Deformations of space curves lying on a del Pezzo surface

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Today's slide

Plan of Talk

1. Hilbert schemes and deformation of flags
2. Kleppe-Ellia conjecture and its generalization (Main result) (cf. arXiv:2501.15788)
3. Applications and Examples

§1 Hilbert schemes and deformation of flags

the Hilbert scheme

Given a projective scheme X, and given Hilbert polynomial P,

$$\mathbf{Hilb}_P\, X = \left\{ C \subset X \;\middle|\; \text{closed subscheme of } P(C) = P \right\}$$

is called the Hilbert scheme of X.The Hilbert scheme has the following nice properties:

- fine moduli scheme, i.e. it has a universal family $\mathscr{C} \subset X \times \mathbf{Hilb}_P\, X$ such that every deformation of C in X derived from $\mathscr{C}$
- projective $(\mathbf{Hilb}_P\, X \hookrightarrow \mathbf{Gr})$
- existence of nice deformation theories, e.g., if C is a loc. c.i., then $H^0(N_{C/X})$ and $H^1(N_{C/X})$ resp. represent the tangent and obstruction spaces at $[C]$.

while it also has “not so nice” properties: e.g.

- it may have bad singularities (e.g. non-reduced components),
- may be highly reducible (for some P).

Dimension of $\mathrm{Hilb}\, X$

For simplicity, we assume C is a c.i. in X. Then

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \leq \dim_{[C]} \mathrm{Hilb}\, X \leq \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}} .$$

However, when $H^1(N_{C/X}) \neq 0$, it is hard to determine the dimension of $\mathrm{Hilb}\, X$ at $[C]$ (depending on whether C is obstructed or not).
To resolve this problem, we take an intermediate variety $C \subset S \subset X$ and use Hilbert-flag scheme

$$\mathrm{HF}\, X = \left\{ (C, S) \,\middle|\, C \subset S \subset X \right\} \subset \mathrm{Hilb}\, X \times \mathrm{Hilb}\, X.$$

Let

$$N_{(C,S)} := N_{C/X} \times_{N_{S/X}\big|_C} N_{S/X}$$

be the normal sheaf of (C, S) in X.

Naive question

Then

$$H^1(N_{(C,S)/X}) = 0 \Longrightarrow \mathbf{HF}\, X \text{ is nonsingular at } (C, S)$$

and if moreover $H^i(N_{(C,S)/X}) = 0$ for all $i > 0$, then $\mathbf{HF}\, X$ is of expected dimension

$$\chi(N_{(C,S)/X}) = \chi(N_{C/S}) + \chi(N_{S/X}).$$

Let $\mathcal{W}_{C,S}$ be an irreducible component passing through (C, S), and $pr_1 : \mathbf{HF}\, X \to \mathbf{Hilb}\, X$, $(C', S') \mapsto [C']$, the 1st projection.

Question 1

When is the image $pr_1(\mathcal{W}_{C,S})$ of $\mathcal{W}_{C,S}$ an irreducible component of $\mathbf{Hilb}\, X$?

Stable degeneration

Let $C \subset S \subset X$ a sequence of closed subvarieties with $H^i(N_{(C,S)/X}) = 0$ for all $i > 0$.

Definition 2

We say C is stably degenerate or stably contained in S, if for every small global deformation C' of C in X, there exists a global deformation S' of S in X such that $S' \subset C'$.

We have the following implications:

$$(1) \Longrightarrow (2) \Longrightarrow (3)$$

where

1. $H^1(\mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$.
2. $pr_1 : \mathrm{HF}\, X \to \mathrm{Hilb}\, X$, $(C', S') \mapsto [C']$ is smooth at (C, S).
3. C is stably degenerate and $pr_1(\mathcal{W}_{C,S})$ is a component of $\mathrm{Hilb}\, X$.

The implication "(2) $\Longrightarrow$ (3)" follows from the fact that $pr_1' = pr_1\big|_{\mathcal{W}_{C,S}}$ is locally surjective at $[C]$.

§2 Kleppe-Ellia conjecture and its generalization

Mumford's example and Kleppe's generalization

Given a projective scheme, we denote by $\mathbf{Hilb}^{sc}\, X$ the Hilbert scheme of smooth connected curves in X, i.e.,

$$\mathbf{Hilb}^{sc}\, X = \left\{ C \subset X \;\middle|\; C\text{: smooth connected curve} \right\}.$$

Theorem 3 (Mumford'62, a pathology)

$\mathbf{Hilb}^{sc}\, \mathbb{P}^3$ contains a generically non-reduced component.

