Obstructions to deforming curves on Fano 3-folds and 4-folds

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Fano 多様体と双有理幾何学

Today's slide arXiv:2501.15788

Plan of Talk

1. Mumford's pathology (Motivation)
2. Hilbert scheme of Fano $\mathbf{3}$-folds
3. Obstructions to deforming space curves lying on a del Pezzo surface (Kleppe-Ellia conjecture and its generalization, cf. arXiv:2501.15788)

§1 Mumford's example (Motivation)

Hilbert scheme

We work over a field $k = \overline{k}$ of $\operatorname{char} k = 0$.

$X \subset \mathbb{P}^n$: a closed subscheme.
$\mathcal{O}_X(1)$: a very ample line bundle on X.
$C \subset X$: a closed subscheme with Hilbert polynomial $P(C) = P$.

Theorem 1 (Grothendieck'60)

There exists a projective scheme $\mathrm{Hilb}_P\, X$ (called the Hilbert scheme of X), parametrizing all closed subschemes C' of X with (the same) Hilbert polynomial P.

Let $\mathrm{Hilb}\, X := \bigsqcup_P \mathrm{Hilb}_P\, X$ (full Hilbert scheme) and let $\mathrm{Hilb}^{sc}\, X$ denote the open and closed subscheme

$$\mathrm{Hilb}^{sc}\, X := \{\text{smooth connected curves } C \subset X\} \subset \mathrm{Hilb}\, X.$$

Today we consider $\mathrm{Hilb}^{sc}\, X$ of a smooth Fano manifold X from the viewpoint of Mumford's example.

Infinitesimal property of Hilbert schemes

- The tangent space of $\mathbf{Hilb}\, X$ at $[C]$ is isomorphic to $H^0(C, N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(C, N_{C/X})$ $\quad(\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathbf{Hilb}\, X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- We say $C \subset X$ is unobstructed if $\mathbf{Hilb}\, X$ is nonsingular at $[C]$.
- $H^1(C, N_{C/X}) = 0 \Longrightarrow C$ is unobstructed. The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*.).

Purpose 2

Determine $\dim_{[C]} \mathbf{Hilb}\, X$ at a singular point $[C]$ of $\mathbf{Hilb}\, X$.

Mumford's example (pathology)

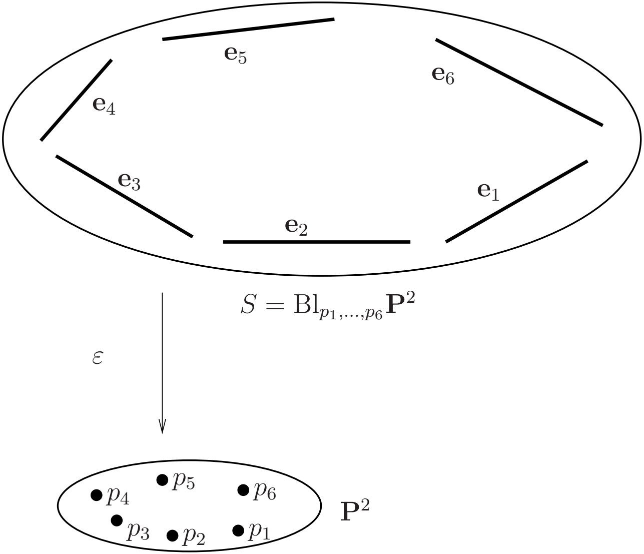
Example 1 (Mumford'62)

$\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W whose general member C satisfies

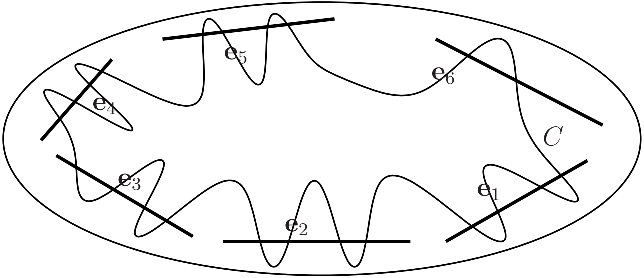
1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ ($\simeq \mathbb{P}^{37}$) on S.
3. $\dim W = 56$, $h^0(C, N_{C/\mathbb{P}^3}) = 57$, and C is of degree 14 and genus 24.

$$\begin{array}{ccccc}
C & \in & W^{(56)} & \subset & \mathbf{Hilb}^{sc}\,\mathbb{P}^3 \\
\downarrow & & \downarrow \mathbb{P}^{37}\text{-bundle} & & \\
(E, S) & \in & U^{(19)} & \subset & G \times |\mathcal{O}_{\mathbb{P}^3}(3)| \\
\downarrow & & \downarrow \mathbb{P}^{15}\text{-bundle} & & \downarrow \\
E & \in & \mathrm{Gr}(2,4)^{(4)} & = & \{\text{lines in } \mathbb{P}^3\}
\end{array}$$

Smooth cubic and Blow-up of $\mathbb{P}^2$

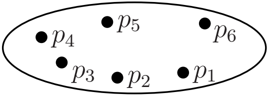


Curves on a smooth cubic (Mumford's ex.)



