Instrumental Variables based DREM for Online Asymptotic Identification of Perturbed Linear Systems

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***Abstract*— Existing online continuous-time parameter estimation laws provide exact (asymptotic/exponential or finite/fixed time) identification of dynamical linear/nonlinear systems parameters only if the external perturbations are equaled to zero or represented as a vanishing and absolutely integrable function of time. However, real systems are generally affected by non-zero and non-decaying disturbances, in the presence of which the above-mentioned identification approaches ensure only boundedness of a parameter estimation error. The main goal of this study is to close this gap and develop a novel online continuous-time parameter estimator, which guarantees exact asymptotic identification of unknown parameters of linear systems in the presence of unknown but bounded perturbations. To achieve the aforementioned goal, it is proposed to augment the deeply investigated Dynamic Regressor Extension and Mixing (DREM) procedure with the novel Instrumental Variables (IV) based extension scheme with averaging. Such an approach allows one to obtain a set of scalar regression equations with asymptotically vanishing perturbation even if the initial disturbance that affects the plant is only bounded. It is rigorously proved that a gradient estimation law designed on the basis of such scalar regressions ensures online asymptotic identification of the parameters of the perturbed linear systems if the disturbance and control input do not include signals with common frequencies, which is a weak assumption for applications. Theoretical results are illustrated and supported with adequate numerical simulations.**

***Index Terms*— adaptive control, identification, unknown parameters, perturbations, extension and mixing.**

I. Introduction

THE aim of the adaptive control methods is a real-time control of dynamical systems with unknown parameters [1]–[3]. As far as the methodology of this theory is concerned, a control law is designed in two stages. At the first one, an ideal control law is chosen that ensures the achievement of the control goal under the assumption that the system parameters are known. At the second stage, in accordance with the principle of certainty equivalence, the unknown parameters in the chosen control law are substituted with their dynamic estimates obtained with the help of adaptive laws, which are designed using direct or indirect approaches. According to the direct method, the adaptive law is designed to adjust the controller parameters directly. In accordance with the indirect approach, first, the system parameters are identified, and then they are recalculated into the controller parameters using algebraic equations. To design the adaptative law, in both cases the second Lyapunov method [2] and parametric identification algorithms (the gradient law [4], various variants of the recursive least squares method [5]) are used.

Long standing question in the sense of the above-mentioned identification and adaptive control problems is to define conditions to guarantee robustness of a parameter estimator or/and overall closed-loop adaptive control system to external bounded perturbations, unknown dynamics and system parameter variations [6]–[9]. It is well known [6] that the basic adaptive and identification laws ensure uniform ultimate boundedness (UUB) of the tracking and parameter estimation errors in the case when the persistence of excitation (PE) requirement with sufficiently large level of excitation is met. However, these results are impractical due to two following main problems.

MP1. Considering the linear systems, the PE condition is met if and only if the system input includes n not equal to each other frequencies, which is restrictive for the practical scenarios.

MP2. Only UUB of the tracking and parameter estimation errors is ensured in the presence of external perturbations even if the PE condition is satisfied.

To solve the first problem, the adaptive control community has proposed many robust modifications (σ, e, projection operator, dead zone etc.), which ensure required UUB [1], [2], [6]–[9] even if the PE condition is not met. Moreover, to close **MP1** a great effort has recently been made [4], [10] to relax the PE condition to exponential stability requirement, which is sufficient [3, p. 327] to ensure robustness (UUB) of the closed-loop adaptive control system in the presence of external perturbations. However, all approaches under consideration do not ensure convergence of the parameter estimation and/or tracking errors to zero even if an arbitrary small time-dependent non-vanishing perturbation affects the system, but this is *a vita* necessary to achieve acceptable control quality and identification accuracy in practice. Considering the adaptive control framework, to cope with the external perturbations, a methodology [3], [11] has been developed that uses the internal model principle to parameterize a perturbation

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in the form of a linear regression equation with respect to some *new* unknown parameters with higher dimension. However, such approach has three main drawbacks: *i*) the overall closed-loop system or/and system parameterization are complicated due to *overparametrization*, *ii*) the internal models can be used for description of relatively small class of perturbations (constants, harmonic and exponentially decaying signals, and their combinations), and *iii*) *a priori* knowledge of the disturbance type is needed to apply the internal model principle.

Generally, the origin of sensitivity of the adaptive control systems to unknown but bounded time-dependent perturbations is rooted in the properties of the adaptive and estimation laws, which are applied to solve the adaptive control and identification problems. Particularly, to the best of authors' knowledge, considering the continuous time adaptive control literature [1]–[3], there is no online identification law, which provides exact asymptotic identification of the unknown parameters (for example, of the linear time-invariant plants) in the presence of external unknown but bounded perturbations. Recently proposed robust identifiers with finite/fixed convergence time and relaxed excitation requirement [12], [13] also provide only boundedness of the parametric error in the presence of an external disturbance. So, the main topic of this study is to close this gap and propose continuous-time online estimation law, which ensures exact asymptotic identification of the unknown parameters of linear systems. It is necessary to remember that, if the law with such property is obtained, then in the future it could potentially be *mutatis mutandis* modified and applied to solve any kind of adaptive and identification problems for continuous-time not necessarily linear systems.

Now we give some small description of the mechanisms and approaches that will be applied in the paper to achieve the stated goal. First of all, using the state variables filters that are well-known in the adaptive control literature [1]–[3], a linear time-invariant system affected by a disturbance is parameterized in the form of a linear regression equation (LRE) with unknown but bounded perturbation [1, p. 99]. Then, the obtained LRE is extended with the help of a novel extension scheme, which is obtained via a combination of: *i*) the sliding window extension scheme [14, Lemma 1], *ii*) the instrumental variables method [15]–[19] and *iii*) the filtering with averaging [20]. The next ingredient of the proposed estimation design procedure is the mixing step [4] from the DREM method, which transforms the extended linear regression equation into a set of separate scalar regression equations. If the system disturbance and control input do not include signals with common frequencies, the instrumental variable approach and filtering with averaging ensure that the new perturbations in such scalar equations converge asymptotically to zero with the rate $\frac{1}{t+F_0}$, $F_0 > 0$. Being applied to the obtained set of scalar regression equations, the gradient-descent-based estimator allows one to guarantee the asymptotic convergence of the unknown parameters estimation error to zero.

The rest of the study is organized as follows. Section II introduces the main definitions and notation. A rigorous problem formulation is shown in Section III. The essence of instrumental variable approach is discussed in Section IV. In Section V the main result is elucidated. The simulation results are listed in Section VI. Appendix A is to give the main definitions of the harmonic analysis. Proofs of the main results of the study are presented in Appendix B.

Notation and Definitions. The below-given definitions and notation are used to present the main result of this study.

Definition 1. *The regressor $\varphi(t) \in \mathbb{R}^n$ is persistently exciting ($\varphi \in \mathrm{PE}$) if $\exists T > 0$ and $\alpha > 0$ such that $\forall t \geq t_0 \geq 0$ the following inequality holds:*

$$\int\limits_{t}^{t+T} \varphi(\tau)\varphi^{\mathrm{T}}(\tau)\, d\tau \geq \alpha I_n. \tag{1}$$

Definition 2. *Signals $f(t) \in \mathbb{R}$ and $g(t) \in \mathbb{R}$ are called independent if it holds that:*

$$\left| \int\limits_{t_0}^{t} f(s)\, g(s)\, ds \right| < \infty. \tag{2}$$

Otherwise, the signals $f(t) \in \mathbb{R}$ and $g(t) \in \mathbb{R}$ are called dependent.

Let us illustrate what signals satisfy definition 2. For that it is assumed that $f(t) \in \mathbb{R}$ and $g(t) \in \mathbb{R}$ are defined as follows:

$$\begin{array}{l} f(t) = \sum\limits_{i=1}^{n_f} \alpha_i^f \sin\left(\omega_i^f t\right) + \beta_i^f \cos\left(\omega_i^f t\right), \\ g(t) = \sum\limits_{i=1}^{n_g} \alpha_i^g \sin\left(\omega_i^g t\right) + \beta_i^g \cos\left(\omega_i^g t\right). \end{array} \tag{3}$$

Equation (3) is, in fact, the Fourier series with a finite number of terms. Then, as the following inequalities

$$\begin{array}{l} \left| \int\limits_{t_0}^{t} \alpha_i^f \sin\left(\omega_i^f s\right) \alpha_i^g \sin\left(\omega_i^g s\right) ds \right| < \infty, \\ \left| \int\limits_{t_0}^{t} \beta_i^f \cos\left(\omega_i^f s\right) \beta_i^g \cos\left(\omega_i^g s\right) ds \right| < \infty \end{array} \tag{4}$$

hold when $\omega_i^f \neq \omega_i^g$, then it is sufficient to meet the condition $\omega_i^f \neq \omega_j^g$, $\forall i \in \overline{1,\, n_f}$, $j \in \overline{1,\, n_g}$ to ensure that the signals $f(t) \in \mathbb{R}$ and $g(t) \in \mathbb{R}$ are independent. It should be noted that this is a sufficient condition as, even if there exist i and j such that $\omega_i^f = \omega_j^g$, then nevertheless the signals can meet the condition (2) owing to the combination of their amplitudes. As an example, the following signals:

$$\begin{array}{l} f(t) = -\sin\left(\omega_1^f t\right) + \cos\left(\omega_1^f t\right), \\ g(t) = \sin\left(\omega_1^g t\right) + \cos\left(\omega_1^g t\right) \end{array} \tag{5}$$

are independent despite the fact that the above-mentioned sufficient condition is not met. It is further considered in this study that for arbitrary signals of the form (3), a situation similar to (5) is almost surely impossible.

Further the following notation is used: $|.|$ is the absolute value, $\|.\|$ is the suitable norm of $(.)$, $I_{n\times n} = I_n$ is an identity $n \times n$ matrix, $0_{n\times n}$ is a zero $n \times n$ matrix, 0_n stands for a zero vector of length n, $\det\{.\}$ stands for a matrix determinant, $\mathrm{adj}\{.\}$ represents an adjoint matrix.

