Photon absorption and scattering cross-section

Electrical Engineering Preliminary Exam Supplemental Report
Professor Chandrashekhar Joshi

University of California, Los Angeles
Electrical Engineering

Name : Hiroki Fujii

January 14, 2019

1. Overview

This report summarizes the dominant absorption and scattering mechanisms and the cross-sections for each energy range of photons incident on a slab material shown in Figure 1. As shown in Figure 1, the photoelectric absorption ($\sim$100 keV), Compton scattering (100 keV$\sim$10 MeV, and the electron-positron pair production (10 MeV$\sim$) are the dominant process in each energy range.

In the following derivation, all the energy and momentum are normalized by m_0c^2 and m_0c, where m_0 is the electron mass at rest and c is the speed of light in vacuum respectively.

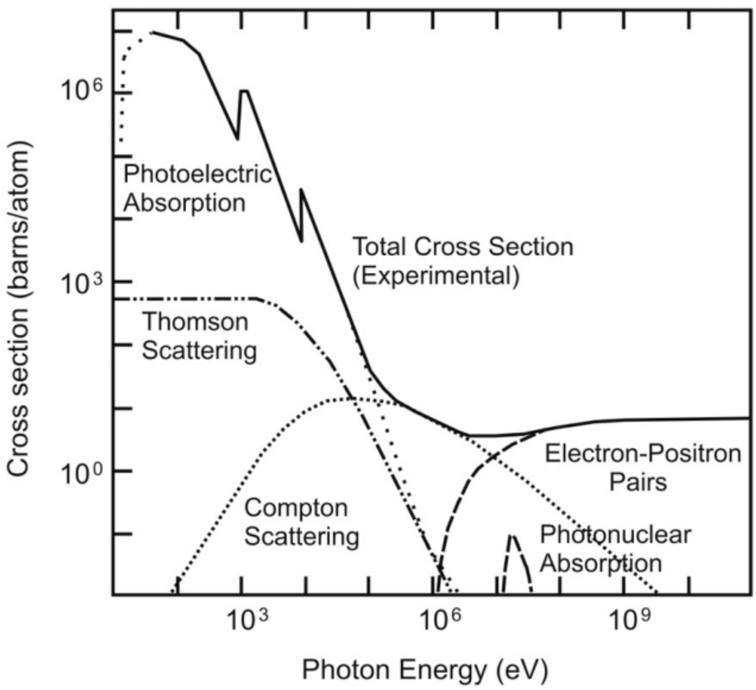

Figure 1: Cross-sections of the photoelectric absorption, the Thomson scattering, the photonuclear absorption, and the production of electron-positron pairs for Cu as a function of energy from 10 eV to 100 GeV [1].

2. Photoelectric absorption

The photoelectric effect is the dominant process in the energy range up to sub-MeV. If the photon energy $h\nu$ is greater than the ionization energy I of the atom, the electron becomes free from the atom. The kinetic energy of this free electron K after leaving the atom is determined by Einstein's equation

$$T = h\nu - I. \tag{2.1}$$

Since I is characteristic constant of the atom, the absorption spectrum is continuous. The free electron can be observed as a current when integrated in a electrical circuit. Figure 2 shows the schematic diagram of the photo electric effect for photons with various energy obliquely incident upon a Potassium. Emitted electron has a momentum in the polarization direction of photons. The energy of the electron are calculated based on Eq. (2.1).

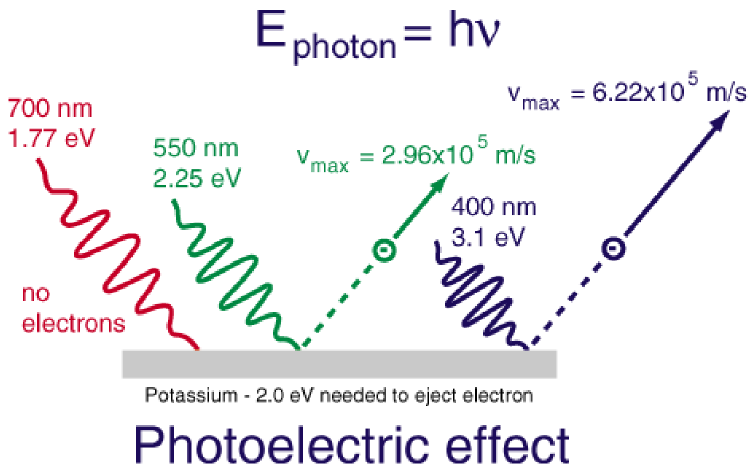

Figure 2: Schematic diagram of the photoelectric effect. In the figure, an electrons with various energy obliquely incident upon a Potassium whose working function I is 2.0 eV[2].