ε

$$S_3 = \mathrm{Bl}_{p_1,\dots,p_6}\mathbf{P}^2$$
$$[C] = (12; 4, 4, 4, 4, 4, 2) \in \mathrm{Pic} S_3 = \mathbf{Z}^7$$
$$d = 3 \cdot 12 - 4 - 4 - 4 - 4 - 4 - 2 = 14$$
$$g = \frac{(12-1)(12-2)}{2} - 6 - 6 - 6 - 6 - 6 - 3 = 24$$



$\mathbf{P}^2$

Example 1 (Mumford'62)

$\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W whose general member C satisfies

1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line a line E on S such that C belongs to a linear system $\Lambda := |-4K_S + 2E| \simeq \mathbb{P}^{37}$ on S.
3. $\dim W = 56$, $h^0(C, N_{C/\mathbb{P}^3}) = 57$, and C is of degree 14 and genus 24.

- The above example appeared in a paper "Further pathologies in algebraic geometry".
- Here C and $\mathbb{P}^3$ are geometrically innocent-looking (a pathology).
- Later many non-reduced components of $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, Fløystad'93, N'05, Kleppe-Ottem'15, etc. and also those of $\mathrm{Hilb}^{sc}\,\mathbb{P}^n$ $(n > 3)$ have been more recently found by Choi-Iliev-Kim'22,23.

Murphy's law in AG

Moreover, to the question:"How bad can the deformation space of an object be?", R. Vakil has answered:

Law 3 (*Murphy's law in AG*)

Unless there is some a priori reason otherwise, the deformation space may be as bad as possible

Theorem 4 (Vakil'06)

The following moduli spaces satisfy Murphy's law,i.e., they have every singularity type of finite type over $\mathbb{Z}$! :

- the Hilbert scheme of smooth connected curves $C \subset \mathbb{P}^r$ $(r \geq 4)$
- the versal deformation spaces of smooth n-folds X (with very ample K_X, $n \geq 2$)
- the Hilbert scheme of smooth surfaces $S \subset \mathbb{P}^r$ $(r \geq 4)$
- ...

A generalization of Mumford's example (with Mukai)

We have found that in Mumford's example, (-1)-curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

Theorem 5 (Mukai-N'09, $\mathrm{char}\, k \geq 0$)

Let X be a smooth projective 3-fold satisfying the following:

1. there exists a smooth rational curve E on X s.t. $N_{E/X}$ is globally generated, and
2. there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathrm{Hilb}^{sc} X$ has infinitely many generically non-reduced components (GNRC).

Remark 6

1. In Mumford's ex., $X = \mathbb{P}^3$, S is a smooth cubic, E is a line.
2. Many uniruled 3-folds X satisfy the assumption of the theorem.

The idea of the proof

- Let $\varepsilon : S \to F$ be the contraction of the (-1)-curve E and $\Delta \geq 0$ a sufficiently general divisor on F. We consider a linear system $|\varepsilon^*\Delta - K_X\big|_S + 2E|$ on S and its general member C (i.e. a smooth curve on S).
- We consider an irreducible component $\mathcal{W}_{C,S}$ of the Hilbert-flag scheme

$$\mathrm{HF}\, X = \left\{(C', S') \;\middle|\; \text{two closed subschemes of } X \text{ s.t. } C' \subset S'\right\}$$

  passing through the point $[(C, S)]$, and let $W_{C,S}$ be it image in $\mathrm{Hilb}^{sc}\, X$.
- For every general $C \in W_{C,S}$, there exists a first order infinitesimal deformation $\tilde{C}$ of C in X not contained in any $\tilde{S}$ of S in X. We prove its obstruction $\mathbf{ob}(\tilde{C})$ is nonzero (which will be explained later).

§2 Hilbert schemes of Fano 3-folds

Hilbert scheme of Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index r of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \operatorname{Pic} X$.

