

# Obstructions to deforming curves lying on a K3 surface

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Today's slide

# Plan of Talk

- 1 Hilbert schemes and Mumford's example (Motivation)
- 2 Deformation of curves lying on a K3 surface
- 3 An example of non-reduced components of  $\mathbf{Hilb}^{sc} \mathbb{P}^5$

# §1 Hilbert schemes and Mumford's example (Motivation)

# Hilbert schemes

Today, we work over a field  $k = \bar{k}$  of  $\text{char } k = 0$ .

## Theorem 1 (Grothendieck'60)

Given a closed subscheme  $X \subset \mathbb{P}^n$  and a closed subscheme  $C \subset X$  with Hilbert polynomial  $P(C) = P$ , there exists a projective scheme  $\text{Hilb}_P X$ , parametrizing all closed subschemes  $C'$  of  $X$  with (the same) Hilbert polynomial  $P$ .

$\text{Hilb } X := \bigsqcup_P \text{Hilb}_P X$  is called the Hilbert scheme of  $X$ . Today we consider the open and closed subscheme

$$\text{Hilb}^{sc} X := \{\text{smooth connected curves } C \subset X\} \subset \text{Hilb } X,$$

that is, the Hilbert scheme of curves in  $X$ .

## Infinitesimal property of Hilbert schemes

- The **tangent space** of  $\text{Hilb } X$  at  $[C]$  is isomorphic to  $H^0(C, N_{C/X})$ .
- $C \subset X$ : a locally complete intersection  $\implies$  **every obstruction** to deforming  $C$  in  $X$  is contained in  $H^1(C, N_{C/X}) \subset \text{Ext}^1(I_C, \mathcal{O}_C)$  and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(= \chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \text{Hilb } X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- We say  $C \subset X$  is **unobstructed** if  $\text{Hilb } X$  is **nonsingular** at  $[C]$ .
- $H^1(C, N_{C/X}) = 0 \implies C$  is **unobstructed**. The converse is not true (e.g. c.i. curves  $C \subset \mathbb{P}^3$  may have large  $H^1(N_{C/\mathbb{P}^3})$  but *unobstructed*).

### Purpose 2

Determine  $\dim_{[C]} \text{Hilb } X$  at a **singular** point  $[C]$  of  $\text{Hilb } X$ .

## Mumford's example (pathology)

The following example appeared in a famous paper “Further pathologies in algebraic geometry [Mumford'62]”.

### Example 1 (Mumford)

$\text{Hilb}^{sc} \mathbb{P}^3$  contains a **generically non-reduced irreducible component**  $W$  of dimension **56**, whose general member  $C$  satisfies

- ①  $C$  is contained in a **smooth cubic surface**  $S \subset \mathbb{P}^3$ .
- ② There exists **a line**  $E$  on  $S$  such that  $C$  belongs to a complete linear system  $\Lambda := |-4K_S + 2E| (\simeq \mathbb{P}^{37})$  on  $S$ .

### Remark 3

- $C$  and  $\mathbb{P}^3$  are **innocent-looking (a pathology)**.
- $C$  is of degree **14** and genus **24**, and  $h^1(N_{C/\mathbb{P}^3}) = 1$  and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \text{Hilb}^{sc} \mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

## Generalization of Mumford's example

- Later **many non-reduced components** of  $\text{Hilb}^{sc} \mathbb{P}^3$  were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of  $\text{Hilb}^{sc} \mathbb{P}^n$  ( $n > 3$ ) have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves  $C$  corresponding to the generic point of the non-reduced components were contained in some surface  $S \subset \mathbb{P}^n$ , e.g.,

	a surface $S$ containing $C$
Mumford['62]	smooth cubic
Gruson-Peskine'82	non-normal cubic
Kleppe'87	smooth cubic
Kleppe-Ottem'15	smooth quartic
Choi-Iliev-Kim'24-1, '24-2	ruled surface

- Vakil's result ['06] on *Murphy's law in AG*: unless there is some a priori reason otherwise, **the deformation space may be as bad as possible**.

