Lecture III: Dynamics of cosmic expansion

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I. INTRODUCTION

Having discussed the description of the expanding Universe, we now focus on the dynamics. We will derive the Friedmann equation – the basic governing equation of cosmic expansion – using the principles from relativity, but we will do so using undergraduate mathematics (so without invoking curvature tensors or other ideas you learn about in GR class).

The basic postulates in GR, are:

I. Physics is the same regardless of the choice of coordinates used to describe it. (This is a generalization of the ideas from special relativity.)

II. Spacetime – or more specifically the metric components appearing in ds^2 – governed by partial differential equations of the second order (no more than 2 spatial or time derivatives) that are linear in the second derivatives.

III. The density (and flux) of energy (and momentum) are the sources of gravity, and appear linearly in the aforementioned PDEs.

IV. The equations describe Newtonian gravity in the non-relativistic limit.

V. The equations *imply* conservation of energy and momentum, in the same way that Maxwell's equations imply conservation of charge ($\dot{\rho} + \nabla \cdot \mathbf{J} = 0$).

VI. (Optional) Flat spacetime (i.e. the spacetime of special relativity) is a solution of the equations with no matter present.

We will see in this lecture how these postulates allow us to derive the Friedmann equation, even though we won't make use of GR. In fact these postulates uniquely allow you to derive GR, so all of the other features GR (even black holes) could be derived from the above postulates! It is also not obvious that there *is* a theory of gravity satisfying the above constraints – we won't prove it in this class, but in any GR class you should learn this.

The cosmological constant (Λ) requires us to revisit the last postulate. In fact, "GR+Λ" is equivalent to dropping Postulate VI, in which case Λ describes some aspect of the curvature of empty spacetime.

In this lecture, we will set $c = 1$.

Note — In writing these notes, I thought for a long time about how much detail to include. I decided on the present choice because I wanted to explain why pressure is a source of gravity, and hence how negative pressure can cause cosmic acceleration. Moreover, I believe this approach to the Friedmann equation – starting from the postulates of GR – is actually more intuitive and is better for understanding the result than starting from the Einstein field equations.

II. ENERGY, MOMENTUM, AND STRESS

In order to discuss the dynamics of cosmic expansion, we first need to consider the source of gravity. In Newtonian mechanics, you learned that *mass* is the source of gravity, but in relativity energy and momentum need to be considered sources as well. We'll flush out these ideas a bit (in the context of special relativity) and then proceed to the postulates of Einstein's theory of gravity.

In Newtonian mechanics, gravity is described by a "gravitational potential" Φ or "gravitational field" $\mathbf{g} = -\boldsymbol{\nabla}\Phi$. This obeys the equation:

$$\nabla^2\Phi = -\boldsymbol{\nabla} \cdot \mathbf{g} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho, \tag{1}$$

where ρ is the mass density and G is the **gravitational constant** that describes the strength of gravity. Newton's gravity is not consistent with the principles of relativity, because: (i) it has only spatial derivatives but no time derivatives; and (ii) the "source" – the matter density ρ – depends on the reference frame.

To see how this might work in relativity, we recall that in electromagnetism (which *is* consistent with special relativity), the sources in Maxwell's equations are the charge density ρ (units: C/m^3) and the 3 components of charge flux $\mathbf{J}$ (units: C/m^2/s; better known as the current density). These can be arranged in a 4-vector:

$$\begin{pmatrix} \rho_q \\ J_x \\ J_y \\ J_z \end{pmatrix}, \tag{2}$$

which can be expressed in a different reference frame via a Lorentz transformation:

$$\begin{pmatrix} \rho'_q \\ J'_x \\ J'_y \\ J'_z \end{pmatrix} = \mathbf{L} \begin{pmatrix} \rho_q \\ J_x \\ J_y \\ J_z \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{3}$$

(for a primed frame moving at speed β in the z-direction and with $\gamma = 1/\sqrt{1-\beta^2}$, as usual).

