

Note: Order parameters of the Ising model for ferromagnetism

Hironao Yamato

Department of Physics, Osaka University

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I. MAGNETIZATION AND MAGNETIC SUSCEPTIBILITY

In dealing with the Ising model for ferromagnetism, let us first define magnetization and magnetic susceptibility. Magnetization, which is often denoted as M , is defined as follows.

$$M \equiv \sum_i \langle S_i \rangle. \quad (1)$$

Since M is an extensive quantity, as you can see from its definition, it is preferable to define m , which is the magnetization per site.

$$m \equiv \frac{M}{N} = \frac{1}{N} \sum_i \langle S_i \rangle. \quad (2)$$

As thermodynamics suggest, m can be calculated using the following relation.

$$m = -\frac{\partial g}{\partial h}, \quad (3)$$

where g is the free energy per site of the system under consideration. This can be easily proven without deviating the framework of the canonical ensemble:

$$g = -\frac{1}{\beta N} \log Z = -\frac{1}{\beta N} \log \left\{ \sum \exp \left(\beta J \sum_{(i,j)} S_i S_j + \beta h \sum_i S_i \right) \right\} \quad \text{and} \quad (4)$$

$$-\frac{\partial g}{\partial h} = \frac{1}{N} \sum \left\{ \left(\sum_i S_i \right) \frac{1}{Z} \exp \left(\beta J \sum_{(i,j)} S_i S_j + \beta h \sum_i S_i \right) \right\} = \frac{1}{N} \sum_i \langle S_i \rangle = m. \quad (5)$$

Magnetic susceptibility, which is often denoted as χ , is defined as follows.

$$\chi \equiv \frac{\beta}{N} \left\{ \left\langle \left(\sum_i S_i \right)^2 \right\rangle - \left\langle \sum_i S_i \right\rangle^2 \right\}. \quad (6)$$

It is worthwhile to know that χ can be written in the following form.

$$\chi = \frac{\beta}{N} \left\{ \left\langle \left(\sum_i S_i \right) \left(\sum_j S_j \right) \right\rangle - \left(\sum_i \langle S_i \rangle \right) \left(\sum_j \langle S_j \rangle \right) \right\} \quad (7)$$

$$= \frac{\beta}{N} \left\{ \sum_i \sum_j \langle S_i \rangle \langle S_j \rangle - \left(\sum_i \langle S_i \rangle \right) \left(\sum_j \langle S_j \rangle \right) \right\} \quad (8)$$

$$= \frac{\beta}{N} \sum_i \sum_j \{ \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \} \quad (9)$$