On the Exact Parameter Estimation for Robot Manipulators Without Persistence of Excitation

Marco A. Arteaga, *Member, IEEE*

***Abstract*—Adaptive control is one of the most employed techniques to achieve trajectory tracking of robot manipulators. Although it is desirable to obtain exact parameter estimation, most adaptive schemes need the persistency of excitation (PE) condition on the regressor to be satisfied. In the recent years, the so-called dynamic regressor extension and mixing (DREM) procedure was developed to provide an alternative in the design of adaptive laws with conditions different from PE. When met, the improvement in parameter estimation is remarkable, but when not, adaptation can simply stop, which might be unacceptable for control purposes. This article proposes for the first time a composite scheme, which combines the standard gradient adaptive law with a DREM-based additional term with the following properties: 1) Trajectory tracking for joint desired positions and velocities is guaranteed; 2) in the absence of PE, if a new condition that can partially be verified online for the additional DREM-based term is matched, then exact parameter estimation takes place in finite time. Simulation results are in good accordance with the developed theory.**

***Index Terms*—Dynamic regressor extension and mixing (DREM), finite time, persistence of excitation, robot parameter estimation, trajectory tracking.**

I. Introduction

In theory, it is possible to describe a robot dynamics accurately by using some standard approaches, such as the Euler–Lagrange equations of motion. However, in practice, the resulting model presents parameter uncertainties, which makes position and velocity tracking imprecise when it is used for the implementation of control schemes. This drawback can be compensated by taking into account some structural properties, e.g., it is possible to rewrite the robot model as the product of a matrix, called the regressor, and a vector of constant parameters. Based on this property, in [1], a simple but effective gradient adaptive law is proposed to achieve exact position tracking. However, parameter estimation is only guaranteed if a condition of persistent excitation (PE) is fulfilled. In fact, in [2], it was shown that the necessary and sufficient condition to guarantee parameter convergence is that the regressor matrix be PE along the reference trajectories, so that it can be verified a priori. In [3], the Mazenc construction is used to design a simple strict Lyapunov function for passivity-based adaptive controllers for Lagrangian systems for the first time.

Recently, considerable efforts have been made to relax the PE condition, e.g., by assuming interval excitation (IE). One of the most remarkable techniques is the so-called *dynamic regressor extension and mixing* (DREM) introduced in [4]. The approach consists of two stages. First, the generation of new regression forms via the application of a dynamic operator to the data of the original regression. Second, a suitable mix of these new data is to obtain the final desired regression form to which standard parameter estimation techniques is applied. The authors show that the new convergence condition is not necessarily related to the PE one, and that much better outcomes can be gotten in comparison, for example, with the standard gradient algorithm. This new strategy has been interpreted in [5] in terms of nonlinear systems and observation of linear functionals for time-varying systems of the classical Luenberger's state observer. In [6], the basic DREM algorithm is modified to obtain finite-time parameter convergence with a more relaxed condition than that mentioned in [4]. In [7], it is proven that if the regressor is PE, then so is the scalar regressor produced by the DREM algorithm. In [8], a solution to the problem of parameter estimation of nonlinearly parameterized regressions is proposed and it is applied for system identification and adaptive control without overparameterization and with the feature that parameter convergence is ensured without a persistency of excitation (PE) assumption. In [9], a simpler parameterization is obtained from the power balance equation of Euler–Lagrange systems to be used for their identification and adaptive control even in a scenario with insufficient excitation.

Despite the advantages of the DREM procedure, to the best of the author's knowledge, it has been scarcely analyzed in the case when the excitation conditions do not hold at all. This contribution introduces a composite adaptive scheme that combines the standard gradient algorithm with a DREM-based adaptive term with the following properties.

1) Position and velocity tracking errors tend asymptotically to zero even if no exact parameter estimation takes place.
2) A new condition for the DREM-based term is provided.
3) The new condition is original in the sense that it is not expressed in terms of an integral or the $\mathcal{L}_2$ norm as usual for any other kind of adaptive schemes.
4) An advantage of the new convergence condition is that it can be (partially) verified online.

Simulation results are in good accordance with the developed theory. The article is organized as follows. Section II provides some preliminaries and the motivation for combining a standard adaptive law with a DREM-based term. Section III introduces the new adaptive law and the stability analysis while simulation results are given in Section IV. Section V concludes this article.

II. Preliminares and Motivation

Consider the following well-known relationship:

$$x = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{\theta} \tag{1}$$

where $x \in \mathbb{R}$ is a scalar output, $\boldsymbol{y} \in \mathbb{R}^p$ is the *regressor*, and $\boldsymbol{\theta} \in \mathbb{R}^p$ is a constant vector of parameters. x and $\boldsymbol{y}$ are known while $\boldsymbol{\theta}$ is to be estimated. One of the best known schemes to get an estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is

Manuscript received 20 February 2023; accepted 29 March 2023. Date of publication 21 April 2023; date of current version 29 December 2023. This work was supported by the DGAPA–UNAM under Grant IN102723. Recommended by Associate Editor M. E. Broucke.

The author is with the Departamento de Control y Robótica, DIE–FI, UNAM, México, D. F. 04510, Mexico (e-mail: marteagp@unam.mx).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2023.3269359.
Digital Object Identifier 10.1109/TAC.2023.3269359

the gradient algorithm

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\boldsymbol{y}\left(\boldsymbol{y}^{\mathrm{T}}\hat{\boldsymbol{\theta}} - x\right) \tag{2}$$

where $\boldsymbol{\Psi} \in \mathbb{R}^{p\times p}$ is a positive-definite diagonal matrix. The corresponding error dynamics can be easily computed as

$$\dot{\tilde{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{\theta}} \tag{3}$$

in view of the fact that $\boldsymbol{\theta}$ is constant and where

$$\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \tag{4}$$

is the parameter error. It is well known that for $\tilde{\boldsymbol{\theta}}$ satisfying (3), the zero equilibrium is (uniformly) exponentially stable if and only if the PE condition on the regressor is satisfied, i.e., [10]

$$\int_t^{t+T} \boldsymbol{y}(\vartheta)\boldsymbol{y}^{\mathrm{T}}(\vartheta)\mathrm{d}\vartheta \geq \mu\boldsymbol{I} \qquad \forall\, t \geq t_0 \tag{5}$$

for some $\mu, T > 0$. Since the PE condition is difficult to satisfy, in [4], the DREM approach is proposed to overcome it. First, $p-1$ linear $\mathcal{L}_\infty$ stable operators $G_j : \mathcal{L}_\infty \to \mathcal{L}_\infty$ for $j = 2, \ldots, p$ are defined, which may be as simple as pure time delays or linear time-invariant filters of the form

$$\dot{x}_{\mathrm{f}j} = -b_j x_{\mathrm{f}j} + a_j x \tag{6}$$

where $a_j \neq 0$ and $b_j > 0$. Note that the same input x [given as the output in (1)] is used for all filters. With these operators at hand, $p-1$ filtered versions of the output (1) can be obtained

$$x_{\mathrm{f}j} = \boldsymbol{y}_{\mathrm{f}j}^{\mathrm{T}}\boldsymbol{\theta}. \tag{7}$$

By defining $x_{\mathrm{f}1} \equiv x$ and $\boldsymbol{y}_{\mathrm{f}1} \equiv \boldsymbol{y}$, a vector equation

$$\boldsymbol{x}_{\mathrm{f}} = \boldsymbol{Y}_{\mathrm{f}}\boldsymbol{\theta} \tag{8}$$

can be formed, where $\boldsymbol{x}_{\mathrm{f}} = [x_{\mathrm{f}1}\, x_{\mathrm{f}2} \cdots x_{\mathrm{f}p}]^{\mathrm{T}}$, and $\boldsymbol{Y}_{\mathrm{f}} = [\boldsymbol{Y}_{\mathrm{f}1}\, \boldsymbol{Y}_{\mathrm{f}2} \cdots \boldsymbol{Y}_{\mathrm{f}p}]^{\mathrm{T}}$. Consider now the relationship [11]

$$\mathrm{adj}(\boldsymbol{Y}_{\mathrm{f}})\boldsymbol{Y}_{\mathrm{f}} = \boldsymbol{Y}_{\mathrm{f}}\mathrm{adj}(\boldsymbol{Y}_{\mathrm{f}}) = \phi\boldsymbol{I} \tag{9}$$

