Dynamic Economic Modelling
Tutorial Exercises (Solutions)

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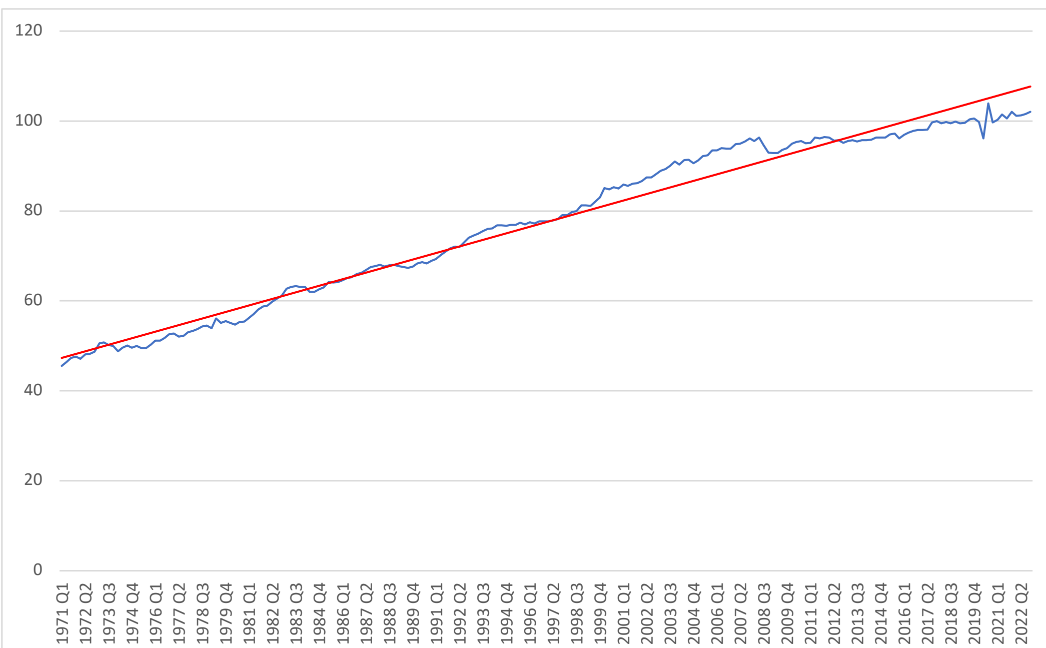
1 Data and the current economy

Download data on UK "Labour Productivity" from the website of the Office for National Statistics (ONS). The correct data should have the name "PRDY".

(a) Using a spreadsheet or other computer program (**Note:** Try and use MATLAB or Python or R if you can!), make a graph of the column labelled "UK Whole Economy: Output per hour worked SA: Index 2018 = 100" over time (code = LZVB). Make sure you only use the quarterly data from 1971Q1 to as close to the present day as possible.

Figure 1: UK Whole Economy: Output per Hour Worked (Index 2017 Q4 = 100)



(Note: This exercise is very easy to do in R, Python, or MATLAB. I would recommend that you take the opportunity to practice one of these languages.)

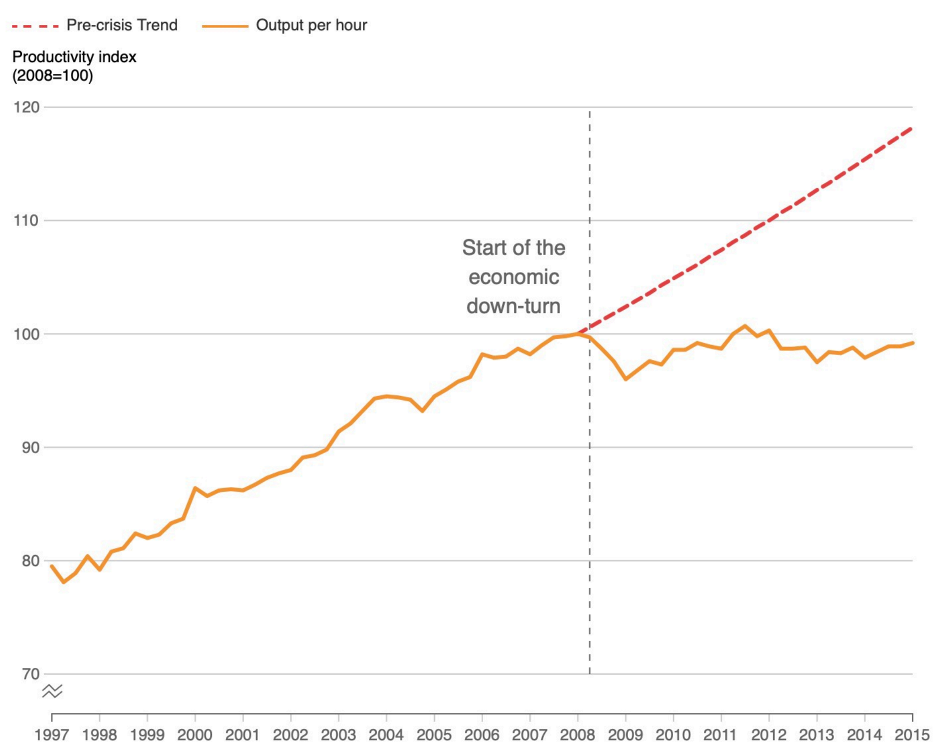
(b) Discuss the shape of the graph after the 2008 financial crisis.

