Homework #3 – Solutions

1. Helium abundance. [10 points; 5 points each]
Using the arguments discussed in class, estimate how much the primordial helium abundance Y would change if:

- *There were a 4th species of neutrino;*
- *The baryon abundance η were doubled.*

4th species of neutrino – The key issue here is that neutrino decoupling occurs when

$$G_F^2 T^5 \sim H \sim g_\star^{1/2} G^{1/2} T^2 \tag{1}$$

(in units where $\hbar = c = k_B = 1$). This means that the neutrino decoupling temperature scales as

$$T_{\rm dec} \sim g_\star^{1/6}. \tag{2}$$

Adding a 4th neutrino means that $g_\star$ increases from its fiducial value of

$$g_\star(N_\nu = 3) = 2 + 4 \times \frac{7}{8} + 3 \times 2 \times \frac{7}{8} = 10.75 \tag{3}$$

to

$$g_\star(N_\nu = 4) = 2 + 4 \times \frac{7}{8} + 4 \times 2 \times \frac{7}{8} = 12.5, \tag{4}$$

so the neutrino decoupling temperature increases by a factor of

$$\frac{T_{\rm dec}(N_\nu = 4)}{T_{\rm edc}(N_\nu = 3)} = \left(\frac{12.5}{10.75}\right)^{1/6} = 1.025. \tag{5}$$

Since the neutron:proton ratio $X_n/(1 - X_n)$ at freezeout is proportional to $e^{-Q/T}$ (where $Q = 1.293$ MeV is the neutron-proton mass difference), we conclude that at freeze-out

$$\ln \frac{X_n^{\rm new}}{1 - X_n^{\rm new}} = \frac{1}{1.025} \ln \frac{X_n^{\rm old}}{1 - X_n^{\rm old}}. \tag{6}$$

Setting $X_n^{\rm old} = 0.15$, we find that the right-hand side is -1.6922, and then $X_n^{\rm new} = e^{-1.6922}/(1 + e^{-1.6922}) = 0.1555$. That is, there is a change in the neutron fraction

$$\frac{\Delta X_n}{X_n} = \frac{0.0055}{0.15} = 0.036. \tag{7}$$

Therefore, we expect the final helium abundance to change in the same way (since it is set by the number of neutrons):

$$\frac{\Delta Y}{Y} = 0.036. \tag{8}$$

This means $\Delta Y = 0.036Y = 0.036 \times 0.24 = +0.0086$.

Equation (13) of Cyburt et al. (2015) [arXiv:1505.01076] gives $\Delta Y = +0.0119$, so our simple estimate was remarkably close. In general, we see that adding another neutrino species increases the amount of helium.

Doubling the baryon abundance – Here the key idea is that the baryons can combine to make deuterium (and then heavier nuclei) a little bit earlier in the history of the Universe, because the equilibrium arguments favor D over $p+n$ at a higher temperature if η is increased. We recall from Lecture VII that the equilibrium relation deuterium was

$$\frac{X_D}{X_n X_p} = 3\sqrt{2} \left(\frac{2\pi}{m_N T}\right)^{3/2} n_b e^{B_D/T}. \tag{9}$$

If we fix the left-hand side, we can see that the temperature T at which deuterium formation becomes favorable satisfies

$$\text{const} = n_b T^{-3/2} e^{B_D/T} \propto \eta T^{3/2} e^{B_D/T} \quad \to \quad \text{const} = \ln \eta + \frac{3}{2} \ln T + \frac{B_D}{T}, \tag{10}$$

where we used the fact that $n_b \propto \eta T^3$. If we change the baryon-to-photon ratio η, we can see that the temperature at which deuterium formation becomes favorable changes by

$$0 = \Delta \ln \eta + \left(\frac{3}{2} - \frac{B_D}{T} \right) \frac{\Delta T}{T}, \tag{11}$$

or

$$\frac{\Delta T}{T} = \frac{\Delta \ln \eta}{-\frac{3}{2} + \frac{B_D}{T}}. \tag{12}$$

