Deformations of space curves lying on a del Pezzo surface

Hirokazu Nasu

Tokai University

June 12, 2025
2025 Algebraic Geometry Conference in Jeonju Hanok Village

Today's slide

Plan of Talk

1. Kleppe-Ellia conjecture
2. Deformation of curves on del Pezzo surface
3. Main result (cf. arXiv:2501.15788)

§1 Kleppe-Ellia conjecture (curves on cubic surface)

the Hilbert scheme

Given a projective scheme X, and given Hilbert polynomial P,

$$\mathbf{Hilb}_P X = \left\{ C \subset X \mid \text{closed subscheme of } P(C) = P \right\}$$

is called the Hilbert scheme of X. The Hilbert scheme has nice properties:

1. fine moduli scheme, i.e. it has a universal family $\mathcal{C} \subset X \times \mathbf{Hilb}_P X$ such that every deformation of C in X derived from $\mathcal{C}$.
2. projective ($\mathbf{Hilb}_P X \hookrightarrow \mathbf{Gr}$), and
3. it has nice deformation theory, e.g., if C is a loc. c.i., then $H^0(N_{C/X})$ and $H^1(N_{C/X})$ resp. represent the tang.sp., and obst.sp.

§3 Obstructions to deforming space curves lying on a del Pezzo surface

Primary obstructions

Let X be a projective scheme over k, C a loc. c. i. closed subscheme of X, and $k[\varepsilon] := k[t]/(t^2)$ (the ring of dual numbers).

- A first order (infinitesimal) deformation of C is a deformation $\tilde{C}$ ($\subset X \times \mathrm{Spec}\, k[\varepsilon]$) of C in X over $k[\varepsilon]$.
- $\tilde{C}$ naturally corresponds to $\alpha \in H^0(C, N_{C/X})$ ($\simeq \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$).
- There is an element $\mathrm{ob}(\alpha)$ in $H^1(C, N_{C/X})$ (called the primary obstruction of α) such that

$$\mathrm{ob}(\alpha) = 0 \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$

- $\mathrm{ob}(\alpha)$ can be expressed as a cup product, and

$$\mathrm{ob}(\alpha) = \alpha \cup \mathbf{e} \cup \alpha \qquad \text{in } \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

  where $\mathbf{e} := [0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0]$.

- $\mathrm{ob}(\alpha) \neq 0$ for some α implies that $\mathrm{Hilb}\, X$ is singular at $[C]$ by infinitesimal lifting property of smoothness.

Obstructedness Criterion (with Mukai)

[Mukai-N'09] gave a sufficient condition for $\mathrm{ob}(\alpha) \neq 0$ when $\dim X = 3$.
Let C be an irreducible curve on a 3-fold X.

Assumption 1

- there exists an intermediate surface $C \subset S \subset X$ s.t. $C \hookrightarrow S$ and $S \hookrightarrow X$ are regular embeddings.
- there exists an irreducible curve E ($\neq C$) on S s.t. $[\mathcal{O}_S \hookrightarrow \mathcal{O}_S(E)] \otimes \mathcal{O}_S(mE)$ induces the injection

$$H^1(S, \mathcal{O}_S(mE)) \hookrightarrow H^1(S, \mathcal{O}_S((m+1)E))$$

for all $m > 0$. (e.g. E=(-1)-$\mathbb{P}^1$ on S)

Obstructedness Criterion (Continued)

Let $\alpha \in H^0(N_{C/X})$ be a first order deformation of C in X and $\mathbf{ob}(\alpha) \in H^1(N_{C/X})$ its primary obstruction. We consider the *"exterior" components*

$$\pi_{C/S}(\alpha) := H^0(\pi_{C/S})(\alpha), \qquad \mathbf{ob}_S(\alpha) := H^1(\pi_{C/S})(\mathbf{ob}(\alpha)).$$

by the projection

$$\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C.$$

Assumption 2

- Suppose $\pi_{C/S}(\alpha)$ lifts to a global section β of $N_{S/X}(E)$.

