

# Deformations of space curves lying on a del Pezzo surface

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Today's slide

## Photo 1 (at Prof. Miyazaki's 65, 2025)



## Photo 2 (at Kumamoto, 2025)



## Photo 3 (at a workshop on IBS, 2024)



# Plan of Talk

- ➊ Hilbert schemes and deformation of flags
- ➋ Kleppe-Ellia conjecture and its generalization (Main result)  
(cf. [arXiv:2501.15788](https://arxiv.org/abs/2501.15788))
- ➌ Applications and Examples

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# §1 Hilbert schemes and deformation of flags

# the Hilbert scheme

Given a projective scheme  $X$ , and given Hilbert polynomial  $P$ ,

$$\mathbf{Hilb}_P X = \{C \subset X \mid \text{closed subscheme of } P(C) = P\}$$

is called the **Hilbert scheme** of  $X$ . The Hilbert scheme has the following nice properties:

- fine moduli scheme, i.e. it has a **universal family**  $\mathcal{C} \subset X \times \mathbf{Hilb}_P X$  such that every deformation of  $C$  in  $X$  derived from  $\mathcal{C}$
- **projective** ( $\mathbf{Hilb}_P X \hookrightarrow \mathbf{Gr}$ )
- existence of nice **deformation theories**, e.g., if  $C$  is a loc. c.i., then  $H^0(N_{C/X})$  and  $H^1(N_{C/X})$  resp. represent the tangent and obstruction spaces at  $[C]$ .

while it also has “not so nice” properties: e.g.

- it may have **bad singularities** (e.g. non-reduced components),
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## Dimension of $\text{Hilb } X$

For simplicity, we assume  $C$  is a c.i. in  $X$ . Then

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.} (= \chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \text{Hilb } X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

However, when  $H^1(N_{C/X}) \neq 0$ , it is hard to determine the dimension of  $\text{Hilb } X$  at  $[C]$  (depending on whether  $C$  is obstructed or not).

To resolve this problem, we take an intermediate variety  $C \subset S \subset X$  and use [Hilbert-flag scheme](#)

$$\text{HF } X = \{(C, S) \mid C \subset S \subset X\} \subset \text{Hilb } X \times \text{Hilb } X.$$

Let

$$N_{(C,S)} := N_{C/X} \times_{N_{S/X}|_C} N_{S/X}$$

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## Naive question

Then

$$H^1(N_{(C,S)/X}) = 0 \implies \text{HF } X \text{ is nonsingular at } (C, S)$$

and if moreover  $H^i(N_{(C,S)/X}) = 0$  for all  $i > 0$ , then  $\text{HF } X$  is of expected dimension

$$\chi(N_{(C,S)/X}) = \chi(N_{C/S}) + \chi(N_{S/X}).$$

Let  $\mathcal{W}_{C,S}$  be an irreducible component passing through  $(C, S)$ , and  $pr_1 : \text{HF } X \rightarrow \text{Hilb } X$ ,  $(C', S') \mapsto [C']$ , the 1st projection.

### Question 1

When is the image  $pr_1(\mathcal{W}_{C,S})$  of  $\mathcal{W}_{C,S}$  an irreducible component of  $\text{Hilb } X$ ?

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## Stable degeneration

Let  $C \subset S \subset X$  a sequence of closed subvarieties with  $H^i(N_{(C,S)/X}) = 0$  for all  $i > 0$ .

### Definition 2

We say  $C$  is **stably degenerate** or **stably contained in  $S$** , if for every small global deformation  $C'$  of  $C$  in  $X$ , there exists a global deformation  $S'$  of  $S$  in  $X$  such that  $S' \subset C'$ .

We have the following implications:

$$(1) \Rightarrow (2) \Rightarrow (3)$$

where

- ①  $H^1(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$ .
- ②  $pr_1 : \text{HF } X \rightarrow \text{Hilb } X$ ,  $(C', S') \mapsto [C']$  is smooth at  $(C, S)$ .
- ③  $C$  is stably degenerate and  $pr_1(\mathcal{W}_{C,S})$  is a component of  $\text{Hilb } X$ .

The implication “ $(2) \Rightarrow (3)$ ” follows from the fact that  $pr'_1 = pr_1|_{\mathcal{W}_{C,S}}$  is locally surjective at  $[C]$ .