## Another generalization (with Mukai)

We found that in Mumford's example, **(-1)-curves**  $E \simeq \mathbb{P}^1$  (on smooth cubics) *play an important role*.

### Theorem 4 (Mukai-N'09, $\text{char } k \geq 0$ )

Let  $X$  be a smooth projective 3-fold satisfying the following:

- ① there exists a smooth **rational curve**  $E$  on  $X$  s.t.  $N_{E/X}$  is globally generated, and
- ② there exists a **smooth surface**  $S$  s.t.  $E \subset S \subset X$ ,  $E^2 = -1$  on  $S$ , and  $H^1(S, N_{S/X}) = p_g(S) = 0$ .

Then the Hilbert scheme  $\text{Hilb}^{sc} X$  has infinitely many **generically non-reduced components (GNRC)**.

### Remark 5

In Mumford's ex.,  $X = \mathbb{P}^3$ ,  $S$  is a **smooth cubic**,  $E$  is a **line**.



## Hilbert scheme of Fano 3-folds

- A **Fano manifold** is a smooth projective variety  $X$  with ample  $-K_X$ .
- The **index  $r$**  of a Fano manifold  $X$  is the maximal integer  $r$  such that  $-K_X \sim rH$  with some  $H \in \text{Pic } X$ .

Let  $X$  be a smooth Fano 3-fold of index  $r$ .

- $X \simeq \mathbb{P}^3$  if  $r = 4$  and  $X \simeq Q^3 \subset \mathbb{P}^4$  if  $r = 3$ , and  $X$  is called **del Pezzo** if  $r = 2$ , and **prime** if  $r = 1$  and  $\rho = 1$ .
- If we restrict  $X$  with  $\rho = 1$ , then there exist **17** deformation equivalence classes of  $X$  (Fujita, Iskovskikh'77,'78):

$r$	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

**Table:** the number of deformation equivalence classes of  $X$

Applying Theorem 4, we obtain

### Example 2 (N'10)

If  $r > 1$  (and of any  $\rho(X)$ ), then  $\text{Hilb}^{sc} X$  contains a **generically non-reduced component**  $W$  satisfying:

- ① every general member  $C$  of  $W$  is contained in a smooth **del Pezzo surface**  $S$  ( $\sim -\frac{r-1}{r}K_X$ ), and
- ② there exists a **(good) line**  $E$  on  $S$  and  $C \sim -K_X|_S + 2E$  in  $\text{Pic } S$ .
- ③  $h^0(C, N_{C/X}) = \dim W + 1$ .

Here

- A curve  $E \subset X$  is a **line**  $\stackrel{\text{def}}{\iff} E \simeq \mathbb{P}^1$  and  $-\frac{1}{r}K_X.E = 1$ .
- A line  $E \subset X$  is **good**  $\stackrel{\text{def}}{\iff} N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$  (for  $r = 2, 3$ ).
- $\dim W = 56, 42$  and  $(-K_X)^3/2 + 4$  for  $r = 4, 3, 2$ , respectively.

### Question 6

What about the case for  $\text{Hilb}^{sc} X$  of prime Fano 3-fold  $X$ ?

If  $X$  is prime ( $r = 1$ ), then there exists NO del Pezzo surface  $S \subset X$ .

## §2 Deformation of curves lying on a K3 surface

## Setting

In this section, we recall my previous results on the deformations of smooth curves  $C$  on a smooth Fano 3-fold  $X$  with the assumption that  $C$  is contained in a smooth K3 surface  $S \subset X$ , i.e., we have

$$C \subset S_{K3} \subset X_{Fano}.$$

### Definition 7

A smooth projective surface  $S$  is a **K3 surface**, if  $K_S \sim 0$  and  $H^1(S, \mathcal{O}_S) = 0$ .

