Obstructions to deforming curves lying on a K3 surface

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Today's slide

Plan of Talk

1. Hilbert schemes and Mumford's example (Motivation)
2. Deformation of curves lying on a K3 surface
3. An example of non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$

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§1 Hilbert schemes and Mumford's example (Motivation)

Hilbert schemes

We work over a field $k = \bar{k}$ of $\mathbf{char}\, k = 0$.

Theorem 1 (Grothendieck'60)

Given a closed subscheme $X \subset \mathbb{P}^n$ and a closed subscheme $C \subset X$ with Hilbert polynomial $P(C) = P$, there exists a projective scheme $\mathbf{Hilb}_P\, X$, parametrizing all closed subschemes C' of X with (the same) Hilbert polynomial P.

$\mathbf{Hilb}\, X := \bigsqcup_P \mathbf{Hilb}_P\, X$ is called the Hilbert scheme of X. Today we consider the open and closed subscheme

$$\mathbf{Hilb}^{sc}\, X := \{\text{smooth connected curves } C \subset X\} \subset \mathbf{Hilb}\, X,$$

that is, the Hilbert scheme of curves in X.

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Infinitesimal property of Hilbert schemes

- The tangent space of $\mathbf{Hilb}\,X$ at $[C]$ is isomorphic to $H^0(C, N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(C, N_{C/X})$ $(\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X})\text{ if } C \text{ is a curve})} \le \dim_{[C]} \mathbf{Hilb}\,X \le \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- We say $C \subset X$ is unobstructed if $\mathbf{Hilb}\,X$ is nonsingular at $[C]$.
- $H^1(C, N_{C/X}) = 0 \Longrightarrow C$ is unobstructed. The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*.).

Purpose 2

Determine $\dim_{[C]} \mathbf{Hilb}\,X$ at a singular point $[C]$ of $\mathbf{Hilb}\,X$.

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Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry [Mumford'62]".

Example 1 (Mumford)

$\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W of dimension 56, whose general member C satisfies:

1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ ($\simeq \mathbb{P}^{37}$) on S.

Remark 3

- C and $\mathbb{P}^3$ are innocent-looking (a pathology).
- C is of degree 14 and genus 24, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

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Generalization of Mumford's example

- Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of $\mathbf{Hilb}^{sc}\,\mathbb{P}^n$ $(n > 3)$ have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves C corresponding to the generic point of the non-reduced components were contained in some surface $S \subset \mathbb{P}^n$, e.g.,

	a surface S containing C
Mumford['62]	smooth cubic
Gruson-Peskine'82	non-normal cubic
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Another generalization (with Mukai)

We found that in Mumford's example, (-1)-curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

Theorem 4 (Mukai-N'09, $\operatorname{char} k \geq 0$)

Let X be a smooth projective 3-fold satisfying the following:

1. there exists a smooth rational curve E on X s.t. $N_{E/X}$ is globally generated, and
2. there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathrm{Hilb}^{sc}\,X$ has infinitely many generically non-reduced components (GNRC).

Remark 5

In Mumford's ex., $X = \mathbb{P}^3$, S is a smooth cubic, E is a line.

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Hilbert scheme of Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index r of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \operatorname{Pic} X$.

Let X be a smooth Fano 3-fold of index r.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Hilbert scheme of Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index r of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \mathrm{Pic}\,X$.

Let X be a smooth Fano 3-fold of index r.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

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Applying Theorem 4, we obtain

Example 2 (N'10)

If $r(X) > 1$, then $\mathrm{Hilb}^{sc} X$ contains a a generically non-reduced component W satisfying:

1. every general member C of W is contained in a smooth del Pezzo surface S $(\sim -\frac{r-1}{r}K_X)$, and
2. there exists a (good) line E on S and $C \sim -K_X\big|_S + 2E$ in $\mathrm{Pic}\,S$.
3. $h^0(C, N_{C/X}) = \dim W + 1$.

Here

- A curve $E \subset X$ is a line $\overset{\text{def}}{\Longleftrightarrow} E \simeq \mathbb{P}^1$ and $-\frac{1}{r}K_X.E = 1$.
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§2 Deformation of curves lying on a K3 surface

Curves on K3 surface

In this section, we study the deformations of smooth curves on a smooth Fano 3-fold, under the assumption that the curve is contained in a smooth $K3$ surface.