$$\begin{array}{ccccccc} & & & H^0(N_{S/X}) & \subset & H^0(N_{S/X}(E)) & \ni \beta \\ & & & \downarrow & & \downarrow & \\ \alpha \in & H^0(N_{C/X}) & \xrightarrow{\pi_{C/S}} & H^0(N_{S/X}\big|_C) & \subset & H^0(N_{S/X}(E)\big|_C) & \end{array}$$

Here β is called an infinitesimal deformation with pole:

Obstructedness Criterion (Continued)

Theorem 1 (Mukai-N'09)

$\mathbf{ob}_S(\alpha)$ is nonzero if

1. $\Delta.E = 2(-E^2 + g(E) - 1)$, where $\Delta := C + K_X\big|_S - 2E$ in $\operatorname{Pic} S$.
2. Let $\beta\big|_E$ be the principal part of β along E. Then $\beta\big|_E \cup \mathbf{k}_E \neq 0$ in $H^1(E, \mathcal{O}_E(2E))$, where

$$\begin{aligned} \mathbf{k}_E &: = [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}\big|_E \longrightarrow 0] \\ &\in \operatorname{Ext}^1_E(N_{S/X}\big|_E, N_{E/S}). \end{aligned}$$

3. the restriction map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

Remark 2

In [Mukai-N'09], this criterion was applied to the proof of Thm. **??**. We obtained the generically non-reduced components explained in §**??** by this criterion.

Stable degeneration

Toward a generalization into higher dimensions, we study the deformations of space curves lying on a del Pezzo surface of degree. Let

$$C \subset S \subset X$$

be a flag of algebraic varieties.

Definition 3

We say $C \subset X$ is stably degnerate (or stably contained in S), if every small deformation C' of C in X is contained in a deformation S' of S in X.

If there exists a component $\mathcal{W}_{C,S}$ of $\mathbf{HF}\, X$ passing through (C, S) such that the first projection

$$pr_1 : \mathcal{W}_{C,S} \to \mathbf{Hilb}\, X, \qquad (C', S') \mapsto [C']$$

is locally surjective at $[C] \in \mathbf{Hilb}\, X$, then $C \subset X$ is *stably degenerate*.

Kleppe-Ellia conjecture

Conjecture (Kleppe'87, modified by Ellia'87)

Let $C \subset S_3 \subset \mathbb{P}^3$ be a smooth connected curve of degree d and genus g lying on a smooth cubic surface $S_3 \subset \mathbb{P}^3$. Then C is *stably degenerate* if

1. $g \geq 3d - 18$,
2. C is linearly normal, i.e. $H^1(\mathcal{I}_C(1)) = 0$,
3. $d > 9$ and C is general in $[C] \in \operatorname{Pic} S_3$.

Remark 4

1. the first two assumptions are necessary for the conclusion.
2. The conjecture is known to be true if
   - C is 3-normal, i.e. $H^1(\mathcal{I}_C(3)) = 0$ (Kleppe'87),
   - C is not 3-normal and $g >> d$ (Kleppe'87 and Ellia'87), or
   - C is 2-normal, i.e. $H^1(\mathcal{I}_C(2)) = 0$ (N'23)

Generalized Kleppe-Ellia conjecture

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a smooth del Pezzo surface $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is *stably degenerate* if

1. $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$,
2. C is linearly normal,
3. $\deg(C) > 9$ for $n = 3$ and $\deg(C) > 2n$ for $n \geq 4$, and C is general in $[C] \in \operatorname{Pic} S_n$.

Remark 5

The first assumption is equivalent to that

$$\dim_{(C,S)} \mathrm{HF}\,\mathbb{P}^n = \chi(N_{(C,S)/\mathbb{P}^n}) \geq \chi(N_{C/\mathbb{P}^n}) = (\text{exp.dim.of } \mathrm{Hilb}\, X \text{ at } [C]),$$

where $N_{(C,S)/X} := N_{C/X} \times_{N_{S/X}|_C} N_{S/X}$.

Application

Proposition 6

If the generalized K-E conjecture is true, then

$$\dim_{[C]} \operatorname{Hilb} \mathbb{P}^n = d + g + n^2 + 9 \; (= \dim_{(C,S)} \operatorname{HF} \mathbb{P}^n).$$

Thus we can determine the dimension of $\operatorname{Hilb} \mathbb{P}^n$ at (even singular) point $[C]$.

