

# Deformations of space curves lying on a del Pezzo surface

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Today's slide

# Plan of Talk

- 1 Hilbert schemes and deformation of flags
- 2 Kleppe-Ellia conjecture and its generalization (Main result)  
(cf. [arXiv:2501.15788](https://arxiv.org/abs/2501.15788))
- 3 Applications and Examples

# §1 Hilbert schemes and deformation of flags

## the Hilbert scheme

Given a projective scheme  $X$ , and given Hilbert polynomial  $P$ ,

$$\mathbf{Hilb}_P X = \{C \subset X \mid \text{closed subscheme of } P(C) = P\}$$

is called the **Hilbert scheme** of  $X$ . The Hilbert scheme has the following nice properties:

- fine moduli scheme, i.e. it has a **universal family**  $\mathcal{C} \subset X \times \mathbf{Hilb}_P X$  such that every deformation of  $C$  in  $X$  derived from  $\mathcal{C}$
- **projective** ( $\mathbf{Hilb}_P X \hookrightarrow \mathbf{Gr}$ )
- existence of nice **deformation theories**, e.g., if  $C$  is a loc. c.i., then  $H^0(N_{C/X})$  and  $H^1(N_{C/X})$  resp. represent the tangent and obstruction spaces at  $[C]$ .

while it also has “not so nice” properties: e.g.

- it may have **bad singularities** (e.g. non-reduced components),
- may be **highly reducible** (for some  $P$ ).

## Hilbert-flag scheme

Given a sequence  $C \subset S \subset X$  of closed subscheme, Hilbert-flag scheme represents

$$\mathbf{HF} X = \{(C, S) \mid C \subset S \subset X\} \subset \mathbf{Hilb} X \times \mathbf{Hilb} X.$$

Let

$$N_{(C,S)} := N_{C/X} \times_{N_{S/X}|_C} N_{S/X}$$

be the normal sheaf of  $(C, S)$  in  $X$ .

$$H^i(N_{(C,S)/X}) = 0 \text{ for all } i > 0 \implies \mathbf{HF} X \text{ is nonsingular at } (C, S)$$

and of expected dimension.

## Stable degeneration

Let  $C \subset S \subset X$  a sequence of closed subvarieties with  $H^i(N_{(C,S)/X}) = 0$  for all  $i > 0$ .

### Definition 1

We say  $C$  is **stably degenerate** or **stably contained in  $S$** , if for every small global deformation  $C'$  of  $C$  in  $X$ , there exists a global deformation  $S'$  of  $S$  in  $X$  such that  $S' \subset C'$ .

We have the following implications:

$$(1) \implies (2) \implies (3)$$

where

- ①  $H^1(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$ .
- ②  $pr_1 : \mathbf{HF} X \rightarrow \mathbf{Hilb} X, (C', S') \mapsto [C']$  is smooth at  $(C, S)$ .
- ③  $C$  is stably degenerate

## **§2 Kleppe-Ellia conjecture and its generalization**

## Kleppe-Ellia conjecture

He proposed a conjecture, which can be reformulated as follows:

Conjecture (Kleppe'87, modified by Ellia'87)

Let  $C \subset S_3 \subset \mathbb{P}^3$  be a smooth curve of degree  $d \geq 14$  and genus  $g$  lying on a smooth cubic surface  $S_3 \subset \mathbb{P}^3$ . Then  $C$  is stably degenerate if

- ①  $\chi(I_C(3)) \geq 1$  ( $\Leftrightarrow g \geq 3d - 18$ ),
- ②  $C$  is linearly normal, and
- ③  $C$  is general in  $[C] \in \text{Pic } S_3$ .