where $\phi = \det(\boldsymbol{Y}_{\mathrm{f}})$, and $\mathrm{adj}(\cdot)$ denotes the adjugate matrix of $(\cdot)$. If (8) is multiplied by $\mathrm{adj}(\boldsymbol{Y}_{\mathrm{f}})$, then p-decoupled equations of the form $x_{\mathrm{e}i} = \phi\theta_i$ are gotten for $i = 1, \ldots, p$, where θ_i is the ith element of $\boldsymbol{\theta}$ and $x_{\mathrm{e}i}$ is the ith element of $\boldsymbol{x}_{\mathrm{e}} = \mathrm{adj}(\boldsymbol{Y}_{\mathrm{f}})\boldsymbol{x}_{\mathrm{f}}$. Based on $x_{\mathrm{e}i}$, p-decoupled estimators

$$\dot{\hat{\theta}}_i = -\gamma_i\phi\left(\phi\hat{\theta}_i - x_{\mathrm{e}i}\right) \tag{10}$$

can be programmed, with $\gamma_i > 0$. Once again, since $\boldsymbol{\theta}$ is constant, the parameter error dynamics satisfies

$$\dot{\tilde{\theta}}_i = -\gamma_i\phi^2\tilde{\theta}_i \tag{11}$$

where $\tilde{\theta}_i$ is the ith element of the parameter vector error given in (4). In [4], it is shown that

$$\phi \notin \mathcal{L}_2 \quad \Rightarrow \quad \lim_{t\to\infty}\tilde{\theta}_i = 0 \tag{12}$$

for $i = 1, \ldots, p$, and a discussion is provided to show the pros and cons of conditions (5) and (12). Since in [7], it has been shown that it is always possible to find operators G_j so that if $\boldsymbol{Y}$ is PE then so is ϕ and consequently (12) holds, then it can be concluded that the DREM approach is always the better option for exact parameter estimation.

The risk that parameter estimation stops abruptly if ϕ becomes zero may not be acceptable for control purposes, as illustrated in Section IV-B. This is because using a DREM technique does not guarantee the control objective is met regardless of excitation conditions, unlike the gradient algorithm. The motivation of this article is to combine the gradient adaptive algorithm and a DREM-based adaptation law to guarantee exact position tracking even if no exact parameter estimation takes place, but, at the same time, the latter can be achieved with a condition less restrictive than the usual PE.

III. Combining the Gradient and the DREM Algorithms

Consider an n-degrees-of-freedom rigid robot manipulator whose dynamics can be described by [12]

$$\boldsymbol{H}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{13}$$

where $\boldsymbol{q} \in \mathbb{R}^n$ is the vector of generalized joint coordinates, $\boldsymbol{H}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the symmetric positive-definite inertia matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the vector of Coriolis and centrifugal torques, $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ is a symmetric positive-semidefinite matrix of joint viscous friction coefficients, $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ is the vector of gravitational torques, and $\boldsymbol{\tau} \in \mathbb{R}^n$ is the vector of input torques acting at the joints. Assume for simplicity's sake that the robot has revolute joints only and that velocity measurements are available. Some useful model properties are listed as follows [12].

Property III.1: It holds $\lambda_{\mathrm{h}}\|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}(\boldsymbol{q})\boldsymbol{x} \leq \lambda_{\mathrm{H}}\|\boldsymbol{x}\|^2 \quad \forall\ \boldsymbol{q} \in \mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{R}^n$, and $0 < \lambda_{\mathrm{h}} \leq \lambda_{\mathrm{H}} < \infty$, with $\lambda_{\mathrm{h}} = \min_{\boldsymbol{q}\in\mathbb{R}^n} \lambda_{\min}(\boldsymbol{H}(\boldsymbol{q}))$ and $\lambda_{\mathrm{H}} = \max_{\boldsymbol{q}\in\mathbb{R}^n} \lambda_{\max}(\boldsymbol{H}(\boldsymbol{q}))$. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a symmetric matrix, respectively. △

Property III.2: By using the Christoffel symbols of the first kind to compute $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$, the matrix $\dot{\boldsymbol{H}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is skew-symmetric. △

Property III.3: With a proper definition of parameters, the left-hand side of model (13) can be written as

$$\boldsymbol{H}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\theta} \tag{14}$$

where $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \in \mathbb{R}^{n\times p}$ is the *regressor* and $\boldsymbol{\theta} \in \mathbb{R}^p$ is a constant vector of parameters. △

Given a bounded desired trajectory $\boldsymbol{q}_{\mathrm{d}}$ with at least bounded first and second derivatives, the control problem consists in designing an adaptive law so that the tracking error given by

$$\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_{\mathrm{d}} \tag{15}$$

tends to zero asymptotically under the assumption that the parameter vector $\boldsymbol{\theta}$ is uncertain. Perhaps, the best known solution is proposed in [1], based on a gradient adaptive law. This scheme can be implemented by defining

$$\dot{\boldsymbol{q}}_{\mathrm{r}} = \dot{\boldsymbol{q}}_{\mathrm{d}} - \boldsymbol{\Lambda}\boldsymbol{e} \tag{16}$$

$$\boldsymbol{s} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_{\mathrm{r}} = \dot{\boldsymbol{e}} + \boldsymbol{\Lambda}\boldsymbol{e} \tag{17}$$

where $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ is a diagonal positive-definite matrix. The definition (17) is equivalent to the following stable linear filter:

$$\dot{\boldsymbol{e}} = -\boldsymbol{\Lambda}\boldsymbol{e} + \boldsymbol{s} \tag{18}$$

so that if $\boldsymbol{s}$ is bounded and tends to zero, so do $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$. The control law proposed in [1] is

$$\begin{aligned}\boldsymbol{\tau} &= \hat{\boldsymbol{H}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{D}}\dot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{g}}(\boldsymbol{q}) - \boldsymbol{K}_{\mathrm{v}}\boldsymbol{s} \\ &= \boldsymbol{Y}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_{\mathrm{r}},\ddot{\boldsymbol{q}}_{\mathrm{r}})\hat{\boldsymbol{\theta}} - \boldsymbol{K}_{\mathrm{v}}\boldsymbol{s}\end{aligned} \tag{19}$$

where Property 3.3 has been used, $\boldsymbol{K}_{\mathrm{v}} \in \mathbb{R}^{n\times n}$ is a diagonal positive-definite matrix, and the time t is included to take explicitly into consideration that $\boldsymbol{q}_{\mathrm{d}}$, $\dot{\boldsymbol{q}}_{\mathrm{d}}$, and $\ddot{\boldsymbol{q}}_{\mathrm{d}}$ are time-varying trajectories. The gradient adaptive law is given by

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\boldsymbol{Y}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{s} \tag{20}$$

where $\boldsymbol{\Psi} \in \mathbb{R}^{p\times p}$ is a diagonal positive-definite matrix, and

$$\boldsymbol{Y}_{\mathrm{a}} = \boldsymbol{Y}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_{\mathrm{r}}, \ddot{\boldsymbol{q}}_{\mathrm{r}}) \tag{21}$$

is used to avoid writing the regressor's arguments. The control law (19) in conjunction with the adaptive law (20) has the following properties [1], [2].

1) $\boldsymbol{s}$, $\boldsymbol{e}$, and $\dot{\boldsymbol{e}} \in \mathcal{L}_\infty$ and tend to zero as $t \to \infty$.
2) $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ remain bounded for all $t \geq t_0$.
3) $\tilde{\boldsymbol{\theta}} \to \boldsymbol{0}$ as $t \to \infty$ if and only if $\boldsymbol{Y}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}, \ddot{\boldsymbol{q}}_{\mathrm{d}})$ is PE.

