

# Obstructions to deforming space curves lying on complete intersection of quadrics

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Today's slide

## Plan of Talk

- ① Hilbert schemes of Fano 3-folds
- ② Deformation of space curves lying on a complete intersection of quadrics (an attempt to generalize known results into higher dimensional varieties)

## **§1 Hilbert schemes of Fano threefolds**

## Hilbert schemes

We work over a field  $k = \overline{k}$  of  $\text{char } k = 0$ .

Let  $X$  be a projective scheme over  $k$ . We denote by  $\mathbf{Hilb} X$  the Hilbert scheme of  $X$ . Today we consider the **open** and **closed** subscheme

$$\mathbf{Hilb}^{sc} X := \{\text{smooth connected curves } C \subset X\} \subset \mathbf{Hilb} X,$$

that is, the Hilbert scheme of curves in  $X$ .

## Fano 3-folds

- A Fano manifold is a smooth projective variety  $X$  with ample  $-K_X$ .
- The index  $i$  of a Fano manifold  $X$  is the maximal integer  $i$  such that  $-K_X \sim iH$  with some  $H \in \text{Pic } X$ .

Let  $X$  be a smooth Fano 3-fold of index  $i$ .

- $X \simeq \mathbb{P}^3$  if  $i = 4$  and  $X \simeq Q^3 \subset \mathbb{P}^4$  if  $i = 3$ , and  $X$  is called del Pezzo if  $i = 2$ , and prime if  $i = 1$  and  $\rho = 1$ .
- If we restrict  $X$  with  $\rho = 1$ , then there exist 17 deformation equivalence classes of  $X$  (Fujita, Iskovskikh'77,'78):

$i$	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of  $X$

# Hilbert scheme of Fano 3-folds

The Hilbert scheme of lines and conics have been studied widely. We can find the following survey in [Kuznetsov-Prokhorov-Shramov'18].

## Theorem 1 (classical)

Let  $X$  be a smooth Fano 3-fold of  $\rho = 1$  and  $i$ . Let  $\Sigma(X)$  and  $S(X)$  denote the Hilbert scheme of lines and conics, respectively. If  $X$  is general and not a quartic double solid ( $i = d = 2$ ), then  $\Sigma(X)$  and  $S(X)$  are generically smooth.

$i$	$X$	$\Sigma(X)$	$S(X)$
4	$\mathbb{P}^3$	$\text{Gr}(2, 4)$	$\mathbb{P}(\text{Sym}^2 U^*)$
3	$Q = (2) \subset \mathbb{P}^4$	$\simeq \mathbb{P}^3$	$\text{Bl}_{\text{OG}(3,5)} \text{Gr}(3, 5)$
2 $(d \geq 3)$	$(3) \subset \mathbb{P}^4$ $(2) \cap (2) \subset \mathbb{P}^5$ $V_5 \subset \mathbb{P}^6$	surface of gen.type abelian surface $\mathbb{P}^2$	$\mathbb{P}^2$ -b'dle/surface $\mathbb{P}^3$ -b'dle/curve $\mathbb{P}^4$
1	prime (general)	smooth curve	irred. surface

## $g = 0$ vs $g > 0$

Let  $X$  be a smooth Fano 3-fold. Then

- The Hilbert scheme  $\Sigma(X)$  and  $S(X)$  of lines and conics on  $X$  **behaves well**, except for low degree case of  $r = 2$  and special prime Fano.
- More generally, the Hilbert scheme  $\mathbf{Hilb}_{d,0}^{sc} X$  of smooth rational curves on  $X$  **behaves well** by *vanishing of obstructions* (deformation theory).
- However, the geometry of the Hilbert scheme  $\mathbf{Hilb}_{d,g}^{sc} X$  for higher genus  $g > 0$  becomes more complicated, due to their **non-vanishings**.