Definition 6

A smooth projective surface S with $K_S \sim 0$ and $H^1(S, \mathcal{O}_S) = 0$ is called a $K3$ surface.

Let

$$C \subset S_{K3} \subset X_{Fano},$$

a sequence of a curve, a $K3$ surface, a Fano 3-fold. We will see the following curves on S control the deformations C in X:

- (-2)-curves $E \simeq \mathbb{P}^1$,
- elliptic curves F

Actually, they play a role very similar to that of (-1)-curve $E \simeq \mathbb{P}^1$ on the smooth cubic $S_3 \subset \mathbb{P}^3$ in Mumford's example!

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Hilbert-flag scheme

A main tool of our studies is $\mathbf{HF}\, X$ the Hilbert-flag scheme of X, i.e.

$$\mathbf{HF}\, X = \left\{(C,S) \;\middle|\; C \subset S \subset X\text{: closed subschemes}\right\}.$$

If $C \hookrightarrow S \hookrightarrow X$ is regular embeddings, then

- the tang. sp. and the obst. sp. of $\mathbf{HF}\, X$ at (C,S) is $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$.
- the normal sheaf $N_{(C,S)/X}$ of (C,S) in X is the fiber product sitting in

$$\begin{tikzcd} N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\ N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C. \end{tikzcd}$$

  Here $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection.
- Similarly to Hilbert schemes,

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathbf{HF}\, X \le h^0(X, N_{(C,S)/X}).$$

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$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathrm{HF}\,X \le h^0(X, N_{(C,S)/X}).$$

Lemma 7

Let $C \subset S_{K3} \subset X_{Fano}$. Then TFAE:

1. $H^1(N_{(C,S)/X}) = 0$, namely $\mathbf{HF}\,X$ is nonsingular at (C,S) of expected dimension $\chi(N_{(C,S)/X})$.
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Proof.

By adjunction, $N_{S/X} \simeq -K_X\big|_S$ and $N_{C/S} \simeq K_C$, which implies $H^i(N_{S/X}) = 0$ for $i > 0$ and $H^1(N_{C/S}) \simeq k$. There exists an exact sequence $0 \to N_{C/S} \to N_{(C,S)/X} \to N_{S/X} \to 0$, inducing

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Example 8

Let X be a prime Fano 3-fold, $S \in |-K_X|$ a $K3$ surface, $E \subset S$ a line on S. Then TFAE:

1. E is of type $(0,-1)$, i.e. $N_{E/X} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$,
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A key lemma

Lemma 10 (char $k = 0$)

Let $S \overset{\iota}{\hookrightarrow} X$ be an embedded $K3$ surface, and let E be a curve on S with $H^1(\mathcal{O}_S(E)) = 0$ and $H^1(X, N_{(E,S)/X}) = 0$. If $C - bE \in \iota^* \operatorname{Pic} X$ for some $b \neq 0$, then $H^1(X, N_{(C,S)/X}) = 0$.

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Since $H^1(X, N_{(E,S)/X}) = 0$, by Lem. 7, there exists a first order deformation $\tilde{S}$ of S in X, to which E does not lift. Then neither does $\mathcal{O}_S(E)$ by $H^1(\mathcal{O}_S(E)) = 0$. Let $\tau \in H^1(S, T_S)$ (abstract def.) correspond to $\tilde{S}$. Then $\tau \cup c(\mathcal{O}_S(E)) \neq 0$ in $H^2(\mathcal{O}_S)$, where $c(*) \in H^1(S, \Omega^1_S)$ denotes the Atiyah-ext. class of $*$. Since $c(\mathcal{O}_S(C)) = c(\mathcal{O}_S(C - bE)) + bc(\mathcal{O}_S(E))$, and $C - bE \in i^* \operatorname{Pic} X$, we have $\tau \cup c(\mathcal{O}_S(C)) \neq 0$, hence $\mathcal{O}_S(C)$ does not lift to $\tilde{S}$, hence neither does C as a closed subscheme of S, and $H^1(X, N_{(C,S)/X}) = 0$. $\square$

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Let E be a curve on S ($\subset X$), and $\pi_{E/S} : N_{E/X} \longrightarrow N_{S/X}\big|_E$ the projection.

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Obstructedness of curves in a $K3$ surface

Let X be a smooth Fano 3-fold, a smooth $K3$ surface $S \in |-K_X|$, $C \subset X$ a smooth curve.

