Lecture VI: The radiation-dominated era

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I. INTRODUCTION

We now begin our study of the thermal history of the Universe. This discussion will start with the radiation-dominated epoch, which is both the earliest period where we understand (or think we understand!) the underlying physics, and also happens to be the simplest. The steps covered today will be:

- Relativistic plasma ($z \geq 10^{10}$).
- Neutrino decoupling ($z \sim 10^{10}$).
- e^+e^- annihilation ($z \sim 2 \times 10^9$; $kT \sim m_e c^2$).
- The non-relativistic plasma ($3000 < z < 2 \times 10^9$).

The radiation is the dominant component of the Universe at these early times, so I will focus on that. We will consider the state of the baryonic matter in the next lecture; in fact, everything in this lecture would work the same way even if there were no excess of baryons over antibaryons in the Universe.

Where appropriate, I will set Boltzmann's constant k, the speed of light c, and the reduced Planck's constant $\hbar$ equal to 1 (i.e., "particle physics" units).

II. RELATIVISTIC PLASMA

We will concern ourselves first with the equation of state for a relativistic plasma with no chemical potentials, and then find the expansion history. The assumptions of no chemical potential and thermal equilibrium are appropriate at early times, and we?ll see when they break down. We further assume the particle masses are small interactions, so that we can use the free particle dispersion relation, which will be a good approximation *except* near the QCD phase transition.

A. Plasma density

Let's consider a gas of bosons and fermions at temperature T. The phase space density of these particles will be:

$$f(\mathbf{q}) = \frac{1}{e^{E(q)/T} \pm 1}, \tag{1}$$

where q is the momentum,

$$E(q) = \sqrt{m^2 + q^2} \tag{2}$$

is the dispersion relation, and the + sign is for fermions and − is for bosons (yes, just like undergraduate statistical mechanics). The total density of these particles is

$$\rho = g \int_{\mathbb{R}^3} \frac{d^3\mathbf{q}}{(2\pi)^3}\, E(q) f(q) = \frac{g}{(2\pi)^3} \int_0^\infty 4\pi q^2 f(q)\, dq = \frac{g}{2\pi^2} \int_0^\infty q^2 f(q)\, dq. \tag{3}$$

where g is the degeneracy of the particle ($2s+1$ for massive particles, 2 for photons, and 1 for scalars or neutrinos that have only one helicity state). We can simplify:

$$\rho = \frac{g}{2\pi^2} \int_0^\infty \frac{q^2 \sqrt{q^2 + m^2}}{\exp[(m^2 + q^2)^{1/2}/T] \pm 1}\, dq. \tag{4}$$

Let?s define $x = q/T$, so

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty \frac{x^2 \sqrt{x^2 + (m/T)^2}}{e^{\sqrt{(m/T)^2 + x^2}} \pm 1}\, dx. \tag{5}$$

We further define $\xi = m/T$ and define the integral to be $I_\pm(\xi)$, so that

$$\rho = \frac{g}{2\pi^2} T^4 I_\pm \left(\frac{m}{T}\right). \tag{6}$$

The total density is the sum of this over all of the species present. A general expression for $I_\pm(\xi)$ is hard but we can do two limiting cases:

- Case I: $\xi \to 0$. In this case the integral reduces to

  $$I_\pm(0) = \int 0^\infty \frac{x^3}{e^x \pm 1}\, dx. \tag{7}$$

  This integral turns out to be analytically solvable: it is

  $$I_-(0) = \frac{\pi^4}{15} \quad \text{or} \quad I_+(0) = \frac{7\pi^4}{120} = \frac{7}{8} I_-(0). \tag{8}$$

  The fermion integral is smaller than the boson integral by a factor of $\frac{7}{8}$, because the fermions must obey the Pauli exclusion principle and cannot "pile up" like radio photons from a thermal source. So if all of the species of interest are effectively massless ($m \ll T$) we can write

  $$\rho_{\rm tot} = \sum_X \rho_X = \sum_X \frac{g_X}{2\pi^2} T^4 I_\pm(0). \tag{9}$$

  Since $I_-(0)/2\pi^2 = \pi^2/30$, this can be written as

  $$\rho_{\rm tot} = \frac{\pi^2 T^4}{30} \sum_X g_X \times \left\{ \begin{array}{ll} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{array} \right. . \tag{10}$$

  This summation is often given the symbol $g_\star$. It is the total number of spin degrees of freedom of the particles with small mass, with a factor of $\frac{7}{8}$ weighting for fermions.