## Infinitesimal property of Hilbert schemes

- The tangent space of  $\text{Hilb } X$  at  $[C]$  is isomorphic to  $H^0(N_{C/X})$ .
- $C \subset X$ : a locally complete intersection  $\Rightarrow$  every obstruction to deforming  $C$  in  $X$  is contained in  $H^1(N_{C/X})$  ( $\subset \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$ ) and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.} (= \chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \text{Hilb } X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}} .$$

- $H^1(N_{C/X}) = 0 \Rightarrow \text{Hilb } X$  is nonsingular at  $[C]$  (We say  $C$  is unobstructed).  
The converse is not true (e.g. c.i. space curves  $C \subset \mathbb{P}$  usually have large obstruction spaces but always unobstructed.).

### Question 2

Determine the local dimension of the Hilbert scheme at a given singular point.

## Mumford's example (pathology)

The following example appeared in a famous paper “Further pathologies in algebraic geometry [Mumford'62]”.

### Example 1 (Mumford)

$\text{Hilb}^{sc} \mathbb{P}^3$  contains a generically non-reduced irreducible component  $W$  of dimension **56**, whose general member  $C$  satisfies:

- ①  $C$  is contained in a smooth cubic surface  $S \subset \mathbb{P}^3$ .
- ② There exists a line  $E$  on  $S$  such that  $C$  belongs to a complete linear system  $\Lambda := |-4K_S + 2E| (\simeq \mathbb{P}^{37})$  on  $S$ .

### Remark 3

$C$  is of degree **14** and genus **24**, and  $h^1(N_{C/\mathbb{P}^3}) = 1$  and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \text{Hilb}^{sc} \mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

## Generalization of Mumford's example

- Later many non-reduced components of  $\text{Hilb}^{sc} \mathbb{P}^3$  were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and more recently, those of  $\text{Hilb}^{sc} \mathbb{P}^n$  ( $n > 3$ ) have been found by Choi-Iliev-Kim'24-1, and '24-2.
- In these examples, generic curves (of the non-reduced component) were contained in some surfaces:

	surfaces
Mumford'62, Kleppe'87, Ellia'87, N'05,23	smooth cubic in $\mathbb{P}^3$
Gruson-Peskine'82	non-normal cubic in $\mathbb{P}^3$
Kleppe-Ottem'15	smooth quartic in $\mathbb{P}^3$
Choi-Iliev-Kim'24-1, '24-2	ruled surface in $\mathbb{P}^n$

## Generalization to smooth Fano 3-folds

Theorem 4 (Mumford'62, Mukai-N'09, N'10, N'19)

If  $X$  is a smooth Fano 3-fold of  $\rho = 1$  and index  $i$ , then  $\mathbf{Hilb}^{sc} X$  contains a generically non-reduced irreducible component, whose general member  $C$  is contained in the following surface  $S \subset X$  and classes  $[C] \in \mathbf{Pic} S$ .

$i$	surface $S$	$[C] \in \mathbf{Pic} S$	$E$
4			
3	del Pezzo $(S \in   - \frac{i-1}{i} K_X  )$	$-K_X _S + 2E$	line
2			
1	$K3$ $(S \in   - K_X  )$	$-2K_X _S + 2E$	conic

## Another generalization (with Mukai)

Theorem 5 (Mukai-N'09,  $\text{char } k \geq 0$ )

Let  $X$  be a smooth projective 3-fold such that

- ① there exists a free rational curve  $E \simeq \mathbb{P}^1$  on  $X$ , and
- ② there exists a smooth surface  $S$  s.t.  $E \subset S \subset X$ ,  $E^2 = -1$  on  $S$ , and  $H^1(S, N_{S/X}) = p_g(S) = 0$ .

Then the Hilbert scheme  $\mathbf{Hilb}^{sc} X$  has infinitely many generically non-reduced irreducible components.

Theorem 6 (Mukai-N'09)

Let  $C$  be a general genus-5 curve,  $X_3$  a general smooth cubic 3-fold, or of Fermat type cubic 3-fold:

$$X_3^{\text{Fermat}} : x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \subset \mathbb{P}^4,$$

then  $\mathbf{Hom}_8(C, X_3)$  has a generically non-reduced irreducible component of dimension 4 and  $h^0(C, T_{X_3}|_C) = 5$ .