Every its general member, i.e., a smooth curve $C \subset \mathbb{P}^3$, was contained in a smooth cubic surface $S_3 \subset \mathbb{P}^3$.
Later, Kleppe['87] generalized this example systematically by using the cooordinate of $(a; b_1, \dots, b_6)$ of $[C]$ in $\mathrm{Pic}\, S_3 \simeq \mathbb{Z}^7$:

$$C \sim a\mathbf{l} - \sum_{i=1}^{6} b_i \mathbf{e}_i \qquad \longleftrightarrow \qquad (a; b_1, \dots, b_6)$$

where $\mathbf{l} = [\mathcal{O}_{\mathbb{P}^2}(1)]$ and $\mathbf{e}_i$ $(i = 1, \dots, 6)$ are 6 exceptional curves.

Kleppe-Ellia conjecture

He proposed a conjecture, which can be reformulated as follows:

Conjecture (Kleppe'87, modified by Ellia'87)

Let $C \subset S_3 \subset \mathbb{P}^3$ be a smooth curve of degree $d \geq 14$ and genus g lying on a smooth cubic surface $S_3 \subset \mathbb{P}^3$. Then C is *stably degenerate* if

1. $\chi(\mathcal{I}_C(3)) \geq 1$ ($\Leftrightarrow g \geq 3d - 18$),
2. C is linearly normal, and
3. C is general in $[C] \in \operatorname{Pic} S_3$.

K-E conj. is non-trivial only if C is not 3-normal ($\Leftrightarrow H^1(\mathcal{I}_C(3)) \neq 0$).In fact, otherwise, $pr_1 : \operatorname{HF} \mathbb{P}^3 \longrightarrow \operatorname{Hilb} \mathbb{P}^3$ is smooth at (C, S) by

$$H^0(N_{(C,S)/\mathbb{P}^3}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^3}) \longrightarrow \underbrace{H^1(N_{S/\mathbb{P}^3}(-C))}_{\simeq H^1(\mathcal{I}_C(3))} \longrightarrow 0.$$

Moreover, the first two assumptions are necessary for the conclusion.

Some remarks

K-E conjecture is Known to be true if

- C is not 3-normal and $g >> d$ (Kleppe'87 and Ellia'87), or
- C is 2-normal, i.e. $H^1(\mathcal{I}_C(2)) = 0$ (N'23)

Del Pezzo surfaces

Definition 4

A smooth projective surface S is called del Pezzo if $-K_S$ is ample.

Every del Pezzo surface S is isomorphic to a blow-up of $\mathbb{P}^2$ (at $9-n$ points) or $\mathbb{P}^1 \times \mathbb{P}^1$.The number $n = (-K_S)^2$ is called the *degree* of S, and $1 \leq n \leq 9$.

Example 1 (del Pezzo surfaces)

degree n	a description of S_n	$-K_S$
$\vdots$	$\vdots$	
3	cubic surface $S_3 \subset \mathbb{P}^3$	
4	quartic c.i. $S_{2,2} \subset \mathbb{P}^4$	
5	lin. section $[\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9] \cap \mathbb{L}^{(5)}$	v.a.
$\vdots$	$\vdots$	

Why on del Pezzo?

Proposition 5 (smoothness of flag-scheme)

Let $C \subset S = S_n \subset \mathbb{P}^n$ be a smooth curve of degree d and genus g lying on a del Pezzo surface S_n $(n \geq 3)$.Then the Hilbert-flag scheme $\mathrm{HF}\,\mathbb{P}^n$ is nonsingular at $(C, S)$of expected dimension

$$\chi(N_{(C,S)/\mathbb{P}^n}) = d + g + n^2 + 9,$$

and $H^i(N_{(C,S)/\mathbb{P}^n}) = 0$ for all $i > 0$.

In fact, $H^i(N_{C/S}) = H^i(N_{S/\mathbb{P}^n}) = 0$ for $i > 0$,which implies $C \subset S$ and $S \subset \mathbb{P}^n$ have nice (unobstructed) deformations and hence so does (C, S) in $\mathbb{P}^n$.

Generalized Kleppe-Ellia conjecture

Toward a generalization, we study the deformations of space curves lying on a del Pezzo surface of degree.

Conjecture (generalized K-E conj.)

Let $C \subset S_n \subset \mathbb{P}^n$ be a smooth connected curve lying on a smooth del Pezzo surface $S_n \subset \mathbb{P}^n$ of degree $n \geq 3$. Then C is *stably degenerate* if

1. $\chi(N_{S/\mathbb{P}^n}(-C)) \geq 0$,
2. C is linearly normal,
3. $\deg(C) > 9$ for $n = 3$ and $\deg(C) > 2n$ for $n \geq 4$, and C is general in $[C] \in \operatorname{Pic} S_n$.

Remark 6

(1) $\Longleftrightarrow \chi(N_{(C,S)/\mathbb{P}^n}) \geq \chi(N_{C/\mathbb{P}^n})$.