- Case II: $\xi \gg 1$, i.e., $m \gg T$. In the absence of a chemical potential we expect very few particles of this species. The integral can be written as:

  $$I_\pm(\xi) = \int_0^\infty \frac{x^2\sqrt{x^2+\xi^2}}{e^{\sqrt{x^2+\xi^2}} \pm 1}\, dx \approx \sqrt{\frac{\pi}{2}}\, \xi^{5/2} e^{-\xi}, \tag{11}$$

  where the approximation arises by taking $\sqrt{x^2+\xi^2} \to \xi + x^2/(2\xi) + ...$ in the exponent in the denominator, dropping the small ± 1, and writing this as a Gaussian integral. From Eq. (8), we can find

  $$\frac{I_\pm(\xi)}{I_-(0)} = \frac{15}{\sqrt{2}\,\pi^{7/2}} \xi^{5/2} e^{-\xi}. \tag{12}$$

  This ratio can be thought of as the contribution of a massive particle ($m \gg T$) to $g_\star$. If $\xi \gg 1$, the contribution is $\ll 1$.

If we are in between cases, we must do the integral numerically. Case I is good to $\sim 10\%$ for $m/T < 0.95$, but for Case II to be valid to this accuracy we need $m/T > 35$. In any case we can define a generalized $g_\star$,

$$g_\star \equiv \sum_X g_X \frac{I_\pm(\xi)}{I_-(0)}, \tag{13}$$

which reduces to the usual equation in the massless case. It is then exactly true that

$$\rho = \frac{\pi^2 g_\star T^4}{30}. \tag{14}$$

B. Plasma pressure and entropy density

For a fully relativistic gas, where the masses are truly zero, we know the pressure is $p = \rho/3$. But we can find the pressure for the general case using the thermodynamic relations. (Also a good review!) Let's define S to be the total entropy of a parcel of gas, V to be its volume, and $s = S/V$ to be its entropy density. Then from the first law of thermodynamics,

$$dU = T\,dS - p\,dV + \sum_X \mu_X\,dN_X, \tag{15}$$

where μ_X is the chemical potential of species X. But we are considering the case of zero chemical potential, so we can drop the last term. Using $U = \rho V$:

$$d(\rho V) = T\,d(sV) - p\,dV \tag{16}$$

or

$$\rho\,dV + V\,d\rho = Ts\,dV + TV\,ds - p\,dV. \tag{17}$$

Now s and ρ depend only on T, not on V, so we can write

$$\rho\,dV + V\frac{d\rho}{dT}\,dT = Ts\,dV + TV\frac{ds}{dT}\,dT - p\,dV. \tag{18}$$

This is a total differential so we can equate coefficients of dT and dV:

$$\rho = Ts - p \quad \text{and} \quad V\frac{d\rho}{dT} = TV\frac{ds}{dT}. \tag{19}$$

The first equation gives the entropy density in terms of pressure:

$$s = \frac{\rho + p}{T} \quad \rightarrow \quad p = Ts - \rho. \tag{20}$$

The second equation can be divided by TV to get:

$$\frac{1}{T}\frac{d\rho}{dT} = \frac{ds}{dT}. \tag{21}$$

We can then see that

$$\frac{d}{dT}\frac{p}{T} = \frac{d}{dT}\left(s - \frac{\rho}{T}\right) = \frac{ds}{dT} - \frac{1}{T}\frac{d\rho}{dT} + \frac{\rho}{T^2} = \frac{\rho}{T^2}. \tag{22}$$

This means that the pressure can be found by integrating:

$$p(T) = T\int_0^T \frac{\rho(T')}{T'^2}\,dT' = \frac{\pi^2}{30}T^4\int_0^1 g_\star(yT)\,y^2\,dy, \tag{23}$$

where we substituted $T' = yT$, and used the fact that $p/T \to 0$ at low temperature (think of a blackbody, since only massless particles will be left as $T \to 0$).