Let X be a smooth Fano 3-fold of index r.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskih'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Applying Theorem 5, we obtain

Example 2 (N'10)

If $r > 1$ (and of any $\rho(X)$), then $\mathrm{Hilb}^{sc} X$ contains a a generically non-reduced component W satisfying:

1. every general member C of W is contained in a smooth del Pezzo surface S ($\sim -\frac{r-1}{r}K_X$), and
2. there exists a (good) line E on S and $C \sim -K_X\big|_S + 2E$ in $\operatorname{Pic} S$.
3. $h^0(C, N_{C/X}) = \dim W + 1$.

Here

- A curve $E \subset X$ is a line $\overset{\text{def}}{\Longleftrightarrow} E \simeq \mathbb{P}^1$ and $-\frac{1}{r}K_X.E = 1$.
- A line $E \subset X$ is good $\overset{\text{def}}{\Longleftrightarrow} N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$ (for $r = 2, 3$).
- $\dim W = 56$, 42 and $(-K_X)^3/2 + 4$ for $r = 4, 3, 2$, respectively.

Hilbert scheme of prime Fano 3-folds ($r = 1$)

If X is prime ($r = 1$), then there exists NO del Pezzo surface $S \subset X$. However, we can make use of $K3$ surfaces $S \subset X$ and (-2)-curves $E \simeq \mathbb{P}^1$ on S instead of (-1)-curves.

Theorem 7 (N'19)

Let X be a prime Fano 3-fold of genus $g := (-K_X)^3/2 + 1$. Then $\mathrm{Hilb}^{sc} X$ contains a generically non-reduced component W with the following properties:

1. Every general member C of W is contained in a $K3$ surface S $(\sim -K_X)$.
2. There exists a good conic $E \simeq \mathbb{P}^1$ on S such that $C \sim -2K_X\big|_S + 2E$.
3. $\dim W = 5g + 1$, $h^0(C, N_{C/X}) = 5g + 2$, and C is of degree $4g$ and genus $4g + 1$.

Here a conic E on X is called good if $N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$.

Another generalization of Mumford's example

Corollary 8

If X is a smooth Fano 3-fold and $\rho(X) = 1$, then $\mathbf{Hilb}^{sc}\, X$ contains a generically non-reduced component.

<table>
<tr><th>3-fold X</th><th>surface S</th><th>$[C] \in \mathrm{Pic}\, S$</th><th>E</th><th></th></tr>
<tr><td>$\mathbb{P}^3$</td><td rowspan="3">del Pezzo</td><td rowspan="3">$-K_X\big|_S + 2E$</td><td rowspan="3">line</td><td>Mumford['62]</td></tr>
<tr><td>$Q^3 \subset \mathbb{P}^4$</td><td>Mukai-N['09]</td></tr>
<tr><td>del Pezzo</td><td>Mukai-N['09], N['10]</td></tr>
<tr><td>prime Fano</td><td>$K3$</td><td>$-2K_X\big|_S + 2E$</td><td>conic</td><td>N['19]</td></tr>
</table>

Table: Generically non-reduced component of Mumford type

Enriques surface and half pencil

Definition 9

A smooth projective surface S is called Enriques if $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$ and $2K_S \sim 0$.

Remark 10

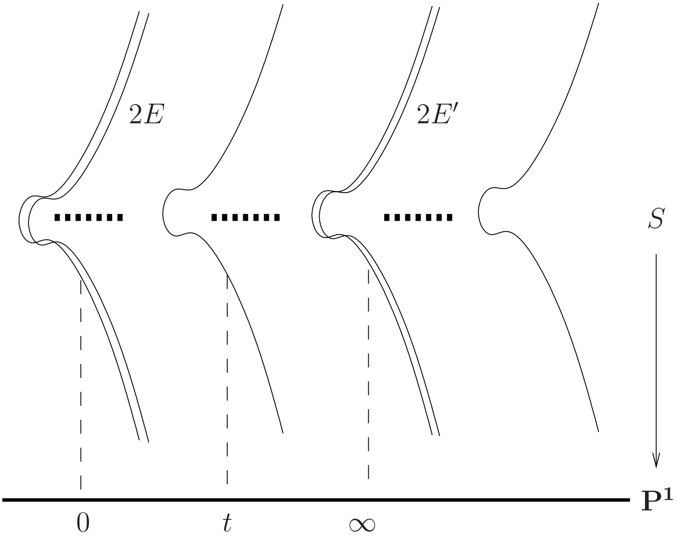
Let S be an Enriques surface. Then

1. $S \simeq X/\varepsilon$ for some $K3$ surface X and a fixed-free involution ε.
2. S admits an elliptic fibration $\varphi : S \longrightarrow \mathbb{P}^1$, which has two double fibers $\varphi^{-1}(0) = 2E$ and $\varphi^{-1}(\infty) = 2E'$.

