Lecture VIII: Introduction to perturbation theory

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I. INTRODUCTION

We now begin our study of the inhomogeneities in the Universe. There are two major tools used in this subject:

- *Perturbation theory* – Perturbations around the homogeneous, isotropic FLRW solution are useful when the density perturbations are $|\delta\rho|/\bar{\rho} \lesssim 1$. Perturbation theory may be linear (1st order) or higher-order. This is the standard description for inflation, the CMB, and the large-scale distribution of matter ($\gtrsim 10$ Mpc) today. Perturbation theory can even be extended to describe the distribution of galaxies on these large scales, even though galaxies themselves have $\delta\rho/\bar{\rho} \gg 1$ – this is known as *biasing theory*, and is based on the idea that galaxy formation, however complicated it might be, is a local process that can be parameterized into a few *biasing coefficients*.

- *Numerical simulations* – On small scales, where perturbations are large, the evolution of matter in the Universe is best treated by numerical simulations. Since most matter is dark, some simulations follow only the equations of motion for dark matter particles (*N-body simulations*). These simulations are fast and mathematically well-understood, but neglect the fact that 17% of the matter in the Universe is baryonic and has non-gravitational interactions on small scales. Simulations that try to describe the baryons (needed for galaxy formation, for example), must solve the hydrodynamic equations and are known as *hydrodynamic simulations*. There are several kinds of hydrodynamic simulations, and they may have different algorithms for incorporating small scale physics such as star formation and feedback that are not fully understood and cannot be modeled from first principles.

In this lecture, we will focus on standard linear perturbation theory in cosmology. We will discuss the other techniques when we start studying galaxies.

II. DECOMPOSITION OF COSMOLOGICAL PERTURBATIONS

We will also focus on the spatially flat universe case, $K = 0$. It is possible to develop the theory on a spatially curved background, but I'm not including that in the scope of this class even though the math is cool (4D spherical harmonics!), because it is a lot of extra work and the Universe we live in is close to flat. I will also use Cartesian coordinates rather than spherical coordinates, since this is better suited to calculations that use translation invariance to simplify the problem.

A. The perturbation variables

In cosmology, as with most subjects where we do perturbation theory, the first step is to define a background and a set of perturbation variables. The background here is of course the expanding FLRW universe. For every aspect of this model that can be perturbed, there is a perturbation variable.

1. *Gravity*

Let us start with gravity. In general relativity, we said that we can write the line element as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) = -dt^2 + a^2(t)\sum_{i,j=1}^{3}\delta_{ij}\,dx^i\,dx^j. \tag{1}$$

In a perturbed Universe, we could write:

$$ds^2 = -(1+2A)dt^2 - 2a(t)\sum_{i=1}^{3}B_i\,dt\,dx^i + a^2(t)\sum_{i,j=1}^{3}\left[(1+2D)\delta_{ij} + 2E_{ij}\right]dx^i\,dx^j, \tag{2}$$

where the scalars A and D, the vector B_i, and the 3×3 traceless-symmetric tensor E_{ij} are functions of space and time. Of these 10 perturbation variables, 4 will be "gauge perturbations," i.e., resulting from a perturbation of the coordinate system but with no change in the actual geometry, while the other 6 perturbation variables correspond to real physical perturbations. In relativistic perturbation theory, we may even choose our coordinate system to make 4 of these perturbation variables zero; such a choice is called a *gauge choice*.

In Newtonian gravity, which is a good description for slow-moving matter on scales small compared to the Hubble scale, we may instead describe a gravitational potential perturbation Φ, which is also a function of space and time.

2. Matter

The matter field is usually described by a density ρ and a velocity vector $\mathbf{v}$ (relative to an observer at fixed spatial coordinates). The density may be written in terms of the background density $\bar{\rho}$ as

$$\rho(\mathbf{r}, t) = \bar{\rho}(t)[1 + \delta_m(\mathbf{r}, t)], \tag{3}$$

where δ_m is the density perturbation. Since $\mathbf{v} = 0$ in the background case, $\mathbf{v}$ is already a perturbation variable. On small scales, or in the early Universe where Thomson scattering is important, we must distinguish baryons from dark matter since they have different equations of motion: we write δ_b and δ_c (where the subscript "c" refers to Cold Dark Matter).

