A Practical Implementation of Stochastic Programming: an application to the evaluation of option contracts in supply chains

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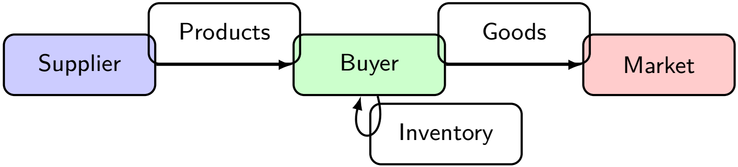
Motivation

- Stochastic programming is hard to implement
- Formulate as linear program
- Large deterministic equivalent: (can be solved nowadays)

Chocolate Material flow



$t = 1, \ldots, 15.$

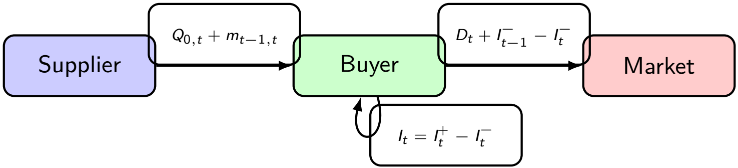
Decision variables: $t = 1, \ldots, T$

- $Q_{0,t}$: firm order, delivered at t
- $M_{0,t}$: # options bought at $t = 0$, can be exercised in t
- $m_{t,t+1}$: # options exercised in t, delivered in $t + 1$

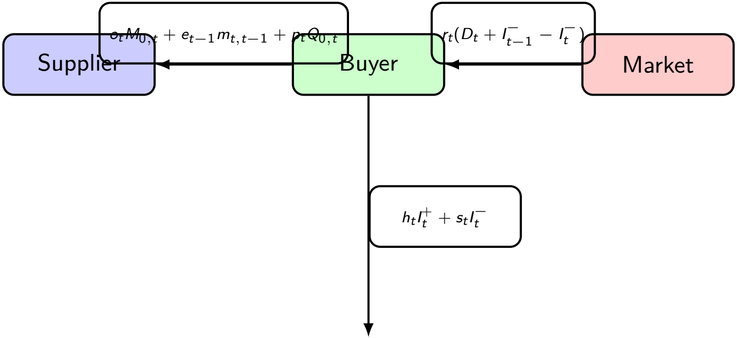
State variables: $t = 0, \ldots, T$

- I_t : goods inventory at the end of t
- I_t^+ : physical goods inventory at the end of t
- I_t^- : backorder of goods inventory at the end of t
- D_t : demand in t

Material Flow



Money flow

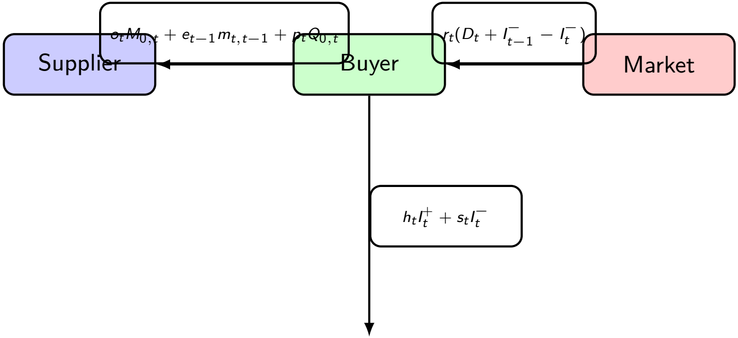

$o_t M_{0,t} + e_{t-1} m_{t,t-1} + p_t Q_{0,t}$
$r_t(D_t + I^-_{t-1} - I^-_t)$
Supplier
Buyer
Market
$h_t I^+_t + s_t I^-_t$

Parameters

- $\bar{M}_t$: bound of available options at t
- v : salvage value of goods
- o_t : price: option that can be exercised in t
- e_t : price: option exercised in t to be delivered in $t+1$
- p_t : price: firm order, delivered in t

- D_t : demand in t
- r_t : selling price of goods in t
- s_t : shortage cost for goods in t
- h_t : 1 period holding cost for goods in t

