Obstructions to deforming space curves lying on complete intersection of quadrics

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Today's slide

Plan of Talk

1. Hilbert schemes of Fano $\mathbf{3}$-folds
2. Deformation of space curves lying on a complete intersection of quadrics (an attempt to generalize known results into higher dimensional varieties)

§1 Hilbert schemes of Fano threefolds

Hilbert schemes

We work over a field $k = \bar{k}$ of $\mathbf{char}\, k = 0$.
Let X be a projective scheme over k. We denote by $\mathbf{Hilb}\, X$ the Hilbert scheme of X. Today we consider the open and closed subscheme

$$\mathbf{Hilb}^{sc}\, X := \{\text{smooth connected curves } C \subset X\} \subset \mathbf{Hilb}\, X,$$

that is, the Hilbert scheme of curves in X.

Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index i of a Fano manifold X is the maximal integer i such that $-K_X \sim iH$ with some $H \in \mathrm{Pic}\, X$.

Let X be a smooth Fano 3-fold of index i.

- $X \simeq \mathbb{P}^3$ if $i = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $i = 3$, and X is called del Pezzo if $i = 2$, and prime if $i = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

i	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Hilbert scheme of Fano 3-folds

The Hilbert scheme of lines and conics have been studied widely. We can find the following survey in [Kuznetsov-Prokohrov-Shramov'18].

Theorem 1 (classical)

Let X be a smooth Fano 3-fold of $\rho = 1$ and i. Let $\Sigma(X)$ and $S(X)$ denote the Hilbert scheme of lines and conics, respectively. If X is general and not a quartic double solid ($i = d = 2$), then $\Sigma(X)$ and $S(X)$ are generically smooth.

i	X	$\Sigma(X)$	$S(X)$
4	$\mathbb{P}^3$	$\mathrm{Gr}(2,4)$	$\mathbb{P}(Sym^2 U^*)$
3	$Q = (2) \subset \mathbb{P}^4$	$\simeq \mathbb{P}^3$	$\mathrm{Bl}_{\mathrm{OG}(3,5)} \mathrm{Gr}(3,5)$
2 $(d \geq 3)$	$(3) \subset \mathbb{P}^4$ $(2) \cap (2) \subset \mathbb{P}^5$ $V_5 \subset \mathbb{P}^6$	surface of gen.type abelian surface $\mathbb{P}^2$	$\mathbb{P}^2$-b'dle/surface $\mathbb{P}^3$-b'dle/curve $\mathbb{P}^4$
1	prime (general)	smooth curve	irred. surface

$g = 0$ vs $g > 0$

Let X be a smooth Fano 3-fold. Then

- The Hilbert scheme $\Sigma(X)$ and $S(X)$ of lines and conics on X behaves well, except for low degree case of $i = 2$ and special prime Fano.
- More generally, the Hilbert scheme of smooth rational curves on X behaves well *by vanishing of obstructions* (deformation theorey).
- However, the geometry of the Hilbert scheme of curves of higher genus $g > 0$ in X becomes more complicated, due to their non-vanishings.

Infinitesimal property of Hilbert schemes

- The tangent space of $\mathbf{Hilb}\, X$ at $[C]$ is isomorphic to $H^0(N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(N_{C/X})$ $\quad (\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(= \chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathbf{Hilb}\, X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}} .$$

- $H^1(N_{C/X}) = 0 \Longrightarrow \mathbf{Hilb}\, X$ is nonsingular at $[C]$ (Then we say C is unobstructed in X). The converse is not true (e.g. c.i. space curves $C \subset \mathbb{P}$ usually have large *obstruction spaces* but always *unobstructed*.).

Question 2

Given singular point $[C]$, determine the local dimension of the Hilbert scheme $\mathbf{Hilb}\, X$.

Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry ['62]".

Example 1 (Mumford)

$\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W such that:

1. every general $C \in W$ is contained in a smooth cubic surface $S \subset \mathbb{P}^3$,
2. for general $C \in W$, there exists a line E on S such that
$$C \sim -4K_S + 2E$$
on S,
3. C is of degree $\mathbf{14}$ and genus $\mathbf{24}$, and
4. $\dim \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^3,[C]} = 56$, while $h^0(N_{C/\mathbb{P}^3}) = 57$.