II. Problem Statement

A perturbed linear dynamical system is considered:

$$\begin{array}{l} y(t) = \frac{Z(\theta,\, s)}{R(\theta,\, s)}\left[u(t) + f(t)\right], \\ Z(\theta,\, s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0, \\ R(\theta,\, s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0 \end{array} \tag{6}$$

together with its parametrization in the form of a linear regression equation [1, p. 99]:

$$z(t) = \varphi^{\mathrm{T}}(t)\,\theta + w(t), \tag{7}$$

where

$$\begin{array}{c} z(t) = \frac{s^n}{\Lambda(s)}y(t), \\ \varphi(t) = \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}y(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}u(t) \end{bmatrix}^{\mathrm{T}}, \\ w(t) = \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix}\frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}f(t), \\ \lambda_{n-1}^{\mathrm{T}}(s) = \begin{bmatrix} s^{n-1} & \cdots & s & 1 \end{bmatrix}, \\ \theta = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & b_0 \end{bmatrix}^{\mathrm{T}} \end{array}$$

and $y(t)$ is a measurable output, $u(t)$ stands for a control signal, $f(t)$ denotes an unknown disturbance, $\theta \in \mathbb{R}^{2n}$ is a vector of unknown parameters, $\varphi(t) \in \mathbb{R}^{2n}$ denotes a measurable regressor, $s := \frac{d}{dt}$, $\Lambda(s)$ is a monic Hurwitz polynomial of the order n. If $Z(\theta, s)$ is of degree $m < n-1$, then the coefficients b_i, $i \in \overline{n-1,\, m+1}$ are equalled to zero.

Considering the control, output and disturbance signals, the following assumptions are adopted.

Assumption 1. *The control $u(t)$, output $y(t)$ and disturbance $f(t)$ are bounded.*

Assumption 2. *The control $u(t)$ and disturbance $f(t)$ are defined as follows:*

$$\begin{array}{l} u(t) = \sum\limits_{k=1}^{n} \rho_k \sin(\omega_k^u t),\, \omega_i^u \neq \omega_j^u\ \forall i \neq j \\ f(t) = \sum\limits_{k=1}^{n_f} \delta_k \sin\left(\omega_k^f t\right),\, \omega_i^u \neq \omega_j^f\ \forall i, j, \end{array} \tag{8}$$

where the number n_f, amplitudes δ_j and frequencies ω_j^f are unknown.

The aim is to design an estimation law, which ensures that the following equality holds:

$$\lim_{t\to\infty}\left|\tilde{\theta}_i(t)\right| = \lim_{t\to\infty}\left|\hat{\theta}_i(t) - \theta_i\right| = 0\ \forall i \in \overline{1, 2n}, \tag{9}$$

where $\hat{\theta}_i(t)$ is an estimate of the i^{th} unknown parameter.

Remark 1. *The above-stated problem* (9) *is well-studied and solved as far as literature on the offline Identification of Continuous-time Models from Sampled Data [17] and finite-frequency identification theory [21] is concerned. However, the algorithms from [17] are based on the sampling of continuous time signals, and the method from [21] requires to solve a set of linear algebraic equations numerically. At the same time, to the best of authors' knowledge, the solution of the stated problem does not exist in the literature on the online adaptive control [1]–[13]. Unlike [17] and [21], in this study the problem* (9) *is solved online in truly continuous-time and in time domain.*

Remark 2. *In contrast to the methods from [3], [11], in this paper the problem* (9) *is solved without: (i) parameterisation of the disturbance $f(t)$ into the form of a linear regression, (ii) knowledge of the value of n_f, and (iii) the need to identify the extended vector of unknown parameters. Moreover, the knowledge of the structure of the perturbation $f(t)$ is necessary rather for the purposes of formal analysis, and, if the below defined conditions are met, then the proposed solution ensures that the goal* (9) *is achieved even if the disturbance has a more complicated structure.*

III. The essence of instrumental variable approach

The aim of this section is to study the feasibility of the stated goal (9). To this end, an offline direct-least-squares-based estimation law is considered, which uses all data that are available along the time axis:

$$\hat{\theta}(t) = \begin{cases} 0, \text{ if } \det\left\{\int\limits_{t_0}^{t}\varphi(s)\,\varphi^{\mathrm{T}}(s)\,ds\right\} = 0, \\ \left[\int\limits_{t_0}^{t}\varphi(s)\,\varphi^{\mathrm{T}}(s)\,ds\right]^{-1}\int\limits_{t_0}^{t}\varphi(s)\,z(s)ds, \text{ else} \end{cases} \tag{10}$$

As the input signal is defined as (8), then the condition $\varphi \in \mathrm{PE}$ is met [22] and the following limit holds:

$$\lim_{t\to\infty}\int\limits_{t_0}^{t}\varphi(s)\,\varphi^{\mathrm{T}}(s)\,ds = \infty,$$

and, consequently, the estimation law (10) ensures that the goal (9) is achieved if the elements of the regressor $\varphi(t)$ and perturbation $w(t)$ are independent, i.e.:

$$\forall i \in \overline{1, 2n}\ \left|\int\limits_{t_0}^{t}\varphi_i(s)\,w(s)\,ds\right| \leq c < \infty\ \forall t \geq t_0, \tag{11}$$

where $c > 0$ is an arbitrary constant.

Proposition 1. *Let assumptions 1 and 2 be met, then there exists $i \in \overline{1, 2n}$ such that:*

$$\lim_{t\to\infty}\left|\int\limits_{t_0}^{t}\varphi_i(s)\,w(s)\,ds\right| = \infty. \tag{12}$$

Proof of proposition 1 is postponed to Appendix.

Thus, since the signals $\varphi_i(t)$ as well as the perturbation $w(t)$ are generated by the disturbance $f(t)$, then there exists an element of the regressor that violates the condition (11). Therefore, in case $\varphi \in \mathrm{PE}$, the law (10) guarantees only boundedness of $\tilde{\theta}_i(t)$. This result is illustrated by a simple example.

Example 1. A scalar system is considered

$$\dot{x}(t) = -x(t) + u(t) + f(t), \tag{13}$$

with $u(t) = \sin(t)$, $f(t) = \sin(\pi t)$.

For the sake of simplicity, it is assumed that $t_0 = 0$, $x(t_0) = 0$ and $\dot{x}(t)$ is measurable. Then the parametrization (7) of the system (13) takes the form:

$$\begin{gathered} z(t) = \dot{x}(t), \\ \varphi(t) = \begin{bmatrix} -x(t) & u(t) \end{bmatrix}^{\mathrm{T}}, w(t) = f(t). \end{gathered} \tag{14}$$

Having solved equation (13), it is obtained after the end of the transient:

$$x(t) = \tfrac{2\sin(\pi t) + \left(1+\pi^2\right)(\sin(t) - \cos(t)) - 2\pi\cos(\pi t)}{2\pi^2+2}, \tag{15}$$

which clearly proves that the condition (11) is not met:

$$\begin{array}{l} \left|\int\limits_{t_0}^{t} -x(s)\, w(s)\, ds\right| \le \tfrac{2}{2\pi^2+2}\left|\int\limits_{t_0}^{t} \sin^2(\pi s)\, ds\right| < \\ < t - \tfrac{1}{2\pi}\sin(2\pi t), \\ \left|\int\limits_{t_0}^{t} u(s)\, w(s)\, ds\right| = \left|\tfrac{\sin(\pi t)\cos(t) - \pi\cos(\pi t)\sin(t)}{\pi^2-1}\right| < \infty. \end{array}$$

The recursive forms of the law (10) are often applied to design adaptive control systems [5], which, as there is no convergence (9), allows one to achieve only the boundedness of the parametric error in case the system is affected by an external disturbance.

To eliminate the dependence between the perturbation and the regressor elements, and hence to ensure that the goal (9) is achieved, in Identification of Continuous-time Models from Sampled Data theory [16], [17] the method of *instrumental variables*, which was first described in the context of identification in [15], is widely applied. According to this method, an instrumental variable $\zeta(t) \in \mathbb{R}^{2n}$ is introduced into consideration such that:

$$\left[\int\limits_{t_0}^{t} \zeta(s)\, \varphi^{\mathrm{T}}(s)\, ds\right]^{-1} \to 0 \text{ as } t \to \infty \tag{16}$$

and

$$\forall i \in \overline{1, 2n} \left|\int\limits_{t_0}^{t} \zeta_i(s)\, w(s)\, ds\right| \le c < \infty \; \forall t \ge t_0. \tag{17}$$

This allows one to introduce the following estimation law:

$$\hat{\theta}(t) = \begin{cases} 0, \text{ if } \det\left\{\int\limits_{t_0}^{t} \zeta(s)\, \varphi^{\mathrm{T}}(s)\, ds\right\} = 0, \\ \left[\int\limits_{t_0}^{t} \zeta(s)\, \varphi^{\mathrm{T}}(s)\, ds\right]^{-1} \int\limits_{t_0}^{t} \zeta(s)\, z(s)\, ds, \text{ else} \end{cases} \tag{18}$$

which, owing to (16) and (17), guarantees that the goal (9) is met.

Example 2. The scalar system (13) and its parametrization (14) are considered. The following instrumental variable model is introduced to obtain the instrumental variable $\zeta(t) \in \mathbb{R}^2$:

$$\begin{gathered} \dot{x}_{iv}(t) = -a x_{iv}(t) + b u(t), \, a > 0, \\ \zeta(t) = \begin{bmatrix} -x_{iv}(t) & u(t) \end{bmatrix}^{\mathrm{T}}, \end{gathered} \tag{19}$$

where (after the end of a transient)

$$x_{iv}(t) = \tfrac{a b\sin(t) - b\cos(t)}{a^2+1}. \tag{20}$$

Then, using simple but tedious mathematical manipulations, it can be proved that the condition (16) is met and shown that (17) is bounded:

$$\begin{array}{l} \left|\int\limits_{t_0}^{t} -x_{iv}(s)\, w(s)\, ds\right| \le \\ \le \tfrac{2}{a^2+1}\left|\int\limits_{t_0}^{t} \sin(\pi s)\left[a b\sin(s) - b\cos(s)\right] ds\right| = \\ = \left|2\tfrac{\sin(\pi t)[b\sin(t) + a b\cos(t)] + \pi(\cos(\pi t)[b\cos(t) - a b\sin(t)] - b)}{(a^2+1)(\pi^2-1)}\right| < \infty, \\ \left|\int\limits_{t_0}^{t} u(s)\, w(s)\, ds\right| = \left|\tfrac{\sin(\pi t)\cos(t) - \pi\cos(\pi t)\sin(t)}{\pi^2-1}\right| < \infty. \end{array}$$

Thus, the method of instrumental variables allows one to obtain the estimation law (18), which ensures that the goal (9) is achieved. However, the law (18) is algebraic and belongs to offline methods, which means that it is not suitable for application to solve online identification problems and design adaptive control systems due to numerical and computational problems. Therefore, there is an actual problem of design of an online recursive estimation law that guarantees the achievement of the goal (9). In the next section of this study such a law is developed on the basis of the dynamic regressor extension and mixing procedure [4], the considered method of instrumental variables [15]–[19] and the filtering with averaging [20].