3. Radiation

The CMB also has perturbations. If an observer sees radiation propagating in direction $\hat{\mathbf{n}}$, then the blackbody temperature of that radiation can be written as

$$T(\mathbf{r}, t, \hat{\mathbf{n}}) = \bar{T}(t) \left[1 + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Theta_{\ell m}(\mathbf{r}, t) Y_{\ell m}(\hat{\mathbf{n}}) \right], \tag{4}$$

where $\Theta_{\ell m}$ are the multipole coefficients and $Y_{\ell m}$ are the spherical harmonics. In principle one might consider radiation that is not a blackbody. However, inhomogeneous blackbody radiation free-streaming through space will remain blackbody radiation because of the non-dispersive nature of electromagnetic waves (the group velocity is constant), and in linear perturbation theory, Thomson scattering will preserve the blackbody spectrum because its cross section is frequency-independent.

Since Thomson scattering produces polarization, we could also write polarization multipoles, but I will do that later. Similarly, one could write an expression for neutrino anisotropies; if the neutrinos are massless (a good approximation in the early Universe, at $T_\nu \gg m_\nu$) then one can write neutrino anisotropies $\Theta^\nu_{\ell m}(\mathbf{r}, t)$ as well. If one wanted to take into account neutrino masses, we would have to write this as a function of the comoving momentum q of the neutrinos (defined as the physical momentum times the scale factor, so that q is fixed in an expanding FLRW universe as the neutrino de Broglie wavelength redshifts).

B. Fourier modes

When we work on a translation-invariant background, it is most useful to do perturbation theory in Fourier space. Any perturbation variable $\psi(\mathbf{r}, t)$ can be written in Fourier space via a transform pair:

$$\psi(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3} \, \tilde{\psi}(\mathbf{k}, t) \, e^{i\mathbf{k}\cdot\mathbf{r}} \quad \leftrightarrow \quad \tilde{\psi}(\mathbf{k}, t) = \int_{\mathbb{R}^3} d^3\mathbf{r} \, \psi(\mathbf{r}, t) \, e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{5}$$

(This placement of the factors of 2π is almost universal in the cosmology literature.) Spatial derivatives in real space are converted to wave vectors in Fourier space, with the usual mapping $\boldsymbol{\nabla} \to i\mathbf{k}$.

When doing linear perturbation theory on a translation invariant background, where spatial information appears only through derivatives, every Fourier mode will evolve independently (Schur's lemma, to be precise). If the background is isotropic, then without loss of generality we may put $\mathbf{k}$ in the z direction. Note that these statements do not apply to 2nd or higher order peturbation theory, though symmetries will still restrict the possible behaviors.

C. Scalar, vector, and tensor perturbations

Even if we consider a single Fourier mode $\mathbf{k}$, and orient $\mathbf{k}$ along the z-direction, we still have not made full use of the symmetry information available in cosmology, because we have not used the fact that we can rotate around the z-axis without changing the problem. In "ordinary" wave physics, you learned that this symmetry behavior gave rise to two types of waves – "longitudinal" waves such as sound waves (whose amplitude is invariant if we rotate around $\mathbf{k}$), and "transverse" waves such as electromagnetic waves (which have two polarizations, and where their combinations $E_x \pm iE_y$ pick up a factor of $e^{\mp i\vartheta}$ if we rotate by ϑ around the z-axis). In a problem with rotational invariance around the z-axis, longitudinal waves and transverse waves are completely separate – one cannot turn into another. The same principle occurs in cosmology.

To be specific, we may take the N perturbation variables in a Fourier mode $\mathbf{k}$ along the z-axis, $\{\psi_1 ... \psi_N\}$. We can say that a variable ψ_j has *helicity* s_j if – under rotation by angle ϑ – it transforms as

$$\psi'_j = e^{-is_j\vartheta}\psi_j. \tag{6}$$

It is always possible to choose a basis for the perturbation variables so that they are helicity eigenstates, i.e., so that each ψ_j has a definite helicity s_j. (It is easiest to see this by thinking about an infinitesimal rotation first, and diagonalizing the rotation matrix.)