Here $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

Generalization of Mumford's example

- Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and more recently, those of $\mathbf{Hilb}^{sc}\,\mathbb{P}^n$ ($n > 3$) have been found by Choi-Iliev-Kim'24-1, and '24-2.
- In these examples, generic curves (of the non-reduced component) were contained in some surfaces:

	surfaces
Mumford'62, Kleppe'87, Ellia'87, N'05,23	smooth cubic in $\mathbb{P}^3$
Gruson-Peskine'82	non-normal cubic in $\mathbb{P}^3$
Kleppe-Ottem'15	smooth quartic in $\mathbb{P}^3$
Choi-Iliev-Kim'24-1, '24-2	ruled surface in $\mathbb{P}^n$

Generalization to smooth Fano 3-folds

Theorem 3 (Mumford'62,Mukai-N'09,N'10,N'19)

If X is a smooth Fano $\mathbf{3}$-fold of $\rho = 1$ and index i, then $\mathbf{Hilb}^{sc}\, X$ contains a generically non-reduced irreducible component, whose general member C is contained in the following surface $S \subset X$ and classes $[C] \in \mathrm{Pic}\, S$.

<table>
<tr><th>i</th><th>surface S</th><th>$[C] \in \mathrm{Pic}\, S$</th><th>E</th></tr>
<tr><td>4</td><td rowspan="3">del Pezzo
$(S \in \lvert -\frac{i-1}{i} K_X \rvert)$</td><td rowspan="3">$-K_X\big|_S + 2E$</td><td rowspan="3">line</td></tr>
<tr><td>3</td></tr>
<tr><td>2</td></tr>
<tr><td>1</td><td>$K3$ $(S \in \lvert -K_X \rvert)$</td><td>$-2K_X\big|_S + 2E$</td><td>conic</td></tr>
</table>

Another generalization (with Mukai)

Theorem 4 (Mukai-N'09, $\mathbf{char}\ k \geq 0$)

Let X be a smooth projective 3-fold such that

1. there exists a free rational curve $E \simeq \mathbb{P}^1$ on X, and
2. there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathbf{Hilb}^{sc}\, X$ has infinitely many generically non-reduced irreducible components.

Theorem 5 (Mukai-N'09)

Let C be a general genus-5 curve, X_3 a general smooth cubic 3-fold, or of Fermat type cubic 3-fold:

$$X_3^{\mathrm{Fermat}} : x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \subset \mathbb{P}^4,$$

then $\mathbf{Hom}_8(C, X_3)$ has a generically non-reduced irreducible component of dimension 4 and $h^0(C, T_{X_3}\big|_C) = 5$.

§2 Deformations of space curves lying on a complete intersection of quadrics

Complete intersection of quadrics

Today we consider deformations of space curves $C \subset \mathbb{P}$ lying on a smooth surface S, when S are complete intersections

$$S_2 \subset \mathbb{P}^3, \quad S_{2,2} \subset \mathbb{P}^4, \quad S_{2,2,2} \subset \mathbb{P}^5$$

of quadrics. For each S and curves C lying on S, we consider the deformations of C in the projective space $\mathbb{P}$ using the flag

$$C \subset S \subset \mathbb{P}.$$

Hilbert-flag scheme

A main tool of our studies is the Hilbert-flag scheme $\mathrm{HF}\, X$ of X, i.e.

$$\mathrm{HF}\, X = \left\{(C,S) \mid C \subset S \subset X\text{: closed subschemes}\right\}.$$

- The normal sheaf $N_{(C,S)/X}$ of (C,S) in X is defined by the fiber product sitting in

$$\begin{tikzcd} N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\ N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C, \end{tikzcd}$$

  where $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection.
- If the two embeddings $C \hookrightarrow S \hookrightarrow X$ are regular, then the tangent space and the obstruction space of $\mathrm{HF}\, X$ at (C,S) is $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$, and

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} \mathrm{HF}\, X \leq h^0(X, N_{(C,S)/X}).$$

- There exist natural projections $pr_i : \mathrm{HF}\, X \to \mathrm{Hilb}\, X$ $(i = 1, 2)$ corresponding to $(C,S) \mapsto [C]$, and $(C,S) \mapsto [S]$.