We can do the same thing for the densities and fluxes of energy and momentum: here we have 1 density component and 3 fluxes for each of the 4 energy-momentum components. This results in a 4×4 object known as the **stress-energy tensor** (or energy-momentum, energy-momentum-stress tensor, etc.):

$$\mathbf{T} = \begin{pmatrix} \rho & S_x & S_y & S_z \\ F_x & T_{xx} & T_{xy} & T_{xz} \\ F_y & T_{yx} & T_{yy} & T_{yz} \\ F_z & T_{zx} & T_{zy} & T_{zz} \end{pmatrix}. \tag{4}$$

As written here, ρ is the energy density; $\mathbf{S}$ is a 3-vector describing the energy flux; $\mathbf{F}$ is the momentum density; and $\mathbf{T}$ is the 3×3 matrix of momentum flux (e.g. T_{xy} is the flux of x-momentum in the y-direction). According to the transformation rules we've described, if we change reference frames, we have to transform *both* the energy and momentum labels, *and* the density and flux labels. That is:

$$\mathbf{T}' = \mathbf{L}\mathbf{T}\mathbf{L}^{\mathrm{T}}. \tag{5}$$

You are probably familiar with the notions of energy density and momentum density, but maybe less so with energy flux and momentum flux. (This sometimes gets covered in the undergraduate courses and sometimes not.) Some examples might help:

- In a gas of particles of mass $m^{(a)}$ and each of velocity $\mathbf{v}^{(a)}$, the energy of a particle is $m^{(a)}\gamma^{(a)}$ and the momentum is $m^{(a)}\mathbf{v}^{(a)}\gamma^{(a)}$. Then the components of $\mathbf{T}$ are:

$$\rho = \frac{1}{V}\sum_a m^{(a)}\gamma^{(a)}; \quad S_i = F_i = \frac{1}{V}\sum_a m^{(a)}\gamma^{(a)}v_i^{(a)}; \quad T_{ij} = \frac{1}{V}\sum_a m^{(a)}\gamma^{(a)}v_i^{(a)}v_j^{(a)}, \tag{6}$$

  where we sum over all particles in some volume V.

- **Pressure** is the force per unit area exerted on a surface, or equivalently the momentum transferred per unit time per unit area. If we consider a surface in the yz-plane then, the amount of x-momentum transferred across the surface per unit time per unit area is the pressure in the x-direction – i.e. $p = T_{xx}$. Under many circumstances (e.g. ideal fluids) the pressure is the same in all directions and so is isotropic, $T_{xx} = T_{yy} = T_{zz}$.

- If $\mathbf{T}$ is off-diagonal, we have **shear stress** – e.g. T_{yx} corresponds to y-momentum being transported in the x-direction, or equivalently a force in the y-direction is being exerted on each element of a surface in the yz-plane. That is, we're applying a force parallel to the surface, instead of perpendicular; this occurs in the case of e.g. friction.

In the above example, $\mathbf{T}$ was symmetric. This is in fact a general principle – it is the relativistic statement of the strong form of Newton's 3rd law: that for every force there is an equal and opposite force that acts along the same line of action. It is straightforward to show that *if* this version of Newton's 3rd law is true, then $\mathbf{T}$ is symmetric. (This is because if the line of action is parallel to, say, the z axis, then the only nonzero spatial component is T_{zz} and hence the 3×3 sub-block of the tensor is symmetric. Since rotations and additions preserve symmetry, then the spatial part of $\mathbf{T}$ overall is symmetric; Lorentz invariance then requires the overall $\mathbf{T}$ to be symmetric.) It turns out that the symmetry of $\mathbf{T}$ is *necessary* in order for the aforementioned principles of GR to lead to a consistent theory. A consequence of this is that the energy flux $\mathbf{S}$ and momentum density $\mathbf{F}$ are equal.

From the perspective of gravity, if we are to extend Newton's theory, we know that energy density (ρ) must gravitate. But under various Lorentz transformations, ρ can be transformed into $\mathbf{S}$, $\mathbf{F}$, and $\mathbf{T}$, so in fact *all* of these must be sources of gravity!

Note that in a homogeneous and isotropic Universe, p is the same at all points and in all directions, i.e. the stress-energy tensor is

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{7}$$

Therefore, the gravitational sources we need to consider are simply the density and pressure, which are functions of time: $\rho(t)$ and $p(t)$.