K-E conj. is non-trivial only if  $C$  is not 3-normal ( $\Leftrightarrow H^1(I_C(3)) \neq 0$ ). In fact, otherwise,  $pr_1 : \mathbf{HF} \mathbb{P}^3 \rightarrow \mathbf{Hilb} \mathbb{P}^3$  is smooth at  $(C, S)$  by

$$H^0(N_{(C,S)/\mathbb{P}^3}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^3}) \rightarrow \underbrace{H^1(N_{S/\mathbb{P}^3}(-C))}_{\simeq H^1(I_C(3))} \rightarrow 0.$$

Moreover, the first two assumptions are necessary for the conclusion.



## Some remarks

K-E conjecture is Known to be true if

- $C$  is not **3**-normal and  $g \gg d$  (Kleppe'87 and Ellia'87), or
- $C$  is **2**-normal, i.e.  $H^1(I_C(2)) = 0$  (N'23)

## Del Pezzo surfaces

### Definition 2

A smooth projective surface  $S$  is called **del Pezzo** if  $-K_S$  is ample.

Every del Pezzo surface  $S$  is isomorphic to a blow-up of  $\mathbb{P}^2$  (at  $9 - n$  points) or  $\mathbb{P}^1 \times \mathbb{P}^1$ . The number  $n = (-K_S)^2$  is called the *degree* of  $S$ , and  $1 \leq n \leq 9$ .

### Example 1 (del Pezzo surfaces)

degree $n$	a description of $S_n$	$-K_S$
$\vdots$	$\vdots$	
<b>3</b>	cubic surface $S_3 \subset \mathbb{P}^3$	v.a.
<b>4</b>	quartic c.i. $S_{2,2} \subset \mathbb{P}^4$	
<b>5</b>	lin. section $[\text{Gr}(2, 5) \hookrightarrow \mathbb{P}^9] \cap \mathbb{L}^{(5)}$	
$\vdots$	$\vdots$	

## Why on del Pezzo?

### Proposition 3 (smoothness of flag-scheme)

Let  $C \subset S = S_n \subset \mathbb{P}^n$  be a smooth curve of degree  $d$  and genus  $g$  lying on a del Pezzo surface  $S_n$  ( $n \geq 3$ ). Then the Hilbert-flag scheme  $\mathbf{HF} \mathbb{P}^n$  is nonsingular at  $(C, S)$  of expected dimension

$$\chi(N_{(C,S)/\mathbb{P}^n}) = d + g + n^2 + 9,$$

and  $H^i(N_{(C,S)/\mathbb{P}^n}) = 0$  for all  $i > 0$ .

## Generalized Kleppe-Ellia conjecture

Toward a generalization, we study the deformations of space curves lying on a del Pezzo surface of degree.

### Conjecture (generalized K-E conj.)

Let  $C \subset S_n \subset \mathbb{P}^n$  be a smooth connected curve lying on a smooth del Pezzo surface  $S_n \subset \mathbb{P}^n$  of degree  $n \geq 3$ . Then  $C$  is *stably degenerate* if

- 1  $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$ ,
- 2  $C$  is **linearly normal**,
- 3  $\deg(C) > 9$  for  $n = 3$  and  $\deg(C) > 2n$  for  $n \geq 4$ , and  $C$  is general in  $[C] \in \text{Pic } S_n$ .

### Remark 4

$$(1) \iff \chi(N_{(C,S)/\mathbb{P}^n}) \geq \chi(N_{C/\mathbb{P}^n}).$$

## Main result

We focus on the case  $n = 4$ , i.e.  $S \simeq S_4$  is a **smooth complete intersection**

$$S_{2,2} = (2) \cap (2) \subset \mathbb{P}^4.$$

We see that  $N_{S/\mathbb{P}^4} \simeq \mathcal{O}_S(2)^{\oplus 2} \simeq \mathcal{O}_S(-2K_S)^{\oplus 2}$  and hence

$$H^1(N_{S/\mathbb{P}^4}(-C)) \simeq H^1(\mathcal{O}_S(-C - 2K_S))^{\oplus 2} \simeq H^1(I_C(2))^{\oplus 2}.$$

Then it follows from a general theory that  $C$  is stably degenerate if  $C$  is 2-normal.

