Chapter 2.2:
Liability-Driven Investing

Contents

The previous chapters have introduced diversification and hedging as two risk management techniques with well-defined objectives. Diversification aims to efficiently harvest risk premia across and within asset classes, while hedging aims to reduce risk with respect to the present value of liabilities or goals. This chapter argues that these techniques are not mutually exclusive, and should in fact be properly integrated with each other to reconcile the conflicting objectives of having high performance and low risk.

The liability-driven investing (LDI) paradigm, which is supported by portfolio theory and has now become the accepted approach to asset-liability management, recommends that investors should in general hold a combination of a well-diversified performance-seeking portfolio (PSP), which targets diversification, and a safe liability/goal-hedging portfolio (LHP/GHP), which takes care of hedging. Moreover, the relative weights should be a function of market conditions, as well as strategic objectives and constraints faced by investors.

The analysis of the LDI paradigm in these Lecture Notes starts with a discussion of *heuristic* strategies that allocate a fixed percentage of the investor's assets to the PSP and the LHP. We then point that while the LDI principle advocates separation between performance and hedging, investor's welfare not only depends on the PSP performance and the hedging qualities of the LHP, but also on the hedging properties of the PSP and the LHP performance. This *interaction result* suggests in particular that an investor may benefit from improving liability-hedging characteristics of their performance-seeking portfolios. We finally move on to the analysis of *optimal* LDI strategies that maximize investor's welfare according to a liability-driven objective. To this end, we introduce multi-period portfolio optimization models, which are naturally suited to the analysis of long-term portfolio decisions. These models differ from the one-period optimization models used in the previous chapters in that they reflect investors' ability to rebalance their portfolios over the investment horizon.

1. A Basic Form of Liability-Driven Investing (LDI) Strategies

> Liability-driven investing (LDI) strategies combine two building blocks, a liability-hedging portfolio (LHP) that tracks the value of liabilities in order to reduce relative risk, and a performance-seeking portfolio (PSP) aiming to generate upside potential. A simple form of LDI strategies is a fixed-mix combination of the two building blocks, in which the allocation to the PSP is adjusted in order to reach a target relative risk level.

We introduce in this section a basic form of liability-driven investing strategy, which is a fixed-mix combination of a PSP and a LHP. Beyond the choice of the building blocks and the relative allocation to them, one of the key implications of the LDI paradigm is that one should make a distinction between two different levels in asset allocation decisions: allocation decisions involved in the design of the PSP and the LHP, and asset allocation decisions involved in the split between the two building blocks. While each level of analysis involves its own challenges, some of which have been discussed in the previous chapters, the LDI paradigm is now widely adopted in institutional money management, and its counterpart in individual money management, the goal-based investing (GBI) paradigm, is increasingly regarded as the relevant approach to investment decisions for individual investors.

Building Blocks

Consider an institutional investor who faces liabilities, the market value of which is denoted by L. For a defined-benefit pension fund, L would represent the present value of promised pension payments to beneficiaries. There are two main concerns when it comes to managing assets in relation to liabilities. On the one hand, performance is needed to increase funding ratio levels without requiring excessively high levels of contributions. This quest for performance gives an incentive for liability-driven investors to turn to equity markets and other risky asset classes. On the other hand, risk management is needed to ensure that the likelihood and severity of a possible drop in funding ratio remains within acceptable limits, which gives an incentive for liability-driven investors to turn to fixed-income markets and hold duration-matched bond portfolios. In the end, investors facing liabilities are left with the usual dilemma between performance and risk, where risk is measured here against the present value of liabilities.

To reconcile the two conflicting objectives, the liability-driven investing (LDI) principle proposes to combine two building blocks, each of which is the best attempt to respond to one of the two concerns. The safety motive justifies the introduction of a customized liability-hedging portfolio (LHP), the purpose of which is to replicate the present value of liabilities: this is usually done by matching the exposures of liabilities to the main risk factors, most notably interest rate and inflation risks. In parallel, the performance motive justifies the introduction of a performance-seeking portfolio (PSP), which aims to provide efficient access to risk premia across and within asset classes. It can be constructed by any of the diversification methods introduced in Chapters 1.1 to 1.3. At this stage, it is assumed that each building block is designed with only one concern in mind, namely replicating liabilities or delivering long-term performance, regardless of how the two building block portfolios interact with each other. We defer until Section 3 below the analysis of the question whether investors having the choice between several PSPs may benefit from picking the one with the best liability-hedging properties.