Real GDP fell dramatically and unemployment rose after the 2008 financial crisis. At first sight we cannot then tell what happened to labour productivity. As time goes on, though, unemployment fell whereas real GDP never recovered the full extent of its initial loss. The evidence is therefore consistent, in that real GDP per worker must have fallen. To properly test for consistency it would be useful to know what happened to hours worked – it may be that these have fallen, in which case productivity could have stayed at its initial level.

(c) Add a trend line to the graph to more precisely pin down how UK labour productivity behaved after the 2008 financial crisis. What is the big picture? Why does the trend line in your graph look different to that produced by the ONS at http://visual.ons.gov.uk/productivity-puzzle/? Which graph gives the most accurate description of reality, yours or the one by the ONS?

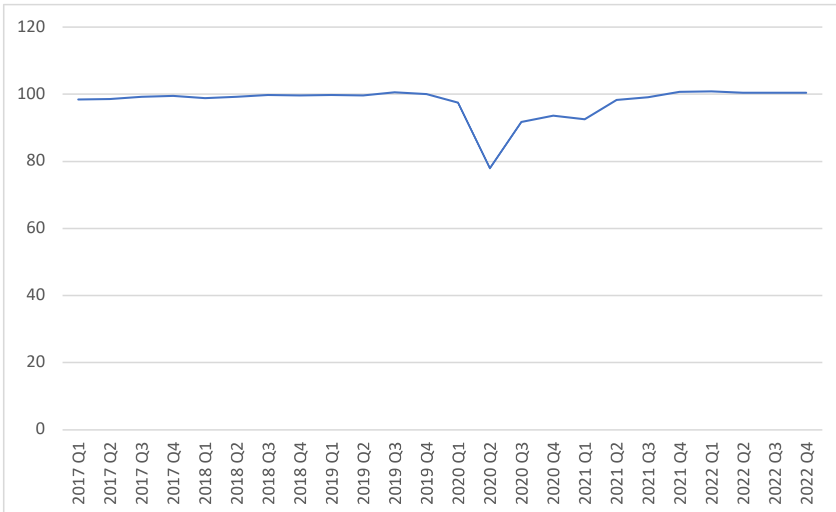
The trend line was already plotted in Figure 1. The big picture is that labour productivity has fallen below trend and has not recovered to its pre-crisis levels – the "productivity puzzle". See also Figure 3.

Figure 2: Productivity, UK, January to March 1997 to January to March 2015



Source: ONS UK

Figure 3: Output per Worker: Whole Economy



The trend line in the ONS graph is calculated using data only up until 2008, whereas ours includes data from the financial crisis period and beyond. Compared to the ONS graph, our graph paints a different picture in which the crisis saw a reversal of the "excessive" productivity seen in the years before. Both graphs do though show that productivity post crisis is low. The problem here is the familiar one of one-sided versus two-sided filters. It's not clear even ex post what the best graph is. In real time of course policymakers have to rely on one-sided filters and do not have the benefit of hindsight.

(d) Why might UK labour productivity have behaved as it has since the 2008 financial crisis? Have a look at the Bank of England's Quarterly Bulletin 2014 Q2 and the 2014 speech by Martin Weale "The UK productivity puzzle: an international perspective" for background reading. They are available at: i) "The UK productivity puzzle: an international perspective", ii) Speech by Martin Weale

From the Bank of England QB:

Overview

Since the onset of the 2007–08 financial crisis, labour productivity in the United Kingdom has been exceptionally weak. Despite some modest improvements in 2013, whole-economy output per hour remains around 16% below the level implied by its pre-crisis trend. Even taking into account possible measurement issues and secular changes in some sectors, this shortfall is large — and often referred to as the 'productivity puzzle'.

Measures of productivity can be used to inform estimates of an economy's ability to grow without generating excessive inflationary pressure, which makes understanding recent movements important for the conduct of monetary policy. In this context a key challenge has been to understand better how much of the weakness in productivity has been due to (i) cyclical explanations related to demand conditions, compared to (ii) other more persistent causes related to the financial crisis. This article sets out some of the factors that might help to explain the UK productivity puzzle, grouped into these two categories. Based on recent research by Bank staff, the available evidence suggests that there is more likely to have been a range of factors at play rather than any one single explanation (see **summary table**).

During the initial phases of the recession, companies appear to have acted flexibly by holding on to labour and lowering levels of factor utilisation in response to weak demand conditions. Other cyclical explanations, such as having to work harder to win new business, are also likely to have played a role. But the protracted weakness in productivity and the strength in employment growth over the past two years suggest that other factors are likely to be having a more persistent impact on the level of productivity. These factors are likely to have manifested themselves in reduced investment in both physical and intangible capital, such as innovation, and impaired resource allocation from low to high productive uses.

But there remains a large degree of uncertainty around any interpretation of the weakness in productivity. The explanations covered in this article are unlikely to be exhaustive and are unable to explain the full extent of the productivity shortfall.

