Lecture XIII: Gravitational waves: generation and propagation

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I. OVERVIEW

We are now ready to consider gravitational waves, again in the weak field regime. This is material from §§9.3 and 9.1, but I will present it a bit differently – we will start from linearized gravity and consider what happens far from a region containing matter. Most of this lecture will be in 3+1 notation, unless we specifically invoke a Greek index.

II. THE FAR FIELD OF AN OSCILLATING SOURCE IN LINEARIZED GRAVITY

Let's now suppose that we have a region containing matter whose stress-energy tensor has an oscillatory component:

$$T^{\alpha\beta}(x^j,t) = S^{\alpha\beta}(x^j)\mathrm{e}^{-\mathrm{i}\Omega t}, \tag{1}$$

with $\Omega > 0$ (it is important here that $\Omega \neq 0$; because only the real part will be taken to have physical significance, we can restrict to positive frequencies). By the principle of superposition, in linearized gravity, we can consider the stationary and dynamic parts of the stress-energy tensor separately. If the stress-energy tensor of the source has higher harmonics (e.g., a binary star on an eccentric orbit) then we can consider each frequency separately.

A. The basic equations, and regularization of the singularity

We start from the equation of linearized gravity,

$$\Box\bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta}, \tag{2}$$

or – with time written separately –

$$(\nabla^2 - \partial_t^2)\bar{h}^{\alpha\beta}(x^j,t) = -16\pi S^{\alpha\beta}(x^j)\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{3}$$

Now since the right-hand side is oscillatory, we expect the left-hand side to have the same oscillatory time dependence. Thus we might expect to write:

$$(\nabla^2 + \Omega^2)\bar{h}^{\alpha\beta}(x^j,t) = -16\pi S^{\alpha\beta}(x^j)\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{4}$$

Going to Fourier space using the 3D Fourier transform rule:

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\tilde{f}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \tilde{f}(\boldsymbol{k}) = \int \boldsymbol{x}\,f(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5}$$

we find

$$(-k^2 + \Omega^2)\tilde{\bar{h}}^{\alpha\beta}(\boldsymbol{k},t) = -16\pi\tilde{S}^{\alpha\beta}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{6}$$

The problem with this is that it leads to a singularity if $k = \Omega$ (where k is the magnitude of $\boldsymbol{k}$); the basic reason is that at $k = \Omega$, there can be a propagating wave in the metric even in the absence of any matter. Such a wave is called a *gravitational wave*.

The correct way to handle the $k = \Omega$ singularity in Eq. (6) depends on the physical situation. However, the usual case of interest is that one starts from a situation where there were no gravitational waves in the distant past. This can be mimicked by replacing $\Omega \to \Omega + \mathrm{i}\epsilon$ in Eq. (1), so that the oscillatory part has an extremely small $\propto \mathrm{e}^{\epsilon t}$ dependence. With this replacement, and taking $\epsilon \to 0^+$, we have

$$[-k^2 + (\Omega + \mathrm{i}\epsilon)^2]\tilde{\bar{h}}^{\alpha\beta}(\boldsymbol{k}) = -16\pi\tilde{S}^{\alpha\beta}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{7}$$

(We only keep ϵ on the left-hand side since this is where it will matter.) Then:

$$\tilde{\bar{h}}^{\alpha\beta}(\boldsymbol{k},t) = \frac{16\pi}{k^2 - (\Omega + \mathrm{i}\epsilon)^2}\tilde{S}^{\alpha\beta}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{8}$$

This is the formal solution for gravitational wave generation from any distribution of stress-energy that has oscillatory time dependence. Our next step is to make the source compact in space, which is the most relevant case in astrophysics.

B. A compact source

Let us now suppose that we are far away from a source contained within a small region of space (radius R). This means that the function $\tilde{S}^{\alpha\beta}(\boldsymbol{k})$ is a smooth function, varying on a scale $\Delta\boldsymbol{k} \sim 1/R$. We may write Eq. (8) in real space again:

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \int \frac{16\pi}{k^2-(\Omega+\mathrm{i}\epsilon)^2}\tilde{S}^{\alpha\beta}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}. \tag{9}$$

If we write $\boldsymbol{x} = r\hat{\boldsymbol{n}}$ and $\boldsymbol{k} = k\hat{\boldsymbol{k}}$ (3-vector equations!) then

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \int_0^\infty \int_{S^2} \frac{16\pi}{k^2-(\Omega+\mathrm{i}\epsilon)^2}\tilde{S}^{\alpha\beta}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{e}^{\mathrm{i}kr\mu}\,\frac{\mathrm{d}^2\hat{\boldsymbol{k}}\,k^2\,\mathrm{d}k}{(2\pi)^3}, \tag{10}$$

where $\mu = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}$ is the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{x}$, and S^2 is the unit sphere.