Allocation Policy

For simplicity, we assume a fixed-mix allocation to the two building blocks, and we denote by x the fraction of assets allocated to the PSP, and $1-x$ the fraction invested in the LHP. This strategy is not intended to be optimal in any sense, but it can be regarded as a simple combination of two building blocks that are useful in asset-liability management. In practice, it is also possible to use cash as a third building block, which allows in particular for leveraged forms of LDI strategies where a pension fund would allocate say 100% of the available assets to the LHP and 30% to the PSP, thus implying a 30% leverage. From an operational standpoint, leverage can be implemented via borrowing, or via the use of derivatives-based implementation of the LHP which would leave some of the assets available for investment in the PSP.

The parameter x is a degree of freedom that controls the relative risk of the portfolio with respect to liabilities. With $x = 0\%$, the strategy is fully invested in the LHP and aims to replicate liabilities, that is to minimize relative risk. On the other hand, $x = 100\%$ implies no concern over hedging. In a formal model of investor's preferences leading to the derivation of a welfare-maximizing strategy (see Section 4 below), x can be related to the "risk tolerance": a greater risk tolerance implies a larger allocation to the PSP, which, in the theoretical optimal strategy, is the maximum Sharpe ratio portfolio. However, risk tolerance is not observable for individual investors, and is not even well-defined for institutional investors. Indeed, considering the case of a corporate defined-benefit pension plan for example, the multiplicity of stakeholders (regulators, pension fund managers, beneficiaries, trustees, shareholders of the sponsor company, bondholders of the sponsor company, etc.), who often have conflicting interests, would make it impossible to define a single risk-aversion parameter even if it was possible to measure risk-aversion for any one of these stakeholders.

In practice, the value of x is chosen by fixing *risk budgets*, represented as target levels of tracking error with respect to liabilities, shortfall probabilities, expected shortfalls or values-at-risk at various horizons. An advantage of the tracking error is that it does not involve any expected return estimate, while non-symmetric risk measures like shortfall indicators, which only penalize downside risk, are sensitive to the assumptions on the future performance of the PSP and the present value of liabilities.

2. Measuring Investor's Welfare for Heuristic LDI Strategies

Investor's welfare can be measured through a quadratic utility function of the final funding ratio. This measure aggregates the preference for high funding levels and aversion for uncertainty over the funding ratio.

We now want to have a quantitative measure of welfare for an investor following the fixed-mix LDI strategy, so as to identify which characteristics of the PSP and the LHP have an impact on this welfare. As a general rule, a welfare measure has to capture appetite for performance and aversion for risk, in both cases relative to liabilities. This can be achieved via a suitable extension of mean-variance analysis where the quantity of interest is not wealth, but the value of assets relative to liabilities, measured with the funding ratio $F = A/L$. Mathematically, welfare measures therefore rely on the expectation and the variance of the funding ratio, which can be obtained by modeling asset and liability returns.

Modeling Assets and Liabilities

Current date is denoted as date 0, and the horizon (in years) at which assets and liabilities are modeled is denoted by T. For mathematical tractability, it is convenient to assume that the fixed-mix portfolio is rebalanced *continuously*, rather than monitored at discrete dates (e.g., at the quarterly frequency) as is the case in practice. We assume that the PSP has the same annualized expected logarithmic return m_S and the same annualized volatility σ_S at all horizons. Assuming away horizon effects means in particular that returns measured over non-overlapping periods are statistically independent, so there is no predictability in returns. Predictability will be introduced in Chapter 2.3. Similarly, the present value of liabilities has expected return m_L and volatility σ_L. Finally, the correlation between PSP returns and liability returns, also assumed to be independent of the measurement horizon, is a constant ρ. The last assumption is that the LHP perfectly replicates liabilities, so it has expected return m_L and volatility σ_L.

