

Obstructions to deforming curves lying on a K3 surface

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Today's slide

Plan of Talk

- 1 Hilbert schemes and Mumford's example (Motivation)
- 2 Deformation of curves lying on a K3 surface
- 3 An example of non-reduced components of $\mathbf{Hilb}^{sc} \mathbb{P}^5$

§1 Hilbert schemes and Mumford's example (Motivation)

Hilbert schemes

We work over a field $k = \bar{k}$ of $\text{char } k = 0$.

Theorem 1 (Grothendieck'60)

Given a closed subscheme $X \subset \mathbb{P}^n$ and a closed subscheme $C \subset X$ with Hilbert polynomial $P(C) = P$, there exists a projective scheme $\text{Hilb}_P X$, parametrizing all closed subschemes C' of X with (the same) Hilbert polynomial P .

$\text{Hilb } X := \bigsqcup_P \text{Hilb}_P X$ is called the Hilbert scheme of X . Today we consider the open and closed subscheme

$$\text{Hilb}^{sc} X := \{\text{smooth connected curves } C \subset X\} \subset \text{Hilb } X,$$

that is, the Hilbert scheme of curves in X .

Infinitesimal property of Hilbert schemes

- The **tangent space** of $\text{Hilb } X$ at $[C]$ is isomorphic to $H^0(N_{C/X})$.
- $C \subset X$: a locally complete intersection \implies **every obstruction** to deforming C in X is contained in $H^1(N_{C/X})$ ($\subset \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$) and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim. (= } \chi(N_{C/X}) \text{ if } C \text{ is a curve)}} \leq \dim_{[C]} \text{Hilb } X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}} .$$

- We say $C \subset X$ is **unobstructed** if $\text{Hilb } X$ is **nonsingular** at $[C]$.
- $H^1(N_{C/X}) = 0 \implies C$ is **unobstructed**. The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*).

Purpose 2

Determine $\dim_{[C]} \text{Hilb } X$ at a **singular** point $[C]$ of $\text{Hilb } X$.

Mumford's example (pathology)

The following example appeared in a famous paper “Further pathologies in algebraic geometry [Mumford'62]”.

Example 1 (Mumford)

$\text{Hilb}^{sc} \mathbb{P}^3$ contains a **generically non-reduced irreducible component** W of dimension **56**, whose general member C satisfies:

- ① C is contained in a **smooth cubic surface** $S \subset \mathbb{P}^3$.
- ② There exists **a line** E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E| (\simeq \mathbb{P}^{37})$ on S .

Remark 3

- C and \mathbb{P}^3 are **innocent-looking (a pathology)**.
- C is of degree **14** and genus **24**, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \text{Hilb}^{sc} \mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

Generalization of Mumford's example

- Later **many non-reduced components** of $\text{Hilb}^{sc} \mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of $\text{Hilb}^{sc} \mathbb{P}^n$ ($n > 3$) have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves C corresponding to the generic point of the non-reduced components were contained in some surface $S \subset \mathbb{P}^n$, e.g.,

	a surface S containing C
Mumford['62]	smooth cubic
Gruson-Peskine'82	non-normal cubic
Kleppe'87	smooth cubic
Kleppe-Ottem'15	smooth quartic
Choi-Iliev-Kim'24-1, '24-2	ruled surface

Another generalization (with Mukai)

We found that in Mumford's example, (-1) -curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

Theorem 4 (Mukai-N'09, $\text{char } k \geq 0$)

Let X be a smooth projective 3-fold satisfying the following:

- ① there exists a smooth rational curve E on X s.t. $N_{E/X}$ is globally generated, and
- ② there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S , and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\text{Hilb}^{sc} X$ has infinitely many generically non-reduced components (GNRC).

Remark 5

In Mumford's ex., $X = \mathbb{P}^3$, S is a smooth cubic, E is a line.

