Homework 1

(Dated: Due: January 25, 2019)

1. Geometry in curved space. [21 points] This problem is a series of investigations of the geometry in space of positive and negative curvature. Their purpose is to shed some light on how these cases differ from Euclidean geometry. In each case, I'm hoping that you will see how a familiar formula from Euclidean geometry changes in open or closed space. Some of the problems are most easily solved directly by thinking about the line element, and some are most easily solved by thinking of S^3 as being embedded in Euclidean $\mathbb{R}^4$ and using familiar Euclidean geometry (and then using analytic continuation to look at the hyperbolic case).

[Parts a–c have increasing levels of difficulty.]

(a) [6 points] Consider a sphere of radius χ. Show that the surface area of the sphere is given by

$$A = 4\pi R^2 \sin^2 \frac{\chi}{R} = 4\pi\chi^2 - \frac{4}{3}\pi K \chi^4 + ..., \tag{1}$$

where the last expression is the beginning of the Taylor expansion.

(b) [6 points] Consider a right triangle with legs a and b, and a hypotenuse of length c. Let A, B, and C be the vertices opposite these sides, respectively. Show that these lengths are related by

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}. \tag{2}$$

[*Hint*: Work in Euclidean $\mathbb{R}^4$ with the coordinate axes chosen conveniently. Then use the fact that the dot product of two vectors is the product of their lengths times the cosine of the angle between them; apply this fact to the vectors OA, OB, and OC, where O is the origin of $\mathbb{R}^4$.]

Then show by Taylor expansion that the leading (order K) correction to the Pythagorean theorem is

$$c^2 = a^2 + b^2 - \frac{1}{3}Ka^2b^2 + ... \tag{3}$$

(c) [7 points] Consider the right triangle of part (b), and let α, β, and $\gamma = \pi/2$ be the angles at the vertices A, B, and C. Show that

$$\sin\frac{c}{R}\sin\alpha = \sin\frac{a}{R}. \tag{4}$$

[*Hint*: For the first equality, work in Euclidean $\mathbb{R}^4$, and orient the triangle so that it is in the xyz-hyperplane, with A and C in the xy-plane. Then write down two different expressions for the z-coordinate of B.]

Then show that the sum of the internal angles of the triangle differs from π:

$$\alpha + \beta + \gamma = \sin^{-1}\frac{\sin(a/R)}{\sin(c/R)} + \sin^{-1}\frac{\sin(b/R)}{\sin(c/R)} + \frac{\pi}{2} = \pi + K\frac{ab}{2} + .., \tag{5}$$

where the last expression is the first-order Taylor expansion in K.

Comment: There are a few ways to prove this, but you might want to first show, from your answer in (b), that

$$\sin^2\frac{c}{R} = \left(\sin^2\frac{a}{R} + \sin^2\frac{b}{R}\right)\left(1 - K\frac{a^2b^2}{a^2+b^2} + ...\right) \tag{6}$$

and then prove the following identity for small ϵ by direct Taylor expansion:

$$\sin^{-1}\sqrt{\frac{x}{1+\epsilon}} + \sin^{-1}\sqrt{\frac{1-x}{1+\epsilon}} = \frac{\pi}{2} - \frac{\epsilon}{2\sqrt{x(1-x)}} + ... \tag{7}$$

(d) [2 points] Summarize your results in a table showing how the sign of the deviations from the Euclidean results (the formula for the surface area of a sphere, the Pythagorean theorem, and the sum of the internal angles in a triangle) relates to the spatial curvature.