Correspondingly, there exist two natural exact sequences

$$0 \longrightarrow \mathcal{I}_{C/S} \otimes_S N_{S/X} \longrightarrow N_{(C,S)/X} \xrightarrow{\pi_1} N_{C/X} \longrightarrow 0,$$
$$0 \longrightarrow N_{C/S} \longrightarrow N_{(C,S)/X} \xrightarrow{\pi_2} N_{S/X} \longrightarrow 0.$$

which inudces the tangent maps of projections pr_1 and pr_2.

Lemma 6

The Hilbert-flag scheme $\mathbf{HF}\,X$ is nonsingular at (C, S) of expected dimension

$$\chi(N_{(C,S)/X}) = \chi(N_{C/S}) + \chi(N_{S/X})$$

if

1. C is a curve, S is del Pezzo surface and X is a Fano 3-fold, or
2. C is a curve on an anti-polarlized del Pezzo surface S in $X = \mathbb{P}^n$.

Proof.

Suppose e.g. X is Fano a 3-fold and S is del Pezzo. Then by adjunction, $N_{S/X} \simeq -K_X\big|_S + K_S$ and $N_{C/S} \simeq -K_S\big|_C + K_C$, where $-K_X$ and $-K_S$ are ample. This implies the higher cohomology groups of $N_{S/X}$ and $N_{C/S}$ vanish, so does that of $N_{(C,S)/X}$. □

Smooth quadric surface S_2 in $\mathbb{P}^3$

We start from curves on smooth quadrics in $\mathbb{P}^3$.
Let $S = S_2 \subset \mathbb{P}^3$. Then

$$S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \qquad \text{and} \qquad \operatorname{Pic} S \simeq (\operatorname{Pic} \mathbb{P}^1)^{\oplus 2} = \mathbb{Z}^2.$$

Thus every curve C on S corresponds to *bidegree* $(a, b) \in \mathbb{Z}^2$ of C. Then C is of degree $a + b$ and genus $(a - 1)(b - 1)$.

Proposition 7 (Tannenbaum'78, Kleppe'87, etc)

Let $d > 4$ and $g \geq 0$. Let $C \subset \mathbb{P}^3$ be a smooth connected curve of degree d and genus g contained in a smooth quadric surface. Then the maximal irreducible family $W(a, b)$ of such curves $C \subset \mathbb{P}^3$ of bidegree (a, b) is of dimension $g + 2d + 8$, and

1. $\operatorname{Hilb} \mathbb{P}^3$ is smooth along $W(a, b)$.
2. If $g \geq 2d - 8$, then the closure $\overline{W}(a, b) \subset \operatorname{Hilb}^{sc} \mathbb{P}^3$ becomes a generically smooth component.
3. If $g < 2d - 8$, then $\overline{W}(a, b)$ is a proper closed subset of a component of $\operatorname{Hilb}^{sc} \mathbb{P}^3$ of codimension $2d - 8 - g$.

Sketch of proof of Proposition

Let $C \subset \mathbb{P}^3$ be contained in a smooth quadric $S = S_2 \subset \mathbb{P}^3$. Then $H^1(N_{C/S}) = 0$ and hence $\mathbf{HF}\,\mathbb{P}^3$ is nonsingular at (C, S) of expected dimension

$$\chi(N_{(C,S)}) = 2d + g + 8.$$

We see that either

[i] $H^1(\mathcal{I}_C(2)) = 0$ or [ii] $H^1(\mathcal{O}_C(2)) = 0$

holds. If $H^1(\mathcal{I}_C(2)) = 0$, then the tangent map

$$p_1 : \mathcal{T}_{\mathbf{HF}\,\mathbb{P}^3,(C,S)} \longrightarrow \mathcal{T}_{\mathbf{Hilb}\,\mathbb{P}^3,[C]}$$

of the 1st projection pr_1 is surjective at (C, S). Then pr_1 is smooth at (C, S). Since smooth morphism preserves nonsingularity, C is unobstructed.
If $H^1(\mathcal{O}_C(2)) = 0$, then by the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \xrightarrow{\pi_{C/S}} N_{S/\mathbb{P}^3}\big|_C \simeq \mathcal{O}_C(2) \longrightarrow 0,$$

we get $H^1(N_{C/\mathbb{P}^3}) = 0$. Thus $\mathbf{Hilb}\,\mathbb{P}^3$ is nonsingular at $[C]$. □