IV. Main Result

The description of the main result of this study is divided into two parts. Subsection A discusses the schemes to generate dynamical instrumental variables. In Subsection B a modified dynamic regressor extension and mixing scheme is proposed on the basis of the idea of regressor extension with the help of instrumental variable and filtering with averaging. According to this procedure, the regression equation (7) is reduced to a set of scalar ones $\mathcal{Y}_i(t) = \Delta(t)\theta_i + \mathcal{W}_i(t)$, in which the perturbation $\mathcal{W}_i(t)$ is asymptotically decreasing if the condition (17) is satisfied. The gradient estimation law is applied to identify the parameters of the obtained regression equations.

Some basic definitions from harmonic analysis (see Appendix A) are used to derive the results of this section.

A. Dynamic generation of instrumental variable

In accordance with Section III, the instrumental variable needs to meet the conditions (16) and (17) to ensure achievement of the goal (9). In the literature on Identification of Continuous-time Models from Sampled Data [16], [17], about four methods of instrumental variable generation have been proposed and proved. In this study, the continuous-time analogues of two of them are described.

1) Linear time-invariant instrumental model. According to this approach, it is proposed to form $\zeta(t)$ with the help of the following auxiliary filter:

$$\begin{gathered} y_{iv}(t) = \tfrac{Z(\theta_{iv}, s)}{R(\theta_{iv}, s)} u(t), \\ \zeta(t) = \begin{bmatrix} -\tfrac{\alpha_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} y_{iv}(t) & \tfrac{\alpha_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u(t) \end{bmatrix}^{\mathrm{T}}, \end{gathered} \tag{21}$$

where θ_{iv} are known parameters of the instrumental model such that $R(\theta_{iv}, s)$ is a Hurwitz polynomial.

The following propositions hold for the instrumental variable (21).

Proposition 2. *Let $u(t)$ be stationary and assume that*

1) $H(j\omega_1), \ldots, H(j\omega_{2n})$ *are linearly independent in $\mathbb{C}^{2n}$ for all $\omega_1, \ldots, \omega_{2n}$ such that $\omega_i \neq \omega_k$ for $i \neq k$;*
2) $rank\{[M(j\omega_1), \ldots, M(j\omega_{2n})]\} = 2n$ *for all $\omega_1, \ldots, \omega_{2n}$ such that $\omega_i \neq \omega_k$ for $i \neq k$.*

Here

$$H(s) = \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s)Z(\theta,\, s)}{\Lambda(s)R(\theta,\, s)} & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} \end{bmatrix}^{\mathrm{T}},$$
$$M(s) = \begin{bmatrix} G(s)I_n & 0 \\ 0 & I_n \end{bmatrix},$$

and $G(s)$ is such that $\frac{G(s)Z(\theta,\, s)}{R(\theta,\, s)} = \frac{Z(\theta_{iv},\, s)}{R(\theta_{iv},\, s)}$.

Then there exist $T>0$ and $\alpha>0$ such that

$$\left|\det\left\{\int\limits_{t}^{t+T} \zeta(\tau)\varphi^{\mathrm{T}}(\tau)\,d\tau\right\}\right| \geq \alpha > 0,\ \forall t \geq t_0, \tag{22}$$

if and only if $u(t)$ is sufficiently rich of order $2n$.

Proof of proposition 2 is presented in Appendix

Proposition 3. *Let assumptions 1 and 2 be met, then the instrumental variable* (21) *meets the condition* (17).

Taking into consideration the proof of proposition 1, it immediately follows that proposition 3 holds.

Proposition 2 is an analog of the classical result [22], in which necessary and sufficient conditions for the regressor $\varphi(t)$ to be persistently exciting were formulated. In comparison with the condition $\varphi \in$ PE, the existence of a controller $G(s)$ providing $\frac{G(s)Z(\theta,\, s)}{R(\theta,\, s)} = \frac{Z(\theta_{iv},\, s)}{R(\theta_{iv},\, s)}$, as well as the independence of $M(j\omega_1), \ldots, M(j\omega_{2n})$ are additionally required to satisfy the inequality (22) and then (16). These new conditions are not restrictive. On the other hand, in proposition 3 the *sufficient* conditions to satisfy (17) are formulated.

The fact that the premises of propositions 1 and 2 hold allows one to *bona fide* use the instrumental variable (21) for further design of the estimation law.

2) Adaptive instrumental model. Using model (21), the error $\tilde{\zeta}(t) = \zeta(t) - \varphi(t)$ that prevents the condition (22) from being satisfied in case $\zeta \in$ PE is caused by the difference between the parameters of model (21) and system (6), and the presence of a perturbation $f(t)$ in (6). In the course of pursuing the goal (9), it is possible to asymptotically reduce $\tilde{\zeta}(t)$, which motivates the application of an adaptive instrumental model:

$$\begin{array}{l} y_{iv}(t) = \frac{Z(\hat{\theta},\, s)}{R(\hat{\theta},\, s)}u(t), \\ \zeta(t) = \begin{bmatrix} -\frac{\alpha_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}y_{iv}(t) & \frac{\alpha_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}u(t) \end{bmatrix}^{\mathrm{T}}. \end{array} \tag{23}$$

The fact that the conditions similar to (16), (17) are met when the model (23) is used was proved for Continuous-time Models with Sampled Data, for example, in [18], [19]. Considering the sampling time to be infinitesimal, these results can be directly transferred to the continuous-time case under consideration. Therefore, in this study the proof that the conditions (16) and (17) are met when (23) is in use is omitted.

Thus, based on the dynamical instrumental models (21) or (23), the instrumental variable $\zeta(t)$, which satisfies the conditions (17) and (22), can be calculated.

B. Instrumental variables based dynamic regressor extension and mixing procedure

In recent years, the procedure of dynamic regressor extension and mixing (DREM) has attracted great attention [4], as it allows one to design the estimation law with relaxed convergence conditions in comparison with the classical approaches and provides an improved transient quality of the parameter estimates. DREM consists of two main stages: dynamic regressor extension and mixing. In the first step, the original equation is transformed by linear operations and dynamical operators into a regression equation with a new regressor, which is a square matrix. At the second stage, an algebraic transformation is applied to transform the equation obtained at the extension stage into a set of scalar independent from each other equations with respect to the components of the vector θ. In this study, it is proposed to use a novel mixing scheme in DREM procedure obtained as a combination of the sliding window extension scheme [14, Lemma 1] with the instrumental variables approach [15]–[19]:

$$\begin{array}{l} \dot{\vartheta}(t) = \zeta(t)z(t) - \zeta(t-T)z(t-T), \\ \dot{\psi}(t) = \zeta(t)\varphi^{\mathrm{T}}(t) - \zeta(t-T)\varphi^{\mathrm{T}}(t-T), \\ \vartheta(t_0) = 0_{2n},\, \psi(t_0) = 0_{2n\times 2n}, \end{array} \tag{24a}$$

and filtering with averaging [20]:

$$\begin{array}{l} \dot{Y}(t) = -\frac{1}{F(t)}\dot{F}(t)\left(Y(t) - \vartheta(t)\right), \\ \dot{\Phi}(t) = -\frac{1}{F(t)}\dot{F}(t)\left(\Phi(t) - \psi(t)\right), \\ \dot{F}(t) = pt^{p-1}, \\ Y(t_0) = 0_{2n},\, \Phi(t_0) = 0_{2n\times 2n},\, F(t_0) = F_0 > 0, \end{array} \tag{24b}$$

where $T>0$ denotes a sliding window width, $p \geq 1$, $F_0 > 0$ stand for the filter parameters.

Proposition 4. *Let $\theta = const$, then the signals $Y(t)$ and $\Phi(t)$ form the following LRE:*

$$Y(t) = \Phi(t)\theta + W(t), \tag{25}$$

where the new disturbance $W(t)$ satisfies the equations:

$$\begin{array}{l} \dot{W}(t) = -\frac{1}{F(t)}\dot{F}(t)\left(W(t) - \varepsilon(t)\right), \\ \dot{\varepsilon}(t) = \zeta(t)w(t) - \zeta(t-T)w(t-T), \\ W(t_0) = 0_{2n},\, \varepsilon(t_0) = 0_{2n}. \end{array} \tag{26}$$

Proof of proposition is given in Appendix.

Considering the mixing step, the left- and right-hand sides of equation (25) are multiplied by an adjoint matrix $\mathrm{adj}\{\Phi(t)\}$, which, owing to $\mathrm{adj}\{\Phi(t)\}\Phi(t) = \det\{\Phi(t)\}I_{2n\times 2n}$, allows one to obtain a set of scalar separable regression equations:

$$\mathcal{Y}_i(t) = \Delta(t)\theta_i + \mathcal{W}_i(t), \tag{27}$$

where

$$\mathcal{Y}(t) := \operatorname{adj}\{\Phi(t)\} Y(t),\ \Delta(t) := \det\{\Phi(t)\},$$
$$\mathcal{W}(t) := \operatorname{adj}\{\Phi(t)\} W(t),$$
$$\mathcal{Y}(t) = \begin{bmatrix} \mathcal{Y}_1(t) & \dots & \mathcal{Y}_{i-1}(t) & \dots & \mathcal{Y}_{2n}(t) \end{bmatrix}^{\mathrm{T}},$$
$$\mathcal{W}(t) = \begin{bmatrix} \mathcal{W}_1(t) & \dots & \mathcal{W}_{i-1}(t) & \dots & \mathcal{W}_{2n}(t) \end{bmatrix}^{\mathrm{T}}.$$

The mixing scheme (24a) in case $\tilde{\zeta}(t) = 0$ can be reduced to the sliding window extension scheme from [14]. However, the application of the averaging filters (24b) and choice of the instrumental variable as (21) or (23) provide the regressor $\Delta(t)$ and disturbances $\mathcal{W}_i(t)$ with the new properties, which are described in the following proposition.