Theorem 13 (N'17)

Put

$$D := C + K_X\big|_S.$$

a divisor on S. If $H^1(N_{(C,S)/X}) = 0$ and $D \geq 0$, then

1. If there exist no (-2)-curves and no elliptic curves on S, or if $H^1(S, D) = 0$, then C is unobstructed.
2. If $D^2 \geq 0$ and there exists a (-2)-curve E on S such that $E.D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover, the π-map $\pi_{E/S}(E)$ is not surjective, then C is obstructed.
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Obstructedness of curves in a $K3$ surface

Let X be a smooth Fano 3-fold, a smooth $K3$ surface $S \in |-K_X|$, $C \subset X$ a smooth curve.

Theorem 13 (N'17)

Put

$$D := C + K_X\big|_S.$$

a divisor on S. If $H^1(N_{(C,S)/X}) = 0$ and $D \geq 0$, then

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Analogy of Mumford's ex. in the case $r = 1$

Applying the previous theorem to a $K3$ surfaces $S \subset X$ and a good conic $E \simeq \mathbb{P}^1$ on S, we obtained:

Theorem 14 (N'19)

Let X be a prime Fano 3-fold of genus $g := (-K_X)^3/2 + 1$. Then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component W with the following properties:

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For the proof, we need the following facts:

- The prime Fano 3-fold $X = X_{2g-2}$ contains a conic $E \subset V$ (cf. [Shokurov'79], [Reid'80]).
- If $E \subset X$ is general, then E is a good conic (cf. [Iskovskih'78]) if $\mathrm{char}\, k = 0$.
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Corollary 15

If X is a smooth Fano 3-fold and $\rho(X) = 1$, then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component.

3-fold X	surface S	$[C] \in \operatorname{Pic} S$	E	
$\mathbb{P}^3$	del Pezzo	$-K_X\big\|_S + 2E$	line	Mumford['62]
$Q^3 \subset \mathbb{P}^4$				Mukai-N['09]
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Table: Generically non-reduced component of Mumford type

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§3 An example of non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$

Main result

Toward a further generazation, we compute the obstruction to deforming curves lying on a complete intersection $K3$ surface in $\mathbb{P}^5$.

Theorem 16 (Main)

The Hilbert scheme $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components W_n ($n \geq 2$) with the following properties:

1. every general C of W_n is a smooth connected curve contained in a smooth complete intersection $K3$ surface
$$S = S_{2,2,2} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5.$$
2. C is linearly equivalent to $n(2\mathbf{h} + E)$, where $\mathbf{h} = [\mathcal{O}_S(1)]$ in $\operatorname{Pic} S$, and E is a line on S.
3. C is of degree $17n$ and genus $17n^2 + 1$.
4. $\dim W = 17n^2 + 54$ $(= g + 53)$, while $h^0(N_{C/\mathbb{P}^5}) = 17n^2 + 57$, thus $h^0(N_{C/\mathbb{P}^5}) - \dim W = 3$.

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Construction

We see $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$ and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(g+53)} \arrow[d, "\mathbb{P}^g\text{-bundle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^5 \\
(E, S) \arrow[d, mapsto] & \in & U^{(53)} \arrow[d, "\mathrm{Gr}(3,18)\text{-bundle}"] & \subset & G \times \mathrm{Gr}(3, V) \arrow[d] \\
E & \in & \mathrm{Gr}(2,6)^{(8)} & = & \left\{\text{lines in } \mathbb{P}^5\right\},
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Key Lemma

Let X be a projective scheme, and let $\mathbf{HF}\,X$ be the Hilbert-flag scheme of X. There exists a projection

$$pr_1 : \mathbf{HF}\,X \to \mathbf{Hilb}\,X, \qquad (C, S) \mapsto [C],$$

which induces the tangent map $p_1 : H^0(X, N_{(C,S)/X}) \to H^0(C, N_{C/X})$.

Lemma 17 (Key Lemma, N['23], Lem. 2.17)

We have $\dim_{(C,S)} \mathbf{HF}\,X = \dim_{[C]} \mathbf{Hilb}\,X$ if we have

1. $H^1(X, N_{(C,S)/X}) = H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
2. for every $\alpha \in H^0(C, N_{C/X}) \setminus \operatorname{im} p_1$, the (primary) obstruction $\mathbf{ob}(\alpha)$ (to extend α a second order deformation over $k[t]/(t^3)$) is nonzero.