The photo electric cross-section versus the photon energy shows the characteristic sawtooth absorption edges as shown in Figure 1 since the binding energy of each electron shell is surpassed and this process is allowed to occur for the inner shells. The K-shell electrons are most tightly bound and therefore have the most significant contribution on the cross-section in the higher energy range. The cross-section is also affected by the Z number of atoms as it determines the binding energy of electrons.

For non-relativistic case, the total cross-section of photoelectric absorption is calculated by considering the collision between a photon with momentum $\mathbf{k}$ and the electron in the K-shell while transferring the momentum to the atom. The total cross-section for the ejection of photoelectrons in any direction is obtained as, according to §21, eq. (14) [3],

$$\phi_K = \phi_0 \frac{Z^5}{137^4} 4\sqrt{2} (\frac{\mu}{k})^{7/2} \tag{2.2}$$

where μ is the rest energy of the electron, $\phi_0 = 8\pi r_0^2/3$ is the cross-section for the Thomson scattering,

$$r_0 \equiv e^2/m_0c^2 = 2.818 \times 10^{-13}\text{cm}$$

is the classical electron radius. The absorption coefficient decreases rapidly as far as T is much larger than I and relativistic corrections are not important, i.e., $h\nu \ll mc^2$.

If the energy of primary electron is of the order mc^2 or larger, the wave function is replaced by the relativistic one. For extremely high energies, $k \gg \mu$, the total cross-section is, according to §21, eq. (18) [3],

$$\phi_K = \phi_0 \frac{3}{2} \frac{Z^5}{137^4} \frac{\mu}{k}. \tag{2.3}$$

which decreases proportionally to μ/k instead of to $(\mu/k)^{7/2}$ for $k \ll \mu$.

3. Compton scattering

In Compton scattering, photon $\mathbf{k_0}$ collides with an free electron, and thus loses some energy and deflected from its original direction of travel with energy $\mathbf{k}$. The schematic diagram of Compton scattering is shown in Figure 3.

In the following derivation, the electron is assumed to be initially at rest with momentum $\mathbf{p_0}$, and energy $E_0 = \mu$ ($\mu = mc^2$), but has energy E and momentum

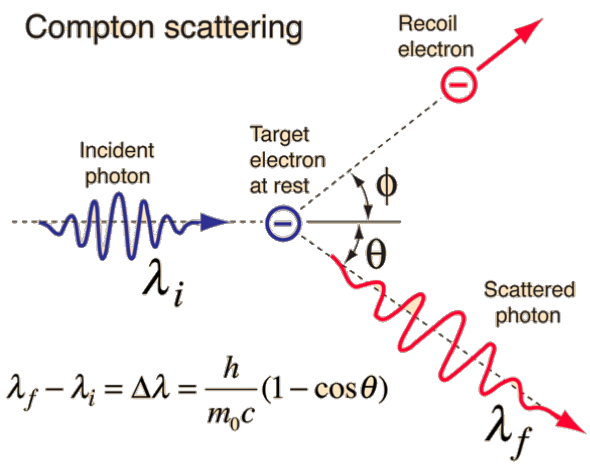

Figure 3: Compton scattering by the incident photon of wavelength λ_i with the electron at rest. The resultant scattered photon with wavelength λ_f and recoil electron with momentum $\mathbf{p}$ are depicted [2].