Hilbert scheme of Fano 3-folds

- A **Fano manifold** is a smooth projective variety X with ample $-K_X$.
- The **index r** of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \text{Pic } X$.

Let X be a smooth Fano 3-fold of index r .

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called **del Pezzo** if $r = 2$, and **prime** if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist **17** deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	\mathbb{P}^3	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Applying Theorem 4, we obtain

Example 2 (N'10)

If $r(X) > 1$, then $\text{Hilb}^{sc} X$ contains a **a generically non-reduced component** W satisfying:

- ① every general member C of W is contained in a smooth **del Pezzo surface** S ($\sim -\frac{r-1}{r}K_X$), and
- ② there exists a **(good) line** E on S and $C \sim -K_X|_S + 2E$ in $\text{Pic } S$.
- ③ $h^0(C, N_{C/X}) = \dim W + 1$.

Here

- A curve $E \subset X$ is a **line** $\stackrel{\text{def}}{\iff} E \simeq \mathbb{P}^1$ and $-\frac{1}{r}K_X.E = 1$.
- A line $E \subset X$ is **good** $\stackrel{\text{def}}{\iff} N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$, or $\mathcal{O}_E \oplus \mathcal{O}_E(1)$ (for $r = 2, 3$).
- $\dim W = 56, 42$ and $(-K_X)^3/2 + 4$ for $r = 4, 3, 2$, respectively.

However, if X is prime ($r = 1$), then **there exists NO del Pezzo surface** $S \subset X$.

§2 Deformation of curves lying on a K3 surface

Curves on K3 surface

In this section, we study the deformations of smooth curves on a smooth Fano 3-fold, under the assumption that the curve is contained in a smooth **K3** surface.

Definition 6

A smooth projective surface S with $K_S \sim \mathbf{0}$ and $H^1(S, \mathcal{O}_S) = \mathbf{0}$ is called a **K3 surface**.

Let

$$C \subset S_{K3} \subset X_{Fano3},$$

a sequence of a curve, a **K3** surface, a Fano 3-fold. We will see the following curves on S control the deformations C in X :

- **(-2) -curves $E \simeq \mathbb{P}^1$,**
- **elliptic curves F**

Actually, they play a role very similar to that of **(-1) -curve $E \simeq \mathbb{P}^1$** on the smooth cubic $S_3 \subset \mathbb{P}^3$ in **Mumford's example**!

Hilbert-flag scheme

A main tool of our studies is **HF** X the Hilbert-flag scheme of X , i.e.

$$\mathbf{HF} X = \{(C, S) \mid C \subset S \subset X: \text{closed subschemes}\}.$$

If $C \hookrightarrow S \hookrightarrow X$ is regular embeddings, then

- the tang. sp. and the obst. sp. of **HF** X at (C, S) is $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$.
- the normal sheaf $N_{(C,S)/X}$ of (C, S) in X is the fiber product sitting in

$$\begin{array}{ccc} N_{(C,S)/X} & \xrightarrow{\pi_2} & N_{S/X} \\ \pi_1 \downarrow & \square & \downarrow |_{\mathcal{C}} \\ N_{C/X} & \xrightarrow{\pi_{C/S}} & N_{S/X}|_C. \end{array}$$

Here $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \rightarrow N_{S/X}|_C$ is the natural projection.

- Similarly to Hilbert schemes,

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} \mathbf{HF} X \leq h^0(X, N_{(C,S)/X}).$$

Lemma 7

Let $C \subset S_{K3} \subset X_{Fano3}$. Then TFAE:

- ① $H^1(N_{(C,S)/X}) = 0$, namely **HF** X is **nonsingular** at (C, S) of **expected dimension** $\chi(N_{(C,S)/X})$.
- ② There exists a first order deformation \tilde{S} of S in X , to which C does not lift.

Proof.

