Lecture XI: Transfer functions and small-scale structure

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I. INTRODUCTION

Last time, we determined that the matter density at low redshift was

$$\tilde{\delta}_m(\mathbf{k},a) = \frac{2k^2}{5\Omega_m H_0^2}\tilde{\zeta}(\mathbf{k})T(k)G_+(a). \tag{1}$$

For Fourier modes that entered the horizon after recombination, we have $T(k) = 1$. For smaller scales that entered the horizon earlier, we must include corrections that will result in $T(k) < 1$. This lecture considers that physics.

II. DARK MATTER

Let's first consider the case where we can neglect the baryons. This is a reasonable first approximation since $\Omega_b \ll \Omega_m$, but of course it won't work at the level of modern precision cosmology. In this case, recombination won't have much of an impact, and rather the important question is whether a Fourier mode entered the horizon before or after matter-radiation equality. We define

$$k_{\rm eq} = a_{\rm eq}H_{\rm eq} = a_{\rm eq}H_0\sqrt{\Omega_m a_{\rm eq}^{-3} + \Omega_r a_{\rm eq}^{-4}} = H_0\sqrt{\frac{2\Omega_m^2}{\Omega_r}} = 0.072\Omega_m h^2\,{\rm Mpc}^{-1}, \tag{2}$$

where we substituted $a_{\rm eq} = \Omega_r/\Omega_m$ and neglected the contribution of Λ this early in the Universe. We also used $H_0 = 100h$ km/s/Mpc= $0.000334c$/Mpc, and $\Omega_r h^2 = 1.69\Omega_{\rm CMB}h^2 = 1.69 \times 2.57 \times 10^{-5}$ (recall the factor of 1.69 is for neutrinos).

In the dark matter only case, we can estimate the transfer function at an order of magnitude level as follows. We suppose that if a Fourier mode enters the horizon at $a_{\rm hor}$, it will have a density perturbation at horizon entry that is of order $\tilde{\zeta}(\mathbf{k})$. (This must be the right order of magnitude: there are no dimensionless constants in the problem to put in front.) Then the mode will grow logarithmically until the epoch of equality. We thus conclude that

$$\tilde{\delta}_m(\mathbf{k},a_{\rm eq}) \sim C\ln\frac{a_{\rm eq}}{a_{\rm hor}}\,\tilde{\zeta}(\mathbf{k}), \tag{3}$$

where C is a constant of order unity. Comparing this to Eq. (1), we find

$$\frac{2k^2}{5\Omega_m H_0^2}\tilde{\zeta}(\mathbf{k})T(k)a_{\rm eq} \sim C\ln\frac{a_{\rm eq}}{a_{\rm hor}}\,\tilde{\zeta}(\mathbf{k}), \tag{4}$$

or

$$T(k) \sim \frac{5\Omega_m H_0^2 C}{2a_{\rm eq}k^2}\ln\frac{a_{\rm eq}}{a_{\rm hor}}. \tag{5}$$

Now since $aH \propto t^{1/2}/t \propto t^{-1/2} \propto a^{-1}$ in the radiation-dominated era, we may replace the ratio of scale factors with the ratio of aH:

$$T(k) \sim \frac{5\Omega_m H_0^2 C}{2a_{\rm eq}k^2}\ln\frac{(aH)_{\rm hor}}{(aH)_{\rm eq}} = \frac{5\Omega_m H_0^2 C}{2a_{\rm eq}k^2}\ln\frac{k}{k_{\rm eq}}. \tag{6}$$

(Remember: horizon crossing is *defined* by the condition $k = aH$.) Finally, the machinery of Eq. (2) tells us that

$$k_{\rm eq}^2 = \frac{2\Omega_m^2 H_0^2}{\Omega_r} = \frac{2\Omega_m H_0^2}{a_{\rm eq}}, \tag{7}$$

so

$$T(k) \sim \frac{5\Omega_m H_0^2 C}{2a_{\rm eq}k^2}\ln\frac{(aH)_{\rm hor}}{(aH)_{\rm eq}} = \frac{5Ck_{\rm eq}^2}{4k^2}\ln\frac{k}{k_{\rm eq}}. \tag{8}$$

