Notes for master's thesis

Foundations & applications of generalised symmetries

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[1]

1 Foundations

1.1 Conventional symmetries *

Most parts of this section are adopted from the matirials below.

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory"

[2] M. Srednicki, "Quantum field theory"

[3] P. R. S. Gomes, "An introduction to higher-form symmetries", Section 3

1.1.1 Schwinger-Dyson equation & conversation laws

Proposition$^{\mathrm{ph}}$ 1.1. We have following relation, which is well-known as Schwinger-Dyson equation.

$$\left\langle \frac{\partial S[\varphi]}{\partial \varphi(x)} \varphi(x_1) \cdots \varphi(x_n) \right\rangle = i \sum_{i=1}^{n} \left\langle \varphi(x_1) \cdots \delta^D(x - x_i) \cdots \varphi(x_n) \right\rangle . \tag{1.1}$$

Proof. First of all, we prove the general relation

$$\int \mathscr{D}\varphi \, \frac{\delta}{\delta\varphi(x)} \Big(F[\varphi] e^{iS[\varphi]} \Big) = 0. \tag{1.2}$$

We start from the obvious relation

$$\int \mathscr{D}\varphi \, F[\varphi] e^{iS[\varphi]} = \int \mathscr{D}\varphi' \, F[\varphi'] e^{iS[\varphi']}, \tag{1.3}$$

[1]The sections with asterisk " * " are materials that is not directly related to generalised symmetries but supplemental.

which is just a renaming of a dummy variable of the integration. Now we perform a change of variables $\varphi'(x) \longmapsto \varphi(x) + \varepsilon(x)$. Then we have

$$\begin{aligned}
\int \mathscr{D}\varphi\, F[\varphi] e^{iS[\varphi]} &= \int \mathscr{D}\varphi\, F[\varphi+\varepsilon] e^{iS[\varphi+\varepsilon]} \\
&= \int \mathscr{D}\varphi \left\{ F[\varphi] + \int \mathrm{d}^D x\, \frac{\delta F[\varphi]}{\delta\varphi(x)} \varepsilon(x) \right\} \exp\left\{ iS[\varphi] + i \int \mathrm{d}^D x\, \frac{\delta S[\varphi]}{\delta\varphi(x)} \varepsilon(x) \right\} \\
&= \int \mathscr{D}\varphi \left\{ F[\varphi] + \int \mathrm{d}^D x\, \frac{\delta F[\varphi]}{\delta\varphi(x)} \varepsilon(x) \right\} \left\{ 1 + i \int \mathrm{d}^D x\, \frac{\delta S[\varphi]}{\delta\varphi(x)} \varepsilon(x) \right\} e^{iS[\varphi]} \\
&= \int \mathscr{D}\varphi\, F[\varphi] e^{iS[\varphi]} + \int \mathscr{D}\varphi \int \mathrm{d}^D x \left\{ \frac{\delta F[\varphi]}{\delta\varphi(x)} e^{iS[\varphi]} + F[\varphi] i \frac{\delta S[\varphi]}{\delta\varphi(x)} e^{iS[\varphi]} \right\} \varepsilon(x), \quad {}^\forall\varepsilon(x), \quad (1.4)
\end{aligned}$$

which leads us to the identity

$$\int \mathscr{D}\varphi \left\{ \frac{\delta F[\varphi]}{\delta\varphi(x)} e^{iS[\varphi]} + F[\varphi] i \frac{\delta S[\varphi]}{\delta\varphi(x)} e^{iS[\varphi]} \right\} = 0. \tag{1.5}$$

Now, by substituting $F[\varphi]$ with $\varphi(x_1)\cdots\varphi(x_n)$ we get

$$i \int \mathscr{D}\varphi\, \frac{\delta S[\varphi]}{\delta\varphi(x)} \varphi(x_1)\cdots\varphi(x_n) e^{iS[\varphi]} = -\sum_{i=1}^{n} \int \mathscr{D}\varphi\, \varphi(x_1)\cdots\delta^D(x-x_i)\cdots\varphi(x_n) e^{iS[\varphi]}, \tag{1.6}$$

which we simply denote

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \varphi(x_1)\cdots\varphi(x_n) \right\rangle = i \sum_{i=1}^{n} \left\langle \varphi(x_1)\cdots\delta^D(x-x_i)\cdots\varphi(x_n) \right\rangle. \tag{1.7}$$

□

Remark 1.1. Schwinger-Dyson equation is the generalisation of Ward-Takahashi identity.