To simplify this, I will use the fact that for a smooth function f on the sphere, in the limit of large kr, we have

$$\begin{aligned}
\int_{S^2} f(\hat{\boldsymbol{k}})\,\mathrm{e}^{\mathrm{i}kr\mu}\,\mathrm{d}^2\hat{\boldsymbol{k}} &\approx \int_0^{2\pi}\int_{-1}^{1} f(\hat{\boldsymbol{k}})\,\mathrm{e}^{\mathrm{i}kr\mu}\,\mathrm{d}\mu\,\mathrm{d}\psi \\
&= \int_0^{2\pi}\frac{1}{\mathrm{i}kr}\int_{-1}^{1} f(\hat{\boldsymbol{k}})\,\frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{e}^{\mathrm{i}kr\mu}\,\mathrm{d}\mu\,\mathrm{d}\psi \\
&= \int_0^{2\pi}\left[\frac{1}{\mathrm{i}kr}f(\hat{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}kr\mu}|_{\mu=-1}^{1} - \frac{1}{\mathrm{i}kr}\int_{-1}^{1}\frac{\mathrm{d}f(\hat{\boldsymbol{k}})}{\mathrm{d}\mu}\mathrm{e}^{\mathrm{i}kr\mu}\,\mathrm{d}\mu\right]\mathrm{d}\psi \\
&\approx \int_0^{2\pi}\frac{1}{\mathrm{i}kr}f(\hat{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}kr\mu}|_{\mu=-1}^{1}\,\mathrm{d}\psi \\
&= \frac{2\pi}{\mathrm{i}kr}[f(\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}kr} - f(-\hat{\boldsymbol{n}})\mathrm{e}^{-\mathrm{i}kr}],
\end{aligned} \tag{11}$$

where we set ψ to be the longitude of $\hat{\boldsymbol{k}}$ and used the fact that for large enough r, $1/(\mathrm{i}kr)\times\mathrm{d}f(\hat{\boldsymbol{k}})/\mathrm{d}\mu$ can be neglected compared to $f(\hat{\boldsymbol{k}})$ (the integration by parts could be repeated to generate a power series in $1/r$). We thus turn Eq. (10) into

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \frac{4}{\pi\mathrm{i}r}\int_0^\infty \frac{1}{k^2-(\Omega+\mathrm{i}\epsilon)^2}[\tilde{S}^{\alpha\beta}(k\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}kr} - \tilde{S}^{\alpha\beta}(-k\hat{\boldsymbol{n}})\mathrm{e}^{-\mathrm{i}kr}]\mathrm{e}^{-\mathrm{i}\Omega t}\,k\,\mathrm{d}k. \tag{12}$$

We may use the fact that the integrand is even to extend the range of integration:

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \frac{2}{\pi\mathrm{i}r}\mathrm{e}^{-\mathrm{i}\Omega t}\int_{-\infty}^\infty \frac{1}{k^2-(\Omega+\mathrm{i}\epsilon)^2}[\tilde{S}^{\alpha\beta}(k\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}kr} - \tilde{S}^{\alpha\beta}(-k\hat{\boldsymbol{n}})\mathrm{e}^{-\mathrm{i}kr}]\,k\,\mathrm{d}k, \tag{13}$$

and then the second term is redundant:

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \frac{2}{\pi\mathrm{i}r}\mathrm{e}^{-\mathrm{i}\Omega t}\int_{-\infty}^\infty \frac{2k}{k^2-(\Omega+\mathrm{i}\epsilon)^2}\tilde{S}^{\alpha\beta}(k\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}k. \tag{14}$$

