Year 1 Laboratory Manual
Main Cycles:
Resonance in an LCR circuit

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1. Investigating Resonance in an LCR Circuit

1.1 Introduction

This experiment uses electrical circuits to demonstrate some of the important features of resonance between a driving force and a damped electrical oscillator. You will plot out the resonance curve for this system and observe how the response of the oscillator peaks sharply when the driving frequency is the same as its resonant frequency. In order to do this, you will set up an LCR circuit consisting of an inductor (L), a capacitor (C) and a resistor (R), which you will then drive with an oscillating voltage signal.

Resonance, which occurs when we drive a oscillating system close to it natural oscillation frequency is a crucial phenomena throughout physics. Some examples of where resonance plays an important role include: pushing a child on a swing, a loudspeaker driven by an amplifier, and atoms in a molecule driven by an electromagnetic wave.

1.2 Physical Principles

The LCR circuit is the electrical equivalent of a damped mechanical oscillator, for example a mass on a spring (see Figure 1.2). In the case of an idealised simple harmonic oscillator, when the system is displaced from its equilibrium position, it feels a restoring force that is proportional to this displacement. This effect is described by the equation

$$F = -kx, \tag{1.1}$$

where k represents the spring constant for the example of a mass on a spring. However, real harmonic oscillators experience damping or energy loss (typically friction for a mechanical oscillator) that works to reduce the oscillations of the system. This results in an additional term in the equation of motion

$$F = -kx - b\frac{dx}{dt}, \tag{1.2}$$

where c is known as the viscous damping coefficient. For the case of the mass on a spring, the frictional damping force is proportional to the velocity of the motion (dx/dt). Rewriting $F = ma = m\frac{d^2x}{dt^2}$, the equation of motion for the damped mechanical oscillator can be written as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.3}$$

If we try to force an oscillator to oscillate, by applying an external, time varying force $F(t)$, the equation of motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)\,. \tag{1.4}$$

The amplitude of a driven oscillator depends on how close the frequency of the driver is to the natural oscillation frequency of the oscillator. Driving close to the natural frequency produces a strong response, which we call *resonance*. The amplitude of the oscillation is given by:

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}}, \tag{1.5}$$

where A is the amplitude of oscillation at angular frequency ω; F_0 is the amplitude of the driving force; $\omega_0 = \sqrt{k/m}$ is the resonant angular frequency; and $\gamma = b/m$ is a term describing the the damping in the system.

In a driven oscillator, the phase of the driver and the phase of the oscillator are not the always the same. At low frequency ($\omega \ll \omega_0$) the oscillation is in phase with the driver. At high frequency ($\omega \gg \omega_0$) the oscillation and the driver are out of phase. The phase difference between the driver and the oscillator is given by:

$$\phi(\omega) = \arctan\left(\frac{-\gamma\omega}{\omega_0^2 - \omega^2}\right). \tag{1.6}$$

These equations are plotted in figure 1.1.

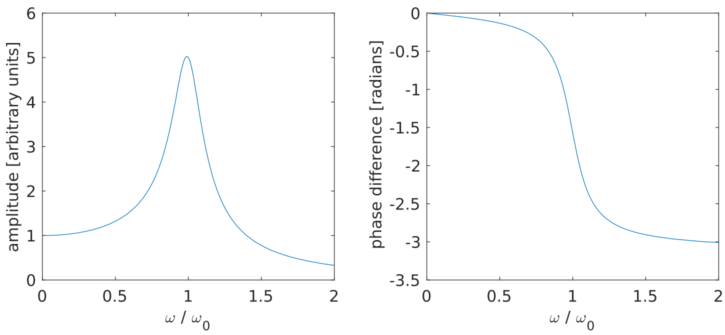

Figure 1.1: Amplitude and Phase of a Forced Oscillator as a function of frequency

The above discussion was about a mechanical oscillator. An LCR circuit is an *electrical* oscillator and behaves analogously to the damped mechanical oscillator. The inductor is analogous to the mass, the capacitor is analogous to the spring and the resistor is analogous to the viscous damping. The Appendix shows how the same equation that governs a driven mechanical oscillator also applies to the LCR circuit. In this experiment you will use an LCR circuit to investigate some of the important features of resonance between a driving force and an oscillator.