Definition 11

Such effective divisors E and E' are called a half pencil on S.

Elliptic fibration on Enriques surface



Enriques-Fano 3-folds

Definition 12

An Enriques-Fano 3-fold (EF3, for short) is a projective 3-fold $X \subset \mathbb{P}^N$ containing an Enriques surface S as a hyperplane section, and such that X is not a cone over S.

Remark 13

Every EF3 X has isolated sings (Conte-Murre'85). X has only cyc. quot. term. sings if and only if $X \simeq Y/\theta$ for a smooth Fano Y and an involution θ on Y (classified by Bayle'94 and Sano'95).

Example 3 (EF3 of $g = 9$)

Let $Y \subset \mathbb{P}^5$ be a smooth complete intersection of 2 quadrics

$$q_i(x_0, x_1, x_2) + q_i'(x_3, x_4, x_5) = 0 \quad (i = 1, 2), \qquad (\heartsuit)$$

where $x_0, \ldots, x_5$ are coordinates of $\mathbb{P}^5$. Consider the involution on Y

$$\theta : (x_0, x_1, x_2, x_3, x_4, x_5) \longmapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5).$$

Then $X := Y/\theta$ is an EF3. In fact, Y contains a smooth $K3$ on which θ acts freely.

Theorem 14 (N'21)

Let X be an Enriques-Fano 3-fold, S an Enriques surface in X. The $\mathrm{Hilb}^{sc}\, X$ contains a generically non-reduced component if there exists a half pencil E on S such that

1. $(-K_X.E)_X \geq 2$, and
2. $H^1(E, N_{E/X}(E)) = 0$, where $N_{E/X}(E) := N_{E/X} \otimes_E N_{E/S}$,

Generalization of Mumford's example

We have obtained the following non-reduced components so far:

3-fold X	surface S	$[C] \in \mathrm{Pic}\, S$	E	
$\mathbb{P}^3$	del Pezzo	$-K_X\big\|_S + 2E$	line	Mumford['62]
$Q^3 \subset \mathbb{P}^4$				Mukai-N['09]
del Pezzo				Mukai-N['09], N['10]
$\mathbb{P}^3$ or $X_4 \subset \mathbb{P}^4$	$K3$		elliptic curve	N['17]
prime Fano		$-2K_X\big\|_S + 2E$	conic	N['19]
Enriques-Fano	Enriques	$-K_X\big\|_S + 2E$	half pencil	N['21]

Table: Generically non-reduced component of Mumford type

Question 15

(non-reduced comp. of $\mathrm{Hilb}\, X$) $\overset{\text{relation?}}{\longleftrightarrow}$ ($\mathbb{P}^1$ or elliptic curves on X)

§3 Obstructions to deforming space curves lying on a del Pezzo surface

Primary obstructions

Let X be a projective scheme over k, C a loc. c. i. closed subscheme of X, and $k[\varepsilon] := k[t]/(t^2)$ (the ring of dual numbers).

- A first order (infinitesimal) deformation of C is a deformation $\tilde{C}$ ($\subset X \times \mathbf{Spec}\, k[\varepsilon]$) of C in X over $k[\varepsilon]$.
- $\tilde{C}$ naturally corresponds to $\alpha \in H^0(C, N_{C/X})$ ($\simeq \mathbf{Hom}(\mathcal{I}_C, \mathcal{O}_C)$).
- There is an element $\mathbf{ob}(\alpha)$ in $H^1(C, N_{C/X})$ (called the primary obstruction of α) such that

$$\mathbf{ob}(\alpha) = \mathbf{0} \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$

- $\mathbf{ob}(\alpha)$ can be expressed as a cup product, and

$$\mathbf{ob}(\alpha) = \alpha \cup \mathbf{e} \cup \alpha \qquad \text{in } \mathbf{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

  where $\mathbf{e} := [0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0]$.
- $\mathbf{ob}(\alpha) \neq \mathbf{0}$ for some α implies that $\mathbf{Hilb}\, X$ is singular at $[C]$ by infinitesimal lifting property of smoothness.