Summary table Factors contributing to the weakness in UK labour productivity by 2013 Q4

Shortfall in labour productivity relative to pre-crisis trend in 2013 Q4	16pp
Measurement issues Including: potential mismeasurement of output and changes to trend rates of growth in some sectors	Around 4pp
Actual shortfall to explain	**12pp**
Hypothesis I: cyclical explanations Including: measured spare capacity within firms, other cyclical factors reflecting changing demand conditions	Uncertain, but little evidence of spare capacity from business surveys and employment outturns
Hypothesis II: more persistent factors Including: reduced investment in physical and intangible capital, and impaired resource allocation and unusually high firm survival rates	Likely to be significant in recent years, contributing around 6 to 9pp
Total explained	**Around 6 to 9pp**

(e) Examine also the series "Output per Worker: Whole Economy SA: Index 2018=100: UK" (code = A4YM). Are you surprised by the behaviour of the productivity series in 2020 Q2 when the pandemic hit? If so, why? If not, why not?

See Figure 3.

(f) Can you plot similar data for your home country?

Try and see if you can do this for your home country. If you can't find adequate data sources, then practice this exercise with US data.

2 Simple representative agent problem

Suppose a representative agent has utility function involving consumption and labour supply of the form

$$U = \ln C - 2L^2,$$

where $\ln C$ is the natural logarithm.

(a) The agent is a "yeoman farmer" that produces their own output with a production function $C = Y = AL^\alpha$, where Y is output per head. Derive the optimal level of labour supply, and comment on what it implies for the relationship between labour supply and productivity growth.

Labour supply solves

$$\max \left\{ \ln(AL^\alpha) - 2L^2 \right\}.$$

The FOC with respect to L is $\alpha/L - 4L = 0$. Hence, labour supply is $L = \sqrt{\alpha}/2$. This is independent of the productivity parameter A, which is a result of utility being logarithmic in consumption and the production being a power function. Intuitively, the income and substitution effects of technological progress on labour supply exactly offset each other. As technology progresses, we want to consume more production goods and leisure (the income effect) but the increased demand for leisure is exactly offset by a desire to consume more production goods relative to leisure (the substitution effect). Such a property is unlikely to hold in general, although labour supply relative to leisure is quite stable in the data.

(b) The agent is instead a worker, who receives a real wage w. Their budget constraint is $C = wL + \pi$, where π are profits distributed by firms, and both w and π are assumed to be exogenous by the worker. Derive an expression for optimal labour supply as a function of w and C.

The worker solves

$$\max \left\{ \ln(wL + \pi) - 2L^2 \right\}.$$

The FOC with respect to L is $w/c - 4L = 0$, and so labour supply is $L = w/(4c)$.

(c) An individual, perfectly competitive firm maximises profits $\pi = AN^\alpha - wN$, where N is the number of workers it employs. Derive the labour demand curve for this firm. What is the relationship between the marginal product of labour and the real wage?

The profit maximisation problem of the firm is

$$\max \left\{ AN^\alpha - wN \right\}.$$

The FOC wrt N is $\alpha AN^{\alpha-1} - w = 0$, so labour demand is given by $N = [w/(\alpha A)]^{1/(\alpha-1)}$. The FOC is the classic perfect factor markets condition that wage equals the marginal product of labour.

(d) Since each worker works L hours and each firm wishes to employ a total of N hours, equilibrium requires that $L = nN$ where n is the number of firms per worker (or, equivalently, the reciprocal of the number of workers per firm) . As consumption per head is equal to output per head, $C = nY = nAN^\alpha$. Use this, plus the expression for the real wage, in the labour supply equation to derive the equilibrium value for L in terms of model parameters. Compare this to your answer to (a), and comment.

The first part of this is maths:

$$L = \frac{w}{4C} = \frac{\alpha A N^{\alpha-1}}{4nAN^{\alpha}} = \frac{\alpha}{4nN} = \frac{\alpha}{4L}.$$

The equilibrium at which labour supply equals labour demand is hence $L = \sqrt{\alpha}/2$, exactly as in the yeoman farmer in the first part of this question. This should not be a surprise – the conditions for the welfare theorems apply here so the decentralised equilibrium will be Pareto efficient and have the same allocation as the yeoman farmer/-social planner.

(e) Suppose the number of agents in the economy increases (because of immigration, for example), but the new agents are just like the existing ones. As a result, the number of firms n falls. Show what happens to real wages, output per head and consumption per head.

We know that labour supply per worker is fixed at $L = \sqrt{\alpha}/2$, so given the number of firms n falls it must be from $L = nN$ that labour demand per firm rises. With $n \downarrow$ and $N \uparrow$ it must be that the real wage $w = \alpha A N^{\alpha-1}$ falls and output and consumption per head $nY = C = nAN^{\alpha} = LAN^{\alpha-1}$ falls too. intuitively, the firms that remain are in a better position relative to the size of the market.

(f) By deriving an expression for profits per firm in terms of L and n, comment on what might happen in the long run as n changes.

It is useful to see what happens to firm profits. Since they are given by $\pi = AN^{\alpha} - WN = AN^{\alpha} - \alpha A N^{\alpha-1} N = (1-\alpha)AN^{\alpha}$, they must rise due to the number of agents in the economy. With profits rising, it is reasonable to expect more firms to enter the market, which cases $n \uparrow$ and wages, output, and consumption to return to their initial values in the long-run.