Smooth complete intersection $S_{2,2} \subset \mathbb{P}^4$

Let $S = S_{2,2} \subset \mathbb{P}^4$ be a smooth complete intersection. Then S is a del Pezzo surface of degree 4, isomorphic to $\mathbf{Bl}_{P_1,\dots,P_5}\,\mathbb{P}^2$ and $\mathrm{Pic}\,S \simeq \mathbb{Z}^6$ is generated by 5 exceptional curves $\mathbf{e}_i$ ($1 \le i \le 5$) and the pull back of $[\mathcal{O}_{\mathbb{P}^2}(1)]$ by $S \to \mathbb{P}^2$. Then every divisor D on S corresponds to a 6-tuple $(a; b_1, \dots, b_5)$ of integers $a, b_1, \dots, b_5$ by

$$[D] = a\mathbf{l} - \sum_{i=1}^{5} b_i \mathbf{e}_i.$$

We can take the standard coordinate $(a; b_1, \dots, b_5)$ of $[D]$ in $\mathrm{Pic}\,S_{2,2}$ so that they satisfies

$$b_1 \ge \cdots \ge b_5 \quad \text{and} \quad a \ge b_1 + b_2 + b_3.$$

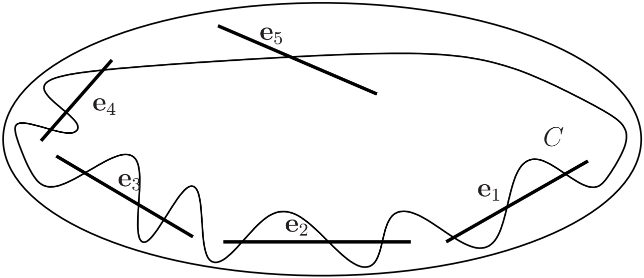
Example 2

If $C \sim -3K_S + 2E$, then

$$[C] = 3(3; 1, 1, 1, 1, 1) + 2(0; 0, 0, 0, 0, -1) = (9; 3, 3, 3, 3, 1)$$

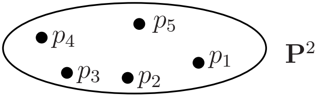
in $\mathrm{Pic}\,S$.

Curves on $S_{2,2} \subset \mathbb{P}^4$



ε

$$S_{2,2} = \mathrm{Bl}_{p_1,\ldots,p_5}\mathbf{P}^2$$
$$[C] = (9;3,3,3,3,1) \in \mathrm{Pic}S_{2,2} = \mathbf{Z}^6$$
$$d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$
$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$



Stable degeneration

Suppose that

$$C \subset S \subset \mathbb{P}$$

is a flag of a space curve C and a surface S.

Definition 3

We say C is stably degenerate if every small global deformation of C in $\mathbb{P}$ is contained in a deformation of S in $\mathbb{P}$, i.e.,

$$C \subset S \text{ and } C \rightsquigarrow C' \Longrightarrow S \rightsquigarrow {}^{\exists}S' \text{ s.t. } S' \supset C'$$

Theorem 8 (Main1)

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth curve of degree $d \geq 10$ of genus $g \geq 2d - 12$, contained in a smooth c.i. $S_{2,2}$ in $\mathbb{P}^4$. Let $(a; b_1, \ldots, b_5)$ be the standard coordinate of $[C]$ in $\mathbf{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$. Then

1. If $b_5 \geq 2$, then C is unobstructed and stably degenerate.
2. If $b_5 = 1$ and $b_4 \geq 2$. then C is obstructed and stably degenerate.
3. If $b_5 = 0$, then C is not stably degenerate.