As an example, the baryon velocity vector $\mathbf{v}_b$ has 3 components. The components v_z and $v_x \pm iv_y$ have helicity 0 and ± 1 since

$$v'_z = v_z \quad \text{and} \quad v'_x \pm iv'_y = e^{\mp i\vartheta}(v_x + iv_y). \tag{7}$$

We can see that the equations of motion of perturbation parameters for every helicity must decouple. To see this, let's write a general linear perturbation evolution

$$\dot{\psi}_j = \sum_{k=1}^{N} A_{jk}\psi_k, \tag{8}$$

where A_{jk} are some coefficients expressing how ψ_k affects the evolution of ψ_j. Then by using the same equation in the primed frame:

$$\dot{\psi}_j = e^{is_j\vartheta}\dot{\psi}'_j = e^{is_j\vartheta}\sum_{k=1}^{N} A_{jk}\psi'_k = e^{is_j\vartheta}\sum_{k=1}^{N} A_{jk}e^{-is_k\vartheta}\psi_k = \sum_{k=1}^{N} A_{jk}e^{i(s_j-s_k)\vartheta}\psi_k. \tag{9}$$

The equations of motion are the same in the two frames if

$$A_{jk} = A_{jk}e^{i(s_j-s_k)\vartheta}, \tag{10}$$

which only works if $s_j = s_k$ (same helicity) or $A_{jk} = 0$. (Note this is a special case of Schur's lemma.) Thus we can consider linear perturbations in cosmology by thinking about one helicity sector at a time:

- The perturbations with $s = 0$ are called *scalar perturbations*; these include density perturbations, are the most important perturbations in cosmology, and are the only ones known to have been present in the initial conditions. In your undergraduate wave courses you would have called them longitudinal waves, but we haven't said yet whether they really propagate as waves or not.

- The perturbations with $s = \pm 1$ are called *vector perturbations* or transverse waves. We don't normally think about them in cosmology because all vector perturbation modes in the early Universe turn out to be decaying, so we would not expect to see them even if they were initially present.

- The perturbations with $s = \pm 2$ are called *tensor perturbations*. The most important tensor perturbation modes, and the only ones that are non-decaying in the early Universe, are gravitational waves.

- Perturbations with $|s| > 2$ can exist in the multipoles of the radiation field, but in the early Universe where the plasma is optically thick to scattering these should be washed out. Therefore we do not consider them here.

III. GROWTH OF SUB-HORIZON PERTURBATIONS

We will now illustrate the discussion of cosmological perturbations with the simplest application: the growth of linear perturbations in the matter on sub-horizon scales, under the assumption that other forms of energy are either smoothly distributed or negligible. This is also the most important application to understand, since most aspects of large-scale structure are either described by this model, or are simple extensions based on it.

A. Newtonian equations of motion

In this problem, we will consider the perturbations in the matter, δ_m and $\mathbf{v}_m$ (one type for now!), and gravity via the Newtonian gravitational potential Φ. We will start in real space. We will also use the fluid equations; the gas really does behave like a fluid, but cold dark matter will also behave like a fluid with zero pressure until we get to small enough scales for stream crossing to be important (i.e., particles from different regions of the initial conditions cross paths, so there is not a single velocity anymore).

1. *Navier-Stokes equations*

We recall from ordinary Newtonian physics that the density and velocity of a fluid behave according to the Navier-Stokes equations:

$$\frac{d\rho}{dt} = -\rho \boldsymbol{\nabla}\cdot\mathbf{u} \quad \text{and} \quad \frac{d\mathbf{u}}{dt} = -\frac{\boldsymbol{\nabla}p}{\rho} - \boldsymbol{\nabla}\Phi_{\rm tot}. \tag{11}$$

Here the d/dt is known as a *convective derivative* and differs from the regular derivative in the sense that it follows the motion of a particle instead of being fixed to the coordinate system:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}. \tag{12}$$

2. *Conversion to cosmology*

In cosmology, the same thing will be true, but we have to be careful about a few things. First, the gradients in Eq. (11) are physical gradients, and we need a factor of $1/a$ to convert them to gradients in comoving coordinates. Second, "$\mathbf{u}$" is a physical velocity measured relative to an observer at the origin, so it is related to the peculiar velocity $\mathbf{v}_m$ by

$$\mathbf{u} = aH\mathbf{r} + \mathbf{v}_m, \tag{13}$$

where we wrote $a\mathbf{r}$ instead of $\mathbf{r}$ to account for the fact that $\mathbf{r}$ is a comoving position. Finally, "$\Phi_{\rm tot}$" in Eq. (11) is the total gravitational potential, but in cosmology we will usually use "Φ" to denote a perturbation to the background potential: thus we make the replacement $\Phi_{\rm tot} \to \bar\Phi + \Phi$.

