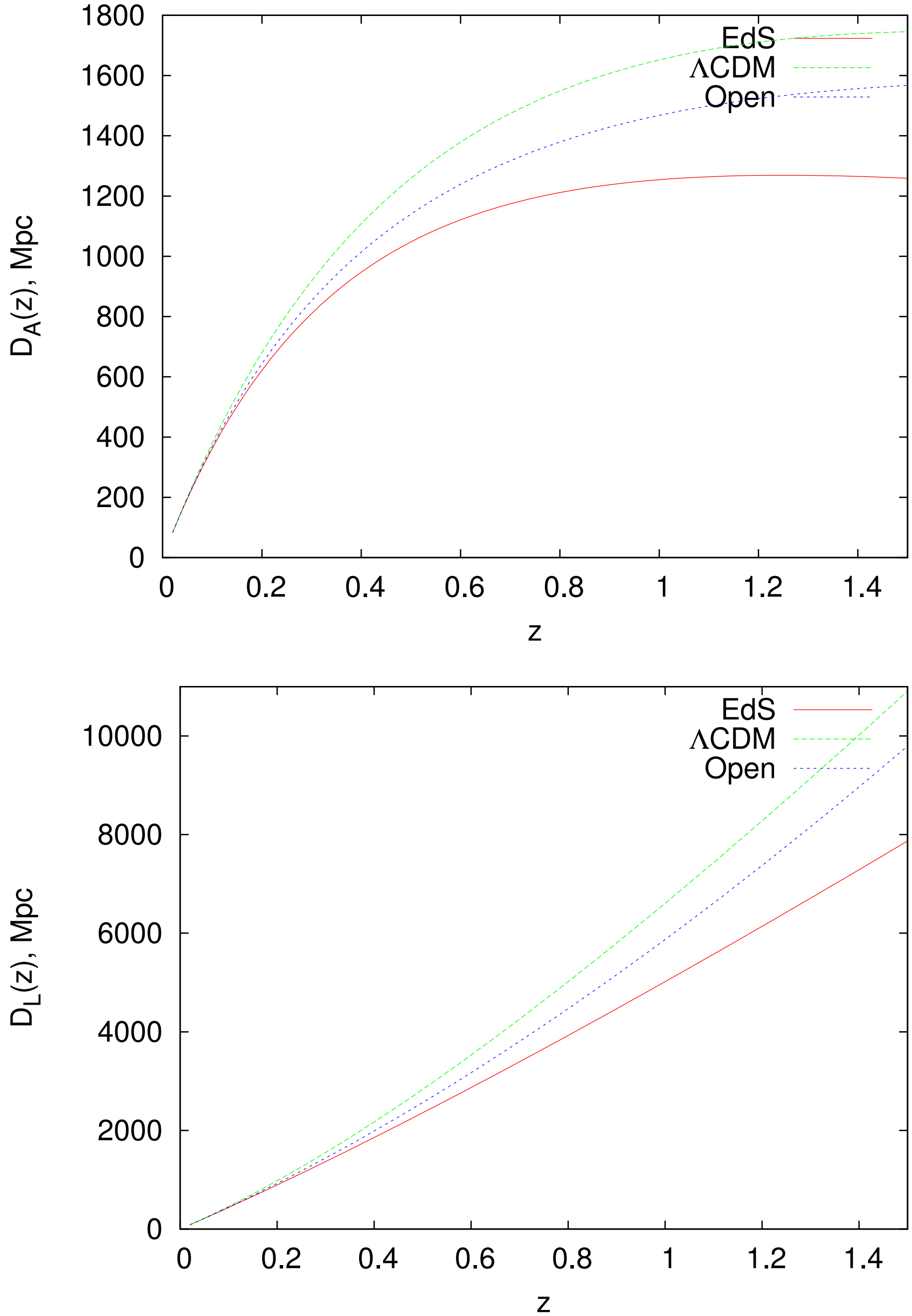

FIG. 1: Plots of $D_A(z)$ (top) and $D_L(z)$ (bottom) for the three cosmologies considered.