Obstructions to deforming curves lying on a K3 surface

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Today's slide

Plan of Talk

1. Hilbert schemes and Mumford's example (Motivation)
2. Deformation of curves lying on a K3 surface
3. An example of non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$

§1 Hilbert schemes and Mumford's example (Motivation)

Hilbert schemes

We work over a field $k = \bar{k}$ of $\mathbf{char}\, k = 0$.

Theorem 1 (Grothendieck'60)

Given a closed subscheme $X \subset \mathbb{P}^n$ and a closed subscheme $C \subset X$ with Hilbert polynomial $P(C) = P$, there exists a projective scheme $\mathbf{Hilb}_P\, X$, parametrizing all closed subschemes C' of X with (the same) Hilbert polynomial P.

$\mathbf{Hilb}\, X := \bigsqcup_P \mathbf{Hilb}_P\, X$ is called the Hilbert scheme of X. Today we consider the open and closed subscheme

$$\mathbf{Hilb}^{sc}\, X := \{\text{smooth connected curves } C \subset X\} \subset \mathbf{Hilb}\, X,$$

that is, the Hilbert scheme of curves in X.

Infinitesimal property of Hilbert schemes

- The tangent space of $\mathrm{Hilb}\,X$ at $[C]$ is isomorphic to $H^0(N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(N_{C/X})$ $\quad (\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathrm{Hilb}\,X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- We say $C \subset X$ is unobstructed if $\mathrm{Hilb}\,X$ is nonsingular at $[C]$.
- $H^1(N_{C/X}) = 0 \Longrightarrow C$ is unobstructed. The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*.).

Purpose 2

Determine $\dim_{[C]} \mathrm{Hilb}\,X$ at a singular point $[C]$ of $\mathrm{Hilb}\,X$.

Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry [Mumford'62]".

Example 1 (Mumford)

$\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W of dimension $\mathbf{56}$, whose general member C satisfies:

1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ ($\simeq \mathbb{P}^{37}$) on S.

Remark 3

- C and $\mathbb{P}^3$ are innocent-looking (a pathology).
- C is of degree $\mathbf{14}$ and genus $\mathbf{24}$, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathbf{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

Generalization of Mumford's example

- Later many non-reduced components of $\mathbf{Hilb}^{sc}\, \mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of $\mathbf{Hilb}^{sc}\, \mathbb{P}^n$ $(n > 3)$ have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves C corresponding to the generic point of the non-reduced components were contained in some surface $S \subset \mathbb{P}^n$, e.g.,

	a surface S containing C
Mumford['62]	smooth cubic
Gruson-Peskine'82	non-normal cubic
Kleppe'87	smooth cubic
Kleppe-Ottem'15	smooth quartic
Choi-Iliev-Kim'24-1, '24-2	ruled surface

Another generalization (with Mukai)

We found that in Mumford's example, (-1)-curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

Theorem 4 (Mukai-N'09, $\mathrm{char}\, k \geq 0$)

Let X be a smooth projective 3-fold satisfying the following:

1. there exists a smooth rational curve E on X s.t. $N_{E/X}$ is globally generated, and
2. there exists a smooth surface S s.t. $E \subset S \subset X$, $E^2 = -1$ on S, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathbf{Hilb}^{sc}\, X$ has infinitely many generically non-reduced components (GNRC).

Remark 5

In Mumford's ex., $X = \mathbb{P}^3$, S is a smooth cubic, E is a line.

Hilbert scheme of Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index r of a Fano manifold X is the maximal integer r such that $-K_X \sim rH$ with some $H \in \mathrm{Pic}\, X$.

Let X be a smooth Fano 3-fold of index r.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and X is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

r	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

Applying Theorem 4, we obtain

Example 2 (N'10)

If $r(X) > 1$, then $\mathbf{Hilb}^{sc} X$ contains a a generically non-reduced component W satisfying:

1. every general member C of W is contained in a smooth del Pezzo surface S ($\sim -\frac{r-1}{r}K_X$), and
2. there exists a (good) line E on S and $C \sim -K_X\big|_S + 2E$ in $\mathrm{Pic}\, S$.
3. $h^0(C, N_{C/X}) = \dim W + 1$.

Here

- A curve $E \subset X$ is a line $\overset{\text{def}}{\Longleftrightarrow} E \simeq \mathbb{P}^1$ and $-\frac{1}{r}K_X.E = 1$.
- A line $E \subset X$ is good $\overset{\text{def}}{\Longleftrightarrow} N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$, or $\mathcal{O}_E \oplus \mathcal{O}_E(1)$ (for $r = 2, 3$).
- $\dim W = 56$, 42 and $(-K_X)^3/2 + 4$ for $r = 4, 3, 2$, respectively.