Under these conditions, it is shown in Section 1 of the Technical Supplement that the logarithmic return of the asset portfolio is the weighted sum of the logarithmic returns of the two constituents plus a *rebalancing premium* Λ. If A denotes asset value, we have[1]

$$\log \frac{A_T}{A_0} = x \log \frac{S_T}{S_0} + [1 - x] \log \frac{L_T}{L_0} + \Lambda. \tag{2.1}$$

The rebalancing premium is always positive, and an explicit expression is given in Section 1 of the Technical Supplement.

Computing Investor's Welfare

The investor is concerned with the relative performance of its assets with respect to liabilities, as measured by the funding ratio $F = A/L$. The twofold concern over upside potential and relative risk is captured by a quadratic utility function, which values high expected funding levels and penalizes funding ratio uncertainty. For mathematical convenience, the function is applied to the logarithm of the funding ratio, rather than the funding ratio itself, so it is given by

$$U = \mathbb{E}\left[\log F_T\right] - \frac{\gamma - 1}{2} \mathbb{V}\left[\log F_T\right],$$

where γ is the risk aversion parameter, $\mathbb{E}$ denotes expectation and $\mathbb{V}$ is the variance operator. To interpret the parameter γ, consider a joint change in the expected funding ratio and in the variance, ΔE and ΔV. It is easy to check that investor welfare is unaffected by these changes if, and only if, we have $\Delta E = [\gamma - 1]/2 \times \Delta V$. If risk increases, expected performance must increase too, and the re-

1 – Some authors adopt a different definition for the rebalancing premium, as the excess return of a rebalanced portfolio over a buy-and-hold portfolio.

quired change in expected return is proportional to risk aversion. In other words, more risk averse investors require a higher compensation to be indifferent to an increase in risk.

It is shown in Section 1 of the Technical Supplement that the variance and the expected value of the logarithmic funding ratio are

$$\mathbb{V}[\log F_T] = x^2 TE_S^2 T, \tag{2.2}$$

$$\mathbb{E}[\log F_T] = \log F_0 + x m_{S/L} T - \frac{1}{2} x^2 TE_S^2 T, \tag{2.3}$$

where TE_S is the tracking error of the PSP with respect to liabilities and $m_{S/L}$ is a constant equal to the expected excess logarithmic return of the PSP over the LHP plus one half of the squared tracking error, and as such can be interpreted as a risk-adjusted expected excess log return of the PSP with respect to the LHP. As expected, the funding ratio variance in Equation (2.2) is zero when $x = 0$, that is if the portfolio is fully invested in the LHP.

Substituting Equations (2.2) and (2.3) in the definition of quadratic utility, we obtain the following quantitative measure for investor's welfare:

$$U = \log F_0 + x m_{S/L} T - \frac{\gamma}{2} x^2 TE_S^2 T. \tag{2.4}$$

It is then possible to find the allocation x that maximizes welfare. This is done in Section 1 of the Technical Supplement, and the optimal allocation is shown to be

$$x^* = \frac{m_{S/L}}{\gamma TE_S^2}.$$

So, the optimal allocation to the PSP is decreasing in risk aversion, which makes sense because more risk-averse investors are more concerned with the relative risk of their asset portfolio. It is also decreasing with the PSP tracking error with respect to the liabilities, which means that if the investor is given the choice between two PSPs with identical upside potential – so that the term $m_{S/L}$ is the same for both – but different tracking errors, then he/she can allocate more to the one with the lower tracking error. This observation is related to the "interaction result" that we discuss in Section 3 below.

3. A "Fund Interaction" Result

> The "fund interaction result" says that investor's welfare can be increased by decreasing the tracking error of the PSP with respect to the present value of liabilities, unless this better alignment of the PSP with liabilities is achieved at too high a cost in terms of PSP performance.

In this section, we show that although the PSP is intended to generate long-term outperformance with respect to liabilities, not to hedge liabilities, investors can benefit in theory from improving the correlation between the PSP and the present value of their liabilities. This property is called an "interaction result" because it involves the interactions of the performance-seeking building block with liabilities.