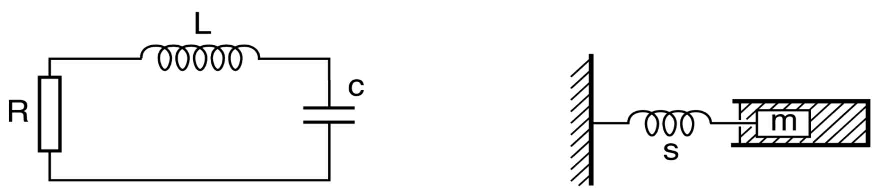

Figure 1.2: A simple LCR circuit (*left*) and a simple mechanical oscillator (*right*).

1.3 Experimental Procedure

The Circuit

In this experiment you will plot out a resonance curve for a driven LCR circuit and observe how the response of the oscillator reacts to changes in the driving frequency. In particular, you will see how the response of the oscillator peaks sharply when the driving frequency is close to the resonant frequency of the circuit. In order to do this, you will need to drive the circuit and then detect the amplitude of oscillation of charge in the circuit.

Set up the circuit as shown in Figure 1.3 using the 1 mH inductor, the 100 nF capacitor and the 1 Ω resistor. Connect the two-way coaxial splitter to the signal generator output of the oscilloscope, then use a coaxial cable to connect the output directly into channel 1 so that you can observe the driving signal. Connect channel 2 of the oscilloscope across the capacitor, so that the measured signal is directly proportional to the charge on the capacitor.

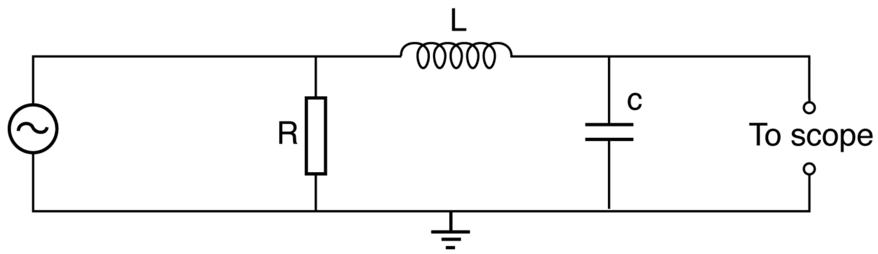

Figure 1.3: Circuit diagram showing the setup of the LCR circuit.

Estimating the Resonant Frequency

In order to work out the range of driving frequencies over which you will need to take measurements to plot out the resonance curve, you will first need to calculate the expected resonant frequency of the circuit where the response of the oscillator should peak strongly. The resonant frequency is given theoretically by the expression

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}. \tag{1.7}$$

Consider the strategy you will use for taking your measurements. Over what frequency range are you going to take measurements? How many measurements are you going to take? Are you going to take your measurements in equal increments across the entire frequency range, or are there certain regions that will require data to be taken at larger increments?

Measuring the Resonance Curves

- Set up the signal generator on the oscilloscope to generate a sinusoidal wave with a frequency at the lower end of your chosen frequency range, with a peak-peak voltage of 2V.
- Set the oscilloscope to measure the peak-peak amplitude the voltage across the capacitor.
- Set the oscilloscope to measure the phase difference between the drive signal and the voltage across the capacitor.
- record the amplitude of the voltage across the capacitor and phase difference for a range of drive frequencies to map the resonance curves.
- As always it is good practice to plot your measurements as you go so (e.g. in Python or Excel) so that you can easily keep track of the shape of your resonance curves and the

measurements you've taken and make any adjustments to your data collection strategy that are necessary.

Resonance and Damping

In order to evaluate how your resonance curve is affected by the damping in the circuit, you will now repeat the measurement above using the $2\,\Omega$ resistor rather than the $1\,\Omega$ resistor. Retake the measurements and plot out the resonance curve again. If you have time, repeat the experiment again with the $3\,\Omega$ resistor.

Once you have the measurements for your all your resonance curves make sure you save the data in a format that you are comfortable reading into Python scripts.

