

Physics 6820 – Homework 6

(Dated: Due: October 25, 2019)

1. Gravitational wave detection by pulsar timing. [30 points]

Gravitational waves with periods of the order of years are expected to be emitted by the supermassive black holes in the centers of galaxies when they merge. These periods are too long for laser interferometers such as LIGO, or even planned space-based interferometers with “arms” millions of kilometers long (LISA). However, there are searches for these gravitational waves with pulsar timing.

For the sake of this problem, let’s imagine a plane gravitational wave traveling through our galaxy in the $+z$ direction and with the $+$ polarization: $h_+ = A \cos \Omega(t - z - t_0)$, $h_\times = 0$, where A is the strain amplitude and t_0 is the arrival time of a crest at Earth. We will place Earth at the origin and at rest in the coordinate frame in TT gauge, and place a pulsar at some distance L from the Earth at spatial coordinates $(L \sin \theta_\star \cos \phi_\star, L \sin \theta_\star \sin \phi_\star, L \cos \theta_\star)$. We will allow for any relation of L to Ω , i.e., in this problem do not assume anything about the numerical value of ΩL until the end.

The pulsar is rapidly rotating and through a still-unknown mechanism sends pulses of radio waves, one after each rotation period P . Each pulse travels toward the Earth, where it is received and assigned a time stamp. To a first approximation, these pulses arrive one after another, separated by time P , but there is a residual δt_{arr} created by the gravitational wave. The objective of this problem is to determine δt_{arr} in the idealized situation described above. You may work the entire problem to first order in A (or in the metric perturbation).

(a) [4 points] First take the unperturbed Minkowski metric. Show that the signal arriving at Earth at time t_{arr} left the pulsar at time $t_{\text{arr}} - L$, and describe the path $x^i(t)$ taken by the radio pulse.

(b) [3 points] Consider the path of a photon, $x^\alpha(\sigma)$, that is a null geodesic in the background Minkowski space, starting from σ_{start} and ending at σ_{end} . Now suppose that in the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the photon takes a perturbed path $x^\alpha(\sigma) + \delta x^\alpha(\sigma)$. Show that

$$0 = h_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + 2\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{d\delta x^\nu}{d\sigma}. \quad (1)$$

(c) [9 points] Now suppose that we consider the case of part (b), using $\sigma = t$ on the unperturbed path, and then integrate Eq. (1) over σ from the start to the end point. Show that the integral of the second term reduces to a boundary term in terms of displacements δx^α at the stars and end points. If we assume no perturbation to the start point $\delta x^\alpha(\text{start}) = 0$, and that at the end point only the arrival time is perturbed, $\delta t(\text{end}) = \delta t_{\text{arr}}$ but $\delta x^i(\text{end}) = 0$ (because “arrival” is always measured at Earth), then

$$\delta t_{\text{arr}} = \frac{1}{2} \int_{t_{\text{arr}} - L}^{t_{\text{arr}}} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu dt. \quad (2)$$

(d) [9 points] Plug into your result from part (c) the specific form of the metric perturbation and do the integral to find that

$$\delta t_{\text{arr}} = 2A \sin^2 \theta_\star \cos 2\phi_\star \frac{\sin[\Omega(1 + \cos \theta_\star) \frac{L}{2}]}{\Omega(1 + \cos \theta_\star)} \cos \left[\Omega \left(t_{\text{arr}} - \frac{1 + \cos \theta_\star}{2} L - t_0 \right) \right]. \quad (3)$$

[You will have to substitute in the unperturbed trajectory to get $z(t)$, and at the end use the sum-to-product trigonometric identity.]

In the next part of this problem, we will consider the *semi-amplitude* of δt_{arr} , i.e., half of the maximum minus the minimum:

$$\text{SemiAmp}[\delta t_{\text{arr}}] = \left| 2A \sin^2 \theta_\star \cos 2\phi_\star \frac{\sin[\Omega(1 + \cos \theta_\star) \frac{L}{2}]}{\Omega(1 + \cos \theta_\star)} \right|. \quad (4)$$

(e) [5 points] For a pulsar at $L = 100$ pc, a period of $2\pi/\Omega = 1$ yr, and a strain of $A = 10^{-15}$, make a plot of $\text{SemiAmp}[\delta t_{\text{arr}}]$ as a function of θ_\star for $0 \leq \theta_\star \leq \pi$, assuming zero longitude ($\phi_\star = 0$). Plot the vertical axis in an SI unit with prefix (s, ms, μ s, ns, ps ... as you see fit). You might need a lot of resolution on the x -axis!