Contents

1 Mathmatical preparations

We begin by estabilishing the mathematical framework. In dealing with symmetries, higher-form symmetries in particular, differential forms are a very convenient tool. First we will introduce differential forms and in relation to that, we will introduce de Rham cohomology groups.

1.1 Differential forms

We will introduce differential forms in a very intuitive way, sacrificing the mathematical rigor. The most part of this section are adapted from [1].

1.1.1 Introduction: Differential forms in 2 dimensional Euclid space

Here we are dealing with a 2 dimensional Euclid space M. And we consider a integral of a vector field $(a_x(x,y), a_y(x,y))$ on a curve $\gamma(t) = (x(t), y(t))$. Here $t \in [0,1]$. In order to define the integral we first introduce a *1-form* ω:

$$\omega := a_x(x,y)\mathrm{d}x + a_y(x,y)\mathrm{d}y. \tag{1.1}$$

Then we define the integral as follows.

$$\int_\gamma \omega := \int_0^1 \mathrm{d}t \left\{ a_x(x(t),y(t))\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_y(x(t),y(t))\frac{\mathrm{d}y(t)}{\mathrm{d}t} \right\}. \tag{1.2}$$

In the case that ω is the total derivative, which means that $a_x(x,y) = \partial_x f(x,y)$ and $a_y(x,y) = \partial_y f(x,y)$ with ${}^\exists f(x,y)$, we write

$$\int_\gamma \omega = \int_\gamma \mathrm{d}f = f(\gamma(1)) - f(\gamma(0)). \tag{1.3}$$

Next, we introduce $\partial\gamma := \gamma(1) - \gamma(0)$. And we also define a integral

$$\int_{\partial\gamma} f := f(\gamma(1)) - f(\gamma(0)). \tag{1.4}$$

Combining (1.3) and (1.4), we get

$$\int_\gamma \mathrm{d}f = \int_{\partial\gamma} f. \tag{1.5}$$

This is the 1 dimensional version of Stokes's theorem.

Next, we introduce a *2-form* $\mathrm{d}\omega$ as

$$\mathrm{d}\omega := \partial_x a_y(x,y)\mathrm{d}x \wedge \mathrm{d}y + \partial_y a_x(x,y)\mathrm{d}y \wedge \mathrm{d}x. \tag{1.6}$$

Here, we introduced the wedge product $\cdot \wedge \cdot$

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x. \tag{1.7}$$

In the case of $\omega = \mathrm{d}f$, we get

$$\mathrm{d}\omega = \partial_x\partial_y f(x,y)\mathrm{d}x \wedge \mathrm{d}y + \partial_y\partial_x f(x,y)\mathrm{d}y \wedge \mathrm{d}x = 0. \tag{1.8}$$

1.1.2 Generalization: Differential forms in D dimensional Euclid space

Now we generalize the properties we got from the 2 dimensional case to D dimensional case.

Definition 1.1 We define the wedge product $\cdot \wedge \cdot$ as

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu. \tag{1.9}$$

Introducing the associativity, we generalize this to the case of more than two terms.

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho = -\mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\rho. \tag{1.10}$$

Definition 1.2 We define a *p-form* as

$$X := \frac{1}{p!} X_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{1.11}$$

And we write the entire set of p-form as $\Omega^p(M)$.

Definition 1.3 We define the *exterior derivative* as

$$\mathrm{d} : \Omega^p(M) \longrightarrow \Omega^{p+1}(M); \quad \frac{1}{p!} X_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \longmapsto \frac{1}{p!} \partial_\nu X_{\mu_1\cdots\mu_p} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{1.12}$$

1.2 de Rham cohomology groups

References

[1] Yoshimasa Hidaka, *Introduction to Higher Symmetries*.