Theorem A (Unobstructedness)

We focus on the case $n = 4$, i.e. $S \simeq S_4$ is a smooth complete intersection $S_{2,2}$ in $\mathbb{P}^4$. We say $C \subset \mathbb{P}^4$ is 2-normal if $H^1(\mathcal{I}_C(2)) = 0$, and 2-nonspecial if $H^1(\mathcal{O}_C(2)) = 0$.

Theorem A

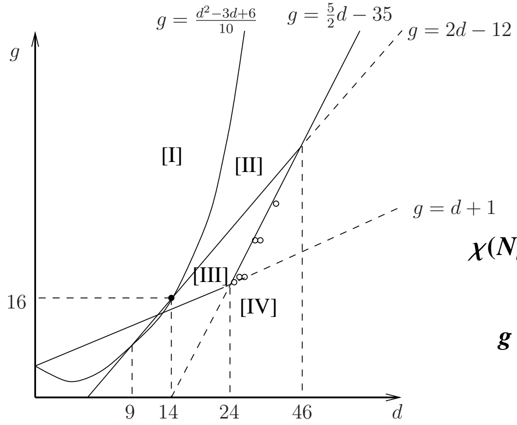
Let $C \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 8$ and genus g contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$. Then

1. Such curves C are parametrised by a *finite union of locally closed irreducible subsets* $W \subset \mathrm{Hilb}^{sc}\,\mathbb{P}^4$ of the same dimension $d + g + 25$.
2. If C is 2-normal, then the closure $\overline{W}$ of W in $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ is a genericaly smooth component of $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$.
3. If C is 2-nonspecial, then $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ is smooth along W and $\overline{W}$ is a (proper) closed subset of $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ of codimension $2h^1(\mathcal{I}_C(2))$.

Theorem A (continued)

Theorem A (continued)

4. C is 2-normal (resp. 2-nonspecial) if (d, g) belongs to the region [I] (resp. [IV] except the 6 pairs corresponding to $\circ$).



$$\chi(N_{S/\mathbb{P}^4}(-C)) \geq 0$$
$$\Updownarrow$$
$$g \geq 2d - 12.$$

Theorem B (Obstructedness)

We expect that if $(d, g) \in$ [II] and C is not 2-normal, then $\overline{W}$ corresponds to a generically non-reduced component of $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$.

Theorem B

Let W be a maximal irreducible family of smooth connected curve $C \subset \mathbb{P}^4$ of degree d and genus g contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$. If $d > 8$, $g \geq 2d - 12$ and $h^1(\mathcal{I}_C(2)) = 1$ (then $(d, g) \in$ [II]), then

1. every general member C of W is stably degenerate and obstructed,
2. $\overline{W}$ is a component of $(\mathrm{Hilb}^{sc}\,\mathbb{P}^4)_{\mathrm{red}}$, and
3. $\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ is generically non-reduced along $\overline{W}$.

Corollary 7

Generalized K-E conjecture holds to be true, if $n = 4$ and $h^1(\mathcal{I}_C(2)) = 1$.

Analogy of Mumford's example

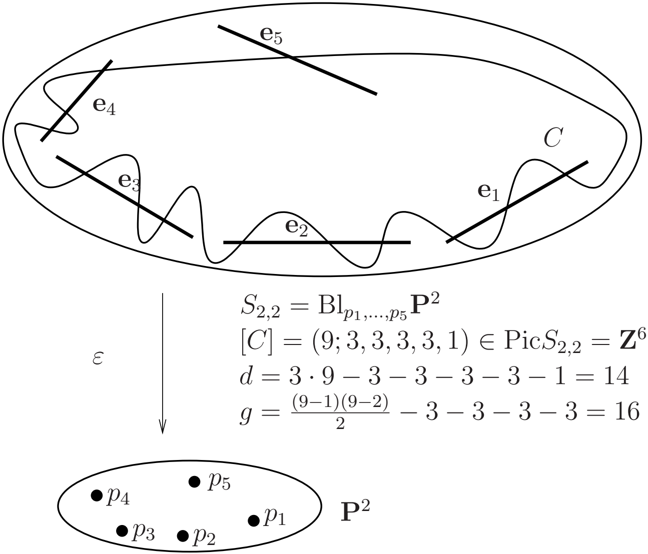
Example 1

$\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W whose general member C satisfies

