Obstructions to deforming space curves lying on complete intersection of quadrics

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January 26, 2026
Birational and Affine Geometry (Incheon Songdo)

Today's slide

Plan of Talk

1. Hilbert schemes of Fano 3-folds
2. Deformation of space curves lying on a complete intersection of quadrics (an attempt to generalize known results into higher dimensional varieties)

§1 Hilbert schemes of Fano threefolds

Hilbert schemes

We work over a field $k = \bar{k}$ of $\mathrm{char}\, k = 0$.
Let X be a projective scheme over k. We denote by $\mathrm{Hilb}\, X$ the Hilbert scheme of X. Today we consider the open and closed subscheme

$$\mathrm{Hilb}^{sc} X := \{\text{smooth connected curves } C \subset X\} \subset \mathrm{Hilb}\, X,$$

that is, the Hilbert scheme of curves in X.

Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index r of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \mathrm{Pic}\, X$.

Let X be a smooth Fano 3-fold of index r.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Hilbert scheme of Fano 3-folds

The Hilbert scheme of lines and conics have been studied widely. We can found the following result in a survey by Kuznetsov-Prokohrov-Shramov.

Theorem 1 (classical)

Let X be a smooth Fano 3-fold of $\rho(X) = 1$ and $r(X) = r$. Let $\Sigma(X)$ and $S(X)$ denote the Hilbert scheme of lines and conics, respectively. If X is general and X is not a quartic double solid ($r = d = 2$), then $\Sigma(X)$ and $S(X)$ are generically smooth.

<table>
<tr><th>r</th><th>X</th><th>$\Sigma(X)$</th><th>$S(X)$</th></tr>
<tr><td>4</td><td>$\mathbb{P}^3$</td><td>$\mathrm{Gr}(2,4)$</td><td>$\mathbb{P}(Sym^2 U^*)$</td></tr>
<tr><td>3</td><td>$Q = (2) \subset \mathbb{P}^4$</td><td>$\simeq \mathbb{P}^3$</td><td>$\mathrm{Bl}_{OG(3,5)} \mathrm{Gr}(3,5)$</td></tr>
<tr><td rowspan="3">2
$(d \geq 3)$</td><td>$(3) \subset \mathbb{P}^4$</td><td>surface of gen.type</td><td>$\mathbb{P}^2$-b'dle/surface</td></tr>
<tr><td>$(2) \cap (2) \subset \mathbb{P}^5$</td><td>abelian surface</td><td>$\mathbb{P}^3$-b'dle/curve</td></tr>
<tr><td>$V_5 \subset \mathbb{P}^6$</td><td>$\mathbb{P}^2$</td><td>$\mathbb{P}^4$</td></tr>
<tr><td>1</td><td>prime Fano</td><td>(reduced) curve</td><td>(reduced) surface</td></tr>
</table>

Rational vs Higher genus

Let X be a smooth Fano 3-fold. Then

- The Hilbert scheme $\Sigma(X)$ and $S(X)$ of lines and conics on X behaves well, except for low degree case of $r = 2$ and special prime Fano.
- More generally, the Hilbert scheme $\mathrm{Hilb}^{sc}_{d,0} X$ of smooth rational curves on X behaves well *by vanishing of obstructions* (deformation theorey).
- However, the geometry of the Hilbert scheme $\mathrm{Hilb}^{sc}_{d,g} X$ for higher genus $g > 0$ becomes more cmplecated, due to their non-vanishings.

Infinitesimal property of Hilbert schemes

- The tangent space of $\mathbf{Hilb}\, X$ at $[C]$ is isomorphic to $H^0(N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(N_{C/X})$ $(\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathbf{Hilb}\, X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- $H^1(N_{C/X}) = 0 \Longrightarrow \mathbf{Hilb}\, X$ is nonsingular at $[C]$ (We say C is unobstructed). The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*.).

Purpose 2

Determine $\dim_{[C]} \mathbf{Hilb}\, X$ at a singular point $[C]$ of $\mathbf{Hilb}\, X$.

Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry [Mumford'62]".

Example 1 (Mumford)

$\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W of dimension 56, whose general member C satisfies:

1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ $(\simeq \mathbb{P}^{37})$ on S.

