Homework #4 – Solutions

1. Growth function in ΛCDM. [30 points]

In this problem, we will solve for the growth function as a function of scale factor, $G_+(a)$, in ΛCDM cosmology. We will assume the Universe to be flat, $\Omega_\Lambda + \Omega_m = 1$, and work at scale factors large enough for the radiation to be negligible, $a \gg a_{\rm eq}$.

Recall that the growth function satisfies

$$\frac{d^2G_+}{dt^2} + 2H\frac{dG_+}{dt} - 4\pi G\bar\rho_m G_+ = 0. \tag{1}$$

We will appropriately re-arrange this equation, and then do a numerical solution.

(a) [6 points] *Let's define the new variable*

$$F_+ = \frac{1}{H}\frac{dG_+}{dt}. \tag{2}$$

Show that G_+ and F_+ satisfy the 1st-order system of differential equations:

$$\frac{dG_+}{da} = \frac{F_+}{a} \quad \text{and} \quad \frac{dF_+}{da} = -\frac{dH}{da}\frac{F_+}{H} - 2\frac{F_+}{a} + \frac{4\pi G\bar\rho_m}{aH^2}G_+. \tag{3}$$

The first one is straightforward – since $da/dt = aH$, we have

$$\frac{dG_+}{da} = \frac{dG_+/dt}{da/dt} = \frac{HF_+}{aH} = \frac{F_+}{a}. \tag{4}$$

For the second equation, we use da/dt again and the usual rules of calculus:

$$\begin{aligned}
\frac{dF_+}{da} = \frac{dF_+/dt}{da/dt} &= \frac{1}{aH}\frac{dF_+}{dt} \\
&= \frac{1}{aH}\frac{d}{dt}\left(\frac{dG_+/dt}{H}\right) \\
&= \frac{1}{aH}\frac{H\,d^2G_+/dt^2 - (dG_+/dt)(dH/dt)}{H^2} \\
&= \frac{1}{aH}\frac{H\,(-2H\,dG_+/dt + 4\pi G\bar\rho_m G_+) - (dG_+/dt)(dH/dt)}{H^2} \\
&= \frac{-2}{aH}\frac{dG_+}{dt} + \frac{4\pi G\bar\rho_m}{aH^2}G_+ - \frac{dH/dt}{aH^3}\frac{dG_+}{dt} \\
&= \frac{-2}{a}F_+ + \frac{4\pi G\bar\rho_m}{aH^2}G_+ - \frac{(dH/da)(da/dt)}{aH^2}F_+ \\
&= \frac{-2}{a}F_+ + \frac{4\pi G\bar\rho_m}{aH^2}G_+ - \frac{dH/da}{H}F_+.
\end{aligned} \tag{5}$$

(b) [6 points] *For the ΛCDM cosmology, show that the differential equation for F_+ can be reduced to*

$$\frac{dF_+}{da} = -\frac{4\Omega_\Lambda + \Omega_m a^{-3}}{2(\Omega_\Lambda + \Omega_m a^{-3})}\frac{F_+}{a} + \frac{3\Omega_m a^{-3}}{2(\Omega_\Lambda + \Omega_m a^{-3})}\frac{G_+}{a}. \tag{6}$$

We need to simplify Eq. (3). We use the substitutions:

$$\frac{4\pi G\bar\rho_m}{H^2} = \frac{4\pi G\bar\rho_m}{\frac{8}{3}\pi G\bar\rho} = \frac{3\bar\rho_m}{2\bar\rho} = \frac{3\rho_{c,0}\Omega_m a^{-3}}{2\rho_{c,0}(\Omega_m a^{-3} + \Omega_\Lambda)} = \frac{3\Omega_m a^{-3}}{2(\Omega_m a^{-3} + \Omega_\Lambda)}, \tag{7}$$

where $\rho_{c,0}$ is the critical density today. We also see that

$$\begin{aligned}
\frac{dH/da}{H} &= \frac{d}{da}\ln H \\
&= \frac{d}{da}\ln\left(H_0\sqrt{\Omega_m a^{-3}+\Omega_\Lambda}\right) \\
&= \frac{1}{2}\frac{d}{da}\ln(\Omega_m a^{-3}+\Omega_\Lambda) \\
&= \frac{1}{2}\frac{-3\Omega_m a^{-4}}{\Omega_m a^{-3}+\Omega_\Lambda}.
\end{aligned} \tag{8}$$