Money flow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$h_t I^+_t + s_t I^-_t$

$$\mathcal{R}(I^-, I^+) = \underbrace{r_1(D_1 - I_1^-)}_{\text{Sales in } t=1} + \sum_{t=2}^{T} \underbrace{r_t(D_t + I_{t-1}^- - I_t^-)}_{\text{Sales in } t=2,\ldots,T} + \underbrace{vI_T^+}_{\text{Salvage value of inventory}}$$

$$\mathrm{E}(\mathrm{I}^-, I^+, m, Q, M) =$$
$$\sum_{t=1}^{T} \underbrace{(p_t Q_{0,t})}_{\text{firm order}} + \sum_{t=1}^{T} \underbrace{(h_t I_t^+ + s_t I_t^- + p_t Q_{0,t})}_{\text{storage, shortage}} +$$
$$\sum_{t=1}^{T-1} \underbrace{(e_t m_{t,t+1} + o_t M_{0,t})}_{\text{Options bought and executed}}$$

Deterministic Buyer Problem

$$
\begin{aligned}
\max \quad & \mathcal{R}(I^-, I^+) - \mathcal{E}(I^-, I^+, m, Q, M) && \text{(Profit Maximization)} \\
\text{s.t.} \quad & I_t = I_t^+ - I_t^- && \text{(Inventory Defintion)} \\
& I_1 = Q_{0,1} - D_1 && \text{(Initial Inventory)} \\
& I_t = I_{t-1} + Q_{0,t} + m_{t-1,t} - D_t, && \text{(Inventory Evolution)} \\
& 0 \leq m_{t,t+1} \leq M_{t,t+1} && \text{(Feasibility of Options)} \\
& 0 \leq M_{0,t} \leq \bar{M}_t && \text{(Bounded Option Availability)} \\
& Q_{0,t}, I_t^+, I_t^- \geq 0 && \text{(Nonnegativity Constraints)}
\end{aligned}
$$

Outline

Stochastic Problem

The Deterministic Equivalent

Numerical results

Finance

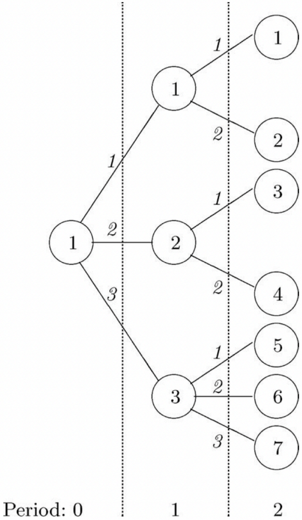

Fig. 1. Example of event tree representation.

Characteristics

- Discrete Stochastic process that is independent of the state and decision variables $\{D_t\}_{t=1}^{T}$
- State of the world determined by the time period $t \in \{0, \ldots, T\}$ and node $n \in \{1, \ldots, N[t]\}$, where $N[t]$ is the number of nodes at t
- Easily applicable to the computer languages

Variables

- $N[t]$: number of nodes at t
- $f[t, n]$: number of branches from (t, n)
- $a[t, n, k]$: predecessor function. Returns the node at $t - k$ that led to (t, n)

Notation: Let T be the final time period.

- $b[t, n]$: returns the node at t that resulted in (T, n)
- $\ell[t, n]$: transition index. Defines the transition from $(t - 1, a[t, n, 1])$ to (t, n)

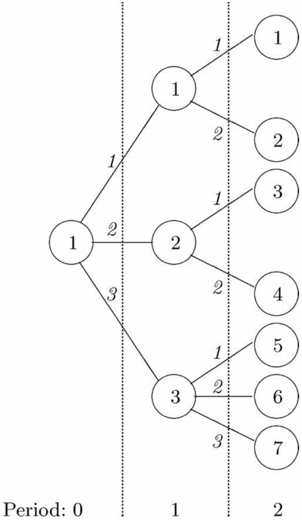

Fig. 1. Example of event tree representation.