We will see that

- **$(-2)$ -curves  $E \simeq \mathbb{P}^1$** , and
- **elliptic curves  $F$**

on  $S$  control the deformations  $C$  in  $X$ . They play a role very similar to that of  **$(-1)$ -curve  $E \simeq \mathbb{P}^1$**  on the smooth cubic  $S_3 \subset \mathbb{P}^3$  in **Mumford's example**!

## Hilbert-flag scheme

The **Hilbert-flag scheme**  $\mathbf{HF} X$  of  $X$  parametrises all pairs  $(C, S)$  of closed subschemes  $C$  and  $S$  of  $X$  satisfying  $C \subset S$ . If  $C \hookrightarrow S \hookrightarrow X$  is regular embeddings, then  $H^0(N_{(C,S)/X})$  and  $H^1(N_{(C,S)/X})$  respectively represents the tangent sp. and the obstruction sp. of  $\mathbf{HF} X$  at  $(C, S)$ , where the normal sheaf  $N_{(C,S)/X}$  of  $(C, S)$  in  $X$  is defined by the Cartesian diagram

$$\begin{array}{ccc} N_{(C,S)/X} & \xrightarrow{\pi_2} & N_{S/X} \\ \pi_1 \downarrow & \square & \downarrow |_{\mathcal{C}} \\ N_{C/X} & \xrightarrow{\pi_{C/S}} & N_{S/X}|_C. \end{array}$$

Here  $|_{\mathcal{C}}$  is the restriction of sheaves, and  $\pi_{C/S} : N_{C/X} \rightarrow N_{S/X}|_C$  is the natural projection. As in the case of Hilbert schemes, we have

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} \mathbf{HF} X \leq h^0(X, N_{(C,S)/X}).$$

## Lemma 8

Under our setting, the following are equivalent:

- ①  $H^1(N_{(C,S)/X}) = 0$ , namely **HF**  $X$  is **nonsingular** at  $(C, S)$  of **expected dimension**  $\chi(N_{(C,S)/X})$ .
- ② There exists a first order deformation  $\tilde{S}$  of  $S$  in  $X$ , to which  $C$  does not lift.

## Proof.

By adjunction,  $N_{S/X} \simeq -K_X|_S$  and  $N_{C/S} \simeq K_C$ , which implies  $H^i(N_{S/X}) = 0$  for  $i > 0$  and  $H^1(N_{C/S}) \simeq k$ . There exists an exact sequence  $0 \rightarrow N_{C/S} \rightarrow N_{(C,S)/X} \rightarrow N_{S/X} \rightarrow 0$ , inducing

$$H^0(N_{(C,S)/X}) \xrightarrow{p_2} H^0(N_{S/X}) \rightarrow \underbrace{H^1(N_{C/S})}_{\simeq k} \rightarrow H^1(N_{(C,S)/X}) \rightarrow 0.$$

$H^1(N_{(C,S)/X}) = 0$  iff  $p_2$  is not surjective, which is equivalent to the second condition. □

### Example 9

Let  $X$  be a prime Fano 3-fold,  $E$  a line on  $X$ . Then  $E$  is of type, either  $(0, -1)$  (good) or  $(1, -2)$  (bad). If  $E$  is contained in a smooth K3 surface  $S$  in  $X$ , then

**HF**  $X$  is nonsingular at  $(E, S)$  (of exp. dim.)  $\iff E$  is good.

In fact, **Hilb**  $X$  is nonsingular at  $[E]$  if and only if  $E$  is good, and the first projection  $pr_1 : (E, S) \rightarrow [E]$  is a smooth morphism at  $(E, S)$ .

### Lemma 10

If  $X$  is prime, and  $E$  is a good line or a good conic on  $X$  contained in a smooth  $S \in |-K_X|$ , then  $H^1(X, N_{(E,S)/X}) = 0$ .

## A key lemma

### Lemma 11 ( $\text{char } k = 0$ )

Let  $i : S \hookrightarrow X$  denote the closed embedding, and let  $E$  be an effective Cartier div on  $S$  with  $H^1(S, \mathcal{O}_S(E)) = 0$ . If  $H^1(X, N_{(E,S)/X}) = 0$  and if  $C - bE \in i^* \text{Pic } X$  with  $b \neq 0$ , then  $H^1(X, N_{(C,S)/X}) = 0$ .