Even though achieving $\boldsymbol{e}, \dot{\boldsymbol{e}} \to \boldsymbol{0}$ is the main goal, transient performance represents also an important challenge, and exact parameter estimation is desirable because it can improve it. The main objective of this section is to design an adaptive law to get parameter error convergence to zero without necessarily satisfying the PE condition in order to improve tracking transient performance. For this, the algorithm should combine a gradient adaptive law with a term based on the DREM method. First of all, the following modification is made to the original control law (19):

$$\begin{aligned}\boldsymbol{\tau} &= \hat{\boldsymbol{H}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{D}}\dot{\boldsymbol{q}}_{\mathrm{r}} + \hat{\boldsymbol{g}}(\boldsymbol{q}) - \boldsymbol{K}_{\mathrm{v}}\lfloor\boldsymbol{s}\rceil^{\boldsymbol{\alpha}_{\mathrm{s}}}\\ &= \boldsymbol{Y}_{\mathrm{a}}\hat{\boldsymbol{\theta}} - \boldsymbol{K}_{\mathrm{v}}\lfloor\boldsymbol{s}\rceil^{\boldsymbol{\alpha}_{\mathrm{s}}}\end{aligned} \tag{22}$$

where $\boldsymbol{\alpha}_{\mathrm{s}} \in \mathbb{R}^n$ and

$$\lfloor\boldsymbol{s}\rceil^{\boldsymbol{\alpha}_{\mathrm{s}}} = \begin{bmatrix}\lfloor s_1\rceil^{\alpha_{\mathrm{s}1}}\\ \vdots\\ \lfloor s_n\rceil^{\alpha_{\mathrm{s}n}}\end{bmatrix} = \begin{bmatrix}|s_1|^{\alpha_{\mathrm{s}1}}\,\mathrm{sign}(s_1)\\ \vdots\\ |s_n|^{\alpha_{\mathrm{s}n}}\,\mathrm{sign}(s_n)\end{bmatrix} \tag{23}$$

where s_i is the ith element of $\boldsymbol{s}$ and $\alpha_{\mathrm{s}i} \in (0.5, 1)$ is the ith element of $\boldsymbol{\alpha}_{\mathrm{s}}$. For $x \in \mathbb{R}$, it is defined

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0\\ 0 & \text{if} \quad x = 0\\ -1 & \text{if} \quad x < 0\end{cases}. \tag{24}$$

The rationale behind the modification on the control law can be found in Remark III.3. Note that contrary to the regressor given in (1), the one given in (14) contains $\ddot{\boldsymbol{q}}$. Therefore, the first step is to define the filtered regressor $\boldsymbol{Y}_{\varphi 1}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times p}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y}_{\varphi 1} = -\lambda_\varphi\boldsymbol{Y}_{\varphi 1} + \lambda_\varphi\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), \quad \boldsymbol{Y}_{\varphi 1}(t_0) = \boldsymbol{O} \tag{25}$$

with $\lambda_\varphi > 0$. The filtered input $\boldsymbol{\tau}_\varphi \in \mathbb{R}^n$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\tau}_\varphi = -\lambda_\varphi\boldsymbol{\tau}_\varphi + \lambda_\varphi\boldsymbol{\tau} \tag{26}$$

where it is set $\boldsymbol{\tau}_\varphi(t_0) = \boldsymbol{0}$. According to the Swapping Lemma [10], (25) and (26) yield to

$$\boldsymbol{\tau}_\varphi = \boldsymbol{Y}_{\varphi 1}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{\theta} \tag{27}$$

where the zero initial conditions have been set.

Remark III.1: The unaware reader may assume that $\ddot{\boldsymbol{q}}$ is necessary to compute $\boldsymbol{Y}_{\varphi 1}$ in (25) but the regressor is filtered to avoid that. Prior to implementation, integration by parts must be carried out (see an example in [12, page 177]). △

The next step in the design of the DREM-based algorithm is to get a relationship of form (8). For this goal, the available $\boldsymbol{\tau}_\varphi$ and $\boldsymbol{Y}_{\varphi 1}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are used. Note that the matrix $\boldsymbol{Y}_{\mathrm{f}} \in \mathbb{R}^{p\times p}$ in (8) is square but it is not necessary, in general, to define $p-1$ stable operators because mostly $n > 1$. Assume without loss of generality that $n < p$. To form the square matrix $\boldsymbol{Y}_{\mathrm{f}}$, the following procedure is proposed.

1) Define the first n rows of $\boldsymbol{Y}_{\mathrm{f}}$ as the n rows of $\boldsymbol{Y}_{\varphi 1}$, so that $p-n$ extra rows need still to be defined.
2) Define as $\tilde{n}$ the (integer) total number of matrices $\boldsymbol{Y}_{\varphi 1}, \ldots, \boldsymbol{Y}_{\varphi\tilde{n}}$ required to form $\boldsymbol{Y}_{\mathrm{f}}$, so that $\tilde{n}-1$ extra matrices are necessary to get $\boldsymbol{Y}_{\mathrm{f}}$ as shown in (29).
3) If

$$\tilde{n} = \frac{p}{n} \tag{28}$$

is an integer number, then $\tilde{n}-1$ matrices $\boldsymbol{Y}_{\varphi 2}, \ldots, \boldsymbol{Y}_{\varphi\tilde{n}} \in \mathbb{R}^{n\times p}$ must be created by using $(\tilde{n}-1)$ linear $\mathcal{L}_\infty$ stable operators $G_j : \mathcal{L}_\infty \to \mathcal{L}_\infty$ for $j = 2, \ldots, \tilde{n}$ as before to get

$$\boldsymbol{Y}_{\mathrm{f}} = \begin{bmatrix}\boldsymbol{Y}_{\varphi 1}\\ \vdots\\ \boldsymbol{Y}_{\varphi\tilde{n}}\end{bmatrix} \in \mathbb{R}^{p\times p}. \tag{29}$$

4) If the quotient $\tilde{n} = \frac{p}{n}$ is not an integer, the DIV operator can be used since it delivers the integer part of a division (e.g., $\frac{6}{4} = 1.5$ but 6 DIV $4 = 1$). This means that

$$\tilde{n} = (p\ \mathrm{DIV}\ n) + 1. \tag{30}$$

In that case, only the last matrix $\boldsymbol{Y}_{\varphi\tilde{n}}$ in (29) is of a different dimension given by $(p\,\mathrm{MOD}\,n) \times p$, where the MOD operator gives the remainder of an integer division (e.g., 6 MOD $4 = 2$).
5) Keeping in mind the previous subdivisions and dimensions, for $j = 2, \ldots, \tilde{n}$, it is chosen

$$\dot{\boldsymbol{Y}}_{\varphi j} = -b_j\boldsymbol{Y}_{\varphi j} + a_j\boldsymbol{Y}_{\varphi 1} \qquad \boldsymbol{Y}_{\varphi j}(t_0) = \boldsymbol{O} \tag{31}$$

where $a_j \neq 0$, $b_j > 0$.

Once $\boldsymbol{Y}_{\mathrm{f}}$ is gotten, it remains to get $\boldsymbol{\tau}_{\mathrm{f}}$, i.e.

$$\boldsymbol{\tau}_{\mathrm{f}} = \begin{bmatrix}\tau_{\mathrm{f}1} & \cdots & \tau_{\mathrm{f}n} & \tau_{\mathrm{f}(n+1)} & \cdots & \tau_{\mathrm{f}p}\end{bmatrix}^{\mathrm{T}} \tag{32}$$

by choosing $\tau_{\mathrm{f}1}$ to $\tau_{\mathrm{f}n}$ equal to the elements of $\boldsymbol{\tau}_\varphi$ in (26) while $\tau_{\mathrm{f}(n+1)}$ to $\tau_{\mathrm{f}p}$ are obtained by applying to the filtered input $\boldsymbol{\tau}_\varphi$ the same $(\tilde{n}-1)$ linear $\mathcal{L}_\infty$ stable operators $G_j : \mathcal{L}_\infty \to \mathcal{L}_\infty$ in the same order used to get $\boldsymbol{Y}_{\mathrm{f}}$. Then

$$\boldsymbol{\tau}_{\mathrm{f}} = \boldsymbol{Y}_{\mathrm{f}}\boldsymbol{\theta} \tag{33}$$

holds, which is equivalent to (8). Note that if $\tilde{n} = \frac{p}{n}$ is not an integer, then not all the elements of $\boldsymbol{\tau}_\varphi$ will be employed for the last operator $G_{\tilde{n}}$. Once (33) has been computed, the DREM approach can be employed exactly as for the single output case to get p decoupled equations of the form

$$\tau_{\mathrm{e}i} = \phi\theta_i \tag{34}$$

for $i = 1, \ldots, p$, where θ_i is the ith element of $\boldsymbol{\theta}$ and $\tau_{\mathrm{e}i}$ is the ith element of

$$\boldsymbol{\tau}_{\mathrm{e}} = \mathrm{adj}(\boldsymbol{Y}_{\mathrm{f}})\boldsymbol{\tau}_{\mathrm{f}} \tag{35}$$

and as before

$$\phi = \det(\boldsymbol{Y}_{\mathrm{f}}). \tag{36}$$

Now, based on the work in [6], for $i = 1, \ldots, p$ define the ith element of a vector $\boldsymbol{f}_\theta \in \mathbb{R}^p$ as

$$f_{\theta i} = \gamma_i\lfloor\phi(\phi\hat{\theta}_i - \tau_{\mathrm{e}i})\rceil^{\alpha_i} \tag{37}$$

where, as before, $\lfloor\cdot\rceil^{\alpha_i} = |\cdot|^{\alpha_i}\mathrm{sign}(\cdot)$, $\alpha_i \in (0.5, 1)$ and $\gamma_i > 0$ is the ith element of a positive-definite diagonal matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{p\times p}$. Then, consider the following modification to the adaptive law in (20)