However, if X is prime ($r = 1$), then there exists NO del Pezzo surface $S \subset X$.

§2 Deformation of curves lying on a K3 surface

Curves on K3 surface

In this section, we study the deformations of smooth curves on a smooth Fano 3-fold, under the assumption that the curve is contained in a smooth $K3$ surface.

Definition 6

A smooth projective surface S with $K_S \sim 0$ and $H^1(S, \mathcal{O}_S) = 0$ is called a $K3$ surface.

Let

$$C \subset S_{K3} \subset X_{Fano3},$$

a sequence of a curve, a $K3$ surface, a Fano 3-fold. We will see the following curves on S control the deformations C in X:

- (-2)-curves $E \simeq \mathbb{P}^1$,
- elliptic curves F

Actually, they play a role very similar to that of (-1)-curve $E \simeq \mathbb{P}^1$ on the smooth cubic $S_3 \subset \mathbb{P}^3$ in Mumford's example!

Hilbert-flag scheme

A main tool of our studies is $\mathbf{HF}\,X$ the Hilbert-flag scheme of X, i.e.

$$\mathbf{HF}\,X = \left\{(C,S)\;\middle|\; C \subset S \subset X\text{: closed subschemes}\right\}.$$

If $C \hookrightarrow S \hookrightarrow X$ is regular embeddings, then

- the tang. sp. and the obst. sp. of $\mathbf{HF}\,X$ at (C,S) is $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$.
- the normal sheaf $N_{(C,S)/X}$ of (C,S) in X is the fiber product sitting in

$$\begin{tikzcd}
N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\
N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C .
\end{tikzcd}$$

  Here $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection.
- Similarly to Hilbert schemes,

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathbf{HF}\,X \le h^0(X, N_{(C,S)/X}).$$

Lemma 7

Let $C \subset S_{K3} \subset X_{Fano3}$. Then TFAE:

1. $H^1(N_{(C,S)/X}) = 0$, namely $\mathbf{HF}\,X$ is nonsingular at (C,S) of expected dimension $\chi(N_{(C,S)/X})$.
2. There exists a first order deformation $\tilde{S}$ of S in X, to which C does not lift.

Proof.

By adjunction, $N_{S/X} \simeq -K_X\big|_S$ and $N_{C/S} \simeq K_C$, which implies $H^i(N_{S/X}) = 0$ for $i > 0$ and $H^1(N_{C/S}) \simeq k$. There exists an exact sequence $0 \to N_{C/S} \to N_{(C,S)/X} \to N_{S/X} \to 0$, inducing

$$H^0(N_{(C,S)/X}) \xrightarrow{p_2} H^0(N_{S/X}) \longrightarrow \underbrace{H^1(N_{C/S})}_{\simeq k} \longrightarrow H^1(N_{(C,S)/X}) \longrightarrow 0.$$

$H^1(N_{(C,S)/X}) = 0$ iff p_2 is not surjective, which is the second condition. □

Example 8

Let X be a prime Fano 3-fold, $S \in |-K_X|$ a $K3$ surface, $E \subset S$ a line on S. Then TFAE:

1. E is of type $(0,-1)$, i.e. $N_{E/X} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$,
2. $\mathbf{Hilb}\, X$ is nonsingular at $[E]$ (of exp. dim.),
3. $\mathbf{HF}\, X$ is nonsingular at (E, S) (of exp. dim.).

A line E on X is called good if E is of type $(0,-1)$, otherwise (that is of type $(1,-2)$) called bad.

Lemma 9

If X is prime, and E is a good line or a good conic on X contained in a smooth $K3$ surface $S \in |-K_X|$, then there exists a first order deformation $\tilde{S}$ of S to which E does not lift, i.e., $H^1(N_{(E,S)/X}) = 0$.

A key lemma

Lemma 10 (char $k = 0$)

Let $S \overset{\iota}{\hookrightarrow} X$ be an embedded $K3$ surface, and let E be a curve on S with $H^1(\mathcal{O}_S(E)) = 0$ and $H^1(N_{(E,S)/X}) = 0$. If $C - bE \in \iota^* \operatorname{Pic} X$ for some $b \neq 0$, then $H^1(N_{(C,S)/X}) = 0$.

Proof.