III. ENERGY CONSERVATION IN AN EXPANDING UNIVERSE

In accordance with Postulate V above, equations describing the geometry must imply the law of conservation of energy. Our next task is thus to understand how energy conservation works in cosmology. The resulting equation is then a hint as to the equations describing gravity.

In a static (time-independent) Universe, with a =constant, the notion of energy conservation is simple: we would simply have a constant total energy ρV, and hence (since V is constant), ρ would be constant. In an expanding Universe, where the background spacetime is time-dependent, total energy is not conserved. However, the first law of thermodynamics still holds:

$$dU = d(\rho V) = -p\,dV, \tag{8}$$

where p is the pressure and $U = \rho V$ is the total energy. The above equation implies:

$$\rho\dot{V} + \dot{\rho}V = -p\dot{V}, \tag{9}$$

and using that $V \propto a^3$ so $\dot{V}/V = 3\dot{a}/a$:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -3H(\rho + p). \tag{10}$$

This is the rule for energy conservation in cosmology, and it relates the change in density ($\dot{\rho}$) to the expansion rate of the Universe (H).

We can write the above equation in a few different forms. One makes use of the **equation of state** of the matter in the Universe, which is defined as the ratio of pressure to energy density:

$$w = \frac{p}{\rho}. \tag{11}$$

Then Eq. (10) can be written as

$$\dot{\rho} = -3(1 + w)\frac{\dot{a}}{a}\rho, \tag{12}$$

or – dividing by ρ and integrating –

$$\ln \rho = -3 \int (1 + w)\, \frac{da}{a}. \tag{13}$$

If w is a constant, this implies

$$\rho \propto a^{-3(1+w)}. \tag{14}$$

In particular, if the Universe contains only nonrelativistic matter particles ($p \ll \rho$, or $w \approx 0$), then $\rho \propto a^{-3}$. That's about what you would expect: the density decreases in inverse proportion to the volume. Another common possibility is radiation, where the pressure is equal to $\frac{1}{3}$ of the energy density: $w = \frac{1}{3}$. In that case, the density scales as $\rho \propto a^{-4}$. In practice, the Universe contains *both* nonrelativistic matter and radiation, and the density of each of these obeys its own power law. This is very important – as we will see later, the energy density of the *early* Universe was dominated by radiation.

IV. THE DYNAMICAL EQUATIONS

We are at last ready to build the dynamical equations describing cosmic expansion. In the FRW case, the kinematic variables available are the function $a(t)$ and the spatial curvature K. The sources are $\rho(t)$ and $p(t)$, and the energy conservation law is given by Eq. (10).

In accordance with Postulates II and III, we need dynamical equations with $\rho(t)$ and $p(t)$ as sources (right-hand side) and a left-hand side that is linear in second derivatives of the coefficients in the line element. Since those PDEs must be valid everywhere, we can look at the origin and see that the spatial curvature appears multiplied by r^2; it is thus already a second derivative. Thus the left-hand sides of the equations must look something like:

$$f_1(a)\ddot{a} + f_2(a, \dot{a}) + f_3(a)K = \rho. \tag{15}$$

[In accordance with Postulate I, t cannot appear explicitly.] Now in accordance with Postulate I, if we re-scale the entire function $a(t)$ by some constant C, nothing happens. That is, we should have the same dynamical equation if we make a new function $\tilde{a}(t) = a(t)/C$, and re-scale the radial coordinate $\tilde{r} = Cr$, and correspondingly the radius of curvature and the spatial curvature as $\tilde{K} = K/C^2$, then the equation should look the same. Doing this implies the proportionalities $f_1(a) \propto 1/a$, $f_3(a) \propto 1/a^2$, and $f_2(a, \dot{a}) = f_2(H)$, where we recall that $H = \dot{a}/a$. Thus the governing equations become

$$b_1\frac{\ddot{a}}{a} + f_2(H) + b_3\frac{K}{a^2} = \rho \quad \text{and} \quad c_1\frac{\ddot{a}}{a} + g_2(H) + c_3\frac{K}{a^2} = p. \tag{16}$$

So ... all we need to do is figure out the functions f_2 and g_2, and the constants b_1, b_3, c_1, and c_3, and we're done!