Demand process $\{D_t\}_{t=1,\dots,T}$.

- μ_t, σ_t be the unconditional mean and variance of D_t
- $\rho_t = \rho(D_t, D_{t+1})$
- Satisfies

$$E(D_{t+1} \mid d_t) = \mu_{t+1} + \rho_t \frac{\sigma_{t+1}}{\sigma_t} (d_t - \mu_t)$$

$$Var(D_{t+1}|d_t) = \sigma_{t+1}^2 (1 - \rho_t^2)$$

Discretized conditional demand

$$D_{t+1} \mid d_t = E(D_{t+1} \mid d_t) + \varepsilon_{t+1} \sqrt{Var(D_{t+1} \mid d_t)}.$$

- Discrete state space $V_\varepsilon = \{\varepsilon_1, \dots, \varepsilon_{N_\varepsilon}\}$
- N_ε cardinality
- Probability distribution $P_\varepsilon = \{p_{\varepsilon_1}, \dots, p_{\varepsilon N_\varepsilon}\}$

Discussion?

- Finer grid, better approximation
- Problem size growth exponential
- Main interest: first period decisions

- $f[t, n]$: number of branches from (t, n) = cardinality of the discretized demand variable ε from node (t, n)
- Corresponding discretized rv $\varepsilon[t, n]$ with cardinality $f[t, n]$ and corresponding state space $\{\varepsilon[t, n, 1], \dots, \varepsilon[t, n, f[t, n]]\}$ and probability distribution $\{p[t, n, 1], \dots, p[t, n, f[t, n]]\}$

Demand at node $(1, n)$

$$D[1,n] = \mu[1] + \varepsilon[0,1,\ell[1,n]]\sigma[1]$$

Demand at node (t, n)

$$\begin{aligned} D[t,n] = \quad & \underbrace{\mu[t]}_{\text{Unconditional mean}} \\ & + \underbrace{\rho[t-1]\frac{\sigma[t]}{\sigma[t-1]}(D[t-1,a[t,n,1]] - \mu[t-1])}_{\text{Autoregressive component}} \\ & + \underbrace{\varepsilon[t-1,a[t,n,1],\ell[t,n]]\sigma[t]\sqrt{1-\rho[t-1]^2}}_{\text{Random component}} \end{aligned}$$

- $p[t, n, j]$: conditional transition probabilities from (t, n) to $(t+1, m)$ with $n = a[t+1, m, 1], \ell[t+1, m] = j$
- From our assumptions, these probabilities depend on t, n, j.
- $P[t, n]$: unconditional occurrence probability of node (t, n)

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$$P[t, n] = p[t-1, a[t, n, 1], \ell[t, n]]\; P[t-1, a[t, n, 1]],$$

$$P[0, 1] = 1$$

Outline

Stochastic Problem

The Deterministic Equivalent

Numerical results

Finance

- $\{D_t\}$, decisions $\{m_{t-1,t}\}$ and state variables (I, I^+, I^-) stochastic
- M, Q fixed at the beginning

Enforce this into each node of the event tree

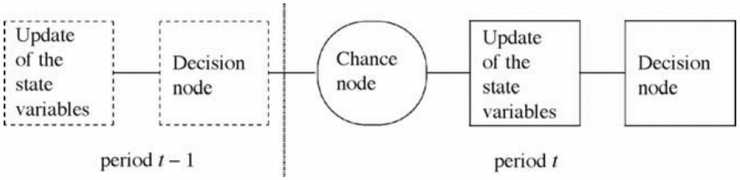

Fig. 2. Sequence in the decision process.

$$
\begin{aligned}
\max \quad & E[R(I^-, I^+) - E(I^-, I^+, m, Q, M)] && \text{(Profit Maximization)} \\
\text{s.t.} \quad & I_t = I_t^+ - I_t^- && \text{(Inventory Defintion)} \\
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& I_t = I_{t-1} + Q_{0,t} + m_{t-1,t} - D_t, && \text{(Inventory Evolution)} \\
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& 0 \leq M_{0,t} \leq \bar{M}_t && \text{(Bounded Option Availability)} \\
& Q_{0,t}, I_t^+, I_t^- \geq 0 && \text{(Nonnegativity Constraints)}
\end{aligned}
$$

Transform this into linear program by enforcing constraints for all nodes (T, n)!