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\left(\boldsymbol{Y}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{s} + \boldsymbol{f}_{\theta}\right). \tag{38}$$

Since the parameter vector is constant and in view of (4), it can be concluded that

$$\dot{\tilde{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\left(\boldsymbol{Y}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{s} + \boldsymbol{f}'_{\theta}\right) \tag{39}$$

where the ith element of $\boldsymbol{f}'_{\theta}$ is given by

$$f'_{\theta i} = \gamma_i\phi^{2\alpha_i}\left|\tilde{\theta}_i\right|^{\alpha_i}\mathrm{sign}(\tilde{\theta}_i) \tag{40}$$

for $i = 1,\ldots,p$. To get (40), from (34), (37) becomes

$$f_{\theta i} = \gamma_i\left\lfloor\phi\left(\phi\tilde{\theta}_i\right)\right\rceil^{\alpha_i} = \gamma_i\left|\phi^2\tilde{\theta}_i\right|^{\alpha_i}\mathrm{sign}\left(\phi^2\tilde{\theta}_i\right). \tag{41}$$

By taking into consideration the sign definition in (24), then it is clear that $\mathrm{sign}\left(\phi^2\tilde{\theta}_i\right) = \mathrm{sign}\left(\tilde{\theta}_i\right)$ if $\phi \neq 0$ while if $\phi \equiv 0$, then $f_{\theta i} \equiv 0$, so that (40) can be gotten from (41). Note that even though $\boldsymbol{f}_{\theta} \equiv \boldsymbol{f}'_{\theta}$, $\boldsymbol{f}_{\theta}$ is implementable while $\boldsymbol{f}'_{\theta}$ is not. Therefore, the latter is used only for stability analysis.

The following error dynamics for $\boldsymbol{s}$ can be computed by substituting (22) into (13)

$$\boldsymbol{H}(\boldsymbol{q})\dot{\boldsymbol{s}} = -\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} - \boldsymbol{D}\boldsymbol{s} - \boldsymbol{K}_{\mathrm{v}}\lfloor\boldsymbol{s}\rceil^{\boldsymbol{\alpha}_{\mathrm{s}}} + \boldsymbol{Y}_{\mathrm{a}}\tilde{\boldsymbol{\theta}}. \tag{42}$$

Define the state $\boldsymbol{x} \in \mathbb{R}^{n+p}$ of (39) and (42) as

$$\boldsymbol{x} = \begin{bmatrix}\boldsymbol{s}^{\mathrm{T}} & \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \tag{43}$$

and consider the following positive-definite function

$$V(t,\boldsymbol{x}(t)) = \frac{1}{2}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{H}(\boldsymbol{q})\boldsymbol{s} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\tilde{\boldsymbol{\theta}} \tag{44}$$

which satisfies

$$\lambda_1\|\boldsymbol{x}\|^2 \leq V(t,\boldsymbol{x}(t)) \leq \lambda_2\|\boldsymbol{x}\|^2 \tag{45}$$

with $\lambda_1 = \frac{1}{2}\min\left\{\lambda_{\mathrm{h}}, \frac{1}{\psi_{\max}}\right\}$ and $\lambda_2 = \frac{1}{2}\max\left\{\lambda_{\mathrm{H}}, \frac{1}{\psi_{\min}}\right\}$, where λ_{h} and λ_{H} are defined in Property 3.1, $\psi_{\min} = \lambda_{\min}(\boldsymbol{\Psi})$ and $\psi_{\max} = \lambda_{\max}(\boldsymbol{\Psi})$. Then, the following theorem establishes some properties of (39) and (42).

Theorem III.1: Consider the closed-loop error dynamics (39) and (42) generated by substituting the control law (22) into the robot dynamics (13) and by using the adaptive algorithm (38). The following holds.

a) $\boldsymbol{s}$, $\boldsymbol{e}$, $\dot{\boldsymbol{e}}$, and $\tilde{\boldsymbol{\theta}}$ remain bounded for all $t \geq t_0$.
b) $\boldsymbol{s}$, $\boldsymbol{e}$ and $\dot{\boldsymbol{e}} \to \boldsymbol{0}$ as $t \to \infty$.
c) Choose $\alpha_j = \alpha_{si} = \alpha$ for all $j = 1,\ldots,p$ and for all $i = 1,\ldots,n$ with $\alpha \in (0.5,1)$, and set $k_{\mathrm{vm}} = \lambda_{\min}(\boldsymbol{K}_{\mathrm{v}})$ and $\gamma_{\min} = \lambda_{\min}(\boldsymbol{\Gamma})$. Furthermore, define for any $t_{\mathrm{s}} \geq t_0$

$$\kappa = \frac{k_{\mathrm{vm}}}{\lambda_2^{\frac{1+\alpha}{2}}} \tag{46}$$

$$T^* = \frac{2}{\kappa(1-\alpha)}V^{\frac{1-\alpha}{2}}(t_{\mathrm{s}}) \tag{47}$$

$$t_{\mathrm{e}} = t_{\mathrm{s}} + T^*. \tag{48}$$

If for some $t_{\mathrm{s}} \geq t_0$ it holds

$$\phi^2 \geq \left(\frac{k_{\mathrm{vm}}}{\gamma_{\min}}\right)^{\frac{1}{\alpha}} \tag{49}$$

for all $t \in [t_{\mathrm{s}}, t_{\mathrm{e}}]$, then

$$\tilde{\boldsymbol{\theta}} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{s} = \boldsymbol{0} \quad \forall t \geq t_{\mathrm{e}}. \tag{50}$$

Proof:

a) By using Property 3.2, the derivative of $V = V(t,\boldsymbol{x}(t))$ along (39) and (42) can be shown to satisfy

$$\dot{V} = -\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{s} - \boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{v}}\lfloor\boldsymbol{s}\rceil^{\boldsymbol{\alpha}_{\mathrm{s}}} - \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{f}'_{\theta}. \tag{51}$$

From (23) and (40), one has

$$\dot{V} = -\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{s} - \sum_{i=1}^{n}k_{\mathrm{v}i}|s_i|^{1+\alpha_{\mathrm{s}i}} - \sum_{j=1}^{p}\gamma_j\phi^{2\alpha_j}|\tilde{\theta}_j|^{1+\alpha_j} \tag{52}$$

where s_i is the ith element of $\boldsymbol{s}$ and $k_{\mathrm{v}i}$ is the ith element of the diagonal of $\boldsymbol{K}_{\mathrm{v}}$ for $i = 1,\ldots,n$. Also, $\tilde{\theta}_j$ is the jth element of $\tilde{\boldsymbol{\theta}}$ for $j = 1,\ldots,p$. Since $V(t,\boldsymbol{x}(t))$ is positive definite and $\dot{V} \leq 0$ for all $t \geq t_0$, then $\tilde{\boldsymbol{\theta}}, \boldsymbol{s} \in \mathcal{L}_{\infty}$. But since $\boldsymbol{s}$ is the input of the linear filter given by (18), this, in turn, means that $\dot{\boldsymbol{e}}, \boldsymbol{e} \in \mathcal{L}_{\infty}$.

b) To show that $\boldsymbol{s} \to \boldsymbol{0}$ and as a consequence $\dot{\boldsymbol{e}}, \boldsymbol{e} \to \boldsymbol{0}$, it must be shown that $\dot{V} \to 0$ as $t \to \infty$. For this goal, take into account that V must have a finite limit as $t \to \infty$ because it is positive definite and $\dot{V} \leq 0$. According to Barbalat's Lemma, $\dot{V} \to 0$ if it is uniformly continuous [13]. A sufficient condition to show this is to prove that $\ddot{V}$ is bounded. In view of (52), $\dot{V}$ has three basic terms. The first one is $-\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{s}$ with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(-\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{s}) = -2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\dot{\boldsymbol{s}}. \tag{53}$$

Clearly, $\frac{\mathrm{d}}{\mathrm{d}t}(-\boldsymbol{s}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{s})$ will be bounded if $\dot{\boldsymbol{s}}$ is bounded. By taking into account that $\boldsymbol{H}(\boldsymbol{q})$ is invertible, then from (42), $\dot{\boldsymbol{s}}$ will be bounded as long as $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are bounded since it has already been shown that $\boldsymbol{e}, \dot{\boldsymbol{e}}, \boldsymbol{s}$, and $\tilde{\boldsymbol{\theta}}$ are bounded, and therefore, $\dot{\boldsymbol{q}}_{\mathrm{r}}$ and $\ddot{\boldsymbol{q}}_{\mathrm{r}}$ must be bounded because $\boldsymbol{q}_{\mathrm{d}}$, $\dot{\boldsymbol{q}}_{,\mathrm{d}}$ and $\ddot{\boldsymbol{q}}_{\mathrm{d}}$ are bounded by assumption. By taking into account that $\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_{\mathrm{d}}$ and $\dot{\boldsymbol{e}} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_{\mathrm{d}}$, then $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ must be bounded as well. Note that $\boldsymbol{Y}_{\mathrm{a}}$ is also bounded in view of the previous analysis and by considering (21). Therefore, $\dot{\boldsymbol{s}}$ is bounded for all $t \geq t_0$ and so must be (53).

