# Obstructions to deforming curves lying on a K3 surface

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Today's slide

## Plan of Talk

# §1 Hilbert schemes and Mumford's example (Motivation)

# Hilbert schemes

Today, we work over a field $k = \bar{k}$ of $\mathrm{char}\, k = 0$.

**Theorem 1 (Grothendieck'60)**

Given a closed subscheme $X \subset \mathbb{P}^n$ and a closed subscheme $C \subset X$ with Hilbert polynomial $P(C) = P$, there exists a projective scheme $\mathrm{Hilb}_P\, X$, parametrizing all closed subschemes $C'$ of $X$ with (the same) Hilbert polynomial $P$.

$\mathrm{Hilb}\, X := \bigsqcup_P \mathrm{Hilb}_P\, X$ is called the Hilbert scheme of $X$. Today we consider the open and closed subscheme

$$\mathrm{Hilb}^{sc}\, X := \{\text{smooth connected curves } C \subset X\} \subset \mathrm{Hilb}\, X,$$

that is, the Hilbert scheme of curves in $X$.

# Infinitesimal property of Hilbert schemes

- The tangent space of $\mathrm{Hilb}\,X$ at $[C]$ is isomorphic to $H^0(C, N_{C/X})$.
- $C \subset X$: a locally complete intersection $\Longrightarrow$ every obstruction to deforming $C$ in $X$ is contained in $H^1(C, N_{C/X}) \quad (\subset \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C))$ and

$$\underbrace{h^0(C, N_{C/X}) - h^1(C, N_{C/X})}_{\text{exp.dim.}(=\chi(N_{C/X}) \text{ if } C \text{ is a curve})} \le \dim_{[C]} \mathrm{Hilb}\,X \le \underbrace{h^0(C, N_{C/X})}_{\text{tangential dimension}}.$$

- We say $C \subset X$ is unobstructed if $\mathrm{Hilb}\,X$ is nonsingular at $[C]$.
- $H^1(C, N_{C/X}) = 0 \Longrightarrow C$ is unobstructed. The converse is not true (e.g. c.i. curves $C \subset \mathbb{P}^3$ may have large $H^1(N_{C/\mathbb{P}^3})$ but *unobstructed*.).

**Purpose 2**

Determine $\dim_{[C]} \mathrm{Hilb}\,X$ at a singular point $[C]$ of $\mathrm{Hilb}\,X$.

# Mumford's example (pathology)

The following example appeared in a famous paper "Further pathologies in algebraic geometry [Mumford'62]".

## Example 1 (Mumford)

$\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ contains a generically non-reduced irreducible component $W$ of dimension $56$, whose general member $C$ satisfies

1. $C$ is contained in a smooth cubic surface $S \subset \mathbb{P}^3$.
2. There exists a line $E$ on $S$ such that $C$ belongs to a complete linear system $\Lambda := |-4K_S + 2E|$ ($\simeq \mathbb{P}^{37}$) on $S$.

## Remark 3

- $C$ and $\mathbb{P}^3$ are innocent-looking (a pathology).
- $C$ is of degree $14$ and genus $24$, and $h^1(N_{C/\mathbb{P}^3}) = 1$ and

$$\chi(N_{C/\mathbb{P}^3}) = 56 = \dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^3 < h^0(N_{C/\mathbb{P}^3}) = 57.$$

# Generalization of Mumford's example

- Later many non-reduced components of $\mathrm{Hilb}^{sc}\,\mathbb{P}^3$ were found by e.g. Gruson-Peskine'82, Kleppe'87, Ellia'87, N'05, Kleppe-Ottem'15, etc. and also those of $\mathrm{Hilb}^{sc}\,\mathbb{P}^n$ $(n > 3)$ have been more recently found by Choi-Iliev-Kim'24-1, and '24-2.
- Curves $C$ corresponding to the generic point of the non-reduced components were contained in some surface $S \subset \mathbb{P}^n$, e.g.,

| | a surface $S$ containing $C$ |
|---|---|
| Mumford['62] | smooth cubic |
| Gruson-Peskine'82 | non-normal cubic |
| Kleppe'87 | smooth cubic |
| Kleppe-Ottem'15 | smooth quartic |
| Choi-Iliev-Kim'24-1, '24-2 | ruled surface |

- Vakil's result ['06] on *Murphy's law in AG*: unless there is some a priori reason otherwise, the deformation space may be as bad as possible.

