

# Note: Exact Solution of Two Dimensional Ising Model

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## I. THE SETUP

In this document we will be dealing with two-dimensional Ising model, which is the statistical-mechanical model of a system with a Hamiltonian

$$\mathcal{H} = -J \sum_{j,k} s_j s_k.$$

Here we defined the spin variable  $s_j$  which takes  $\pm 1$ . For convenience, let us now put these spins on the two-dimensional lattice and label them by the column and row number  $(n, m)$ . Then the Hamiltonian can be rewritten in the following form.

$$\mathcal{H} = -J \sum_{n,m} s_{n,m} s_{n,m+1} - J \sum_{n,m} s_{n,m} s_{n+1,m}.$$

Now, our job is to calculate the partition function for this system

$$Z = \sum_{\{s\}} \exp(-\beta \mathcal{H}) = \sum_{\{s\}} \exp \left[ \beta J \sum_{n,m} (s_{n,m} s_{n,m+1} + s_{n,m} s_{n+1,m}) \right].$$

The exact solution for the problem was first given by Onsager and subsequently given by many other physicists including the Japanese physicist Nambu. The solution we will see in this document is neither Onsager's nor Nambu's solution, but the one given by Vdovicheko.

## II. HIGH-TEMPERATURE EXPANSION

First of all, let us rewrite the partition function in the following form.

$$\begin{aligned} Z &= \sum_{\{s\}} \prod_{n,m} e^{\beta J (s_{n,m} s_{n,m+1} + s_{n,m} s_{n+1,m})} \\ &= \cosh^{2N}(\beta J) \sum_{\{s\}} \prod_{n,m} (1 + s_{n,m} s_{n,m+1} z)(1 + s_{n,m} s_{n+1,m} z) \quad \text{where } z := \tanh(\beta J). \end{aligned}$$

Here, we used the identity

$$e^{xss'} = \cosh x + ss' \sinh x.$$

In the end, the partition function can be written in the following form.

$$Z = 2^N \cosh^{2N}(\beta J) \tilde{Z}, \quad \tilde{Z} := \sum_{k=0}^N g(k) z^k$$

Here we defined  $g(k)$ , which is the number of diagrams comprised of  $k$  edges.

### III. LOOP EXPANSION

By interpreting  $\tilde{Z}$  as a sum of undirected loops instead of a sum of diagrams, we can map the system to a random walk problem. But the problem is that the sum of undirected loop does not necessarily equal the sum of diagrams. We can avoid this problem by summing up with a weight  $(-1)^t$  where  $t$  is the number of unavoidable self-intersections.

Now classifying the loops by the number of bonds and shape, the sum can be expressed in the following form.

$$\begin{aligned} \tilde{Z} &= 1 + \sum \square + \sum (\square)^2 + \sum (\square)^3 + \dots \\ &\quad + 1 + \sum \square \square + \sum (\square \square)^2 + \sum (\square \square)^3 + \dots \\ &\quad + 1 + \sum \square \square \square + \sum (\square \square \square)^2 + \sum (\square \square \square)^3 + \dots \\ &= \exp \left( \sum \square + \sum \square \square + \dots \right) \end{aligned}$$

For simplicity, we only consider the argument of exponential.

$$\log \tilde{Z} =: - \sum_{k=0}^{\infty} f(k) z^k$$

### IV. MAPPING THE SYSTEM TO A RANDOM WALK PROBLEM