By adjunction, $N_{S/X} \simeq -K_X|_S$ and $N_{C/S} \simeq K_C$, which implies $H^i(N_{S/X}) = 0$ for $i > 0$ and $H^1(N_{C/S}) \simeq k$. There exists an exact sequence $0 \rightarrow N_{C/S} \rightarrow N_{(C,S)/X} \rightarrow N_{S/X} \rightarrow 0$, inducing

$$H^0(N_{(C,S)/X}) \xrightarrow{p_2} H^0(N_{S/X}) \rightarrow \underbrace{H^1(N_{C/S})}_{\simeq k} \rightarrow H^1(N_{(C,S)/X}) \rightarrow 0.$$

$H^1(N_{(C,S)/X}) = 0$ iff p_2 is not surjective, which is the second condition. □

Example 8

Let X be a prime Fano 3-fold, $S \in |-K_X|$ a K3 surface, $E \subset S$ a line on S . Then TFAE:

- ① E is of type $(0, -1)$, i.e. $N_{E/X} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$,
- ② $\text{Hilb } X$ is nonsingular at $[E]$ (of exp. dim.),
- ③ $\text{HF } X$ is nonsingular at (E, S) (of exp. dim.).

A line E on X is called **good** if E is of type $(0, -1)$, otherwise (that is of type $(1, -2)$) called **bad**.

Lemma 9

If X is prime, and E is a **good line** or a **good conic** on X contained in a smooth K3 surface $S \in |-K_X|$, then there exists a first order deformation \tilde{S} of S to which E does not lift, i.e., $H^1(N_{(E,S)/X}) = 0$.

A key lemma

Lemma 10 (char $k = 0$)

Let $S \xhookrightarrow{\iota} X$ be an embedded K3 surface, and let E be a curve on S with $H^1(\mathcal{O}_S(E)) = 0$ and $H^1(N_{(E,S)/X}) = 0$. If $C - bE \in \iota^* \text{Pic } X$ for some $b \neq 0$, then $H^1(N_{(C,S)/X}) = 0$.

Proof.

Since $H^1(N_{(E,S)/X}) = 0$, by Lem. 7, there exists a first order deformation \tilde{S} of S in X , to which E does not lift. Then neither does $\mathcal{O}_S(E)$ by $H^1(\mathcal{O}_S(E)) = 0$. Let $\tau \in H^1(T_S)$ (abstract def.) correspond to \tilde{S} . Then $\tau \cup c(\mathcal{O}_S(E)) \neq 0$ in $H^2(\mathcal{O}_S)$, where $c(*) \in H^1(\Omega_S^1)$ denotes the Atiyah-ext. class of $*$. Since $c(\mathcal{O}_S(C)) = c(\mathcal{O}_S(C - bE)) + bc(\mathcal{O}_S(E))$, and $C - bE \in i^* \text{Pic } X$, we have $\tau \cup c(\mathcal{O}_S(C)) \neq 0$, hence $\mathcal{O}_S(C)$ does not lift to \tilde{S} , hence neither does C as a closed subscheme of S , and $H^1(N_{(C,S)/X}) = 0$. □

π -map

Let E be a curve on $S (\subset X)$, and $\pi_{E/S} : N_{E/X} \rightarrow N_{S/X}|_E$ the projection.

Definition 11 (π -map)

The homomorphism $\pi_{E/S}(E) = \pi_{E/S} \otimes \mathcal{O}_E(E)$ of sheaves on E induces a map

$$\pi_{E/S}(E) : H^0(E, N_{E/X}(E)) \rightarrow H^0(E, N_{S/X}(E)|_E),$$

on the global sections. We call this map **the π -map** for (E, S) .

Example 12

Let E be a conic on a prime Fano 3-fold X , contained in a smooth $K3$ $S \in |-K_X|$. Then $\mathcal{O}_E(E) \simeq \mathcal{O}_E(-2)$, and $N_{S/X}|_E \simeq \mathcal{O}_E(2)$. If E is good (i.e. $N_{E/X}$ is trivial), then the π -map

$$\pi_{E/S}(E) : H^0(E, \mathcal{O}_E(-2)^2) \rightarrow H^0(E, \mathcal{O}_E)$$

for (E, S) is **zero map** (and hence not surjective).

