Homework #1 – Solutions

1. Geometry in curved space. [21 points] *This problem is a series of investigations of the geometry in space of positive and negative curvature. Their purpose is to shed some light on how these cases differ from Euclidean geometry. In each case, I'm hoping that you will see how a familiar formula from Euclidean geometry changes in open or closed space. Some of the problems are most easily solved directly by thinking about the line element, and some are most easily solved by thinking of S^3 as being embedded in Euclidean $\mathbb{R}^4$ and using familiar Euclidean geometry (and then using analytic continuation to look at the hyperbolic case).*

[The three parts have increasing levels of difficulty.]

(a) [6 points] *Consider a sphere of radius χ. Show that the surface area of the sphere is given by*

$$A = 4\pi R^2 \sin^2\frac{\chi}{R} = 4\pi\chi^2 - \frac{4}{3}\pi K\chi^4 + ..., \tag{1}$$

where the last expression is the beginning of the Taylor expansion.

The line element

$$ds^2 = d\chi^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2}$$

tells us that at fixed χ, an infinitesimal rectangular area in (θ,ϕ) has sides $r\,d\theta$ and $r\sin\theta\,d\phi$. Thus the area is

$$A = \int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\,d\theta\,d\phi = 4\pi r^2 = 4\pi\left(R\sin\frac{\chi}{R}\right)^2 = 4\pi R^2\sin^2\frac{\chi}{R}. \tag{3}$$

We can do the Taylor expansion to order χ^4:

$$A = 4\pi R^2\sin^2\frac{\chi}{R} = 4\pi R^2\left(\frac{\chi}{R} - \frac{\chi^3}{6R^3} + \mathcal{O}(\chi^5)\right)^2 = 4\pi R^2\left(\frac{\chi^2}{R^2} - \frac{\chi^4}{3R^4} + \mathcal{O}(\chi^6)\right) = 4\pi\chi^2 - \frac{4\pi\chi^4}{3R^2} + \mathcal{O}(R^6). \tag{4}$$

Using $K = 1/R^2$, the conclusion follows.

(b) [6 points] *Consider a right triangle with legs a and b, and a hypotenuse of length c. Let A, B, and C be the vertices opposite these sides, respectively. Show that these lengths are related by*

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}. \tag{5}$$

*[*Hint*: Work in Euclidean $\mathbb{R}^4$ with the coordinate axes chosen conveniently. Then use the fact that the dot product of two vectors is the product of their lengths times the cosine of the angle between them; apply this fact to the vectors OA, OB, and OC, where O is the origin of $\mathbb{R}^4$.]*

Then show by Taylor expansion that the leading correction to the Pythagorean theorem is

$$c^2 = a^2 + b^2 - \frac{1}{3}Ka^2b^2 + ... \tag{6}$$

Let's put the origin O at the center of the hypersphere, so the equation of the hypersphere is $w^2+x^2+y^2+z^2 = R^2$. Then without loss of generality, we may place C at the North Pole: $C = (0,0,0,R)$. Since this is a right triangle, we may place A in the xz-plane and B in the yz-plane. The angle $\angle COA$ has measure b/R, so then

$$A = \left(0, R\sin\frac{b}{R}, 0, R\cos\frac{b}{R}\right), \tag{7}$$

and similarly we find

$$B = \left(0, 0, R\sin\frac{a}{R}, R\cos\frac{a}{R}\right). \tag{8}$$

Now the angle $\angle AOB$ has measure c/R, and A and B are both distance R from O, so

$$\overrightarrow{OA}\cdot\overrightarrow{OB} = R^2\cos\frac{c}{R}. \tag{9}$$

But direct numerical evaluation of $\overrightarrow{OA} \cdot \overrightarrow{OB}$ from Eqs. (7,8) gives $R^2 \cos(a/R)\cos(b/R)$, so we conclude that

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}. \tag{10}$$

The second part of this question asks us to find c^2. Let's begin by expanding the right-hand side of Eq. (10) in inverse powers of R: to order R^{-4},

$$\cos\frac{c}{R} = \left(1 - \frac{a^2}{2R^2} + \frac{a^4}{24R^4} - \mathcal{O}(R^{-6})\right)\left(1 - \frac{b^2}{2R^2} + \frac{b^4}{24R^4} - \mathcal{O}(R^{-6})\right) = 1 - \frac{a^2+b^2}{2R^2} + \frac{a^4+b^4+6a^2b^2}{24R^4} + \mathcal{O}(R^{-6}). \tag{11}$$