### Proof.

Since  $H^1(X, N_{(E,S)/X}) = 0$ , by Lem. 8, there exists a first order deformation  $\tilde{S}$  of  $S$  in  $X$ , to which  $E$  does not lift. Let  $\alpha \in H^0(S, N_{S/X})$  and  $\tau \in H^1(S, T_S)$  (abstract def.) correspond to  $\tilde{S}$ . Then  $\tau \cup c(\mathcal{O}_S(E)) \neq 0$  in  $H^2(\mathcal{O}_S)$ , where  $c(*) \in H^1(S, \Omega_S^1)$  denotes the Atiyah-ext. class of  $*$ . Since  $c(\mathcal{O}_S(C)) = c(\mathcal{O}_S(C - bE)) + bc(\mathcal{O}_S(E))$ , and  $C - bE \in i^* \text{Pic } X$ , we have  $\tau \cup c(\mathcal{O}_S(C)) \neq 0$ , hence  $\mathcal{O}_S(C)$  does not lift to  $\tilde{S}$ , hence neither does  $C$  as a closed subscheme of  $S$ , and  $H^1(X, N_{(C,S)/X}) = 0$ .  $\square$



## $\pi$ -map

Let  $E$  be a curve on  $S (\subset X)$ , and  $\pi_{E/S} : N_{E/X} \rightarrow N_{S/X}|_E$  the projection.

### Definition 12 ( $\pi$ -map)

The homomorphism  $\pi_{E/S}(E) = \pi_{E/S} \otimes \mathcal{O}_E(E)$  of sheaves on  $E$  induces a map

$$\pi_{E/S}(E) : H^0(E, N_{E/X}(E)) \rightarrow H^0(E, N_{S/X}(E)|_E),$$

on the global sections. We call this map **the  $\pi$ -map** for  $(E, S)$ .

### Example 13

Let  $E$  be a conic on a prime Fano 3-fold  $X$ , contained in a smooth  $S \in |-K_X|$ . Then  $\mathcal{O}_E(E) \simeq \mathcal{O}(-2)$ , and  $N_{S/X}|_E \simeq \mathcal{O}(2)$ . If  $E$  is good, then the  $\pi$ -map

$$\pi_{E/S}(E) : H^0(E, \mathcal{O}(-2)^2) \rightarrow H^0(E, \mathcal{O})$$

for  $(E, S)$  is **zero** (hence not surjective).

## Obstructedness of curves in a K3 surface

Let  $X$  be a smooth Fano 3-fold,  $C \subset X$  a smooth curve contained in a smooth K3 surface  $S \subset X$ . Let  $D$  be a divisor on  $S$  defined by

$$D := C + K_X|_S.$$

### Theorem 14 (N'17)

If  $H^1(N_{(C,S)/X}) = 0$  and  $D \geq 0$ , then

- ① If there exist no  $(-2)$ -curves and no elliptic curves on  $S$ , or if  $H^1(S, D) = 0$ , then  $C$  is **unobstructed**.
- ② If  $D^2 \geq 0$  and there exists a  $(-2)$ -curve  $E$  on  $S$  such that  $E \cdot D = -2$  and  $H^1(S, D - 3E) = 0$ , then we have  $h^1(S, D) = 1$ . If moreover, the  $\pi$ -map  $\pi_{E/S}(E)$  is not surjective, then  $C$  is **obstructed**.
- ③ If there exists an elliptic curve  $F$  on  $S$  such that  $D \sim mF$  for  $m \geq 2$ , then we have  $h^1(S, D) = m - 1$ . If moreover,  $\pi_{F/S}(F)$  is not surjective, then  $C$  is **obstructed**.

## Analogy of Mumford's ex. in the case $r = 1$

Applying Theorem 14 to a **K3 surfaces**  $S \subset X$  and a **good conic**  $E \simeq \mathbb{P}^1$  on  $S$ , we can prove the following.