A. Fixing b_1: automatic conservation of energy

To continue, let's invoke Postulate V, which tells us that the gravitational equations must imply the energy conservation equation, Eq. (10). We should imagine plugging Eq. (16) into both $\dot{\rho}$ and $-3H(\rho + p)$ and showing that they are necessarily equal regardless of K and $a(t)$. This leads us to

$$b_1\frac{a\dddot{a} - \dot{a}\ddot{a}}{a^2} + f_2'(H)\frac{a\ddot{a} - \dot{a}^2}{a^2} - 2b_3\frac{K}{a^3}\dot{a} = -3H\left[(b_1 + c_1)\frac{\ddot{a}}{a} + f_2(H) + g_2(H) + (b_3 + c_3)\frac{K}{a^2}\right]. \tag{17}$$

This equation looks messy, but the key point is that if $b_1 \neq 0$, then we have a $\dddot{a}$ on the left-hand side but not the right-hand side. This can't be true for any function a, so this must mean that:

$$b_1 = 0. \tag{18}$$

Isn't that remarkable? You should view it as a spectacular indicator of the power of the principles outlined at the beginning of lecture. [Aside: if you do this in a GR class from the field equations, it is much less obvious why b_1 had to vanish; it wasn't until my last year of grad school that I understood there was a deeper reason than "the $\dddot{a}$ terms just happen to cancel after 6 pages of math." The fact that only first derivatives of a appear in a dynamical equation is an example of a *constraint* and something similar exists in all gauge theories, including in E&M (Gauss's law: $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$) and the weak and strong forces.]

B. Finding $f_2(H)$: the Milne Universe

To push further, we'll use Postulate I in a very non-obvious way. It turns out that the flat spacetime of special relativity – **Minkowski spacetime** – can also be viewed as (the analytic continuation of) an open universe. Minkowski spacetime can be described in polar coordinates as

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - dt^2. \tag{19}$$

Now let's change the coordinates. Define a new time variable $\tilde{t} = \sqrt{t^2 - r^2}$ that is the proper time from the origin to the point in question. By Lorentz invariance, a surface of constant $\tilde{t}$ should be homogeneous and isotropic; we will see that it is an open space. We'll also define a new radial coordinate $\tilde{r} = r/\tilde{t}$. With this change:

$$r = \tilde{r}\tilde{t}; \qquad t = \tilde{t}\sqrt{1 + \tilde{r}^2}, \tag{20}$$

we can see that

$$dr = \tilde{r}\,d\tilde{t} + \tilde{t}\,d\tilde{r}; \qquad dt = \sqrt{1+\tilde{r}^2}\,d\tilde{t} + \frac{\tilde{t}\tilde{r}}{\sqrt{1+\tilde{r}^2}}\,d\tilde{r}. \tag{21}$$

Substitution into Eq. (19) yields:

$$\begin{aligned} ds^2 &= (\tilde{r}\,d\tilde{t} + \tilde{t}\,d\tilde{r})^2 + (\tilde{r}\tilde{t})^2(d\theta^2 + \sin^2\theta\,d\phi^2) - \left(\sqrt{1+\tilde{r}^2}\,d\tilde{t} + \frac{\tilde{t}\tilde{r}}{\sqrt{1+\tilde{r}^2}}\,d\tilde{r}\right)^2 \\ &= \tilde{r}^2\,d\tilde{t}^2 + \tilde{t}^2\,d\tilde{r}^2 + 2\tilde{r}\tilde{t}\,d\tilde{r}\,d\tilde{t} + \tilde{t}^2\tilde{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2) - (1+\tilde{r}^2)\,d\tilde{t}^2 - \frac{\tilde{t}^2\tilde{r}^2}{1+\tilde{r}^2}\,d\tilde{r}^2 - 2\tilde{r}\tilde{t}\,d\tilde{r}\,d\tilde{t} \\ &= -d\tilde{t}^2 + \tilde{t}^2\left[\frac{d\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \end{aligned} \tag{22}$$

