Lecture XV: Inflation and cosmic acceleration

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I. INTRODUCTION

In the final lecture, we will study inflation – the paradigm that an early epoch of exponential inflation lay the seeds for structure formation in the cosmos. We will investigate what inflation predicts for the power spectrum of fluctuations. Finally, we will consider the present era of cosmic acceleration, which is similar to inflation in many ways, and what quantum mechanics has to say about the fate of the Universe.

As usual, $\hbar = c = k_\mathrm{B} = 1$.

A useful reference if you want to learn more about this material is Liddle & Lyth, *Cosmological inflation and large-scale structure.*

II. NEAR-EXPONENTIAL EXPANSION

Let us suppose that at some early time in the Universe, the cosmos was dominated not by radiation, but by a vacuum energy with density ρ_I. During such an epoch, the Friedmann equations say that the Universe should have accelerated exponentially with a rate

$$H_\mathrm{I} = \sqrt{\frac{8\pi G\rho_\mathrm{I}}{3}}. \tag{1}$$

The expansion history has the trivial solution

$$a = a_1 e^{H_\mathrm{I}t}. \tag{2}$$

The inflationary paradigm holds that the Universe went through such a period of accelerated expansion in the past, with unimaginably large H_I. The Hubble radius H_I^{-1} would have been so small that quantum mechanics would have been necessary to describe it; indeed in inflation, the $\sim 10^{-5}$ "primordial" curvature fluctuations are the result of quantum fluctuations. The field responsible for ρ_I would then have subsequently decayed into other fields, and ultimately thermalized to become the radiation of the radiation-dominated epoch.

The conformal time during inflation is

$$\eta = \int \frac{dt}{a} = \int \frac{1}{a_1} e^{-H_\mathrm{I}t}\,dt = -\frac{1}{a_1 H_\mathrm{I}} e^{-H_\mathrm{I}t}. \tag{3}$$

I set this up so that $\eta \to 0$ at large $t \gg H_\mathrm{I}^{-1}$ (the end of inflation!). Note that $\eta \to -\infty$ at small t; this means that at early times, all modes were inside the horizon. To be specific:

$$aH = a_1 H_\mathrm{I} e^{H_\mathrm{I}t} = -\frac{1}{\eta}, \tag{4}$$

so at early times or $\eta \to -\infty$, we have $aH \to 0$ and $k \gg aH$ for any k.

III. SCALAR FIELDS AND QUANTUM FLUCTUATIONS

Let's now suppose that there was a scalar field ϕ present during inflation. (In "single-field inflation" – the simplest model that works – ϕ is the only important field and also the field responsible for the energy density ρ_I.)

A. Scalar fields in flat spacetime

In flat spacetime, we write the action for a scalar field as:

$$S = \int L\,dt = \int \left[\frac{1}{2}\dot\phi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] d^3\mathbf{x}\,dt, \tag{5}$$

where the integral of $\frac{1}{2}\dot{\phi}^2$ is the "kinetic" energy, and $V(\phi)$ is the "potential energy density." The gradient term is necessary for relativistic invariance (it is just like the $\dot{\phi}^2$ term, with the obligatory $-$ sign for spatial coordinates). Because it has no time derivatives, it acts from the Lagrangian mechanics perspective like a potential term. The total energy density is the sum of kinetic and potential terms:

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi). \tag{6}$$

By varying the action, one can see that

$$0 = \delta S = \int \left[\dot{\phi}\,\delta\dot{\phi} - \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\delta\phi - \frac{dV(\phi)}{d\phi}\,\delta\phi\right] d^3\mathbf{x}\,dt = \int\left[-\ddot{\phi}\,\delta\phi + \nabla^2\phi\,\delta\phi - \frac{dV(\phi)}{d\phi}\,\delta\phi\right] d^3\mathbf{x}\,dt, \tag{7}$$

so the equation of motion is:

$$\ddot{\phi} = -\nabla^2\phi - \frac{dV(\phi)}{d\phi}. \tag{8}$$

B. Scalar fields in FLRW spacetime

In the FLRW universe, we have to include both the factor of a^3 in the volume in the action, and the factor of $1/a$ in converting the comoving to physical gradient:

$$S = \int L\,dt = \int a^3\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)\right] d^3\mathbf{r}\,dt. \tag{9}$$

This time when we vary the action, we need to remember that a depends on t; this means

$$0 = \delta S = \int\left[a^3\dot{\phi}\,\delta\dot{\phi} - a\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\delta\phi - a^3\frac{dV(\phi)}{d\phi}\,\delta\phi\right] d^3\mathbf{x}\,dt = \int\left[-(a^3\ddot{\phi} + 3a^2\dot{a}\dot{\phi})\,\delta\phi + a\nabla^2\phi\,\delta\phi - a^3\frac{dV(\phi)}{d\phi}\,\delta\phi\right] d^3\mathbf{x}\,dt. \tag{10}$$

Using $\dot{a} = aH$, we find the new equation of motion:

$$\ddot{\phi} = -3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi - \frac{dV(\phi)}{d\phi}. \tag{11}$$

The "$3H$" term is just like the $2/r$ term in the formula for the Laplacian in spherical coordinates or the $1/r$ term in cylindrical coordinates; it represents the way that the coordinate system is curved.