### Theorem 15 (N'19)

Let  $X$  be a prime Fano 3-fold of genus  $g := (-K_X)^3/2 + 1$ . Then  $\text{Hilb}^{sc} X$  contains a **generically non-reduced component**  $W$  with the following properties:

- ① Every general member  $C$  of  $W$  is contained in a **K3 surface**  $S$  ( $\sim -K_X$ ).
- ② There exists a **good conic**  $E \simeq \mathbb{P}^1$  on  $S$  such that  $C \sim -2K_X|_S + 2E$ .
- ③  $\dim W = 5g + 1$ ,  $h^0(C, N_{C/X}) = 5g + 2$ , and  $C$  is of degree  $4g$  and genus  $4g + 1$ .

Here a conic  $E$  on  $X$  is called **good** if  $N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$ .

## Some remarks

We used the following facts:

- The prime Fano 3-fold  $X = X_{2g-2}$  contains a conic  $E \subset V$  (cf. [Shokurov'79], [Reid'80]).
- If  $E \subset X$  is general, then  $E$  is a good conic (cf. [Iskovskih'78]) if  $\text{char } k = 0$ .
- For every conic  $E$ , there is a smooth K3 surface  $S \in |-K_V|$  containing  $E$  (cf. [Iskovskih]).

## Corollary 16

If  $X$  is a smooth Fano 3-fold and  $\rho(X) = 1$ , then  $\text{Hilb}^{sc} X$  contains a **generically non-reduced component**.

3-fold $X$	surface $S$	$[C] \in \text{Pic } S$	$E$	
$\mathbb{P}^3$	del Pezzo	$-K_X _S + 2E$	line	Mumford['62]
$Q^3 \subset \mathbb{P}^4$				Mukai-N['09]
del Pezzo				Mukai-N['09], N['10]
prime Fano	$K3$	$-2K_X _S + 2E$	conic	N['19]

Table: Generically non-reduced component of Mumford type

## §3 An example of non-reduced components of $\text{Hilb}^{sc} \mathbb{P}^5$

## Main result

Toward a further generalization, we compute the obstruction to deforming curves lying on a complete intersection **K3** surface in  $\mathbb{P}^5$ .

### Theorem 17 (Main)

The Hilbert scheme  $\text{Hilb}^{sc} \mathbb{P}^5$  contains a **generically non-reduced** components  $W_n$  ( $n \geq 2$ ) with the following properties:

- ① every general  $C$  of  $W_n$  is a smooth connected curve contained in a **smooth complete intersection K3 surface**

$$S = S_{2,2,2} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5.$$

- ②  $C$  is linearly equivalent to  $n(2h + E)$ , where  $h = [O_S(1)]$  in  $\text{Pic } S$ , and  $E$  is a line on  $S$ .
- ③  $C$  is of degree  $17n$  and genus  $17n^2 + 1$ .
- ④  $\dim W = 17n^2 + 54$  ( $= g + 53$ ), while  $h^0(N_{C/\mathbb{P}^5}) = 17n^2 + 57$ , thus  $h^0(N_{C/\mathbb{P}^5}) - \dim W = 3$ .

# Construction

We see  $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$  and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{array}{ccccc}
 C & \in & W^{(g+53)} & \subset & \mathbf{Hilb}^{sc} \mathbb{P}^5 \\
 \downarrow & & \downarrow \mathbb{P}^g\text{-bundle} & & \\
 (E, S) & \in & U^{(53)} & \subset & G \times \mathrm{Gr}(3, V) \\
 \downarrow & & \downarrow \mathrm{Gr}(3, 18)\text{-bundle} & & \downarrow \\
 E & \in & \mathrm{Gr}(2, 6)^{(8)} & = & \{\text{lines in } \mathbb{P}^5\},
 \end{array}$$

where  $g + 53 = 17n^2 + 54$  and  $V = H^0(\mathbb{P}^5, \mathcal{O}(2))$



## Hilbert-flag scheme and Key Lemma

Let  $X$  be a projective scheme. Then there exists a projective scheme  
(Hilbert-flag scheme)

$$\mathbf{HF} X := \{(C, S) \mid C \subset S \subset X\} \subset \mathbf{Hilb} X \times \mathbf{Hilb} X.$$

Let

$$N_{(C,S)/X} := N_{C/X} \times_{N_{S/X}|_C} N_{S/X}$$

be the **normal sheaf** of  $(C, S) \in \mathbf{HF} X$ .