Remark 3

- C and $\mathbb{P}^3$ are innocent-looking (a pathology).
- C is of degree 14 and genus 24, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

Generalization of Mumford's example

- Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of $\mathbf{Hilb}^{sc}\,\mathbb{P}^n$ $(n > 3)$ have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves C corresponding to the generic point of the non-reduced components were contained in some surface $S \subset \mathbb{P}^n$, e.g.,

	a surface S containing C
Mumford['62]	smooth cubic
Gruson-Peskine'82	non-normal cubic
Kleppe'87	smooth cubic
Kleppe-Ottem'15	smooth quartic
Choi-Iliev-Kim'24-1, '24-2	ruled surface

Generalization to smooth Fano 3-folds

Theorem 4 (Mumford'62,Mukai-N'09,N'10,N'19)

If X is a smooth Fano $\mathbf{3}$-fold and $\rho(X) = 1$, then $\mathbf{Hilb}^{sc} X$ contains a generically non-reduced component such that

1. its general member C is contained in the following surface $S \subset X$, and
2. C belongs to the following class $[C] \in \operatorname{Pic} S$:

3-fold X	surface S	$[C] \in \operatorname{Pic} S$	E
$\mathbb{P}^3$	del Pezzo	$-K_X\big\|_S + 2E$	line
$Q^3 \subset \mathbb{P}^4$			
del Pezzo			
prime Fano	$K3$	$-2K_X\big\|_S + 2E$	conic

Another generalization (with Mukai)

In Mumford's example, (-1)-curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

Theorem 5 (Mukai-N'09, $\mathrm{char}\, k \geq 0$)

Let X be a smooth projective 3-fold satisfying the following:

1. there exists a smooth rational curve E on X s.t. $N_{E/X}$ is globally generated, and
2. there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathrm{Hilb}^{sc} X$ has infinitely many generically non-reduced components (GNRC).

Remark 6

In Mumford's ex., $X = \mathbb{P}^3$, S is a smooth cubic, E is a line.

Another example (Hom scheme)

An example of Hom scheme, which has a generically non-reduced irreducible component, was given.

Theorem 7 (Mukai-N'09)

Let X be a general genus-5 curve, V_3 be a general smooth cubic 3-fold, or of Fermat type cubic 3-fold:

$$V_3^{\mathrm{Fermat}} : x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \subset \mathbb{P}^4,$$

then $\mathbf{Hom}_8(X, V)$ has a genrically non-reduced component of dimension 4.

§2 Deformations of space curves lying on a complete intersection of quadrics

Complete intersection of quadrics

Today we consider deformations of space curves $C \subset \mathbb{P}$ lying on a smooth surface S, when S are complete intersections

$$S_2 \subset \mathbb{P}^3, \quad S_{2,2} \subset \mathbb{P}^4, \quad S_{2,2,2} \subset \mathbb{P}^5$$

of quadrics. For each S and curves C lying on S, we consider the deformations of C in the projective space $\mathbb{P}$ using the flag

$$C \subset S \subset \mathbb{P}.$$

Hilbert-flag scheme

A main tool of our studies is $\mathbf{HF}\,X$ the Hilbert-flag scheme of X, i.e.

$$\mathbf{HF}\,X = \left\{(C,S) \;\middle|\; C \subset S \subset X\text{: closed subschemes}\right\}.$$

If $C \hookrightarrow S \hookrightarrow X$ is regular embeddings, then

- the tang. sp. and the obst. sp. of $\mathbf{HF}\,X$ at (C,S) is $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$.
- the normal sheaf $N_{(C,S)/X}$ of (C,S) in X is the fiber product sitting in

$$\begin{tikzcd} N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\ N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C. \end{tikzcd}$$

  Here $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection.
- Similarly to Hilbert schemes,

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathbf{HF}\,X \le h^0(X, N_{(C,S)/X}).$$

Smooth quadric surface S_2 in $\mathbb{P}^3$

Let $S = S_2 \subset \mathbb{P}^3$. Then $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Pic}\, S \simeq (\mathrm{Pic}\, \mathbb{P}^1)^{\oplus 2} = \mathbb{Z}^2$. Thus every curve C on S corresponds to $(a, b) \in \mathbb{P}^2$ (called *bidgree* of C). Then $d(C) = a + b$ and $g(C) = (a-1)(b-1)$.