## **§2 Deformations of space curves lying on a complete intersection of quadrics**

## Complete intersection of quadrics

Today we consider deformations of space curves  $C \subset \mathbb{P}$  lying on a smooth surface  $S$ , when  $S$  are complete intersections

$$S_2 \subset \mathbb{P}^3, \quad S_{2,2} \subset \mathbb{P}^4, \quad S_{2,2,2} \subset \mathbb{P}^5$$

of quadrics. For each  $S$  and curves  $C$  lying on  $S$ , we consider the deformations of  $C$  in the projective space  $\mathbb{P}$  using the flag

$$C \subset S \subset \mathbb{P}.$$

## Hilbert-flag scheme

A main tool of our studies is  $\mathbf{HF} X$  the Hilbert-flag scheme of  $X$ , i.e.

$$\mathbf{HF} X = \{(C, S) \mid C \subset S \subset X: \text{closed subschemes}\}.$$

If  $C \hookrightarrow S \hookrightarrow X$  is regular embeddings, then

- the tang. sp. and the obst. sp. of  $\mathbf{HF} X$  at  $(C, S)$  is  $H^0(N_{(C,S)/X})$  and  $H^1(N_{(C,S)/X})$ .
- the normal sheaf  $N_{(C,S)/X}$  of  $(C, S)$  in  $X$  is the fiber product sitting in

$$\begin{array}{ccc} N_{(C,S)/X} & \xrightarrow{\pi_2} & N_{S/X} \\ \pi_1 \downarrow & \square & \downarrow |_C \\ N_{C/X} & \xrightarrow{\pi_{C/S}} & N_{S/X}|_C. \end{array}$$

Here  $|_C$  is the restriction of sheaves, and  $\pi_{C/S} : N_{C/X} \rightarrow N_{S/X}|_C$  is the natural projection.

- Similarly to Hilbert schemes,

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} \mathbf{HF} X \leq h^0(X, N_{(C,S)/X}).$$

If  $C \hookrightarrow S \hookrightarrow X$  is a sequence of regular embeddings of closed subschemes, then there exists a natural exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{(C,S)/X} \xrightarrow{\pi_2} N_{S/X} \longrightarrow 0.$$

### Lemma 7

The Hilbert-flag scheme  $\mathbf{HF} X$  is nonsingular at  $(C, S)$  of expected dimension

$$\chi(N_{(C,S)/X}) = \chi(N_{C/S}) + \chi(N_{S/X})$$

if  $C$  is a curve and

- ①  $X$  is Fano a 3-fold,  $S$  is del Pezzo, or
- ②  $X = \mathbb{P}^n$ ,  $S$  is del Pezzo (embedded into  $\mathbb{P}^n$  by  $| -K_S |$ ).

### Proof.

Suppose e.g.  $X$  is Fano a 3-fold,  $S$  is del Pezzo. Then by adjunction,  $N_{S/X} \simeq -K_X|_S + K_S$  and  $N_{C/S} \simeq -K_S|_C + K_C$ , where  $-K_X$  and  $-K_S$  are ample. This implies the higher cohomology groups of  $N_{S/X}$  and  $N_{C/S}$  vanish, so does that of  $N_{(C,S)/X}$ . □

## Smooth quadric surface $S_2$ in $\mathbb{P}^3$

We start from curves on smooth quadrics in  $\mathbb{P}^3$ .

Let  $S = S_2 \subset \mathbb{P}^3$ . Then

$$S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{and} \quad \mathbf{Pic} S \simeq (\mathbf{Pic} \mathbb{P}^1)^{\oplus 2} = \mathbb{Z}^2.$$

Thus every curve  $C$  on  $S$  corresponds to  $(a, b) \in \mathbb{P}^2$  (called *bidegree* of  $C$ ). Then  $C$  is of degree  $d = a + b$  and genus  $g = (a - 1)(b - 1)$ .