Benefits of Liability-Friendly PSP with Heuristic LDI Strategies

What should the investor do if given the choice between several PSPs? Because the PSP is primarily intended to outperform liabilities in the long run, it seems to make sense to pick the one with the highest expected return or Sharpe ratio, but what eventually matters is the utility derived from the fixed-mix portfolio invested in the PSP and the LHP. So, the PSP should be chosen so as to maximize the quadratic utility function. This prescription is incomplete, however, because utility also depends on how much is allocated to the PSP versus the LHP. More generally, welfare does not only depend on the intrinsic properties of the building blocks, but also on their correlations and on the allocation strategy.

Suppose that there are two PSPs. The "base case" one can be thought of as a standard equity index, such as the cap-weighted S&P 500 index, and the "alternative one" is an alternative equity index, which may be based on a selection of stocks and/or a weighting scheme different from capitalization weighting. Define two strategies, 0 and 1, that start from the same initial capital, and proceed as follows:

- Strategy 0 allocates the weight x_0 to the base case equity index;

- Strategy 1 allocates the weight x_1 to an alternative equity index.

There are two degrees of freedom, corresponding to the weights x_0 and x_1. To fix one, let us choose x_1 in such a way as to equate the variances of the funding ratios in the two strategies. By Equation (2.2), this "variance-matching allocation" is given by

$$x_1 = \frac{TE_{S0}}{TE_{S1}} x_0. \tag{3.1}$$

Equation (3.1) shows that for a given risk budget, formally defined here as the variance of the log terminal funding ratio, one can allocate more to the new equity index if it has lower tracking error with respect to liabilities than the base case one. Under the condition of equal variances, the change in quadratic utility reduces to the difference between the expected logarithmic returns on the two funding ratios, and, by Equation (2.3), it can be rewritten as

$$\begin{aligned} \Delta U &= \mathbb{E}[\log F_T^1] - \mathbb{E}[\log F_T^0] \\ &= x_0 \left[\frac{TE_{S0}}{TE_{S1}} m_{S1/L} - m_{S0/L} \right] T. \end{aligned}$$

Thus, the sign of ΔU depends only on the tracking errors and the expected returns of the two equity indices with respect to the present value of liabilities. If the new equity index has lower tracking error, then the investor benefits from the change if the risk-adjusted expected log return on the new index is greater than or equal to TE_{S0}/TE_{S1} times that of the original index.[2] As a consequence, the investor can tolerate a loss in expected return as long as this loss does not offset the gain in tracking error. If the new index happens to have both lower tracking error and higher risk-adjusted expected log return, then the utility gain is unambiguously positive.

Practical Implications

> The tracking error of an equity portfolio with respect to liabilities can be reduced by selecting stocks that are more "bond-like" than the average, such as low volatility or high dividend yield stocks. Alternative weighting schemes like portfolio variance minimization also help.

Since equities typically form a large fraction of the PSP and interest rates are a major source of risk in the present value of liabilities, especially those with long durations, one might wonder if cross-sectional differences in interest-rate sensitivities of stock returns can be used to construct more "liability-friendly" equity benchmarks. Coqueret,

2 – The risk-adjusted expected log return is defined in Section 2 as the expected log return on the index plus one half of the tracking error with respect to liabilities.

Table 1: Comparison of cap-weighted index and alternative equity benchmarks.

Strategy	Tracking error (%)	Volatility (%)	Correlation (%)	Ann. return (%)	Sharpe ratio	Turnover (%)
All / CW	18.8	17.3	1.5	10.9	0.42	4.4
All / MV-C	16.1	14.3	2.4	13.5	0.69	30.8
Low Vol / EW	14.6	13.0	7.7	13.2	0.73	27.5
Low Vol / MV-C	14.1	12.5	8.0	13.0	0.74	51.9
High DY / EW	17.9	16.2	1.9	n.a.	0.62	23.3
High DY / MV-C	15.6	14.0	5.5	n.a.	0.73	43.5
DY200-VOL100 / EW	15.0	13.5	6.9	13.6	0.74	24.9
DY200-VOL100 / MV-C	14.1	12.5	8.2	13.6	0.79	46.0

