Lecture IX: Statistical field theory

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I. INTRODUCTION

In order to describe the cosmic density field (or curvature perturbations, or CMB anisotropies), it is convenient to use a statistical approach. That is, we use theory not to predict the specific density field in the Universe, but its statistical properties. This is a common approach in the study of thermalized systems, quantum systems, etc. In fact, it is what you do when you describe the "intensity" of light from the Sun instead of describing the electric field $\mathbf{E}(\mathbf{r})$ of the electromagnetic waves it emits.

This lecture will cover first the basics of statistical fields, and then we will apply this to the perturbations in the Universe.

II. CORRELATION FUNCTIONS AND POWER SPECTRA

Let's consider a random field $\phi(\boldsymbol{x})$ in N dimensions, i.e. $\boldsymbol{x} \in \mathbb{R}^N$. In general, averaged over realizations of the random process, one may define the nth moment to be the expectation value:

$$\langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)...\phi(\boldsymbol{x}_n)\rangle. \tag{1}$$

As always, a distribution is completely described by the hierarchy of all possible moments, since the expectation value of any functional $f[\phi]$ can then (formally) be obtained by a Taylor expansion.

We will focus our investigation on *statistically homogeneous* media, that is, one in which the moments are translation-invariant: for any $\boldsymbol{r} \in \mathbb{R}^N$,

$$\langle \phi(\boldsymbol{x}_1+\boldsymbol{r})\phi(\boldsymbol{x}_2+\boldsymbol{r})...\phi(\boldsymbol{x}_n+\boldsymbol{r})\rangle = \langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)...\phi(\boldsymbol{x}_n)\rangle. \tag{2}$$

In this case, the medium itself is not homogeneous, because there may be an eddy here or there; but averaged over realizations of the process (in the case of a fluid, this could mean averaging over many convection times) any region is as likely to contain an eddy as any other. We now consider the various moments in turn.

In such a medium, the mean ($n = 1$) $\langle\phi(\boldsymbol{x})\rangle$ is the same everywhere, $\bar\phi$. Normally we will take $\phi(\boldsymbol{x})$ to be a *fluctuation* in which case $\bar\phi = 0$. If not, then we can simply subtract the mean without loss of generality to simplify what follows.

The next moment ($n = 2$) corresponds to the covariance matrix of ϕ. Translation invariance guarantees that it depends only on the separation of the two points; we call this covariance matrix the 2-point *correlation function*:

$$\langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\rangle = \xi(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{3}$$

Note that ξ is an even function: $\xi(\boldsymbol{r}) = \xi(-\boldsymbol{r})$, and that $\xi(\boldsymbol{0})$ is the one-point variance of the field, $\mathrm{Var}\,\phi(\boldsymbol{x})$.

For Gaussian fields, all higher moments can be obtained via Wick's theorem. Non-Gaussian fields may however have nontrivial 3, 4, etc.-point correlation functions, e.g. there is a 3-point correlation function ζ:

$$\langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\phi(\boldsymbol{x}_3)\rangle = \zeta(\boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{x}_3 - \boldsymbol{x}_1). \tag{4}$$

We won't study those in this class.

It is also valid to perform a similar analysis in the Fourier domain, using the fields:

$$\tilde\phi(\boldsymbol{k}) = \int_{\mathbb{R}^N} \phi(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^N\boldsymbol{x} \quad \leftrightarrow \quad \phi(\boldsymbol{x}) = \int_{\mathbb{R}^N} \tilde\phi(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \frac{d^N\boldsymbol{k}}{(2\pi)^N}. \tag{5}$$

If ϕ is a real field then the Fourier transform has the property $\tilde\phi(-\boldsymbol{k}) = \tilde\phi^*(\boldsymbol{k})$. The correlation of two Fourier modes is – with the substitution $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ –

$$\begin{aligned}
\langle \tilde\phi(\boldsymbol{k}_1)\tilde\phi(\boldsymbol{k}_2)\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \langle\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\rangle\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\, e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}_2}\, d^N\boldsymbol{x}_1\, d^N\boldsymbol{x}_2 \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \xi(\boldsymbol{r})\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\, e^{-i\boldsymbol{k}_2\cdot(\boldsymbol{x}_1-\boldsymbol{r})}\, d^N\boldsymbol{x}_1\, d^N\boldsymbol{r} \\
&= \int_{\mathbb{R}^N} \xi(\boldsymbol{r}) e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}}\, d^N\boldsymbol{r} \int_{\mathbb{R}^N} e^{-i(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x}_1}\, d^N\boldsymbol{x}_1 \\
&= \tilde\xi(-\boldsymbol{k}_2)\,(2\pi)^N\delta^{(N)}(\boldsymbol{k}_1+\boldsymbol{k}_2).
\end{aligned} \tag{6}$$