There exists a projection

$$pr_1 : \mathbf{HF} X \rightarrow \mathbf{Hilb} X, \quad (C, S) \mapsto [C],$$

which induces the tangent map  $p_1 : H^0(X, N_{(C,S)/X}) \rightarrow H^0(C, N_{C/X})$ .

Lemma 18 (Key Lemma, N[23], Lem. 2.17)

We have  $\dim_{(C,S)} \mathbf{HF} X = \dim_{[C]} \mathbf{Hilb} X$  if we have

- ①  $H^1(X, N_{(C,S)/X}) = H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$ , and
- ②  $\text{ob}(\alpha) \neq 0$  for any  $\alpha \in H^0(C, N_{C/X}) \setminus \text{im } p_1$ .

## Primary obstructions

Let  $X$  be a projective scheme over  $k$ ,  $C$  a loc. c. i. closed subscheme of  $X$ , and  $k[\varepsilon] := k[t]/(t^2)$  (the ring of dual numbers).

- A **first order (infinitesimal) deformation** of  $C$  is a deformation  $\tilde{C}$  ( $\subset X \times \text{Spec } k[\varepsilon]$ ) of  $C$  in  $X$  **over**  $k[\varepsilon]$ .
- $\tilde{C}$  naturally corresponds to  $\alpha \in H^0(C, N_{C/X})$  ( $\simeq \text{Hom}(I_C, O_C)$ ).
- There is an element **ob**( $\alpha$ ) in  $H^1(C, N_{C/X})$  (called the **primary obstruction** of  $\alpha$ ) such that

$$\text{ob}(\alpha) = 0 \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$

- **ob**( $\alpha$ ) can be expressed as a **cup product**, and

$$\text{ob}(\alpha) = \alpha \cup e \cup \alpha \quad \text{in } \text{Ext}^1(I_C, O_C)$$

where  $e := [0 \rightarrow I_C \rightarrow O_X \rightarrow O_C \rightarrow 0]$ .

- **ob**( $\alpha$ )  $\neq 0$  for some  $\alpha$  implies that **Hilb**  $X$  is **singular** at  $[C]$  by infinitesimal lifting property of smoothness.

## Sketch of Proof of Main thm.

Let  $C \subset \mathbb{P}^5$  be a smooth connected curve lying on a complete intersection **K3** surface  $S_{2,2,2} \subset \mathbb{P}^5$ , and such that  $C \sim 2nh + nE$  in  $\text{Pic } S$  for  $n \geq 2$ , where  $E$  is a line on  $S$

- Since  $d = 17n > 16 = 2h^2$ ,  $S$  is uniquely determined by  $C$ .
- Then for all  $i > 0$ ,  $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$  by proj. normality and  $H^i(N_{E/\mathbb{P}^5}) = 0$  by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

that  $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$ , which implies there exists a first order deformation of  $\tilde{S}$  of  $S$ , to which  $E$  (and hence  $C$ ) does not lift.

- Then  $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$  for  $i > 0$  and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

is exact. Here  $p_1$  is the tangent map of  $pr_1 : \text{HF } \mathbb{P}^5 \rightarrow \text{Hilb } \mathbb{P}^5$  at  $(C, S)$  and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2h - C))^{\oplus 3} \simeq k^3.$$

## Sketch of Proof of Main thm.

- We note that  $H^1(N_{S/\mathbb{P}^5}(E - C)) = H^1(-L^{\oplus 3}) = 0$ , where

$$L = C - 2h - E = (n - 1)(2h + E)$$

is nef and big.