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is nef and big.

- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \mathrm{im}\, p_1$, its *exterior component* $\pi_{C/S}(\alpha)$, i.e., the image of α in $H^0(N_{S/\mathbb{P}^5}(E)\big|_C)$, lifts to a global section β of $N_{S/\mathbb{P}^5}(E)$. (Here β is called an infinitesimal deformation with poles.)

- Applying a *"modification"* of the obstructedness criterion [Mukai-N'09] to β, we obtain $\mathrm{ob}(\alpha) \neq 0$. This implies

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Thank you very much for listening!

Primary obstructions

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- A first order (infinitesimal) deformation of C is a deformation $\tilde{C}$ ($\subset X \times \mathbf{Spec}\,k[\varepsilon]$) of C in X over $k[\varepsilon]$.
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Obstructedness Criterion (with Mukai)

[Mukai-N'09] gave a sufficient condition for $\mathbf{ob}(\alpha) \neq 0$ when $\dim X = 3$.

Let C be an irreducible curve on a 3-fold X.

Assumption 1

- there exists an intermediate surface $C \subset S \subset X$ s.t. $C \hookrightarrow S$ and $S \hookrightarrow X$ are regular embeddings.
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for all $m > 0$. (e.g. E=(-1)-$\mathbb{P}^1$ on S)

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$$\begin{tikzcd}
 & & H^0(N_{S/X}) \arrow[d] \arrow[r, phantom, "\subset"] & H^0(N_{S/X}(E)) \arrow[d] \arrow[r, phantom, "\ni \beta"] & {} \\
\alpha \in H^0(N_{C/X}) \arrow[rr, "\pi_{C/S}"] & & H^0(N_{S/X}\big|_C) \arrow[r, phantom, "\subset"] & H^0(N_{S/X}(E)\big|_C) &
\end{tikzcd}$$

Here β is called an infinitesimal deformation with pole:

Obstructedness Criterion (Continued)

Let $\alpha \in H^0(N_{C/X})$ be a first order deformation of C in X and $\mathbf{ob}(\alpha) \in H^1(N_{C/X})$ its primary obstruction. We consider the *"exterior" components*

$$\pi_{C/S}(\alpha) := H^0(\pi_{C/S})(\alpha), \qquad \mathbf{ob}_S(\alpha) := H^1(\pi_{C/S})(\mathbf{ob}(\alpha)).$$

by the projection

$$\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C.$$

Assumption 2

- Suppose $\pi_{C/S}(\alpha)$ lifts to a global section β of $N_{S/X}(E)$.

$$\begin{array}{ccccccc} & & & H^0(N_{S/X}) & \subset & H^0(N_{S/X}(E)) & \ni \beta \\ & & & \downarrow & & \downarrow & \\ \alpha \in & H^0(N_{C/X}) & \xrightarrow{\pi_{C/S}} & H^0(N_{S/X}\big|_C) & \subset & H^0(N_{S/X}(E)\big|_C) & \end{array}$$

Here β is called an infinitesimal deformation with pole:

Obstructedness Criterion (Continued)

Theorem 18 (Mukai-N'09)

$\mathbf{ob}_S(\alpha)$ is nonzero if

1. $\Delta.E = 2(-E^2 + g(E) - 1)$, where $\Delta := C + K_X\big|_S - 2E$ in $\operatorname{Pic} S$.
2. Let $\beta\big|_E$ be the principal part of β along E. Then $\beta\big|_E \cup \mathrm{k}_E \neq 0$ in $H^1(E, \mathcal{O}_E(2E))$, where
$$\begin{aligned}\mathrm{k}_E &:= [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}\big|_E \longrightarrow 0] \\ &\in \operatorname{Ext}^1_E(N_{S/X}\big|_E, N_{E/S}).\end{aligned}$$
3. the restriction map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

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further questions

1. Generalization to the deformations of space curves lying on a non complete intersection surface $S_n \subset \mathbb{P}^n$ for $n \geq 4$.
2. Are there relationships between the obstructed curves $C \subset \mathbb{P}^n$ and the projections $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$? (cf. [Y. Choi–H. Iliev–S. Kim'22])
3. More generally, study Hilbert schemes from the viewpoint of morphisms (N.B. $X \mapsto \mathbf{Hilb}\,X$ is not functorial!)
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