The second term of $\dot{V}$ is the elements $-k_{\mathrm{v}i}|s_i|^{1+\alpha_{\mathrm{s}i}}$ for $i = 1,\ldots,n$. By taking into account that $|s_i| = \sqrt{s_i^2} = (s_i^2)^{\frac{1}{2}}$, it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-k_{\mathrm{v}i}|s_i|^{1+\alpha_{\mathrm{s}i}}\right) = -k_{\mathrm{v}i}(1+\alpha_{\mathrm{s}i})|s_i|^{\alpha_{\mathrm{s}i}}\mathrm{sign}(s_i)\dot{s}_i. \tag{54}$$

Since it has already been shown that $\dot{\boldsymbol{s}}$ is bounded, so must be its ith element $\dot{s}_i$. Thus, the derivative in (54) is also bounded for all $t \geq t_0$. Finally, it remains to analyze the terms $-\gamma_i\phi^{2\alpha_i}|\tilde{\theta}_i|^{1+\alpha_i}$ of $\dot{V}$ for $i = 1,\ldots,p$. To compute its derivative, it is taken into account once again that $|\tilde{\theta}_i| = \sqrt{\tilde{\theta}_i^2}$ to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\gamma_i\phi^{2\alpha_i}|\tilde{\theta}_i|^{1+\alpha_i}\right) = &-\gamma_i 2\alpha_i\phi^{2\alpha_i-1}|\tilde{\theta}_i|^{1+\alpha_i}\dot{\phi}\\ &-\gamma_i\phi^{2\alpha_i}(1+\alpha_i)|\tilde{\theta}_i|^{\alpha_i}\mathrm{sign}(\tilde{\theta}_i)\dot{\tilde{\theta}}_i.\end{aligned} \tag{55}$$

The terms $-\gamma_i 2\alpha_i\phi^{2\alpha_i-1}|\tilde{\theta}_i|^{1+\alpha_i}\dot{\phi}$ will be bounded if both ϕ and $\dot{\phi}$ are bounded because it has already been shown that $\tilde{\boldsymbol{\theta}}$ is bounded. Also, since $\alpha_i \in (0.5,1)$ for all $i = 1,\ldots,p$, then $2\alpha_i - 1 > 0$ so that $\phi^{2\alpha_i-1} = 0$ if $\phi = 0$. Consider now that the determinant of any matrix can be obtained by using the Laplace expansion as explained in [11], which shows that the determinant ϕ is a sum of products of the elements of $\boldsymbol{Y}_{\mathrm{f}}$, and thus, it will be bounded if so are those elements. To show the boundedness of $\boldsymbol{Y}_{\mathrm{f}}$, consider first the filtered regressor $\boldsymbol{Y}_{\varphi 1}$ given in (25), which is the state of a linear first-order stable filter, and thus, it and its derivative $\dot{\boldsymbol{Y}}_{\varphi 1}$ will be bounded as long as the filter's input, the regressor $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, is bounded. But $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ must be bounded because it has already been shown that both $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are so while $\ddot{\boldsymbol{q}}$ must be bounded as well since $\ddot{\boldsymbol{q}} = \dot{\boldsymbol{s}} + \ddot{\boldsymbol{q}}_{\rm d} - \boldsymbol{\Lambda}\dot{\boldsymbol{e}}$ holds. By construction, the first n rows of $\boldsymbol{Y}_{\rm f}$ in (29) are those of $\boldsymbol{Y}_{\varphi 1}$ while the rest of them are computed according to the stable linear filters given in (31) for $j = 2, \ldots, \tilde{n}$. Since the inputs of those linear filters are the rows of the bounded filtered regressor $\boldsymbol{Y}_{\varphi 1}$, then this implies that both $\boldsymbol{Y}_{\rm f}$ and $\dot{\boldsymbol{Y}}_{\rm f}$ must be bounded. This shows that $\phi = \det(\boldsymbol{Y}_{\rm f})$ as defined in (36) is bounded. Now it is quite direct to prove the boundedness of $\dot{\phi}$ by using Jacobi's formula [14], i.e., $\dot{\phi} = \mathrm{trace}(\mathrm{adj}(\boldsymbol{Y}_{\rm f})\dot{\boldsymbol{Y}}_{\rm f})$. This shows that $\dot{\phi}$ is bounded because it has been shown that $\boldsymbol{Y}_{\rm f}$ and $\dot{\boldsymbol{Y}}_{\rm f}$ are bounded.

In fact, the boundedness of ϕ implies that of $\dot{\tilde{\boldsymbol{\theta}}}$ as well, as can be concluded from (39) and (40). This means that the terms $-\gamma_i \phi^{2\alpha_i}(1+\alpha_i)|\tilde{\theta}_i|^{\alpha_i}\mathrm{sign}(\tilde{\theta}_i)\dot{\tilde{\theta}}_i$ in (55) are bounded, which implies that the third term of $\ddot{V}$ given by (55) is bounded. In conclusion, the derivative $\ddot{V}$ of $\dot{V}$ is bounded, which according to Barbalat's Lemma means that $\dot{V} \to 0$. In turn, from (52), this guarantees that $\boldsymbol{s} \to \boldsymbol{0}$ as $t \to \infty$, and thus, $\boldsymbol{e}, \dot{\boldsymbol{e}} \to \boldsymbol{0}$ as $t \to \infty$. Note that despite (52) containing the elements of $\tilde{\boldsymbol{\theta}}$ as well, this does not necessarily mean that $\tilde{\boldsymbol{\theta}} \to \boldsymbol{0}$ since the case can be that $\phi \equiv 0$.

c) Let $\alpha_j = \alpha_{{\rm s}i} = \alpha \in (0.5, 1)$ for all $j=1, \ldots, p$ and for all $i=1, \ldots, n$. Then, by taking into account that $-\boldsymbol{s}^{\rm T}\boldsymbol{D}\boldsymbol{s} \le 0$ and that $k_{\rm vm} = \min\{k_{\rm v1}, \ldots, k_{{\rm v}n}\}$ and $\gamma_{\min} = \min\{\gamma_1, \ldots, \gamma_p\}$, $\dot{V}$ in (52) satisfies

$$\dot{V} \le -\sum_{i=1}^{n} k_{\rm vm}|s_i|^{1+\alpha} - \sum_{j=1}^{p} \gamma_{\min}\phi^{2\alpha}|\tilde{\theta}_j|^{1+\alpha}. \tag{56}$$

Now, assume that condition (49) holds for all $t \in [t_{\rm s}, t_{\rm e}]$ for some $t_{\rm s} \ge t_0$ and $t_{\rm e} = t_{\rm s} + T^* < \infty$, with T^* given in (47). For that case, $\dot{V}$ satisfies for all $t \in [t_{\rm s}, t_{\rm e}]$

$$\dot{V} \le -\sum_{i=1}^{n} k_{\rm vm}(s_i^2)^{\frac{1+\alpha}{2}} - \sum_{j=1}^{p} k_{\rm vm}(\tilde{\theta}_j^2)^{\frac{1+\alpha}{2}}. \tag{57}$$