# Another generalization (with Mukai)

We found that in Mumford's example, $(-1)$-curves $E \simeq \mathbb{P}^1$ (on smooth cubics) *play an important role*.

## Theorem 4 (Mukai-N'09, $\mathrm{char}\, k \geq 0$)

Let $X$ be a smooth projective $3$-fold satisfying the following:

1. there exists a smooth rational curve $E$ on $X$ s.t. $N_{E/X}$ is globally generated, and
2. there exists a smooth surface $S$ s.t. $E \subset S \subset X$, $E^2 = -1$ on $S$, and $H^1(S, N_{S/X}) = p_g(S) = 0$.

Then the Hilbert scheme $\mathbf{Hilb}^{sc}\, X$ has infinitely many generically non-reduced components (GNRC).

## Remark 5

In Mumford's ex., $X = \mathbb{P}^3$, $S$ is a smooth cubic, $E$ is a line.

# Hilbert scheme of Fano 3-folds

- A Fano manifold is a smooth projective variety $X$ with ample $-K_X$.
- The index $r$ of a Fano manifold $X$ is the maximal integer $r$ such that $-K_X \sim rH$ with some $H \in \operatorname{Pic} X$.

Let $X$ be a smooth Fano 3-fold of index $r$.

- $X \simeq \mathbb{P}^3$ if $r = 4$ and $X \simeq Q^3 \subset \mathbb{P}^4$ if $r = 3$, and $X$ is called del Pezzo if $r = 2$, and prime if $r = 1$ and $\rho = 1$.
- If we restrict $X$ with $\rho = 1$, then there exist $17$ deformation equivalence classes of $X$ (Fujita, Iskovskikh'77,'78):

| $r$ | $4$ | $3$ | $2$ | $1$ |
|---|---|---|---|---|
| the number of cls. | $1$ | $1$ | $5$ | $10$ |
| variety / cls. | $\mathbb{P}^3$ | $Q^3 \subset \mathbb{P}^4$ | del Pezzo | prime Fano |

Table: the number of deformation equivalence classes of $X$

Applying Theorem 4, we obtain

**Example 2 (N'10)**

If $r > 1$ (and of any $\rho(X)$), then $\mathrm{Hilb}^{sc}\, X$ contains a a generically non-reduced component $W$ satisfying:

1. every general member $C$ of $W$ is contained in a smooth del Pezzo surface $S$ $(\sim -\frac{r-1}{r}K_X)$, and
2. there exists a (good) line $E$ on $S$ and $C \sim -K_X\big|_S + 2E$ in $\mathrm{Pic}\, S$.
3. $h^0(C, N_{C/X}) = \dim W + 1$.

Here

- A curve $E \subset X$ is a line $\overset{\text{def}}{\Longleftrightarrow} E \simeq \mathbb{P}^1$ and $-\frac{1}{r}K_X.E = 1$.
- A line $E \subset X$ is good $\overset{\text{def}}{\Longleftrightarrow} N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$ (for $r = 2, 3$).
- $\dim W = 56$, $42$ and $(-K_X)^3/2 + 4$ for $r = 4, 3, 2$, respectively.

**Question 6**

What about the case for $\mathrm{Hilb}^{sc}\, X$ of prime Fano $3$-fold $X$?

If $X$ is prime $(r = 1)$, then there exists NO del Pezzo surface $S \subset X$.

# §2 Deformation of curves lying on a K3 surface

# Setting

In this section, we recall my previous results on the deformations of smooth curves $C$ on a smooth Fano $3$-fold $X$ with the assumption that $C$ is contained in a smooth $K3$ surface $S \subset X$, i.e., we have

$$C \subset S_{K3} \subset X_{Fano}.$$

**Definition 7**

A smooth projective surface $S$ is a $K3$ surface, if $K_S \sim 0$ and $H^1(S, \mathcal{O}_S) = 0$.

We will see that

- $(-2)$-curves $E \simeq \mathbb{P}^1$, and
- elliptic curves $F$

on $S$ control the deformations $C$ in $X$. They play a role very similar to that of $(-1)$-curve $E \simeq \mathbb{P}^1$ on the smooth cubic $S_3 \subset \mathbb{P}^3$ in Mumford's example!