Since $H^1(N_{(E,S)/X}) = 0$, by Lem. 7, there exists a first order deformation $\tilde{S}$ of S in X, to which E does not lift. Then neither does $\mathcal{O}_S(E)$ by $H^1(\mathcal{O}_S(E)) = 0$. Let $\tau \in H^1(T_S)$ (abstract def.) correspond to $\tilde{S}$. Then $\tau \cup c(\mathcal{O}_S(E)) \neq 0$ in $H^2(\mathcal{O}_S)$, where $c(*) \in H^1(\Omega^1_S)$ denotes the Atiyah-ext. class of $*$. Since $c(\mathcal{O}_S(C)) = c(\mathcal{O}_S(C - bE)) + bc(\mathcal{O}_S(E)))$, and $C - bE \in i^* \operatorname{Pic} X$, we have $\tau \cup c(\mathcal{O}_S(C)) \neq 0$, hence $\mathcal{O}_S(C)$ does not lift to $\tilde{S}$, hence neither does C as a closed subscheme of S, and $H^1(N_{(C,S)/X}) = 0$. □

π-map

Let E be a curve on S ($\subset X$), and $\pi_{E/S} : N_{E/X} \longrightarrow N_{S/X}\big|_E$ the projection.

Definition 11 (π-map)

The homomorphism $\pi_{E/S}(E) = \pi_{E/S} \otimes \mathcal{O}_E(E)$ of sheaves on E induces a map

$$\pi_{E/S}(E) : H^0(E, N_{E/X}(E)) \longrightarrow H^0(E, N_{S/X}(E)\big|_E),$$

on the global sections. We call this map the π-map for (E, S).

Example 12

Let E be a conic on a prime Fano 3-fold X, contained in a smooth $K3$ $S \in |-K_X|$. Then $\mathcal{O}_E(E) \simeq \mathcal{O}_E(-2)$, and $N_{S/X}\big|_E \simeq \mathcal{O}_E(2)$. If E is good (i.e. $N_{E/X}$ is trivial), then the π-map

$$\pi_{E/S}(E) : H^0(E, \mathcal{O}_E(-2)^2) \longrightarrow H^0(E, \mathcal{O}_E)$$

for (E, S) is zero map (and hence not surjective).

Obstructedness of curves in a $K3$ surface

Let $C \subset S_{K3} \subset X_{Fano3}$ be as above and suppose that $H^1(N_{(C,S)}/X) = 0$. Then

$$\mathrm{coker}\, p_1 \simeq H^1(N_{S/X}(-C)) \simeq H^1(-D),$$

where $D := C + K_X\big|_S$ is a divisor on S.

Theorem 13 (N'17)

1. If $D \geq 0$ and there exist no (-2)-curves and no elliptic curves on S, or if $H^1(S, D) = 0$, then C is unobstructed.
2. If $D \geq 0$, $D^2 \geq 0$ and there exists a (-2)-curve E on S such that $E.D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover, the π-map $\pi_{E/S}(E)$ is not surjective, then C is obstructed.
3. If there exists an elliptic curve F on S such that $D \sim mF$ for $m \geq 2$, then we have $h^1(S, D) = m - 1$. If moreover, $\pi_{F/S}(F)$ is not surjective, then C is obstructed.

Analogy of Mumford's ex. in the case $r = 1$

Applying the previous theorem to a $K3$ surfaces $S \subset X$ and a good conic $E \simeq \mathbb{P}^1$ on S, we obtained:

Theorem 14 (N'19)

Let X be a prime Fano 3-fold of genus $g := (-K_X)^3/2 + 1$. Then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component W with the following properties:

1. Every general member C of W is contained in a $K3$ surface S $(\sim -K_X)$.
2. There exists a good conic $E \simeq \mathbb{P}^1$ on S such that $C \sim -2K_X\big|_S + 2E$.
3. $\dim W = 5g + 1$, $h^0(C, N_{C/X}) = 5g + 2$, and C is of degree $4g$ and genus $4g + 1$.

Some remarks

For the proof, we need the following facts:

- The prime Fano 3-fold $X = X_{2g-2}$ contains a conic $E \subset V$ (cf. [Shokurov'79], [Reid'80]).
- If $E \subset X$ is general, then E is a good conic (cf. [Iskovskih'78]) if $\mathrm{char}\, k = 0$.
- For every conic E, there is a smooth $K3$ surface $S \in |-K_V|$ containing E (cf. [Iskovskikh]).

Corollary 15

If X is a smooth Fano $\mathbf{3}$-fold and $\rho(X) = \mathbf{1}$, then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component.

3-fold X	surface S	$[C] \in \mathbf{Pic}\,S$	E	
$\mathbb{P}^3$	del Pezzo	$-K_X\big\|_S + 2E$	line	Mumford['62]
$Q^3 \subset \mathbb{P}^4$				Mukai-N['09]
del Pezzo				Mukai-N['09], N['10]
prime Fano	$K3$	$-2K_X\big\|_S + 2E$	conic	N['19]

Table: Generically non-reduced component of Mumford type

§3 An example of non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$

Main result

Toward a further generazation, we compute the obstruction to deforming curves lying on a complete intersection $K3$ surface in $\mathbb{P}^5$.