Substituting this into Eq. (3) gives

$$\begin{aligned}
\frac{dF_+}{da} &= -\frac{dH}{da}\frac{F_+}{H} - 2\frac{F_+}{a} + \frac{4\pi G\bar{\rho}_m}{aH^2}G_+ \\
&= \left(-\frac{1}{2}\frac{-3\Omega_m a^{-4}}{\Omega_m a^{-3}+\Omega_\Lambda} - \frac{2}{a}\right)F_+ + \frac{3\Omega_m a^{-3}}{2a(\Omega_m a^{-3}+\Omega_\Lambda)}G_+ \\
&= -\frac{4\Omega_\Lambda+\Omega_m a^{-3}}{2(\Omega_m a^{-3}+\Omega_\Lambda)}\frac{F_+}{a} + \frac{3\Omega_m a^{-3}}{2(\Omega_m a^{-3}+\Omega_\Lambda)}\frac{G_+}{a}.
\end{aligned} \tag{9}$$

(c) [4 points] *Using your knowledge of the growth function in the matter-dominated era, explain why we can initialize the integration of the growth function ODEs at some small value of a ($a_{\rm init}$) with*

$$G_+(a_{\rm init}) = F_+(a_{\rm init}) = a_{\rm init}. \tag{10}$$

In the matter-dominated era, the growth function has two solutions, $\propto a$ and $\propto a^{-3/2}$. The former is the growing solution, and we **can** normalize it to $G_+(a) = a$. Then in the matter-dominated era,

$$F_+(a) = a\frac{dG_+}{da} = a\times 1 = a. \tag{11}$$

(d) [10 points] *Numerically integrate the aforementioned ODEs for $\Omega_m = 0.1$, 0.3, and 1.0. Make plots of the results.*

I wrote a C program to compute the growth function for Ω_m as a command-line argument:

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>

/* parameter list is pars[] = {Omega_m} */

/* get d(F,G)/da at a given a */
void get_derivs(double *dFG, double *FG, double a, double *pars) {
  double oma3 = pars[0]/a/a/a;
  double ol = 1-pars[0];
  dFG[0] = -(4.*ol + oma3)/2./(ol+oma3)*FG[0]/a + 1.5*oma3/(ol+oma3)*FG[1]/a;
  dFG[1] = FG[0]/a;
}

/* 2nd order ODE solver */
void odestep(double *FG, double a, double *pars, double da) {
  int i,N=2;
  double *FGtemp, *y1, *y2;
  FGtemp = (double*)malloc((size_t)(3*N*sizeof(double)));
  y1 = FGtemp+N;
  y2 = FGtemp+N;
```

```
  get_derivs(y1, FG, a, pars);
  for(i=0;i<N;i++) FGtemp[i] = FG[i] + da*y1[i];
  get_derivs(y2, FGtemp, a+da, pars);
  for(i=0;i<N;i++) FG[i] += da*(y1[i]+y2[i])/2.;

  free((char*)FGtemp);
}

int main(int argc, char **argv) {
  long ia;
  double a;
  double a_init = 0.01;
  long Na = 9901;
  double da;
  double pars[1], FG[2];

  sscanf(argv[1], "%lg", pars);
  da = (1.-a_init)/(Na-1);

  a=a_init;
  FG[0] = FG[1] = a_init;
  for(ia=0;ia<Na;ia++) {
    printf("%11.5lE %11.5lE %11.5lE\n", a, FG[0], FG[1]);
    odestep(FG,a,pars,da);
    a += da;
  }

  return(0);
}
```

This generates tables of a, $F_+(a)$, and $G_+(a)$:

```
$ gcc growth.c -lm -Wall -o growth.x
$ ./growth.x .3
1.00000E-02 1.00000E-02 1.00000E-02
1.01000E-02 1.01000E-02 1.01000E-02
1.02000E-02 1.02000E-02 1.02000E-02
```

...