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The implication “(2)  $\Rightarrow$  (3)” follows from the fact that  $pr'_1 = pr_1|_{W_{C,S}}$  is locally surjective at  $[C]$ .

## Stable degeneration

Let  $C \subset S \subset X$  a sequence of closed subvarieties with  $H^i(N_{(C,S)/X}) = 0$  for all  $i > 0$ .

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We say  $C$  is **stably degenerate** or **stably contained in  $S$** , if for every small global deformation  $C'$  of  $C$  in  $X$ , there exists a global deformation  $S'$  of  $S$  in  $X$  such that  $S' \subset C'$ .

We have the following implications:

$$(1) \Rightarrow (2) \Rightarrow (3)$$

where

- ➊  $H^1(I_{C/S} \otimes_S N_{S/X}) = 0$ .
- ➋  $pr_1 : \mathbf{HF} X \rightarrow \mathbf{Hilb} X, (C', S') \mapsto [C']$  is smooth at  $(C, S)$ .
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## §2 Kleppe-Ellia conjecture and its generalization

## Mumford's example and Kleppe's generalization

Given a projective scheme, we denote by  $\mathbf{Hilb}^{sc} X$  the Hilbert scheme of smooth connected curves in  $X$ , i.e.,

$$\mathbf{Hilb}^{sc} X = \{C \subset X \mid C: \text{smooth connected curve}\}.$$

Theorem 3 (Mumford'62, a pathology)

$\mathbf{Hilb}^{sc} \mathbb{P}^3$  contains a generically non-reduced component.

Every its general member, i.e., a smooth curve  $C \subset \mathbb{P}^3$ , was contained in a smooth cubic surface  $S_3 \subset \mathbb{P}^3$ .

Later, Kleppe['87] generalized this example systematically by using the coordinate of  $(a; b_1, \dots, b_6)$  of  $[C]$  in  $\mathbf{Pic} S_3 \simeq \mathbb{Z}^7$ :

$$C \sim al - \sum_{i=1}^6 b_i e_i \quad \longleftrightarrow \quad (a; b_1, \dots, b_6)$$

where  $l = [\mathcal{O}_{\mathbb{P}^2}(1)]$  and  $e_i$  ( $i = 1, \dots, 6$ ) are 6 exceptional curves.

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He proposed a conjecture, which can be reformulated as follows:

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# Del Pezzo surfaces

## Definition 4

A smooth projective surface  $S$  is called **del Pezzo** if  $-K_S$  is ample.

Every del Pezzo surface  $S$  is isomorphic to a blow-up of  $\mathbb{P}^2$  (at  $9 - n$  points) or  $\mathbb{P}^1 \times \mathbb{P}^1$ . The number  $n = (-K_S)^2$  is called the *degree* of  $S$ , and  $1 \leq n \leq 9$ .

## Example 1 (del Pezzo surfaces)

degree $n$	a description of $S_n$	$-K_S$
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4	quartic c.i. $S_{2,2} \subset \mathbb{P}^4$	
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### Proposition 5 (smoothness of flag-scheme)

Let  $C \subset S = S_n \subset \mathbb{P}^n$  be a smooth curve of degree  $d$  and genus  $g$  lying on a del Pezzo surface  $S_n$  ( $n \geq 3$ ). Then the Hilbert-flag scheme  $\text{HF } \mathbb{P}^n$  is nonsingular at  $(C, S)$  of expected dimension

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and  $H^i(N_{(C,S)/\mathbb{P}^n}) = 0$  for all  $i > 0$ .

In fact,  $H^i(N_{C/S}) = H^i(N_{S/\mathbb{P}^n}) = 0$  for  $i > 0$ , which implies  $C \subset S$  and  $S \subset \mathbb{P}^n$  have nice (unobstructed) deformations and hence so does  $(C, S)$  in  $\mathbb{P}^n$ .

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Toward a generalization, we study the deformations of space curves lying on a del Pezzo surface of degree.