# Hilbert-flag scheme

The Hilbert-flag scheme $\mathrm{HF}\,X$ of $X$ parametrises all pairs $(C, S)$ of closed subschemes $C$ and $S$ of $X$ satisfying $C \subset S$. If $C \hookrightarrow S \hookrightarrow X$ is regular embeddings, then $H^0(N_{(C,S)/X})$ and $H^1(N_{(C,S)/X})$ respectively represents the tangent sp. and the obstruction sp. of $\mathrm{HF}\,X$ at $(C, S)$, where the normal sheaf $N_{(C,S)/X}$ of $(C, S)$ in $X$ is defined by the Cartesian diagram

$$\begin{tikzcd} N_{(C,S)/X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & N_{S/X} \arrow[d, "|_C"] \\ N_{C/X} \arrow[r, "\pi_{C/S}"] & N_{S/X}\big|_C . \end{tikzcd}$$

Here $|_C$ is the restriction of sheaves, and $\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C$ is the natural projection. As in the case of Hilbert schemes, we have

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \le \dim_{(C,S)} \mathrm{HF}\,X \le h^0(X, N_{(C,S)/X}).$$

## Lemma 8

Under our setting, the following are equivalent:

1. $H^1(N_{(C,S)/X}) = 0$, namely $\mathbf{HF}\, X$ is nonsingular at $(C, S)$ of expected dimension $\chi(N_{(C,S)/X})$.
2. There exists a first order deformation $\tilde{S}$ of $S$ in $X$, to which $C$ does not lift.

## Proof.

By adjunction, $N_{S/X} \simeq -K_X\big|_S$ and $N_{C/S} \simeq K_C$, which implies $H^i(N_{S/X}) = 0$ for $i > 0$ and $H^1(N_{C/S}) \simeq k$. There exists an exact sequence $0 \to N_{C/S} \to N_{(C,S)/X} \to N_{S/X} \to 0$, inducing

$$H^0(N_{(C,S)/X}) \xrightarrow{p_2} H^0(N_{S/X}) \longrightarrow \underbrace{H^1(N_{C/S})}_{\simeq k} \longrightarrow H^1(N_{(C,S)/X}) \longrightarrow 0.$$

$H^1(N_{(C,S)/X}) = 0$ iff $p_2$ is not surjective, which is equivalent to the second condition.

□

## Example 9

Let $X$ be a prime Fano $3$-fold, $E$ a line on $X$. Then $E$ is of type, either $(0,-1)$ (good) or $(1,-2)$ (bad). If $E$ is contained in a smooth $K3$ surface $S$ in $X$, then

$$\mathbf{HF}\,X \text{ is nonsingular at } (E,S) \text{ (of exp. dim.)} \iff E \text{ is good.}$$

In fact, $\mathbf{Hilb}\,X$ is nonsingular at $[E]$ if and only if $E$ is good, and the first projection $pr_1 : (E,S) \to [E]$ is a smooth morphism at $(E,S)$.

## Lemma 10

If $X$ is prime, and $E$ is a good line or a good conic on $X$ contained in a smooth $S \in |-K_X|$, then $H^1(X, N_{(E,S)/X}) = 0$.

# A key lemma

**Lemma 11 (char $k = 0$)**

Let $i : S \hookrightarrow X$ denote the closed embedding, and let $E$ be an effective Cartier div on $S$ with $H^1(S, \mathcal{O}_S(E)) = 0$. If $H^1(X, N_{(E,S)/X}) = 0$ and if $C - bE \in i^* \operatorname{Pic} X$ with $b \neq 0$, then $H^1(X, N_{(C,S)/X}) = 0$.

**Proof.**

Since $H^1(X, N_{(E,S)/X}) = 0$, by Lem. 8, there exists a first order deformation $\tilde{S}$ of $S$ in $X$, to which $E$ does not lift. Let $\alpha \in H^0(S, N_{S/X})$ and and $\tau \in H^1(S, T_S)$ (abstract def.) correspond to $\tilde{S}$. Then $\tau \cup c(\mathcal{O}_S(E)) \neq 0$ in $H^2(\mathcal{O}_S)$, where $c(*) \in H^1(S, \Omega^1_S)$ denotes the Atiyah-ext. class of $*$. Since $c(\mathcal{O}_S(C)) = c(\mathcal{O}_S(C - bE)) + bc(\mathcal{O}_S(E)))$, and $C - bE \in i^* \operatorname{Pic} X$, we have $\tau \cup c(\mathcal{O}_S(C)) \neq 0$, hence $\mathcal{O}_S(C)$ does not lift to $\tilde{S}$, hence neither does $C$ as a closed subscheme of $S$, and $H^1(X, N_{(C,S)/X}) = 0$. $\square$

# $\pi$-map

Let $E$ be a curve on $S$ ($\subset X$), and $\pi_{E/S} : N_{E/X} \longrightarrow N_{S/X}\big|_E$ the projection.