Proposition 5. *The following statements hold:*

1) *if* (22) *is met, then* $|\Delta(t)| \geq \Delta_{\mathrm{LB}} > 0\ \forall t \geq T$,
2) *if assumption 1 is satisfied, then* $|\Delta(t)| \leq \Delta_{\mathrm{UB}}\ \forall t \geq t_0$,
3) *if inequalities* (17) *hold and assumption 1 is met, then for all* $i \in \overline{1, 2n}$ *there exists* $c_{\mathcal{W}} > 0$, *such that:*

$$|\mathcal{W}_i(t)| \leq \frac{\dot{F}(t) c_{\mathcal{W}}}{F(t)} < \infty. \tag{28}$$

Proof of proposition 5 is presented in Appendix.

Thus, the dynamic regressor extension and mixing procedure based on instrumental variables approach and filtering with averaging allows one to obtain a set of scalar regression equations (27), in which the perturbation $\mathcal{W}_i(t)$ satisfies the condition (28), i.e., it asymptotically decreases. Using the obtained set of equations (27), an estimation law can be introduced that ensures that the goal (9) is achieved.

Theorem 1. *Define the estimation law as follows:*

$$\dot{\hat{\theta}}_i(t) = -\gamma \Delta(t) \Big(\Delta(t) \hat{\theta}_i(t) - \mathcal{Y}_i(t)\Big),\ \hat{\theta}_i(t_0) = \hat{\theta}_{0i}. \tag{29}$$

Assume that the following conditions are met (see Propositions 2-5):

1) $|\Delta(t)| \geq \Delta_{\mathrm{LB}} > 0\ \forall t \geq T$ *and* $\Delta_{\mathrm{UB}} \geq |\Delta(t)|\ \forall t \geq t_0$,
2) $|\mathcal{W}_i(t)| \leq \frac{\dot{F}(t) c_{\mathcal{W}}}{F(t)} < \infty$.

Then the stated goal (9) *is achieved.*

Proof of theorem can be found in Appendix.

Therefore, the proposed regressor extension and mixing scheme based on the method of instrumental variables allows one to obtain a regression equation with asymptotically decreasing perturbation, which is used to design the estimation law (29) that ensures the achievement of the goal (9). Unlike the existing online identification laws [1]–[14], the proposed one (21) + (24a) + (24b) + (29) is capable of identifying the exact values of the unknown parameters of linear systems affected by external disturbances. The first condition of exact asymptotic convergence (22) is referred to the modified notion of the persistence of excitation, and the second condition (17) requires independence (in the sence of definition 1) of the perturbation $w(t)$ and instrumental variables $\zeta_i(t)$ from each other. Particularly, how it can be seen from propositions 1, 3 and assumption 2, the independence requirement (17) is not restrictive and satisfied when the control input $u(t)$ and plant perturbation $f(t)$ do not contain common frequencies.

Remark 3. *The state of filters* (24b) *lose their awareness to new values of signals* $\vartheta(t)$ *and* $\psi(t)$ *at the rate of* $\frac{\dot{F}(t)}{F(t)}$. *Completely preventing such loss is not possible at the moment, since the perturbation* $\varepsilon(t)$ *is averaged with the help of the coefficient* $\frac{\dot{F}(t)}{F(t)}$ *ensuring that the condition* (28) *is met.*

However, with the help of periodic reset procedure (t_0, $t_1, \dots, t_k$, $k \in \mathbb{N}$ *is some a priori given sequence):*

$$\dot{F}(t) = p t^{p-1},\ F(t_i) = F_0 > 0,$$

it is possible to regain awareness of the filters (24b).

C. Extension to identification in closed loop

The above-proposed identifier is applicable to the identification of system parameters in an open loop. In case of the closed-loop system, in which the controller is feedback-based one, the control signal $u(t)$ depends on the disturbance, which results in the common frequencies in their spectra and the violation of condition (17) in case the instrumental model (21) is used. In this subsection a method to choose an instrumental model is proposed, which provides condition (17) satisfaction in case of closed-loop identification.

It is assumed that the control signal $u(t)$ is formed as follows:

$$u(t) = \frac{P_y(\kappa, s)}{Q_y(\kappa, s)} y(t) + \frac{P_r(\kappa, s)}{Q_r(\kappa, s)} r(t), \tag{30}$$

where $\kappa \in \mathbb{R}^{n_\kappa}$ are known time-invariant parameters of the control law, $r(t)$ stands for the reference signal, $m_y \leq n_y$ and $m_r \leq n_r$ are orders of the couples of polynomials $P_y(\kappa, s)$, $Q_y(\kappa, s)$ and $P_r(\kappa, s)$, $Q_r(\kappa, s)$, respectively.

The following assumption is adopted in regard with the reference $r(t)$ and disturbance $f(t)$ signals, which is similar to (3).

Assumption 3. *The reference* $r(t)$ *and disturbance* $f(t)$ *signals are defined as follows:*

$$\begin{aligned} r(t) &= \sum_{k=1}^{n} \rho_k \sin(\omega_k^r t),\ \omega_i^r \neq \omega_j^r\ \forall i \neq j, \\ f(t) &= \sum_{k=1}^{n_f} \delta_k \sin\left(\omega_k^f t\right),\ \omega_i^r \neq \omega_j^f\ \forall i, j, \end{aligned} \tag{31}$$

where the value of n_f, *amplitudes* δ_j *and frequencies* ω_j^f *are unknown.*

The instrumental variable $\zeta(t)$ is proposed to be chosen as:

$$\begin{aligned} y_{iv}(t) &= \tfrac{Z(\theta_{iv}, s)}{R(\theta_{iv}, s)} u_{iv}(t), \\ u_{iv}(t) &= \tfrac{P_y(\kappa, s)}{Q_y(\kappa, s)} y_{iv}(t) + \tfrac{P_r(\kappa, s)}{Q_r(\kappa, s)} r(t), \\ \zeta(t) &= \Big[-\tfrac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} y_{iv}(t) \quad \tfrac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u_{iv}(t)\Big]^{\mathrm{T}}, \end{aligned} \tag{32}$$

where θ_{iv} are known parameters of the instrumental model such that the closed-loop model (32) is stable.

In contrast to (21), the instrumental variable (32) does not depend on the control signal $u(t)$, which allows one to ensure that the condition (17) is met when the system (6) is inside a closed loop. Further the analogs of propositions 2 and 3 are formulated for the closed-loop system (6) + (30).

Proposition 6. *Let* $r(t)$ *be stationary and assume that:*

1) *$H(j\omega_1), \ldots, H(j\omega_{2n})$ are linearly independent in $\mathbb{C}^{2n}$ for all $\omega_1, \ldots, \omega_{2n}$ such that $\omega_i \neq \omega_k$ for $i \neq k$;*
2) *$rank\{[M(j\omega_1), \ldots, M(j\omega_{2n})]\} = 2n$ for all $\omega_1, \ldots, \omega_{2n}$ such that $\omega_i \neq \omega_k$ for $i \neq k$.*

Here

$$H(s) = \left[-\frac{\lambda_{n-1}^{\mathrm{T}}(s)W_{cl}(\theta, s)Z(\theta, s)}{\Lambda(s)} \quad \frac{\lambda_{n-1}^{\mathrm{T}}(s)W_{cl}(\theta, s)R(\theta, s)}{\Lambda(s)}\right]^{\mathrm{T}},$$
$$M(s) = \begin{bmatrix} G_1(s) & 0 \\ 0 & G_2(s) \end{bmatrix},$$
$$W_{cl}(\theta, s) = \frac{P_r(\kappa, s)Q_y(\kappa, s)}{Q_r(\kappa, s)[Q_y(\kappa, s)R(\theta, s) - Z(\theta, s)P_y(\kappa, s)]}.$$

and $G_1(s), G_2(s)$ are such that

$$G_1(s)\lambda_{n-1}(s)W_{cl}(\theta, s)Z(\theta, s) = = \lambda_{n-1}(s)W_{cl}(\theta_{iv}, s)Z(\theta_{iv}, s),$$
$$G_2(s)\lambda_{n-1}(s)W_{cl}(\theta, s)R(\theta, s) = = \lambda_{n-1}(s)W_{cl}(\theta_{iv}, s)R(\theta_{iv}, s).$$

Then there exists $T > 0$ and $\alpha > 0$ such that (22) *is met if and only if $r(t)$ is sufficiently rich of order $2n$.*

Proof of proposition 6 is given in Appendix.

Proposition 7. *Let Assumptions 1 and 3 be met, then the condition* (17) *is satisfied.*

Proof of this proposition coincides with the one for proposition 3.

Therefore, when the instrumental model is chosen as (32), the proposed estimation law (24a) + (24b) + (29) ensures exact asymptotic identification of the unknown parameters of system (6) even in a closed-loop case.

Remark 4. *It should be mentioned, that if $Z(\theta, s)$ is stable, then, based on the obtained unknown parameters estimates, the perturbation $f(t)$ can be approximately estimated through the following simple certainty equivalence adaptive disturbance observer:*

$$\hat{f}(t) := \frac{P_f(s)R(\hat{\theta}, s)}{Q_f(s)Z(\hat{\theta}, s)}y(t) - \frac{P_f(s)}{Q_f(s)}u(t),$$

where $P_f(s), Q_f(s)$ are chosen such that the transfer function $\frac{P_f(s)R(\theta, s)}{Q_f(s)Z(\theta, s)}$ is proper and satisfy $\frac{P_f(0)}{Q_f(0)} = 1$.

V. Numerical Simulation

The below-given linear dynamical system of the second order with the piecewise-constant parameters has been considered:

$$y(t) = \begin{cases} \frac{-2s-1}{s^2+s+2}[u(t) + f(t)], & \text{if} \quad t \leq 50 \\ \frac{-4s-2}{s^2+2s+4}[u(t) + f(t)], & \text{if} \quad t \geq 50 \end{cases}, \tag{33}$$

where the control and disturbance signals were defined as follows:

$$\begin{gathered} u(t) = \sin(2\pi t) + \cos(3t), \\ f(t) = 0.25\sin(0.1\pi t) + \sin(4t) + 1. \end{gathered} \tag{34}$$

It should be noted that the disturbance chosen for the experiment was more complex in comparison with the one used in the theoretical analysis (8), which should demonstrate the robustness of the proposed solution. The system parameter switch (33) was to demonstrate that the algorithm (29) is capable of tracking the system parameters change, which is the main motivation for the application of the online identification methods.