$\mathbf{p}$ in the final state. Since the momentum is conserved,

$$\mathbf{p} = \mathbf{k_0} - \mathbf{k}. \tag{3.1}$$

The conservation of energy states that

$$E + k = k_0 + \mu \tag{3.2}$$

Using the relativistic relation between momentum and energy $p^2 = E^2 - \mu^2$, and by denoting the angle between $\mathbf{k_0}$ and $\mathbf{k}$ by θ, we obtain

$$k = \frac{k_0\mu}{\mu + k_0(1 - \cos\theta)}. \tag{3.3}$$

The differential cross-section is well-known Klein-Nishina formula, according to §22, eq. (35) [3],

$$\frac{d\phi}{d\Omega} = \frac{1}{4} r_0^2 \frac{k^2}{k_0^2} \left[\frac{k_0}{k} + \frac{k}{k_0} - 2 + 4\cos^2\Theta \right]. \tag{3.4}$$

where r_0 is the classical electron radius, Θ is the angle between the directions of polarization of the primary and secondary electrons. This formula gives the intensity of the scattered radiation at a given solid angle $d\Omega$ and with a given direction of polarization for all primary light quanta of a given frequency and polarization. This formula can be expressed as a function of θ, k_0 and Θ by substituting Eq. 3.3. If the primary wave is unpolarized, we can average over Θ and obtain

$$\overline{\sin^2\Theta} = \frac{1}{2}(1+\cos^2\theta), \tag{3.5}$$

$$\frac{d\phi}{d\Omega} = \frac{1}{2}r_0^2\frac{k^2}{k_0^2}\left[\frac{k_0}{k}+\frac{k}{k_0}-\sin^2\theta\right]. \tag{3.6}$$

This angular distribution for typical energies are shown in Figure 4.

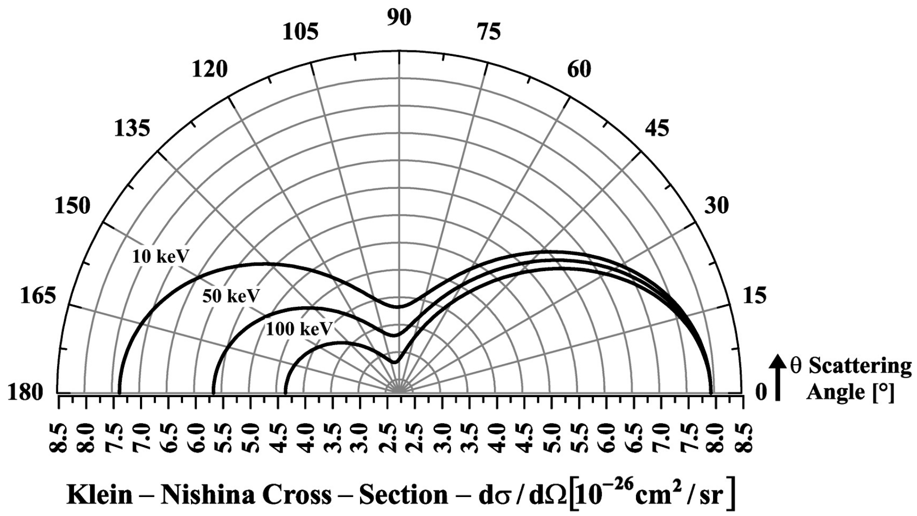

Figure 4: Differential cross-section of Compton scattering [4].

In the low energy limit of Klein-Nishina formula,

$$\frac{\phi(\theta)}{d\Omega} = \frac{1}{2}r_0^2(1+\cos^2\theta) \tag{3.7}$$

which is **Thomson differential scattering cross-section** derived by J. J. Thomson. After integrating over all scattered angle θ and letting $k \sim k_0$, we obtain the cross-section

$$\phi = \frac{8\pi}{3}r_0^2 = 6.65\times10^{-25}\text{cm}^2. \tag{3.8}$$

Thus, the cross-section for this scattering is a universal constant.

4. Electron-Positron Pair Production

As shown in Figure 1, the pair production is the dominant process of the absorption of high energy (10 MeV $\sim$) photons. This process involves the interaction of an incident gamma ray with a nucleus by its Coulomb field.

4.1 Hole theory

The relativistic wave equation (Klein-Gorden Equation)

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi + m^2 c^4 \phi \tag{4.1}$$

have four solutions whose energy

$$E = \pm\sqrt{p^2 + \mu^2}$$

and spin are either positive or negative. The interpretation of this negative energy states is given by Dirac's Hole theory[3]. In this theory, we make two fundamental assumptions, according to § 11 Heitler,

- All negative states with energies ranging from $-mc^2$ to $-\infty$ are filled with electrons. No electron can jump into one of these occupied states (exclusion principle).
- The electrons filling up the negative energy states do not produce an external field and do not give any contribution to the total charge, energy, and momentum of this system. The 'zero point' for the charge, energy and momentum is represented by the electron distribution in which all negative energy states and no positive energy states are occupied. This state will be called the *electron vacuum*.