3 Simplified Romer model

Consider a two period version of a Romer model. The technology at the beginning of period 1 is given at a given exogenous level, A_1. In period 1, agents spend proportion $1 - l_1$ of their time in productive activities with production function:

$$Y_1 = A_1(1 - l_1)L,$$

where $L \geq 1$ is the total labour supply in the economy. Agents spend the remaining proportion l_1 of their time in period 1 producing ideas, according to the production function:

$$A_2 - A_1 = zA_1 l_1 L,$$

with $z \geq 1$ a parameter measuring the productivity of workers producing ideas. In the final period 2, workers spend proportion $1 - l_2$ of their time in productive activities according to:

$$Y_2 = A_2(1 - l_2)L.$$

(a) Assuming that agents only value output produced in periods 1 and 2, what would be the best proportion of time to spend producing ideas in period 2?

$l_2 = 0$, as there is no point producing ideas for use beyond period 1.

(b) Make a plot of the combinations of Y_1 and Y_2 that are possible when agents choose different proportions of their time to spend on production in period 1. To start you off, what happens to Y_1 and Y_2 if $l_1 = 0$ so that all time in period 1 is spent on production? What happens if $l_1 = 1$ so that all time in period 1 is spent producing ideas? What about intermediate cases where l_1 is between 0 and 1, for example 0.5?

If $l_1 = 0$ then $Y_1 = A_1$ and $Y_2 = A_1$, which maximises output in period 1. Technology is stagnant since no time is spent producing ideas in period 1, the economy does not grow and output in period 2 is the same as that in period 1. If $l_1 = 1$, then $Y_1 = 0$ and $Y_2 = (1 + zL)A_1$ and output in period 2 is maximised. No goods are produced in period 1 as all time is spent producing ideas, which maximises growth in technology. Intermediate values of l_1 give a weighted average of the two possibilities for output:

$$\begin{aligned} Y_1 &= (1 - l_1)A_1 + l_1(0), \\ Y_2 &= (1 - l_1)A_1 + l_1(1 + zL)A_1. \end{aligned}$$

For example, when $l_1 = 1/2$ we have that $Y_1 = A_1/2$ and $Y_2 = A_1/2 + (1 + zL)A_1/2$.

(c) Agents ideally want to consume production goods in both periods 1 and 2. One way to model this is through a multiplicative utility function:

$$U = Y_1 Y_2.$$

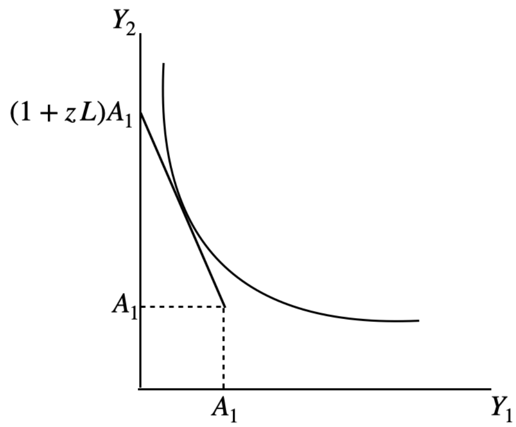
Use the three production functions given in the introduction to this question to solve out for output per head Y_1 and Y_2 as functions of l_1 and A_1, z, L alone. What amount of time should be spent producing ideas in period 1 to maximise utility? Add indifference curves to your plot in part (b) to illustrate your answer.

Substitution of the three production functions into the utility function gives:

$$U = Y_1 Y_2 = [(1 - l_1)A_1]\,[(1 - l_1)A_1 + l_1(1 + zA_1 L_1)A_1]\,.$$

Ignoring multiplicative constants and simplifying means that utility maximisation is a question of maximising $(1 + l_1)[1 + l_1(zL)]$. The FOC wrt l_1 is:

$$l_1 = \frac{zL - 1}{2zL}.$$



Note that in the diagram we need $zL > 1$ (which it is by assumption) to avoid a corner solution.

(d) How does the optimal allocation of time in period 1 change as the productivity z of hours spent producing ideas increases?

The time l_1 spent producing ideas in period 1 is increasing in zL. This makes intuitive sense as agents find it beneficial to spend more time producing ideas if they are good at it or if there are a lot of people in the economy (note that this is because of non-rivalry in the production of ideas – if a lot of people are doing research then growth will be worthwhile even if they do not each spend much time doing so).

(e) Without doing any extra calculations, discuss what you would expect to happen if agents start valuing goods produced in period 1 more that those produced in period 2 (why might they do this?). For example, their utility function might be $U = Y_1^{\phi} Y_2$, where $\phi > 1$. Answer intuitively, using a diagram where that would be helpful.

Agents will value period 1 goods over period 2 goods if they discount the future. We would expect that this would reduce the incentives to spend time producing ideas in period 1. This means that l_1 would fall. In the diagram, the budget constraint stays in the same place (the technology for producing output and ideas has not changed) but the indifference curves are distorted (the preferences across goods has changed). At each point, the indifference curves become steeper as more of the period 2 good is needed to compensate for loss of period 1 goods.

4 Filtering and business cycles

In this exercise you need to replicate some business cycle data work that you studied in class using any programming language of your choice. You are also encouraged to work with your classmates in completing these coding exercises. In class, I will ask a few of you to present your results, so try your best to make your codes and plotss as neat as possible.

Download quarterly data for US real GDP, consumption, and investment.[1] Take logs of the data and make sure that your data is seasonally adjusted.