1.4 Data Analysis

The Resonance Curves

Load your data into Python and plot a graph of the peak-peak voltage vs. driving frequency for each of the resistors that you used, commenting on the shape of your resonance curves. How does the response of the oscillator relate to the phase difference between the driven and response signals? How do the shapes of the resonance curves depend on the damping provided by the different resistors? Why has the shape changed in this way?

The general expression for the resonance curve of an arbitrary driven, damped oscillator follows the functional form

$$A(\omega) = \frac{N}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}}, \tag{1.8}$$

where A represents the amplitude of the resonance curve as a function of the angular frequency ω, N is an amplitude factor that depends on the amplitude of the driver, ω_0 is the angular resonant frequency of the circuit and γ is a term relating to the damping in the system.

Using SciPy functions, fit this functional form to your resonance curves. One of the fit parameters should be the resonant frequency agree with the theoretical value that you calculated earlier?

Repeat a similar analysis but for the phase difference between the driver and oscillator. We expect the phase difference to be

$$\phi(\omega) = \arctan\left(\frac{-\gamma\omega}{\omega_0^2 - \omega^2}\right), \tag{1.9}$$

using SciPy functions, fit this functional form to your phase difference curves. Does the

Calculating the Quality Factor

The Quality factor of an oscillator, Q, describes how 'sharp' the peak in the resonance curve is. More formally, Q relates the energy stored in the circuit to the energy dissipated during each oscillation. Theoretically, the Q factor of an LCR circuit is given by

$$Q = \frac{\omega_0 L}{R}. \tag{1.10}$$

The Q factor can be determined experimentally from measurements of the resonant frequency ω_0 and the width of the resonance curve $\Delta\omega$ (defined as the distance in frequency between two points where the amplitude has reduced to $1/\sqrt{2}$ of its maximal value):

$$Q = \frac{\omega_0}{\Delta\omega}. \tag{1.11}$$

First, calculate the theoretical Q factor for your circuit from equation 1.10 for each resistor. Then, use the best fitting resonance curves you produced in the previous part to measure $\delta\nu$ and ν_0 for your circuit, and hence measure Q from equation 1.11 for each resistor. Compare the experimental values and the theoretical values, commenting on whether the agreement is good or not, and why you think this is the case.

Now consider the different Q factors you calculated for the different resistors in the circuit. How does the Q factor vary as a function of the damping in the circuit? Is this as you expected?

Often in physics experiments we have a mathematical model of our experiment which seems sound. However, when we compare our model to the data, we find unexpected differences. One major cause of such discrepancies is that the assumptions made in our model are not accurate – the experiment is more complicated than we initially realised.

Appendix: Physical Principles

Equation of Motion for an LCR Circuit

An LCR circuit is the electrical analogue of a damped mechanical oscillator. This becomes apparent if you consider the total voltage across the circuit. According to Kirchhoff's Voltage Law, the total voltage around a loop is equal to the sum of the voltage drops within the loop. For a simple LCR circuit, such as the one shown in Figure 1.2, all of the components are in series, and so Kirchhoff's Voltage Law implies that:

$$V_T = V_R + V_L + V_C. \tag{1.12}$$

The equations governing the voltage drops across our individual components are $V_R = IR$ for the resistor, $V_L = LdI/dt$ for the inductor and $V_C = Q/C$ for the capacitor. Rewriting V_C in terms of the current using $Q = \int I(t)dt$ and substituting these into the above equation yields

$$V_T = IR + L\frac{dI}{dt} + V_0 + \frac{1}{C}\int I(t)dt, \tag{1.13}$$

where V_0 is the initial voltage across the capacitor at time $t = 0$. Taking the time derivative of this equation yields

$$\frac{dV_T}{dt} = R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{I}{C}. \tag{1.14}$$

For this experiment, we are interested in the case of an LCR circuit driven by a sinusoidal voltage signal. In this case, $V_T = V_0 sin(\omega t)$, and substituting this into the above equation yields

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \omega V_0 cos(\omega t) \tag{1.15}$$

Compare the terms in this equation to equation 1.4 for the driven damped mechanical oscillator. What are the analagous roles of the different components in the LCR circuit compared to the damped mechanical oscillator? This should make it more clear why an LCR circuit can be considered the electrical analogue of the damped mechanical oscillator.