### Proposition 8 (Tannenbaum'78, Kleppe'87, etc)

Let  $d > 4$  and  $g \geq 0$ . Let  $C \subset \mathbb{P}^3$  be a smooth connected curve of degree  $d$  and genus  $g$  contained in a smooth quadric surface. The maximal irreducible family  $W(a, b)$  of such curves  $C \subset \mathbb{P}^3$  of bidegree  $(a, b)$  is of dimension  $g + 2d + 8$ . Then

- ①  $\mathbf{Hilb} \mathbb{P}^3$  is smooth along  $W(a, b)$ .
- ② The closure  $\overline{W}(a, b) \subset \mathbf{Hilb}^{sc} \mathbb{P}^3$  becomes a generically smooth component if  $g \geq 2d - 8$ , otherwise  $\overline{W}(a, b)$  is a proper closed subset of a component of  $\mathbf{Hilb}^{sc} \mathbb{P}^3$  of codimension  $2d - 8 - g$ .

## Sketch of proof

If  $C \subset \mathbb{P}^3$  is contained in a smooth quadric  $S = S_2 \subset \mathbb{P}^3$ , then  $H^1(N_{C/S}) = 0$ . Then  $\mathbf{HF}\mathbb{P}^3$  is nonsingular at  $(C, S)$  of expected dimension

$$\chi(N_{(C,S)}) = 2d + g + 8.$$

We see that either  $H^1(\mathcal{I}_C(2)) = 0$  or  $H^1(\mathcal{O}_C(2)) = 0$  holds.

If  $H^1(\mathcal{I}_C(2)) = 0$ , then the tangent map

$$p_1 : \mathcal{T}_{\mathbf{HF}\mathbb{P}^3, (C,S)} \longrightarrow \mathcal{T}_{\mathbf{Hilb}\mathbb{P}^3, [C]}$$

is surjective and the 1st projection  $pr_1 : (C, S) \mapsto [C]$  is smooth at  $(C, S)$ .

If  $H^1(\mathcal{O}_C(2)) = 0$ , then by  $\mathcal{O}_C(2) \simeq N_{S/\mathbb{P}^3}|_C$  and the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3}|_C \longrightarrow 0,$$

$H^1(N_{C/\mathbb{P}^3})$  vanishes. Thus  $\mathbf{Hilb}\mathbb{P}^3$  is nonsingular at  $[C]$ .

□

## Smooth complete intersection $S_{2,2} \subset \mathbb{P}^3$

Let  $S = S_{2,2} \subset \mathbb{P}^4$  be a smooth complete intersection. Then  $S$  is a del Pezzo surface of degree 4, isomorphic to  $\text{Bl}_{P_1, \dots, P_5} \mathbb{P}^2$  and  $\text{Pic } S \simeq \mathbb{Z}^7$ . Moreover, every divisor  $D$  on  $S$  corresponds to a 7-tuple  $(a; b_1, \dots, b_5)$  of integers  $a, b_1, \dots, b_5$  by

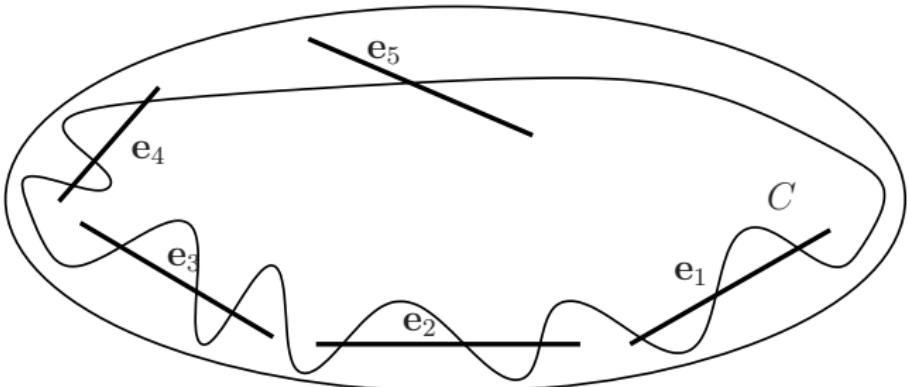
$$[D] = a\mathbf{l} - \sum_{i=1}^5 b_i \mathbf{e}_i,$$