Source: Tables 10 and 12 in Coqueret et al. (2014).
The investment universe is the S&P 500 over the 1975-2012 period (data is from CRSP). "All" refers to the universe of the S&P 500; "Low Vol" to the selection of the 100 least volatile stocks; "High DY" to the selection of the 100 stocks with the highest dividend yield; "DY200-VOL100" to the double-sort selection of the 200 stocks with the highest dividend yield and then the 100 least volatile amongst these 200 high dividend yield stocks. "CW" denotes the cap-weighted portfolio; "MV-C" is a minimum variance portfolio with weights constrained to lie between $1/(3 \times 500)$ *and* $3/500$*; "EW" is an equally-weighted portfolio. Liabilities are represented by a bond with constant maturity of 15 years, i.e. a roll-over of 15-year zero-coupon bonds. Bond returns are calculated as in Section 4.1 of Campbell, Chan, and Viceira (2003). Values marked as "n.a." are not reported in the study.*

Martellini, and Milhau (2017) address this question by showing that it is indeed possible to construct equity portfolios with better liability-hedging properties, as measured by the tracking error or the correlation with liabilities, than a broad equity index weighted by market capitalization of equally weighted. These portfolios are based on a selection of stocks and/or an alternative weighting scheme.

Table 1 shows examples borrowed from the long version of the study, by Coqueret et al. (2014). The first selection criterion is low volatility. Indeed, despite their long duration, liabilities are in general less volatile than stocks, like bonds are. So, stocks with lower volatilities are expected to be closer to liabilities, and to make an equity portfolio more "liability-friendly", it is natural to seek to tilt it towards low volatility stocks. This proves to be effective at reducing the tracking error with respect to liabilities, but also at increasing the correlation. In addition, the investor benefits from the "low volatility anomaly" documented by Ang et al. (2006) and Ang et al. (2009): low volatility stocks tend to outperform the broad market. Here, the low volatility portfolios outperform the broad cap-weighted index (the S&P 500 index) both in average performance and in Sharpe ratio.

The second selection criterion is the dividend yield, and is motivated by the premise that stocks that pay substantial dividends on a regular basis behave somewhat like fixed-income securities. Selection of high dividend stocks also reduces the tracking error with respect to the broad cap-weighted index, but not as much as the selection of low volatility securities. It is of course possible to combine the two selection criteria, as is done in the last two rows of the table. Finally, changing the weighting scheme to variance minimization, as opposed to equal weighting, always leads to a reduction in tracking error and an increase in correlation, while still improving the Sharpe ratio.

Coqueret, Martellini, and Milhau (2017) also test other selection criteria like the leverage ratio, the firm size, the book-to-market, the market beta or the empirical beta with respect to interest rate changes (an empirical measure of the "equity duration"), some of which also prove to be useful to identify stocks with the best liability-hedging abilities. Finally, they show how the distribution of the funding ratio for a pension fund is improved by using such alternative equity indices in place of the default S&P 500 index in the PSP.

4. From Heuristic to Optimal LDI Strategies

> The LDI strategy that maximizes expected utility for a risk-averse investor when investment opportunities (volatilities, expected returns and correlations) are constant is a fixed-mix LDI strategy whose building blocks are the maximum Sharpe ratio (MSR) portfolio, the portfolio that maximizes the squared correlation with the present value of liabilities, and the cash account. The optimal percentage allocation to the MSR portfolio is decreasing in risk aversion.

We conclude this chapter by presenting an optimal (in the sense of utility maximization) LDI strategy, which is a fixed-mix strategy of three funds: a PSP, which is the maximum Sharpe ratio portfolio; a LHP, which is the correlation-maximizing portfolio; and a cash account. This section keeps the mathematical burden to a minimum, and technical details are banned to Section 2 of the Technical Supplement. An accessible presentation of the mathematical theory can be found in reference textbooks such as Cvitanić and Zapatero (2004) for example (see their Chapter I.4).

Framework and Investment Universe

To calculate optimal portfolios with explicit mathematical expressions, the continuous-time framework introduced by Merton (1969) is commonly used. The main assumptions are that:

- There are no frictions such as transaction costs, limited market capacity or restrictions on short sales and leverage;
- Investors are able to trade *continuously*, e.g. every second if needed.