Obstructedness Criterion (with Mukai)

[Mukai-N'09] gave a sufficient condition for $\mathrm{ob}(\alpha) \neq 0$ when $\dim X = 3$.
Let C be an irreducible curve on a 3-fold X.

Assumption 1

- there exists an intermediate surface $C \subset S \subset X$ s.t. $C \hookrightarrow S$ and $S \hookrightarrow X$ are regular embeddings.
- there exists an irreducible curve E ($\neq C$) on S s.t. $[\mathcal{O}_S \hookrightarrow \mathcal{O}_S(E)] \otimes \mathcal{O}_S(mE)$ induces the injection

$$H^1(S, \mathcal{O}_S(mE)) \hookrightarrow H^1(S, \mathcal{O}_S((m+1)E))$$

for all $m > 0$. (e.g. E=(-1)-$\mathbb{P}^1$ on S)

Obstructedness Criterion (Continued)

Let $\alpha \in H^0(N_{C/X})$ be a first order deformation of C in X and $\mathrm{ob}(\alpha) \in H^1(N_{C/X})$ its primary obstruction. We consider the *"exterior" components*

$$\pi_{C/S}(\alpha) := H^0(\pi_{C/S})(\alpha), \qquad \mathrm{ob}_S(\alpha) := H^1(\pi_{C/S})(\mathrm{ob}(\alpha)).$$

by the projection

$$\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C.$$

Assumption 2

- Suppose $\pi_{C/S}(\alpha)$ lifts to a global section β of $N_{S/X}(E)$.

$$\begin{array}{ccccccc} & & & H^0(N_{S/X}) & \subset & H^0(N_{S/X}(E)) & \ni \beta \\ & & & \downarrow & & \downarrow & \\ \alpha \in & H^0(N_{C/X}) & \xrightarrow{\pi_{C/S}} & H^0(N_{S/X}\big|_C) & \subset & H^0(N_{S/X}(E)\big|_C) & \end{array}$$

Here β is called an infinitesimal deformation with pole:

Obstructedness Criterion (Continued)

Theorem 16 (Mukai-N'09)

$\mathbf{ob}_S(\alpha)$ is nonzero if

1. $\Delta.E = 2(-E^2 + g(E) - 1)$, where $\Delta := C + K_X\big|_S - 2E$ in $\operatorname{Pic} S$.
2. Let $\beta\big|_E$ be the principal part of β along E. Then $\beta\big|_E \cup \mathbf{k}_E \neq 0$ in $H^1(E, \mathcal{O}_E(2E))$, where

$$\mathbf{k}_E := [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}\big|_E \longrightarrow 0] \in \operatorname{Ext}^1_E(N_{S/X}\big|_E, N_{E/S}).$$

3. the restriction map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

Remark 17

In [Mukai-N'09], this criterion was applied to the proof of Thm. 5. We obtained the generically non-reduced components explained in §2 by this criterion.

Stable degeneration

Toward a generalization into higher dimensions, we study the deformations of space curves lying on a del Pezzo surface of degree. Let

$$C \subset S \subset X$$

be a flag of algebraic varieties.

Definition 18

We say $C \subset X$ is stably degnerate (or stably contained in S), if every small deformation C' of C in X is contained in a deformation S' of S in X.

If there exists a component $\mathcal{W}_{C,S}$ of $\mathrm{HF}\, X$ passing through (C, S) such that the first projection

$$pr_1 : \mathcal{W}_{C,S} \to \mathrm{Hilb}\, X, \qquad (C', S') \mapsto [C']$$

is locally surjective at $[C] \in \mathrm{Hilb}\, X$, then $C \subset X$ is *stably degenerate*.

Kleppe-Ellia conjecture

Conjecture (Kleppe'87, modified by Ellia'87)

Let $C \subset S_3 \subset \mathbb{P}^3$ be a smooth connected curve of degree d and genus g lying on a smooth cubic surface $S_3 \subset \mathbb{P}^3$. Then C is *stably degenerate* if

1. $g \geq 3d - 18$,
2. C is linearly normal, i.e. $H^1(\mathcal{I}_C(1)) = 0$,
3. $d > 9$ and C is general in $[C] \in \operatorname{Pic} S_3$.