Proposition$^{\mathrm{ph}}$ 1.2. We have following relation, which is a correlation function representation of the conversation law of Noether current.

$$\langle \{\partial_\mu j^\mu(x)\} \varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n) \rangle = -i \sum_{i=0}^{n} \left\langle \varphi_{a_1}(x_1)\cdots\delta^D(x-x_i)\delta\varphi_{a_i}(x_i)\cdots\varphi_{a_n}(x_n) \right\rangle. \tag{1.8}$$

Proof. Now, we consider the transformation

$$\varphi_a(x) \longmapsto \varphi'_a(x) = \varphi_a(x) + \delta_{\varepsilon(x)}\varphi_a(x) = \varphi_a(x) + \varepsilon(x) g_a(x). \tag{1.9}$$

We first prove the relation

$$\delta_{\varepsilon(x)} S[\varphi] = -\int \mathrm{d}^D x\, \varepsilon(x) \partial_\mu \left\{ \sum_{a=1}^{N} g_a(x) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi_a(x))} - K^\mu[\varphi] \right\} = -\int \mathrm{d}^D x\, \varepsilon(x) \partial_\mu j^\mu(x) \tag{1.10}$$

where we defined $j^\mu(x)$ as

$$j^\mu(x) = \sum_{a=1}^{N} g_a(x) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi_a(x))} - K^\mu[\varphi]. \tag{1.11}$$

The direct calculation of $\delta_{\varepsilon(x)}S[\varphi]$ is as follows.

$$\begin{aligned}\delta_{\varepsilon(x)}S[\varphi] &= \int \mathrm{d}^D x\, \delta_{\varepsilon(x)}\mathscr{L}[\varphi, \partial_\mu\varphi] = \int \mathrm{d}^D x\, \{\mathscr{L}[\varphi(x)+\varepsilon(x)\varphi(x), \partial_\mu\{\varphi(x)+\varepsilon(x)g(x)\}] - \mathscr{L}[\varphi(x), \partial_\mu\varphi(x)]\} \\ &= \int \mathrm{d}^D x\, \{\mathscr{L}[\varphi(x)+\varepsilon(x)\varphi(x), \partial_\mu\varphi(x)+\partial_\mu\varepsilon(x)g(x)+\varepsilon(x)\partial_\mu g(x)] - \mathscr{L}[\varphi(x), \partial_\mu\varphi(x)]\} \\ &= \int \mathrm{d}^D x \left\{\varepsilon(x)\sum_{a=1}^N g_a(x)\frac{\partial\mathscr{L}}{\partial\varphi_a(x)} + \varepsilon(x)\sum_{a=1}^N \partial_\mu g_a(x)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_a(x))} + \partial_\mu\varepsilon(x)\sum_{a=1}^N g_a(x)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_a(x))}\right\} \\ &= \int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu K^\mu(x) + \int \mathrm{d}^D x\, \partial_\mu\varepsilon(x)\sum_{a=1}^N g_a(x)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_a(x))} \end{aligned} \tag{1.12}$$

Here, we assumed

$$\sum_{a=1}^N g_a(x)\frac{\partial\mathscr{L}}{\partial\varphi_a(x)} + \sum_{a=1}^N \partial_\mu g_a(x)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_a(x))} = \partial_\mu K^\mu[\varphi(x)], \quad {}^\exists K^\mu[\varphi(x)]. \tag{1.13}$$