(a) Plot US real GDP from 1950 Q1 to 2023 Q1. If possible, try to also plot NBER defined recession periods. You can be as creative as you want when it comes to labels, colours, font, and etc. Just make sure that the plots look nice and are easy to understand.

(b) With the data you have for US real GDP, fit a line of linear trend – i.e., show the data and linear trend line on a single graph. Following from what we did in previous classes, how would you interpret the performance of the US economy since the early 2000s?

(c) Now, try and extract fluctuations or deviations of US real GDP from the linear trend you found previously.

(d) Now, apply a Hodrick-Prescott (HP) filter to your US real GDP series, setting the HP filter smoothing parameter to 1600. Compare the fluctuations you get from the HP filter to the fluctuations you get from a simple [log] linear filter. Comment on the fluctuations since the early 2000s.

(e) Now, apply the HP filter to real consumption and real investment. Plot your series. What do you notice when you compare the HP filtered series of real GDP, consumption, and investment.

(f) Compare the US real GDP data series and its HP filtered series, but also add in two more HP filtered series: one where the HP smoothing parameter is set to 400, and another one with the parameter set to 800. Then focus your graph so that it starts from the early 2000s. Looking at the original data and the HP filtered data, what are some of the potential pitfalls of relying on the HP filter to determine if we're in a boom or a bust?

Below is my MATLAB code – again, don't take it as gospel. There is probably a far more efficient and elegant way of doing this, especially in other programming languages.

```
%% Retrieve and plot data using FRED and getFredData function
% https://github.com/robertdkirkby/getfreddata-matlab
% getFredData(series_id, observation_start, observation_end, units, frequency, aggregation_method, ondate, realtime_end)

%% US Real GDP
```

[1]Note, to get consumption as far back as 1950, you may need to get a bit creative and use consumption as a ratio of GDP and the real GDP series.

```
GDPC1 = getFredData('GDPC1','1950-01-01','2023-03-31'); % seasonally adjusted
time = datetime(GDPC1.Data(:,1),'ConvertFrom','datenum','Format','yyyy-QQQ');
gdp = GDPC1.Data(:,2);
gdp_df = timetable(time,gdp);

figure
plot(gdp_df.time,gdp_df.gdp,'b-','LineWidth',1);
recessionplot;
ylabel('Billions of Chained 2012 Dollars');
set(gcf,'Color','w','position',[500,500,700,400]);
%export_fig c1_us_gdp.pdf %install the export_fig package

lngdp = log(gdp);
lngdp_fluc = detrend(lngdp);
lngdp_tr = lngdp - lngdp_fluc;
lngdp_df = timetable(time,lngdp,lngdp_tr,lngdp_fluc);
figure
plot(lngdp_df.time,lngdp_df.lngdp,'b-','LineWidth',1);
hold on
plot(lngdp_df.time,lngdp_df.lngdp_tr,'r-','LineWidth',1);
recessionplot;
ylabel('Log Units');
set(gcf,'Color','w','position',[500,500,700,400]);
% export_fig c1_us_loggdp_trend.pdf
% Get trendline slope
periods = linspace(1,length(lngdp_df.time),length(lngdp_df.time));
lngdp_fit = polyfit(periods,lngdp_tr,1);
% Plot fluctuations from trend
figure
plot(lngdp_df.time,100*lngdp_df.lngdp_fluc,'b-','LineWidth',1);
recessionplot;
yline(0,'--','LineWidth',1);
ylabel('Percent Deviation from Trend');
set(gcf,'Color','w','position',[500,500,700,400]);
%export_fig c1_us_loggdp_deviations.pdf

% HP filter
[y_hp,dy_hp] = one_sided_hp_filter_serial(lngdp,1600);
figure
plot(lngdp_df.time,lngdp_df.lngdp,'b-','LineWidth',1);
hold on
plot(lngdp_df.time,y_hp,'r-','LineWidth',1);
recessionplot;
ylabel('Log Units');
```

```
    legend('Data','HP Filter','interpreter','latex','fontsize'
        ,10,'Location','northwest');
    set(gcf,'Color','w','position',[500,500,700,400]);
    %export_fig c1_gdp_hp.pdf
    % HP filter fluctuations
    figure
    plot(lngdp_df.time,100*dy_hp,'r-','LineWidth',1);
    hold on
    plot(lngdp_df.time,100*lngdp_df.lngdp_fluc,'b-','LineWidth'
        ,1);
    recessionplot;
    yline(0,'--','LineWidth',1);
    ylabel('Percent Deviation from Trend');
    legend('HP Filter','Log-Linear','interpreter','latex','
       fontsize',10,'Location','northwest');
    set(gcf,'Color','w','position',[500,500,700,400]);
    %export_fig c1_gdp_hp_deviations.pdf

    % HP filter for consumption and investment
    DPCERE1Q156NBEA = getFredData('DPCERE1Q156NBEA','1950-01-01
       ','2023-03-31','lin','q'); %real consumption (share of
       GDP)
    GPDIC1 = getFredData('GPDIC1','1950-01-01','2023-03-31','
       lin','q'); %real investment
    cons = DPCERE1Q156NBEA.Data(:,2).*gdp/100;
    c_df = timetable(time,cons);
    inv = GPDIC1.Data(:,2);
    inv_df = timetable(time,inv);
    [c_hp,dc_hp] = one_sided_hp_filter_serial(log(cons),1600);
    [i_hp,di_hp] = one_sided_hp_filter_serial(log(inv),1600);
    figure
    plot(c_df.time,100*dc_hp,'b-','LineWidth',1);
    hold on
    plot(inv_df.time,100*di_hp,'k-','LineWidth',1);
    recessionplot;
    ylabel('Percent Deviation from Trend');
    legend('Consumption','Investment','interpreter','latex','
       fontsize',10,'Location','northeast');
    set(gcf,'Color','w','position',[500,500,700,400]);
    %export_fig c1_c_i_hp_deviations.pdf

    % Comparison of different filters
    [y_hp_400,dy_hp_400] = one_sided_hp_filter_serial(lngdp
        ,400);
    [y_hp_800,dy_hp_800] = one_sided_hp_filter_serial(lngdp
        ,800);
    figure
    plot(lngdp_df.time,lngdp_df.lngdp,'k-','LineWidth',1);
```

```
    hold on
    plot(lngdp_df.time,y_hp_400,'b-','LineWidth',1);
    hold on
    plot(lngdp_df.time,y_hp_800,'g-','LineWidth',1);
    hold on
    plot(lngdp_df.time,y_hp,'r-','LineWidth',1);
    recessionplot;
    ylabel('Log Units');
    t1 = string({'2000-Q1'});
    t2 = string({'2023-Q3'});
    t1 = datetime(t1,'Format','yyyy-QQQ');
    t2 = datetime(t2,'Format','yyyy-QQQ');
    xlim([t1 t2]);
    ylim([9.45 9.95]);
    legend('No Filter','$\lambda = 400$','$\lambda = 800$','$\lambda = 1600$','interpreter','latex','fontsize',10,'Location','northwest');
    set(gcf,'Color','w','position',[500,500,700,400]);
    %export_fig c1_y_hp_comparisons.pdf

    % Historic co-movement between GDP and consumption PER CAPITA
    fred_lgdppc = getFredData('A939RX0Q048SBEA','1950-01-01','2023-03-31','log','q');
    fred_lconpc = getFredData('A796RX0Q048SBEA','1950-01-01','2023-03-31','log','q');
    time = datetime(fred_lgdppc.Data(:,1),'ConvertFrom','datenum','Format','yyyy-QQQ');
    ypc_df = timetable(time,fred_lgdppc.Data(:,2));
    cpc_df = timetable(time,fred_lconpc.Data(:,2));
    [ypc_hp,dypc_hp] = one_sided_hp_filter_serial(ypc_df.Var1,1600);
    [cpc_hp,dcpc_hp] = one_sided_hp_filter_serial(cpc_df.Var1,1600);
    figure
    plot(ypc_df.time,100*dypc_hp,'b-','LineWidth',1);
    hold on
    plot(cpc_df.time,100*dcpc_hp,'r-','LineWidth',1);
    yline(0,'--','LineWidth',1);
    ylabel('Percent Deviation');
    recessionplot;
    legend('$y$','$c$','interpreter','latex','fontsize',10,'Location','northwest');
    set(gcf,'Color','w','position',[500,500,700,400]);
    %export_fig c1_y_c_hp.pdf
```