The transfer function thus depends on the dimensionless ratio $q = k/k_{\rm eq}$. When $q \ll 1$, we use the limit of Lecture X, $T(k) \to 1$. However when $q \gg 1$, we have $T(k) \propto q^{-2}\ln q$. The standard fit to numerical results for dark matter only cosmology (no baryons) is the Bardeen et al. (1986) fit, which gives the numerical prefactor as:

$$T(k) \approx \frac{12.3}{q^2}\ln q. \tag{9}$$

As q becomes large, the transfer function becomes small.

A. Overall shape of the matter power spectrum

Related to the matter density equation, Eq. (1), is the result for the matter power spectrum,

$$P_{\delta_m}(k,a) = \frac{4k^4}{25(\Omega_m H_0^2)^2}[T(k)]^2[G_+(a)]^2 P_\zeta(k). \tag{10}$$

Using the rule that $\Omega_m H_0^2 = \frac{1}{2}k_{\rm eq}^2 a_{\rm eq}$, we get

$$P_{\delta_m}(k,a) = \frac{16k^4}{25k_{\rm eq}^4}[T(k)]^2\left[\frac{G_+(a)}{a_{\rm eq}}\right]^2 P_\zeta(k). \tag{11}$$

For a power law primordial power spectrum,

$$\frac{k^3 P_\zeta(k)}{2\pi^2} = A_s\left(\frac{k}{k_\star}\right)^{n_s-1}, \tag{12}$$

we can conclude that the standard deviation per unit logarithmic range in wavenumber is

$$\Delta_{\delta_m}(k) \equiv \sqrt{\frac{k^3 P_{\delta_m}(k,a)}{2\pi^2}} = \frac{4}{5}A_s^{1/2}\frac{k^2}{k_{\rm eq}^2}T(k)\frac{G_+(a)}{a_{\rm eq}}\left(\frac{k}{k_\star}\right)^{(n_s-1)/2}. \tag{13}$$

For $n_s \approx 1$ (as observed), this goes to zero as $\propto k^2$ at small k (i.e., $T(k) \to 1$): that is, at a comoving length scale $r \sim 1/k$, the RMS density fluctuation scales as $\propto r^{-2}$. The Universe is very smooth on large scales. In contrast, at $k \gg k_{\rm eq}$, Eq. (9) gives

$$\Delta_{\delta_m}(k) \equiv \sqrt{\frac{k^3 P_{\delta_m}(k,a)}{2\pi^2}} = 9.8A_s^{1/2}\ln\frac{k}{k_{\rm eq}}\,\frac{G_+(a)}{a_{\rm eq}}\left(\frac{k}{k_\star}\right)^{(n_s-1)/2}. \tag{14}$$

The density perturbations per $\ln k$ increase logarithmically as we go to high k. Since in fact $(n_s-1)/2 = -0.018\pm0.002$, there is formally a maximum at some absurdly large value of k (at $k \sim e^{-1/0.018}k_{\rm eq} \sim 10^{23}\,{\rm Mpc}^{-1}$), but this is at too small scales to be relevant for cosmology (and the assumptions of our model would break down anyway). So in practical situations, we can say that the density perturbations continue to increase as we go to smaller scales.

Since the primordial fluctuation amplitude is $A_s^{1/2} \approx 5\times10^{-5}$, we would expect that $\Delta_{\delta_m}(k) \sim 1$ at $k \sim 1\,{\rm Mpc}^{-1} \sim 100k_{\rm eq}$ when $a \sim G_+(a) = 400a_{\rm eq} = 400/3000 = 0.13$, i.e., $z \sim 6$. Thus the smallest scales (Mpc and below) will break perturbation theory at all observable redshifts, and larger scales (~ 20 Mpc) would break perturbation theory today. We will discuss non-linear structure formation later.

III. INCLUSION OF BARYONS

The next step in our equations is to include the presence of baryons. We first investigate the behavior of a general fluid, and then consider the regimes where the baryons are important.