Then the entropy is

$$s(T) = \frac{\pi^2}{30}T^3\left[g_\star(T) + \int_0^1 g_\star(yT)\,y^2\,dy\right] \tag{24}$$

and the equation of state is:

$$w(T) = \frac{p(T)}{\rho(T)} = \int_0^1 \frac{g_\star(yT)}{g_\star(T)}\,y^2\,dy. \tag{25}$$

If $g_\star$ is a constant, which is true when all species are either massless ($m \ll T$) or irrelevant ($m \gg T$) then $w = 1/3$, and then

$$p \approx \frac{\pi^2}{90}g_\star T^4 \quad \text{and} \quad s \approx \frac{2\pi^2}{45}g_\star T^3. \tag{26}$$

Note that since these equations come from basic thermodynamics, they are true even for strongly interacting particles, e.g., during the QCD phase transition, so long as the function $g_\star(T)$ (as defined by $\rho = \pi^2 g_\star T^4/30$) can be computed.

C. Cases of cosmological interest

Let's consider the function $g_\star$ at early times.

- $T \sim$few MeV: we have photons $g_\gamma = 2$, electrons/positrons ($g_e = 4$), and neutrinos ($g_\nu = 6$ species, one helicity each for ν and $\bar{\nu}$, and 3 flavors). The latter two are fermions so

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(6) = \frac{43}{4} = 10.75 \tag{27}$$

- At $T \approx 120$ MeV, somewhat below QCD transition, we now also have $\mu^\pm$ pairs ($g_\mu = 4$), and pions (which are spin 0 bosons, with three isospin possibilities π^+, π^0, and π^- – thus $g_\pi = 3$). If the latter were massless, would have

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(6) + \frac{7}{8}(4) + 3 = \frac{69}{4} = 17.25. \tag{28}$$

  (This is not perfect since muons/pions have significant mass, and kaons are not negligible. Also we've left out pion self-interactions.)

- At $T \sim$few hundred MeV, above the QCD phase transition: now the massless particles are photons (g_γ), $e^\pm$ pairs ($g_e = 4$), $\mu^\pm$ pairs ($g_\mu = 4$), neutrinos ($g_\nu = 6$), quarks (3 flavors up/down/strange, 3 colors, 2 spin states, and anti-particles, so $g_q = 36$), and gluons (8 colors, 2 polarizations, so $g_g = 16$). Then

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(4) + \frac{7}{8}(6) + \frac{7}{8}(36) + 16 = \frac{247}{4} = 61.75. \tag{29}$$

- As we increase the temperature, we can add the remaining standard model particles; when we get to $T \gtrsim 100$ GeV, we have all of them (the other quark flavors charm/bottom/top, the $\tau^\pm$ leptons, W and Z bosons, and the Higgs). This gets us to $g_\star = 107.75$.

If you think there are more heavy particles, then at even higher temperatures you can add them to your running total of $g_\star$. For example the Minimal Supersymmetric Standard Model converges to $g_\star = 228.75$ at high temperature.