5 The natural rate of unemployment

Consider a bathtub model of unemployment. Let E_t denote the employment level and U_t the unemployment level in period t. Also, let L denote the (constant) labour force. Then, the model consists of the following two equations:

$$\Delta U_{t+1} = sE_t - fU_t,$$
$$E_t + U_t = L,$$

where f denotes the probability with which unemployed people find jobs in a certain period, and s denotes the probability with which employed people lose their jobs in a certain period. In other words, s is the separation rate and f is the job finding rate.

(a) Consider a steady state situation, where neither employment nor unemployment change over time. Describe the unemployment rate in the steady state, as a function of s and f.

Steady state satisfies $U_{t+1} = 0 = sE_t - fU_t = s(L - U_t) - fU_t$. The unemployment rate in the steady state is:

$$\frac{U}{L} = \frac{s}{s+f}.$$

Now suppose that the steady state unemployment is described by the formula you provided in part (a), but f and s are not constant. Here is how these variables are determined: The **job finding rate** is given by $f(e) = e$ where e takes values in $[0, 1]$ and denotes the worker's effort. This effort is, in turn, given by $e(b) = 0.5 - 0.05b$, where b denotes the level of unemployment benefits. Assume that b takes values in $[0, 10]$. The **job separation rate** is given by $s(c) = 0.1 - 0.02c$, where c is the fee that a firm has to pay in order to terminate a work relationship (also known as a firing cost). Assume that c takes values in $[0, 5]$.

(b) What is the economic intuition behind the determination of the job finding rate and the job separation rate above?

The probability of finding a job depends positively on the effort that the unemployed expend looking for jobs – the more effort they make the higher the job finding rate. The efforts the unemployed make to find jobs depends in turn negatively on the level of unemployment benefits. When unemployment benefits are high there is less incentive for someone unemployed to search for a job. The job separation rate depends on the cost the firm has to pay when it makes someone redundant. The assumption here is that higher firing costs will reduce the job separation rate as it is less attractive to fire workers.