The convective derivative is now

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}_m\cdot\boldsymbol{\nabla}. \tag{14}$$

(Technical note on partial derivatives: you will note that if we are rigorous in writing Eq. 14 there should be a term of the form $H\mathbf{r}\cdot\boldsymbol{\nabla}$. However, in Eq. 12, we wrote the partial derivative $\partial/\partial t$ at fixed physical position $a\mathbf{r}$, whereas in Eq. 14 we wrote it at fixed $\mathbf{r}$. This change cancels the $H\mathbf{r}\cdot\boldsymbol{\nabla}$ term.)

With these replacements, the density equation becomes

$$\frac{d\rho_m}{dt} = -\rho_m\frac{1}{a}\boldsymbol{\nabla}\cdot(aH\mathbf{r}+\mathbf{v}_m) = -3H\rho_{\rm m} - \rho_m\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_m. \tag{15}$$

Writing $\rho_m = \bar\rho_m(1+\delta_m)$, we can re-write this as

$$\frac{d\bar\rho_m}{dt}(1+\delta_m) + \bar\rho_m\frac{d\delta_m}{dt} = -3H\bar\rho_{\rm m}(1+\delta_m) - \bar\rho_m(1+\delta_m)\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_m; \tag{16}$$

simplifying, and using $d\bar\rho_m/dt = -3H\bar\rho_m$, gives

$$\frac{d\delta_m}{dt} = -(1+\delta_m)\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_m. \tag{17}$$

The velocity equation becomes

$$\frac{d(aH)}{dt}\mathbf{r} + aH\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{v}_m}{dt} = -\frac{\boldsymbol{\nabla}p}{a\bar\rho_m(1+\delta_m)} - \frac{1}{a}\boldsymbol{\nabla}\bar\Phi - \frac{1}{a}\boldsymbol{\nabla}\Phi. \tag{18}$$

Of course, we know that the rate of change of comoving coordinate is $d\mathbf{r}/dt = \mathbf{v}_m/a$. We can further see by considering the homogeneous Universe case (no perturbations: $\mathbf{v}_m = 0$, $\boldsymbol{\nabla} p = 0$, $\Phi = 0$) that the $d(aH)/dt\,\mathbf{r}$ term must cancel the $(1/a)\boldsymbol{\nabla}\bar{\Phi}$ term. (Comment: yes, this means that in Newtonian physics in a homogeneous Universe, $\bar{\Phi} \propto r^2$. There is a minimum of the potential! But every observer thinks they are at the minimum of the potential.) Simplifying then gives

$$H\mathbf{v}_m + \frac{d\mathbf{v}_m}{dt} = -\frac{\boldsymbol{\nabla} p}{a\bar{\rho}_m(1+\delta_m)} - \frac{1}{a}\boldsymbol{\nabla}\Phi. \tag{19}$$

Finally, the gravitational potential perturbation is sourced by the matter density perturbations by the usual Poisson equation:

$$\frac{1}{a^2}\nabla^2\Phi = 4\pi G\bar{\rho}_m\delta_m. \tag{20}$$

Combining these three results gives the following system of equations for perturbations in the matter:

$$\begin{aligned}
\frac{\partial \delta_m}{\partial t} &= -\frac{1}{a}\mathbf{v}_m\cdot\boldsymbol{\nabla}\delta_m - (1+\delta_m)\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_m, \\
\frac{\partial \mathbf{v}_m}{\partial t} &= -\frac{1}{a}(\mathbf{v}_m\cdot\boldsymbol{\nabla})\mathbf{v}_m - H\mathbf{v}_m - \frac{\boldsymbol{\nabla} p}{a\bar{\rho}_m(1+\delta_m)} - \frac{1}{a}\boldsymbol{\nabla}\Phi, \quad \text{and} \\
\nabla^2\Phi &= 4\pi G a^2\bar{\rho}_m\delta_m.
\end{aligned} \tag{21}$$