For comparison purposes, we also implemented the gradient-based estimation law [1]–[3] and the one designed in accordance with the basic DREM procedure [4]:

$$\dot{\hat{\theta}}(t) = -\Gamma\varphi(t)\left(\varphi^{\mathrm{T}}(t)\hat{\theta}(t) - z(t)\right), \hat{\theta}(t_0) = \hat{\theta}_0, \tag{35a}$$

$$\begin{gathered} \dot{\hat{\theta}}_i(t) = -\gamma\Delta(t)\left(\Delta(t)\hat{\theta}_i(t) - \mathcal{Y}_i(t)\right), \\ \mathcal{Y}(t) := \operatorname{adj}\{\Phi(t)\}Y(t), \Delta(t) := \det\{\Phi(t)\}, \\ \dot{Y}(t) = -lY(t) + \varphi(t)z(t), Y(t_0) = 0_{2n}, \\ \dot{\Phi}(t) = -lY(t) + \varphi(t)\varphi^{\mathrm{T}}(t), \Phi(t_0) = 0_{2n\times 2n}, \end{gathered} \tag{35b}$$

and in addition to this the law on the basis of the signals (24a) mixing was also implemented:

$$\begin{gathered} \dot{\hat{\theta}}_i(t) = -\gamma\Delta(t)\left(\Delta(t)\hat{\theta}_i(t) - \mathcal{Y}_i(t)\right), \\ \mathcal{Y}(t) := \operatorname{adj}\{\psi(t)\}\vartheta(t), \Delta(t) := \det\{\psi(t)\}. \end{gathered} \tag{35c}$$

The parameters of filters (7), (24a), (24b), (35b), instrumental model (21) and estimation laws (29), (35a), (35b), (35c) were picked as follows:

$$\begin{gathered} \Lambda(s) = R(\theta_{iv}, s) = s^2 + 20s + 100, \; F_0 = 0.01, \\ Z(\theta_{iv}, s) = 20s + 100, \; T = 5, \; p = 10, \\ \gamma = \begin{cases} 10^{26}, & \text{for } (29) \\ 10^{20}, & \text{for } (35\text{b}) \\ 10^{26}, & \text{for } (35\text{c}) \end{cases}, \; \Gamma = 100, \; l = 0.1. \end{gathered} \tag{36}$$

Figure 1 depicts the behavior of the regressor $\Delta(t)$ and disturbance $\|\mathcal{W}(t)\|$ obtained with the help of equations (24a), (24b)-(27).

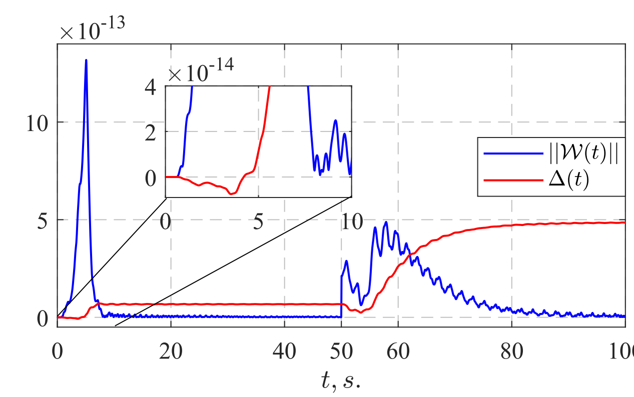

Fig. 1. Behavior of $\boldsymbol{\Delta}(t)$ and $\|\boldsymbol{\mathcal{W}}(t)\|$.

The simulation results presented in Fig. 1 allows one to make the following conclusions:

i) $\Delta_{\mathrm{UB}} \geq |\Delta(t)| \; \forall t \geq 0$,
ii) the condition $|\Delta(t)| \geq \Delta_{\mathrm{LB}} > 0$ is met $\forall t \geq T$,
iii) $\|\mathcal{W}(t)\|$ decreases asymptotically, which is required to meet the inequalities $|\mathcal{W}_i(t)| \leq \frac{\dot{F}(t)c_{\mathcal{W}}}{F(t)} < \infty$.

Figure 2 presents the comparison of behavior of $\hat{\theta}(t)$ obtained with the help of the estimation laws (29) and (35a), (35b), (35c).

The simulation results confirm the theoretical conclusions made in the propositions and theorem 1, and show that,

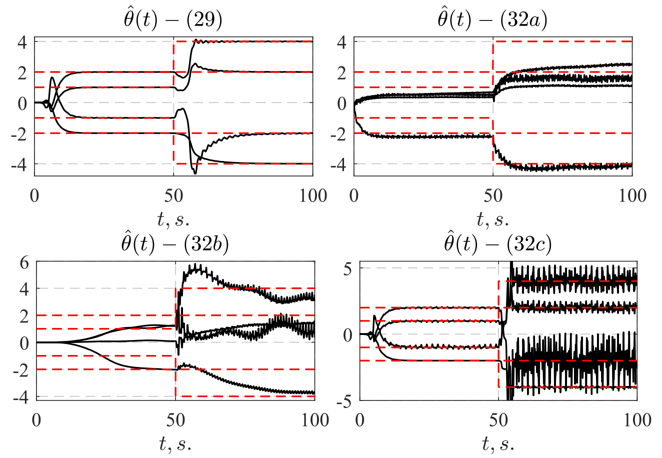

Fig. 2. Behavior of $\hat{\boldsymbol{\theta}}\,(t)$ obtained using estimation laws (29) and (35a), (35b), (35c).

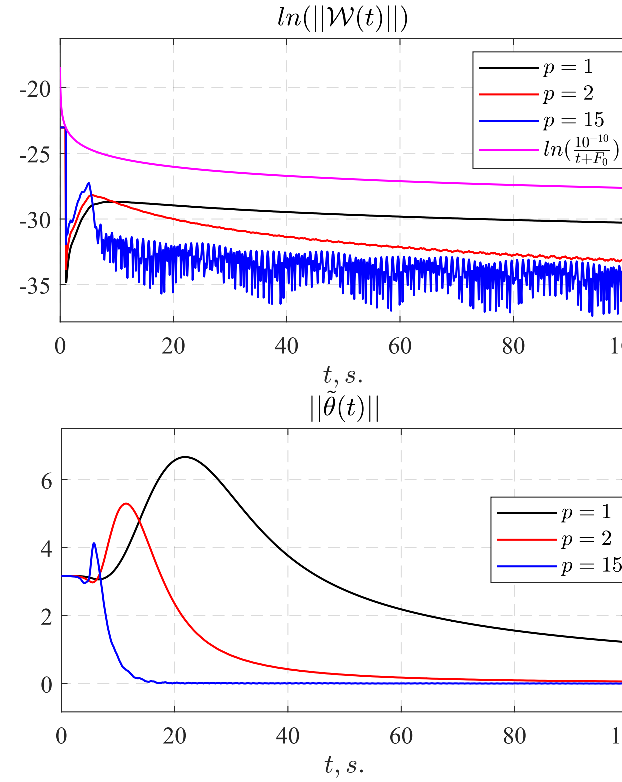

Fig. 3. Behavior of $\left\|\tilde{\boldsymbol{\theta}}\,(t)\right\|$ and $\|\boldsymbol{\mathcal{W}}\,(t)\|$ for different values of p.

considering the laws (29) and (35a), (35b), (35c), only the proposed one (29) provides exact asymptotic estimation of the unknown parameters θ in the presence of an unknown external perturbation $f\,(t)$. From comparison of the transients generated by the laws (29) and (35c) the effect and fundamental role of filtering with averaging (24b) is clearly seen.

According to the theoretical analysis, the convergence rate of $\left\|\tilde{\theta}\,(t)\right\|$ is independent of the parameter p value and does not exceed $\frac{1}{t+F_0}$. However, this parameter can significantly affect the quality of transients. Therefore, an experiment with the law (29) with different values of the parameter p (Fig. 3) was conducted. For simplicity of the results analysis, it was assumed that $\theta = const$ for all $t \geq t_0$.

The simulation results show that the increase of the parameter p value allows one also to increase the convergence of the errors $\left\|\tilde{\theta}\,(t)\right\|$ and $\|\mathcal{W}\,(t)\|$ to zero at the initial stage of transients. In addition to this, the conducted experiment validates the theoretical bound of the parametric error convergence rate.

VI. Conclusion and Further Research

A new online estimation law for the parameters of linear systems is proposed, which provides asymptotic convergence of the estimation error in the presence of an unknown but bounded perturbation signal, which spectrum does not have common frequencies with the control/reference signal one.

The scope of future research is to: *i*) find a way to increase the rate of convergence and improve alertness to the change of plant unknown parameters, *ii*) apply the proposed estimation law to the problems of indirect adaptive control and adaptive observers design.

Appendix A

This appendix contains some basic facts from the harmonic analysis [22], [23], and also includes the proof of one new lemma, which result is used in the proof of proposition 3.

Definition A1. *A signal $u\,(t) \in \mathbb{R}^n$ is said to be stationary if the following limit exists uniformly in t_0:*

$$R_u\,(t) = \lim_{T\to\infty} \frac{1}{T} \int\limits_{t_0}^{t_0+T} u\,(\tau)\, u^{\mathrm{T}}\,(t+\tau)\, d\tau, \tag{A1}$$

where $R_u\,(t)$ stands for the autocovariance of $u\,(t)$.

Definition A2. *A spectral measure of $u\,(t) \in \mathbb{R}^n$ is the result of the Fourier transform of the autocovariance matrix $R_u\,(t)$:*

$$S_u\,(\omega) = \int\limits_{-\infty}^{+\infty} e^{-j\omega\tau} R_u\,(\tau)\, d\tau. \tag{A2}$$

Proposition A1. *Let $y\,(t) = H\,(s)\,u\,(t)$, where $H\,(s)$ is a proper stable $m \times n$ matrix transfer function with real impulse response $H\,(t)$. Then, if $u\,(t) \in \mathbb{R}^n$ is stationary, then the autocovariance and spectral measure between $u\,(t)$ and $y\,(t)$ are defined as follows:*

$$R_{yu}\,(t) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} H\,(\tau_1)\, R_u\,(t+\tau_1-\tau_2)\, H^{\mathrm{T}}\,(\tau_2)\, d\tau_1 d\tau_2,$$

$$S_y\,(\omega) = H^*\,(j\omega)\, S_u\,(\omega)\, H^{\mathrm{T}}\,(j\omega) = H\,(-j\omega)\, S_u\,(\omega)\, H^{\mathrm{T}}\,(j\omega)\,,$$

where $H^*\,(j\omega)$ is a Hermitian matrix.

Proof is given in [23, p.42].