III. EXPANSION HISTORY IN THE RADIATION-DOMINATED EPOCH

The Universe is flat to a good approximation at early times when the horizon is small compared to the radius of curvature of the Universe. An alternative way to put this is that in the Friedmann equation,

$$\frac{8}{3}\pi G\rho = H^2 + \frac{K}{a^2}, \tag{30}$$

the left hand side $\rho \propto a^{-4}$ for radiation domination (approximately) so this dominates over the curvature term a^{-2}. So we?ll drop the curvature. Using equation for ρ in terms of $g_\star$: ($\rho = \pi^2 g_\star T^4/30$)

$$\frac{4}{45}\pi^3 G g_\star T^4 = H^2, \tag{31}$$

so the Hubble rate is

$$H = \frac{2\pi^{3/2}}{3\sqrt{5}} G^{1/2} g_\star^{1/2} T^2. \tag{32}$$

This is one equation relating the expansion history to temperature, but we need one more equation to close the system for $a(t)$ and $T(t)$ (we have 2 unknowns, so need 2 equations). Assuming the Universe remains in thermal equilibrium, which is true prior to neutrino decoupling, the total entropy of the Universe remains fixed (adiabatic expansion). Then entropy density declines as 1/volume: $s \propto a^{-3}$. We can thus write

$$s = \frac{s_e}{a^3}, \tag{33}$$

where se is the extrapolated entropy density of the Universe, i.e. the entropy density today if there were no new sources of entropy. [Warning: the actual entropy today is $> s_e$ due to non-equilibrium processes, to be discussed later.] So we know:

$$\frac{s_e}{a^3} = \frac{\pi^2}{30} T^3 \left[g_\star(T) + \int_0^1 g_\star(yT)\, y^2\, dy \right]. \tag{34}$$

Let?s consider the case where $g_\star$ is constant over a reasonable range in temperature, so the second integral is $g_\star/3$. Then

$$\frac{s_e}{a^3} = \frac{2\pi^2}{45} g_\star T^3, \tag{35}$$

and

$$T = \left(\frac{45 s_e}{2\pi^2 g_\star} \right)^{1/3} \frac{1}{a}. \tag{36}$$

From Eq. (32):

$$\frac{1}{a}\frac{da}{dt} = H = \frac{2\pi^{3/2}}{3\sqrt{5}} \left(\frac{45 s_e}{2\pi^2 g_\star} \right)^{2/3} G^{1/2} g_\star^{1/2} a^{-2} = (180\pi)^{1/6} G^{1/2} g_\star^{-1/6} s_e^{2/3} a^{-2}. \tag{37}$$

We can re-arrange to get

$$a\, da = (180\pi)^{1/6} G^{1/2} g_\star^{-1/6} s_e^{2/3}\, dt, \tag{38}$$

or – integrating and using the initial condition $a = 0$ at $t = 0$ (the Big Bang):

$$a = 2^{1/2} (180\pi)^{1/12} G^{1/4} g_\star^{-1/12} s_e^{1/3} t^{-1/2}. \tag{39}$$

We see that the expansion history is the usual result ($a \propto t^{1/2}$), except that there?s a change in the coefficient every time the Universe cools off enough for another species to become “effectively massive” and $g_\star$ changes. The change in coefficient is small, though, since $g_\star$ has a small exponent ($-\frac{1}{12}$).

The temperature as a function of time is then

$$T = \frac{45^{1/4}}{2\pi^{3/4}} G^{-1/4} g_\star^{-1/4} t^{-1/2}. \tag{40}$$

In less clumsy units (i.e. putting in factors of $\hbar$ and c):

$$kT = 1.56 g_\star^{-1/4} \sqrt{\frac{1\,\mathrm{s}}{t}}\,\mathrm{MeV}. \tag{41}$$

Alternatively, we may write the equation for the time:

$$t = 1 g_\star^{-1/2} \left(\frac{1.56\,\mathrm{MeV}}{T} \right)^2 \mathrm{s}. \tag{42}$$

IV. PHASE TRANSITIONS

We can see that the QCD phase transition ($g_\star \sim 17.25 - 61.75$ and $T \sim 170\,\mathrm{MeV}$) should have occurred at 10–20 μs after the Big Bang. At one point it was thought that this might be a first-order phase transition, in which case the Universe could have become quite lumpy around that time (with “bubbles” of quark+gluon plasma amid regions with restored chiral symmetry containing pions and kaons, analogous to water condensing from gas to liquid phase). However, this phase transition is now understood to be a crossover, and no such lumpiness is believed to have occurred.

