Obstructions to deforming space curves lying on complete intersection of quadrics

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Today's slide

Plan of Talk

1. Hilbert schemes of Fano 3-folds
2. Deformation of space curves lying on a complete intersection of quadrics (an attempt to generalize known results into higher dimensional varieties)

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§1 Hilbert schemes of Fano threefolds

Hilbert schemes

We work over a field $k = \bar{k}$ of $\mathbf{char}\, k = 0$.

Let X be a projective scheme over k. We denote by $\mathbf{Hilb}\, X$ the Hilbert scheme of X. Today we consider the open and closed subscheme

$$\mathbf{Hilb}^{sc} X := \{\text{smooth connected curves } C \subset X\} \subset \mathbf{Hilb}\, X,$$

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Fano 3-folds

- A Fano manifold is a smooth projective variety X with ample $-K_X$.
- The index i of a Fano manifold X is the maximal integer i such that $-K_X \sim iH$ with some $H \in \mathrm{Pic}\, X$.

Let X be a smooth Fano 3-fold of index i.

- $X \simeq \mathbb{P}^3$ if $i = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $i = 3$, and X is called del Pezzo if $i = 2$, and prime if $i = 1$ and $\rho = 1$.
- If we restrict X with $\rho = 1$, then there exist 17 deformation equivalence classes of X (Fujita, Iskovskikh'77,'78):

i	4	3	2	1
the number of cls.	1	1	5	10
variety / cls.	$\mathbb{P}^3$	$Q^3 \subset \mathbb{P}^4$	del Pezzo	prime Fano

Table: the number of deformation equivalence classes of X

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The Hilbert scheme of lines and conics have been studied widely. We can find the following survey in [Kuznetsov-Prokohrov-Shramov'18].

Theorem 1 (classical)

Let X be a smooth Fano 3-fold of $\rho = 1$ and i. Let $\Sigma(X)$ and $S(X)$ denote the Hilbert scheme of lines and conics, respectively. If X is general and not a quartic double solid ($i = d = 2$), then $\Sigma(X)$ and $S(X)$ are generically smooth.

i	X	$\Sigma(X)$	$S(X)$
4	$\mathbb{P}^3$	$\mathrm{Gr}(2,4)$	$\mathbb{P}(Sym^2 U^*)$
3	$Q = (2) \subset \mathbb{P}^4$	$\simeq \mathbb{P}^3$	$\mathrm{Bl}_{\mathrm{OG}(3,5)} \mathrm{Gr}(3,5)$
2 ($d \geq 3$)	$(3) \subset \mathbb{P}^4$ $(2) \cap (2) \subset \mathbb{P}^5$ $V_5 \subset \mathbb{P}^6$	surface of gen.type abelian surface $\mathbb{P}^2$	$\mathbb{P}^2$-b'dle/surface $\mathbb{P}^3$-b'dle/curve $\mathbb{P}^4$
1	prime (general)	smooth curve	irred. surface

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Let X be a smooth Fano 3-fold. Then

- The Hilbert scheme $\Sigma(X)$ and $S(X)$ of lines and conics on X behaves well, except for low degree case of $i = 2$ and special prime Fano.
- More generally, the Hilbert scheme of smooth rational curves on X behaves well *by vanishing of obstructions* (deformation theorey).
- However, the geometry of the Hilbert scheme of curves of higher genus $g > 0$ in X becomes more complicated, due to their non-vanishings.

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Infinitesimal property of Hilbert schemes

- The tangent space of $\mathbf{Hilb}\, X$ at $[C]$ is isomorphic to $H^0(N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming C in X is contained in $H^1(N_{C/X}) \quad (\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathrm{Hilb}\, X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- $H^1(N_{C/X}) = 0 \Longrightarrow \mathrm{Hilb}\, X$ is nonsingular at $[C]$ (Then we say C is unobstructed in X). The converse is not true (e.g. c.i. space curves $C \subset \mathbb{P}$ usually have large *obstruction spaces* but always *unobstructed*.).