To proceed, we need to be able to invert the series for the cosine. Writing

$$\cos\theta = 1 - \frac{y}{2}, \tag{12}$$

we want to solve for θ^2. Let's expand the left-hand side to get

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \mathcal{O}(\theta^6) = 1 - \frac{y}{2}, \tag{13}$$

so that

$$\theta^2 - \frac{\theta^4}{12} + \mathcal{O}(\theta^6) = y. \tag{14}$$

Now by squaring this, we see that $\theta^4 + \mathcal{O}(\theta^6) = y^2$, so:

$$\theta^2 + \mathcal{O}(\theta^6) = y + \frac{y^2}{12}, \tag{15}$$

and repeating the procedure we find the reversed series for θ^2:

$$\theta^2 = y + \frac{y^2}{12} + \mathcal{O}(y^3). \tag{16}$$

Applying this to Eq. (11), we see that y takes the place of $(a^2+b^2)/R^2 - (a^4+b^4-6a^2b^2)/(12R^4)$ and θ^2 takes the place of c^2/R^2, so:

$$\frac{c^2}{R^2} = \frac{a^2+b^2}{R^2} - \frac{a^4+b^4+6a^2b^2}{12R^4} + \frac{[(a^2+b^2)/R^2]^2}{12} + \mathcal{O}(R^{-6}). \tag{17}$$

Multiplying through by R^2 and replacing R^{-2} with K, we find:

$$c^2 = a^2 + b^2 + \left[-\frac{a^4+b^4+6a^2b^2}{12} + \frac{(a^2+b^2)^2}{12}\right] K + \mathcal{O}(K^2). \tag{18}$$

Simplifying leads to Eq. (6).

(c)* [7 points] *Consider the right triangle of part (b), and let α, β, and $\gamma = \pi/2$ be the angles at the vertices A, B, and C. Show that*

$$\sin\frac{c}{R}\sin\alpha = \sin\frac{a}{R}. \tag{19}$$

[Hint: For the first equality, work in Euclidean $\mathbb{R}^4$, and orient the triangle so that it is in the xyz-hyperplane, with A and C in the xy-plane. Then write down two different expressions for the z-coordinate of B.]

Then show that the sum of the internal angles of the triangle differs from π:

$$\alpha + \beta + \gamma = \sin^{-1}\frac{\sin(a/R)}{\sin(c/R)} + \sin^{-1}\frac{\sin(b/R)}{\sin(c/R)} + \frac{\pi}{2} = \pi + K\frac{ab}{2} + .., \tag{20}$$

where the last expression is the first-order Taylor expansion in K. There are a few ways to prove this, but you might want to first show, from your answer in (b), that

$$\sin^2\frac{c}{R} = \left(\sin^2\frac{a}{R} + \sin^2\frac{b}{R}\right)\left(1 - K\frac{a^2b^2}{a^2+b^2} + ...\right) \tag{21}$$

and then prove the following identity for small ϵ by direct Taylor expansion:

$$\sin^{-1}\sqrt{\frac{x}{1+\epsilon}}+\sin^{-1}\sqrt{\frac{1-x}{1+\epsilon}}=\frac{\pi}{2}-\frac{\epsilon}{2\sqrt{x(1-x)}}+... \tag{22}$$

Let's first focus on proving Eq. (19). Taking the hint given, we see that in $\mathbb{R}^4$, the w-coordinates of O, A, B, and C are all zero, so we can work in ordinary $\mathbb{R}^3$ with only the x, y, and z coordiantes. We can find the z-coordinate of B first by noting that the plane OBA is tilted by angle α from the xy-plane. In this plane, project a ray $\overrightarrow{OP}$ perpendicular to $\overline{OB}$. As vectors, since $\angle BOA$ has measure c/R,

$$\overrightarrow{OB}=R\cos\frac{c}{R}\widehat{OA}+R\sin\frac{c}{R}\widehat{OP}, \tag{23}$$

where $\widehat{OP}$ denotes a unit vector. Now the z-component of $\widehat{OA}$ is 0 (since O and A are in the xy-plane) and that of $\widehat{OP}$ is $\sin\alpha$. This means that the z-coordinate of B is

$$z(B)=R\sin\frac{c}{R}\sin\alpha. \tag{24}$$

A similar argument but using OBC instead of OBA gives

$$z(B)=R\sin\frac{a}{R}\sin\gamma, \tag{25}$$

where $\gamma=\pi/2$ since this is a right triangle. Equating the two expressions for $z(B)$ gives Eq. (19).

To prove Eq. (20), I'll follow the outline provided by the hints. We need to prove two "hint" equations. For the first, squaring the result from part (b) gives

$$\cos^2\frac{c}{R}=\cos^2\frac{a}{R}\cos^2\frac{b}{R}. \tag{26}$$

Using the Pythagorean identity:

$$\left(1-\sin^2\frac{c}{R}\right)=\left(1-\sin^2\frac{a}{R}\right)\left(1-\sin^2\frac{b}{R}\right), \tag{27}$$

and some algebraic rearrangement gives

$$\sin^2\frac{c}{R}=\left(\sin^2\frac{a}{R}+\sin^2\frac{b}{R}\right)\left(1-\frac{\sin^2\frac{a}{R}\sin^2\frac{b}{R}}{\sin^2\frac{a}{R}+\sin^2\frac{b}{R}}\right). \tag{28}$$