The electroweak phase transition at $g_\star = 107.75$ and $T \sim 150$ GeV should have occurred at $\sim 10^{-11}$ s after the Big Bang. This transition is when the Higgs field acquires a non-zero vacuum expectation value and breaks the SU(2)$\times$U(1) electroweak symmetry to U(1). This transition is also a crossover in the Standard Model (this is possible, even though a symmetry is being broken, because this is a gauge symmetry and the global average $\langle \phi \rangle$ of the Higgs is not a meaningful concept).

It is possible that other phase transitions occur at even higher temperatures earlier in the Universe (for example, a GUT phase transition), but of course we don’t know the correct theory at those temperatures.

V. NEUTRINO DECOUPLING, e^+e^- ANNIHILATION, AND THE COSMIC NEUTRINO BACKGROUND

We now want to understand how the relations in the previous section relate to present-day observables, and how we can normalize s_e. This is a bit complicated because at about 1 s after the Big Bang, the Universe becomes transparent to neutrinos. Thereafter, we will need to separately keep track of the electromagnetically interacting particles and the neutrinos. We will treat these effects in order.

A. Neutrino decoupling

Neutrinos are kept in thermal equilibrium with the rest of the Universe at early times by reactions such as

$$\nu_x + \bar{\nu}_x \leftrightarrow e^+ + e^- \quad (x = e, \mu, \tau). \tag{43}$$

However, these reactions may not be in thermal equilibrium if the neutrinos don't interact often enough. So we need to address the question of when neutrinos "decouple," i.e., when the Universe becomes transparent to them.

The cross section for a neutrino interaction is given at order of magnitude level by

$$\sigma_{\rm weak} \sim G_{\rm F}^2 E_{\rm cm}^2, \tag{44}$$

where $E_{\rm cm}$ is the center of mass energy (assumed to be $E_{\rm cm} \ll m_W$) and $G_{\rm F} = 1.166 \times 10^{-5}\,{\rm GeV}^{-2}$ is the Fermi coupling constant describing the strength of the weak interaction (actually a combination of the weak fine structure constant and the W mass). The number density of targets is $n \sim T^3$ (I'm not putting in the $g_\star$, since only particles of the right flavor are possible targets) and the typical collision energy is $\sim T$, so the interaction rate is $\Gamma = \sigma_{\rm weak} n \sim G_{\rm F}^2 T^5$. If we divide this by the Hubble rate, we get

$$\frac{\Gamma}{H} \sim \frac{G_{\rm F}^2 T^5}{G^{1/2} g_\star^{1/2} T^2} \sim g_\star^{-1/2} G_{\rm F}^2 G^{-1/2} T^3. \tag{45}$$

This ratio is large at early times or high temperatures, i.e., the neutrinos collide many times before the Universe has a chance to expand, but at low temperatures the neutrino interaction rate is longer than the age of the Universe. The transition $\Gamma \sim H$ occurs at the decoupling temperature,

$$T_{\rm dec} \sim g_\star^{1/6} G_{\rm F}^{-2/3} G^{1/6}. \tag{46}$$

In Planck units, $G = m_{\rm Pl}^{-2} = 6.7 \times 10^{-39}\,{\rm GeV}^{-2}$, and if we set $g_\star = 10.75$ (small errors in $g_\star$ will hardly matter!) then we find $T_{\rm dec} = 1.3$ MeV. This is in fact about right (the "real" temperature of decoupling is closer to 1 MeV).