## Definition 12 ($\pi$-map)

The homomorphism $\pi_{E/S}(E) = \pi_{E/S} \otimes \mathcal{O}_E(E)$ of sheaves on $E$ induces a map

$$\pi_{E/S}(E) : H^0(E, N_{E/X}(E)) \longrightarrow H^0(E, N_{S/X}(E)\big|_E),$$

on the global sections. We call this map the $\pi$-map for $(E, S)$.

## Example 13

Let $E$ be a conic on a prime Fano $3$-fold $X$, contained in a smooth $S \in |-K_X|$. Then $\mathcal{O}_E(E) \simeq \mathcal{O}(-2)$, and $N_{S/X}\big|_E \simeq \mathcal{O}(2)$. If $E$ is good, then the $\pi$-map

$$\pi_{E/S}(E) : H^0(E, \mathcal{O}(-2)^2) \longrightarrow H^0(E, \mathcal{O})$$

for $(E, S)$ is zero (hence not surjective).

# Obstructedness of curves in a $K3$ surface

Let $X$ be a smooth Fano $3$-fold, $C \subset X$ a smooth curve contained in a smooth $K3$ surface $S \subset X$. Let $D$ be a divisor on $S$ defined by

$$D := C + K_X\big|_S.$$

## Theorem 14 (N'17)

If $H^1(N_{(C,S)/X}) = 0$ and $D \geq 0$, then

1. If there exist no $(-2)$-curves and no elliptic curves on $S$, or if $H^1(S, D) = 0$, then $C$ is unobstructed.
2. If $D^2 \geq 0$ and there exists a $(-2)$-curve $E$ on $S$ such that $E.D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover, the $\pi$-map $\pi_{E/S}(E)$ is not surjective, then $C$ is obstructed.
3. If there exists an elliptic curve $F$ on $S$ such that $D \sim mF$ for $m \geq 2$, then we have $h^1(S, D) = m - 1$. If moreover, $\pi_{F/S}(F)$ is not surjective, then $C$ is obstructed.

# Analogy of Mumford's ex. in the case $r = 1$

Applying Theorem 14 to a $K3$ surfaces $S \subset X$ and a good conic $E \simeq \mathbb{P}^1$ on $S$, we can prove the following.

**Theorem 15 (N'19)**

Let $X$ be a prime Fano 3-fold of genus $g := (-K_X)^3/2 + 1$. Then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component $W$ with the following properties:

1. Every general member $C$ of $W$ is contained in a $K3$ surface $S$ $(\sim -K_X)$.
2. There exists a good conic $E \simeq \mathbb{P}^1$ on $S$ such that $C \sim -2K_X\big|_S + 2E$.
3. $\dim W = 5g + 1$, $h^0(C, N_{C/X}) = 5g + 2$, and $C$ is of degree $4g$ and genus $4g + 1$.

Here a conic $E$ on $X$ is called good if $N_{E/X} \simeq \mathcal{O}_E^{\oplus 2}$.

# Some remarks

We used the following facts:

- The prime Fano 3-fold $X = X_{2g-2}$ contains a conic $E \subset V$ (cf. [Shokurov'79], [Reid'80]).
- If $E \subset X$ is general, then $E$ is a good conic (cf. [Iskovskih'78]) if $\mathrm{char}\, k = 0$.
- For every conic $E$, there is a smooth $K3$ surface $S \in |-K_V|$ containing $E$ (cf. [Iskovskih]).