Definition A3. *A stationary signal $u\,(t) \in \mathbb{R}^n$ is called sufficiently rich of order n, if the support of the spectral measure $S_u\,(\omega)$ of $u\,(t)$ contains at least n points.*

Definition A4. *The cross correlation between stationary signals $u(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ is defined to be the following limit, uniform in t_0*

$$R_{yu}(t) = \lim_{T\to\infty} \tfrac{1}{T} \int\limits_{t_0}^{t_0+T} y(\tau) u^{\mathrm{T}}(t+\tau) d\tau, \tag{A3}$$

where $R_{yu}(t)$ denotes the cross correlation between the signals $u(t)$ and $y(t)$.

Definition A5. *The cross spectral measure of the stationary signals $u(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ is called the Fourier transform of the cross correlation matrix:*

$$S_{yu}(\omega) = \int\limits_{-\infty}^{+\infty} e^{-j\omega\tau} R_{yu}(\tau) d\tau. \tag{A4}$$

Proposition A2. *Let $y(t) = H(s) u(t)$, where $H(s)$ is a proper stable $m \times n$ matrix transfer function with impulse response $H(t)$. Then, if the signal $u(t) \in \mathbb{R}^n$ is stationary, then the cross correlation and cross spectral measure of $u(t)$ and $y(t)$ are defined as follows:*

$$R_{yu}(t) = \int\limits_{-\infty}^{+\infty} H(\tau_1) R_u(t+\tau_1) d\tau_1,$$

$$S_{yu}(\omega) = H^*(j\omega) S_u(\omega),$$

where $H^*(j\omega)$ is a Hermitian matrix.

Proof is presented in [23, p. 44].

Lemma A1. *Let $u(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ be stationary signals. Then, if and only if $rank\{R_{yu}(0)\} = n$, there exist $T > 0$ and $\alpha > 0$ such that*

$$\left|\det\left\{\int\limits_{t}^{t+T} y(\tau) u^{\mathrm{T}}(\tau) d\tau\right\}\right| \geq \alpha > 0, \forall t \geq t_0, \tag{A5}$$

Proof of Lemma A1.

IF: The definition of the cross correlation matrix implies that there exist $T_0 > 0$ and scalars ℓ_1, ℓ_2 of the same sign such that

$$\ell_1 R_{yu}(0) \leq \tfrac{1}{T_0} \int\limits_{t_0}^{t_0+T_0} y(\tau) u^{\mathrm{T}}(\tau) d\tau \leq \ell_2 R_{yu}(0), \tag{A6}$$

from which

$$\begin{gathered}\ell_1^n \det\{R_{yu}(0)\} \leq \tfrac{1}{T_0^n} \det\left\{\int\limits_{t_0}^{t_0+T_0} y(\tau) u^{\mathrm{T}}(\tau) d\tau\right\} \leq \\ \leq \ell_2^n \det\{R_{yu}(0)\},\end{gathered}$$

and therefore, when $rank\{R_{yu}(0)\} = n$, then equation (A5) holds.

IFF: For all $T > 0$ there exists a constant $\alpha_0 > 0$ such that $\alpha = \alpha_0 T^n > 0$, which allows one to rewrite the inequality (A5) in the following form:

$$\left|\det\left\{\int\limits_{t}^{t+T} y(\tau) u^{\mathrm{T}}(\tau) d\tau\right\}\right| \geq \alpha_0 T^n > 0, \forall t \geq t_0. \tag{A7}$$

The division of the left- and right-hand sides of equation (A7) by T^n yields:

$$\frac{1}{T^n}\left|\det\left\{\int\limits_{t}^{t+T} y(t) u^{\mathrm{T}}(t) d\tau\right\}\right| \geq \alpha_0 \tag{A8}$$

and, consequently,

$$\begin{gathered}|\det\{R_{yu}(0)\}| = \lim_{T\to\infty} \tfrac{1}{T^n} \left|\det\left\{\int\limits_{t}^{t+T} y(t) u^{\mathrm{T}}(t) d\tau\right\}\right| \\ \geq \lim_{T\to\infty} \alpha_0 = \alpha_0 > 0,\end{gathered}$$

from which $rank\{R_{yu}(0)\} = n$.

Appendix B

This appendix contains proofs of Theorem 1 and Propositions 1, 2, 4-6.

Proof of Propostion 1. As the polynomial $\Lambda(s)$ is Hurwitz one, the disturbance $w(t)$ is written as follows:

$$w(t) = \sum_{k=1}^{n_f} \delta_k \left[\alpha_k^w \sin\left(\omega_k^f t\right) + \beta_k^w \cos\left(\omega_k^f t\right)\right] + \varepsilon_w(t), \tag{B1}$$

where $\varepsilon_w(t)$ stands for an exponentially vanishing disturbance, which is caused by the initial conditions and transients, and α_k^w, β_k^w are some scalars.

The following scalar signal is introduced

$$\begin{gathered}\phi(t) = \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix} \varphi_y(t) = \\ = -\begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix} \tfrac{\lambda_{n-1}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} [u(t) + f(t)] = \\ = -\begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix} \tfrac{\lambda_{n-1}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} u(t) - \\ - \tfrac{Z(\theta,\, s)}{R(\theta,\, s)} w(t),\end{gathered}$$

where $\varphi_y^{\mathrm{T}}(t) = \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix} \varphi(t) = -\frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} y(t)$.

As, according to assumption 1, the polynomial $\Lambda(s) R(\theta, s)$ is Hurwitz one, then $\phi(t)$ is written as:

$$\begin{gathered}\phi(t) = \sum_{k=1}^{n} \rho_k \left[\alpha_k^u \sin(\omega_k^u t) + \beta_k^u \cos(\omega_k^u t)\right] - \\ - \sum_{k=1}^{n_f} \delta_k \left[\alpha_k^{\overline{w}} \sin\left(\omega_k^f t\right) + \beta_k^{\overline{w}} \cos\left(\omega_k^f t\right)\right] + \varepsilon_{\overline{\varphi}}(t),\end{gathered} \tag{B2}$$

where $\varepsilon_{\overline{\varphi}}(t)$ is an exponentially vanishing term, and α_k^u, β_k^u, $\alpha_k^{\overline{w}}$, $\beta_k^{\overline{w}}$ are some scalars.

As the signals (B2) and (B1) include terms with the same frequencies, then, according to (3)-(4), it can be concluded that:

$$\lim_{t\to\infty} \left|\int\limits_{t_0}^{t} \phi(s) w(s) ds\right| = \infty, \tag{B3}$$

which, as

$$\phi(t) = \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_0 \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix} \varphi(t), \tag{B4}$$

ensures existence of such $i \in \overline{1,\, 2n}$ that equation (12) holds.

Proof of Propostion 2. Taking into account equations (6), (7), (21), the multiplication $\zeta(t)\varphi^{\mathrm{T}}(t)$ is written as follows:

$$\begin{array}{l}
\zeta(t)\varphi^{\mathrm{T}}(t) = \\
= \begin{bmatrix} -\frac{\lambda_{n-1}(s)}{\Lambda(s)} y_{iv}(t) \\ \frac{\lambda_{n-1}(s)}{\Lambda(s)} u(t) \end{bmatrix} \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} y(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u(t) \end{bmatrix} = \\
= \begin{bmatrix} -\frac{\lambda_{n-1}(s) Z(\theta_{iv},\, s)}{\Lambda(s) R(\theta_{iv},\, s)} u(t) \\ \frac{\lambda_{n-1}(s)}{\Lambda(s)} u(t) \end{bmatrix} \times \\
\qquad \times \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} u(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u(t) \end{bmatrix} + \\
+ \begin{bmatrix} -\frac{\lambda_{n-1}(s) Z(\theta_{iv},\, s)}{\Lambda(s) R(\theta_{iv},\, s)} u(t) \\ \frac{\lambda_{n-1}(s)}{\Lambda(s)} u(t) \end{bmatrix} \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} f(t) & 0 \end{bmatrix} = \\
= \zeta(t)\overline{\varphi}^{\mathrm{T}}(t) + \zeta(t) \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} f(t) & 0 \end{bmatrix},
\end{array}$$

where

$$\begin{array}{l}
\zeta(t) = \begin{bmatrix} -G(s) \frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} u(t) \\ \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u(t) \end{bmatrix} = \begin{bmatrix} G(s) I_n & 0 \\ 0 & I_n \end{bmatrix} \overline{\varphi}(t), \\
\overline{\varphi}(t) = \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} u(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)} u(t) \end{bmatrix}^{\mathrm{T}}.
\end{array}$$

IF: As $\zeta(t)$ and $\varphi(t)$ are stationary, then $R_{\zeta\varphi}(0)$ is uniform with respect to t_0, which allows one to write:

$$\begin{array}{l}
R_{\zeta\varphi}(0) = \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \zeta(\tau)\varphi^{\mathrm{T}}(\tau)\, d\tau = \\
= \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \zeta(\tau)\overline{\varphi}^{\mathrm{T}}(\tau)\, d\tau + \\
+ \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \zeta(\tau) \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} f(\tau) & 0 \end{bmatrix} d\tau = \\
= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} S_{\zeta\overline{\varphi}}(\omega)\, d\omega + \\
+ \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \zeta(\tau) \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} f(\tau) & 0 \end{bmatrix} d\tau,
\end{array} \tag{B5}$$

where $S_{\zeta\overline{\varphi}}(\omega)$ stands for a cross spectral measure defined as (Proposition A2):

$$S_{\zeta\overline{\varphi}}(\omega) = M^*(j\omega) S_{\overline{\varphi}}(\omega), \tag{B6}$$

where $S_{\overline{\varphi}}(\omega)$ is a spectral measure of $\overline{\varphi}(t)$, which is defined as follows (Proposition A1):

$$S_{\overline{\varphi}}(\omega) = H(-j\omega) S_u(\omega) H^{\mathrm{T}}(j\omega). \tag{B7}$$

As the signal $u(t)$ has spectral lines at $2n$ points, then the spectral measure $S_u(\omega)$ is defined as follows:

$$S_u(\omega) = \sum_{i=1}^{2n} f_u(\omega_i) \delta(\omega - \omega_i), \tag{B8}$$

where $f_u(\omega_i) > 0$.