Integrating the second term by parts, we get

$$\delta_{\varepsilon(x)}S[\varphi] = \int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu K^\mu(x) - \int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu\left\{g_a(x)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_a(x))}\right\} = -\int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu j^\mu(x). \tag{1.14}$$

Now, we move on to the proof of the relation in consider. We start from the trivial relation

$$\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]} = \int\left(\prod_{a=1}^N \mathscr{D}\varphi'_a\right)\varphi'_{a_1}(x_1)\cdots\varphi'_{a_n}(x_n)e^{iS[\varphi]}. \tag{1.15}$$

Here, we perform the change of variable $\varphi'_a(x) = \varphi_a(x) + \varepsilon(x)g_a(x)$ and we get

$$\begin{aligned}\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]} &= \int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]} \\ &\quad + i\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\left\{-\int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu j^\mu(x)\right\}e^{iS[\varphi]} \\ &\quad + \sum_{i=1}^n\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varepsilon(x_i)g_{a_i}(x_i)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]},\end{aligned} \tag{1.16}$$

which leads us to

$$\begin{aligned}& -i\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\left\{\int \mathrm{d}^D x\, \varepsilon(x)\partial_\mu j^\mu(x)\right\}e^{iS[\varphi]} \\ &\qquad = \sum_{i=1}^n\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varepsilon(x_i)g_{a_i}(x_i)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]} = 0.\end{aligned} \tag{1.17}$$

Now, we substitute $\varepsilon(x_i)$ in RHS with a trivial relation $\varepsilon(x_i) = \int \mathrm{d}^D x\, \varepsilon(x)\delta^D(x - x_i)$, we get

$$\begin{aligned}& -i\int \mathrm{d}^D x\, \varepsilon(x)\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\{\partial_\mu j^\mu(x)\}e^{iS[\varphi]} \\ &\qquad = \int \mathrm{d}^D x\, \varepsilon(x)\sum_{i=1}^n\int\left(\prod_{a=1}^N \mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\delta^D(x-x_i)g_{a_i}(x_i)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]}, \quad {}^\forall\varepsilon(x).\end{aligned} \tag{1.18}$$

Then we get

$$i\int\left(\prod_{a=1}^{N}\mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\{\partial_\mu j^\mu(x)\}e^{iS[\varphi]}=\sum_{i=1}^{n}\int\left(\prod_{a=1}^{N}\mathscr{D}\varphi_a\right)\varphi_{a_1}(x_1)\cdots\delta^D(x-x_i)g_{a_i}(x_i)\cdots\varphi_{a_n}(x_n)e^{iS[\varphi]}, \tag{1.19}$$

which means that

$$\langle\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\{\partial_\mu j^\mu(x)\}\rangle=-i\sum_{i=0}^{n}\left\langle\varphi_{a_1}(x_1)\cdots\delta^D(x-x_i)g_{a_i}(x_i)\cdots\varphi_{a_n}(x_n)\right\rangle. \tag{1.20}$$

□

1.1.2 Reformulation in differential forms

Fact$^{\text{ph}}$ 1.1. Mathematically speaking, Noether current $j^\mu(x)$ are coefficients of a vector field on the spacetime manifold $\mathscr{M}$, which is a section of the tangent bundle $T\mathscr{M}$. And equivaletly $j_\mu(x)$ are coefficients of a covector field of a section of the cotangent bundle $T^*\mathscr{M}$. Let us stick to the representation on cotangent bundle in this notes, then we can write

$$j=j_\mu(x)\mathrm{d}x^\mu. \tag{1.21}$$

And obviously, we have the Hodge dual of it.

$$*j=\frac{1}{(D-1)!}j_{\mu_0}(x)\epsilon^{\mu_0}{}_{\mu_1\cdots\mu_{D-1}}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{D-1}}. \tag{1.22}$$