The easiest way to do this integral is by contour integration: the integral is taken to be along a path $\mathcal{C}$ from $-\infty$ to ∞ along the real line, but we imagine the real line to be part of the complex plane. Since there is a factor of $\mathrm{e}^{\mathrm{i}kr}$ that goes exponentially to zero if k has a positive imaginary part, we close the contour in the upper half plane. Now as long as a function is analytic [1], its integral does not depend on the path chosen; we could take the contour and close it down to a point and then the integral is zero. However, the function written here fails to be analytic at $k = \pm(\Omega+\mathrm{i}\epsilon)$. The failure at $-(\Omega+\mathrm{i}\epsilon)$ is not important since it is in the lower half plane and does not interfere with our ability to shrink the contour to a point. However the failure at $\Omega+\mathrm{i}\epsilon$ is an issue. If we replace Eq. (14) with a contour that is a tiny little circle around $k = \Omega+\mathrm{i}\epsilon$, then $\tilde{S}^{\alpha\beta}(k\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}kr}$ can be approximated by its value at $k = \Omega+\mathrm{i}\epsilon$ and pulled out of the integral. Then:

$$\int_{\mathcal{C}} \frac{2k}{k^2-(\Omega+\mathrm{i}\epsilon)^2}\,\mathrm{d}k = \ln[k^2-(\Omega+\mathrm{i}\epsilon)]|_{\mathrm{start}}^{\mathrm{finish}} = 2\pi\mathrm{i}, \tag{15}$$

where we have noted that $\ln x = \ln|x| + \mathrm{i}\arg x$ increases by $2\pi\mathrm{i}$ if x goes around zero counterclockwise (the phase $\arg x$ increases by 2π). This leads to

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) = \frac{4}{r}\mathrm{e}^{-\mathrm{i}\Omega t}\tilde{S}^{\alpha\beta}(\Omega\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}\Omega r} \quad \text{(large } r\text{).} \tag{16}$$

This has the form of an outward-traveling wave going at the speed of light: it is proportional to $\mathrm{e}^{\mathrm{i}\Omega(r-t)}$. Note further that the amplitude is $\propto 1/r$, as one expects for a wave expanding in 3 dimensions (where the flux scales as $\propto 1/r^2$).

C. Radiation from a non-relativistic source

Of special interest is radiation from a non-relativistic source, where the velocities are $v \ll 1$. It follows that the frequencies of interest are $\Omega \sim v/R \ll 1/R$, where R is the length scale of the system. Thus $\Omega R \ll 1$ and when we write the Fourier transform of the source we can expand the exponential:

$$\tilde{S}^{\alpha\beta}(\Omega\hat{\boldsymbol{n}}) = \int_{\mathbb{R}^3} S^{\alpha\beta}(\boldsymbol{x})\left(1 - \mathrm{i}\Omega\hat{n}_i x^i - \frac{1}{2}\Omega^2\hat{n}_i\hat{n}_j x^i x^j + ...\right)\mathrm{d}^3\boldsymbol{x} \tag{17}$$

and thus take the first few terms (the first few moments of $\tilde{S}^{\alpha\beta}$). Let us consider each of these moments, going through order Mv^2 where M is the total mass. We further make use of the fact that since $T^{\alpha\beta}{}_{,\beta} = 0$, we have

$$S^{\alpha i}{}_{,i} - \mathrm{i}\Omega S^{\alpha 0} = 0 \quad \to S^{\alpha 0} = -\mathrm{i}\Omega^{-1}S^{\alpha i}{}_{,i}. \tag{18}$$

We first consider the 00 component of $\tilde{S}^{\alpha\beta}$. This is:

$$\tilde{S}^{00}(\Omega\hat{\boldsymbol{n}}) = \int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} - \mathrm{i}\Omega\hat{n}_i\int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})x^i\,\mathrm{d}^3\boldsymbol{x} - \frac{1}{2}\Omega^2\hat{n}_i\hat{n}_j\int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})x^i x^j\,\mathrm{d}^3\boldsymbol{x} + \dots. \tag{19}$$

The first term is of order M, and subsequent terms have additional factors of v. However, using Eq. (18) and integration by parts, we find:

$$\int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = -\mathrm{i}\Omega^{-1}\int_{\mathbb{R}^3} S^{0i}{}_{,i}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = 0. \tag{20}$$

(This is saying that the total energy can't oscillate.) Furthermore,

$$\int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})x^i\,\mathrm{d}^3\boldsymbol{x} = -\frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{0j}{}_{,j}(\boldsymbol{x})x^i\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{0j}(\boldsymbol{x})x^i{}_{,j}\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{kj}{}_{,k}(\boldsymbol{x})\delta^i_j\,\mathrm{d}^3\boldsymbol{x} = 0. \tag{21}$$

