Lecture XII: Cosmic microwave background anisotropies

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I. INTRODUCTION

Having covered the growth of structure in the Universe, we will now turn our attention for the remainder of the course to the probes of structure growth. The first such probe will be the anisotropies in the cosmic microwave background. These notes cover the theory of temperature anisotropies, then we will discuss the observational status (and qualitatively consider polarization) in the slides.

II. FORMALISM

The cosmic microwave background has a power spectrum, just like any 2D random field. A 2D field on a flat surface has a correlation function

$$\langle \Delta T(\mathbf{x})\Delta T(\mathbf{x}')\rangle = \xi(r), \quad r = |\mathbf{x}-\mathbf{x}'|, \quad \xi(r) = \int_{\mathbb{R}^2} P(k)\, e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{d^2\mathbf{k}}{(2\pi)^2} = \int_0^\infty \frac{k\,dk}{2\pi}\,P(k)J_0(kr). \tag{1}$$

Since the CMB is a 2D surface on a curved sky, we use slightly different formalism. There is still a correlation function $\xi(\vartheta)$ (now there is an angle between two points instead of a distance in Mpc). We may then decompose the temperature perturbations into spherical harmonics,

$$\Delta T(\hat{\mathbf{n}}) = \sum_{\ell=0}^{\infty} a_{\ell m}Y_{\ell m}(\hat{\mathbf{n}}), \quad a_{\ell m} = \int_{S^2} \Delta T(\hat{\mathbf{n}})Y^*_{\ell m}(\mathbf{n})\,d^2\hat{\mathbf{n}}. \tag{2}$$

We will decompose the correlation function this time as

$$\xi(\vartheta) = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}C_\ell P_\ell(\cos\vartheta), \tag{3}$$

which looks a lot like Eq. (1) if you recall the identity $P_\ell(\cos\vartheta)\approx J_0(\ell\vartheta)$ (valid in the limit $\ell\vartheta$ fixed, with $\ell\to\infty$ and $\vartheta\to 0$) and identify C_ℓ as the power spectrum. The covariance of the spherical harmonic moments is the power spectrum,

$$\begin{aligned}
\langle a_{\ell m}a^*_{\ell' m'}\rangle &= \int_{S^2}\int_{S^2}\langle \Delta T(\hat{\mathbf{n}})\Delta T(\hat{\mathbf{n}}')\rangle Y^*_{\ell m}(\mathbf{n})Y_{\ell' m'}(\mathbf{n}')\,d^2\hat{\mathbf{n}}\,d^2\hat{\mathbf{n}}' \\
&= \int_{S^2}\int_{S^2}\xi(\vartheta)Y^*_{\ell m}(\mathbf{n})Y_{\ell' m'}(\mathbf{n}')\,d^2\hat{\mathbf{n}}\,d^2\hat{\mathbf{n}}' \\
&= \sum_{L=0}^{\infty}\int_{S^2}\int_{S^2}\frac{2\ell+1}{4\pi}C_L P_L(\cos\vartheta)Y^*_{\ell m}(\mathbf{n})Y_{\ell' m'}(\mathbf{n}')\,d^2\hat{\mathbf{n}}\,d^2\hat{\mathbf{n}}' \\
&= \sum_{L=0}^{\infty}\int_{S^2}\int_{S^2}C_L\sum_{M=-L}^{L}Y^*_{LM}(\mathbf{n})Y_{LM}(\mathbf{n}')Y^*_{\ell m}(\mathbf{n})Y_{\ell' m'}(\mathbf{n}')\,d^2\hat{\mathbf{n}}\,d^2\hat{\mathbf{n}}' \\
&= \sum_{L=0}^{\infty}C_L\sum_{M=-L}^{L}\delta_{L\ell}\delta_{L\ell'}\delta_{Mm}\delta_{Mm'} \\
&= C_\ell\delta_{\ell\ell'}\delta_{mm'},
\end{aligned} \tag{4}$$

where we used the spherical harmonic addition theorem in the 4th equality.

The total variance of the CMB temperature is

$$\langle \Delta T^2\rangle = \xi(0) = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}C_\ell \approx \int\frac{d\ell}{\ell}\,\frac{\ell(\ell+1)C_\ell}{2\pi}, \tag{5}$$

where the last approximation treats ℓ as a continuous variable; thus you will often see the "variance per $\ln \ell$" or $\Delta^2 \equiv \ell(\ell+1)C_\ell/2\pi$ plotted.

The basic objective of CMB theory studies is to predict the power spectrum for a given set of cosmological parameters; this can then be compared to the data. We will focus on large scales first, and then go to small scales. We also assume only adiabatic perturbations.