B. Linearization and Fourier decomposition

We may now approximate Eq. (21) to linear order, dropping any terms that are second order and higher in the density or velocity perturbations. We will also neglect the pressure terms. This leads to

$$\frac{\partial \delta_m}{\partial t} = -\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}_m, \quad \frac{\partial \mathbf{v}_m}{\partial t} = -H\mathbf{v}_m - \frac{1}{a}\boldsymbol{\nabla}\Phi, \quad \text{and} \quad \nabla^2\Phi = 4\pi G a^2\bar{\rho}_m\delta_m. \tag{22}$$

Taking a Fourier wave mode in the z-direction, this becomes

$$\frac{\partial \tilde{\delta}_m}{\partial t} = -\frac{ik}{a}\tilde{v}_{m,z}, \quad \frac{\partial \tilde{v}_{m,z}}{\partial t} = -H\tilde{v}_{m,z} - \frac{ik}{a}\tilde{\Phi}, \quad \text{and} \quad -k^2\tilde{\Phi} = 4\pi G a^2\bar{\rho}_m\tilde{\delta}_m. \tag{23}$$

The x and y components of the velocity are vector modes, and de-couple from the scalar density perturbations as promised. They obey $\partial\tilde{v}_{m,x}/\partial t = -H\tilde{v}_{m,x}$, etc., so they simply decay with time. In what follows, we focus only on the scalar modes.

Equation (23) is two first-order ODEs, but can be combined into a single second-order ODE. Using the first equation to write $\tilde{v}_{m,z} = (ia/k)\partial\tilde{\delta}_m/\partial t$, and substituting this into the second equation, we get

$$\frac{\partial}{\partial t}\left(\frac{ia}{k}\frac{\partial\tilde{\delta}_m}{\partial t}\right) = -H\frac{ia}{k}\frac{\partial\tilde{\delta}_m}{\partial t} - \frac{ik}{a}\frac{-1}{k^2}4\pi G a^2\bar{\rho}_m\tilde{\delta}_m. \tag{24}$$

Algebraic simplification, and using $da/dt = aH$, gives

$$\frac{\partial^2\tilde{\delta}_m}{\partial t^2} = -2H\frac{\partial\tilde{\delta}_m}{\partial t} + 4\pi G\bar{\rho}_m\tilde{\delta}_m. \tag{25}$$

The amazing fact about Eq. (25) is that it doesn't depend on k. The equation simply says that a density perturbation grows as

$$\tilde{\delta}_m(k,t) = \delta_+(k)G_+(t) + \delta_-(k)G_-(t), \tag{26}$$

where $G_\pm(t)$ are the two linearly independent solutions to Eq. (25). These solutions are called *growth functions* and describe how matter perturbations grow with time. Normally of the two linearly independent solutions, one defines $G_+(t)$ to be the growing solution, and $G_-(t)$ to be the decaying solution; one expects $G_+(t)$ to dominate in the late Universe, and often only this function is called the "growth function."

C. Growth functions for different scenarios

We may evaluate the growth function for a few simple cases of interest.

1. Einstein-de Sitter Universe

The simplest case we consider is the Einstein-de Sitter Universe, with $\Omega_m = 1$, $a \propto t^{2/3}$, and then

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}, \quad \text{and} \quad \bar{\rho}_m = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G t^2}. \tag{27}$$

The growth equation, Eq. (25), is then

$$\ddot{G} = -\frac{4}{3t}\dot{G} + \frac{2}{3t^2}G. \tag{28}$$

This is a special type of ODE – it is a dimensionally homogeneous equation and it has two solutions of the form $G \propto t^\mu$. If we substitute t^μ into the equation, we get

$$\mu(\mu-1)t^{\mu-2} = -\frac{4}{3t}\mu t^{\mu-1} + \frac{2}{3t^2}t^\mu \quad \rightarrow \quad \mu^2 + \frac{1}{3}\mu - \frac{2}{3} = 0. \tag{29}$$

This is a quadratic equation with two solutions, $\mu = \frac{2}{3}$ and $\mu = -1$. Thus we have

$$G_+(t) \propto t^{2/3} \propto a \quad \text{and} \quad G_-(t) \propto t^{-1} \propto a^{-3/2}. \tag{30}$$

We can see that matter density perturbations will grow in proportion to the scale factor.