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Given singular point $[C]$, determine the local dimension of the Hilbert scheme $\mathrm{Hilb}\, X$.

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Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry ['62]".

Example 1 (Mumford)

$\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component W of dimension $\mathbf{56}$, whose general member C satisfies:

1. C is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line E on S such that C belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ ($\simeq \mathbb{P}^{37}$) on S.

Remark 3

C is of degree $\mathbf{14}$ and genus $\mathbf{24}$, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

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Generalization of Mumford's example

- Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and more recently, those of $\mathbf{Hilb}^{sc}\,\mathbb{P}^n$ ($n > 3$) have been found by Choi-Iliev-Kim'24-1, and '24-2.
- In these examples, generic curves (of the non-reduced component) were contained in some surfaces:

	surfaces
Mumford'62, Kleppe'87, Ellia'87, N'05,23	smooth cubic in $\mathbb{P}^3$
Gruson-Peskine'82	non-normal cubic in $\mathbb{P}^3$
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- Later many non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and more recently, those of $\mathbf{Hilb}^{sc}\,\mathbb{P}^n$ $(n > 3)$ have been found by Choi-Iliev-Kim'24-1, and '24-2.
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Generalization to smooth Fano 3-folds

Theorem 4 (Mumford'62,Mukai-N'09,N'10,N'19)

If X is a smooth Fano 3-fold of $\rho = 1$ and index i, then $\mathbf{Hilb}^{sc} X$ contains a generically non-reduced irreducible component, whose general member C is contained in the following surface $S \subset X$ and classes $[C] \in \operatorname{Pic} S$.

i	surface S	$[C] \in \operatorname{Pic} S$	E
4	del Pezzo	$-K_X\big\vert_S + 2E$	line
3	$(S \in \lvert -\frac{i-1}{i} K_X \rvert)$		
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§2 Deformations of space curves lying on a complete intersection of quadrics

Complete intersection of quadrics

Today we consider deformations of space curves $C \subset \mathbb{P}$ lying on a smooth surface S, when S are complete intersections

$$S_2 \subset \mathbb{P}^3, \quad S_{2,2} \subset \mathbb{P}^4, \quad S_{2,2,2} \subset \mathbb{P}^5$$

of quadrics. For each S and curves C lying on S, we consider the deformations of C in the projective space $\mathbb{P}$ using the flag

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Hilbert-flag scheme

A main tool of our studies is the Hilbert-flag scheme $\mathbf{HF}\,X$ of X, i.e.

$$\mathbf{HF}\,X = \left\{(C,S) \,\middle|\, C \subset S \subset X\text{: closed subschemes}\right\}.$$

- The normal sheaf $N_{(C,S)/X}$ of (C,S) in X is defined by the fiber product sitting in

$$\begin{tikzcd} N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\ N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C, \end{tikzcd}$$

  where $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection.
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$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathrm{HF}\,X \le h^0(X, N_{(C,S)/X}).$$

- There exist natural projections $pr_i : \mathrm{HF}\,X \to \mathrm{Hilb}\,X$ $(i = 1, 2)$ corresponding to $(C,S) \mapsto [C]$, and $(C,S) \mapsto [S]$.

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Lemma 7

The Hilbert-flag scheme $\mathrm{HF}\, X$ is nonsingular at (C, S) of expected dimension

$$\chi(N_{(C,S)/X}) = \chi(N_{C/S}) + \chi(N_{S/X})$$

if C is a curve and

1. X is a Fano 3-fold and S is del Pezzo, or
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Proof.

Suppose e.g. X is Fano a 3-fold and S is del Pezzo. Then by adjunction, $N_{S/X} \simeq -K_X\big|_S + K_S$ and $N_{C/S} \simeq -K_S\big|_C + K_C$, where $-K_X$ and $-K_S$ are ample. This implies the higher cohomology groups of $N_{S/X}$ and $N_{C/S}$ vanish, so does that of $N_{(C,S)/X}$. □