Optimization Problem

$$
\begin{aligned}
\max \quad & \sum_{n=1}^{N[T]} P[T,n](\mathcal{R}[n] - \mathcal{E}[n]) && \text{(Profit Maximization)} \\
\text{s.t.} \quad & I[t,n] = I^{+}[t,n] - I^{-}[t,n] && \text{(Inventory Defintion)} \\
& I[1,n] = Q[0,1,1] - D[1,n] && \text{(Initial Inventory)} \\
& I[t,n] = I[t-1, a[t,n,1]] + Q[0,1,t] \\
& \quad + m[t-1, a[t,n,1], t] - D[t,n] && \text{(Inventory Evolution)} \\
& 0 \leq m[t,n,t+1] \leq M[0,1,t+1] && \text{(Feasibility of Options)} \\
& 0 \leq M[0,1,t] \leq \bar{M}_t && \text{(Bounded Option Availability)} \\
& I^{+}[t,n], I^{-}[t,n], Q[0,1,t], M[0,1,t] \geq 0 && \text{(Nonnegativity Constraints)}
\end{aligned}
$$

Revenue:

$$\mathcal{R}[n] = r_1\big(D[1, b[1, n]] - I^-[1, b[1, n]]\big) + v^{bf} I^+[T, n] + \sum_{t=2}^{T} r_t\big(D[t, b[t, n]] + I^-[t-1, b[t-1, n]] - I^-[t, b[t, n]]\big)$$

Expenses:

$$
\begin{aligned}
\mathcal{E}[n] = & \sum_{t=1}^{T} p_t Q[0,1,t] \\
& + \sum_{t=1}^{T} \left(h_t I^+[t, b[t,n]] + s_t I^-[t, b[t,n]] \right) \\
& + \sum_{t=1}^{T-1} \left(e_t m[t, b[t,n], t+1] + o_t M[0,1,t] \right)
\end{aligned}
$$

Optimization Problem

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\begin{aligned}
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\end{aligned}
$$

Numerical Results

- $(\mu_t, \sigma_t, \rho_t) = (1500, 330, 0.5)$
- $(c_t, h_t, r_t, p_t, o_t, e_t, s_t, v) = (3, 0.5, 12, 8, 1.5, 8, 6, 2)$
- Constraint parameters on the option right level $\bar{M}_t = 10000$

Base Points in each t : 81, tree with 6642 nodes. Deterministic equivalent (19.929, 13.285)

Finer $\forall t$: 321, tree with 103,362 nodes. Deterministic equivalent (200.000, 300.000)

Fineness Senstivitiy Analysis

Table 1
Simultaneous refinement of the 1st and 2nd stage grids

Grid	Number of nodes	First stage decisions M_1	$Q_{0,1}$	$Q_{0,2}$	Expected profit
5 × 5	25	753	1853	924	8604
11 × 11	121	486	2027	863	8369
21 × 21	441	482	1969	913	8342
41 × 41	1681	469	1983	911	8327
81 × 81	6561	470	1965	933	8324
161 × 161	25 921	471	1968	929	8323
321 × 321	1 03 041	470	1969	928	8323

Table 2
Refinement of the 2nd stage grid alone

Grid	Number of nodes	First stage decisions M_1	$Q_{0,1}$	$Q_{0,2}$	Expected profit
321 × 5	1605	538	1969	883	8452
321 × 11	3531	469	1969	929	8331
321 × 21	6741	478	1969	924	8331
321 × 41	13 161	471	1969	927	8325
321 × 81	26 001	471	1969	927	8323
321 × 161	51 681	471	1969	928	8323
321 × 321	1 03 041	470	1969	928	8323

What more?