Main result

We focus on the case $n = 4$, i.e. $S \simeq S_4$ is a smooth complete intersection

$$S_{2,2} = (2) \cap (2) \subset \mathbb{P}^4.$$

We see that $N_{S/\mathbb{P}^4} \simeq \mathcal{O}_S(2)^{\oplus 2} \simeq \mathcal{O}_S(-2K_S)^{\oplus 2}$and hence

$$H^1(N_{S/\mathbb{P}^4}(-C)) \simeq H^1(\mathcal{O}_S(-C-2K_S))^{\oplus 2} \simeq H^1(\mathcal{I}_C(2))^{\oplus 2}.$$

Then it follows from a general theory that C is stably degenerate if C is 2-normal.

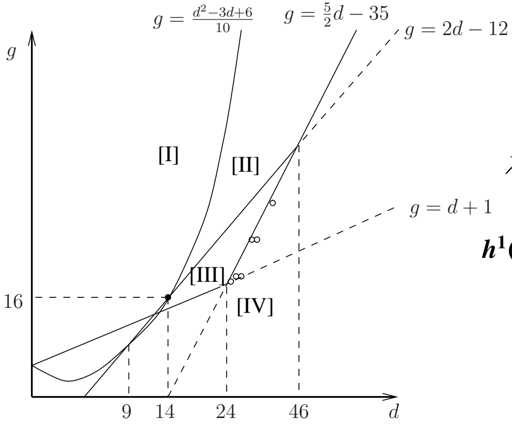
Theorem 7 (N'25)

Let $C \subset \mathbb{P}^4$ be a smooth connected curve of degree $d > 8$ contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$. Then

1. If C is 2-normal, then C is unobstructed and stably degenerate.
2. If $\mathcal{O}_C(2)$ is non-special and C is not 2-normal, then C is unobstructed, but not stably degenerate.
3. If $h^1(\mathcal{O}_C(2)) \geq h^1(\mathcal{I}_C(2)) = 1$ and C is general in $[C] \in \operatorname{Pic} S$, then C is obstructed and stably degenerate.

Main result (continued)

C is 2-normal (resp. $\mathcal{O}_C(2)$ is nonspecial) if (d, g) belongs to the region [I] (resp. [IV] except the 6 pairs corresponding to $\circ$).



$$\chi(N_{S/\mathbb{P}^4}(-C)) \geq 0$$
$$\Updownarrow$$
$$h^1(\mathcal{O}_C(2)) \geq h^1(\mathcal{I}_C(2))$$
$$\Updownarrow$$
$$g \geq 2d - 12.$$

Applications

Corollary 8

Generalized K-E conjecture holds to be true, if $n = 4$ and $h^1(\mathcal{I}_C(2)) = 1$.

Let $C \subset S = S_{2,2} \subset \mathbb{P}^4$ be as in Theorem 7, and $\mathcal{W}_{C,S}$ the irreducible component of $\mathbf{HF}\,\mathbb{P}^4$ passing through (C, S),and put

$$W_{C,S} := pr_1(\mathcal{W}_{C,S}) \cap \mathbf{Hilb}^{sc}\,\mathbb{P}^4.$$

Then we have the following 3 possibilities for $W_{C,S}$:

degeneration of C	Is C obstructed?	$W_{C,S} \subset \mathbf{Hilb}^{sc}\,\mathbb{P}^4$
stable	NO	gen.smooth component
stable	YES	gen.non-reduced component
unstable	YES/NO	not a component

Analogy of Mumford's example

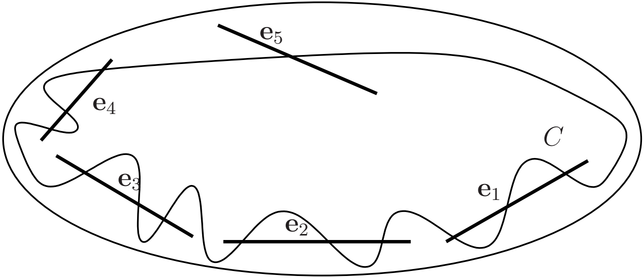
Example 2

$\mathbf{Hilb}^{sc}\,\mathbb{P}^4$ contains a generically non-reduced irreducible component W whose general member C satisfies

1. C is contained in a smooth c.i. $S = S_{2,2} \subset \mathbb{P}^4$,
2. there exists a line E on S such that C belongs to a complete linear system $\Lambda := |-3K_S + 2E|$ ($\simeq \mathbb{P}^{29}$) on S, and
3. $\dim W = 55$, $h^0(C, N_{C/\mathbb{P}^4}) = 57$, and C is of degree 14 and genus 16.