## Corollary 16

If $X$ is a smooth Fano 3-fold and $\rho(X) = 1$, then $\mathbf{Hilb}^{sc}\,X$ contains a generically non-reduced component.

| 3-fold $X$ | surface $S$ | $[C] \in \mathrm{Pic}\,S$ | $E$ | |
|---|---|---|---|---|
| $\mathbb{P}^3$ | del Pezzo | $-K_X\big\|_S + 2E$ | line | Mumford['62] |
| $Q^3 \subset \mathbb{P}^4$ | | | | Mukai-N['09] |
| del Pezzo | | | | Mukai-N['09], N['10] |
| prime Fano | $K3$ | $-2K_X\big\|_S + 2E$ | conic | N['19] |

Table: Generically non-reduced component of Mumford type

# §3 An example of non-reduced components of $\mathbf{Hilb}^{sc}\,\mathbb{P}^5$

# Main result

Toward a further generazation, we compute the obstruction to deforming curves lying on a complete intersection $K3$ surface in $P^5$.

## Theorem 17 (Main)

The Hilbert scheme $\mathrm{Hilb}^{sc}\,\mathbb{P}^5$ contains a generically non-reduced components $W_n$ ($n \geq 2$) with the following properties:

1. every general $C$ of $W_n$ is a smooth connected curve contained in a smooth complete intersection $K3$ surface
$$S = S_{2,2,2} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5.$$
2. $C$ is linearly equivalent to $n(2\mathbf{h} + E)$, where $h = [\mathcal{O}_S(1)]$ in $\mathrm{Pic}\,S$, and $E$ is a line on $S$.
3. $C$ is of degree $17n$ and genus $17n^2 + 1$.
4. $\dim W = 17n^2 + 54$ ($= g + 53$), while $h^0(N_{C/\mathbb{P}^5}) = 17n^2 + 57$, thus $h^0(N_{C/\mathbb{P}^5}) - \dim W = 3$.

# Construction

We see $h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{5+2}{2} = 21$ and

$$h^0(\mathbb{P}^5, \mathcal{I}_E(2)) = 21 - h^0(E, \mathcal{O}_E(2)) = 18.$$

Then

$$\begin{tikzcd}
C \arrow[d, mapsto] & \in & W^{(g+53)} \arrow[d, "\mathbb{P}^g\text{-bundle}"] & \subset & \mathrm{Hilb}^{sc}\,\mathbb{P}^5 \\
(E, S) \arrow[d, mapsto] & \in & U^{(53)} \arrow[d, "\mathrm{Gr}(3,18)\text{-bundle}"] & \subset & G \times \mathrm{Gr}(3, V) \arrow[d] \\
E & \in & \mathrm{Gr}(2,6)^{(8)} & = & \left\{\text{lines in } \mathbb{P}^5\right\},
\end{tikzcd}$$

where $g + 53 = 17n^2 + 54$ and $V = H^0(\mathbb{P}^5, \mathcal{O}(2))$

# Hilbert-flag scheme and Key Lemma

Let $X$ be a projective scheme. Then there exists a projective scheme (Hilbert-flag scheme)

$$\mathrm{HF}\,X := \left\{(C,S) \,\middle|\, C \subset S \subset X\right\} \subset \mathrm{Hilb}\,X \times \mathrm{Hilb}\,X.$$

Let

$$N_{(C,S)/X} := N_{C/X} \times_{N_{S/X}\big|_C} N_{S/X}$$

be the normal sheaf of $(C,S) \in \mathrm{HF}\,X$.
There exists a projection

$$pr_1 : \mathrm{HF}\,X \to \mathrm{Hilb}\,X, \qquad (C,S) \mapsto [C],$$

which induces the tangent map $p_1 : H^0(X, N_{(C,S)/X}) \to H^0(C, N_{C/X})$.

**Lemma 18 (Key Lemma, N['23], Lem. 2.17)**

We have $\dim_{(C,S)} \mathrm{HF}\,X = \dim_{[C]} \mathrm{Hilb}\,X$ if we have

1. $H^1(X, N_{(C,S)/X}) = H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, and
2. $\mathrm{ob}(\alpha) \neq 0$ for any $\alpha \in H^0(C, N_{C/X}) \setminus \mathrm{im}\,p_1$.

# Primary obstructions

Let $X$ be a projective scheme over $k$, $C$ a loc. c. i. closed subscheme of $X$, and $k[\varepsilon] := k[t]/(t^2)$ (the ring of dual numbers).