You will often see this written in the alternative form with the complex conjugate, using $\tilde{\phi}(\boldsymbol{k}_2) = \tilde{\phi}^*(-\boldsymbol{k}_2)$:

$$\langle \tilde{\phi}^*(\boldsymbol{k})\tilde{\phi}(\boldsymbol{k}')\rangle = (2\pi)^N P(\boldsymbol{k})\delta^{(N)}(\boldsymbol{k}-\boldsymbol{k}'), \tag{7}$$

where $P(\boldsymbol{k})$ is the *power spectrum* and is defined by $P(\boldsymbol{k}) = \tilde{\xi}(\boldsymbol{k})$.

For non-Gaussian fields one must also specify the Fourier-space versions of the higher-order correlation functions. For example, one may write the *bispectrum* B:

$$\langle \tilde{\phi}(\boldsymbol{k}_1)\tilde{\phi}(\boldsymbol{k}_2)\tilde{\phi}(\boldsymbol{k}_3)\rangle = (2\pi)^N B(\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(N)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3), \quad B(\boldsymbol{k},\boldsymbol{k}') = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \zeta(\boldsymbol{r},\boldsymbol{r}')\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}'}\, d^N\boldsymbol{r}\, d^N\boldsymbol{r}' = \tilde{\zeta}^*(\boldsymbol{k},\boldsymbol{k}'). \tag{8}$$

We won't focus on these however for the purposes of this class.

For statistically isotropic fields, we have 2-point correlation functions and power spectra that depend only on the magnitude of separation r or wave vector k, not the direction. In this case, we can do the angular integrals in the Fourier transform. In 2 dimensions, if $\boldsymbol{r}$ is placed in the x-direction, we have

$$\xi(r) = \int P(k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{d^2\boldsymbol{k}}{(2\pi)^2} = \int_0^\infty \int_0^{2\pi} P(k)\, e^{ikr\cos\phi}\,\frac{1}{(2\pi)^2}\, d\phi\, k\, dk = \frac{1}{2\pi}\int_0^\infty P(k) J_0(kr)\, k\, dk, \tag{9}$$

and similarly

$$P(k) = 2\pi \int_0^\infty \xi(r)\, J_0(kr)\, r\, dr. \tag{10}$$

In 3 dimensions, we have

$$\xi(r) = \int P(k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{d^3\boldsymbol{k}}{(2\pi)^2} = \int_0^\infty \int_0^{\pi} \int_0^{2\pi} P(k)\, e^{ikr\cos\theta}\,\frac{1}{(2\pi)^3}\, d\phi\, \sin\theta\, d\theta\, k^2\, dk = \frac{1}{2\pi^2}\int_0^\infty P(k)\frac{\sin(kr)}{kr}\, k^2\, dk \tag{11}$$

and

$$P(k) = 4\pi \int_0^\infty \xi(r)\,\frac{\sin(kr)}{kr}\, r^2\, dr. \tag{12}$$

III. EXAMPLES

A few examples of correlation function − power spectrum pairs in 3D should illustrate the above points.

- The *power law* model with index γ $(0 < \gamma < 3)$ and correlation length r_0:

$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma} \quad \leftrightarrow \quad P(k) = 4\pi\Gamma(2-\gamma)\sin\frac{\pi\gamma}{2}\, r_0^\gamma k^{\gamma-3}. \tag{13}$$

  [Recall that the Γ-function has a simple pole at zero, so the above expression is well-behaved at $\gamma = 2$.]

- A *Gaussian* correlation function maps into a Gaussian power spectrum (note: do not confuse the Gaussian shape of the correlation function with whether the underlying field ϕ is Gaussian-distributed − one is Gaussian in the independent variable and the other in the dependent variable!):

$$\xi(r) = \sigma_\phi^2 e^{-r^2/2\sigma_r^2} \quad \leftrightarrow \quad P(k) = (2\pi)^{3/2}\sigma_r^3\sigma_\phi^2 e^{-\sigma_r^2k^2/2}. \tag{14}$$

- A *Lorentzian* in the power spectrum:

$$\xi(r) = \frac{Ak_0^2 e^{-k_0 r}}{4\pi r} \quad \leftrightarrow \quad P(k) = \frac{A}{1+(k/k_0)^2}; \tag{15}$$

  note that here the correlation function blows up as $r \to 0$ (there is infinite variance in ϕ).