Let's consider homogeneous ($\nabla\phi = 0$) solutions to this equation. If H is big enough – as was presumed to have occurred during inflation – then the $\ddot{\phi}$ term may be neglected, and the kinetic energy may be neglected compared to the potential energy. Then we have $H = H_{\rm I}$, $V(\phi) = \rho_{\rm I}$, and

$$\dot{\phi} \approx -\frac{1}{3H_{\rm I}}\frac{dV(\phi)}{d\phi} \approx -\frac{1}{\sqrt{24\pi G V(\phi)}}\frac{dV(\phi)}{d\phi}. \tag{12}$$

This situation is called *slow-roll inflation*, and is valid when V is large enough and the derivatives of V are small enough. We commonly write

$$\epsilon = \frac{2\dot{\phi}^2}{3V(\phi)} \approx \frac{[dV(\phi)/d\phi]^2}{16\pi G[V(\phi)]^2}, \tag{13}$$

and slow-roll inflation requires $\epsilon \ll 1$.

The key property of slow-roll inflation is that ϕ evolves toward lower values of the potential $V(\phi)$; if the minimum is at $V(\phi_\star) = 0$, then inflation must end as ϕ approaches $\phi_\star$ and the slow-roll conditions are violated. At any given value of ϕ, we can compute the number N of e-folds of inflation remaining by the integral

$$N(\phi) = \int H\,(-dt) = \int_{\phi_\star}^{\phi} H_{\rm I}\,\frac{d\phi}{-\dot{\phi}} = \int_{\phi_\star}^{\phi} 3H_{\rm I}^2\,\frac{d\phi}{dV(\phi)/d\phi} = \int_{\phi_\star}^{\phi}\frac{8\pi G V(\phi)}{dV(\phi)/d\phi}\,d\phi. \tag{14}$$

C. Quantum fluctuations

In an expanding universe, the key point is what happens to the scalar field as a mode k exits the horizon. Classically, there is a solution where ϕ is spatially uniform (no dependence on $\mathbf{x}$). But quantum mechanically, there is always a fluctuation in the field. We can compute this by recalling that at horizon exit, any mode will have a frequency of order $H_{\rm I}$, and so should have a zero-point energy of $\sim \frac{1}{2}\omega \sim H_{\rm I}$. The physical volume in a horizon region is $\sim H_{\rm I}^{-3}$. Thus we have

$$H_{\rm I}^{-3}\dot{\delta\phi}^2 \sim H_{\rm I} \quad \rightarrow \quad \dot{\delta\phi} \sim H_{\rm I}^2. \tag{15}$$

But since the frequency is $\sim H_{\rm I}$, we have $\dot{\delta\phi} \sim H_{\rm I}\delta\phi$, and

$$\delta\phi \sim H_{\rm I}. \tag{16}$$

This is indeed the correct order of magnitude; a more detailed calculation gives

$$\Delta_{\delta\phi}(k) = \frac{H_{\rm I}}{2\pi}. \tag{17}$$

The "$H_{\rm I}$" here is to be evaluated when the mode in question exits the horizon, which is later (so slightly smaller $H_{\rm I}$) for larger k.

Now because of Eq. (14), a fluctuation in the scalar field means that the inflationary Universe expands a bit more or less in this or that region, depending on the value of $\delta\phi$. The conversion is of course

$$\delta N = \frac{dN}{d\phi}\delta\phi = \frac{8\pi G V(\phi)}{dV(\phi)/d\phi}\,\delta\phi = \sqrt{\frac{4\pi G}{\epsilon}}\,\delta\phi. \tag{18}$$

We finally identify the variation in the amount of expansion as the curvature perturbation: $\zeta = \delta N$. Thus:

$$\Delta_\zeta(k) = \sqrt{\frac{4\pi G}{\epsilon}}\,\Delta_{\delta\phi}(k) = \sqrt{\frac{4\pi G}{\epsilon}}\,\frac{H_{\rm I}}{2\pi}. \tag{19}$$

Thus a near-exponential expansion will give a near-scale-invariant curvature perturbation.