1. C is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. there exists a line E on S such that C belongs to a complete linear system $\Lambda := |-3K_S + 2E|$ ($\simeq \mathbb{P}^{29}$) on S, and
3. $\dim W = 55$, $h^0(C, N_{C/\mathbb{P}^4}) = 57$, and C is of degree 14 and genus 16.

$$\begin{tikzcd}[column sep=small]
C \arrow[d, mapsto] & \in & W^{(55)} \arrow[d, "\mathbb{P}^{29}\text{-b'dle}"] & \subset & \mathbf{Hilb}^{sc}\,\mathbb{P}^4 \\
(E, S) \arrow[d, mapsto] & \in & U^{(26)} \arrow[d, "G(2,12)\text{-b'dle}"] & \subset & G \times G(2, H^0(\mathcal{O}_{\mathbb{P}^4}(2))) \arrow[d] \\
E & \in & G(2,5)^{(6)} & = & \{\text{lines in } \mathbb{P}^4\}\,.
\end{tikzcd}$$

Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)



$$S_{2,2} = \mathrm{Bl}_{p_1,\dots,p_5}\mathbf{P}^2$$
$$[C] = (9;3,3,3,3,1) \in \mathrm{Pic} S_{2,2} = \mathbf{Z}^6$$
$$d = 3\cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$
$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$

Sketch of Proof of Thm. ??

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 8$ and genus g lying on a smooth complete intersection $S_{2,2}$ in $\mathbb{P}^4$.

- $S = S_{2,2}$ s.t $C \subset S \subset \mathbb{P}^4$ is uniquely determined by $d > 8$.
- Since $H^i(N_{C/S}) = H^i(N_{S/\mathbb{P}^4}) = 0$ for $i > 0$, it follows from an exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{(C,S)/\mathbb{P}^4} \xrightarrow{\pi_2} N_{S/\mathbb{P}^4} \longrightarrow 0$$

  that $H^i(N_{(C,S)/\mathbb{P}^4}) = 0$ for $i > 0$, which implies $\mathrm{HF}\,\mathbb{P}^4$ is nonsingular and of expected dimension at (C, S).
- There exists another exact sequence

$$0 \longrightarrow N_{S/\mathbb{P}^4}(-C) \longrightarrow N_{(C,S)/\mathbb{P}^4} \xrightarrow{\pi_1} N_{C/\mathbb{P}^4} \longrightarrow 0,$$

  where π_1 induces the tangent map p_1 of $pr_1 : \mathrm{HF}\,\mathbb{P}^4 \to \mathrm{Hilb}\,\mathbb{P}^4$ at (C, S) and we obtain

$$H^0(N_{(C,S)/\mathbb{P}^4}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^4}) \longrightarrow \underbrace{H^1(N_{S/\mathbb{P}^4}(-C))}_{\simeq H^1(\mathcal{I}_C(2))^{\oplus 2}} \longrightarrow 0.$$

Sketch of Proof of Thm. ??

- Suppose now that $g \geq 2d - 12$ and $h^1(\mathcal{I}_C(2)) = 1$. Then $\dim \operatorname{coker} p_1 = 2$ and there exists a line E on S such that
$$|C + 2K_S| = |C + 2K_S - E| + E. \qquad \text{(Zariski decomp.)}$$
- We note $N_{S/\mathbb{P}^4} \simeq \mathcal{O}_S(-2K_S)^{\oplus 2}$ and
$$H^1(N_{S/\mathbb{P}^4}(E - C)) = H^1(-L^{\oplus 2}) = 0,$$
because $L := C + 2K_S - E$ is nef and big (by $g \geq 2d - 12$).
- For every $\alpha \in H^0(N_{C/\mathbb{P}^4}) \setminus \operatorname{im} p_1$, its *exterior component* $\pi_{C/S}(\alpha)$ in $H^0(N_{S/\mathbb{P}^4}\big|_C)$ lifts to a global section β of $N_{S/\mathbb{P}^4}(E)$ (after admitting a pole along E).
- Applying a *"modification"* of the obstructedness criterion to the infinitesimal deformation β with poles, we obtain $\operatorname{ob}(\alpha) \neq 0$. This implies
$$\dim_{[C]} \operatorname{Hilb}^{sc} \mathbb{P}^4 = \dim_{(C,S)} \operatorname{HF} \mathbb{P}^4 = d + g + 25,$$
and thereby C is obstructed and stably degenerate. □