Moreover, the coordinate  $(a; b_1, \dots, b_5)$  satisfies

$$b_1 \geq \dots \geq b_5 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \tag{1}$$

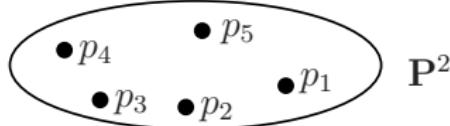
Here  $(a; b_1, \dots, b_5)$  is called the **standard coordinate** of  $[D]$  in  $\text{Pic } S_{2,2}$ .

## Curves on $S_{2,2} \subset \mathbb{P}^4$



$$\begin{array}{l}
 S_{2,2} = \text{Bl}_{p_1, \dots, p_5} \mathbb{P}^2 \\
 [C] = (9; 3, 3, 3, 3, 1) \in \text{Pic}S_{2,2} = \mathbf{Z}^6 \\
 d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14 \\
 g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16
 \end{array}$$

$\downarrow$



# Stable degeneration

Suppose that

$$C \subset S \subset \mathbb{P}$$

is a flag of a space curve  $C$  and a surface  $S$ .

## Definition 2

We say  $C$  is **stably degenerate** if every small global deformation of  $C$  in  $\mathbb{P}$  is contained in a deformation of  $S$  in  $\mathbb{P}$ .

## Theorem 9 (Main1)

Let  $C \subset S_{2,2} \subset \mathbb{P}^4$  be a smooth curve of degree  $d \geq 10$  of genus  $g \geq 2d - 12$ , contained in a smooth c.i.  $S_{2,2}$  in  $\mathbb{P}^4$ . Let  $(a; b_1, \dots, b_5)$  be the standard coordinate of  $[C]$  in  $\text{Pic } S_{2,2} \simeq \mathbb{Z}^6$ . Then

- ① If  $b_5 \geq 2$ , then  $C$  is **unobstructed** and **stably degenerate**.
- ② If  $b_5 = 1$  and  $b_4 \geq 2$ , then  $C$  is **obstructed** and **stably degenerate**.
- ③ If  $b_5 = 0$ , then  $C$  is **not stably degenerate**.

## Examples

Table: curves on  $S_{2,2}$  and stable degeneration

$(d, g)$	the class of $C$	the max. irred. family of $C$ ( $\subset \text{Hilb}^{sc} \mathbb{P}^4$ )
$(14, 16)$	$(8; 2, 2, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 4, 3, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 3, 3, 3, 3, 1)$	generically non-reduced component
$(15, 18)$	$(9; 4, 2, 2, 2, 2)$	generically smooth component
$(15, 18)$	$(9; 3, 3, 3, 2, 1)$	generically non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18, 24)$	$(9; 2, 2, 2, 2, 1)$	generically non-reduced component
$(18, 24)$	$(10; 4, 3, 3, 1, 1)$	unknown ( $h^1(\mathcal{I}_C(2)) = 2$ )
$(18, 24)$	$(10; 3, 3, 3, 3, 0)$	a proper closed subset of a component
$(18, 24)$	$(11; 6, 3, 2, 2, 2)$	generically smooth component
$\vdots$	$\vdots$	$\vdots$

## Analogy of Mumford's example

### Example 3

$\text{Hilb}^{sc} \mathbb{P}^4$  contains a generically non-reduced irreducible component  $W$  whose general member  $C$  satisfies

- ①  $C$  is contained in a smooth c.i.  $S = S_{2,2} \subset \mathbb{P}^4$ ,
- ② there exists a line  $E$  on  $S$  such that  $C$  belongs to a complete linear system  $\Lambda := |-3K_S + 2E| (\simeq \mathbb{P}^{29})$  on  $S$ , and
- ③  $\dim_{[C]} O_{\text{Hilb } \mathbb{P}^4} = 55$ ,  $h^0(C, N_{C/\mathbb{P}^4}) = 57$ , and  $C$  is of degree **14** and genus **16**.