Correspondingly to projections pr_i ($i = 1, 2$), if $C \hookrightarrow S \hookrightarrow X$ are regular, then there exist two natural exact sequences

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The Hilbert-flag scheme $\mathbf{HF}\, X$ is nonsingular at (C, S) of expected dimension

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We start from curves on smooth quadrics in $\mathbb{P}^3$.
Let $S = S_2 \subset \mathbb{P}^3$. Then

$$S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \qquad \text{and} \qquad \operatorname{Pic} S \simeq (\operatorname{Pic} \mathbb{P}^1)^{\oplus 2} = \mathbb{Z}^2.$$

Thus every curve C on S corresponds to *bidegree* $(a, b) \in \mathbb{Z}^2$ of C. Then C is of degree $a + b$ and genus $(a - 1)(b - 1)$.

Proposition 8 (Tannenbaum'78, Kleppe'87, etc)

Let $d > 4$ and $g \geq 0$. Let $C \subset \mathbb{P}^3$ be a smooth connected curve of degree d and genus g contained in a smooth quadric surface. Then the maximal irreducible family $W(a, b)$ of such curves $C \subset \mathbb{P}^3$ of bidegree (a, b) is of dimension $g + 2d + 8$, and

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Sketch of proof

Let $C \subset \mathbb{P}^3$ be contained in a smooth quadric $S = S_2 \subset \mathbb{P}^3$. Then $H^1(N_{C/S}) = 0$ and hence $\mathbf{HF}\,\mathbb{P}^3$ is nonsingular at (C, S) of expected dimension

$$\chi(N_{(C,S)}) = 2d + g + 8.$$

We see that either

$$\text{[i] } H^1(\mathcal{I}_C(2)) = 0 \text{ or [ii] } H^1(\mathcal{O}_C(2)) = 0$$

holds. If $H^1(\mathcal{I}_C(2)) = 0$, then the tangent map

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Smooth complete intersection $S_{2,2} \subset \mathbb{P}^4$

Let $S = S_{2,2} \subset \mathbb{P}^4$ be a smooth complete intersection. Then S is a del Pezzo surface of degree 4, isomorphic to $\mathbf{Bl}_{P_1,\dots,P_5}\,\mathbb{P}^2$ and $\mathbf{Pic}\,S \simeq \mathbb{Z}^6$ is generated by 5 exceptional curves $\mathbf{e}_i$ $(1 \le i \le 5)$ and the pull back of $[\mathcal{O}_{\mathbb{P}^2}(1)]$ by $S \to \mathbb{P}^2$. Then every divisor D on S corresponds to a 7-tuple $(a; b_1, \dots, b_5)$ of integers $a, b_1, \dots, b_5$ by

$$[D] = a\mathbf{l} - \sum_{i=1}^{5} b_i \mathbf{e}_i.$$

Moreover, the coordinate $(a; b_1, \dots, b_5)$ satisfies

$$b_1 \ge \cdots \ge b_5 \quad \text{and} \quad a \ge b_1 + b_2 + b_3. \tag{1}$$

Here $(a; b_1, \dots, b_5)$ is called the standard coordinate of $[D]$ in $\mathbf{Pic}\,S_{2,2}$.

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Let $S = S_{2,2} \subset \mathbb{P}^4$ be a smooth complete intersection. Then S is a del Pezzo surface of degree 4, isomorphic to $\mathbf{Bl}_{P_1,\dots,P_5}\,\mathbb{P}^2$ and $\mathrm{Pic}\,S \simeq \mathbb{Z}^6$ is generated by 5 exceptional curves $\mathbf{e}_i$ $(1 \le i \le 5)$ and the pull back of $[\mathcal{O}_{\mathbb{P}^2}(1)]$ by $S \to \mathbb{P}^2$. Then every divisor D on S corresponds to a 7-tuple $(a; b_1, \dots, b_5)$ of integers $a, b_1, \dots, b_5$ by