Proposition$^{\text{ph}}$ 1.3. The conversation law $\partial_\mu j^\mu(x)=0$ can be written as

$$\mathrm{d}*j=0. \tag{1.23}$$

Proof. We will see the equivalence of the two representations $\partial_\mu j^\mu=\partial^\mu j_\mu=0$ and $\mathrm{d}*j=0$. We start from

$$\begin{aligned}\mathrm{d}*j&=\frac{1}{(D-1)!}\partial_\alpha j_{\mu_0}(x)\epsilon^{\mu_0}{}_{\mu_1\cdots\mu_{D-1}}\mathrm{d}x^\alpha\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{D-1}}=\frac{1}{(D-1)!}\partial_\alpha j_{\mu_0}(x)\epsilon^{\mu_0}{}_{\mu_1\cdots\mu_{D-1}}\epsilon^{\alpha\mu_1\cdots\mu_{D-1}}\\&=(-1)^{D-1}\partial_\alpha j_{\mu_0}(x)\eta^{\mu_0\alpha}\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}=(-1)^{D-1}\partial^\mu j_\mu(x)\mathrm{d}x^0\wedge\cdots\mathrm{d}x^{D-1}.\end{aligned} \tag{1.24}$$

Now, if we assume $\mathrm{d}*j=0$, obviously $\partial_\mu j^\mu(x)=0$ because $\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\neq0$. Also if we assume $\partial_\mu j^\mu(x)=0$, obviously $\mathrm{d}*j=0$ from the relation we just derived. □

Remark 1.2. In the Euclidean theory, which we will be using so often in this note, we simply have

$$\mathrm{d}*j=\partial_\mu j^\mu(x)\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}. \tag{1.25}$$

Proposition$^{\text{ph}}$ 1.4. The definition of Noether charge $Q=\displaystyle\int\mathrm{d}^{D-1}x\,j_0(x)$ is translated to differential forms as follows.

$$Q=\int_\Sigma *j, \tag{1.26}$$

where, Σ is the spatial submanifold of the spacetime manifold $\mathscr{M}$.

Proof. We start by rewriting the integral.

$$
\begin{aligned}
Q &= \int_\Sigma *j = \int_\Sigma \frac{1}{(D-1)!} j_{\mu_0}(x)\epsilon^{\mu_0}{}_{\mu_1\cdots\mu_{D-1}} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{D-1}} \\
&= \int_\Sigma \frac{1}{(D-1)!} j_{\mu_0}(x)\epsilon^{\mu_0}{}_{\mu_1\cdots\mu_{D-1}} \epsilon^{0\mu_1\cdots\mu_{D-1}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{D-1} = \int_\Sigma j_{\mu_0}(x)(-1)^{D-1}\eta^{\mu_0 0}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{D-1}.
\end{aligned}
\tag{1.27}
$$

Obviously, (1.27) $= 0$ for $\mu_0 \neq 0$. Then we have

$$
Q = (-1)^{D-1} \int_\Sigma j_0(x)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{D-1}, \tag{1.28}
$$

which has the same meaning as $\int \mathrm{d}^{D-1}x\, j_0(x)$.[2] □

Proposition$^{\mathrm{ph}}$ 1.5.

1.2 Definition of higher-form symmetries

Most parts of this section are adopted from the materials below.

[3] P. R. S. Gomes, "An introduction to higher-form symmetries"

2 Applications

References

[1] M. E. Peskin and D. V. Schroeder, Addison-Wesley, 1995, ISBN 978-0-201-50397-5, 978-0-429-50355-9, 978-0-429-49417-8 doi:10.1201/9780429503559

[2] M. Srednicki, Cambridge University Press, 2007, ISBN 978-0-521-86449-7, 978-0-511-26720-8 doi:10.1017/CBO9780511813917

[3] P. R. S. Gomes, SciPost Phys. Lect. Notes **74** (2023), 1 doi:10.21468/SciPostPhysLectNotes.74 [arXiv:2303.01817 [hep-th]].

[2]Here, we ignore the factor $(-1)^{D-1}$ because it does not affect the physical invariant.