## Smooth complete intersection $S_{2,2,2} \subset \mathbb{P}^5$

Let  $S_{2,2,2} \subset \mathbb{P}^5$  be a smooth complete intersection. Then  $S$  is a **K3** of degree 8 (and genus 5). Since  $N_{S/\mathbb{P}^5} \simeq \mathcal{O}_S(2)^{\oplus 2}$ , we have

$$H^1(N_{S/\mathbb{P}^5}(-C)) = H^1(S, -(C - 2\mathbf{h}))^{\oplus 2},$$

where  $\mathbf{h} = \mathcal{O}_S(1)$ . Thus if  $C - 2\mathbf{h}$  is nef and big and  $H^1(N_{(C,S)/\mathbb{P}^5}) = \mathbf{0}$ , then  $C$  is stably degenerate and unobstructed.

### Theorem 10 (Main2)

The Hilbert scheme  $\text{Hilb}^{sc} \mathbb{P}^5$  contains a generically non-reduced components  $W_n$  ( $n \geq 2$ ) with the following properties:

- ① every general  $C$  of  $W_n$  is a smooth connected curve contained in a smooth complete intersection **K3** surface  $S_{2,2,2} \subset \mathbb{P}^5$ .
- ②  $C$  is linearly equivalent to  $n(2\mathbf{h} + E)$ , where  $E$  is a line on  $S$ .
- ③  $C$  is of degree  $d = 17n$  and genus  $g = 17n^2 + 1$ .
- ④  $\dim_{[C]} \mathcal{O}_{\text{Hilb} \mathbb{P}^5} = g + 53$ , while  $h^0(N_{C/\mathbb{P}^5}) = g + 56$ , thus  $h^0(N_{C/\mathbb{P}^5}) = \dim_{[C]} \mathcal{O}_{\text{Hilb} \mathbb{P}^5} + 3$ .

# Construction

We consider the family

$$W_n := \{C \subset \mathbb{P}^5 \mid C \subset S \text{ for some } S = S_{2,2,2} \subset \mathbb{P}^5 \text{ and } C \sim n(2\mathbf{h} + \mathbf{E})\}$$

of curves in  $\mathbb{P}^5$ . Then

$$\begin{array}{ccccccc} C & \in & W^{(g+53)} & \subset & \mathbf{Hilb}^{sc} \mathbb{P}^5 \\ \downarrow & & \downarrow \mathbb{P}^g\text{-bundle} & & & & \\ (\mathbf{E}, S) & \in & U^{(53)} & \subset & G \times \mathrm{Gr}(3, V) \\ \downarrow & & \downarrow \mathrm{Gr}(3, 18)\text{-bundle} & & & & \downarrow \\ \mathbf{E} & \in & \mathrm{Gr}(2, 6)^{(8)} & = & \{\text{lines in } \mathbb{P}^5\}, & & \end{array}$$

where  $V = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ .

## Key Lemma

Let  $X$  be a projective scheme, and let  $\mathbf{HF} X$  be the Hilbert-flag scheme of  $X$ . Let

$$pr_1 : \mathbf{HF} X \rightarrow \mathbf{Hilb} X, \quad (C, S) \mapsto [C],$$

be the first projection morphism, which induces

$$p_1 : H^0(N_{(C,S)/X}) \rightarrow H^0(N_{C/X}).$$

Lemma 11 (Key Lemma, cf. N[’23])

We have  $\dim_{(C,S)} \mathbf{HF} X = \dim_{[C]} \mathbf{Hilb} X$  if

- ①  $H^1(N_{(C,S)/X}) = H^0(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$ , and
- ② for every  $\alpha \in H^0(N_{C/X}) \setminus \text{im } p_1$ , the (primary) obstruction  $\mathbf{ob}(\alpha)$  (to extend  $\alpha$  a second order deformation over  $k[t]/(t^3)$ ) is nonzero.