$$[D] = a\mathbf{l} - \sum_{i=1}^{5} b_i \mathbf{e}_i.$$

Moreover, the coordinate $(a; b_1, \dots, b_5)$ satisfies

$$b_1 \ge \dots \ge b_5 \quad \text{and} \quad a \ge b_1 + b_2 + b_3. \tag{1}$$

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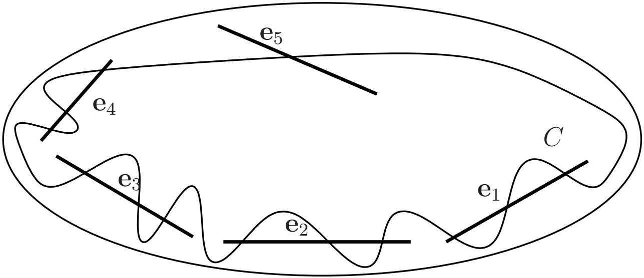


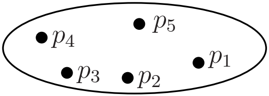
ε

$S_{2,2} = \mathrm{Bl}_{p_1,\dots,p_5}\mathbf{P}^2$

$[C] = (9; 3, 3, 3, 3, 1) \in \mathrm{Pic} S_{2,2} = \mathbf{Z}^6$

$d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$

$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$



$\mathbf{P}^2$

Stable degeneration

Suppose that

$$C \subset S \subset \mathbb{P}$$

is a flag of a space curve C and a surface S.

Definition 2

We say C is stably degenerate if every small global deformation of C in $\mathbb{P}$ is contained in a deformation of S in $\mathbb{P}$, i.e.,

$$C \subset S \text{ and } C \rightsquigarrow C' \Longrightarrow S \rightsquigarrow {}^{\exists}S' \text{ s.t. } S' \supset C'$$

Theorem 9 (Main1)

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth curve of degree $d \geq 10$ of genus $g \geq 2d - 12$, contained in a smooth c.i. $S_{2,2}$ in $\mathbb{P}^4$.Let $(a; b_1, \ldots, b_5)$ be the standard coordinate of $[C]$ in $\operatorname{Pic} S_{2,2} \simeq \mathbb{Z}^6$. Then

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Examples

Table: curves on $S_{2,2}$ and stable degeneration

(d, g)	the class of C	the max. irred. family of C ($\subset \mathbf{Hilb}^{sc}\, \mathbb{P}^4$)
$(14, 16)$	$(8; 2, 2, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 4, 3, 2, 2, 2)$	generically smooth component
$(14, 16)$	$(9; 3, 3, 3, 3, 1)$	generically non-reduced component
$(15, 18)$	$(9; 4, 2, 2, 2, 2)$	generically smooth component
$(15, 18)$	$(9; 3, 3, 3, 2, 1)$	generically non-reduced component
$\vdots$	$\vdots$	$\vdots$
$(18, 24)$	$(9; 2, 2, 2, 2, 1)$	generically non-reduced component
$(18, 24)$	$(10; 4, 3, 3, 1, 1)$	unknown ($h^1(\mathcal{I}_C(2)) = 2$)
$(18, 24)$	$(10; 3, 3, 3, 3, 0)$	a proper closed subset of a component
$(18, 24)$	$(11; 6, 3, 2, 2, 2)$	generically smooth component
$\vdots$	$\vdots$	$\vdots$

Analogy of Mumford's example

Example 3

$\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W,

1. whose general member C is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. for general $C \in W$, there exists a line E on S such that
$$C \sim -3K_S + 2E$$
(Here $|C| \simeq \mathbb{P}^{29}$), and
3. C is of degree 14 and genus 16.
4. $\dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^4} = 55$, while $h^0(C, N_{C/\mathbb{P}^4}) = 57$.

Thus $h^0(N_{C/\mathbb{P}^4}) = \dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^4} + 2$.