### Theorem 5 (N'25)

Let  $C \subset \mathbb{P}^4$  be a smooth connected curve of degree  $d > 8$  contained in a smooth c.i.  $S = S_{2,2} \subset \mathbb{P}^4$ . Then

- ① If  $C$  is 2-normal, then  $C$  is **unobstructed and stably degenerate**.
- ② If  $\mathcal{O}_C(2)$  is non-special and  $C$  is not 2-normal, then  $C$  is **unobstructed**, but not stably degenerate.
- ③ If  $h^1(\mathcal{O}_C(2)) \geq h^1(I_C(2)) = 1$  and  $C$  is general in  $[C] \in \mathbf{Pic} S$ , then  $C$  is **obstructed and stably degenerate**.

## Applications

### Corollary 6

Generalized K-E conjecture holds to be true, if  $n = 4$  and  $h^1(I_C(2)) = 1$ .

Let  $C \subset S = S_{2,2} \subset \mathbb{P}^4$  be as in Theorem 5, and  $\mathcal{W}_{C,S}$  the irreducible component of  $\mathbf{HF} \mathbb{P}^4$  passing through  $(C, S)$ , and put

$$W_{C,S} := pr_1(\mathcal{W}_{C,S}) \cap \mathbf{Hilb}^{sc} \mathbb{P}^4.$$

Then we have the following 3 possibilities for  $W_{C,S}$ :

degeneration of $C$	Is $C$ obstructed?	$W_{C,S} \subset \mathbf{Hilb}^{sc} \mathbb{P}^4$
stable	NO	gen.smooth component
stable	YES	gen.non-reduced component
unstable	YES/NO	not a component

## Analogy of Mumford's example

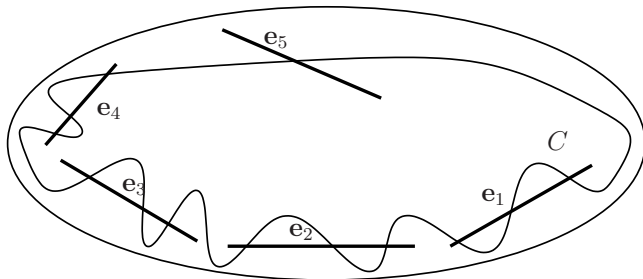
### Example 2

$\mathbf{Hilb}^{sc} \mathbb{P}^4$  contains a **generically non-reduced** irreducible component  $W$  whose general member  $C$  satisfies

- ①  $C$  is contained in a **smooth c.i.**  $S = S_{2,2} \subset \mathbb{P}^4$ ,
- ② there exists a **line**  $E$  on  $S$  such that  $C$  belongs to a complete linear system  $\Lambda := |-3K_S + 2E| (\simeq \mathbb{P}^{29})$  on  $S$ , and
- ③  $\dim W = 55$ ,  $h^0(C, N_{C/\mathbb{P}^4}) = 57$ , and  $C$  is of degree 14 and genus 16.

$$\begin{array}{ccccc}
 C & \in & W^{(55)} & \subset & \mathbf{Hilb}^{sc} \mathbb{P}^4 \\
 \downarrow & & \downarrow \mathbb{P}^{29}\text{-b'dle} & & \\
 (E, S) & \in & U^{(26)} & \subset & G \times G(2, H^0(\mathcal{O}_{\mathbb{P}^4}(2))) \\
 \downarrow & & \downarrow G(2, 12)\text{-b'dle} & & \downarrow \\
 E & \in & G(2, 5)^{(6)} & = & \{\text{lines in } \mathbb{P}^4\}.
 \end{array}$$

## Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)