Remark 19

1. the first two assumptions are necessary for the conclusion.
2. The conjecture is known to be true if
   - C is 3-normal, i.e. $H^1(\mathcal{I}_C(3)) = 0$ (Kleppe'87),
   - C is not 3-normal and $g >> d$ (Kleppe'87 and Ellia'87), or
   - C is 2-normal, i.e. $H^1(\mathcal{I}_C(2)) = 0$ (N'23)

Generalized Kleppe-Ellia conjecture

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a smooth del Pezzo surface $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is *stably degenerate* if

1. $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$,
2. C is linearly normal,
3. $\deg(C) > 9$ for $n = 3$ and $\deg(C) > 2n$ for $n \geq 4$, and C is general in $[C] \in \operatorname{Pic} S_n$.

Remark 20

The first assumption is equivalent to that

$$\dim_{(C,S)} \mathrm{HF}\, \mathbb{P}^n = \chi(N_{(C,S)/\mathbb{P}^n}) \geq \chi(N_{C/\mathbb{P}^n}) = (\text{exp.dim.of } \mathbf{Hilb}\, X \text{ at } [C]),$$

where $N_{(C,S)/X} := N_{C/X} \times_{N_{S/X}\big|_C} N_{S/X}$.

Application

Proposition 21

If the generalized K-E conjecture is true, then

$$\dim_{[C]} \operatorname{Hilb} \mathbb{P}^n = d + g + n^2 + 9 \; (= \dim_{(C,S)} \operatorname{HF} \mathbb{P}^n).$$

Thus we can determine the dimension of $\operatorname{Hilb} \mathbb{P}^n$ at (even singular) point $[C]$.

Theorem A (Unobstructedness)

We focus on the case $n = 4$, i.e. $S \simeq S_4$ is a smooth complete intersection $S_{2,2}$ in $\mathbb{P}^4$. We say $C \subset \mathbb{P}^4$ is 2-normal if $H^1(\mathcal{I}_C(2)) = 0$, and 2-nonspecial if $H^1(\mathcal{O}_C(2)) = 0$.

Theorem A

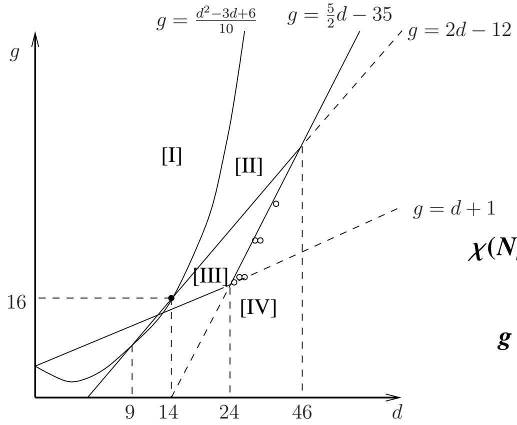
Let $C \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 8$ and genus g contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$. Then

1. Such curves C are parametrised by a *finite union of locally closed irreducible subsets* $W \subset \mathbf{Hilb}^{sc}\,\mathbb{P}^4$ of the same dimension $d + g + 25$.
2. If C is 2-normal, then the closure $\overline{W}$ of W in $\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ is a genericaly smooth component of $\mathbf{Hilb}^{sc}\,\mathbb{P}^4$.
3. If C is 2-nonspecial, then $\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ is smooth along W and $\overline{W}$ is a (proper) closed subset of $\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ of codimension $2h^1(\mathcal{I}_C(2))$.

Theorem A (continued)

Theorem A (continued)

4. C is 2-normal (resp. 2-nonspecial) if (d, g) belongs to the region [I] (resp. [IV] except the 6 pairs corresponding to $\circ$).



$$\chi(N_{S/\mathbb{P}^4}(-C)) \geq 0$$
$$\Updownarrow$$
$$g \geq 2d - 12.$$

Theorem B (Obstructedness)

We expect that if $(d, g) \in$ [II] and C is not 2-normal, then $\overline{W}$ corresponds to a generically non-reduced component of $\mathbf{Hilb}^{sc}\, \mathbb{P}^4$.

Theorem B

Let W be a maximal irreducible family of smooth connected curve $C \subset \mathbb{P}^4$ of degree d and genus g contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$. If $d > 8$, $g \geq 2d - 12$ and $h^1(\mathcal{I}_C(2)) = 1$ (then $(d, g) \in$ [II]), then

1. every general member C of W is stably degenerate and obstructed,
2. $\overline{W}$ is a component of $(\mathbf{Hilb}^{sc}\, \mathbb{P}^4)_{\mathbf{red}}$, and
3. $\mathbf{Hilb}^{sc}\, \mathbb{P}^4$ is generically non-reduced along $\overline{W}$.

Corollary 22

Generalized K-E conjecture holds to be true, if $n = 4$ and $h^1(\mathcal{I}_C(2)) = 1$.