Plugging in $\Delta \ln \eta = \ln 2$ (changing the baryon abundance by a factor of 2), $B_D = 2.22$ MeV, and $T = 80$ keV (the temperature we argued is appropriate for nucleosynthesis), we find

$$\frac{\Delta T}{T} = 0.026. \tag{13}$$

Since $T \propto t^{-1/2}$, this implies the time for BBN changes by

$$\frac{\Delta t}{t} = -2 \times 0.026 = -0.053 \quad \Delta t = -0.053 \times 200\,\text{s} = -10.6\,\text{s}. \tag{14}$$

At this earlier time, there will be more neutrons left over due to the finite neutron lifetime. This implies

$$\frac{\Delta X_n}{X_n} = -\frac{\Delta t}{\tau_n} = -\frac{-10.6\,\text{s}}{880\,\text{s}} = 0.012. \tag{15}$$

This implies a change to the helium abundance

$$\Delta Y = \frac{\Delta Y}{Y} Y = \frac{\Delta X_n}{X_n} Y = 0.012 \times 0.24 = +0.0029. \tag{16}$$

Equation (13) of Cyburt et al. (2015) [arXiv:1505.01076] gives $\Delta Y = +0.0068$. The real change is about a factor of 2 larger than our simple estimate, primarily because this is not a purely equilibrium problem. At higher baryon abundance, not only does formation of D happen earlier but also the reactions $D+p$, $D+n$, etc. happen earlier (they have another factor of baryon density: we should hold fixed $n_b^2 e^{B_D/T}$ rather than $n_b e^{B_D/T}$).

2. Short problems. [20 points; 5 points each]

(a) *Practice with magnitudes: Suppose someone were a distance D from Earth, and shone a laser pointer directly at you (the observer). To an order of magnitude, what is the maximum distance D at which your eye could see the laser? What if we used the Sloan Digital Sky Survey (which scanned much of the Northern Hemisphere sky to a depth of 22nd magnitude) instead of your eye?*

Let's suppose the laser has a power of $P = 2$ mW (the regulatory limit in the US is 5 mW), a wavelength of $\lambda = 532$ nm (the most common green laser), an exit pupil of $s = 1$ mm diameter, and is diffraction-limited. The beam diffracts into an angle of width $\theta = \lambda/s$, so at distance D the power P is spread over an area $A = (D\lambda/s)^2$. The spectral flux, averaged over the visible bandpass $\Delta\nu$ (the laser is monochromatic but the eye's response is broadband), is

$$\bar{F}_\nu = \frac{P}{(D\lambda/s)^2\,\Delta\nu}. \tag{17}$$

Solving for the distance gives

$$D = \frac{s}{\lambda} \sqrt{\frac{P}{\bar{F}_\nu\,\Delta\nu}}. \tag{18}$$

Now the visible bandpass is 400-700 nm, which corresponds to frequencies $\nu = c/\lambda$ of $(4.3 - 7.5) \times 10^{14}$ Hz, so we take $\Delta\nu = 3.2 \times 10^{14}$ Hz. The eye can see a 6th magnitude star, which corresponds to a flux of

$$\bar{F}_\nu = 3.631 \times 10^{-23} \frac{\mathrm{W}}{\mathrm{m^2\,Hz}}\, 10^{-0.4(6)} = 1.4 \times 10^{-25} \frac{\mathrm{W}}{\mathrm{m^2\,Hz}}. \tag{19}$$

Plugging these numbers in gives

$$D = 1.2 \times 10^7\,\mathrm{m} = 12,000\,\mathrm{km}. \tag{20}$$

For the case of the Sloan Digital Sky Survey, one uses 22nd magnitude instead of 6th. That's a factor of $10^{0.4(6-22)} = 4 \times 10^{-7}$ in brightness or, taking the inverse square root, $(4 \times 10^{-7})^{-1/2} = 1600$ in distance. This means that SDSS could see the laser pointer from a distance of

$$D = 2 \times 10^{10}\,\mathrm{m} = 2 \times 10^7\,\mathrm{km}. \tag{21}$$

(Yes, that's 50 times the distance to the Moon!)