A. Fluids, waves, and the Jeans length

Looking at the linearized equations from Lecture VIII, including the pressure, we find for a fluid f,

$$\frac{\partial\delta_f}{\partial t} = -\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_f, \quad \frac{\partial\mathbf{v}_f}{\partial t} = -H\mathbf{v}_f - \frac{1}{a}\frac{\boldsymbol{\nabla}p_f}{\bar\rho_f} - \frac{1}{a}\boldsymbol{\nabla}\Phi. \tag{15}$$

Now we suppose that perturbations in the pressure of the fluid are related to density perturbations by the *effective sound speed* c_s:

$$\delta p_f = c_s^2 \delta \rho_f \quad \rightarrow \quad \frac{\boldsymbol{\nabla} p_f}{\bar{\rho}_f} = c_s^2 \boldsymbol{\nabla} \delta_f. \tag{16}$$

(For most of this class, we may assume c_s is the adiabatic sound speed that you studied in undergraduate classes, though when we talk about heating and cooling of the IGM this may not strictly be the case.) Then in Fourier space the aforementioned equations become:

$$\frac{\partial \tilde{\delta}_f}{\partial t} = -\frac{ik}{a}\tilde{v}_{f,z}, \quad \frac{\partial \tilde{v}_{f,z}}{\partial t} = -H\tilde{v}_{f,z} - \frac{ik}{a}\tilde{\delta}_f - \frac{ik}{a}\Phi, \tag{17}$$

or – repeating the steps of Lecture VIII –

$$\frac{\partial^2 \tilde{\delta}_f}{\partial t^2} = -2H\frac{\partial \tilde{\delta}_f}{\partial t} - \frac{c_s^2 k^2}{a^2}\tilde{\delta}_f + 4\pi G a^2 \bar{\rho}_m \tilde{\delta}_m. \tag{18}$$

Now it is clear that there are two opposite limits of what should happen, depending on how k compares to the *Jeans scale*:

$$k_{\rm J} \equiv \sqrt{\frac{2}{3}}\frac{aH}{c_s} \quad \text{or} \quad L_{\rm J} = \frac{\pi}{k_{\rm J}} = \sqrt{\frac{3}{2}}\pi\frac{c_s}{aH}. \tag{19}$$

This may be a function of time; the factor of $\sqrt{2/3}$ is a numerical convention that makes some results simpler, and the factor of π makes $L_{\rm J}$ a half-wavelength.

In the limit where $k \ll k_{\rm J}$, we will have $c_s k/a \ll H$, and so the sound speed term in Eq. (18) is irrelevant. On these scales, the fluid simply clusters in the same way as cold dark matter.

On the other hand, in the case $k \gg k_{\rm J}$, we have $c_s k/a \gg H$. In this case, Eq. (18) looks like a harmonic oscillator with a frequency $\omega = c_s k/a \gg H$. Thus the correct behavior is that the density oscillates as a function of time. The mean of the fluid density averaged over a wave oscillation cycle is the equilibrium position of the oscillator, obtained by setting the time derivatives to zero in Eq. (18):

$$\frac{c_s^2 k^2}{a^2}\tilde{\delta}_{f,\rm equil} = 4\pi G a^2 \bar{\rho}_m \tilde{\delta}_m \quad \rightarrow \quad \tilde{\delta}_{f,\rm equil} = \frac{4\pi G a^4 \bar{\rho}_m}{c_s^2 k^2}\tilde{\delta}_m = \frac{k_{\rm J}^2}{k^2}\Omega_m(z)\tilde{\delta}_m \tag{20}$$

(using the relation between $\bar{\rho}_m$ and H.) Thus at $k \gg k_{\rm J}$, the fluid overdensity is small compared to the total matter overdensity. It is thus likely – given our discussion on the matter power spectrum – that there are very small dark matter structures throughout the cosmos, but if they have always been smaller than $L_{\rm J}$, they never collected gas and hence are not directly visible.

