Note: Order parameters of the Ising model for ferromegnetism

Hironao Yamato

Department of Physics, Osaka University

(Dated: October 12, 2024)

I. MAGNETIZATION AND MAGNETIC SUSCEPTIBILITY

In dealing with the Ising model for ferromagnetism, let us first define magnetization and magnetic susceptibility. Magnetization, which is often denoted as M, is defined as follows.

$$M \equiv \sum_i \langle S_i \rangle . \tag{1}$$

Since M is a extensive quantity, as you can see from its definition, it is preferable to define m, which is the magntization per site.

$$m \equiv \frac{M}{N} = \frac{1}{N} \sum_i \langle S_i \rangle . \tag{2}$$

As thermodynamics suggests, m can be calculated using the following relation.

$$m = -\frac{\partial g}{\partial h}, \tag{3}$$

where g is the free energy per site of the system under consideration. This can be easily proven without deviating the framework of the canonical ensemble:

$$g = -\frac{1}{\beta N} \log Z = -\frac{1}{\beta N} \log \left\{ \sum \exp \left(\beta J \sum_{(i,j)} S_i S_j + \beta h \sum_i S_i \right) \right\} \quad \text{and} \tag{4}$$

$$-\frac{\partial g}{\partial h} = \frac{1}{N} \sum \left\{ \left(\sum_i S_i \right) \frac{1}{Z} \exp \left(\beta J \sum_{(i,j)} S_i S_j + \beta h \sum_i S_i \right) \right\} = \frac{1}{N} \sum_i \langle S_i \rangle = m. \tag{5}$$

Magnetic susceptibility, which is often denoted as χ, is defined as follows.

$$\chi \equiv \frac{\beta}{N} \left\{ \left\langle \left(\sum_i S_i \right)^2 \right\rangle - \left\langle \sum_i S_i \right\rangle^2 \right\}. \tag{6}$$

It is worthwhile to know that χ can be written in the following form.

$$\chi = \frac{\beta}{N}\left\{\left\langle\left(\sum_i S_i\right)\left(\sum_j S_j\right)\right\rangle - \left(\sum_i \langle S_i\rangle\right)\left(\sum_j \langle S_j\rangle\right)\right\} \tag{7}$$

$$= \frac{\beta}{N}\left\{\sum_i \sum_j \langle S_i S_j\rangle - \left(\sum_i \langle S_i\rangle\right)\left(\sum_j \langle S_j\rangle\right)\right\} \tag{8}$$

$$= \frac{\beta}{N}\sum_i \sum_j \left\{\langle S_i S_j\rangle - \langle S_i\rangle \langle S_j\rangle\right\} \tag{9}$$