III. LARGE SCALES – THE SACHS-WOLFE EFFECT

Let's first consider large scales, which were outside the horizon at the time of recombination. This should mean comoving scales greater than $\sim \eta_{\rm rec}$, or angular scales greater than $\sim \eta_{\rm rec}/\eta_0$. In Fourier space, this corresponds to

$$\ell < \frac{\pi}{\eta_{\rm rec}/\eta_0} \approx \frac{\pi}{a_{\rm rec}^{1/2}} \approx \pi\sqrt{1100} \approx 100, \tag{6}$$

where we used $\eta \propto a^{1/2}$ in the matter-dominated era. These large scales are called the *Sachs-Wolfe regime*.

It turns out that the Sachs-Wolfe regime can be analytically understood. To do so, we will work in Lagrangian coordinates (technically *synchronous gauge* in relativistic perturbation theory), where we consider the CMB perturbations as measured in the local rest frame of the dark matter. At the time of recombination, the perturbation was outside the horizon, and a local observer does not even know the perturbation exists: then $\Delta T/T = 0$.

In the Lagrangian picture, temperature perturbations accumulate because – as a photon travels through space, passing one observer and then the next and then the next – there is a slight Doppler shift between the observers. For a photon traveling in direction $-\hat{\mathbf{n}}$, this is

$$z_{\rm D} = -\hat{\mathbf{n}}\cdot\Delta\mathbf{v} = \Delta r \sum_{ij} \frac{\partial v_i^{\rm tot}}{\partial r_j}\hat{n}_i\hat{n}_j = a\Delta r \sum_{ij}(H\delta_{ij} + S_{ij})\hat{n}_i\hat{n}_j. \tag{7}$$

Here we define the 3×3 peculiar velocity gradient tensor

$$S_{ij} = \frac{1}{a}\frac{\partial v_i}{\partial r_j}. \tag{8}$$

Now since the velocity is the gradient of a scalar (for scalar perturbations), S_{ij} is the second derivative of a scalar. Moreover, we see that in linear perturbation theory,

$$\mathrm{Tr}\,\mathbf{S} = \frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v} = -\dot{\delta}_m = \frac{2\dot{G}_+(a)}{5\Omega_m H_0^2}\nabla^2\zeta. \tag{9}$$

Thus:

$$S_{ij} = \frac{2\dot{G}_+(a)}{5\Omega_m H_0^2}\nabla_i\nabla_j\zeta. \tag{10}$$

Adding up the little peculiar contributions to the redshift along the whole path that the photon takes from the epoch of recombination (at $r_{\rm LSS}$) to us, $\Delta z_{\rm D}$ (i.e., not including the mean Hubble expansion), we find

$$\begin{aligned}\frac{\Delta T}{T} &= -\sum \Delta z_{\rm D} = -\int_0^{r_{\rm LSS}} a\,dr \sum_{ij} S_{ij}\hat{n}_i\hat{n}_j = -\int_0^{r_{\rm LSS}} \sum_{ij}\frac{2a\dot{G}_+(a)}{5\Omega_m H_0^2}[\nabla_i\nabla_j\zeta(r\hat{\mathbf{n}})]\hat{n}_i\hat{n}_j\,dr \\ &= -\int_0^{r_{\rm LSS}} \frac{2a\dot{G}_+(a)}{5\Omega_m H_0^2}\frac{\partial^2\zeta(r\hat{\mathbf{n}})}{\partial r^2}\,dr. \end{aligned}\tag{11}$$

At this point, we have an expression for the CMB anisotropies!

At this point, we will make the simplifying assumption that the Universe was matter-dominated for most of the history, so $G_+(a) \approx a$ and $\dot{G}_+(a) \approx aH$. (You saw on the homework that this assumption is only broken by 20% even today.) Moreover, a is related to the conformal time by

$$\eta = \int\frac{dt}{a} = \int\frac{da}{a^2H} \approx \int\frac{da}{a^2H_0\Omega_m^{1/2}a^{-3/2}} = 2\frac{a^{1/2}}{\Omega_m^{1/2}H_0}, \tag{12}$$

so

$$\frac{2a\dot{G}_+(a)}{5\Omega_m H_0^2} \approx \frac{2a^2 H}{5\Omega_m H_0^2} \approx \frac{2a^2 H_0\sqrt{\Omega_m a^{-3}}}{5\Omega_m H_0^2} \approx \frac{2a^{1/2}}{5\Omega_m^{1/2} H_0} = \frac{\eta}{5} = \frac{r_{\rm LSS}-r}{5}. \tag{13}$$

