Homework #4

Due: Friday, March 29, 2019

1. Growth function in ΛCDM. [30 points]

In this problem, we will solve for the growth function as a function of scale factor, $G_+(a)$, in ΛCDM cosmology. We will assume the Universe to be flat, $\Omega_\Lambda + \Omega_m = 1$, and work at scale factors large enough for the radiation to be negligible, $a \gg a_{\rm eq}$.

Recall that the growth function satisfies

$$\frac{d^2G_+}{dt^2} + 2H\frac{dG_+}{dt} + 4\pi G\bar{\rho}_m G_+ = 0. \tag{1}$$

We will appropriately re-arrange this equation, and then do a numerical solution.

(a) [6 points] Let's define the new variable

$$F_+ = \frac{1}{H}\frac{dG_+}{dt}. \tag{2}$$

$da/dt = aH$

Show that G_+ and F_+ satisfy the 1st-order system of differential equations:

$$\frac{dG_+}{da} = \frac{F_+}{a} \quad \text{and} \quad \frac{dF_+}{da} = -\frac{dH}{da}\frac{F_+}{H} - 2\frac{F_+}{a} + \frac{4\pi G\bar{\rho}_m}{aH^2}G_+. \tag{3}$$

(b) [6 points] For the ΛCDM cosmology, show that the differential equation for F_+ can be reduced to

$$\frac{dF_+}{da} = -\frac{4\Omega_\Lambda + \Omega_m a^{-3}}{2(\Omega_\Lambda + \Omega_m a^{-3})}\frac{F_+}{a} + \frac{3\Omega_m a^{-3}}{2(\Omega_\Lambda + \Omega_m a^{-3})}\frac{G_+}{a}. \tag{4}$$

(c) [4 points] Using your knowledge of the growth function in the matter-dominated era, explain why we can initialize the integration of the growth function ODEs at some small value of a ($a_{\rm init}$) with

$$G_+(a_{\rm init}) = F_+(a_{\rm init}) = a_{\rm init}. \tag{5}$$

(d) [10 points] Numerically integrate the aforementioned ODEs for $\Omega_m = 0.1$, 0.3, and 1.0. Make plots of the results.

(e) [4 points] For each of these values of Ω_m, what is the ratio of the growth function at $z = 1$ to that at $z = 0$?

Comment – The answer to part (e) should explain why we use the growth of structure as a test for the composition of the Universe.