```
1.00000E+00 3.99409E-01 7.78907E-01
```

Plots of the results for $\Omega_m = 0.1$, 0.3, and 1 are shown in Fig. 1.

(e) [4 points] *For each of these values of Ω_m, what is the ratio of the growth function at $z = 1$ to that at $z = 0$?*

Looking at the file outputs, we see that:

$$\frac{G_+(z=1)}{G_+(z=0)} = \frac{G_+(a=0.5)}{G_+(a=1)} = \begin{cases} \frac{0.4306}{0.5908} = 0.729 & \Omega_m = 0.1 \\ \frac{0.4766}{0.7789} = 0.612 & \Omega_m = 0.3 \\ \frac{0.5000}{1.0000} = 0.500 & \Omega_m = 1 \end{cases} . \tag{12}$$

Comment – *The answer to part (e) should explain why we use the growth of structure as a test for the composition of the Universe.*

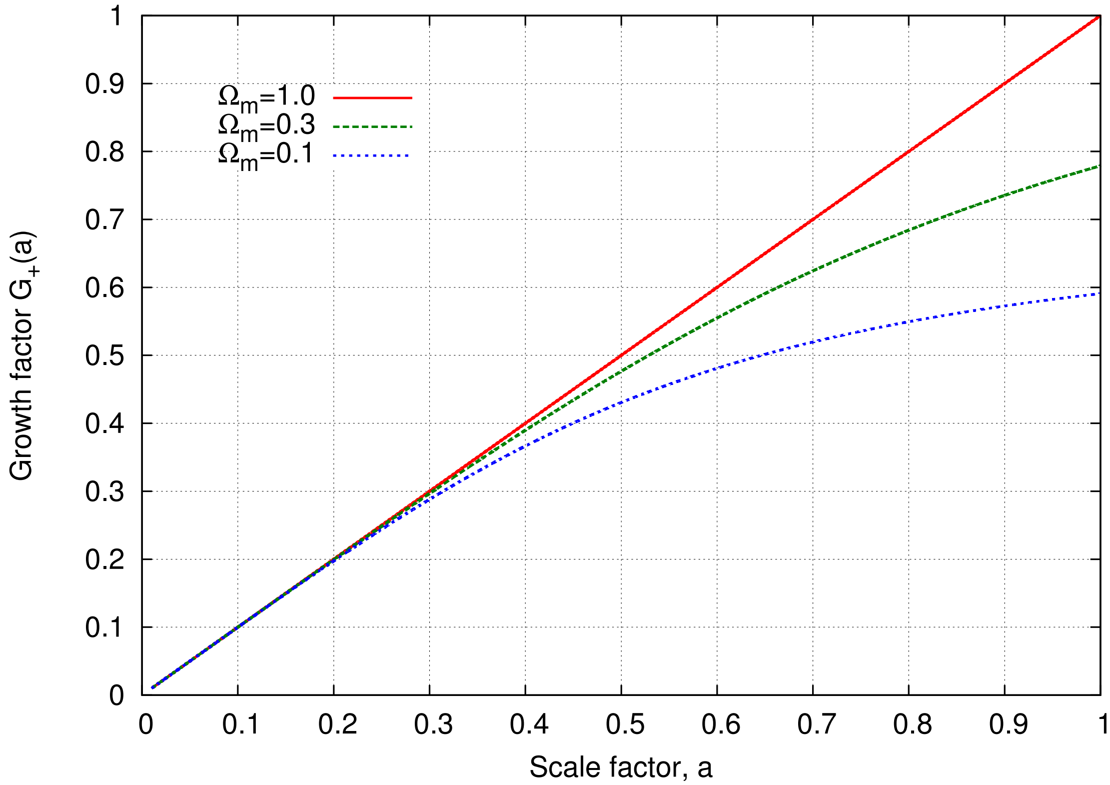

FIG. 1: The growth functions $G_+(a)$ for $\Omega_m = 0.1$, 0.3, and 1.