Note: Fisher zeros for two dimensional Ising model lie on unit circle in the complex $w := \sinh \beta J$ plane.

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I. THE SETUP AND THE PROOF

The exanct expression of the partition function of a finite two dimensional Ising model is as follows.

$$Z = 2^N \prod_{j=1}^{L} \prod_{k=1}^{L} \left\{ \left(\frac{1+z^2}{1-z^2} \right)^2 - \frac{2z}{1-z^2} \left(\cos \frac{2\pi j}{L} + \cos \frac{2\pi k}{L} \right) \right\}^{1/2}, \quad z := \tanh \beta J \tag{1}$$

Now let us rewrite this expression in terms of a new variable $w := \sinh \beta J$. Since

$$\left(\frac{1+z^2}{1-z^2} \right)^2 = \left(\frac{2}{1-z^2} - 1 \right)^2 = \left(2\cosh^2 \beta J - 1 \right)^2 = (\cosh 2\beta J)^2 = 1 + \sinh^2 2\beta J, \tag{2}$$

$$\frac{2z}{1-z^2} = 2\cosh^2 \beta J \tanh \beta J = 2 \cosh \beta J \sinh \beta J = \sinh 2\beta J, \tag{3}$$

then

$$Z = 2^N \prod_{j=1}^{L} \prod_{k=1}^{L} \left\{ 1 + w^2 - w \left(\cos \frac{2\pi j}{L} + \cos \frac{2\pi k}{L} \right) \right\}^{1/2}, \quad w := \sinh 2\beta J. \tag{4}$$

For simplicity, we now write

$$a_L(j,k) := \cos \frac{2\pi j}{L} + \cos \frac{2\pi k}{L}, \tag{5}$$

and

$$Z = 2^N \prod_{j=1}^{L} \prod_{k=1}^{L} \left\{ w^2 - a_L(j,k) w + 1 \right\}^{1/2}, \quad w := \sinh 2\beta J. \tag{6}$$

Now we can easily find the zeros of Z in terms of w by solving the equation

$$w^2 - a_L(j,k) w + 1 = 0 \tag{7}$$

for all j and k. Using the quadratic formula we get

$$w = \frac{a_L(j,k)}{2} \pm i \frac{\sqrt{4 - a_L^2(j,k)}}{2} \tag{8}$$

As one can easily see from the form of the solution, w lie on unit circle $|w| = 1$.

$$|w|^2 = \left\{ \frac{a_L(j,k)}{2} \right\}^2 + \left\{ \frac{\sqrt{4 - a_L^2(j,k)}}{2} \right\}^2 = \frac{a_L^2(j,k)}{4} + \frac{4 - a_L^2(j,k)}{4} = 1 \tag{9}$$

In the case of a system described by two dimensional Ising model, we can see that some singularityies appear on the real axis, and some of the w give a real inverse temperature. It can be obtained by the following procedure. First of all, the real w that satisify the realation $|w| = 1$ are ± 1. Then

$$w = \sinh 2\beta J = \frac{e^{2\beta J} - e^{-2\beta J}}{2} = \pm 1 \quad \longrightarrow \quad \left(e^{2\beta J}\right)^2 \pm 2e^{2\beta J} - 1 = 0. \tag{10}$$

By solving the equation above (10), we get

$$e^{2\beta J} = \sqrt{2} \pm 1 \quad \longrightarrow \quad e^{\beta J} = \left(\sqrt{2} \pm 1\right)^{1/2} \tag{11}$$

Here we only consider the solutions which gives the real inverse temperature. The $z = \tanh \beta J$ that corresponds to the real w at singularity is as follows.

$$z = \tanh \beta J = \frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} = \sqrt{2} - 1 \,\text{or}\, 1 - \sqrt{2}. \tag{12}$$

According to Fisher, the former solution locate the ferromagnetic temperature and the latter locate the antiferromagnetic temperature.

II. COMPARISON WITH THE CLASSICAL PREDICTION

Classicaly, we can predict the transition point by taking the thermodynamic limit of the free energy per site and looking for a point where the free energy gets singular. In this case, the free energy per site in the thermodynamic limit is as follows.

$$f = -\frac{1}{\beta} \log \frac{2}{1 - z^2} - \frac{1}{2\beta} \int_0^{2\pi} \frac{d\omega_j}{2\pi} \int_0^{2\pi} \frac{d\omega_k}{2\pi} \log \left\{ (1 + z^2)^2 - 2z(1 - z^2)(\cos \omega_j + \cos \omega_k) \right\} \tag{13}$$

Here the integrand is minmized, in term of ω_j and ω_k, at the point where satisify $\cos \omega_j + \cos \omega_k = 2$ and

$$(1 + z^2)^2 - 4z(1 - z^2) = (z^2 + 2z - 1)^2 \geq 0, \tag{14}$$

then we get

$$(1+z^2)^2 - 2z(1-z^2)(\cos\omega_j + \cos\omega_k) \geq 0. \tag{15}$$

This implies that at the singular points, $z^2+2z-1=0$ has to be satisified. For ferromagnetic model, by taking the positive solution, we can predict the transition point

$$z = \sqrt{2} - 1 \tag{16}$$

which is the same value as the one predicted above using the method of Lee and Yang.