(This is saying that the center of mass position can't oscillate.) Thus the leading term is the last term. We define, for future reference, the quadrupole moment tensor:

$$Q^{ij} = \int_{\mathbb{R}^3} S^{00}(\boldsymbol{x})x^i x^j\,\mathrm{d}^3\boldsymbol{x}, \tag{22}$$

with

$$\tilde{S}^{00}(\Omega\hat{\boldsymbol{n}}) = -\frac{1}{2}\Omega^2\hat{n}_i\hat{n}_j Q^{ij} = \frac{1}{2}\hat{n}_i\hat{n}_j\ddot{Q}^{ij}. \tag{23}$$

Now let's consider the $0i$ component of $\tilde{S}^{\alpha\beta}$. Here:

$$\tilde{S}^{0i}(\Omega\hat{\boldsymbol{n}}) = \int_{\mathbb{R}^3} S^{0i}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} - \mathrm{i}\Omega\hat{n}_j\int_{\mathbb{R}^3} S^{0i}(\boldsymbol{x})x^j\,\mathrm{d}^3\boldsymbol{x} + \dots. \tag{24}$$

The first term is Eq. (21) and vanishes. To understand the second term, define

$$M^{ij} \equiv \int_{\mathbb{R}^3} S^{0i}(\boldsymbol{x})x^j\,\mathrm{d}^3\boldsymbol{x} = -\frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{ki}{}_{,k}(\boldsymbol{x})x^j\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{ki}(\boldsymbol{x})x^j{}_{,k}\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{ji}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}. \tag{25}$$

Thus M^{ij} is symmetric. (This is equivalent to saying that the angular momentum can't oscillate.) Now note that

$$Q^{ij} = -\frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{0k}{}_{,k}(\boldsymbol{x})x^i x^j \,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{0k}(\boldsymbol{x})(x^i x^j)_{,k}\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{i}}{\Omega}\int_{\mathbb{R}^3} S^{0k}(\boldsymbol{x})(\delta^i_k x^j + \delta^j_k x^i)\,\mathrm{d}^3\boldsymbol{x} = 2\frac{\mathrm{i}}{\Omega}M^{jk}. \tag{26}$$

We conclude that to the relevant order:

$$\tilde{S}^{0i}(\Omega\hat{\boldsymbol{n}}) = -\frac{1}{2}\Omega^2\hat{n}_j Q^{ij} = \frac{1}{2}\hat{n}_j\ddot{Q}^{ij}. \tag{27}$$

Finally, we consider the ij component:

$$\tilde{S}^{ij}(\Omega\hat{\boldsymbol{n}}) = \int_{\mathbb{R}^3} S^{ij}(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = \mathrm{i}\Omega M^{ij} = -\frac{1}{2}\Omega^2 Q^{ij} = \frac{1}{2}\ddot{Q}^{ij}. \tag{28}$$

We thus find the metric perturbation, written in matrix form:

$$\bar{h}^{\alpha\beta}(\boldsymbol{x},t) \to \frac{2}{r}\mathrm{e}^{\mathrm{i}\Omega(r-t)}\left(\begin{array}{c|c} \hat{\boldsymbol{n}}^{\mathrm{T}}\ddot{\mathbf{Q}}\hat{\boldsymbol{n}} & \hat{\boldsymbol{n}}^{\mathrm{T}}\ddot{\mathbf{Q}} \\ \hline \ddot{\mathbf{Q}}\hat{\boldsymbol{n}} & \ddot{\mathbf{Q}} \end{array}\right). \tag{29}$$

D. Transverse traceless gauge

Just because there is a metric perturbation that propagates away from the source does not imply there is a physical wave; for all we know, it could just be a wave in the coordinate system with no physical consequence (i.e., a pure gauge mode). To see what portions of the wave can be eliminated by a gauge transformation, let's first consider a generic gauge perturbation around flat spacetime: we see that

$$x^\alpha = x^{\bar{\alpha}} + \xi^\alpha, \tag{30}$$

where $\boldsymbol{\xi}$ is a 4-vector function of spacetime. This perturbation results in a change in the metric tensor,

$$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}}\frac{\partial x^\nu}{\partial x^{\bar{\beta}}}\eta_{\mu\nu} = (\delta^\mu_{\bar{\alpha}} + \xi^\mu{}_{,\bar{\alpha}})(\delta^\nu_{\bar{\beta}} + \xi^\nu{}_{,\bar{\beta}})\eta_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}} + \xi_{\bar{\beta},\bar{\alpha}} + \xi_{\bar{\alpha},\bar{\beta}}. \tag{31}$$

