Physics 6820 – Homework 3

(Dated: Due: September 13, 2019)

1. Maxwell's equations. [32 points]

In this problem, we consider the action for some particles and the electromagnetic field in special relativity:

$$S = \sum_{\text{particles}} \left[-m \int d\tau + q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu \right] + \int \frac{1}{4} F_{\gamma\mu} F^{\gamma\mu} \mathrm{d}^4\boldsymbol{x}. \tag{1}$$

Here $\boldsymbol{A}$ is a 1-form, and in the sum over particles, m is the mass of the particle and q is its electric charge. The 2-form electromagnetic field tensor is $\boldsymbol{F} = \tilde{\boldsymbol{d}}\boldsymbol{A}$. We have used units where $\epsilon_0 = 1$ (and since $c = 1$, we will also have $\mu_0 = 1$).

On Homework #2, you showed that the particle acceleration satisfied $ma_\gamma = qF_{\gamma\mu}u^\mu$, and that the field components were

$$F_{\gamma\mu} \to \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{array} \right). \tag{2}$$

(a) [4 points] Show that the $\sum_{\text{particles}} q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu$ term in the action can be written as

$$\int A_\mu J^\mu \, \mathrm{d}^4\boldsymbol{x}, \tag{3}$$

where the *4-current density* $\boldsymbol{J}$ for a swarm of particles is:

$$J^\mu(\boldsymbol{y}) = \sum_{\text{particles}} \int \delta^{(4)}(\boldsymbol{x}(\sigma) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma} \, \mathrm{d}\sigma. \tag{4}$$

(b) [3 points] Explain (maybe with words and a few equations) why J^0 is the usual charge density and J^i are the components of the usual current density.

(c) [6 points] Now let's consider the variation δS of the action to first order when there is a small change δA_μ in the 4-vector potential. Show that

$$\delta S = \int \left(J^\mu \delta A_\mu + F^{\gamma\mu} \partial_\gamma \delta A_\mu \right) \mathrm{d}^4\boldsymbol{x}. \tag{5}$$

Conclude, using integration by parts, that the equation of motion is

$$F^{\gamma\mu}{}_{,\gamma} = J^\mu. \tag{6}$$

(d) [4 points] By explicitly writing the components, show that the equations of motion from (c) correspond to Gauss's law and to Ampère's law (including displacement current).

(e) [4 points] Explain why the 3-form $\tilde{\boldsymbol{d}}\boldsymbol{F} = 0$. Show that the 4 independent components of this equation correspond to the divergencelessness of the magnetic field and to Faraday's law of induction.

(f) [4 points] Show that the equation of motion from (c) can be written in terms of differential forms as:

$$\tilde{\boldsymbol{d}}({}^\star\boldsymbol{F}) = {}^\star\boldsymbol{J}. \tag{7}$$

What type of form is on both sides of this equation?

(g) [3 points] Show that the equation of motion in (f) *requires* that the 4-divergence of $\boldsymbol{J}$ be zero. This is known as "automatic conservation of the source," and will occur in GR as well. Express this equation both as a differential form and in index notation.

(h) [4 points] Show that the field part of the action (Eq. 1) can be written as

$$S = \int_{\mathbb{R}^4} -\frac{1}{4} \boldsymbol{F} \wedge {}^\star\boldsymbol{F}. \tag{8}$$

where $\mathbb{R}^4$ is the 4-dimensional region over which we do the integral.