(c) Write the steady state unemployment rate as a function of b, c.[2] Does steady state unemployment depend positively or negatively on b and c? Discuss.

$$\frac{U}{L} = \frac{0.1 - 0.02c}{0.6 - 0.02c - 0.05b}.$$

You can then do some basic derivatives to find that increasing unemployment benefit leads to an increase in steady-state unemployment. The increase in benefits leads to

[2] Notice that, by doing so, you have expressed the unemployment rate as a function of two policy variables (that is, two variables that are perfectly controlled by a policymaker).

a lower job search effort and feeds through into a lower job finding rate, which to be consistent with steady-state requires a rise in the unemployment rate. Increasing the firing costs causes the job separation rate to fall, which to be consistent with steady state requires a fall in the unemployment rate.

(d) If you were a policymaker, and your goal was to minimise unemployment, how would you set the policy variables b, c?

The policymaker can achieve zero unemployment by setting c at its upper limit of 5 for any value of b, in which case firms never fire their workers. If the only policy instrument available is unemployment benefit then the best that can be done is set b equal to its lower bound value of 0, which incentivises maximal search for jobs.

(e) If $c = 2.5$, how should the government set b in order to achieve unemployment equal to 10%?

The unemployment benefit must satisfy

$$\frac{U}{L} = 0.1 = \frac{0.1 - 0.2 \times 2.5}{0.6 - 0.02 \times 2.5 - 0.05b}.$$

Solving this for b gives $b = 1$.

6 Simple two period model and consumption Euler equation

Consider the neoclassical consumption model. An individual lives for periods 1 and 2 and supplies labour inelastically. Labour income, before taxes is y_1 in period 1 and y_2 in period 2. The consumption levels of this individual are c_1 and c_2. The individual starts with no wealth, and can borrow and lend freely at the gross real rate of interest $R = 1 + r$. The individual cannot default on debt.

The individual faces two types of tax: proportional taxes on labour income in each period, at rates τ_1^w and τ_2^w; and proportional taxes on consumption at rates τ_1^c and τ_2^c. If the individual earns y_1 and consumes c_1 in period 1 then τ_1^w is paid as income tax and $\tau_1^c c_1$ is the tax payment on consumption. There is no tax on interest earned on assets or paid on debts.

(a) Write down the individual's intertemporal (or lifetime) budget constraint.

$$c_1(1 + \tau 1^c) + \frac{c_2(1 + \tau_2^c}{R} = y_1(1 - \tau_1^w) + \frac{y_2(1 - \tau_2^w}{R}.$$

Present value of consumption expenditure equals present value of after-tax labour income, with taxes on consumption accounted for in consumption expenditure.

(b) Using the lifetime budget constraint, solve for c_2 as a function of c_1. If the individual decreases c_1 by 1 unit, by how much does c_2 adjust? Explain your answer fully.

Solution is:

$$c_2 = -\frac{c_1(1 + \tau_1^c)R}{1 + \tau_2^c} + \frac{y_1(1 - \tau_1^w)R}{1 + \tau_2^c} + \frac{y_2(1 - \tau_2^w)}{1 + \tau_2^c}.$$

Therefore, if c_1 decreases by 1, c_2 increases by

$$\frac{R(1 + \tau_1^c)}{1 + \tau_2^c}.$$

This can be derived by taking the derivative (and remembering the question asks about a decrease in c_1) or by using the intuition looking at the equation. This makes sense because there are intertemporal effects which mean that the change in c_2 may be different from the change in c_1: The interest rate gives a time value of money. If $R > 1$, the interest rate will boost the extra resources available for c_2 (ceteris parabus). The real interest rate could be negative in which case it would reduce the increase in c_2 relative to the decrease in c_1. Secondly, the consumption tax also shifts the effect on consumption. if you reduce c_1, you also save the consumption tax in that period (τ_1^c). But when you spend c_2 you have to pay the τ_2^c tax. If $\tau_1^c = \tau_2^c$, these effects offset one another in terms of the marginal condition (though they would affect the level of consumption) but if $\tau_1^c \neq \tau_2^c$, it creates a wedge in the amount you have to give up in one period compared to the gain in the other.

(c) The individual has the following utility function:

$$U = \ln(c_1) + \beta \ln(c_2),$$

where $\beta \in (0, 1)$ captures how the individual discounts utility from c_2 relative to c_1. The individual chooses consumption in the two periods to maximise utility subject to the intertemporal budget constraint. What is the optimal ratio of consumption in period 2 to that in period 1, c_2/c_1?

Solving this problem gives:

$$MU(\tau_1^c) = \beta R \frac{(1+\tau_1^c)}{1+\tau_2^c} MU(\tau_2^c),$$

or

$$\frac{c_2}{c_1} = \beta R \frac{(1+\tau_1^c)}{1+\tau_2^c}.$$

(d) Comment on how consumption can be expected to react if, suppose, pandemic lockdown policies lead to a reduction in y_1 while y_2 is unaffected.

The effect of the pandemic reducing y_1 is to lower the life-time earnings of the individual. Unless anything else in the changes (which we assume that it doesn't), nothing in the Euler equation is affected. So individuals should maintain the same path across time for consumption but at a lower level. In other words, the neoclassical individual will smooth out the income drop by reducing consumption in both periods.