- A first order (infinitesimal) deformation of $C$ is a deformation $\tilde{C}$ ($\subset X \times \operatorname{Spec} k[\varepsilon]$) of $C$ in $X$ over $k[\varepsilon]$.
- $\tilde{C}$ naturally corresponds to $\alpha \in H^0(C, N_{C/X})$ ($\simeq \mathbf{Hom}(\mathcal{I}_C, \mathcal{O}_C)$).
- There is an element $\mathbf{ob}(\alpha)$ in $H^1(C, N_{C/X})$ (called the primary obstruction of $\alpha$) such that

$$\mathbf{ob}(\alpha) = \mathbf{0} \iff \tilde{C} \text{ is liftable to some } \tilde{\tilde{C}} \text{ over } k[t]/(t^3).$$

- $\mathbf{ob}(\alpha)$ can be expressed as a cup product, and

$$\mathbf{ob}(\alpha) = \alpha \cup \mathbf{e} \cup \alpha \qquad \text{in } \mathbf{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

  where $\mathbf{e} := [0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0]$.
- $\mathbf{ob}(\alpha) \neq \mathbf{0}$ for some $\alpha$ implies that $\mathbf{Hilb}\, X$ is singular at $[C]$ by infinitesimal lifting property of smoothness.

# Sketch of Proof of Main thm.

Let $C \subset \mathbb{P}^5$ be a smooth connected curve lying on a complete intersection $K3$ surface $S_{2,2,2} \subset \mathbb{P}^5$, and such that $C \sim 2n\mathbf{h} + nE$ in $\mathrm{Pic}\,S$ for $n \geq 2$, where $E$ is a line on $S$

- Since $d = 17n > 16 = 2h^2$, $S$ is uniquely determined by $C$.
- Then for all $i > 0$, $H^i(N_{S/\mathbb{P}^5}(-E)) = 0$ by proj. normality and $H^i(N_{E/\mathbb{P}^5}) = 0$ by ampleness. Then it follows from

$$0 \longrightarrow N_{S/\mathbb{P}^5}(-E) \longrightarrow N_{(E,S)/\mathbb{P}^5} \xrightarrow{\pi_1} N_{E/\mathbb{P}^5} \longrightarrow 0$$

  that $H^i(N_{(E,S)/\mathbb{P}^5}) = 0$, which implies there exists a first order deformation of $\tilde{S}$ of $S$, to which $E$ (and hence $C$) does not lift.
- Then $H^i(N_{(C,S)/\mathbb{P}^5}) = 0$ for $i > 0$ and

$$H^0(N_{(C,S)/\mathbb{P}^5}) \xrightarrow{p_1} H^0(N_{C/\mathbb{P}^5}) \longrightarrow H^1(N_{S/\mathbb{P}^5}(-C)) \longrightarrow 0$$

  is exact. Here $p_1$ is the tangent map of $pr_1 : \mathrm{HF}\,\mathbb{P}^5 \to \mathrm{Hilb}\,\mathbb{P}^5$ at $(C, S)$ and its cokernel is of dimension $3$ by

$$H^1(N_{S/\mathbb{P}^5}(-C)) \simeq H^1(\mathcal{O}_S(2\mathbf{h} - C))^{\oplus 3} \simeq k^3.$$

## Sketch of Proof of Main thm.

- We note that $H^1(N_{S/\mathbb{P}^5}(E-C)) = H^1(-L^{\oplus 3}) = 0$, where

$$L = C - 2\mathrm{h} - E = (n-1)(2\mathrm{h} + E)$$

  is nef and big.
- Then for every $\alpha \in H^0(N_{C/\mathbb{P}^5}) \setminus \mathrm{im}\, p_1$, its *exterior component* $\pi_{C/S}(\alpha)$, i.e., the image of $\alpha$ in $H^0(N_{S/\mathbb{P}^5}(E)\big|_C)$, lifts to a global section $\beta$ of $N_{S/\mathbb{P}^5}(E)$. (Here $\beta$ is called an infinitesimal deformation with poles.)
- Applying a *"modification"* of the obstructedness criterion [Mukai-N'09] to $\beta$, we obtain $\mathrm{ob}(\alpha) \neq 0$. This implies

$$\dim_{[C]} \mathrm{Hilb}^{sc}\,\mathbb{P}^5 = \dim_{(C,S)} \mathrm{HF}\,\mathbb{P}^5$$

  by the key lemma. Therefore $C$ is obstructed and parametrised by an open dense subset of a component of $\mathrm{Hilb}^{sc}\,\mathbb{P}^5$.