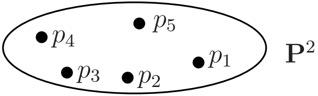
$$
\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(55)} \arrow[d, "\mathbb{P}^{29}\text{-b'dle}"] & \subset & \mathbf{Hilb}^{sc}\,\mathbb{P}^4 \\
(E,S) \arrow[d, mapsto] & \in & U^{(26)} \arrow[d, "G(2,12)\text{-b'dle}"] & \subset & G \times G(2, H^0(\mathcal{O}_{\mathbb{P}^4}(2))) \arrow[d] \\
E & \in & G(2,5)^{(6)} & = & \{\text{lines in } \mathbb{P}^4\}.
\end{tikzcd}
$$

Curves on $S_{2,2} \subset \mathbb{P}^4$ (analogy of Mumford's ex.)



$\varepsilon \Big\downarrow$

$$S_{2,2} = \mathrm{Bl}_{p_1,\ldots,p_5}\mathbf{P}^2$$
$$[C] = (9; 3, 3, 3, 3, 1) \in \mathrm{Pic} S_{2,2} = \mathbf{Z}^6$$
$$d = 3 \cdot 9 - 3 - 3 - 3 - 3 - 1 = 14$$
$$g = \frac{(9-1)(9-2)}{2} - 3 - 3 - 3 - 3 = 16$$



Standard coordinate

Let D be a divisor on a smooth c.i. $S_{2,2} \subset \mathbb{P}^4$, i.e., a quartic del Pezzo surface.Then there exists a suitable blow-up $\varepsilon : S_{2,2} \to \mathbb{P}^2$ such that

$$[D] = a - \sum_{i=1}^{5} b_i \mathbf{e}_i,$$

in $\mathrm{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$,where $\mathbf{l} = [\varepsilon^* \mathcal{O}_{\mathbb{P}^2}(1)]$and e_i's are 5 exceptionals, and such that

$$b_1 \geq \cdots \geq b_5 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \tag{1}$$

Then the set of 6-tuples $(a; b_1, \ldots, b_5)$ of integers is called standard coordinate of $[D]$ in $\mathrm{Pic}\, S_{2,2}$.

A criterion

Theorem 9 (N'25)

Let $C \subset S_{2,2} \subset \mathbb{P}^4$ be a smooth curve of degree $d \geq 10$ of genus $g \geq 2d - 12$, contained in a smooth c.i. $S_{2,2}$ in $\mathbb{P}^4$.Let $(a; b_1, \dots, b_5)$ be the standard coordinate of $[C]$ in $\mathrm{Pic}\, S_{2,2} \simeq \mathbb{Z}^6$. Then

1. If $b_5 \geq 2$, then C is unobstructed and stably degenerate.
2. If $b_5 = 1$ and $b_4 \geq 2$. then C is obstructed and stably degenerate.
3. If $b_5 = 0$, then C is not stably degenerate.

Examples

Table: curves on $S_{2,2}$ and stable degeneration

(d, g)	$(a; b_1, b_2, b_3, b_4, b_5)$	$W(a; b_1, b_2, b_3, b_4, b_5)$
$(14, 16)$	$(8; 2, 2, 2, 2, 2)$	unobstructed and stab.degenerate
$(14, 16)$	$(9; 4, 3, 2, 2, 2)$	unobstructed and stab.degenerate
$(14, 16)$	$(9; 3, 3, 3, 3, 1)$	obstructed and stab.degenerate
$(15, 18)$	$(9; 4, 2, 2, 2, 2)$	unobstructed and stab.degenerate
$(15, 18)$	$(9; 3, 3, 3, 2, 1)$	obstructed and stab.degenerate
$\vdots$	$\vdots$	$\vdots$
$(18, 24)$	$(9; 2, 2, 2, 2, 1)$	obstructed and stab.degenerate
$(18, 24)$	$(10; 4, 3, 3, 1, 1)$	unknown ($h^1(\mathcal{I}_C(2)) = 2$)
$(18, 24)$	$(10; 3, 3, 3, 3, 0)$	not stab.degenerate ($h^1(\mathcal{I}_C(2)) = 3$)
$(18, 24)$	$(11; 6, 3, 2, 2, 2)$	unobstructed and stab.degenerate
$\vdots$	$\vdots$	$\vdots$

further questions

1. Deformations of curves lying on del Pezzo $S_n \subset \mathbb{P}^n$ of degree $n \geq 5$.
2. Deformation of degenerate curves on del Pezzo manifold of higher dimension (> 3).
3. Study the relation to other examples of obstructed curves $C \subset \mathbb{P}^n$ (or non-reduced components of $\mathbf{Hilb}^{sc}\, \mathbb{P}^n$).
   [Y. Choi–H. Iliev–S. Kim'24] have recently proved the existence of many non-reduced components of $\mathbf{Hilb}^{sc}\, \mathbb{P}^n$ of higher dimensional projective space $\mathbb{P}^n$ by using ruled surfaces.
4. ...

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