Lecture X: Initial conditions and super-horizon evolution

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I. INTRODUCTION

We will now try to write a description of the initial conditions of the Universe and their evolution outside the horizon. This complements the approach we have been using to describe matter density perturbations inside the horizon. At the end, we will put these two pieces of information together, and derive a mapping from initial conditions to matter fluctuations at low redshift.

In this lecture, $c = 1$.

II. PERTURBATIONS IN THE EARLY UNIVERSE

Our first step here will be to consider the types of perturbations that may exist outside the horizon. By "outside the horizon," we mean to consider perturbations where light can travel only a small fraction of a wavelength in the time it takes the Universe to expand. The physical wavelength of a perturbation is $2\pi a/k$ (and so the time for light to travel a wavelength is $2\pi a/k$), and the time for the Universe to expand is $1/H$. Thus this condition is satisfied if

$$k \ll aH. \tag{1}$$

We note that for the radiation-dominated era, $aH \propto t^{1/2}/t \propto t^{-1/2}$. We thus conclude that, in the context of the radiation-dominated early Universe,

$$\lim_{t\to 0^+} aH = \infty \tag{2}$$

and thus if we look early enough in the Universe, *all perturbations are outside the horizon.* As the Universe expands, aH drops, and perturbations may "enter the horizon" (when $k \sim aH$).

A. Adiabatic perturbations

Let us now consider the possible types of perturbations in the radiation-dominated Universe. The simplest possibility – and, it turns out, the one chosen by the real Universe – is an *adiabatic perturbation.* This is a perturbation where the spatial geometry of the early Universe is not described by Euclidean 3-space, but by a perturbed spatial geometry:

$$ds^2 = -dt^2 + [a(t)]^2 \left\{ [1 + 2\zeta(\mathbf{r})] \sum_{i=1}^{3} (dx^i)^2 + \sum_{ij} 2h_{ij}(\mathbf{r})\, dx^i\, dx^j \right\}. \tag{3}$$

Here we have separated the 6 possible perturbations into an overall scaling ζ and a symmetric traceless part h_{ij} (with $\sum_{i=1}^{3} h_{ii} = 0$). It turns out that by appropriate choice of the 3 spatial coordinates, we can impose 3 conditions on h_{ij}; normally we choose h_{ij} to be divergenceless,

$$\sum_{j=1}^{3} \frac{\partial h_{ij}}{\partial x^j} = 0. \tag{4}$$

This reduces the number of degrees of freedom of h to 2 (6 for a symmetric matrix, minus 1 for traceless, and minus 3 for divergenceless). The perturbation variable $\zeta(\mathbf{r})$ is called the *curvature perturbation*, since it corresponds to curvature of the spatial geometry in the early Universe.

As long as the perturbation is outside the horizon ($k \ll aH$), each portion of this Universe looks flat. Therefore, $a(t)$ will obey the standard Friedmann equation, the spatial geometry will remain unchanged, and processes such as the evolution of $g_\star$ occur in the same way in every part of the Universe. Indeed, if the perturbation is outside the horizon, two observers A and B located in different parts of the Universe but at the same time t will measure the same physical conditions, and neither can detect the presence of the perturbation.

If we go to Fourier space and place $\mathbf{k}$ on the z-axis, we can see which portions of Eq. (3) correspond to scalar, vector, and tensor perturbations. Clearly ζ is a scalar, and this is the part that will lead to density pertubations. For h, we see that the divergenceless condition says $ik\tilde{h}_{iz} = 0$, or $\tilde{h}_{iz} = 0$; thus the most general possible form of $\tilde{h}$ is

$$\tilde{h}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

These perturbations are tensor perturbations since under a rotation by φ around the z-axis, $h_+ \pm ih_\times$ picks up a factor of $e^{\mp 2i\varphi}$. We will see later that they correspond to primordial gravitational waves.

It can be shown that for a Universe described only by GR and by blackbody radiation (generalized to include the various heavy particles), Eq. (3) is sufficiently general to describe all non-decaying modes.

B. Isocurvature perturbations

However, the real Universe contains baryons and dark matter as well, and their origin is not well understood. We do know, however, that in the radiation-dominated era, they contribute little to the overall energy budget of the universe, and therefore do not determine its geometry or expansion history. Therefore, we could consider a type of perturbation where the radiation and metric of the early Universe are unperturbed, but the density of baryons or of dark matter has a perturbation:

$$\delta_{\rm b} \neq 0 \quad \text{or} \quad \delta_{\rm dm} \neq 0. \tag{6}$$

Such a perturbation has no dynamical effect early on, but as the Universe becomes matter-dominated, these perturbations will start to evolve. Such a type of perturbation is called an *isocurvature perturbation*. Isocurvature perturbations occur in theories where the dark matter is present during inflation (e.g., some axion models). A general perturbation is a superposition of adiabatic and isocurvature perturbations.