2. Cold + hot dark matter

Another possibility is that there is a mixture of *cold dark matter* (negligible pressure) and *hot dark matter* (very fast-moving particles that form a smooth background, at least relative to the scale of interest). In fact, we know there is at least one kind of hot dark matter – the cosmic neutrino background. In this case, we can suppose that during the "matter-dominated" era, there is a fraction of the matter

$$f_{\rm hot} = \frac{\sum_{i=1}^3 m_{\nu,i}}{94\Omega_{\rm m}h^2\,{\rm eV}} \approx \frac{\sum_{i=1}^3 m_{\nu,i}}{11\,{\rm eV}} \tag{31}$$

that is hot. This changes Eq. (28) by inserting a factor of $1 - f_{\rm hot}$ in the density of matter that can cluster, or

$$\ddot{G} = -\frac{4}{3t}\dot{G} - \frac{2}{3t^2}(1 - f_{\rm hot})G. \tag{32}$$

The equation is still dimensionally homogeneous, but now we have

$$\mu^2 + \frac{1}{3}\mu - \frac{2}{3}(1 - f_{\rm hot}) = 0, \tag{33}$$

and the solution to the quadratic equation is

$$\mu = \frac{-1 \pm \sqrt{25 - 24 f_{\rm hot}}}{6}. \tag{34}$$

This means the growing solution is of the form

$$\mu = \frac{-1 + \sqrt{25 - 24 f_{\rm hot}}}{6} \approx \frac{2}{3} - \frac{2}{5}f_{\rm hot} + \ldots \quad (\text{for } f_{\rm hot} \ll 1). \tag{35}$$

We then find that

$$G_+(t) \propto t^{2/3 - (2/5)f_{\rm hot}} \propto a^{1-(3/5)f_{\rm hot}}. \tag{36}$$

Therefore the presence of massive neutrinos will slow down the growth of structure (again, on scales small compared to the distance the neutrinos can travel per Hubble time). Even if the sum of the neutrino masses is 0.06 eV (the smallest possible given oscillation mass splittings), then we have $G_+ \propto a^{0.997}$. With percent-level precision and a few e-folds of cosmic structure growth from the CMB era to galaxy survey measurements, we may be able to measure this in the near future!

3. The radiation-dominated era

We can take the extreme limit of asking about the growth of perturbations in the dark matter during the radiation-dominated era. Now we have $a \propto t^{1/2}$, so $H = 1/(2t)$. The matter density is far below critical so $4\pi G\rho_{\rm m} \ll H^2$, and we will neglect it. Then the growth equation is

$$\ddot{G} = -\frac{1}{t}\dot{G}. \tag{37}$$

This is dimensionally homogeneous and has a repeated root at $\mu = 0$. Either following ODE theory or by inspection, we can see that the solutions are

$$G_+(t) = \ln t = 2\ln a + \text{const} \quad \text{and} \quad G_-(t) = 1. \tag{38}$$

Thus structure in the dark matter grows logarithmically during the radiation-dominated era.

4. Dark energy

Finally, we take the case where the Universe is dominated by dark energy, $\Omega_\Lambda \to 1$. In this case, we may once again ignore the matter, and take

$$H = \frac{8\pi G\rho_\Lambda}{3} = \text{constant}. \tag{39}$$

Then the growth equation becomes

$$\ddot{G} = -2H\dot{G}, \tag{40}$$

and the two modes are

$$G_+(t) = 1 \quad \text{and} \quad G_-(t) = e^{-2Ht} \propto \frac{1}{a^2}. \tag{41}$$

Thus we can see that if dark energy (in the form of Λ) takes over, the growth of structure will freeze. This is already starting to happen; in a ΛCDM universe (such as – apparently – our own), density perturbations initially grow as $\delta \propto a$, and then grow more slowly than this as the dark energy becomes important.

We therefore see how the rate of growth of cosmic structure is one of the most important measurements in our study of dark energy (and neutrinos), and why cosmologists have put so much effort into measuring it.