B. Baryon acoustic oscillations

The most spectacular result of the aforementioned fluid physics in large scale structure is the baryon acoustic oscillations. Prior to recombination, the gas was opaque to the CMB, and so the gas and CMB combined formed a fluid – the *baryon-photon fluid* $(b+\gamma)$. The sound speed in the baryon-photon fluid was

$$c_s = \sqrt{\frac{\partial p}{\partial \rho}} \approx \frac{1}{\sqrt{3}} \tag{21}$$

since both the density and pressure were dominated by radiation and so $p = \rho/3$. (A more detailed calculation would take into account the slight reduction at finite baryon density.) An initial perturbation ζ at horizon entry thus leads to an oscillating behavior in $\tilde{\delta}_{b+\gamma}$. This is

$$\tilde{\delta}_{b+\gamma} \sim C'\tilde{\zeta}\cos\int \omega\,dt \sim C'\tilde{\zeta}\cos\int \frac{c_s k}{a}\,dt \sim C'\tilde{\zeta}\cos\frac{k\eta}{\sqrt{3}}, \tag{22}$$

where C' is a hard-to-guess constant. At recombination, then, there is a perturbation to the baryon density that is of the form

$$\tilde{\delta}_b(\mathbf{k}, a_{\rm rec}) \sim C'\tilde{\zeta}(\mathbf{k})\,\cos\frac{k\eta_{\rm rec}}{\sqrt{3}}. \tag{23}$$

Since for most large scale structure $\ln(k/k_{\rm eq})$ is of order unity, this perturbation is in principle of the same order of magnitude as the dark matter overdensity. However, baryons are only a small fraction of the total matter of the universe. Thus at recombination, the total matter density pertubation should be

$$\tilde{\delta}_m(\mathbf{k}, a_{\rm eq}) \sim \left[\left(1 - \frac{\Omega_b}{\Omega_m}\right) C \frac{a_{\rm rec}}{a_{\rm eq}} \ln \frac{a_{\rm eq}}{a_{\rm hor}} + \frac{\Omega_b}{\Omega_m} C' \cos \frac{k\eta_{\rm rec}}{\sqrt{3}}\right] \tilde{\zeta}(\mathbf{k}). \tag{24}$$

The first term here is copied from our discussion of dark matter, and the second term is the contribution from baryons. (I neglected damping of waves due to the finite mean free path of photons – that is, the *Silk damping* – which causes the oscillating term to die off as we go to higher k.) That contribution is small – it is suppressed by a factor of $\Omega_b a_{\rm eq}/\Omega_m a_{\rm rec} \sim 1/20$ – but with big surveys it is measurable. It leads to an oscillation in the matter power spectrum with regularly spaced peaks at

$$\Delta k \approx \frac{2\pi\sqrt{3}}{\eta_{\rm rec}} \approx 0.04\,{\rm Mpc}^{-1} \approx \frac{2\pi}{150\,{\rm Mpc}}. \tag{25}$$

An oscillation in the power spectrum also corresponds to a localized feature in the correlation function at $r_s = 150$ Mpc. This feature is called the *baryon-acoustic oscillation* (BAO) and can be seen in the correlation function of galaxies in 3D maps (and, at lower signal-to-noise ratio, in 2D maps).

The BAO can be used as a distance indicator in cosmology. If used by itself, it is "relative" in the sense that one does not know *a priori* the value of r_s. The value of r_s does depend on $\eta_{\rm rec}$, which is itself a function of $\Omega_b h^2$ and $\Omega_m h^2$ (and any new physics that you add to the early Universe). Thus in the context of a joint fit of BAO data and other data that constrains $\Omega_b h^2$ and $\Omega_m h^2$ (e.g., CMB), there is absolute distance information in the BAO.

IV. NUMERICAL METHODS

Most computations of the transfer function today are done with numerical perturbation theory codes that follow all of the components (photons, neutrinos, dark matter, and baryons). The most common and well-supported choices today are:

- CAMB (`http://camb.info`) – previously in Fortran90, now there is a Python interface as well.
- CLASS (`http://class-code.net`) – in C.

There are **many** clever tricks that go into making these codes fast, but at their core they are solving the system of ODEs for the perturbation variables we discussed in class. Since they explicitly treat the photons, these codes can compute CMB anisotropy power spectra as well.