Isocurvature perturbations can be evolved using the same codes that evolve adiabatic perturbations, however we won't study them in this class since we know that the main type of perturbation in the early Universe is adiabatic. It is possible that there is a small admixture of isocurvature perturbations, and searching for these is one of the goals of CMB observations.

III. MAPPING OF PRIMORDIAL PERTURBATIONS TO LATE-TIME DENSITY PERTURBATIONS: LARGE SCALES

You are probably wondering how an initial perturbation in the geometry of the Universe ζ maps into a density perturbation δ_m at late times. We will address this first on large scales, where "large" means that the perturbation entered the horizon after recombination (and hence also after matter-radiation equality). That is, we want $k \ll 1/\eta_{\rm rec}$. It turns out that through clever reasoning, we can figure out how to do this mapping without having to solve lots of GR equations. (There will be a few places where you will have to just trust me on some subtle points.) We will also ignore dark energy in this section, because the entire mapping occurs before the dark energy becomes significant.

A. Generalities

Let's begin with some generalities about the curvature perturbation. Recall Eq. (3):

$$ds^2 = -dt^2 + [a(t)]^2[1 + 2\zeta(\mathbf{r})] \sum_{i=1}^{3} (dx^i)^2. \tag{7}$$

You can immediately see from this equation that if we have a constant offset in ζ doesn't do anything except re-scale the coordinate system. Therefore the density δ_m is not related to ζ itself, but probably to some derivative of ζ. Since we want a scalar, the answer can't be $\nabla\zeta$, but δ_m could be proportional to $\nabla^2\zeta$. This will turn out to be correct, and it also turns out you can get the constant of proportionality by a simple mathematical trick without doing full GR calculations.

B. Mapping of a closed universe to a perturbed flat universe

The trick is to use a particular coordinate transformation to represent a closed universe (3-sphere) with a large radius of curvature R (small K) as a perturbation to a flat universe. We already know how to solve the density evolution of a closed universe, so if we can find $\nabla^2\zeta$ for the closed universe we can find the constant of proportionality to δ_m. Recall that the 3D line element for a 3-sphere of radius R is

$$ds_3^2 = d\chi^2 + R^2 \sin^2 \frac{\chi}{R}\,(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{8}$$

This doesn't look like Eq. (7), but it can be put in that form with a coordinate transformation. Define

$$\tilde{r} = 2R \tan\frac{\chi}{2R} \quad \leftrightarrow \quad \chi = 2R \tan^{-1}\frac{\tilde{r}}{2R}, \tag{9}$$

which runs from $\tilde{r} = 0$ at the origin to $\tilde{r} = \infty$ at the antipode ($\chi = \pi R$). Near the origin, $\tilde{r} \approx \chi$. (You may recognize this transformation as *stereographic projection*.) One can show using trigonometric identities that

$$d\chi = \frac{1}{1 + \tilde{r}^2/4R^2}\, d\tilde{r} \quad \text{and} \quad R\sin\frac{\chi}{R} = 2R\sin\frac{\chi}{2R}\cos\frac{\chi}{2R} = \frac{2R\tan\frac{\chi}{2R}}{1+\tan^2\frac{\chi}{2R}} = \frac{\tilde{r}}{1+\tilde{r}^2/4R^2}, \tag{10}$$

so the line element can be written as

$$ds_3^2 = \frac{1}{(1+\tilde{r}^2/4R^2)^2}\left[d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{11}$$

The piece in brackets is the line element of Euclidean space, and therefore in this description a closed universe looks like a flat universe with a curvature perturbation. To lowest order in $\tilde{r}$, the correspondence is

$$1 + 2\zeta = \frac{1}{(1+\tilde{r}^2/4R^2)^2} \quad \rightarrow \quad \zeta = -\frac{\tilde{r}^2}{4R^2} = -\frac{1}{4}K\tilde{r}^2. \tag{12}$$

By taking the Laplacian, we see that

$$\nabla^2\zeta = -\frac{3}{2}K. \tag{13}$$

We thus conclude that – to linear order in K – a closed universe is a perturbation to a flat universe, with spatial curvature corresponding to $-\frac{2}{3}\nabla^2\zeta$.

C. Density evolution in a closed universe

Our next step will be to follow the density evolution in a closed universe, to relate δ_m to K and hence ultimately to ζ. Since we are interested in linear perturbations, we will solve the Friedmann equations to order K. We will work at early enough times that dark energy is unimportant.