Examples

Table: curves on $S_{2,2}$ and stable degeneration

(d,g)	the class of C	the max. irred. family of C ($\subset \mathbf{Hilb}^{sc}\,\mathbb{P}^4$)
$(14,16)$	$(8;2,2,2,2,2)$	generically smooth component
$(14,16)$	$(9;4,3,2,2,2)$	generically smooth component
$(14,16)$	$(9;3,3,3,3,1)$	generically non-reduced component
$(15,18)$	$(9;4,2,2,2,2)$	generically smooth component
$(15,18)$	$(9;3,3,3,2,1)$	generically non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18,24)$	$(9;2,2,2,2,1)$	generically non-reduced component
$(18,24)$	$(10;4,3,3,1,1)$	unknown ($h^1(\mathcal{I}_C(2))=2$)
$(18,24)$	$(10;3,3,3,3,0)$	a proper closed subset of a component
$(18,24)$	$(11;6,3,2,2,2)$	generically smooth component
$\vdots$	$\vdots$	$\vdots$

Analogy of Mumford's example

Example 4

$\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W such that:

1. general member $C \in W$ is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. for general $C \in W$, there exists a line E on S such that
$$C \sim -3K_S + 2E$$
on S (Here $|C| \simeq \mathbb{P}^{29}$),
3. C is of degree 14 and genus 16, and
4. $\dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^4} = 55$, while $h^0(C, N_{C/\mathbb{P}^4}) = 57$.

Thus $h^0(N_{C/\mathbb{P}^4}) = \dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^4} + 2$.

Smooth complete intersection $S_{2,2,2} \subset \mathbb{P}^5$

Let $S_{2,2,2} \subset \mathbb{P}^5$ be a smooth complete intersection. Then S is a $K3$ surface of degree $8 = 2^3$ and genus 5. Since $N_{S/\mathbb{P}^5} \simeq \mathcal{O}_S(2)^{\oplus 2}$, we have

$$H^1(N_{S/\mathbb{P}^5}(-C)) = H^1(S, -(C - 2\mathbf{h}))^{\oplus 2},$$

where $\mathbf{h} = \mathcal{O}_S(1)$. If $C - 2\mathbf{h}$ is nef and big and $H^1(N_{(C,S)/\mathbb{P}^5}) = 0$, then C is stably degenerate and unobstructed. On the other hands,

Theorem 9 (Main2)

For every integer $n \geq 2$, the Hilbert scheme $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components W_n,

1. whose general C is contained in a $S_{2,2,2} \subset \mathbb{P}^5$,
2. for general $C \in W_n$, there exists a line E on S such that
$$C \sim n(2\mathbf{h} + E),$$
3. C is of degree $d = 17n$ and genus $g = 17n^2 + 1$, and
4. $\dim_{[C]} \mathcal{O}_{\mathbf{Hilb}\,\mathbb{P}^5} = g + 53$, while $h^0(N_{C/\mathbb{P}^5}) = g + 56$.

Thus $h^0(N_{C/\mathbb{P}^5}) = \dim_{[C]} \mathcal{O}_{\mathbf{Hilb}\,\mathbb{P}^5} + 3$. Since $\mathbf{h}.E = 1$ and $E^2 = -2$, we have $C.E = 0$ and $(C - 2\mathbf{h}).E = -2$.

Construction

We consider the family

$$W_n := \left\{C \subset \mathbb{P}^5 \;\middle|\; C \subset S \text{ for some } S = S_{2,2,2} \subset \mathbb{P}^5 \text{ and } C \sim n(2\mathrm{h} + E)\right\}$$

of curves in $\mathbb{P}^5$. Then

$$\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(g+53)} \arrow[d, "\mathbb{P}^g\text{-bundle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^5 \\
(E,S) \arrow[d, mapsto] & \in & U^{(53)} \arrow[d, "\mathrm{Gr}(3,18)\text{-bundle}"] & \subset & G \times \mathrm{Gr}(3,V) \arrow[d] \\
E & \in & \mathrm{Gr}(2,6)^{(8)} & = & \left\{\text{lines in } \mathbb{P}^5\right\},
\end{tikzcd}$$

where $V = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$.

Key Lemma

Let X be a projective scheme, $\mathbf{HF}\, X$ the Hilbert-flag scheme of X, and

$$p_1 : H^0(N_{(C,S)/X}) \to H^0(N_{C/X})$$

the tangent map of the first proejction $pr_1 : \mathbf{HF}\, X \to \mathbf{Hilb}\, X, (C,S) \mapsto [C]$.