(b) *Densities*: *The Sun orbits the Milky Way at about 220 km/s, at a distance of about 8.6 kpc from the center. What is the approximate mass interior to the Sun's orbit, and the mean density (in kg/m^3) of that region? [For the sake of this problem, you can neglect the fact that the Milky Way's mass distribution is not spherical.] Compare this to the cosmological mean density of matter; what is the ratio $\rho_{\text{Milky Way}}/\bar{\rho}$?*

We use Newtonian mechanics for circular motion: if v is the Sun's velocity, R is the radius of the orbit, and M is the interior mass, then

$$\frac{v^2}{R} = \frac{GM}{R^2} \quad \rightarrow \quad M = \frac{v^2 R}{G} = \frac{(2.2 \times 10^5\,\mathrm{m/s})^2 (8.6\,\mathrm{kpc})(3.086 \times 10^{19}\,\mathrm{m/kpc})}{6.672 \times 10^{-11}\,\mathrm{m^3/kg/s^2}} = 1.9 \times 10^{41}\,\mathrm{kg}. \tag{22}$$

The density is

$$\rho_{\mathrm{MW}} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.9 \times 10^{41}\,\mathrm{kg}}{\frac{4}{3}\pi \times [(8.6\,\mathrm{kpc})(3.086 \times 10^{19}\,\mathrm{m/kpc}]^3} = 2.4 \times 10^{-21}\,\mathrm{kg/m^3}. \tag{23}$$

Compare to the cosmic mean density

$$\bar{\rho} = 1.88 \times 10^{-26} \Omega_{\mathrm{m}} h^2\,\mathrm{kg/m^3} = 1.88 \times 10^{-26} (0.12)\,\mathrm{kg/m^3} = 2.3 \times 10^{-27}\,\mathrm{kg/m^3}. \tag{24}$$

The Milky Way as a whole may not be dense, but it is 10^6 times denser than an average part of the universe!

(c) *Intergalactic gas*: *We believe that the baryonic matter in the Universe today is mostly ionized intergalactic hydrogen. Under this assumption, and the values given in the notes for $\Omega_b h^2$, compute the number density n_e of intergalactic electrons. What is the plasma frequency ν_p that this corresponds to, and what is the physical interpretation of this number?*

If the baryonic matter is all in ionized hydrogen, then the electron density n_e is the baryon density ρ_b divided by the mass of hydrogen:

$$n_e = \frac{\rho_b}{m_{\mathrm{H}}} = \frac{1.88 \times 10^{-26} \Omega_{\mathrm{b}} h^2\,\mathrm{kg/m^3}}{1.67 \times 10^{-27}\,\mathrm{kg}} = 11.3 \Omega_{\mathrm{b}} h^2\,\mathrm{m^{-3}} = 11.3(0.022)\,\mathrm{m^{-3}} = 0.25\,\mathrm{m^{-3}}. \tag{25}$$

The plasma frequency is related to n_e via fundamental constants:

$$\nu_p = \frac{1}{2\pi} \sqrt{\frac{q_e^2 n_e}{\epsilon_0 m_e}} = 4.5\,\mathrm{Hz}. \tag{26}$$

This is the minimum frequency of an electromagnetic wave that can exist in a plasma.

(d) *Scattering of intergalactic photons*: *The electron density obtained in (c) implies some probability per unit path length dP/dx of scattering due to Thomson scattering. What is this probability in units of Gpc^{-1}? Thomson scattering will make distant objects appear dimmer. Do you think this is (i) a major effect on $D_L(z)$ inferred from Type Ia supernovae; (ii) a small effect that we might have to correct with future telescopes that will observe 1–2 orders of magnitude more supernovae than we are currently using; or (iii) completely negligible for any conceivable experiment?*

The probability of scattering per unit length is

$$\frac{dP}{dx} = n_e \sigma_{\rm T} = (0.25\,{\rm m}^{-3})(6.65\times 10^{-29}\,{\rm m}^2) = 1.7\times 10^{-29}\,{\rm m}^{-1} = 5\times 10^{-4}\,{\rm Gpc}^{-1}. \tag{27}$$

This effect makes distant Type Ia supernovae (or anything else) appear fainter, but at typical distances of say 3 Gpc for a cosmological Type Ia the dimming effect is only 0.15%. The real effect is a bit bigger because the Universe was denser in the past (you will do the integral on the next homework set). Since supernovae have an intrinsic dispersion of about 15% in luminosity, this will become important if we use 10^4 supernovae for cosmology ($15\%/\sqrt{10^4} = 0.15\%$). So it isn't important yet, but in the 2020s we may have samples of this size.