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## Main result

We focus on the case  $n = 4$ , i.e.  $S \simeq S_4$  is a smooth complete intersection

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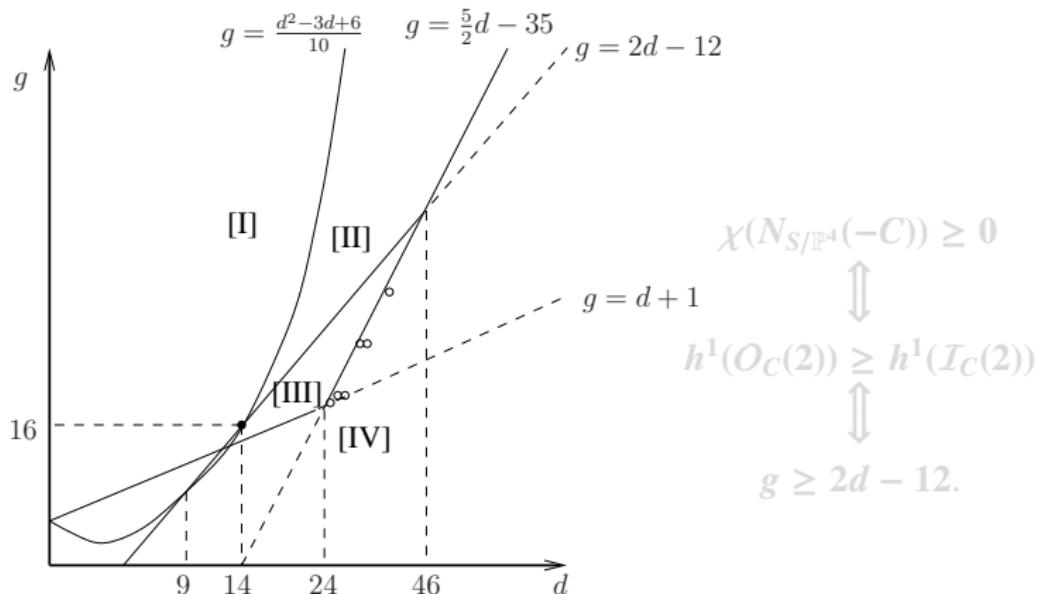
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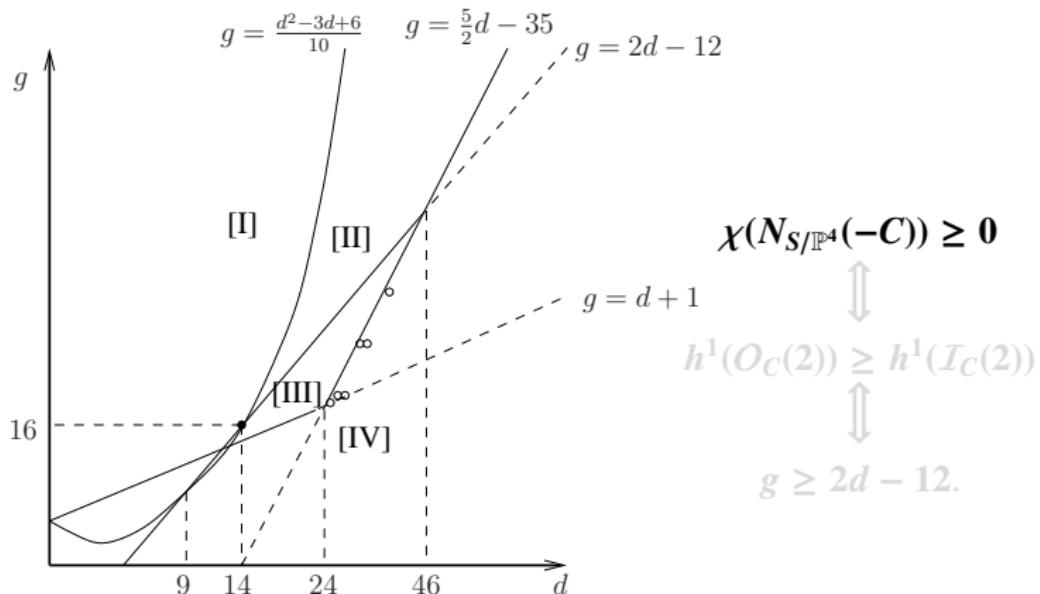