## Obstructedness Criterion (with Mukai)

[Mukai-N'09] gave a sufficient condition for $\mathbf{ob}(\alpha) \neq 0$ when $\dim X = 3$.
Let $C$ be an irreducible curve on a $3$-fold $X$.

**Assumption 1**

- there exists an intermediate surface $C \subset S \subset X$ s.t. $C \hookrightarrow S$ and $S \hookrightarrow X$ are regular embeddings.
- there exists an irreducible curve $E$ ($\neq C$) on $S$ s.t. $[\mathcal{O}_S \hookrightarrow \mathcal{O}_S(E)] \otimes \mathcal{O}_S(mE)$ induces the injection

$$H^1(S, \mathcal{O}_S(mE)) \hookrightarrow H^1(S, \mathcal{O}_S((m+1)E))$$

for all $m > 0$. (e.g. $E$=$(-1)$-$\mathbb{P}^1$ on $S$)

# Obstructedness Criterion (Continued)

Let $\alpha \in H^0(N_{C/X})$ be a first order deformation of $C$ in $X$ and $\mathbf{ob}(\alpha) \in H^1(N_{C/X})$ its primary obstruction. We consider the *"exterior" components*

$$\pi_{C/S}(\alpha) := H^0(\pi_{C/S})(\alpha), \qquad \mathbf{ob}_S(\alpha) := H^1(\pi_{C/S})(\mathbf{ob}(\alpha)).$$

by the projection

$$\pi_{C/S} : N_{C/X} \to N_{S/X}\big|_C.$$

Assumption 2

- Suppose $\pi_{C/S}(\alpha)$ lifts to a global section $\beta$ of $N_{S/X}(E)$.

$$\begin{array}{ccccccc} & & & H^0(N_{S/X}) & \subset & H^0(N_{S/X}(E)) & \ni \beta \\ & & & \downarrow & & \downarrow & \\ \alpha \in & H^0(N_{C/X}) & \xrightarrow{\pi_{C/S}} & H^0(N_{S/X}\big|_C) & \subset & H^0(N_{S/X}(E)\big|_C) & \end{array}$$

Here $\beta$ is called an infinitesimal deformation with pole:

# Obstructedness Criterion (Continued)

**Theorem 19 (Mukai-N'09)**

$\mathbf{ob}_S(\alpha)$ is nonzero if

1. $\Delta.E = 2(-E^2 + g(E) - 1)$, where $\Delta := C + K_X\big|_S - 2E$ in $\mathrm{Pic}\, S$.
2. Let $\beta\big|_E$ be the principal part of $\beta$ along $E$. Then $\beta\big|_E \cup \mathbf{k}_E \neq 0$ in $H^1(E, \mathcal{O}_E(2E))$, where
$$\mathbf{k}_E := [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}\big|_E \longrightarrow 0] \in \mathrm{Ext}^1_E(N_{S/X}\big|_E, N_{E/S}).$$
3. the restriction map $H^0(S, \Delta) \to H^0(E, \Delta\big|_E)$ is surjective,

# References

D. Mumford.
Further pathologies in algebraic geometry.
*Amer. J. Math.*, 84:642–648, 1962.

S. Mukai and H. Nasu.
Obstructions to deforming curves on a 3-fold, I: A generalization of Mumford's example and an application to Hom schemes.
*J. Algebraic Geom.*, 18(4):691–709, 2009.

H. Nasu,
Obstructions to deforming curves on a $3$-fold, II: Deformations of degenerate curves on a del Pezzo 3-fold,
*Annales de L'Institut Fourier*, **60**(2010), no. 4, 1289–1316.

H. Nasu.
Obstructions to deforming curves on a 3-fold, III: Deformations of curves lying on a $K3$ surface.
*Internat. J. Math.*, 28(13):1750099, 30, 2017.

H. Nasu,
Obstructions to deforming curves on a prime Fano 3-fold,
*Mathematische Nachrichten*, **292**(2019), no. 8, 1777–1790.

H. Nasu.
Obstructions to deforming curves on an Enriques-Fano $3$-fold.
*J. Pure Appl. Algebra*, 225(9):106677, 15, 2021.