The investment universe contains N securities said to be "risky" because their returns over a trading interval ranging from date t to date $t+h$ are not known as of date t. In contrast, the cash account is said to be "risk-free" because its return between dates t and $t+h$ is the short-term interest rate that prevails at date t. Were h equal to one quarter, the short-term rate could be thought of as the interest rate on 3-month US Treasury bills. In the continuous-time model, h is infinitesimally small, so a real-world analogy for h is one night, and the short-term interest rate is the overnight rate, e.g. the Federal Funds Rate in the United States or the Libor in the United Kingdom. The cash account is akin to a deposit account that earns the continuously compounded short-term rate. All other assets like stocks, bonds, futures contracts, shares of mutual funds... fit in the category of risky assets.

For each risky security, the return between dates t and $t+h$ is decomposed into an "expected" part and an "unexpected" term. For instance, if S is the price of a security, the realized return is written as

$$\frac{S_{t+h}}{S_t} - 1 = [r_t + \mu_t]h + \varepsilon_{t+h}, \qquad (4.1)$$

where μ_t is the expected excess return over the short-term rate as of date t, expressed in annual terms, and ε_{t+h} is an innovation, the value of which is only revealed at the end of the period, hence the subscript $t+h$. Note that μ_t is the expected excess return *conditional* on the information available at date t. The present value of liabilities can be decomposed in a similar way, whatever the exact nature of these liabilities, which does not need to be specified in the model.

Equation (4.1) is a tautology and does not constitute a model until some restrictions are put on the short-term rate, the expected return and the innovation. For the derivation of the optimal LDI strategy, it is assumed that:

- Conditional expected excess returns (i.e., risk premia) are constant over time;
- The conditional volatilities of innovations are also constant;
- And so are the correlations between the innovations of various securities.

These assumptions apply to all risky assets and to the present value of liabilities. Remarkably, nothing is said of the short-term rate, which, in this model, is allowed to vary randomly over time.

The Optimization Problem

An investor starts with an amount A_0 of assets to be invested in the N risky securities and in the cash account. w_{it} denotes the weight allocated to security S_i at date t, and the fraction invested in the cash account is 1 minus the sum of the other weights. For a more concise notation, the weights of the risky assets are stacked in a column vector $\mathbf{w}_t$.

In asset-liability management, the concern is over the performance of assets relative to liabilities and the relative risk of the portfolio. It can be captured in the "expected utility" of the funding ratio, that is the ratio of assets to liabilities, at the terminal date: Mathematically, the utility function is increasing and concave: monotony means that the investor prefers high to low funding ratios, and concavity is related to risk aversion. Indeed, it can be shown that concavity implies the following property: if an investor is given the choice between two distributions of the future funding ratio with the same expected value, but one is risky and the other one has no risk (so that it takes a single value), then the expected utility of the latter is greater. In other words, an investor considering a risky distribution of the future funding ratio prefers to have the expected value for sure. A non-concave function would imply that the two expected utilities are equal, so that the investor would be indifferent between the risky and the risk-free distributions as long as they have the same mean, and he/she would be "risk-neutral".

A standard specification for the utility function in the literature on optimal portfolio choice is the *constant relative risk aversion* function, defined as

$$U(F) = \frac{F^{1-\gamma}}{1-\gamma}.$$

The parameter γ represents risk aversion. The investor's objective is them to maximize expected utility by finding an optimal investment policy. We emphasize that the solution to the utility maximization program is an investment *policy*, which specifies how to invest from date 0 all the way to date T, not just how to invest at date 0. In general, the weights to implement at a given point in time may depend on the business conditions that prevail at this date, as well as the asset and liability values. Of course, these conditions and values are not known as of date 0, when optimization is performed, but the investment policy specifies how the optimal weights depend on them. However, as we shall see below, the constant relative risk aversion function implies that the optimal weights do not change over time.