Taking into consideration proof of Proposition 1, it follows that:

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \zeta(\tau) \begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s) Z(\theta,\, s)}{\Lambda(s) R(\theta,\, s)} f(\tau) & 0 \end{bmatrix} d\tau = 0. \tag{B9}$$

Then, considering (B6)-(B8), it is obtained:

$$\begin{array}{l}
R_{\xi\overline{\varphi}}(0) = \\
= \frac{1}{2\pi} \sum\limits_{i=1}^{2n} f_u(\omega_i) M^*(j\omega_i) H(-j\omega_i) H^{\mathrm{T}}(j\omega_i) = \\
= \frac{1}{2\pi} \mathcal{M}\mathcal{H},
\end{array} \tag{B10}$$

where

$$\begin{array}{l}
\mathcal{M} = \begin{bmatrix} M^*(j\omega_1) & \cdots & M^*(j\omega_{2n-1}) & M^*(j\omega_{2n}) \end{bmatrix}, \\
\mathcal{H} = \begin{bmatrix} f_u(\omega_1) H(-j\omega_1) H^{\mathrm{T}}(j\omega_1) \\ \vdots \\ f_u(\omega_{2n-1}) H(-j\omega_{2n-1}) H^{\mathrm{T}}(j\omega_{2n-1}) \\ f_u(\omega_{2n}) H(-j\omega_{2n}) H^{\mathrm{T}}(j\omega_{2n}) \end{bmatrix}.
\end{array}$$

According to the premises of proposition under consideration, it holds that

$$\mathcal{H} \in \mathbb{C}^{2n\times 2n},\, rank\{\mathcal{H}\} = 2n,\, rank\{\mathcal{M}\} = 2n, \tag{B11}$$

then

$$rank\{R_{\xi\overline{\varphi}}(0)\} = rank\{\mathcal{M}\} = 2n, \tag{B12}$$

and, therefore, using Lemma A1, it is concluded that the condition (22) holds.

IFF: Let us assume that $u(t)$ is sufficiently rich of order $r < 2n$, then $R_{\xi\overline{\varphi}}(0)$ is written as:

$$R_{\xi\overline{\varphi}}(0) = \frac{1}{2\pi} \mathcal{M}_r \mathcal{H}_r, \tag{B13}$$

where

$$\begin{array}{l}
\mathcal{M}_r = \begin{bmatrix} M^*(j\omega_1) & \cdots & M^*(j\omega_{r-1}) & M^*(j\omega_r) \end{bmatrix}, \\
\mathcal{H}_r = \begin{bmatrix} f_u(\omega_1) H(-j\omega_1) H^{\mathrm{T}}(j\omega_1) \\ \vdots \\ f_u(\omega_{r-1}) H(-j\omega_{r-1}) H^{\mathrm{T}}(j\omega_{r-1}) \\ f_u(\omega_r) H(-j\omega_r) H^{\mathrm{T}}(j\omega_r) \end{bmatrix}.
\end{array}$$

So, the rank of $R_{\xi\overline{\varphi}}(0)$ can be at most $r < 2n$, which contradicts the assumption that $R_{\xi\overline{\varphi}}(0)$ is a full rank matrix.

Proof of Propostion 4. The solutions of the differential equations for $\psi(t)$ and $\vartheta(t)$ are written as:

$$\begin{array}{l}
\psi(t) = \int\limits_{\max(t_0,\, t-T)}^{t} \zeta(\tau)\varphi^{\mathrm{T}}(\tau)\, d\tau, \\
\vartheta(t) = \int\limits_{\max(t_0,\, t-T)}^{t} \zeta(\tau) z(\tau)\, d\tau = \\
= \int\limits_{\max(t_0,\, t-T)}^{t} \zeta(\tau)\varphi^{\mathrm{T}}(\tau)\, d\tau\theta + \\
\qquad + \int\limits_{\max(t_0,\, t-T)}^{t} \zeta(\tau) w(\tau)\, d\tau \\
= \psi(t)\theta + \varepsilon(t).
\end{array} \tag{B14}$$

To complete the proof, the following functions are introduced:

$$\begin{array}{c}
K_1(t) = F(t) Y(t),\, K_2(t) = F(t)\Phi(t), \\
K_3(t) = F(t) W(t)
\end{array} \tag{B15}$$

and their derivatives are written:

$$\begin{array}{l}
\dot{K}_1(t) = \dot{F}(t) Y(t) + F(t)\dot{Y}(t) = \\
= \left(\dot{F}(t) - \dot{F}(t)\right) Y(t) + \dot{F}(t)\vartheta(t) = \dot{F}(t)\vartheta(t) \\
\dot{K}_2(t) = \dot{F}(t)\psi(t),\, \dot{K}_3(t) = \dot{F}(t)\varepsilon(t),
\end{array} \tag{B16}$$

which allows one to obtain:

$$\begin{array}{l} Y\left(t\right)=\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(s\right)\vartheta\left(s\right)ds=\\ =\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(s\right)\psi\left(s\right)\theta ds+\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(s\right)\varepsilon\left(s\right)ds=\\ =\Phi\left(t\right)\theta+W\left(t\right). \end{array} \tag{B17}$$

Proof of Propostion 5. **1)** Using (B14)-(B17), the solution of the differential equation for $\Phi(t)$ is obtained in the following form:

$$\Phi\left(t\right)=\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(s\right)\int\limits_{\max(t_0,\,s-T)}^{s}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau ds. \tag{B18}$$

Owing to the fact that the condition (22) holds, for all $t\geq T$ there exist matrices R_- and R_+ such that:

$$\begin{array}{l} \int\limits_{t}^{t+T}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau\leq R_-,\\ \int\limits_{t}^{t+T}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau\geq R_+, \end{array} \tag{B19}$$

where $\det\{R_-\}=-\alpha,\det\{R_+\}=\alpha$.

From which it is obtained that:

$$\begin{array}{l} \Phi\left(t\right)=\frac{1}{F(t)}\int\limits_{T}^{t}\dot{F}\left(s\right)\int\limits_{\max(t_0,\,s-T)}^{s}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau ds\\ \geq\frac{1}{F(t)}\int\limits_{T}^{t}\dot{F}\left(s\right)dsR_+=\frac{F(t)-F(T)}{F(t)}R_+\geq cR_+,\\ \Phi\left(t\right)=\frac{1}{F(t)}\int\limits_{T}^{t}\dot{F}\left(s\right)\int\limits_{\max(t_0,\,s-T)}^{s}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau ds\\ \leq\frac{1}{F(t)}\int\limits_{T}^{t}\dot{F}\left(s\right)dsR_-=\frac{F(t)-F(T)}{F(t)}R_-\leq cR_-, \end{array} \tag{B20}$$

where $c>0$ is an arbitrary scalar.

Then there exists a scalar $\Delta_{\mathrm{LB}}=c^n\alpha$ such that $|\Delta(t)|\geq\Delta_{\mathrm{LB}}>0$ for all $t\geq T$.

2) As according to assumption 1 the signals $y(t), u(t), f(t)$ are bounded, then, owing to the fact that $\Lambda(s)$ is a Hurwitz polynomial, the signals $\zeta(t), \varphi(t)$ are also bounded.

From which we have:

$$\begin{array}{l} \left\|\Phi\left(t\right)\right\|\leq\left\|\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(s\right)\int\limits_{\max(t_0,\,s-T)}^{s}\zeta\left(\tau\right)\varphi^{\mathrm{T}}\left(\tau\right)d\tau ds\right\|\leq\\ \leq T\frac{F(t)-F(t_0)}{F(t)}\sup\limits_{t}\left\|\zeta\left(t\right)\varphi^{\mathrm{T}}\left(t\right)\right\|, \end{array} \tag{B21}$$

and, therefore, owing to the definition

$$\Delta\left(t\right):=\det\left\{\Phi\left(t\right)\right\},$$

there exists a scalar $\Delta_{\mathrm{UB}}>0$ such that $|\Delta(t)|\leq\Delta_{\mathrm{UB}}$.

3) Owing to proved proposition 3, when the premises of proposition 5 are met, then (17) holds. Thus, since the function $\dot{F}(t)$ is monotonically increasing, the following estimate is obtained:

$$\begin{array}{l} \left\|W\left(t\right)\right\|=\left\|\frac{1}{F(t)}\int\limits_{t_0}^{t}\dot{F}\left(\tau\right)\int\limits_{\max(t_0,\,\tau-T)}^{\tau}\zeta\left(s\right)w\left(s\right)dsd\tau\right\|\leq\\ \leq\left\|\frac{\dot{F}(t)}{F(t)}\int\limits_{t_0}^{t}\int\limits_{\max(t_0,\,\tau-T)}^{\tau}\zeta\left(s\right)w\left(s\right)dsd\tau\right\|=\\ =\left\|\frac{\dot{F}(t)}{F(t)}\int\limits_{\max(t_0,\,t-T)}^{t}\int\limits_{t_0}^{\tau}\zeta\left(s\right)w\left(s\right)dsd\tau\right\|\leq\\ \leq\frac{\dot{F}(t)}{F(t)}T\sup\limits_{t}\left\|\int\limits_{t_0}^{t}\zeta\left(s\right)w\left(s\right)ds\right\|\leq\frac{\dot{F}(t)}{F(t)}c_W<\infty, \end{array} \tag{B22}$$

where $c_W:=T\sup\limits_{t}\left\|\int\limits_{t_0}^{t}\zeta\left(s\right)w\left(s\right)ds\right\|<\infty$.

As the regressor $\Phi(t)$ is bounded (B21), then the matrix $\mathrm{adj}\{\Phi(t)\}$ is also bounded, and therefore, considering (B22), there exists a scalar $c_{\mathcal{W}}>0$ such that the inequality (28) holds.

Proof of Theorem 1. The following signal is introduced:

$$\kappa\left(t\right)=\left(t+F_0\right)\tilde{\theta}_i\left(t\right) \tag{B23}$$

and its derivative is written with respect to (29):

$$\begin{array}{l} \dot{\kappa}\left(t\right)=\tilde{\theta}_i\left(t\right)-\gamma\Delta^2\left(t\right)\left(t+F_0\right)\tilde{\theta}_i\left(t\right)+\\ +\gamma\Delta\left(t\right)\left(t+F_0\right)\mathcal{W}_i\left(t\right)=\\ =-\gamma\Delta^2\left(t\right)\kappa\left(t\right)+\gamma\Delta\left(t\right)\left(t+F_0\right)\mathcal{W}_i\left(t\right)+\tilde{\theta}_i\left(t\right). \end{array} \tag{B24}$$

Following (B23), it can be concluded from the boundedness of $\kappa(t)$ and $\tilde{\theta}_i(t)$ that $\tilde{\theta}_i(t)$ converges to zero.