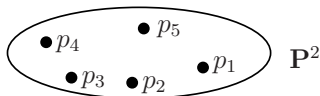


$$S_{2,2} = \text{Bl}_{p_1, \dots, p_5} \mathbb{P}^2$$

$$[C] = (9; 3, 3, 3, 3, 1) \in \text{Pic} S_{2,2} = \mathbb{Z}^6$$

$$d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$

$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$





## Standard coordinate

Let  $D$  be a divisor on a smooth c.i.  $S_{2,2} \subset \mathbb{P}^4$ , i.e., a quartic del Pezzo surface. Then there exists a suitable blow-up  $\varepsilon : S_{2,2} \rightarrow \mathbb{P}^2$  such that

$$[D] = a - \sum_{i=1}^5 b_i e_i,$$

in  $\text{Pic } S_{2,2} \simeq \mathbb{Z}^6$ , where  $\mathbf{1} = [\varepsilon^* \mathcal{O}_{\mathbb{P}^2}(\mathbf{1})]$  and  $e_i$ 's are 5 exceptionals, and such that

$$b_1 \geq \cdots \geq b_5 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \quad (1)$$

Then the set of 6-tuples  $(a; b_1, \dots, b_5)$  of integers is called **standard coordinate** of  $[D]$  in  $\text{Pic } S_{2,2}$ .

## A criterion

### Theorem 7 (N'25)

Let  $C \subset S_{2,2} \subset \mathbb{P}^4$  be a smooth curve of degree  $d \geq 10$  of genus  $g \geq 2d - 12$ , contained in a smooth c.i.  $S_{2,2}$  in  $\mathbb{P}^4$ . Let  $(a; b_1, \dots, b_5)$  be the standard coordinate of  $[C]$  in  $\text{Pic } S_{2,2} \simeq \mathbb{Z}^6$ . Then

- 1 If  $b_5 \geq 2$ , then  $C$  is unobstructed and stably degenerate.
- 2 If  $b_5 = 1$  and  $b_4 \geq 2$ , then  $C$  is obstructed and stably degenerate.
- 3 If  $b_5 = 0$ , then  $C$  is not stably degenerate.

## Examples

Table: curves on  $S_{2,2}$  and stable degeneration

$(d, g)$	$(a; b_1, b_2, b_3, b_4, b_5)$	$W(a; b_1, b_2, b_3, b_4, b_5)$
(14, 16)	(8; 2, 2, 2, 2, 2)	unobstructed and stab.degenerate
(14, 16)	(9; 4, 3, 2, 2, 2)	unobstructed and stab.degenerate
(14, 16)	(9; 3, 3, 3, 3, <b>1</b> )	obstructed and stab.degenerate
(15, 18)	(9; 4, 2, 2, 2, 2)	unobstructed and stab.degenerate
(15, 18)	(9; 3, 3, 3, 2, <b>1</b> )	obstructed and stab.degenerate
⋮	⋮	⋮
(18, 24)	(9; 2, 2, 2, 2, <b>1</b> )	obstructed and stab.degenerate
(18, 24)	(10; 4, 3, 3, <b>1</b> , <b>1</b> )	unknown ( $h^1(\mathcal{I}_C(2)) = 2$ )
(18, 24)	(10; 3, 3, 3, 3, <b>0</b> )	not stab.degenerate ( $h^1(\mathcal{I}_C(2)) = 3$ )
(18, 24)	(11; 6, 3, 2, 2, 2)	unobstructed and stab.degenerate
⋮	⋮	⋮

## further questions

- 1 Deformations of curves lying on del Pezzo  $S_n \subset \mathbb{P}^n$  of degree  $n \geq 5$ .
- 2 Deformation of degenerate curves on del Pezzo manifold of higher dimension ( $> 3$ ).
- 3 Study the relation to other examples of obstructed curves  $C \subset \mathbb{P}^n$  (or non-reduced components of  $\mathbf{Hilb}^{sc} \mathbb{P}^n$ ).  
[Y. Choi–H. Iliev–S. Kim'24] have recently proved the existence of many non-reduced components of  $\mathbf{Hilb}^{sc} \mathbb{P}^n$  of higher dimensional projective space  $\mathbb{P}^n$  by using ruled surfaces.
- 4 ...

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