This is an alternative description of Minkowski space, and it corresponds to an open model with $K = -1$ and scale factor $\tilde{a}(\tilde{t}) = \tilde{t}$. It is an empty space that expands at a constant rate. In this model, space is curved and there was an apparent "Big Bang," but both features are fictitious – the spacetime itself is flat and the Big Bang is an artifact of the coordinate system. This model is known as the **Milne Universe**, and clearly does *not* describe the real Universe. Interest in it is entirely mathematical. Nevertheless, it is important because it must satisfy the dynamical equations governing the expansion of the Universe – i.e. Eq. (16) must hold, with $\rho = 0$ (invoking Postulate VI!), $\tilde{a} = \tilde{t}$, and $\tilde{H} = \dot{\tilde{a}}/\tilde{a} = 1/\tilde{t}$. We thus see that

$$f_2(1/\tilde{t}) - \frac{b_3}{\tilde{t}^2} = 0 \tag{23}$$

and so we must have $f_2(H) = b_3 H^2$. We arrive at the resulting dynamical equation

$$b_3\left(H^2 + \frac{K}{a^2}\right) = \rho. \tag{24}$$

The only remaining question is the value of the constant b_3.

C. Fixing b_3: Correspondence with Newtonian gravity

So the last step is to figure out b_3. This is a dimensionful parameter, and you can see that it is in some sense the coefficient relating curvature (left-hand side of Eq. 24) to the matter content (right-hand side). You are probably guessing that it has something to do with Newton's G, and so to find its value we invoke Postulate IV.

It turns out that the easiest way to do this is actually to take the time derivative of Eq. (24). Noting that

$$\dot{H} = \frac{d}{dt}\frac{\dot{a}}{a} = \frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2, \tag{25}$$

we see that the derivative of Eq. (24) is:

$$b_3\left(2H\dot{H} - \frac{2K\dot{a}}{a^3}\right) = \dot{\rho} = -3H(\rho + p). \tag{26}$$

Simplifying the left-hand side gives

$$b_3\left(2H\frac{\ddot{a}}{a} - 2H^3 - \frac{2KH}{a^2}\right) = -3H(\rho + p). \tag{27}$$

Now if we divide by H and add $2\times$ Eq. (24), we get

$$b_3\left(2\frac{\ddot{a}}{a} - 2H^2 - \frac{2K}{a^2}\right) + 2b_3\left(H^2 + \frac{K}{a^2}\right) = -3(\rho + p) + 2\rho, \tag{28}$$

which simplifies to

$$\frac{\ddot{a}}{a} = -\frac{1}{2b_3}(\rho + 3p). \tag{29}$$

In Newtonian gravity, you learned that a uniform spherical region of physical radius $\mathcal{R}$ and density ρ produces a gravitational acceleration at the surface $g = -\frac{4}{3}\pi G\rho\mathcal{R}$. This means that for nonrelativistic matter (where Newtonian gravity should be valid, and where $p \ll \rho$), the ratio of acceleration to radius is $-\frac{4}{3}\pi G\rho$. We identify this with $\ddot{a}/a$ for matter in free-fall, since the physical radius of a sphere is proportional to a. Therefore we must make the identification:

$$\frac{1}{2b_3} = \frac{4}{3}\pi G. \tag{30}$$

This implies that Eq. (24) becomes

$$H^2 + \frac{K}{a^2} = \frac{8}{3}\pi G\rho \tag{31}$$

and Eq. (29) becomes

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p). \tag{32}$$

Equations (31) and (32) are called the **Friedmann equations** and they control the expansion history of the Universe.

Some simple consequences are:

- A static universe (a constant) with $\rho > 0$ and $p \geq 0$ is impossible.
- Both positive energy density and positive pressure tend to make the Universe decelerate.
- Negative pressure could under some circumstances enable a static ($w = -\frac{1}{3}$) or accelerating ($w < -\frac{1}{3}$) Universe, or even a "Big Bounce" (a decreases, then increases).
- An expanding Universe must be open, flat, or closed depending on whether the density ρ is less than, equal to, or greater than the **critical density**

$$\rho_{\rm cr} = \frac{3H^2}{8\pi G}. \tag{33}$$