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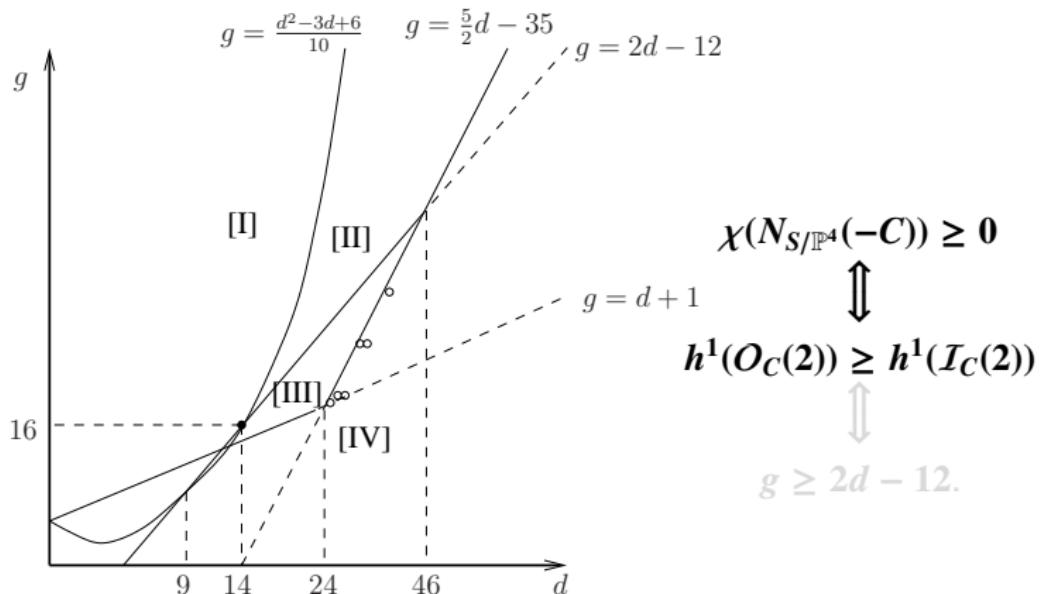
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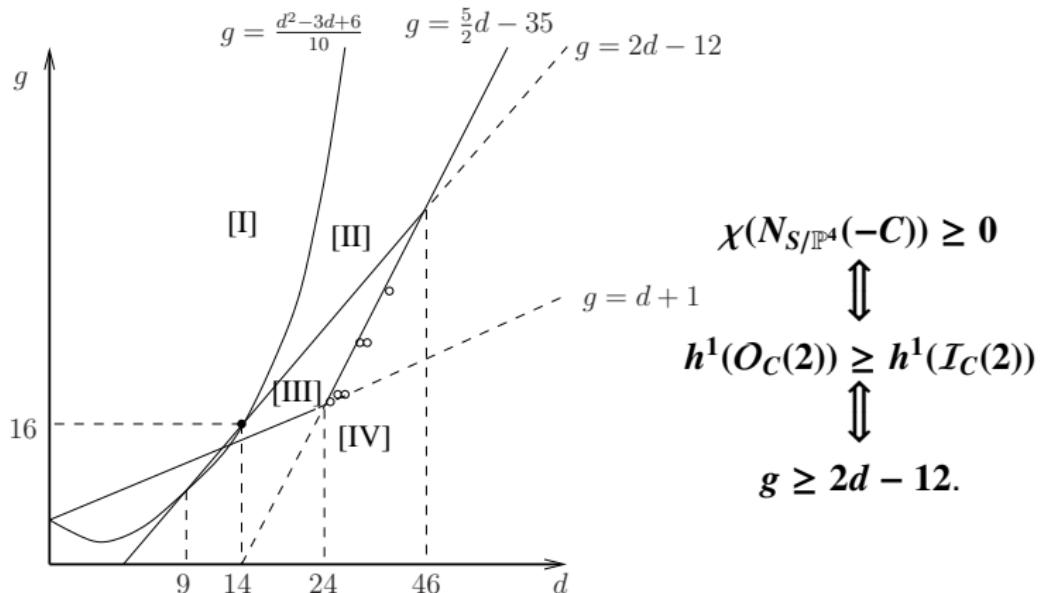
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## Applications