Optimal LDI Strategy: Three-Fund Separation Theorem

The expected utility maximization program can be solved by the duality technique, which is also employed by Martellini and Milhau (2012) to solve a similar portfolio optimization problem. Section 2 in the Technical Supplement goes through the mathematical derivation, and the main results are:

- The optimal weights in the N risky assets are constant over time, so the optimal strategy is a fixed-mix;
- The optimal strategy can also be regarded as a fixed-mix portfolio of three synthetic assets: the maximum Sharpe ratio (MSR) portfolio; a liability-hedging portfolio that maximizes the squared correlation with changes in the present value of liabilities; the cash account.

The second property is a three-fund separation theorem. The exact mathematical statement is as follows. Let $\mathbf{w}^*$ denote the optimal vector of weights in the risky assets at any point in time (it is time-independent), let $\mathbf{w}_{MSR}$ be the MSR portfolio of the risky assets and let $\mathbf{w}_{LHP}$ be the correlation-maximizing portfolio. In fact, the third portfolio maximizes the *squared* correlation, not the correlation itself: in most cases, the two criteria are equivalent because the LHP is positively correlated with liabilities, but for the sake of generality, it is more correct to specify that the criterion that the LHP maximizes is the squared correlation. Let also λ_{MSR} and σ_{MSR} be the Sharpe ratio and the volatility of the MSR portfolio, and $\beta_{L/LHP}$ be the beta of the liabilities with respect to the LHP. Then, we have

$$\mathbf{w}^* = \frac{\lambda_{MSR}}{\gamma \sigma_{MSR}} \mathbf{w}_{MSR} + \left[1 - \frac{1}{\gamma}\right] \beta_{L/LHP} \mathbf{w}_{LHP},$$

As shown in the Technical Supplement, the term $\beta_{LHP}\mathbf{w}_{LHP}$ can be rewritten as $\mathbf{w}_{TE}$, which is the tracking error-minimizing portfolio. Unlike the MSR portfolio and the LHP, which are fully invested in the risky assets, this portfolio may also contain cash.

This result provides formal justification for the liability-driven investing paradigm by showing that an optimal allocation strategy in the presence of liabilities involves a customized LHP and a PSP providing investors with a well-diversified, efficient, access to risk premia across and within asset classes. In practice, the LDI strategy differs from the optimal one in several ways. First, the PSP is a well-diversified portfolio, which is a broader objective than Sharpe ratio maximization, because expected returns are very difficult to estimate. Second, the LHP is often a factor-matching portfolio rather than a correlation-maximizing portfolio: if the risk factors with matched exposures make up for a large fraction of the variances of the portfolio and the present value of liabilities, there is little difference between the two. Third, cash can be excluded from the practical LDI strategy.

It is instructive to look at limit cases for risk aversion. If it is infinite, then the investor maximizing expected utility is fully invested in the LHP and cash. If the beta is 1, cash disappears from the portfolio, so the investor is only concerned with minimizing the tracking error with respect to liabilities, which makes intuitive sense. On the other hand, for a unit risk aversion, the investor holds the MSR portfolio plus some cash, and does not invest in the LHP. For intermediate levels of risk aversion, the optimal policy is a fixed-mix combination of the MSR portfolio, the LHP and the cash account.

The relative weighting of the three funds also depends on some key characteristics of the funds. The allocation to the MSR portfolio is increasing in the Sharpe ratio and decreasing in volatility, which again is in line with intuition. The allocation to the LHP is increasing in the beta: a zero beta would imply that assets are useless for the purpose of hedging liability risk, so the allocation to the LHP would be zero too.

Fund Interaction Result

In Section 3 of the Technical Supplement, we calculate the *indirect utility*, which is the expected utility for an investor who follows the optimal LDI strategy, or, equivalently, the maximum possible welfare. This quantity is different from the quadratic utility of Section 2 evaluated at the optimal fixed-mix policy, because the two frameworks are different: here, a constant relative risk aversion utility function is assumed, instead of a quadratic utility, and the LHP is not assumed to perfectly replicate liabilities.

In spite of these differences, a similarity exists because the expression given in the Technical Supplement shows that the maximum welfare is increasing in the correlation between the MSR portfolio and liabilities. In the quadratic utility case, welfare is decreasing in the tracking error of the PSP with respect to liabilities, so it is increasing in the correlation between PSP returns and liabilities. Thus, a "fund interaction result" is recovered with the optimal strategy as with the heuristic one.

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