Consider the definition of the state $\boldsymbol{x}$ in (43), which obviously implies that

$$\dot{V} \le -\sum_{i=1}^{n+p} k_{\rm vm}(x_i^2)^{\frac{1+\alpha}{2}} \tag{58}$$

where x_i is the ith element of $\boldsymbol{x}$ for $i = 1, \ldots, n+p$. Apply Lemma A.1 by considering $c = \frac{1+\alpha}{2} < 1, m = n+p$ and $a_i = x_i^2$ for $i = 1, \ldots, n+p$ to get

$$\left(\sum_{i=1}^{m} x_i^2\right)^{\frac{1+\alpha}{2}} \le \sum_{i=1}^{m}(x_i^2)^{\frac{1+\alpha}{2}} \tag{59}$$

so that (58) becomes

$$\dot{V} \le -k_{\rm vm}(\|\boldsymbol{x}\|^2)^{\frac{1+\alpha}{2}} = -\frac{k_{\rm vm}}{\lambda_2^{\frac{1+\alpha}{2}}}(\lambda_2\|\boldsymbol{x}\|^2)^{\frac{1+\alpha}{2}}. \tag{60}$$

In view of (45) and (60), it can be concluded that

$$\dot{V} \le -\kappa V^{\frac{1+\alpha}{2}} \tag{61}$$

for all $t \in [t_{\rm s}, t_{\rm e}]$, where κ is given in (46). According to the Comparison Lemma [13], the solution of (61) satisfies for $t \in [t_{\rm s}, t_{\rm e}]$

$$V(t) \le \left(-\kappa\frac{1-\alpha}{2}(t-t_{\rm s}) + V^{\frac{1-\alpha}{2}}(t_{\rm s})\right)^{\frac{2}{1-\alpha}} \tag{62}$$

where the notation $V(t) = V(t, \boldsymbol{x}(t))$ is used for simplicity. Therefore, at $t = t_{\rm e} = t_{\rm s} + T^*$, one has from (47)

$$V(t_{\rm e}) \le 0. \tag{63}$$

Since $V(t) \ge 0$ for all $t \ge t_0$, then necessarily the equality $V(t_{\rm e}) \equiv 0$ holds, which means that

$$V(t, \boldsymbol{x}(t)) = \frac{1}{2}\boldsymbol{s}^{\rm T}\boldsymbol{H}(\boldsymbol{q})\boldsymbol{s} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^{\rm T}\boldsymbol{\Psi}^{-1}\tilde{\boldsymbol{\theta}} \equiv 0 \tag{64}$$

at $t = t_{\rm e} = t_{\rm s} + T^*$. This can be possible if and only if $\boldsymbol{s} \equiv \boldsymbol{0}$ and $\tilde{\boldsymbol{\theta}} \equiv \boldsymbol{0}$, which means that the equilibrium point of (39) and (42) has been reached in finite time. Therefore, (50) holds for all $t \ge t_{\rm e}$, whether condition (49) is satisfied for $t > t_{\rm e}$ or not. △

Remark III.2: To the best of the author's knowledge, condition (49) given in Theorem 3.1 is conceptually different from any other parameter convergence condition in the sense that it is not expressed as an integral but as a lower constant bound for a scalar function to hold for a finite lapse of time. Note that it is *not* equivalent to

$$\int_{t_{\rm s}}^{t_{\rm e}=t_{\rm s}+T^*} \phi^2(\vartheta){\rm d}\vartheta \ge \mu = \left(\frac{k_{\rm vm}}{\gamma_{\min}}\right)^{\frac{1}{\alpha}} T^* \tag{65}$$

which represents the interval excitation (IE) assumption [14]. Indeed, condition (49) is stronger, i.e., more restrictive, since if it holds, then (65) holds as well but the contrary is not true in general. Note that condition (49) is also stronger than the initial excitation condition (denominated IE as well). The reason is that if the IE condition holds, then so does the initial excitation condition [15]. The second IE has the advantage that it can completely be verified online while relationship (49) can only be partially verified online but this is enough to allow returning if necessary. △

Remark III.3: The modification introduced in the control law (22) is made to achieve finite-time parameter estimation, not finite-time tracking error. Even if (49) holds, still one has $\boldsymbol{e}, \dot{\boldsymbol{e}} \to \boldsymbol{0}$ as $t \to \infty$. To see the necessity of modifying the control law to achieve finite-time parameter estimation, assume that $\tilde{\boldsymbol{\theta}} \equiv \boldsymbol{0}$ at some $t_{\rm e} \ge t_0$. By design, this renders the DREM term in (38) zero, i.e., $\boldsymbol{f}_\theta \equiv \boldsymbol{0}$, so that the adaptive term becomes the standard gradient algorithm alone, i.e.

$$\dot{\tilde{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}\boldsymbol{Y}_{\rm a}^{\rm T}\boldsymbol{s}.$$

If $\boldsymbol{s} \ne \boldsymbol{0}$ then $\tilde{\boldsymbol{\theta}} = \boldsymbol{0}$ will not necessarily hold for all $t \ge t_{\rm e}$ because, in general, $\dot{\tilde{\boldsymbol{\theta}}} \ne \boldsymbol{0}$. Thus, in order to *enforce* $\tilde{\boldsymbol{\theta}} \equiv \boldsymbol{0}$ and $\dot{\tilde{\boldsymbol{\theta}}} \equiv \boldsymbol{0}$ for all $t \ge t_{\rm e}$, not only $\tilde{\boldsymbol{\theta}}$ has to become zero at $t_{\rm e} \ge t_0$ but also $\boldsymbol{s}$. This goal is achieved by introducing in the control law (22) the term $\lfloor \boldsymbol{s} \rceil^{\boldsymbol{\alpha}_{\rm s}}$. △

Remark III.4: Setting $\alpha = 1$ yields to the simplified version

$$\boldsymbol{\tau} = \boldsymbol{Y}_{\rm a}\hat{\boldsymbol{\theta}} - \boldsymbol{K}_{\rm v}\boldsymbol{s} \qquad \dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{\Psi}(\boldsymbol{Y}_{\rm a}^{\rm T}\boldsymbol{s} + \boldsymbol{\Gamma}\phi^2\tilde{\boldsymbol{\theta}})$$

where an abuse of notation is made by writing $\tilde{\boldsymbol{\theta}}$ explicitly. It is straightforward to show that items a) and b) of Theorem 3.1 are still fulfilled while the convergence conditions become different [4]. This simpler version represents an acceptable alternative in case the robot's trajectories are chosen such that enough excitation is provided for parameter estimation. △

IV. Simulation Results

To test the adaptive algorithm (38) proposed in the previous section, a simplified version of the robot model *CRS*-A465 described in [12] is employed. Instead of using the six degrees of freedom, only joints 2, 3, and 5 (renamed as 1, 2, and 3, respectively) are considered while the rest of them are fixed. Furthermore, motors dynamics are excluded for simplicity's sake. The resulting model is given in (66), where the following definitions have been made $s_2 = \sin(q_2)$, $s_3 = \sin(q_3)$, $s_{12} = \sin(q_1 + q_2)$, $s_{23} = \sin(q_2 + q_3)$, $s_{123} = \sin(q_1 + q_2 + q_3)$, $c_1 = \cos(q_1)$, $c_2 = \cos(q_2)$, $c_3 = \cos(q_3)$, $c_{23} = \cos(q_2 + q_3)$, $\dot{q}_{12} = \dot{q}_1 + \dot{q}_2$, $\dot{q}_{23} = \dot{q}_2 + \dot{q}_3$, $\dot{q}_{123} = \dot{q}_1 + \dot{q}_2 + \dot{q}_3$. The corresponding parameters are $\theta_1 = 6.3922\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_2 = 1.4338\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_3 = 0.0706\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_4 = 0.0653\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_5 = 2.4552\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_6 = 0.2868\,[\mathrm{kg}\cdot\ \mathrm{m}^2]$, $\theta_7 = 113.6538\,[\mathrm{N}\cdot\ \mathrm{m}]$, $\theta_8 = 46.1168\,[\mathrm{N}\cdot\ \mathrm{m}]$, $\theta_9 = 2.0993\,[\mathrm{N}\cdot\ \mathrm{m}]$, $\theta_{10} = 2.6\,[\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}]$, $\theta_{11} = 2.5\,[\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}]$, and $\theta_{12} = 1.5\,[\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}]$. The regressor $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ can easily be obtained by direct computation but it is omitted here for lack of room.