Unfortunately, with only knowledge of $\Delta_\zeta \sim 5\times 10^{-5}$, we can't tell whether $H_{\rm I}$ is large and ϵ is large, or if $H_{\rm I}$ is small and ϵ is small: only the combination $H_{\rm I}/\sqrt{\epsilon}$ can be determined. Equivalently, out of two variables $V(\phi)$ and $V'(\phi)$, we only get one measurement. Measuring n_s is interesting but doesn't solve the problem, since we get one more measurement but by taking a derivative $n_s - 1 = d\ln\Delta_\zeta^2(k)/d\ln k = -d\ln\Delta_\zeta^2(k)/d\ln N$, we introduce $V''(\phi)$.

The way in principle that we could break the degeneracy of $V(\phi)$ and $V'(\phi)$ is to measure the primordial tensor perturbations. For gravitational waves, the energy density is $\sim \dot{h}^2/G$, so Eq. (17) applies to h as well as $\delta\phi$ but with a factor of $\sqrt{G}$. With the appropriate factors put in:

$$\Delta_h(k) = \sqrt{32\pi G}\,\frac{H_{\rm I}}{2\pi}, \tag{20}$$

and the tensor-to-scalar ratio is

$$r = \frac{\Delta_h^2(k)}{\Delta_\zeta^2(k)} = 8\epsilon. \tag{21}$$

Thus if $H_{\rm I}$ and ϵ are large, we may see the tensor perturbations soon; but if $\epsilon \ll 10^{-4}$, then we have no real hope of seeing the tensor perturbations, and we will obtain only an upper limit.

IV. DE SITTER SPACETIME AND THE FUTURE

Many of the same concepts involved in inflation apply – at least in principle – to the Λ-dominated epoch that we have recently entered. However, the observable aspects look very different as seen from an observer *in* the accelerating universe instead of looking at a subsequent radiation-dominated phase. *If* dark energy is really a cosmological constant, then the future of the Universe is an inflationary state with $H_\infty = \sqrt{\Lambda/3}$ (the subscript "∞" representing the far future).

The space-time of constant Hubble rate – known as *de Sitter spacetime* – can be written as

$$ds^2 = -dt^2 + e^{2H_\infty t}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{22}$$

The region of space that can send a signal to an observer living at $r = 0$ is

$$r < \eta(\infty) - \eta(t) = \int_t^\infty \frac{dt'}{e^{H_\infty t'}} = \frac{1}{H_\infty} e^{H_\infty t}. \tag{23}$$

To see how it looks to an observer inside, let's make the transformation

$$r = e^{-H_\infty t}\tilde{r}, \quad t = \tilde{t} + \frac{1}{2H_\infty}\ln(1 - H_\infty^2\tilde{r}^2), \tag{24}$$

so that $\tilde{r} = e^{H_\infty t} r = ar$ is the physical distance from the observer to the point. After some algebra,

$$ds^2 = -\left(1 - H_\infty^2\tilde{r}^2\right)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - H_\infty^2\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{25}$$

Now this is a time-independent description ($\tilde{t}$ does not appear explicitly), and the region that can send a signal to an observer at $r = 0$ is $\tilde{r} < 1/H_\infty$. We can say that – to an observer at $r = 0$ – the Universe looks like an inside-out black hole, with the event horizon at $\tilde{r} = 1/H_\infty$. Just like a normal black hole, someone who falls through the event horizon experiences nothing special.

From a quantum point of view, we know there are fluctuations in the density of any field of $\sim \delta\dot{\phi}^2 \sim H_\infty^4$. These fluctuations are in some ways similar to Hawking radiation from the inside-out event horizon, and based on their density should have a temperature of order H_∞. A detailed calculation gives

$$T_\infty = \frac{H_\infty}{2\pi} \sim 2 \times 10^{-30}\,\mathrm{K}. \tag{26}$$

Associated with this is the entropy of the de Sitter horizon, which is $\frac{1}{4G}$ of the horizon area:

$$S_\infty = \frac{1}{4G} \times 4\pi\left(\frac{1}{H_\infty}\right)^2 = \frac{\pi}{GH_\infty^2} = \frac{3\pi}{G\Lambda} \approx 3 \times 10^{122}. \tag{27}$$

If the dark energy really is a cosmological constant, we must face this ultimate entropy of the entire observable Universe. This is the modern notion of a "heat death" – the de Sitter horizon grows every time an object becomes gravitationally unbound from us and is swept away by Λ. (After infinite time, this is *all* objects, due to quantum tunneling.) The entropy S_∞ must have profound philosophical implications as well, since – if the aforementioned arguments are taken seriously – it represents a limit, in principle, on information storage (recall: 1 bit has entropy $\ln 2$), and even on scientific knowledge itself.