Lemma 10 (Key Lemma, cf. N['23])

We have $\dim_{(C,S)} \mathbf{HF}\, X = \dim_{[C]} \mathbf{Hilb}\, X$ if

1. $H^1(N_{(C,S)/X}) = H^0(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
2. for every $\alpha \in H^0(N_{C/X}) \setminus \operatorname{im} p_1$, the (primary) obstruction $\mathbf{ob}(\alpha)$ (to extend α a second order deformation over $k[t]/(t^3)$) is nonzero.

Sketch of Proof of Main2.

Let $C \subset \mathbb{P}^5$ be a smooth connected curve lying on a complete intersection $K3$ surface $S = S_{2,2,2} \subset \mathbb{P}^5$, and such that $C \sim n(2\mathbf{h} + E)$ in $\mathrm{Pic}\, S$ for $n \geq 2$, where E is a line on S.

- Since $d = 17n > 16 = 2h^2$, S is uniquely determined by C.
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by projective normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

  that $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$, which implies there exists a first order deformation of $\tilde{S}$ of S, to which E (and hence C) does not lift.
- Then $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$ for $i > 0$ and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

  is exact. Here p_1 is the tangent map of $pr_1 : \mathrm{HF}\,\mathbb{P}^5 \to \mathrm{Hilb}\,\mathbb{P}^5$ at (C, S) and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2\mathbf{h} - C))^{\oplus 3} \simeq k^3.$$

Sketch of Proof of Main2.(continued)

- We note that $H^1(N_{S/\mathbb{P}^5}(E - C)) = H^1(-L^{\oplus 3}) = 0$, since $L = C - 2\mathrm{h} - E = (n-1)(2\mathrm{h} + E)$ is nef and big, and we have

$$H^0(N_{S/\mathbb{P}^5}(E)) \xrightarrow{|_C} H^0(N_{S/\mathbb{P}^5}(E)\big|_C) \longrightarrow H^1(N_{S/\mathbb{P}^5}(E - C)) = 0.$$

- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \operatorname{im} p_1$, its *exterior component* $\pi_{C/S}(\alpha)$, i.e., the image of α in $H^0(N_{S/\mathbb{P}^5}(E)\big|_C)$, lifts to a global section β of $N_{S/\mathbb{P}^5}(E)$. (Here β is called an infinitesimal deformation with poles.)
- Applying a *"modification"* of the obstructedness criterion [Mukai-N'09] to β, we obtain $\mathbf{ob}(\alpha) \neq 0$. This implies

$$\dim_{[C]} \operatorname{Hilb}^{sc} \mathbb{P}^5 = \dim_{(C,S)} \operatorname{HF} \mathbb{P}^5$$

by the key lemma. Therefore C is obstructed and parametrised by an open dense subset of a component of $\operatorname{Hilb}^{sc} \mathbb{P}^5$ (thus yields to a non-reduced component of $\operatorname{Hilb}^{sc} \mathbb{P}^5$). □

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Primary obstructions

Let X be a projective scheme over k, C a loc. c. i. closed subscheme of X, and $k[\varepsilon] := k[t]/(t^2)$ (the ring of dual numbers).

- A first order (infinitesimal) deformation of C is a deformation $\tilde{C}$ ($\subset X \times \operatorname{Spec} k[\varepsilon]$) of C in X over $k[\varepsilon]$.
- $\tilde{C}$ naturally corresponds to $\alpha \in H^0(C, N_{C/X})$ ($\simeq \operatorname{Hom}(\mathcal{I}_C, \mathcal{O}_C)$).
- There is an element $\mathrm{ob}(\alpha)$ in $H^1(C, N_{C/X})$ (called the primary obstruction of α) such that
$$\mathrm{ob}(\alpha) = 0 \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$
- $\mathrm{ob}(\alpha)$ can be expressed as a cup product, and
$$\mathrm{ob}(\alpha) = \alpha \cup \mathbf{e} \cup \alpha \qquad \text{in } \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$
where $\mathbf{e} := [0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0]$.
- $\mathrm{ob}(\alpha) \neq 0$ for some α implies that $\operatorname{Hilb} X$ is singular at $[C]$ by infinitesimal lifting property of smoothness.