## Sketch of Proof of Main2.

Let  $C \subset \mathbb{P}^5$  be a smooth connected curve lying on a complete intersection  $K3$  surface  $S = S_{2,2,2} \subset \mathbb{P}^5$ , and such that  $C \sim n(2h + E)$  in  $\text{Pic } S$  for  $n \geq 2$ , where  $E$  is a line on  $S$ .

- Since  $d = 17n > 16 = 2h^2$ ,  $S$  is uniquely determined by  $C$ .
- Then for all  $i > 0$ ,  $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$  by proj. normality and  $H^i(N_{E/\mathbb{P}^5}) = 0$  by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

that  $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$ , which implies there exists a first order deformation of  $\tilde{S}$  of  $S$ , to which  $E$  (and hence  $C$ ) does not lift.

- Then  $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$  for  $i > 0$  and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

is exact. Here  $p_1$  is the tangent map of  $pr_1 : \text{HF } \mathbb{P}^5 \rightarrow \text{Hilb } \mathbb{P}^5$  at  $(C, S)$  and its cokernel is of dimension 3 by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(O_S(2h - C))^{\oplus 3} \simeq k^3.$$

## Sketch of Proof of Main2.(continued)

- We note that  $H^1(N_{S/\mathbb{P}^5}(E - C)) = H^1(-L^{\oplus 3}) = 0$ , since  $L = C - 2h - E = (n - 1)(2h + E)$  is nef and big, and we have

$$H^0(N_{S/\mathbb{P}^5}(E)) \xrightarrow{lc} H^0(N_{S/\mathbb{P}^5}(E)|_C) \longrightarrow H^1(N_{S/\mathbb{P}^5}(E - C)) = 0.$$

- Then for every  $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \text{im } p_1$ , its exterior component  $\pi_{C/S}(\alpha)$ , i.e., the image of  $\alpha$  in  $H^0(N_{S/\mathbb{P}^5}(E)|_C)$ , lifts to a global section  $\beta$  of  $N_{S/\mathbb{P}^5}(E)$ . (Here  $\beta$  is called an **infinitesimal deformation with poles**.)
- Applying a “modification” of the **obstructedness criterion** [Mukai-N’09] to  $\beta$ , we obtain  $\text{ob}(\alpha) \neq 0$ . This implies

$$\dim_{[C]} \text{Hilb}^{sc} \mathbb{P}^5 = \dim_{(C,S)} \text{HF} \mathbb{P}^5$$

by the key lemma. Therefore  $C$  is **obstructed** and parametrised by an **open dense subset of a component** of  $\text{Hilb}^{sc} \mathbb{P}^5$  (thus yields to a **non-reduced** component of  $\text{Hilb}^{sc} \mathbb{P}^5$ ). □

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## Primary obstructions

Let  $X$  be a projective scheme over  $k$ ,  $C$  a loc. c. i. closed subscheme of  $X$ , and  $k[\varepsilon] := k[t]/(t^2)$  (the ring of dual numbers).

- A first order (infinitesimal) deformation of  $C$  is a deformation  $\tilde{C}$  ( $\subset X \times \text{Spec } k[\varepsilon]$ ) of  $C$  in  $X$  over  $k[\varepsilon]$ .
- $\tilde{C}$  naturally corresponds to  $\alpha \in H^0(C, N_{C/X})$  ( $\simeq \text{Hom}(\mathcal{I}_C, \mathcal{O}_C)$ ).
- There is an element  $\text{ob}(\alpha)$  in  $H^1(C, N_{C/X})$  (called the primary obstruction of  $\alpha$ ) such that

$$\text{ob}(\alpha) = 0 \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$

- $\text{ob}(\alpha)$  can be expressed as a cup product, and

$$\text{ob}(\alpha) = \alpha \cup e \cup \alpha \quad \text{in } \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

where  $e := [0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0]$ .

- $\text{ob}(\alpha) \neq 0$  for some  $\alpha$  implies that  $\text{Hilb } X$  is singular at  $[C]$  by infinitesimal lifting property of smoothness.