$$
\begin{bmatrix}
\begin{matrix}\theta_1 + 2\theta_2 s_2 + \\ 2\theta_3 c_3 + 2\theta_4 s_{23}\end{matrix} & \begin{matrix}\theta_2 s_2 + 2\theta_3 c_3 + \\ \theta_4 s_{23} + \theta_5\end{matrix} & \theta_3 c_3 + \theta_4 s_{23} + \theta_6 \\
\begin{matrix}\theta_2 s_2 + 2\theta_3 c_3 + \\ \theta_4 s_{23} + \theta_5\end{matrix} & 2\theta_3 c_3 + \theta_5. & \theta_3 c_3 + \theta_6 \\
\theta_3 c_3 + \theta_4 s_{23} + \theta_6 & \theta_3 c_3 + \theta_6 & \theta_6
\end{bmatrix} \ddot{\boldsymbol{q}}
$$

$$
+ \begin{bmatrix}
\begin{matrix}\theta_2 c_2 \dot{q}_2 - \theta_3 s_3 \dot{q}_3 + \\ \theta_4 c_{23} \dot{q}_{23}\end{matrix} & \begin{matrix}\theta_2 c_2 \dot{q}_{12} - \theta_3 s_3 \dot{q}_3 + \\ \theta_4 c_{23} \dot{q}_{123}\end{matrix} & \begin{matrix}-\theta_3 s_3 \dot{q}_{123} + \\ \theta_4 c_{23} \dot{q}_{123}\end{matrix} \\
\begin{matrix}-\theta_2 c_2 \dot{q}_1 - \theta_3 s_3 \dot{q}_3 \\ -\theta_4 c_{23} \dot{q}_1\end{matrix} & -\theta_3 s_3 \dot{q}_3 & -\theta_3 s_3 \dot{q}_{123} \\
\theta_3 s_3 \dot{q}_{12} - \theta_4 c_{23} \dot{q}_1 & \theta_3 s_3 \dot{q}_{12} & 0
\end{bmatrix} \dot{\boldsymbol{q}}
$$

$$
+ \begin{bmatrix}
\theta_7 c_1 + \theta_8 s_{12} + \theta_9 s_{123} \\
\theta_8 s_{12} + \theta_9 s_{123} \\
\theta_9 s_{123}
\end{bmatrix} + \begin{bmatrix}\theta_{10}\dot{q}_1 \\ \theta_{11}\dot{q}_2 \\ \theta_{12}\dot{q}_3\end{bmatrix} = \boldsymbol{\tau}, \tag{66}
$$

A. First Simulation: Not PE Case

Two simulations have been carried out. Recall that in [2], it was shown that the standard gradient adaptive algorithm in [1] achieves parameter estimation if and only if $\boldsymbol{Y}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}, \ddot{\boldsymbol{q}}_{\mathrm{d}})$ is PE. Therefore, it makes just sense to choose

$$\boldsymbol{q}_{\mathrm{d}} = [90^\circ\ -90^\circ\ 0^\circ]^{\mathrm{T}} \tag{67}$$

while obviously $\dot{\boldsymbol{q}}_{\mathrm{d}} = \ddot{\boldsymbol{q}}_{\mathrm{d}} = \mathbf{0}$. By looking at (66), this implies that $\boldsymbol{Y}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}, \ddot{\boldsymbol{q}}_{\mathrm{d}}) \equiv \boldsymbol{O}$. It was set $\boldsymbol{q}(t_0) = [0^\circ\ 0^\circ\ 90^\circ]^{\mathrm{T}}$.

The gains for the controller given by (16)–(17) and (22) have been chosen as $\boldsymbol{\Lambda} = 10\boldsymbol{I}, \boldsymbol{K}_{\mathrm{v}} = 10\boldsymbol{I}$, and $\alpha_{\mathrm{s}1} = \alpha_{\mathrm{s}2} = \alpha_{\mathrm{s}3} = 0.50001$. As to the implementation of the adaptive law (37)–(38), it was set $\boldsymbol{\Psi} = 10\boldsymbol{I}$, $\boldsymbol{\Gamma} = 10\boldsymbol{I}$, and $\alpha_i = 0.50001$ for $i = 1, \ldots, p$. For the filtered regressor in (25), it was set $\lambda_\varphi = 1$. Also, to compute $\boldsymbol{Y}_{\mathrm{f}}$ in (29), it was set in (31) $b_2 = 0.2$, $b_3 = 0.3$, $b_4 = 0.4$, and $a_2 = a_3 = a_4 = 700$. To have a point of comparison, the standard controller and gradient algorithm (19)–(20) were also tested with the same gains. The results for position tracking can be seen in Fig. 1. It can be appreciated that both schemes yield the desired positions in less than 2[s] with practically the very same behavior. On the other hand, Fig. 2 shows the estimation errors. Just as expected, the scheme (19)–(20) was not capable of estimating the actual parameters. On the contrary, the proposed new adaptive law was able to estimate all of them exactly. Condition (49) holds if $\phi^2 \geq \left(\frac{k_{\mathrm{vm}}}{\gamma_{\min}}\right)^{\frac{1}{\alpha}} \equiv 1$. In Fig. 3, it can be seen that this is the case for $t \geq 1.43$[s]. Note that even though most parameter estimation errors actually become zero before, only after (49) holds, all of them vanish.

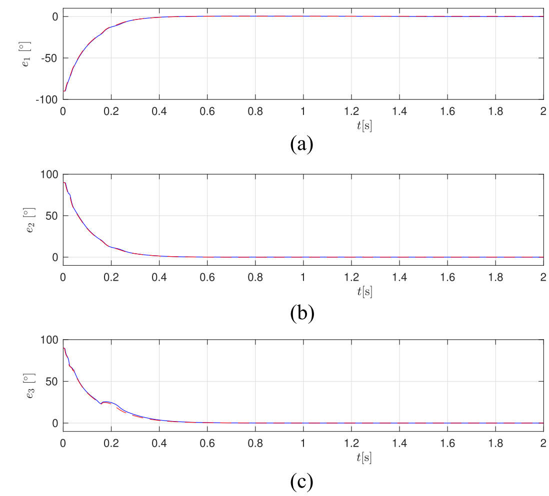

Fig. 1. First simulation. Joints position tracking errors. Proposed algorithm (——) and gradient algorithm (– – –).

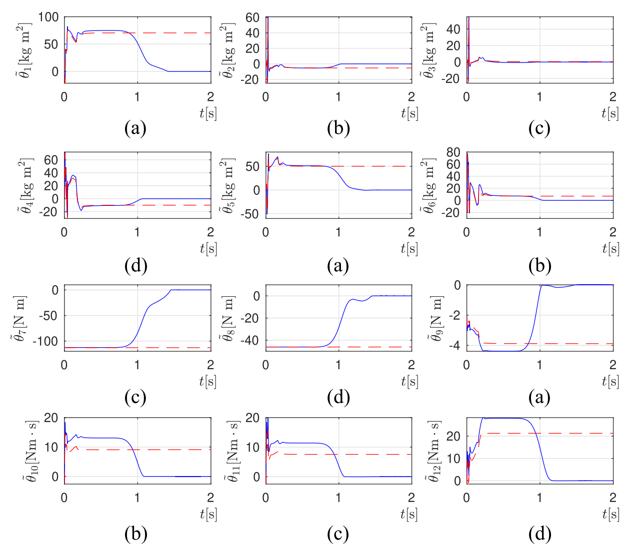

Fig. 2. First simulation. Parameter estimation errors. Proposed algorithm (——) and gradient algorithm (– – –).

Remark IV.1: Just setting $\boldsymbol{K}_{\mathrm{v}}$ small or $\boldsymbol{\Gamma}$ large does not guarantee that (49) will hold. Indeed, due to the definition of $\boldsymbol{Y}_{\mathrm{f}}$, ϕ actually depends on $\boldsymbol{K}_{\mathrm{v}}$ as well, so that changing it affects both sides of (49) in a nonevident way. The same can be said for $\boldsymbol{\Gamma}$. Other factors that can affect whether (49) holds or not are the desired trajectories, meaning that for some of them, it can hold but for some others not. △

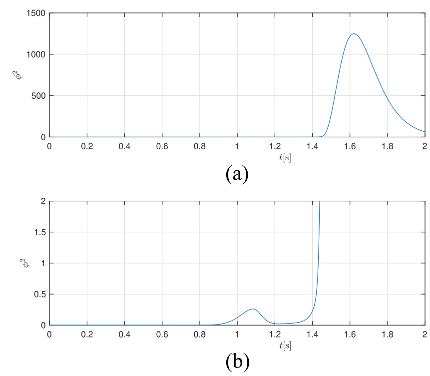

Fig. 3. First simulation. Square determinant ϕ^2. (a) Full scale. (b) Zoom for $\phi^2 \in [0, 2]$.