Theorem 16 (Main)

The Hilbert scheme $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components W_n ($n \geq 2$) with the following properties:

1. every general C of W_n is a smooth connected curve contained in a smooth complete intersection $K3$ surface

$$S = S_{2,2,2} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5.$$

2. C is linearly equivalent to $n(2\mathbf{h} + E)$, where $h = [\mathcal{O}_S(1)]$ in $\operatorname{Pic} S$, and E is a line on S.
3. C is of degree $d = 17n$ and genus $g = 17n^2 + 1$.
4. $\dim W = g + 53$, while $h^0(N_{C/\mathbb{P}^5}) = g + 56$, thus $h^0(N_{C/\mathbb{P}^5}) = \dim W_n + 3$.

Construction

We see $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$ and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{array}{ccccc}
C & \in & W^{(g+53)} & \subset & \mathbf{Hilb}^{sc}\,\mathbb{P}^5 \\
\big\downarrow & & \big\downarrow \mathbb{P}^g\text{-bundle} & & \\
(E, S) & \in & U^{(53)} & \subset & G \times \mathbf{Gr}(3, V) \\
\big\downarrow & & \big\downarrow \mathbf{Gr}(3,18)\text{-bundle} & & \big\downarrow \\
E & \in & \mathbf{Gr}(2,6)^{(8)} & = & \left\{\text{lines in } \mathbb{P}^5\right\},
\end{array}$$

where $V = H^0(\mathbb{P}^5, \mathcal{O}(2))$.

Key Lemma

Let X be a projective scheme, and let $\mathbf{HF}\,X$ be the Hilbert-flag scheme of X. There exists a projection

$$pr_1 : \mathbf{HF}\,X \to \mathbf{Hilb}\,X, \qquad (C, S) \mapsto [C],$$

which induces the tangent map $p_1 : H^0(N_{(C,S)/X}) \to H^0(N_{C/X})$.

Lemma 17 (Key Lemma, N['23], Lem. 2.17)

We have $\mathbf{dim}_{(C,S)}\,\mathbf{HF}\,X = \mathbf{dim}_{[C]}\,\mathbf{Hilb}\,X$ if

1. $H^1(N_{(C,S)/X}) = H^0(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
2. for every $\alpha \in H^0(N_{C/X}) \setminus \mathbf{im}\,p_1$, the (primary) obstruction $\mathbf{ob}(\alpha)$ (to extend α a second order deformation over $k[t]/(t^3)$) is nonzero.

Sketch of Proof of Main thm.

Let $C \subset \mathbb{P}^5$ be a smooth connected curve lying on a complete intersection $K3$ surface $S = S_{2,2,2} \subset \mathbb{P}^5$, and such that $C \sim n(2\mathbf{h} + E)$ in $\operatorname{Pic} S$ for $n \geq 2$, where E is a line on S.

- Since $d = 17n > 16 = 2h^2$, S is uniquely determined by C.
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by proj. normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

  that $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$, which implies there exists a first order deformation of $\tilde{S}$ of S, to which E (and hence C) does not lift.
- Then $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$ for $i > 0$ and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

  is exact. Here p_1 is the tangent map of $pr_1 : \mathrm{HF}\,\mathbb{P}^5 \to \mathrm{Hilb}\,\mathbb{P}^5$ at (C, S) and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2\mathbf{h} - C))^{\oplus 3} \simeq k^3.$$

Sketch of Proof of Main thm.(continued)

- We note that $H^1(N_{S/\mathbb{P}^5}(E-C)) = H^1(-L^{\oplus 3}) = 0$, where

$$L = C - 2\mathrm{h} - E = (n-1)(2\mathrm{h} + E)$$

is nef and big.

- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \mathbf{im}\, p_1$, its *exterior component* $\pi_{C/S}(\alpha)$, i.e., the image of α in $H^0(N_{S/\mathbb{P}^5}(E)\big|_C)$, lifts to a global section β of $N_{S/\mathbb{P}^5}(E)$. (Here β is called an infinitesimal deformation with poles.)
- Applying a *"modification"* of the obstructedness criterion [Mukai-N'09] to β, we obtain $\mathbf{ob}(\alpha) \neq 0$. This implies

$$\mathbf{dim}_{[C]}\, \mathbf{Hilb}^{sc}\,\mathbb{P}^5 = \mathbf{dim}_{(C,S)}\, \mathbf{HF}\,\mathbb{P}^5$$

by the key lemma. Therefore C is obstructed and parametrised by an open dense subset of a component of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$.

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