First, recall the Friedmann equation in the form

$$H^2 + \frac{K}{a^2} = \frac{8}{3}\pi G\rho = \frac{8\pi G\rho_0}{3a^3}. \tag{14}$$

Solving for H gives

$$\frac{da/dt}{a} = H = \left[\frac{8\pi G\rho_0}{3a^3} - \frac{K}{a^2}\right]^{1/2}. \tag{15}$$

Let's solve for dt:

$$\begin{aligned} dt &= \left[\frac{8\pi G\rho_0}{3a^3} - \frac{K}{a^2}\right]^{-1/2}\frac{da}{a} \\ &= \left[\frac{8\pi G\rho_0}{3a^3}\right]^{-1/2}\left[1 - \frac{3Ka}{8\pi G\rho_0}\right]^{-1/2}\frac{da}{a} \\ &\approx \frac{\sqrt{3}}{\sqrt{8\pi G\rho_0}}\left[1 + \frac{3Ka}{16\pi G\rho_0}\right]a^{1/2}\,da, \end{aligned} \tag{16}$$

where the approximation is the first order expansion in K. Integrating, and using the initial condition $t = 0$ at $a = 0$, gives

$$t \approx \frac{\sqrt{3}}{\sqrt{8\pi G\rho_0}}\left[\frac{2}{3}a^{3/2} + \frac{3K}{40\pi G\rho_0}a^{5/2}\right] = \frac{\sqrt{3}}{\sqrt{8\pi G\rho_0}}\frac{2}{3}a^{3/2}\left[1 + \frac{9K}{80\pi G\rho_0}a\right] = \frac{1}{\sqrt{6\pi G\rho}}\left[1 + \frac{9K}{80\pi G\rho_0}a\right]. \tag{17}$$

We may now solve for ρ:

$$\rho \approx \frac{1}{6\pi G t^2}\left[1 + \frac{9K}{80\pi G\rho_0}a\right]^2 \approx \frac{1}{6\pi G t^2}\left[1 + \frac{9K}{40\pi G\rho_0}a\right]. \tag{18}$$

The density perturbation δ_m for the closed universe relative to the "unperturbed" flat universe is then

$$\delta_m = \frac{9K}{40\pi G\rho_0}a = \frac{9(-\frac{2}{3}\nabla^2\zeta)}{40\pi G\rho_0}a = -\frac{3\nabla^2\zeta}{20\pi G\rho_0}a = -\frac{2\nabla^2\zeta}{5\Omega_m H_0^2}a, \tag{19}$$

where we used $\rho_0 = 3\Omega_m H_0^2/(8\pi G)$.

D. Matching to the growing solution

You will note that the solution for δ_m is proportional to a: it is the growing solution. In the case where later on the dark energy becomes important, we should replace a with $G_+(a)$, where the normalization of the growing solution is such that $G_+(a) = a$ before the dark energy becomes important, i.e., $a \ll (\Omega_m/\Omega_\Lambda)^{1/3}$ for the case of a cosmological constant. We thus have our final solution for the density perturbation:

$$\delta_m(\mathbf{r}, a) = -\frac{2\nabla^2\zeta(\mathbf{r})}{5\Omega_m H_0^2}G_+(a), \tag{20}$$

for modes that enter the horizon during the matter-dominated era. In Fourier space, this is

$$\tilde{\delta}_m(\mathbf{k}, a) = \frac{2k^2}{5\Omega_m H_0^2}\tilde{\zeta}(\mathbf{k})G_+(a). \tag{21}$$

Recall that this is only for large-scale modes with $k \ll 1/\eta_{\rm rec}$. For shorter-wavelength modes, the growth function is still correct, but radiation pressure and baryonic effects will be important. We therefore write in general:

$$\tilde{\delta}_m(\mathbf{k}, a) = \frac{2k^2}{5\Omega_m H_0^2}\tilde{\zeta}(\mathbf{k})T(k)G_+(a), \tag{22}$$

where $T(k)$ is called the *matter transfer function*. It satisfies $T(k) \to 1$ on large scales $k \ll 1/\eta_{\rm rec}$, and it encapsulates the radiation physics. We will explore it in Lecture XI.

IV. POWER SPECTRA

The primordial curvature perturbation $\zeta(\mathbf{r})$ has some power spectrum $P_\zeta(k)$, generally known as the *primordial power spectrum*. The primordial power spectrum is clearly telling us something about the very early Universe (probably inflation). The simplest possibility for the primordial power spectrum would be for it to be scale-invariant,

$$\frac{k^3 P_\zeta(k)}{2\pi^2} = A_s, \tag{23}$$

where A_s is a constant independent of k. This is a type of initial condition where the geometry of the universe initially had perturbations of variance A_s (standard deviation $\sqrt{A_s}$) on all scales, and it would be expected if the perturbations were generated by a scale invariant process such as quantum fluctuations during exponential expansion (every e-fold of expansion looks like every previous one). However, observations show that Eq. (23) is not an accurate description. The simplest description that fits is a power law,

$$\frac{k^3 P_\zeta(k)}{2\pi^2} = A_s\left(\frac{k}{k_\star}\right)^{n_s - 1}, \tag{24}$$

where n_s is called the *primordial spectral index* (by convention, 1 corresponds to scale invariance) and $k_\star$ is a reference wavenumber that we use to set the normalization A_s. The *Planck* CMB mission, in its 2018 analysis, reports $A_s = (2.10 \pm 0.03) \times 10^{-9}$ (using a reference point of $k_\star = 0.05$ Mpc^{-1}) and $n_s = 0.9649 \pm 0.0042$.