Consider what happens when one of the electron with the energy $-|E|$ and momentum $\mathbf{p}$ in the vacuum in the negative energy state is removed. The whole system then has a charge, energy, and momentum different from zero.

$$E_+ = -E = |E|, \quad e_+ = -e, \quad \mathbf{p}_+ = -\mathbf{p} \tag{4.2}$$

The hole in the negative energy electrons has therefore a positive charge, positive energy and a momentum and spin opposite to the original electron. Therefore this particle behaves like an ordinary particle with electron mass but with a positive charge,

$$E_+ = +\sqrt{(p_+^2 + \mu^2)} \tag{4.3}$$

The creation and annihilation of a pair of a positive and negative electron is interpreted by the following way. Stating from a state where no particle and hole is present, an external field acting on the electrons in the negative energy state may cause a transition of one of these electrons with energy E and momentum $\mathbf{p}$ to a state with positive energy E' and momentum $\mathbf{p}'$. We then have a pair present with energies and momenta

$$\mathbf{p}_+ = -\mathbf{p}, \quad \mathbf{E}_+ = -\mathbf{E} = |\mathbf{E}|,$$
$$\mathbf{p}_- = -\mathbf{p}', \quad \mathbf{E}_- = \mathbf{E}'. \tag{4.4}$$

The energy required to cause this transition is larger than the rest energy of two particles

$$E' - E = E_+ + E_- \geq 2mc^2 \tag{4.5}$$

On the other hand, if an electron present in a positive energy state initially, the electron can jump into the hole representing the positron and the pair is annihilated.

4.2 Energy and momentum conservation

Figure 5 shows the Feynman diagram of the pair production by an incident gamma ray. The incident gamma ray is shown as a wavy allow traveling from the left to right, which is propagating forward in time, and the straight arrows are either electrons or positrons depending on its direction. The convention in Feynman diagram is that the antiparticle of electron (positron) line is drawn to travel backwards in time.

As the figure shows, the pair production process for incident gamma ray requires an recoil nucleus so that both energy and momentum conservation are satisfied at the same time.

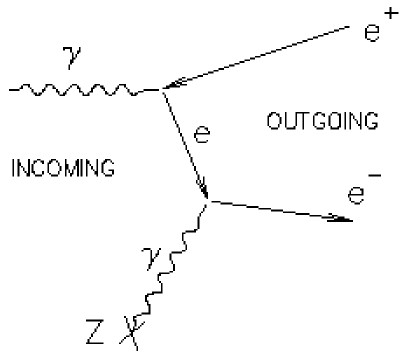

Figure 5: Feynman diagram of the pair production by a gamma ray[5].

By describing the kinetic energy and the momentum of photon k, positron T_+, electron T_-, recoil nucleus T_r, then the energy conservation is expressed as, according to Ch. 3 Devon[6],

$$k = (T_+ + 1) + (T_- + 1) + T_r \tag{4.6}$$

where unity indicates the rest mass of the positron and electron. The corresponding momentum conservation is written as

$$\mathbf{k} = \mathbf{p}_+ + \mathbf{p}_- + \mathbf{q} \tag{4.7}$$

where $\mathbf{q}$ is the recoil momentum of nucleus. The recoil energy E_r for the nucleus with mass m_r is then

$$E_r = \sqrt{q^2 + \frac{m_r^2}{m_e^2}} - \frac{m_r}{m_e} \tag{4.8}$$

By solving for these conservation equations, we obtain the minimum recoil momentum

$$q_{min} = k - (k^2 - 4)^{1/2} \tag{4.9}$$

when all the particles in system is along the direction of the incident photon. The existence of finite positive q value suggests the necessity of the recoil nucleus.

4.3 Cross-section of the pair production

The differential cross-section of the pair production is given by Bethe-Heitler formula. The cross-section is affected by the screening of the Coulomb field by outer electrons of a nucleus. The screening parameter is defined by

$$\mu = \frac{100}{k} \frac{1}{\nu(1-\nu)Z^{1/3}} \tag{4.10}$$

where $\nu = E_+/k$ and E_+, E_- are the total positron and electron energies (PE+KE). The screening effect becomes important when μ is close to 0, less important as it increases, and negligible if $\mu \sim \infty$.