Obstructedness of curves in a K3 surface

Let $C \subset S_{K3} \subset X_{Fano3}$ be as above and suppose that $H^1(N_{(C,S)}/X) = 0$.
Then

$$\text{coker } p_1 \simeq H^1(N_{S/X}(-C)) \simeq H^1(-D),$$

where $D := C + K_X|_S$ is a divisor on S .

Theorem 13 (N'17)

- 1 If $D \geq 0$ and there exist **no (-2)-curves** and **no elliptic curves** on S , or if $H^1(S, D) = 0$, then C is **unobstructed**.
- 2 If $D \geq 0$, $D^2 \geq 0$ and there exists a **(-2)-curve** E on S such that $E \cdot D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover, the π -map $\pi_{E/S}(E)$ is not surjective, then C is **obstructed**.
- 3 If there exists an **elliptic curve** F on S such that $D \sim mF$ for $m \geq 2$, then we have $h^1(S, D) = m - 1$. If moreover, $\pi_{F/S}(F)$ is not surjective, then C is **obstructed**.

Analogy of Mumford's ex. in the case $r = 1$

Applying the previous theorem to a **K3 surfaces** $S \subset X$ and a **good conic** $E \simeq \mathbb{P}^1$ on S , we obtained:

Theorem 14 (N'19)

Let X be a prime Fano 3-fold of genus $g := (-K_X)^3/2 + 1$. Then $\text{Hilb}^{sc} X$ contains a **generically non-reduced component** W with the following properties:

- 1 Every general member C of W is contained in a **K3 surface** S ($\sim -K_X$).
- 2 There exists a **good conic** $E \simeq \mathbb{P}^1$ on S such that $C \sim -2K_X|_S + 2E$.
- 3 $\dim W = 5g + 1$, $h^0(C, N_{C/X}) = 5g + 2$, and C is of degree $4g$ and genus $4g + 1$.

Some remarks

For the proof, we need the following facts:

- The prime Fano 3-fold $X = X_{2g-2}$ contains a conic $E \subset V$ (cf. [Shokurov'79], [Reid'80]).
- If $E \subset X$ is general, then E is a good conic (cf. [Iskovskih'78]) if $\text{char } k = 0$.
- For every conic E , there is a smooth K3 surface $S \in |-K_V|$ containing E (cf. [Iskovskih]).

Corollary 15

If X is a smooth Fano 3-fold and $\rho(X) = 1$, then $\text{Hilb}^{sc} X$ contains a **generically non-reduced component**.

3-fold X	surface S	$[C] \in \text{Pic } S$	E	
\mathbb{P}^3	del Pezzo	$-K_X _S + 2E$	line	Mumford['62]
$Q^3 \subset \mathbb{P}^4$				Mukai-N['09]
del Pezzo				Mukai-N['09], N['10]
prime Fano	$K3$	$-2K_X _S + 2E$	conic	N['19]

Table: Generically non-reduced component of Mumford type

§3 An example of non-reduced components of $\text{Hilb}^{sc} \mathbb{P}^5$

Main result

Toward a further generalization, we compute the obstruction to deforming curves lying on a complete intersection **K3** surface in \mathbb{P}^5 .

Theorem 16 (Main)

The Hilbert scheme $\text{Hilb}^{sc} \mathbb{P}^5$ contains a **generically non-reduced** components W_n ($n \geq 2$) with the following properties:

- ① every general C of W_n is a smooth connected curve contained in a **smooth complete intersection K3 surface**

$$S = S_{2,2,2} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5.$$

- ② C is linearly equivalent to $n(2h + E)$, where $h = [O_S(1)]$ in $\text{Pic } S$, and **E is a line** on S .
- ③ C is of degree $d = 17n$ and genus $g = 17n^2 + 1$.
- ④ $\dim W_n = g + 53$, while $h^0(N_{C/\mathbb{P}^5}) = g + 56$, thus $h^0(N_{C/\mathbb{P}^5}) = \dim W_n + 3$.