Theorem 8 (Tannenbaum'78, Kleppe'87, etc)

Let $d > 4$ and $g \geq 0$. Let $W(a, b)$ be a maximal irreducible family of smooth connected curves $C \subset \mathbb{P}^3$ of bidegree (a, b), contained in a smooth quadric surface $S_2 \subset \mathbb{P}^3$. Then

1. $\mathrm{Hilb}\, \mathbb{P}^3$ is nonsingular along $W(a, b)$.
2. $\dim W(a, b) = 2d + g + 8$.
3. The closure $\overline{W}(a, b)$ of $W(a, b)$ becomes an irreducible component of $\mathrm{Hilb}^{sc}\, \mathbb{P}^3$ if $g \geq 2d - 8$, otherwise $\overline{W}(a, b)$ is a proper closed subset of of $\mathrm{Hilb}^{sc}\, \mathbb{P}^3$ of codimension $2d - 8 - g$.

Sketch of proof

If $C \subset \mathbb{P}^3$ is contained in a smooth quadric $S = S_2 \subset \mathbb{P}^3$, then $H^1(N_{C/S}) = 0$ and this implies $\mathrm{HF}\,\mathbb{P}^3$ is nonsingular at (C, S) of expected dimension

$$\chi(N_{(C,S)}) = \chi(N_{C/S}) + \chi(N_{S/\mathbb{P}^3}) = 2d + g + 8.$$

Then we see that either $H^1(\mathcal{I}_C(2)) = 0$ or $H^1(\mathcal{O}_C(2)) = 0$.
If $H^1(\mathcal{I}_C(2)) = 0$, then the tangent map

$$p_1 : \mathcal{T}_{\mathrm{HF}\,\mathbb{P}^3,(C,S)} \longrightarrow \mathcal{T}_{\mathrm{Hilb}\,\mathbb{P}^3,[C]}$$

is surjective and the 1st projection $pr_1 : (C, S) \mapsto [C]$ is smooth at (C, S).
If $H^1(\mathcal{O}_C(2)) = 0$, then by $\mathcal{O}_C(2) \simeq N_{S/\mathbb{P}^3}\big|_C$ and the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3}\big|_C \longrightarrow 0,$$

$H^1(N_{C/\mathbb{P}^3})$ vanishes. Thus $\mathrm{Hilb}\,\mathbb{P}^3$ is nonsingular at $[C]$. $\square$

Smooth complete intersection $S_{2,2} \subset \mathbb{P}^3$

Let $S = S_{2,2} \subset \mathbb{P}^4$ be a smooth complete intersection. Then S is a del Pezzo surface of degree 4, that is, $\mathrm{Bl}_{P_1,\dots,P_5}\,\mathbb{P}^2$. Then $\mathrm{Pic}\,S \simeq \mathbb{Z}^7$. Moreover, every divisor D on S corresponds to a 7-tuple $(a; b_1, \dots, b_5)$ of integers $a, b_1, \dots, b_5$ by

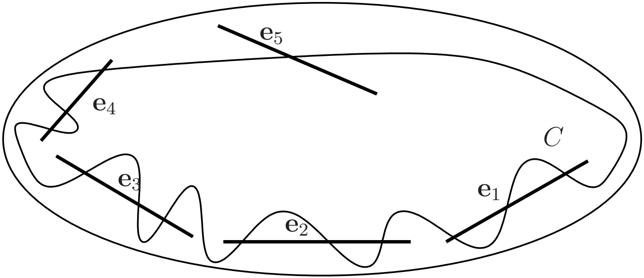
$$[D] = a\mathbf{l} - \sum_{i=1}^{5} b_i \mathbf{e}_i,$$

Moreover, the coordinate $(a; b_1, \dots, b_5)$ satisfies

$$b_1 \geq \cdots \geq b_5 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \tag{1}$$

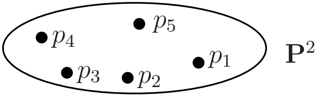
Here $(a; b_1, \dots, b_5)$ is called the standard coordinate of $[D]$ in $\mathrm{Pic}\,S_{2,2}$.

Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)



$\varepsilon \Big\downarrow$

$$S_{2,2} = \mathrm{Bl}_{p_1,\dots,p_5}\mathbf{P}^2$$
$$[C] = (9;3,3,3,3,1) \in \mathrm{Pic} S_{2,2} = \mathbf{Z}^6$$
$$d = 3\cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$
$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$



Stable degeneration

Suppose that

$$C \subset S \subset \mathbb{P}$$

is a flag of a space curve C and a surface S.

Definition 2

We say C is stably degenerate if every small global deformation of C in $\mathbb{P}$ is contained in a deformation of S in $\mathbb{P}$.

Theorem 9 (Main1)

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth curve of degree $d \geq 10$ of genus $g \geq 2d - 12$, contained in a smooth c.i. $S_{2,2}$ in $\mathbb{P}^4$. Let $(a; b_1, \dots, b_5)$ be the standard coordinate of $[C]$ in $\mathbf{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$. Then

1. If $b_5 \geq 2$, then C is unobstructed and stably degenerate.
2. If $b_5 = 1$ and $b_4 \geq 2$. then C is obstructed and stably degenerate.
3. If $b_5 = 0$, then C is not stably degenerate.

Examples

Table: curves on $S_{2,2}$ and stable degeneration

(d, g)	the class of C	the max. irred. family of C ($\subset \mathbf{Hilb}^{sc}\, \mathbb{P}^4$)
$(14, 16)$	$(8; 2, 2, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 4, 3, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 3, 3, 3, 3, 1)$	generically non-reduced component
$(15, 18)$	$(9; 4, 2, 2, 2, 2)$	generically smooth component
$(15, 18)$	$(9; 3, 3, 3, 2, 1)$	generically non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18, 24)$	$(9; 2, 2, 2, 2, 1)$	generically non-reduced component
$(18, 24)$	$(10; 4, 3, 3, 1, 1)$	unknown ($h^1(\mathcal{I}_C(2)) = 2$)
$(18, 24)$	$(10; 3, 3, 3, 3, 0)$	a proper closed subset of a component
$(18, 24)$	$(11; 6, 3, 2, 2, 2)$	generically smooth component
$\vdots$	$\vdots$	$\vdots$

Analogy of Mumford's example

Example 3

$\mathrm{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W whose general member C satisfies

1. C is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. there exists a line E on S such that C belongs to a complete linear system $\Lambda := |-3K_S + 2E|$ $(\simeq \mathbb{P}^{29})$ on S, and
3. $\dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^4} = 55$, $h^0(C, N_{C/\mathbb{P}^4}) = 57$, and C is of degree 14 and genus 16.

$$
\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(55)} \arrow[d, "\mathbb{P}^{29}\text{-b'dle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^4 \\
(E,S) \arrow[d, mapsto] & \in & U^{(26)} \arrow[d, "G(2,12)\text{-b'dle}"] & \subset & G \times G(2, H^0(\mathcal{O}_{\mathbb{P}^4}(2))) \arrow[d] \\
E & \in & G(2,5)^{(6)} & = & \{\text{lines in } \mathbb{P}^4\}.
\end{tikzcd}
$$

Generalized Kleppe-Ellia conjecture

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a anti-polarized *smooth del Pezzo surface* $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is stably degenerate if

1. $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$,
2. C is linearly normal,
3. $\deg(C) > 9$ for $n = 3$ and $\deg(C) > 2n$ for $n \geq 4$, and C is general in $[C] \in \operatorname{Pic} S_n$.

Smooth complete intersection $S_{2,2,2} \subset \mathbb{P}^5$

Let $S_{2,2,2} \subset \mathbb{P}^5$ be a smooth complete intersection. Then S is a $K3$ of degree 8 (and genus 5). Since $N_{S/\mathbb{P}^5} \simeq \mathcal{O}_S(2)^{\oplus 2}$, we have

$$H^1(N_{S/\mathbb{P}^5}(-C)) = H^1(S, -(C - 2\mathbf{h}))^{\oplus 2},$$

where $\mathbf{h} = \mathcal{O}_S(1)$. Thus if $C - 2\mathbf{h}$ is nef and big and $H^1(N_{(C,S)/\mathbb{P}^5}) = 0$, then C is stably degenerate and unobstructed.