Plugging this into Eq. (11) and integrating by parts gives

$$\begin{aligned}\frac{\Delta T}{T} &= -\frac{1}{5}\int_0^{r_{\rm LSS}} (r_{\rm LSS}-r)\frac{\partial^2\zeta(r\hat{\mathbf{n}})}{\partial r^2}\,dr \\ &= -\frac{1}{5}\left[-r_{\rm LSS}\left.\frac{\partial\zeta(r\hat{\mathbf{n}})}{\partial r}\right|_{r=0} + \int_0^{r_{\rm LSS}}\frac{\partial\zeta(r\hat{\mathbf{n}})}{\partial r}\,dr\right] \\ &= -\frac{1}{5}\left[-r_{\rm LSS}\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\zeta(0) + \zeta(r_{\rm LSS}\hat{\mathbf{n}}) - \zeta(0)\right].\end{aligned} \tag{14}$$

Here the first term is a dipole term (unobservable since it corresponds to a change in reference frame), and the last term is a monopole term (unobservable since we don't know the "unperturbed" temperature $\bar{T}$). Thus the outcome is a CMB anisotropy that is actually a map of the primordial curvature perturbation on a shell surrounding us, with a radius of $r_{\rm LSS}$, and a multiplying factor of $-\frac{1}{5}$.

The remarkable simplicity of the Sachs-Wolfe regime allows us to infer that since

$$\frac{\Delta_T}{\bar{T}} = \frac{[\ell(\ell+1)C_\ell/2\pi]^{1/2}}{\bar{T}} \approx 10^{-5}, \tag{15}$$

with no strong dependence on ℓ, the primordial power spectrum amplitude Δ_ζ is $\sim 5\times10^{-5}$ with weak k dependence (at least over the range of scales in the Sachs-Wolfe plateau).

The presence of dark energy causes deviations from Eq. (13); in particular, when we integrate by parts, there are residual terms that involve ζ between us and the surface of last scattering. These terms give rise to the *integrated Sachs-Wolfe effect*, and cause correlations between the CMB and galaxy surveys. Their detection in $\sim$ 2003 was one additional piece of evidence in favor of dark energy.

IV. SMALL SCALES

At $\ell < 100$, Eq. (6) holds, and we can treat the perturbations as being super-horizon at recombination. At smaller scales – ℓ of a few hundred or more – the correct physics is that the baryon-photon fluid underwent acoustic oscillations. At the time of recombination, a wave of wavenumber k has undergone N oscillation cycles, where

$$N = \frac{\int \omega\,dt}{2\pi} = \frac{\int (c_s k/a)\,dt}{2\pi} = \frac{k\eta_{\rm rec}}{2\pi\sqrt{3}}, \tag{16}$$

where we recall $c_s = 1/\sqrt{3}$. At recombination, the baryons then become transparent and the photons are released. We thus expect to see maxima in power if N is an integer or half-integer (so that the density perturbations are maximized), and minima in power if N is a quarter-integer (so that the density perturbations are passing through a null at recombination).

In practice, the situation is more complicated because a given k does not map onto a single ℓ. Rather, the correspondence is

$$\ell = \frac{k_\perp}{r_{\rm LSS}}, \tag{17}$$

where $r_{\rm LSS}$ is the comoving angular diameter distance to the last scattering surface. However, since $k_\perp = k\sin\alpha$ where α is the angle between the line of sight and $\mathbf{k}$, and there is a "pile-up" of modes at $\sin\alpha = 1$ when we do the integral over angles, the power maxima and minima do show up in C_ℓ. The condition for a maximum is that $2N =$ is an integer, or

$$2\frac{k\eta_{\rm rec}}{2\pi\sqrt{3}} \approx \frac{\ell\eta_{\rm rec}}{\pi\sqrt{3}\,r_{\rm LSS}} \in \mathbb{Z}. \tag{18}$$

This leads to peaks with a spacing

$$\Delta\ell \approx \frac{\pi\sqrt{3}\,r_{\rm LSS}}{\eta_{\rm rec}} \approx 300. \tag{19}$$

At higher $\ell \gtrsim 2000$, the CMB power spectrum is strongly damped by photon diffusion. We thus describe the CMB power spectrum as having the *acoustic peak* and *damping tail* regimes.

Because recombination occurred shortly after matter-radiation equality, the peak amplitudes are sensitive to the ratio $a_{\rm rec}/a_{\rm eq}$, which is proportional to $\Omega_m h^2$. Several effects (including the damping, which depends on the photon mean free path) depend on the baryon density as well. Thus the acoustic peak and damping tail structure provide a strong constraint on $\Omega_m h^2$ and $\Omega_b h^2$.