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Generalized K-E conjecture holds to be true, if  $n = 4$  and  $h^1(\mathcal{I}_C(2)) = 1$ .

Let  $C \subset S = S_{2,2} \subset \mathbb{P}^4$  be as in Theorem 7, and  $\mathcal{W}_{C,S}$  the irreducible component of  $\text{HF } \mathbb{P}^4$  passing through  $(C, S)$ , and put

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**Hilb<sup>sc</sup>  $\mathbb{P}^4$**  contains a **generically non-reduced** irreducible component  $W$  whose general member  $C$  satisfies

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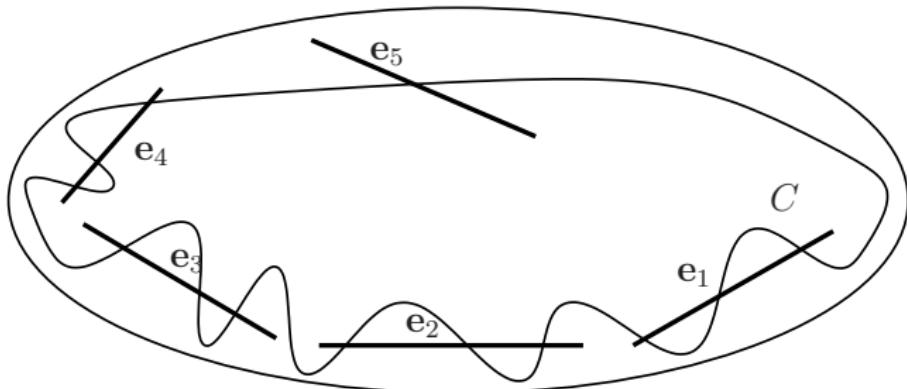
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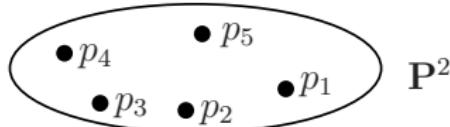
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## Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)



$$\begin{aligned}
 S_{2,2} &= \text{Bl}_{p_1, \dots, p_5} \mathbf{P}^2 \\
 [C] &= (9; 3, 3, 3, 3, 1) \in \text{Pic} S_{2,2} = \mathbf{Z}^6 \\
 d &= 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14 \\
 g &= \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16
 \end{aligned}$$



## Standard coordinate

Let  $D$  be a divisor on a smooth c.i.  $S_{2,2} \subset \mathbb{P}^4$ , i.e., a quartic del Pezzo surface. Then there exists a suitable blow-up  $\varepsilon : S_{2,2} \rightarrow \mathbb{P}^2$  such that

$$[D] = a - \sum_{i=1}^5 b_i e_i,$$

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# Examples

Table: curves on  $S_{2,2}$  and stable degeneration

$(d, g)$	$(a; b_1, b_2, b_3, b_4, b_5)$	$W(a; b_1, b_2, b_3, b_4, b_5)$
(14, 16)	(8; 2, 2, 2, 2, 2)	unobstructed and stab.degnerate
(14, 16)	(9; 4, 3, 2, 2, 2)	unobstructed and stab.degnerate
(14, 16)	(9; 3, 3, 3, 3, 1)	obstructed and stab.degnerate
(15, 18)	(9; 4, 2, 2, 2, 2)	unobstructed and stab.degnerate
(15, 18)	(9; 3, 3, 3, 2, 1)	obstructed and stab.degnerate
⋮	⋮	⋮
(18, 24)	(9; 2, 2, 2, 2, 1)	obstructed and stab.degnerate
(18, 24)	(10; 4, 3, 3, 1, 1)	unknown ( $h^1(\mathcal{I}_C(2)) = 2$ )
(18, 24)	(10; 3, 3, 3, 3, 0)	not stab.degnerate ( $h^1(\mathcal{I}_C(2)) = 3$ )
(18, 24)	(11; 6, 3, 2, 2, 2)	unobstructed and stab.degnerate
⋮	⋮	⋮

## further questions

- ① Deformations of curves lying on del Pezzo  $S_n \subset \mathbb{P}^n$  of degree  $n \geq 5$ .
- ② Deformation of degenerate curves on del Pezzo manifold of higher dimension ( $> 3$ ).
- ③ Study the relation to other examples of obstructed curves  $C \subset \mathbb{P}^n$  (or non-reduced components of  $\text{Hilb}^{sc} \mathbb{P}^n$ ).  
[Y. Choi–H. Iliev–S. Kim'24] have recently proved the existence of many non-reduced components of  $\text{Hilb}^{sc} \mathbb{P}^n$  of higher dimensional projective space  $\mathbb{P}^n$  by using ruled surfaces.
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Thank you very much for listening! and Happy birthday to Prof. Youngook Choi.



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