

Note: Proof of Lee-Yang Theorem ^a

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I. THE SETUP AND LEMMAS

We will be dealing with the following grand partition function in this document.

$$\Xi^G(\{z_j\}) = \sum_{\{n_j=0,1\}} \prod_j (z_j)^{n_j} e^{-4\beta \sum_{(j<k)} V_{jk}(n_j-n_k)^2} \quad (\text{grand partition function}) \quad (1)$$

This grand partition function is for Ising model or lattice gas model.

In order to facilitate the discussion in the next section, let us first introduce two identities and a theorem.

Lemma. The grand partition function (??) satisfies the following identity.

$$\Xi^G(\{z_j\}) = z_1 z_2 \cdots z_N \Xi^G(\{1/z_j\}) \quad (2)$$

Proof. By simply calculating the right-hand side,

$$\begin{aligned} z_1 z_2 \cdots z_N \Xi^G(\{1/z_j\}) &= z_1 z_2 \cdots z_N \sum_{\{n_j=0,1\}} \prod_{j=1}^N \left(\frac{1}{z_j}\right)^{n_j} e^{-4\beta \sum_{(j<k)} V_{jk}(n_j-n_k)^2} \\ &= \sum_{\{n_j=0,1\}} \prod_{j=1}^N z_j^{1-n_j} e^{-4\beta \sum_{(j<k)} V_{jk}(n_j-n_k)^2} \\ &= \sum_{\{m_j=1,0\}} \prod_{j=1}^N z_j^{m_j} e^{-4\beta \sum_{(j<k)} V_{jk}(m_j-m_k)^2} \quad (m_j := 1 - n_j) \end{aligned} \quad (3)$$

Here, n_j and m_j are dummy index, then we are done. ■

Lemma. The grand partition function (??) satisfies the following identity.

$$\Xi^G(\{z_j\}) = \{\Xi^G(\{z_j^*\})\}^* \quad (4)$$

Proof. Obivous.

^a See: <https://github.com/hironaoy/notes/LeeYangTheorem>

II. GENERALISED PROPOSITION AND ITS PROOF

Proposition. Let

$$\Xi^G(\{z_j\}) = \sum_{\{n_j=0,1\}} \prod_j (z_j)^{n_j} e^{-4\beta \sum_{(j<k)} V_{jk}(n_j-n_k)^2} \quad (\text{grand partition function}). \quad (5)$$

Then the grand partition function $\Xi^G(\{z_j\})$ is non-zero in $\bigcap_{j=1}^N \{|z_j| < 1\}$.

Proof. This proposition can be proven by induction on N , which is the number of sites of the system under consideration. **Suppose this proposition is true for the situation which the number of sites are less than or equal to $N - 1$.** The grand partition function can be written in the following form by calculating the sum over n_N in (??):

$$\begin{aligned} \Xi^G(\{z_j\}) &= \sum_{n_N=0,1} (z_N)^{n_N} \sum_{\{n_j; j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k}(n_j-n_k)^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2} \\ &= \sum_{\{n_j; j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k} n_k^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2} \\ &\quad + z_N \sum_{\{n_j; j<N\}} \prod_{j<N} (z_j)^{n_j} e^{-4\beta \sum_{k<N} V_{1k}(1-n_k)^2} e^{-4\beta \sum_{(j<k<N)} (n_j-n_k)^2}. \end{aligned} \quad (6)$$

Since n_k only takes 0 or 1,

$$e^{-4\beta \sum_{k<N} V_{1k} n_k^2} = e^{-4\beta \sum_{k<N} V_{1k} n_k^2}, \quad e^{-4\beta \sum_{k<N} V_{1k}(1-n_k)^2} = e^{4\beta \sum_{k<N} V_{1k} n_k - 4\beta \sum_{k<N} V_{1k}}. \quad (7)$$

Then, setting $a_{jk} := e^{-\beta V_{jk}} \leq 1$,

$$\Xi^G(\{z_j\}) = \Xi^{G \setminus \{N\}}(a_{12}z_2, a_{13}z_3, \dots) + z_1 a_{12} a_{13} \cdots a_{1N} \Xi^{G \setminus \{N\}}(z_2/a_{12}, z_2/a_{13}, \dots). \quad (8)$$

where $\Xi^{G \setminus \{N\}}(\{z_j\})$ is the grand partition function for a system with $N - 1$ sites. In other words, $\Xi^{G \setminus \{N\}}(\{z_j\})$ is the grand partition function for the system that the N th site is removed. Then using (??) and (??), we can write

$$\Xi^G(\{z_j\}) = \Xi^{G \setminus \{N\}}(a_{12}z_2, a_{13}z_3, \dots) + z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots). \quad (9)$$

What we need to show is that $\Xi^G(\{z_j\})$ is non-zero. Suppose (??) be zero, then at least the following condition must be met.

$$\left| \Xi^{G \setminus \{N\}}(a_{12}z_2, a_{13}z_3, \dots) \right| = \left| z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots) \right| \quad (10)$$

Since $\Xi^{G \setminus \{N\}}(a_{12}z_2, a_{13}z_3, \dots)$ is non-zero by hypothesis, this condition can be rewritten in the following form.

$$\left| \frac{z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12}z_2, a_{13}z_3, \dots)} \right| = 1 \quad (11)$$

Now, let us prove that this condition is not met by proving the sufficient condition:

$$\sup_{|a_{1j}| \leq 1} \sup_{|z_j| < 1} \left| \frac{z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| < 1. \quad (12)$$

By the maximum modulus principle, which was introduced in the last section, we only need to consider the boundary with respect to z_j .

$$\begin{aligned} \sup_{|a_{1j}| \leq 1} \sup_{|z_j| < 1} \left| \frac{z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| &< \sup_{|a_{1j}| \leq 1} \sup_{|z_j|=1} \left| \frac{z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| \\ &= \sup_{|a_{1j}| \leq 1} \sup_{|z_j| < 1} \left| \frac{\Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| \end{aligned} \quad (13)$$

Also, since $|z_j| = 1$, we can replace $1/z_j^*$ with z_j . Then we get

$$\sup_{|a_{1j}| \leq 1} \sup_{|z_j| < 1} \left| \frac{z_1 z_2 \cdots z_N \Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| < \sup_{|a_{1j}| \leq 1} \sup_{|z_j|=1} \left| \frac{\Xi^{G \setminus \{N\}}(a_{12}/z_2^*, a_{13}/z_2^*, \dots)}{\Xi^{G \setminus \{N\}}(a_{12} z_2, a_{13} z_3, \dots)} \right| = 1. \quad (14)$$

Here, we proved (??) which proves the negation of (??) which proves that the grand partition function for a system with N sites is non-zero under the hypothesis that grand partition function for a system with $N - 1$ sites is non-zero. Then, we are done. \blacksquare

III. LEE-YANG THEOREM

Now, we specialise to the case when all the z_j are z , and $\Xi^G(z)$ is a polynomial of degree N .

Theorem. The zeros of $\Xi^G(z)$ all lie on the unit circle $\{z; |z| = 1\}$ if exist.

Proof. From the proposition we proved in the last section, the interior of the unit circle is free of zeros. From (??), which with the proposition leads to

$$\Xi^G(z) = z^N \Xi^G(1/z) \neq 0 \quad \text{in} \quad \{z; |z| < 1\}, \quad (15)$$

it is also non-zero in the exterior. In order to check that we can substitute $1/z$ with some other variable w :

$$\frac{\Xi^G(w)}{w^N} \neq 0 \quad \text{in} \quad \{w; |w| > 1\} \quad (16)$$

We are done. \blacksquare