If we expand the last fraction to leading order in K, we find

$$\sin^2\frac{c}{R}=\left(\sin^2\frac{a}{R}+\sin^2\frac{b}{R}\right)\left(1-\frac{K^2a^2b^2+\mathcal{O}(K^3)}{K(a^2+b^2)+\mathcal{O}(K^2)}\right)=\left(\sin^2\frac{a}{R}+\sin^2\frac{b}{R}\right)\left(1-\frac{Ka^2b^2}{a^2+b^2}+\mathcal{O}(K^2)\right). \tag{29}$$

The second "hint" equation – Eq. (22) – comes from the fact that the derivative of $\sin^{-1}y$ is $1/\sqrt{1-y^2}$. Then:

$$\frac{d}{d\epsilon}\sin^{-1}\sqrt{\frac{x}{1+\epsilon}}=\frac{1}{\sqrt{1-x/(1+\epsilon)}}\frac{1}{2}\sqrt{\frac{1+\epsilon}{x}}\frac{-x}{(1+\epsilon)^2}\to-\frac{1}{2}\sqrt{\frac{x}{1-x}}, \tag{30}$$

where the limit is taken as $\epsilon\to 0$. Therefore

$$\sin^{-1}\sqrt{\frac{x}{1+\epsilon}}=\sin^{-1}\sqrt{x}-\frac{\epsilon}{2}\sqrt{\frac{x}{1-x}}+\mathcal{O}(\epsilon^2). \tag{31}$$

Adding this to the same result with x replaced by $1-x$ gives

$$\sin^{-1}\sqrt{\frac{x}{1+\epsilon}}+\sin^{-1}\sqrt{\frac{1-x}{1+\epsilon}}=\sin^{-1}\sqrt{x}+\sin^{-1}\sqrt{1-x}-\frac{\epsilon}{2}\left(\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}\right)+\mathcal{O}(\epsilon^2). \tag{32}$$

TABLE I: Table of deviations from Euclidean geometry in open and closed universes.

Property	Closed ($K > 0$)	Flat ($K = 0$)	Open ($K < 0$)
Surface area of sphere	$A < 4\pi\chi^2$	$A = 4\pi\chi^2$	$A > 4\pi\chi^2$
Pythagorean theorem	$c^2 < a^2 + b^2$	$c^2 = a^2 + b^2$	$c^2 > a^2 + b^2$
Interior angles in triangle	$\alpha + \beta + \gamma > \pi$	$\alpha + \beta + \gamma = \pi$	$\alpha + \beta + \gamma < \pi$

Now if $x = \sin^2\theta$ then $1 - x = \cos^2\theta = \sin^2(\pi/2 - \theta)$, so we have

$$\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{1-x} = \frac{\pi}{2}. \tag{33}$$

Finally, the algebraic combination $\sqrt{x/(1-x)} + \sqrt{(1-x)/x} = 1/\sqrt{x(1-x)}$ yields

$$\sin^{-1}\sqrt{\frac{x}{1+\epsilon}} + \sin^{-1}\sqrt{\frac{1-x}{1+\epsilon}} = \frac{\pi}{2} - \frac{\epsilon}{2\sqrt{x(1-x)}} + \mathcal{O}(\epsilon^2). \tag{34}$$

Finally, we return to the interior angles of the triangle. We see from Eq. (29) that

$$\alpha = \sin^{-1}\frac{\sin(a/R)}{\sin(c/R)} = \sin^{-1}\sqrt{\frac{\sin^2(a/R)}{\sin^2(c/R)}} = \sin^{-1}\sqrt{\frac{x}{1+\epsilon}}, \tag{35}$$

where we make the identifications

$$x = \frac{\sin^2(a/R)}{\sin^2(a/R) + \sin^2(b/R)} \quad \text{and} \quad \epsilon = -\frac{Ka^2b^2}{a^2+b^2} + \mathcal{O}(K^2). \tag{36}$$

Similarly, $\beta = \sin^{-1}\sqrt{(1-x)/(1+\epsilon)}$, and so from Eq. (34),

$$\alpha + \beta = \frac{\pi}{2} - \frac{-Ka^2b^2/(a^2+b^2)}{2\sqrt{\sin^2(a/R)\sin^2(b/R)}\,/[\sin^2(a/R) + \sin^2(b/R)]} + \mathcal{O}(K^2). \tag{37}$$

Simplifying to lowest order in K:

$$\alpha + \beta = \frac{\pi}{2} - \frac{-Ka^2b^2/(a^2+b^2)}{2ab/(a^2+b^2)} + \mathcal{O}(K^2) = \frac{\pi}{2} + \frac{Kab}{2} + \mathcal{O}(K^2). \tag{38}$$

Since $\gamma = \pi/2$, the conclusion follows.

(d) [2 points] *Summarize your results in a table showing how the sign of the deviations from the Euclidean results (the formula for the surface area of a sphere, the Pythagorean theorem, and the sum of the internal angles in a triangle) relates to the spatial curvature.*

See Table I.