In general, when the incident photon energy k is high, the screening parameter μ decreases. In addition to that, μ becomes large when ν is close to either 0 or 1. This corresponds to the case when either E_+ or E_- is very small, and thus they have more chances of being affected by the Coulomb field of nucleus.

The Bethe-Heitler cross-section, $\frac{d\sigma}{dE^+}(k, E_+)$ is defined as the probability producing a positron of energy between (E_+) and $(E_+ + dE_+)$ from a incident gamma-ray with energy k. The cross-sections for a screened point-nucleus with extreme relativistic energies are summarized in §26 of Heitler [3]. The figure 6 plots the Bethe-Heitler cross-section $\frac{d\sigma}{dE^+}(k, E_+)$ as a function of normalized positron energy ν. As shown in the figure 6 (a), at low energies, the probability for each energy ratio is roughly equal from $k = 0.2E_+ - 0.8E+$. Also, the more positron is generated as k increases. Though, for low incident photon energy k, the cross-section becomes maximum when the electrons and positrons obtain equal energy, the antisymmetric electron-positron energy distribution dominates as k increases.

Although the total cross-section increases, as shown in the figure 6 (b), the differential cross-section $d\sigma/dE_+$ decreases due to the prevalent effect of screening for the larger photon energy k as indicated by the screening parameter μ.

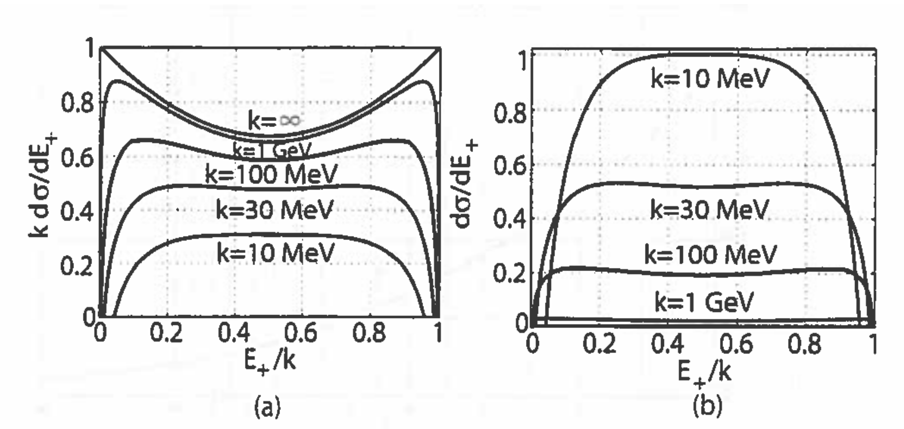

Figure 6: The Bethe Heitler cross-section $\frac{d\sigma}{dE^{+}}(k, E_{+})$ for Tungsten[6]. X-axis is positron energy, E_{+}, to the incident photon energy, k. (a) Y axis is the Bethe-Heitler cross-section normalized by incident photon energy k. (b) Y-axis is only the Bethe-Heitler cross-section.

References

[1] Christian Felix Hermanns. *X-ray absorption studies of metalloporphyrin molecules on surfaces: Electronic interactions, magnetic coupling, and chemical switches*. PhD thesis, Freie Universitat Berlin, 2013.

[2] Hyperphysics. `http://hyperphysics.phy-astr.gsu.edu`. [Accessed: 2019-01-12].

[3] W. Heitler. *The Quantum Theory of Radiation*. Dover Books on Physics and Chemistry. Dover Publications, 1954.

[4] Francisco A. Scannavino and Paulo E. Cruvinel. A graphical tool for an analytical approach of scattering photons by the compton effect. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, Vol. 674, pp. 28 – 38, 2012.

[5] RELATIVISTIC KINEMATICS. conversion of energy into mass. `http://www.jick.net/~jess/hr/skept/EMC2/node9.html`. [Accessed: 2019-01-12].

[6] Devon Kryle Johnson. *Positron Production by X-rays Emitted from Betatron motion in a Plasma Wiggler*. PhD thesis, University of California, Los Angeles, 2006.