- By S.P. analysis, we can know the distribution of expected profits and distribution of the optimal decision variables

Sensitivity Analysis: Option Right and Option Exercise Price

Table 3
Impact of a variation of the option right price

Option price o_t	First stage decisions M_1	$Q_{0,1}$	$Q_{0,2}$	Expected profit
0	2728	1955	0	9876
1	745	1968	782	8624
2	247	1968	1050	8145
3	0	1968	1187	8067

Table 4
Impact of the price of the option exercise

Exercise price e_t	First stage decisions M_1	$Q_{0,1}$	$Q_{0,2}$	Expected profit
6	1653	1833	0	9562
7	804	1968	656	8638
8	471	1968	929	8323
9	280	1968	1055	8165

Absolute Lower Bound on Losses:

$$\mathcal{E}[n] - \mathcal{R}[n] \leq c, \forall n \in N[T]$$

Increase the total number of constraints in the deterministic equivalent by $N[T]$ (number of terminal nodes).

Table 5
Impact of a maximal admissible loss

c	M_1	$Q_{0,1}$	$Q_{0,2}$	Expected profit
−10 000	1495	1735	0	7253
−9000	1939	1497	0	6873
−8000	2587	1215	0	6106
−7000	2664	1237	0	4305
−6000	1368	1904	0	876

Curse of dimensionality

- Exponential growth of nodes, variables, constraints

Sol 1 Coarser grids

Sol 2 Dynamically adjusted grids (if probability of visiting very small, information probably irrelevant)

$$f[t,n] = \begin{cases} FineGridSize[t] \text{ if } \frac{P[t,n]}{1/N[t]} \geq GSL[t] \\ CrudeGridSize[t] \text{ if } \frac{P[t,n]}{1/N[t]} < GSL[t] \end{cases}$$

$GSL[t]$: grid switch level for time period t

- Same 2 grid sizes $\forall t$: crude grid with 5 nodes, fine grid with 321 nodes.

Table 8
Grid refinement as a function of the node probability

GSL	Number of nodes	First stage decisions M_1	$Q_{0,1}$	$Q_{0,2}$	E(profit)
1000	1605	538	1969	883	8452
2.389	3817	538	1969	883	8437
2.365	6977	538	1969	883	8418
2.250	13 297	518	1969	903	8383
1.840	25 937	500	1969	921	8344
0.800	51 849	474	1969	925	8314

Multiperiodic

- Grid size grows in the case (5,11) by either 5 or 11
- Switching factor chosen so that # nodes ≈ 60k

Table 9
Multi-periodic model with different grid definitions

Periods	1	2	3	4	5
Grid scheme 1.	Expected profit: 22 081				
Grid sizes	(5, 11)	(5, 11)	(5, 11)	(5, 11)	(5, 11)
Switching factor	0.2	0.2	0.2	0.2	0.2
M_t	128	377	477	820	—
Q_t	2027	1678	1436	1300	436
Grid scheme 2.	Expected profit: 22 172				
Grid sizes	(5, 21)	(5, 21)	(5, 21)	(5, 21)	(5, 21)
Switching factor	2	2	2	2	2
M_t	119	374	480	837	—
Q_t	1969	1755	1419	1303	425
Grid scheme 3.	Expected profit: 22 271				
Grid sizes	(5, 41)	(5, 41)	(5, 41)	(5, 41)	(5, 41)
Switching factor	4	4	4	4	4
M_t	129	345	459	885	—
Q_t	1983	1700	1446	1330	399
Grid scheme 4.	Expected profit: 22 314				
Grid sizes	(5, 81)	(5, 41)	(5, 21)	(5, 11)	(5, 11)
Switching factor	12	12	12	12	12
M_t	129	361	461	881	—
Q_t	1965	1725	1435	1309	415

Conclusion

- Simple implementation of stochastic programming
- Transform s.p. into linear program by enforcing constraints at each node
- Grid size matters for granularity
- Choice of grid size tailored prioritizing first decisions and potentially more important nodes
- Deals with discrete s.p. but with continuous decision and control variables

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