Modern Portofolio Theory

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1 Markowitz Optimization and the Efficient Frontier

Think of the plane whose x-axis is a variance and y-axis is a expected return.
A set of possible returns of portfolios can be denoted by a curve which pass through the original two portfolios.
By adding a neww asset to the options, it dramatically expands the range of the possible returns and variances.
The optimized portfolio should locate on the upper boundary of the region, and the upper edge is called as **Efficient Portfolio**

2 Applying quadprog to draw the efficient Frontier

Since we use the same plane with the first section, we have to get an expression of an expected return and a variance of the portfolio.

$$R_p = \sum_{i=1}^{k} w_i R_i$$

$$\sigma_p^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} w_i w_j \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{k}\sum_{j=1}^{k} w_i w_j \sigma_{ij}$$

where R_i is an expected return for the asset i and σ_i is a variance for the asset i.
If there are more than 2 assets, the return of the portfolio will be

$$R_p = w^T R$$

where w^T is a weight vector and R is the asset returns.
To find the efficient frontier, we need to solve these equation

$$\min \frac{1}{2} w^T \Sigma w$$

$$s.t. w^T R = r_0, w^T = 1, w \geq 0$$

3 Fund Separation Theorem and the Capital Market Line

Capital Market Line

The efficient frontier changes it's shape dramatically when a risk free asset is introduced.

The **Capital Market Line** is a tangency line of the efficient frontier which pass through the risk-free rate. This line has a mazimized sharp ratio.

Since the sharp ratio is

$$SR_p = \frac{\mu_p - r_f}{\sigma_p} = \frac{\sum_{i=1}^{N} w_i \mu_i - r_f}{\sqrt{\sum_{i,j=1}^{N} w_i w_j \sigma_{ij}}}$$

the weights of a tangency portfolio is given by

$$\arg\max_{w} SR_p$$

4 Lack of robustness of Markowitz analysis

Estimation error is a main challenge of portfolio optimization.

Some uncertaincy always exist in a parameter estimation.

In order to tackle with the estimation error, people think of the global minimum variance portfolio. This GMV is the portfolio with the least variance in the eddicient fromtier, and is least sensitive to estimation errors.