Therefore, to prove the theorem, the boundedness of $\tilde{\theta}_i(t)$ and $\kappa(t)$ are investigated.

To obtain the bound of $\tilde{\theta}_i(t)$, the differential equation (29) is solved:

$$\tilde{\theta}_i\left(t\right)=\phi\left(t,t_0\right)\tilde{\theta}_i\left(t_0\right)+\gamma\int\limits_{t_0}^{t}\phi\left(t,s\right)\Delta\left(s\right)\mathcal{W}_i\left(s\right)ds, \tag{B25}$$

where

$$\phi\left(t,\tau\right)=\exp\left(-\gamma\int_{\tau}^{t}\Delta^2\left(s\right)ds\right). \tag{B26}$$

Further, for all $t\geq t_0$ it is written:

$$\begin{array}{l} \left|\tilde{\theta}_i\left(t\right)\right|\leq\phi\left(t,t_0\right)\left|\tilde{\theta}_i\left(t_0\right)\right|+\\ \qquad+\gamma\int\limits_{t_0}^{t}\left|\phi\left(t,s\right)\Delta\left(s\right)\right|\left|\mathcal{W}_i\left(s\right)\right|ds\leq\\ \leq\phi\left(t,t_0\right)\left|\tilde{\theta}_i\left(t_0\right)\right|+\\ \qquad+\gamma\sqrt{\int\limits_{t_0}^{t}\phi^2\left(t,s\right)\Delta^2\left(s\right)ds\int\limits_{t_0}^{t}\mathcal{W}_i^2\left(s\right)ds}, \end{array} \tag{B27}$$

where the Cauchy–Schwarz inequality is used in the last step.

The equality

$$\frac{\partial}{\partial s}\phi^2\left(t,s\right)=2\gamma\phi^2\left(t,s\right)\Delta^2\left(s\right)$$

allows one to rewrite equation (B27) as follows:

$$\begin{array}{l} \left|\tilde{\theta}_i\left(t\right)\right|\leq\phi\left(t,t_0\right)\left|\tilde{\theta}_i\left(t_0\right)\right|+\\ \qquad+\sqrt{\frac{\gamma}{2}}\sqrt{1-\phi^2\left(t,t_0\right)}\sqrt{\int\limits_{t_0}^{t}\mathcal{W}_i^2\left(s\right)ds}. \end{array} \tag{B28}$$

As it holds that:

$$\begin{array}{l} \phi\left(t,t_0\right)\leq 1,\ \sqrt{1-\phi^2\left(t,t_0\right)}\leq 1,\\ \sqrt{\int\limits_{t_0}^{t}\mathcal{W}_i^2\left(s\right)ds}\leq c_{\mathcal{W}}\sqrt{\int\limits_{t_0}^{t}\frac{\dot{F}^2(s)}{F^2(s)}ds}<\infty, \end{array} \tag{B29}$$

then, owing to (B28), there exists $\tilde{\theta}_{\max} > 0$ such that $\left|\tilde{\theta}_i(t)\right| \leq \tilde{\theta}_{\max}$ for all $t \geq t_0$.

Taking into account the obtained bounds, the boundedness of the variable $\kappa(t)$ is to be proved. For this purpose, a quadratic form is introduced:

$$V = \tfrac{1}{2}\tilde{\kappa}^2. \tag{B30}$$

The derivative of (B30) with respect to (B24) is:

$$\dot{V} = -\gamma\Delta^2\tilde{\kappa}^2 + \gamma\Delta\tilde{\kappa}\,(t+F_0)\,\mathcal{W}_i + \tilde{\kappa}\tilde{\theta}_i. \tag{B31}$$

Using the inequalities:

$$\begin{gathered} (t+F_0)\,\Delta\tilde{\kappa}\mathcal{W}_i \leq \delta^{-1}\Delta^2\tilde{\kappa}^2 + \delta(t+F_0)^2\mathcal{W}_i^2,\ \delta>0 \\ (t+F_0)^2\mathcal{W}_i^2(t) \leq \tfrac{(t+F_0)^2\dot{F}^2c_{\mathcal{W}}^2}{F^2} = \\ = \tfrac{(t+F_0)^2p^2t^{2p-2}c_{\mathcal{W}}^2}{(t^p+F_0)^2} \leq \mathcal{W}_{\mathrm{UB}}^2 < \infty, \\ \tilde{\kappa}\tilde{\theta}_i \leq \rho^{-1}\tilde{\kappa}^2 + \rho\tilde{\theta}_i^2,\ \rho>0 \end{gathered} \tag{B32}$$

it is written that:

$$\begin{gathered} \dot{V} \leq -2\gamma\left(\left(1-\delta^{-1}\right)\Delta^2 - \rho^{-1}\gamma^{-1}\right)V + \\ +\delta\mathcal{W}_{\mathrm{UB}}^2 + \rho\tilde{\theta}_i^2. \end{gathered} \tag{B33}$$

According to the premises of theorem, for all $t \geq T$ it holds that $|\Delta(t)| \geq \Delta_{\mathrm{LB}} > 0$, and therefore, there exist constants $\delta > 0,\ \rho > 0$ and $\eta_{\min} > 0$ such that

$$\gamma\left(\left(1-\delta^{-1}\right)\Delta^2 - \rho^{-1}\gamma^{-1}\right) \geq \eta_{\min} > 0. \tag{B34}$$

Considering (B34), the solution of equation (B33) is obtained as:

$$\begin{gathered} V(t) \leq e^{-2\eta_{\min}(t-T)}V(T) + \\ + \tfrac{1}{2\eta_{\min}}\left[\delta\mathcal{W}_{\mathrm{UB}}^2 + \rho\tilde{\theta}_{\max}^2\right], \end{gathered} \tag{B35}$$

from which it follows that there exists a scalar $\tilde{\kappa}_{\max}$ such that $|\tilde{\kappa}(t)| \leq \tilde{\kappa}_{\max}$ for all $t \geq T$.

As $t + F_0 \to \infty$ when $t \to \infty$, and $|\tilde{\kappa}(t)| \leq \tilde{\kappa}_{\max}$, then from (B23) it follows that (9) holds, which completes the proof.

Proof of Propostion 6. Equation (30) is substituted into (6) and vice versa to obtain:

$$\begin{gathered} u(t) = W_{cl}(\theta,\,s)\,R(\theta,\,s)\left[r(t) + \tfrac{P_y(\kappa,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{Q_y(\kappa,\,s)R(\theta,\,s)P_r(\kappa,\,s)}f(t)\right], \\ y(t) = W_{cl}(\theta,\,s)\,Z(\theta,\,s)\left[r(t) + \tfrac{Q_r(\kappa,\,s)}{P_r(\kappa,\,s)}f(t)\right]. \end{gathered} \tag{B36}$$

Considering (7), (32) and (A1), the multiplication $\zeta(t)\,\varphi^{\mathrm{T}}(t)$ is rewritten as:

$$\begin{aligned} &\zeta(t)\,\varphi^{\mathrm{T}}(t) = \\ &= \begin{bmatrix} -\frac{\lambda_{n-1}(s)}{\Lambda(s)}y_{iv}(t) \\ \frac{\lambda_{n-1}(s)}{\Lambda(s)}u(t) \end{bmatrix}\begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}y(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)}{\Lambda(s)}u(t) \end{bmatrix} = \\ &= \begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)Z(\theta_{iv},\,s)}{\Lambda(s)}r(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)R(\theta_{iv},\,s)}{\Lambda(s)}r(t) \end{bmatrix}\times \\ &\times\begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)}{\Lambda(s)}\left[r(t) + \frac{Q_r(\kappa,\,s)}{P_r(\kappa,\,s)}f(t)\right] \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)R(\theta,\,s)}{\Lambda(s)}\left[r(t) + \frac{P_y(\kappa,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{Q_y(\kappa,\,s)R(\theta,\,s)P_r(\kappa,\,s)}f(t)\right] \end{bmatrix}^{\mathrm{T}} = \\ &= \begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)Z(\theta_{iv},\,s)}{\Lambda(s)}r(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)R(\theta_{iv},\,s)}{\Lambda(s)}r(t) \end{bmatrix}\times \\ &\times\begin{bmatrix} -\frac{\lambda_{n-1}^{\mathrm{T}}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)}{\Lambda(s)}r(t) & \frac{\lambda_{n-1}^{\mathrm{T}}(s)W_{cl}(\theta,\,s)R(\theta,\,s)}{\Lambda(s)}r(t) \end{bmatrix} + \\ &+ \begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)Z(\theta_{iv},\,s)}{\Lambda(s)}r(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta_{iv},\,s)R(\theta_{iv},\,s)}{\Lambda(s)}r(t) \end{bmatrix}\times \\ &\times\begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{\Lambda(s)P_r(\kappa,\,s)}f(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)R(\theta,\,s)P_y(\kappa,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{\Lambda(s)Q_y(\kappa,\,s)R(\theta,\,s)P_r(\kappa,\,s)}f(t) \end{bmatrix}^{\mathrm{T}} = \\ &= \zeta(t)\,\overline{\varphi}^{\mathrm{T}}(t) + \zeta(t)\,d(t)\,. \end{aligned} \tag{B37}$$

where

$$\begin{aligned} \zeta(t) &= \begin{bmatrix} -G_1(s)\,\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)}{\Lambda(s)}r(t) \\ G_2(s)\,\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)R(\theta,\,s)}{\Lambda(s)}r(t) \end{bmatrix} = \\ &= \begin{bmatrix} G_1(s)\,I_n & 0 \\ 0 & G_2(s)\,I_n \end{bmatrix}\overline{\varphi}(t)\,, \\ \overline{\varphi}(t) &= \begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)}{\Lambda(s)}r(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)R(\theta,\,s)}{\Lambda(s)}r(t) \end{bmatrix}, \\ d(t) &= \begin{bmatrix} -\frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{\Lambda(s)P_r(\kappa,\,s)}f(t) \\ \frac{\lambda_{n-1}(s)W_{cl}(\theta,\,s)R(\theta,\,s)P_y(\kappa,\,s)Z(\theta,\,s)Q_r(\kappa,\,s)}{\Lambda(s)Q_y(\kappa,\,s)R(\theta,\,s)P_r(\kappa,\,s)}f(t) \end{bmatrix} \end{aligned} \tag{B38}$$

Further proof is done by analogy with the one of proposition 2.

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