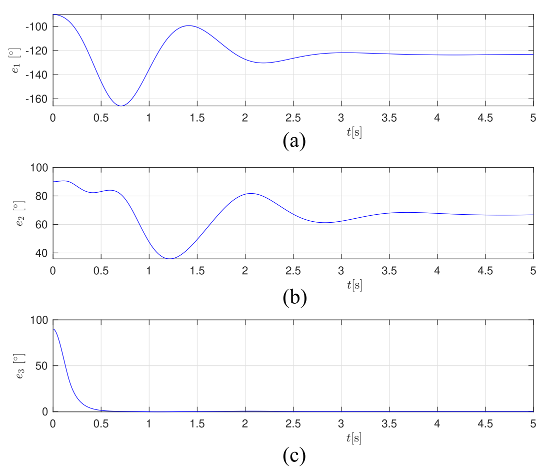

Fig. 4. Second simulation. Joints position tracking errors. Proposed algorithm without gradient term.

B. Second Simulation: DREM Component Alone

The last statement of Section II can be confusing. Looking at Fig. 2, it is clear that it is the DREM term in (38), which achieves parameter convergence. Therefore, the gradient component appears to be unnecessary. Suppose that it is removed and all gains used in Section IV-A are chosen again but for $b_2 = 15$, $b_3 = 20$, and $b_4 = 25$. The result is given in Fig. 4, where it can be appreciated that only e_3 becomes zero but $e_1 \approx -122^\circ$ and $e_2 \approx 65^\circ$. This poor behavior is explained because ϕ^2 becomes 0 without matching (49). The corresponding figure is omitted for lack of room but the fact that $\phi^2 \equiv 0$ automatically stops adaptation. It is well known that position regulation can be achieved by using a PD and gravity compensation. Fig. 5 shows the parameter errors related to the gravity vector only. Just as expected for the DREM approach, $\tilde{\theta}_7, \tilde{\theta}_8$, and $\tilde{\theta}_9$ diminish their values until adaptation stops and the large errors observed in Fig. 4 arise. Even though for this example, it is possible to find a combination of gains to make both parameter and tracking errors tend to zero, it is not guaranteed *a priori* that the same set of gains will work for any possible desired trajectory. This is what is meant for unacceptable and it is in clear contrast with the gradient algorithm alone, which makes all errors tend to zero, despite *not even one single* parameter was estimated (not even $\tilde{\theta}_3$ becomes zero exactly), as shown in Fig. 2. The adaptive scheme in [9] also studies tracking control of Euler–Lagrange systems using an adaptive DREM-based algorithm but it must be assumed that an IE condition holds to guarantee trajectory tracking. It is not discussed the case when it does not.

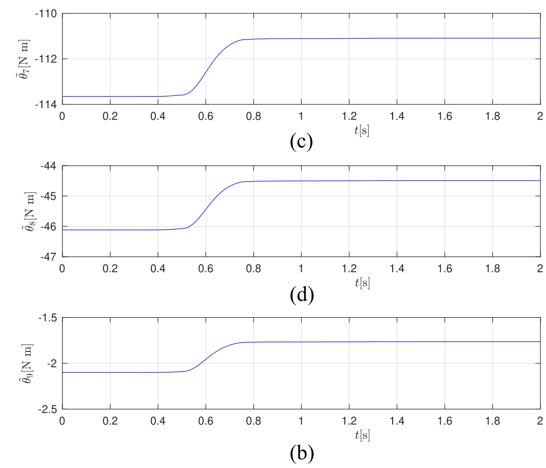

Fig. 5. Second simulation. Parameter estimation errors $\tilde{\theta}_7$, $\tilde{\theta}_8$, and $\tilde{\theta}_9$ for the gravity vector. Proposed algorithm without gradient term.

V. Conclusion

For many robot manipulator tasks, the main objective consists of following position and velocity time-varying trajectories. To achieve this goal, an accurate robot model should be employed but this is hard to obtain in practice. To deal with this problem, adaptive control schemes have been developed where precise parameter estimation has been left as a desirable but not mandatory plus. The reason is that for most schemes, the PE is necessary for this target. In recent years, the so-called DREM procedure was developed to provide an alternative in the design of adaptive laws with conditions different from the PE. When met, there is an improvement in parameter estimation performance, but when not, adaptation can simply stop, which might just be unacceptable for the control of robot manipulators.

This article proposes, for the first time, a composite scheme, which combines the standard gradient adaptive law with a DREM-based additional term with the following properties: 1) Trajectory tracking for joint desired positions and velocities is guaranteed; 2) in the absence of PE, if a new condition for the additional DREM-based term is matched, then exact parameter estimation takes place in finite time. The new condition for exact parameter estimation in finite time is different from previous ones found in the literature and it is more relaxed than the PE condition but stronger than the IE assumptions. It has the advantage that it can be partially verified online, which is useful for gain tuning. Simulation results for a three-degrees-of-freedom robot manipulator with 12 parameters show a very good performance of the new proposal since all the parameters are estimated in finite time. As future research, it remains to carry out a stability analysis for the discretization of the new algorithm, as well as to examine how noisy measurements affect the performance of the present proposal.

Appendix A

Lemma A.1: Assume that for $i = 1, \ldots, m$ one has $a_i \in \mathbb{R}$ satisfying $a_i \geq 0$. Then, the following holds.

a) $\left(\sum_{i=1}^m a_i^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^m a_i$.

b) For $0 < c < 1$, $\left(\sum_{i=1}^m a_i\right)^c \leq \sum_{i=1}^m a_i^c$.

Proof: By direct computation. △

Acknowledgment

The author warmly thanks all unknown reviewers for so many useful comments and suggestions.

References

[1] J. J. E. Slotine and W. Li, "On the adaptive control of robot manipulators," *Int. J. Robot. Res.*, vol. 6, no. 3, pp. 49–59, 1987.

[2] A. Loria, R. Kelly, and A. R. Teel, "Uniform parametric convergence in the adaptive control of mechanical systems," *Eur. J. Control*, vol. 11, pp. 87–100, 2005.

[3] A. Loria, E. Panteley, and M. Maghenem, "Strict Lyapunov functions for model reference adaptive control: Application to Lagrangian systems," *IEEE Trans. Autom. Control*, vol. 64, no. 7, pp. 3040–3045, Jul. 2019.

[4] S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin, "Performance enhancement of parameter estimators via dynamic regressor extension and mixing," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3546–3550, Jul. 2017.

[5] R. Ortega, L. Praly, S. Aranovskiy, B. Yi, and W. Zhang, "On dynamic regressor extension and mixing parameter estimators: Two Luenberger observers interpretations," *Automatica*, vol. 95, no. 6, pp. 548–551, 2018.

[6] J. Wang, D. Efimov, and A. Bobtsov, "On robust parameter estimation in finite-time without persistence of excitation," *IEEE Trans. Autom. Control*, vol. 65, no. 5, pp. 1731–1738, Apr. 2020.

[7] M. Korotina, S. Aranovskiy, R. Ushirobira, and A. Vedyakov, "On parameter tuning and convergence properties of the DREM procedure," in *Proc. Eur. Control Conf.*, 2020, pp. 53–58.

[8] R. Ortega, V. Gromov, E. Nuño, A. Pyrkin, and J. G. Romero, "Parameter estimation of nonlinearly parameterized regressions without overparameterization: Application to adaptive control," *Automatica*, vol. 127, 2021, Art. no. 109544.

[9] J. G. Romero, R. Ortega, and A. Bobtsov, "Parameter estimation and adaptive control of Euler–Lagrange systems using the power balance equation parameterisation," *Int. J. Control*, vol. 96, pp. 475–487, 2021.

[10] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. New York, NY, USA: Dover, 2012.

[11] R. A. Horn and C. A. Johnson, *Matrix Analysis*, 2nd ed. Cambridge, MA, USA: Cambridge Univ. Press, 2013.

[12] M. A. Arteaga, A. Gutiérrez-Giles, and J. Pliego-Jiménez, *Local Stability and Ultimate Boundedness in the Control of Robot Manipulators*. Cham, Switzerland: Springer, 2022.

[13] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[14] R. Ortega, V. Nikiforov, and D. Gerasimov, "On modified parameter estimators for identification and adaptive control. A unified framework and some new schemes," *Annu. Rev. Control*, vol. 50, pp. 278–293, 2020.

[15] S. B. Roy, "Relaxing persistence of excitation for parameter convergence in adaptive control: An initial excitation based approach," Ph.D. dissertation, Dept. Elect. Eng., Indian Inst. Technol., Delhi, New Delhi, India, 2019.