問4.1

(1) $\log \frac{\pi_i}{1-\pi_i} = \log \exp(\beta_0 + \beta_1 x_i)$

$= \beta_0 + \beta_1 x_i$

(2) systematic componet

$$y = \frac{\exp(\beta^T x)}{1+\exp(\beta^T x)}$$

random componet

ベルヌーイ分布 $f(y_i | \pi_i) = \pi_i^{y_i}(1-\pi_i)^{1-y_i}$ $\begin{pmatrix} y_i = 0, 1 \\ i = 1, 2, \cdots, n \end{pmatrix}$

(3) $y_1, \cdots, y_n$ に基づく尤度関数 $L(\pi_1, \pi_2, \cdots, \pi_n)$ は $L(\beta_0, \beta_1)$ とも表せて、

$$L(\beta_0, \beta_1) = \prod_{i=1}^{n} f(y_i | \pi_i) = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{1-y_i}$$

$$= \prod_{i=1}^{n} \left(\frac{\pi_i}{1-\pi_i}\right)^{y_i}(1-\pi_i)$$

$$= \prod_{i=1}^{n} \left[\exp\{y_i(\beta_0+\beta_1 x)\} \left\{\frac{1}{1+\exp(\beta_0+\beta_1 x_i)}\right\}\right].$$

(4) $l(\beta) = \log L(\beta) = \sum_{i=1}^{n} y_i \beta^T x_i - \sum_{i=1}^{n} \log\{1+\exp(\beta^T x_i)\}.$

(5) $\frac{\partial l}{\partial \beta}(\beta) = \sum_{i=1}^{n} \frac{\partial(y_i \beta^T x_i)}{\partial \beta} - \sum_{i=1}^{n} \frac{\partial}{\partial \beta}\left(\log\{1+\exp(\beta^T x_i)\}\right)$

$$= \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} \frac{x_i \cdot \exp(\beta^T x_i)}{1+\exp(\beta^T x_i)} = \sum_{i=1}^{n} (y_i - \pi_i) x_i.$$

2階微分は第1項は寄与せず

$$\frac{\partial^2 l}{\partial \beta \partial \beta^T} = -\sum_{i=1}^{n} \frac{\partial \pi_i}{\partial \beta^T} x_i = -\sum_{i=1}^{n} \pi_i (1-\pi_i) x_i x_i^T.$$

問4.2

(1) $y = F(x)$

$$= \int_{-\alpha}^{x} f(t)dt$$

$$= \int_{-\infty}^{x} \overbrace{\beta_1 \exp\{(\beta_0+\beta_1 t) - \exp(\beta_0+\beta_1 t)\}}^{\boxed{\beta_1 e^{\beta_0+\beta_1 t - e^{\beta_0+\beta_1 t}}}} dt \quad (\beta_1 > 0)$$

ここで $z = \exp(\beta_0+\beta_1 t)$ とおくと、$dz = \beta_1 \exp(\beta_0+\beta_1 t)\, dt$、

t	$-\infty \longrightarrow x$
z	$0 \longrightarrow \exp(\beta_0+\beta_1 x)$

であるから、

$$= \int_{0}^{\exp(\beta_0+\beta_1 x)} \beta_1 \exp(\beta_0+\beta_1 t)\ e^{-z} \left[\beta_1 \exp(\beta_0+\beta_1 t)\right]^{-1} dz$$

$$= \int_{0}^{\exp(\beta_0+\beta_1 x)} e^{-z}\, dz$$

$$= 1 - \exp\{-\exp(\beta_0+\beta_1 x)\}$$ ■

(2) まず $\dfrac{dF}{dx} = \beta_1 \exp(\beta_0+\beta_1 x) \exp[-\exp(\beta_0+\beta_1 x)] > 0$ であり

単調関数であることが言えて、$F(-\infty) = 0$, $F(\infty) = 1$ だから

$(0, 1)$ 区間の値 y を出力する。■

(3) $g(y) = \log\{-\log(1-y)\}$ とすると、

$$g(y) = \log\{-\log(\exp[-\exp(\beta_0+\beta_1 x)])\}$$

$$= \log\{\exp(\beta_0+\beta_1 x)\}$$

$$= \beta_0 + \beta_1 x$$ ■

問4.3

$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}dt$ は $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$

を用いて、($u=t/\sqrt{2}$ とおけば) $\Phi(z)=\frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)$

と表せる。これから、$\Phi^{-1}(z)=\sqrt{2}\,\mathrm{erf}^{-1}(2z-1)$ を得る。

よって、

$$
\begin{aligned}
\Phi^{-1}(y) &= \sqrt{2}\,\mathrm{erf}^{-1}(2y-1) \\
&= \sqrt{2}\,\mathrm{erf}^{-1}\left(2\Phi\left(\frac{x-\mu}{\sigma}\right)-1\right) \\
&= \sqrt{2}\,\mathrm{erf}^{-1}\left(2\left(\frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right)-1\right) \\
&= \sqrt{2}\cdot\frac{x-\mu}{\sqrt{2}\sigma} \\
&= \frac{x-\mu}{\sigma} \quad \blacksquare
\end{aligned}
$$

(erfを定義しなくても、$\Phi^{-1}(y)=\frac{x-\mu}{\sigma}$ は自明では?)