Construction

We see $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$ and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{array}{ccccc}
 C & \in & W^{(g+53)} & \subset & \mathbf{Hilb}^{sc} \mathbb{P}^5 \\
 \downarrow & & \downarrow \mathbb{P}^g\text{-bundle} & & \\
 (E, S) & \in & U^{(53)} & \subset & G \times \mathrm{Gr}(3, V) \\
 \downarrow & & \downarrow \mathrm{Gr}(3, 18)\text{-bundle} & & \downarrow \\
 E & \in & \mathrm{Gr}(2, 6)^{(8)} & = & \{\text{lines in } \mathbb{P}^5\},
 \end{array}$$

where $V = H^0(\mathbb{P}^5, \mathcal{O}(2))$.

Key Lemma

Let X be a projective scheme, and let $\mathbf{HF} X$ be the Hilbert-flag scheme of X . There exists a projection

$$pr_1 : \mathbf{HF} X \rightarrow \mathbf{Hilb} X, \quad (C, S) \mapsto [C],$$

which induces the tangent map $p_1 : H^0(N_{(C,S)/X}) \rightarrow H^0(N_{C/X})$.

Lemma 17 (Key Lemma, N[23], Lem. 2.17)

We have $\dim_{(C,S)} \mathbf{HF} X = \dim_{[C]} \mathbf{Hilb} X$ if

- ① $H^1(N_{(C,S)/X}) = H^0(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
- ② for every $\alpha \in H^0(N_{C/X}) \setminus \text{im } p_1$, the (primary) obstruction $\text{ob}(\alpha)$ (to extend α a second order deformation over $k[t]/(t^3)$) is nonzero.

Sketch of Proof of Main thm.

Let $C \subset \mathbb{P}^5$ be a smooth connected curve lying on a complete intersection $K3$ surface $S = S_{2,2,2} \subset \mathbb{P}^5$, and such that $C \sim n(2h + E)$ in $\text{Pic } S$ for $n \geq 2$, where E is a line on S .

- Since $d = 17n > 16 = 2h^2$, S is uniquely determined by C .
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by proj. normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

that $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$, which implies there exists a first order deformation of \tilde{S} of S , to which E (and hence C) does not lift.

- Then $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$ for $i > 0$ and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

is exact. Here p_1 is the tangent map of $pr_1 : \text{HF } \mathbb{P}^5 \rightarrow \text{Hilb } \mathbb{P}^5$ at (C, S) and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2h - C))^{\oplus 3} \simeq k^3.$$

Sketch of Proof of Main thm.(continued)

- We note that $H^1(N_{S/\mathbb{P}^5}(\mathbf{E} - C)) = H^1(-L^{\oplus 3}) = \mathbf{0}$, where

$$L = C - 2h - \mathbf{E} = (n - 1)(2h + \mathbf{E})$$

is nef and big.

- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \text{im } p_1$, its exterior component $\pi_{C/S}(\alpha)$, i.e., the image of α in $H^0(N_{S/\mathbb{P}^5}(\mathbf{E})|_C)$, lifts to a global section β of $N_{S/\mathbb{P}^5}(\mathbf{E})$. (Here β is called an infinitesimal deformation with poles.)
- Applying a “modification” of the obstructedness criterion [Mukai-N’09] to β , we obtain $\mathbf{ob}(\alpha) \neq \mathbf{0}$. This implies

$$\dim_{[C]} \mathbf{Hilb}^{sc} \mathbb{P}^5 = \dim_{(C,S)} \mathbf{HF} \mathbb{P}^5$$

by the key lemma. Therefore C is obstructed and parametrised by an open dense subset of a component of $\mathbf{Hilb}^{sc} \mathbb{P}^5$.

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