Analogy of Mumford's example

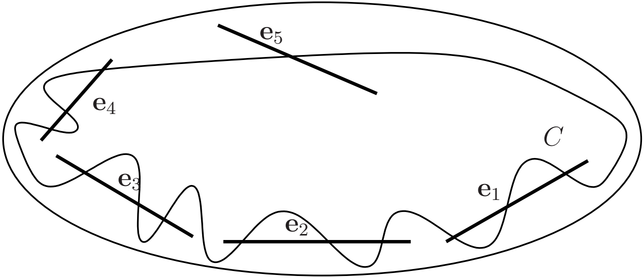
Example 4

$\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W whose general member C satisfies

1. C is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. there exists a line E on S such that C belongs to a complete linear system $\Lambda := |-3K_S + 2E|$ ($\simeq \mathbb{P}^{29}$) on S, and
3. $\dim W = 55$, $h^0(C, N_{C/\mathbb{P}^4}) = 57$, and C is of degree 14 and genus 16.

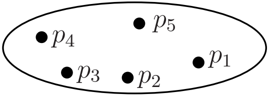
$$
\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(55)} \arrow[d, "\mathbb{P}^{29}\text{-b'dle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^4 \\
(E,S) \arrow[d, mapsto] & \in & U^{(26)} \arrow[d, "G(2,12)\text{-b'dle}"] & \subset & G \times G(2, H^0(\mathcal{O}_{\mathbb{P}^4}(2))) \arrow[d] \\
E & \in & G(2,5)^{(6)} & = & \{\text{lines in } \mathbb{P}^4\}.
\end{tikzcd}
$$

Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)



$\varepsilon \downarrow$

$$S_{2,2} = \mathrm{Bl}_{p_1,\dots,p_5}\mathbf{P}^2$$
$$[C] = (9; 3, 3, 3, 3, 1) \in \mathrm{Pic}\, S_{2,2} = \mathbf{Z}^6$$
$$d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$
$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$



$\mathbf{P}^2$

Sketch of Proof of Thm. B

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 8$ and genus g lying on a smooth complete intersection $S_{2,2}$ in $\mathbb{P}^4$.

- $S = S_{2,2}$ s.t $C \subset S \subset \mathbb{P}^4$ is uniquely determined by $d > 8$.
- Since $H^i(N_{C/S}) = H^i(N_{S/\mathbb{P}^4}) = 0$ for $i > 0$, it follows from an exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{(C,S)/\mathbb{P}^4} \xrightarrow{\pi_2} N_{S/\mathbb{P}^4} \longrightarrow 0$$

  that $H^i(N_{(C,S)/\mathbb{P}^4}) = 0$ for $i > 0$, which implies $\mathrm{HF}\,\mathbb{P}^4$ is nonsingular and of expected dimension at (C, S).
- There exists another exact sequence

$$0 \longrightarrow N_{S/\mathbb{P}^4}(-C) \longrightarrow N_{(C,S)/\mathbb{P}^4} \xrightarrow{\pi_1} N_{C/\mathbb{P}^4} \longrightarrow 0,$$

  where π_1 induces the tangent map p_1 of $pr_1 : \mathrm{HF}\,\mathbb{P}^4 \to \mathrm{Hilb}\,\mathbb{P}^4$ at (C, S) and we obtain

$$H^0(N_{(C,S)/\mathbb{P}^4}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^4}) \longrightarrow \underbrace{H^1(N_{S/\mathbb{P}^4}(-C))}_{\simeq H^1(\mathcal{I}_C(2))^{\oplus 2}} \longrightarrow 0.$$

Sketch of Proof of Thm. B

- Suppose now that $g \geq 2d - 12$ and $h^1(\mathcal{I}_C(2)) = 1$. Then $\dim \operatorname{coker} p_1 = 2$ and there exists a line E on S such that
$$|C + 2K_S| = |C + 2K_S - E| + E. \qquad \text{(Zariski decomp.)}$$
- We note $N_{S/\mathbb{P}^4} \simeq \mathcal{O}_S(-2K_S)^{\oplus 2}$ and
$$H^1(N_{S/\mathbb{P}^4}(E - C)) = H^1(-L^{\oplus 2}) = 0,$$
because $L := C + 2K_S - E$ is nef and big (by $g \geq 2d - 12$).
- For every $\alpha \in H^0(N_{C/\mathbb{P}^4}) \setminus \operatorname{im} p_1$, its *exterior component* $\pi_{C/S}(\alpha)$ in $H^0(N_{S/\mathbb{P}^4}\big|_C)$ lifts to a global section β of $N_{S/\mathbb{P}^4}(E)$ (after admitting a pole along E).
- Applying a *"modification"* of the obstructedness criterion to the infinitesimal deformation β with poles, we obtain $\mathrm{ob}(\alpha) \neq 0$. This implies
$$\dim_{[C]} \operatorname{Hilb}^{sc} \mathbb{P}^4 = \dim_{(C,S)} \operatorname{HF} \mathbb{P}^4 = d + g + 25,$$
and thereby C is obstructed and stably degenerate. □