Thus a pure gauge perturbation is described by

$$h_{\bar{\alpha}\bar{\beta}} = \xi_{\bar{\beta},\bar{\alpha}} + \xi_{\bar{\alpha},\bar{\beta}} \tag{32}$$

or

$$\bar{h}_{\bar{\alpha}\bar{\beta}} = \xi_{\bar{\beta},\bar{\alpha}} + \xi_{\bar{\alpha},\bar{\beta}} - \xi^{\bar{\mu}}{}_{,\bar{\mu}}\eta_{\bar{\alpha}\bar{\beta}}. \tag{33}$$

Note that since this is already a first order perturbation, this can be written in either the original or the perturbed coordinate system. We call this the gauge change "generated by" $\boldsymbol{\xi}$.

Let's consider how this applies to a plane wave going in the $+z$ direction so $\partial_3\xi^\alpha = \partial_0\xi^\alpha = \dot{\xi}^\alpha$. If we have ξ as a function of only z and t, then

$$\Delta\bar{\mathbf{h}} \to \left(\begin{array}{c|ccc} \dot{\xi}^3 - \dot{\xi}^0 & \dot{\xi}^1 & \dot{\xi}^2 & \dot{\xi}^3 - \dot{\xi}^0 \\ \hline \dot{\xi}^1 & -\dot{\xi}^0 - \dot{\xi}^3 & 0 & \dot{\xi}^1 \\ \dot{\xi}^2 & 0 & -\dot{\xi}^0 - \dot{\xi}^3 & \dot{\xi}^2 \\ \dot{\xi}^3 - \dot{\xi}^0 & \dot{\xi}^1 & \dot{\xi}^2 & \dot{\xi}^3 - \dot{\xi}^0 \end{array}\right). \tag{34}$$

Comparing this to Eq. (29) in the far field of the $+z$ direction, where the outgoing wave is essentially a plane wave, tells us that with the choice:

$$\dot{\xi}^\alpha \to -\frac{2}{r}\mathrm{e}^{i\Omega(r-t)}\left(-\frac{\ddot{Q}_{11}+\ddot{Q}_{22}}{4} - \frac{\ddot{Q}_{33}}{2}, \ddot{Q}_{13}, \ddot{Q}_{23}, -\frac{\ddot{Q}_{11}+\ddot{Q}_{22}}{4} + \frac{\ddot{Q}_{33}}{2}\right), \tag{35}$$

we arrive at the metric

$$\bar{\mathbf{h}} \to \frac{2}{r}\mathrm{e}^{i\Omega(r-t)}\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2}(\ddot{Q}_{11} - \ddot{Q}_{22}) & \ddot{Q}_{12} & 0 \\ 0 & \ddot{Q}_{12} & \frac{1}{2}(\ddot{Q}_{22} - \ddot{Q}_{11}) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{36}$$

The new gauge is called *transverse-traceless* (TT) gauge, and is denoted by $\bar{h}^{\mathrm{TT}}_{\alpha\beta}$. It is completely transverse (no metric components in the time or longitudinal direction) and is traceless, so $\bar{h}^{\mathrm{TT}}_{\alpha\beta} = h^{\mathrm{TT}}_{\alpha\beta}$. Using the transverse projection matrix $\boldsymbol{\Pi}_\perp = \mathbb{I}_{3\times 3} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^{\mathrm{T}}$, we write this in a form that doesn't depend on the observer being on the z-axis. The spatial part of the metric perturbation (the only part that is non-zero) is

$$\mathbf{h}^{\mathrm{TT}} = \frac{2}{r}\mathrm{e}^{i\Omega(r-t)}\left(\boldsymbol{\Pi}_\perp \ddot{\mathbf{Q}} \boldsymbol{\Pi}_\perp - \frac{1}{2}\boldsymbol{\Pi}_\perp \mathrm{Tr}[\boldsymbol{\Pi}_\perp \ddot{\mathbf{Q}}]\right). \tag{37}$$

Note, however, that TT gauge only applies to the far field of the source. A gravitational wave detector on Earth observing a distant binary star can work in TT gauge, but an observer within the binary star cannot.

The TT condition completely defines $\boldsymbol{\xi}$, so what is left is a real wave and not its coordinate system.

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[1] A function $f(z)$ is analytic at z if it has a convergent Taylor expansion in an open region containing z.