After neutrino decoupling, the neutrinos continue to redshift as $T \propto 1/a$ (at least ignoring their masses). So let's take the neutrino temperature today to be $T_{\nu 0}$ (note I haven't calculated its value yet!), and from Eq. (36):

$$s_e = \frac{43\pi^2}{90} T_{\nu 0}^3. \tag{47}$$

The scale factor as a function of temperature prior to neutrino decoupling is then

$$a = \left(\frac{43}{4g_\star}\right)^{1/3} \frac{T_{\nu 0}}{T}. \tag{48}$$

B. e^+e^- annihilation

The electrons and positrons annihilate after neutrino decoupling. (Actually they don't completely annihilate since there are a few more e^- than e^+, but this won't concern us yet.) The characteristics of this process are:

- The principal reaction is $e^+ + e^- \leftrightarrow \gamma + \gamma$.
- The neutrinos (almost) don't participate. (A few annihilations go to $e^+ + e^- \to \nu_x + \bar{\nu}_x$, affecting results at the $\sim 1\%$ level.)

- The annihilation is fast. The time to reach equilibrium is $\sim \alpha^2/m_e \sim 10^{-18}$ s, much less than the age of the Universe, t$\sim$ few seconds.

Under these circumstances, the electron-positron-photon plasma adiabatically transitions into a photon-only plasma, i.e. the entropies before and after are the same. Since $g_\star$ for photons is 2 and for $e^\pm + \gamma$ is $\frac{11}{2}$, we have

$$\frac{11}{2} T^3 a^3|_{\text{before ann.}} = 2T_\gamma^3 a^3|_{\text{after ann.}}. \tag{49}$$

Now since the neutrinos don't participate, $T_\nu \propto 1/a$, so the temperature on the left hand side applies to neutrinos even after annihilation. Thus, after e^+e^- annihilation,

$$\frac{11}{2} T_\nu^3 = 2T_\gamma^3 \tag{50}$$

and so

$$T_\nu = \sqrt[3]{\frac{4}{11}}\, T_\gamma. \tag{51}$$

Since today we have $T_{\gamma 0} = 2.73K$, the neutrino temperature is $T_{\nu 0} = 1.95$ K.

In general, we can replace Eq. (48) with something in terms of the photon temperature:

$$a = \left(\frac{43}{11 g_\star}\right)^{1/3} \frac{T_\gamma 0}{T} \quad \text{(before annihilation)}. \tag{52}$$

It is common to define an "effective" $g_{\star,\text{eff}}$, such that

$$\rho = \frac{\pi^2}{30} g_{\star,\text{eff}} T_\gamma^4, \tag{53}$$

such that $g_{\star,\text{eff}} = g_\star$ when all species are in equilibrium. This makes, e.g., the $T(t)$ relation (Eq. 41) valid. To find $g_{\star,\text{eff}}$ after e^+e^- annihilation, we write

$$\rho = \frac{\pi^2}{30} g_{\star,\gamma} T_\gamma^4 + \frac{\pi^2}{30} g_{\star,\nu} T_\nu^4, \tag{54}$$

so that

$$g_{\star,\text{eff}} = g_{\star,\gamma} + g_{\star,\nu}\left(\frac{T_\nu}{T_\gamma}\right)^4 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} \approx 3.36. \tag{55}$$

(Actually this is 3.38 when one takes into account $\mathcal{O}(\alpha)$ QED corrections to the plasma equation of state and $e^+ + e^- \to \nu_x + \bar{\nu}_x$ annihilations.)

C. Importance of neutrino masses

As a final note, all this assumes the neutrinos are massless, which is a good approximation at high z. It is not good today since the neutrino mass is large compared to $T_{\nu 0}$:

- $T_{\nu 0} = 1.95$ K today is 1.7×10^{-4} eV in energy units.
- The neutrino mass splittings are $\Delta m_{21}^2 = (0.009\,\text{eV})^2$ and $|\Delta m_{32}^2| = (0.05\,\text{eV})^2$.

So at least two of the neutrino species are nonrelativistic today. We will discuss the consequences later, but in general when they become non-relativistic it is their *momentum* that scales as $\propto 1/a$ (stretching of the de Broglie wavelength), and their number density is conserved (remember, they are non-interacting).