Note: Proof of Lee-Yang Theorem

Hironao Yamato

(Dated: October 19, 2024)

I. LEE-YANG THEOREM

A. The setup and lemmas

B. Generalised proposition and its proof

Let

$$\Xi^G(\{z_j\}) = \sum_{\{n_j\}} \prod_j (z_j)^{n_j} e^{-4\beta \sum_{(j<k)} V_{jk}(n_j-n_k)^2} \quad \text{(grand partition function).} \tag{1}$$

Then the grand partition function $\Xi^G(\{z_j\})$ is non-zero in $\bigcap_{j=1}^{N} \{|z_j| < 1\}$.

This proposition can be proven by induction on N, which is the number of sites of the system under consideration. **Supose this proposition is true for the situation which the number of sites are less than or equal to $N-1$.** (hypothesis) The grand partition function can be written in the following form by calculating the sum over n_N in (1):

$$\begin{aligned} \Xi^G(\{z_j\}) &= \sum_{n_N=0,1} (z_N)^{n_N} \sum_{\{n_j;\, j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k}(n_j-n_k)^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2} \\ &= \sum_{\{n_j;\, j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k} n_k^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2} \\ &\quad + z_N \sum_{\{n_j;\, j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k}(1-n_k)^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2}. \end{aligned} \tag{2}$$

Since n_k only takes 0 or 1,

$$e^{-4\beta \sum_{k<N} V_{1k} n_k^2} = e^{-4\beta \sum_{k<N} V_{1k} n_k^2}, \quad e^{-4\beta \sum_{k<N} V_{1k}(1-n_k)^2} = e^{4\beta \sum_{k<N} V_{1k} n_k - 4\beta \sum_{k<N} V_{1k}}. \tag{3}$$

Then, setting $a_{jk} := e^{-\beta V_{jk}}$,

$$\Xi^G(\{z_j\}) = \Xi^{G\setminus\{N\}}(a_{12}z_2, a_{13}z_3, \dots) + z_1 a_{12}a_{13}\cdots a_{1N} \Xi^{G\setminus\{N\}}(z_2/a_{12}, z_2/a_{13}, \cdots). \tag{4}$$

where $\Xi^{G\setminus\{N\}}(\{z_j\})$ is the grand partition function for a system with $N-1$ sites. In other words, $\Xi^{G\setminus\{N\}}(\{z_j\})$ is the grand partition function for the system that the Nth site is removed. Then using (**??**) and (**??**), we can write

$$\Xi^G(\{z_j\}) = \Xi^{G\backslash\{N\}}(a_{12}z_2, a_{13}z_3, \dots) + z_1 z_2 \cdots z_N \Xi^{G\backslash\{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \cdots). \tag{5}$$

What we need to show is that $\Xi^G(\{z_j\})$ is non-zero. Suppose (5) be zero, then at least the following condition must be met.

$$\left|\Xi^{G\backslash\{N\}}(a_{12}z_2, a_{13}z_3, \dots)\right| = \left|z_1 z_2 \cdots z_N \Xi^{G\backslash\{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \cdots)\right| \tag{6}$$

Since $\Xi^{G\backslash\{N\}}(a_{12}z_2, a_{13}z_3, \dots)$ is non-zero by hypothesis, this condition can be rewritten in the following form.

$$\left|\frac{z_1 z_2 \cdots z_N \Xi^{G\backslash\{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \cdots)}{\Xi^{G\backslash\{N\}}(a_{12}z_2, a_{13}z_3, \dots)}\right| = 1 \tag{7}$$