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$$H^1(N_{S/\mathbb{P}^5}(-C)) = H^1(S, -(C - 2\mathbf{h}))^{\oplus 2},$$

where $\mathbf{h} = \mathcal{O}_S(1)$. If $C - 2\mathbf{h}$ is nef and big and $H^1(N_{(C,S)/\mathbb{P}^5}) = 0$, then C is stably degenerate and unobstructed. On the other hands,

Theorem 10 (Main2)

For every integer $n \geq 2$, the Hilbert scheme $\mathrm{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components W_n,

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$$C \sim n(2\mathbf{h} + E),$$

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4. $\dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^5} = g + 53$, while $h^0(N_{C/\mathbb{P}^5}) = g + 56$.

Thus $h^0(N_{C/\mathbb{P}^5}) = \dim_{[C]} \mathcal{O}_{\mathrm{Hilb}\,\mathbb{P}^5} + 3$. Since $\mathbf{h}.E = 1$ and $E^2 = -2$, we have $C.E = 0$ and $(C - 2\mathbf{h}).E = -2$.

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We consider the family

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Let X be a projective scheme, $\mathbf{HF}\,X$ the Hilbert-flag scheme of X, and

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- Since $d = 17n > 16 = 2h^2$, S is uniquely determined by C.
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by projective normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

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- We note that $H^1(N_{S/\mathbb{P}^5}(E-C)) = H^1(-L^{\oplus 3}) = 0$, since $L = C - 2\mathrm{h} - E = (n-1)(2\mathrm{h}+E)$ is nef and big, and we have

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Conclusion

- The Hilbert scheme of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ has a generically non-reduced component whose general member is contained in $S_{2,2} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$, respectively.
- The existence of the non-reduced components are somewhat related to the geometry of line (rational curves) on the surfaces.
- As a result, Hilbert schemes of curves of higher genus may have bad components (non-reduced, etc).

Thank you very much for listening!

some non-complete intersection case

Let $S \subset \mathbb{P}^4$ be a cubic scroll $S_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow \mathbb{P}^4$. Then S is isomorphic to a blow up of $\mathbb{P}^2$ at a point, and

$$\operatorname{Pic} S \simeq \mathbb{Z}[C_0] \oplus \mathbb{Z}[f]$$

where C_0 is a negative section, i.e. $C_0^2 = -1$, and f is a fiber of $S \to \mathbb{P}^1$.

Theorem 12 (with A. Yamada)

Let $C \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 6$ and genus g, contained in a cubic scroll $S \subset \mathbb{P}^4$ and such that $C \sim aC_0 + bf$ in $\operatorname{Pic} S$. Then the maximal irreducible family W of such $C \subset \mathbb{P}^4$ is of dimension $d + g + b + 17$ and

1. If $b \geq a + 2$ and $a \geq 5$, then $\overline{W}$ is a generically smooth component of $\operatorname{Hilb}^{sc} \mathbb{P}^4$.
2. If $b = a + 1$ and $a \geq 6$, then $\overline{W}$ is a generically non-reduced component of $\operatorname{Hilb}^{sc} \mathbb{P}^4$.
3. If $b = a$ and $a \geq 5$, then there exists a (strictly larger) irreducible component V of $\operatorname{Hilb}^{sc} \mathbb{P}^4$ such that $\dim V > \dim W$.

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Primary obstructions

Let X be a projective scheme over k, C a loc. c. i. closed subscheme of X, and $k[\varepsilon] := k[t]/(t^2)$ (the ring of dual numbers).

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Generalized Kleppe-Ellia conjecture

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a anti-polarized *smooth del Pezzo surface* $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is stably degenerate if

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Generalized Kleppe-Ellia conjecture

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a anti-polarized *smooth del Pezzo surface* $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is stably degenerate if

1. $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$,
2. C is linearly normal,
3. $\deg(C) > 9$ for $n = 3$ and $\deg(C) > 2n$ for $n \geq 4$, and C is general in $[C] \in \operatorname{Pic} S_n$.