Theorem 10 (Main2)

The Hilbert scheme $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components W_n ($n \geq 2$) with the following properties:

1. every general C of W_n is a smooth connected curve contained in a smooth complete intersection $K3$ surface $S_{2,2,2} \subset \mathbb{P}^5$.
2. C is linearly equivalent to $n(2\mathbf{h} + E)$, where E is a line on S.
3. C is of degree $d = 17n$ and genus $g = 17n^2 + 1$.
4. $\dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^5} = g + 53$, while $h^0(N_{C/\mathbb{P}^5}) = g + 56$, thus $h^0(N_{C/\mathbb{P}^5}) = \dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^5} + 3$.

Construction

We see $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$ and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(g+53)} \arrow[d, "\mathbb{P}^g\text{-bundle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^5 \\
(E, S) \arrow[d, mapsto] & \in & U^{(53)} \arrow[d, "\mathrm{Gr}(3,18)\text{-bundle}"] & \subset & G \times \mathrm{Gr}(3, V) \arrow[d] \\
E & \in & \mathrm{Gr}(2,6)^{(8)} & = & \left\{\text{lines in } \mathbb{P}^5\right\},
\end{tikzcd}$$

where $V = H^0(\mathbb{P}^5, \mathcal{O}(2))$.

Key Lemma

Let X be a projective scheme, and let $\mathbf{HF}\ X$ be the Hilbert-flag scheme of X. There exists a projection

$$pr_1 : \mathrm{HF}\ X \to \mathrm{Hilb}\ X, \qquad (C, S) \mapsto [C],$$

which induces the tangent map $p_1 : H^0(N_{(C,S)/X}) \to H^0(N_{C/X})$.

Lemma 11 (Key Lemma, N['23], Lem. 2.17)

We have $\dim_{(C,S)} \mathrm{HF}\ X = \dim_{[C]} \mathrm{Hilb}\ X$ if

1. $H^1(N_{(C,S)/X}) = H^0(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
2. for every $\alpha \in H^0(N_{C/X}) \setminus \mathrm{im}\ p_1$, the (primary) obstruction $\mathbf{ob}(\alpha)$ (to extend α a second order deformation over $k[t]/(t^3)$) is nonzero.

Sketch of Proof of Main2.

Let $C \subset \mathbb{P}^5$ be a smooth connected curve lying on a complete intersection $K3$ surface $S = S_{2,2,2} \subset \mathbb{P}^5$, and such that $C \sim n(2\mathbf{h} + E)$ in $\mathrm{Pic}\, S$ for $n \geq 2$, where E is a line on S.

- Since $d = 17n > 16 = 2h^2$, S is uniquely determined by C.
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by proj. normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

  that $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$, which implies there exists a first order deformation of $\tilde{S}$ of S, to which E (and hence C) does not lift.
- Then $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$ for $i > 0$ and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

  is exact. Here p_1 is the tangent map of $pr_1 : \mathrm{HF}\, \mathbb{P}^5 \to \mathrm{Hilb}\, \mathbb{P}^5$ at (C, S) and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2\mathbf{h} - C))^{\oplus 3} \simeq k^3.$$

Sketch of Proof of Main2.(continued)

- We note that $H^1(N_{S/\mathbb{P}^5}(E - C)) = H^1(-L^{\oplus 3}) = 0$, where

$$L = C - 2\mathrm{h} - E = (n-1)(2\mathrm{h} + E)$$

  is nef and big.
- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \mathrm{im}\, p_1$, its *exterior component* $\pi_{C/S}(\alpha)$, i.e., the image of α in $H^0(N_{S/\mathbb{P}^5}(E)\big|_C)$, lifts to a global section β of $N_{S/\mathbb{P}^5}(E)$. (Here β is called an infinitesimal deformation with poles.)
- Applying a *"modification"* of the obstructedness criterion [Mukai-N'09] to β, we obtain $\mathrm{ob}(\alpha) \neq 0$. This implies

$$\dim_{[C]} \mathrm{Hilb}^{sc}\, \mathbb{P}^5 = \dim_{(C,S)} \mathrm{HF}\, \mathbb{P}^5$$

  by the key lemma. Therefore C is obstructed and parametrised by an open dense subset of a component of $\mathrm{Hilb}^{sc}\, \mathbb{P}^5$.

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