$S_{2,2}$-maximal families of curves in $\mathbb{P}^4$

Let $C \subset S \subset X$ be a flag of algebraic varieties. We say an irreducible closed subset W of $\mathrm{Hilb}^{sc} X$ is S-maximal if there exists an irreducible component $\mathcal{W}_{C,S}$ of $\mathrm{HF}^{sc} X$ $(:= pr_1^{-1}(\mathrm{Hilb}^{sc} X))$ passing through (C, S) and $pr_1(\mathcal{W}_{C,S}) = W$.

If $d > 8$, then there exists a natural 1-to-1 correspondence between the set of $S_{2,2}$-maximal families in $\mathrm{Hilb}^{sc}_{d,g}\ \mathbb{P}^4$ and the set of 6-tuples of integer $(a; b_1, \dots, b_5)$ satisfying

$$a > b_1 \geq \cdots \geq b_5 \geq 0 \quad \text{and} \quad a \geq b_1 + b_2 + b_3 \qquad (1)$$

and

$$d = 3a - \sum_{i=1}^{5} b_i \quad \text{and} \quad g = \frac{(a-1)(a-2)}{2} - \sum_{i=1}^{5} \frac{b_i(b_i - 1)}{2}, \qquad (2)$$

by coordinates in $\mathrm{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$, i.e.,

$$[C] = a[\varepsilon^* \mathcal{O}_{\mathbb{P}^2}(1)] - \textstyle\sum_{i=1}^{5} b_i \mathbf{e}_i \longleftrightarrow (a; b_1, \dots, b_5).$$

Theorem C (Criterion)

Theorem C

Let $W := W(a; b_1, \dots, b_5) \subset \mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4$ be the $S_{2,2}$-maximal family of smooth connected curves of degree d and genus g in $\mathbb{P}^4$ correponding to $(a; b_1, \dots, b_5)$. Suppose that $d > 10$ and $g \geq 2d - 12$. Then

1. If $b_5 \geq 2$, then W is an irreducible component of $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4$ and $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4$ is generically smooth along W.
2. If $b_5 = 1$ and $b_4 \geq 2$, then W is an irreducible component of $(\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4)_{\mathrm{red}}$ and $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4$ is generically non-reduced along W.
3. If $b_5 = 0$, then W is not an irreducible component of $(\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^4)_{\mathrm{red}}$, i.e., there exists an irreducible component of $V \supsetneq W$.

Examples

Table: $S_{2,2}$-maximal families in $\mathbf{Hilb}^{sc}_{d,g}\,\mathbb{P}^4$

(d,g)	$(a; b_1, b_2, b_3, b_4, b_5)$	$W(a; b_1, b_2, b_3, b_4, b_5)$
$(14,16)$	$(8; 2, 2, 2, 2, 2)$	gen. smooth component
$(14,16)$	$(9; 4, 3, 2, 2, 2)$	gen. smooth component
$(14,16)$	$(9; 3, 3, 3, 3, 1)$	gen. non-reduced component
$(15,18)$	$(9; 4, 2, 2, 2, 2)$	gen. smooth component
$(15,18)$	$(9; 3, 3, 3, 2, 1)$	gen. non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18,24)$	$(9; 2, 2, 2, 2, 1)$	gen. non-reduced component
$(18,24)$	$(10; 4, 3, 3, 1, 1)$	unknown ($h^1(\mathcal{I}_C(2)) = 2$)
$(18,24)$	$(10; 3, 3, 3, 3, 0)$	non-component ($h^1(\mathcal{I}_C(2)) = 3$)
$(18,24)$	$(11; 6, 3, 2, 2, 2)$	gen. smooth component
$\vdots$	$\vdots$	$\vdots$

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