- Then for every  $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \text{im } p_1$ , its exterior component  $\pi_{C/S}(\alpha)$ , i.e., the image of  $\alpha$  in  $H^0(N_{S/\mathbb{P}^5}(E)|_C)$ , lifts to a global section  $\beta$  of  $N_{S/\mathbb{P}^5}(E)$ . (Here  $\beta$  is called an **infinitesimal deformation with poles**.)
- Applying a “modification” of the **obstructedness criterion** [Mukai-N’09] to  $\beta$ , we obtain **ob**( $\alpha$ )  $\neq 0$ . This implies

$$\dim_{[C]} \mathbf{Hilb}^{sc} \mathbb{P}^5 = \dim_{(C,S)} \mathbf{HF} \mathbb{P}^5$$

by the key lemma. Therefore  $C$  is **obstructed** and parametrised by an **open dense subset of a component** of  $\mathbf{Hilb}^{sc} \mathbb{P}^5$ .  $\square$

## Obstructedness Criterion (with Mukai)

[Mukai-N'09] gave a sufficient condition for  $\text{ob}(\alpha) \neq 0$  when  $\dim X = 3$ .  
Let  $C$  be an irreducible curve on a 3-fold  $X$ .

Assumption 1

- there exists an intermediate surface  $C \subset S \subset X$  s.t.  $C \hookrightarrow S$  and  $S \hookrightarrow X$  are regular embeddings.
- there exists an irreducible curve  $E \neq C$  on  $S$  s.t.  
 $[O_S \hookrightarrow O_S(E)] \otimes O_S(mE)$  induces the injection

$$H^1(S, O_S(mE)) \hookrightarrow H^1(S, O_S((m+1)E))$$

for all  $m > 0$ . (e.g.  $E = (-1)\text{-}\mathbb{P}^1$  on  $S$ )

## Obstructedness Criterion (Continued)

Let  $\alpha \in H^0(N_{C/X})$  be a first order deformation of  $C$  in  $X$  and  $\text{ob}(\alpha) \in H^1(N_{C/X})$  its primary obstruction. We consider the “exterior” components

$$\pi_{C/S}(\alpha) := H^0(\pi_{C/S})(\alpha), \quad \text{ob}_S(\alpha) := H^1(\pi_{C/S})(\text{ob}(\alpha)).$$

by the projection

$$\pi_{C/S} : N_{C/X} \rightarrow N_{S/X}|_C.$$

Assumption 2

- Suppose  $\pi_{C/S}(\alpha)$  lifts to a global section  $\beta$  of  $N_{S/X}(\mathbf{E})$ .

$$\begin{array}{ccccccc} & & & H^0(N_{S/X}) & \subset & H^0(N_{S/X}(\mathbf{E})) & \ni \beta \\ & & & \downarrow & & \downarrow & \\ \alpha \in H^0(N_{C/X}) & \xrightarrow{\pi_{C/S}} & H^0(N_{S/X}|_C) & \subset & H^0(N_{S/X}(\mathbf{E})|_C) & & \end{array}$$

Here  $\beta$  is called an infinitesimal deformation with pole:

## Obstructedness Criterion (Continued)

### Theorem 19 (Mukai-N'09)

$\text{ob}_S(\alpha)$  is nonzero if

- ①  $\Delta \cdot E = 2(-E^2 + g(E) - 1)$ , where  $\Delta := C + K_X|_S - 2E$  in  $\text{Pic } S$ .
- ② Let  $\beta|_E$  be the principal part of  $\beta$  along  $E$ . Then  $\beta|_E \cup k_E \neq 0$  in  $H^1(E, \mathcal{O}_E(2E))$ , where

$$k_E := [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}|_E \longrightarrow 0] \\ \in \text{Ext}_E^1(N_{S/X}|_E, N_{E/S}).$$

- ③ the restriction map  $H^0(S, \Delta) \rightarrow H^0(E, \Delta|_E)$  is surjective,

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