Problem Set 1

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We have

$$\begin{aligned}
(\boldsymbol{b}_1', \boldsymbol{b}_2')' &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} \\
&= \left(\begin{pmatrix} \boldsymbol{X}_1' \\ \boldsymbol{X}_2' \end{pmatrix} (\boldsymbol{X}_1, \boldsymbol{X}_2) \right)^{-1} \begin{pmatrix} \boldsymbol{X}_1' \\ \boldsymbol{X}_2' \end{pmatrix} \boldsymbol{y} \\
&= \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{X}_1'\boldsymbol{X}_2 \\ \boldsymbol{X}_2'\boldsymbol{X}_1 & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{pmatrix},
\end{aligned}$$

where the last equality follows from the assumption that $\boldsymbol{X}_1'\boldsymbol{X}_2 = \boldsymbol{0}$ and $\boldsymbol{X}_2'\boldsymbol{X}_1 = \boldsymbol{0}$. Since

$$\begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2'\boldsymbol{X}_2 \end{pmatrix} \begin{pmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1} \end{pmatrix} = \boldsymbol{I}_k,$$

where $\boldsymbol{I}_k$ is a $k \times k$ identity matrix, we have

$$\begin{aligned}
\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} &= \begin{pmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{y} \\ \boldsymbol{X}_2'\boldsymbol{y} \end{pmatrix} \\
&= \begin{pmatrix} (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y} \\ (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{y} \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{\boldsymbol{b}}_1 \\ \boldsymbol{b}_2 \end{pmatrix}.
\end{aligned}$$

□

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Let $e_i = y_i - \hat{y}_i$. We first show $\sum_{i=1}^n \boldsymbol{x}_i e_i = 0$. Since we have

$$\begin{aligned}
\boldsymbol{b} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \\
&= \left((\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \begin{pmatrix} \boldsymbol{x}_1' \\ \vdots \\ \boldsymbol{x}_n' \end{pmatrix} \right)^{-1} (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\
&= \left(\sum_{i=1}^n \boldsymbol{x}_i \boldsymbol{x}_i' \right)^{-1} \left(\sum_{i=1}^n \boldsymbol{x}_i y_i \right),
\end{aligned}$$

It follows that

$$
\begin{aligned}
\sum_{i=1}^{n} \boldsymbol{x}_i e_i &= \sum_{i=1}^{n} \boldsymbol{x}_i (y_i - \hat{y}_i) \\
&= \sum_{i=1}^{n} \boldsymbol{x}_i \left(y_i - \boldsymbol{x}_i' \boldsymbol{b} \right) \\
&= \sum_{i=1}^{n} \boldsymbol{x}_i y_i - \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i' \boldsymbol{b} \\
&= \sum_{i=1}^{n} \boldsymbol{x}_i y_i - \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i' \left(\sum_{j=1}^{n} \boldsymbol{x}_j \boldsymbol{x}_j' \right)^{-1} \left(\sum_{j=1}^{n} \boldsymbol{x}_j y_j \right) \\
&= \sum_{i=1}^{n} \boldsymbol{x}_i y_i - \sum_{i=1}^{n} \boldsymbol{x}_i y_i \\
&= \boldsymbol{0}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \hat{y}_i e_i &= \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{b} e_i \\
&= \sum_{i=1}^{n} \boldsymbol{b}' \boldsymbol{x}_i e_i \\
&= \boldsymbol{b}' \sum_{i=1}^{n} \boldsymbol{x}_i e_i \\
&= 0.
\end{aligned}
$$

Also, let $\bar{\hat{y}} = \frac{1}{n} \sum_{i=1}^{n} \hat{y}_i$. Then we have

$$
\begin{aligned}
\bar{\hat{y}} &= \frac{1}{n} \sum_{i=1}^{n} \hat{y}_i \\
&= \frac{1}{n} \sum_{i=1}^{n} (y_i - e_i) \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{1}{n} \sum_{i=1}^{n} e_i \\
&= \bar{y}.
\end{aligned}
$$

Now, denote the sample correlation coefficient between y_i and $\hat{y}_i$ by r. We have

$$
\begin{aligned}
r^2 &= \left(\frac{\sum_{i=1}^n (y_i - \bar{y})(\hat{y}_i - \bar{\hat{y}})}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{\hat{y}})^2}} \right)^2 \\
&= \left(\frac{\sum_{i=1}^n (y_i - \bar{y})(\hat{y}_i - \bar{y})}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2}} \right)^2 \\
&= \left(\frac{\sum_{i=1}^n (\hat{y}_i + e_i - \bar{y})(\hat{y}_i - \bar{y})}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2}} \right)^2 \\
&= \left(\frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n e_i(\hat{y}_i - \bar{y})}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2}} \right)^2 \\
&= \left(\frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n \hat{y}_i e_i - \bar{y} \sum_{i=1}^n e_i}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2}} \right)^2 \\
&= \frac{\left(\sum_{i=1}^n (\hat{y}_i - \bar{y})^2 \right)^2}{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2} \\
&= \frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
&= R^2.
\end{aligned}
$$