$S_{2,2}$-maximal families of curves in $\mathbb{P}^4$

Let $C \subset S \subset X$ be a flag of algebraic varieties. We say an irreducible closed subset W of $\mathbf{Hilb}^{sc}\, X$ is S-maximal if there exists an irreducible component $\mathcal{W}_{C,S}$ of $\mathbf{HF}^{sc}\, X$ $(:= pr_1^{-1}(\mathbf{Hilb}^{sc}\, X))$ passing through (C, S) and $pr_1(\mathcal{W}_{C,S}) = W$.

If $d > 8$, then there exists a natural 1-to-1 correspondence between the set of $S_{2,2}$-maximal families in $\mathbf{Hilb}^{sc}_{d,g}\, \mathbb{P}^4$ and the set of 6-tuples of integer $(a; b_1, \dots, b_5)$ satisfying

$$a > b_1 \geq \cdots \geq b_5 \geq 0 \quad \text{and} \quad a \geq b_1 + b_2 + b_3 \tag{1}$$

and

$$d = 3a - \sum_{i=1}^{5} b_i \quad \text{and} \quad g = \frac{(a-1)(a-2)}{2} - \sum_{i=1}^{5} \frac{b_i(b_i - 1)}{2}, \tag{2}$$

by coordinates in $\mathbf{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$, i.e.,

$$[C] = a[\varepsilon^* \mathcal{O}_{\mathbb{P}^2}(1)] - \textstyle\sum_{i=1}^{5} b_i \mathbf{e}_i \longleftrightarrow (a; b_1, \dots, b_5).$$

Theorem C (Criterion)

Theorem C

Let $W := W(a; b_1, \ldots, b_5) \subset \mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4$ be the $S_{2,2}$-maximal family of smooth connected curves of degree d and genus g in $\mathbb{P}^4$ correponding to $(a; b_1, \ldots, b_5)$. Suppose that $d > 10$ and $g \geq 2d - 12$. Then

1. If $b_5 \geq 2$, then W is an irreducible component of $\mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4$ and $\mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4$ is generically smooth along W.
2. If $b_5 = 1$ and $b_4 \geq 2$, then W is an irreducible component of $(\mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4)_{\mathrm{red}}$ and $\mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4$ is generically non-reduced along W.
3. If $b_5 = 0$, then W is not an irreducible component of $(\mathbf{Hilb}^{sc}_{d,g} \mathbb{P}^4)_{\mathrm{red}}$, i.e., there exists an irreducible component of $V \supsetneqq W$.

Examples

Table: $S_{2,2}$-maximal families in $\mathbf{Hilb}^{sc}_{d,g}\ \mathbb{P}^4$

(d, g)	$(a; b_1, b_2, b_3, b_4, b_5)$	$W(a; b_1, b_2, b_3, b_4, b_5)$
$(14, 16)$	$(8; 2, 2, 2, 2, 2)$	gen. smooth component
$(14, 16)$	$(9; 4, 3, 2, 2, 2)$	gen. smooth component
$(14, 16)$	$(9; 3, 3, 3, 3, 1)$	gen. non-reduced component
$(15, 18)$	$(9; 4, 2, 2, 2, 2)$	gen. smooth component
$(15, 18)$	$(9; 3, 3, 3, 2, 1)$	gen. non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18, 24)$	$(9; 2, 2, 2, 2, 1)$	gen. non-reduced component
$(18, 24)$	$(10; 4, 3, 3, 1, 1)$	unknown ($h^1(\mathcal{I}_C(2)) = 2$)
$(18, 24)$	$(10; 3, 3, 3, 3, 0)$	non-component ($h^1(\mathcal{I}_C(2)) = 3$)
$(18, 24)$	$(11; 6, 3, 2, 2, 2)$	gen. smooth component
$\vdots$	$\vdots$	$\vdots$

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