IV. FILTERING, VARIANCES, ETC.

The total variance of the field ϕ is:

$$\mathrm{Var}[\phi(\boldsymbol{x})] = \xi(0) = \int \frac{P(k)\,d^3\boldsymbol{k}}{(2\pi)^3} = \int_0^\infty \frac{k^3 P(k)}{2\pi^2}\,\frac{dk}{k}. \tag{16}$$

For this reason, the quantity $\Delta^2(k) \equiv k^3P(k)/(2\pi^2)$ can be thought of as the variance per logarithmic range in k.

Since density fluctuations are very large on small scales – indeed, $\mathrm{Var}[\phi(\boldsymbol{x})]$ may be infinite – it is common to also describe the variance of *smoothed* fields. A smoothed field in a spherical region of radius R is defined by

$$\phi_{\rm sm}(\boldsymbol{x}|R) = \int W_R(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y})\,d^3\boldsymbol{y}, \tag{17}$$

where

$$W_R(r) = \frac{3}{4\pi R^3} \quad \text{if } r < R; \quad 0 \text{ otherwise.} \tag{18}$$

Then according to the convolution theorem,

$$\tilde\phi_{\rm sm}(\boldsymbol{k}|R) = \tilde W_R(\boldsymbol{k})\tilde\phi(\boldsymbol{k}). \tag{19}$$

Here:

$$\begin{aligned} \tilde W_R(k) &= \int_0^\infty 4\pi r^2 W_R(r)\frac{\sin(kr)}{kr}\,dr = \frac{3}{R^3}\int_0^R r^2\frac{\sin(kr)}{kr}\,dr \\ &= \frac{3}{kR^3}\int_0^R r\sin(kr)\,dr = \frac{3}{kR^3}\left[\frac{\sin(kr)}{k^2} - \frac{r\cos(kr)}{k}\right]_0^R = \frac{3[\sin(kR)-kR\cos(kR)]}{k^3R^3}. \end{aligned} \tag{20}$$

You may check by Taylor-expanding the numerator that $\tilde W_R(0) = 1$, and you can see that at $kR \gg 1$, $W_R(k)$ is oscillatory with an envelope that decays as $\propto (kR)^{-2}$. The power spectrum of the smoothed field is

$$P_{\rm sm}(k) = |\tilde W_R(\boldsymbol{k})|^2 P(k) \tag{21}$$

and variance of the smoothed field is

$$\sigma^2_\phi(R) = \mathrm{Var}[\phi_{\rm sm}(\boldsymbol{x}|R)] = \int_0^\infty \Delta^2(k)|\tilde W_R(\boldsymbol{k})|^2\,\frac{dk}{k}. \tag{22}$$

V. APPLICATION TO LINEAR GROWTH

One of the simplest – but most important – aspects of statistical fields is to linear growth in cosmology. Recall that after $z \sim 1000$, the matter density perturbations δ grow as

$$\tilde\delta(\mathbf{k},t) = \tilde\delta_+(\mathbf{k})G_+(t). \tag{23}$$

It follows that the matter density perturbation power spectrum at time t satisfies

$$\langle\tilde\delta^*(\mathbf{k},t)\tilde\delta(\mathbf{k}',t)\rangle = \langle\tilde\delta^*_+(\mathbf{k})\tilde\delta_+(\mathbf{k}')\rangle\,|G_+(t)|^2 \tag{24}$$

or

$$(2\pi)^3P_\delta(k,t)\delta^{(3)}(\mathbf{k}-\mathbf{k}') = (2\pi)^3P_{\delta_+}(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,|G_+(t)|^2\,. \tag{25}$$

Thus the power spectrum grows as the square of the growth function:

$$P_\delta(k,t) = P_{\delta_+}(k)\,|G_+(t)|^2\,. \tag{26}$$

(Although note that G_+ is real.) This is unsurprising: the power spectrum is the variance of each Fourier mode, and as such if the density field is rescaled, it should grow with two powers of the amplitude rescaling $G_+(t)$.