(e) Now suppose the government, in order to boost consumption during the pandemic, lowers the consumption tax rate in period 1 (τ_1^c). In the neoclassical consumption model, what, if anything, does this do to the individual's consumption in each period? What would happen to the individual's consumption if instead the government lowered the income tax rate during the pandemic (τ_1^w)? Which would better achieve the government's objective? Explain your answers.

From the Euler equation, a reduced consumption tax in period 1 ($\tau_1^c \downarrow$) changes the rate of growth of consumption. Essentially, because τ_1^c is lower, it makes the relative cost of period 1 consumption lower and incentivises the individual to consume relatively more during the pandemic. If that is the policy objective, this is an effective policy. Lower income taxes, on the other hand, change the value of the lifetime flow of wages and so the individual will smooth the change in the lifetime income over the 2 periods. It is thus less effective at increasing c_1 even though c_1 does increase. c_2 will also decrease.

7 Pricing a stock

Consider the arbitrage equation for pricing a risk-free stock:

$$P_t = \frac{D_t + \Delta P_{t+1}}{R},$$

where P_t is the price of the stock in period t, D_t is the dividend the stock pays in period t, and $\Delta P_{t+1} = P_{t+1} - P_t$ is the capital gain the stock will make before the beginning of the next period. R is the interest rate available on an alternative investment in a bank account, assumed to be constant and the same in every period for simplicity – you can also abstract from uncertainty in your answers.[3]

(a) Re-write the arbitrage equation so it reads for P_t on the left hand side (LHS) and R, D_t, and P_{t+1} on the right hand side (RHS).

$$P_t = \frac{D_t + P_{t+1}}{1+R}$$

(b) The arbitrage equation holds for every period, so roll your solution for P_t in part (a) forward one period to read for P_{t+1} on the LHS and R, D_{t+1}, and P_{t+2} on the RHS.

$$P_{t+1} = \frac{D_{t+1} + P_{t+2}}{1+R}$$

(c) Substitute the solution for P_{t+1} from part (b) into the expression for P_t from part (a) to obtain an expression that has P_t on the LHS and R, D_t, D_{t+1}, and P_{t+2} on the RHS.

$$\begin{aligned} P_t &= \frac{D_t + \frac{D_{t+1}+P_{t+2}}{1+R}}{1+R} \\ &= \frac{D_t}{1+R} + \left(\frac{1}{1+R}\right)^2 D_{t+1} + \left(\frac{1}{1+R}\right)^2 P_{t+2} \end{aligned}$$

(d) Continue rolling the arbitrage equation forward to repeat steps (b) and (c) until you obtain an expression determining the stock price P_t as a function of only the interest rate R and current and all future dividends. Interpret your results.

$$P_t = \frac{D_t}{1+R} + \left(\frac{1}{1+R}\right)^2 D_{t+1} + \left(\frac{1}{1+R}\right)^3 D_{t+2} + \left(\frac{1}{1+R}\right)^4 D_{t+3} + \left(\frac{1}{1+R}\right)^5 D_{t+4} + \ldots$$

(e) Alphabet Inc. (the parent company of Google) briefly displaced Apple Inc. as the largest company in the world on Tuesday 2nd February 2016. Its shares traded at \$784 each, an implied market capitalisation of \$531 billion. Alphabet has never paid any dividends. What does this imply about the calculations you made earlier? Is it possible to rationalise such a high market capitalisation with a lack of dividends?

The pricing equation for a stock emphasises the importance of future dividends (which will be the expected future dividends in the real world). The experience with Alphabet

[3] In other words, do not worry about taking the expectation of future variables with today's information set, $\mathbb{E}_t$.

Inc. will then be consistent with the theory provided that at some stage they are expected to distribute dividends in the future. Alternatively, it could be that we are in a world of bubbles and non-rational investors in which stock prices diverge from fundamentals. One way to get that story up and running would be a "bigger fool" assumption – an investor may know that Alphabet Inc will never pay dividends but if they think they can sell the stock at a profit in the future then it makes sense to hold it now.

Of course, with the benefit of hindsight (and data), you can test this theory for yourself.

8 OLG model with population growth

Consider a standard two-period overlapping generations (OLG) model with log utility, $\ln C_t$, and Cobb-Douglas production technology, $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Capital depreciates completely after a period and households only work when young (they inelastically supply labour normalised to 1 when young).

(a) Assume that there is no population growth, so that K_{t+1}^1 is the savings per young $t+1$ household made at time t and the aggregate capital stock per young household at time $t+1$. Write down the optimisation problem of the household when it is young and solve for the optimal consumption and savings decisions it makes, taking the wage rate w_t and the return to capital $R_{t+1} = 1 + r_{t+1}$ as given. Write down the problem of the perfectly competitive firm and show that market clearing implies a law of motion for capital of the form:

$$K_{t+1} = \frac{1}{2}(1-\alpha)K_t^{\alpha}.$$

Solve for the steady state capital stock. *** **Bonus** *** Log-linearise the law of motion for capital.

The basic OLG is described as follows. Let there be an infinite sequence of time, $t = 0, 1, 2, ..., \infty$. Each generation born in period t is referred to as generation t. There are $N(t)$ members of generation t, and people live for 2 periods, generation t is young in t and old in $t+1$. Generation t does not exist in period $t+2$.

A member h of generation t has utility

$$u_t^h(c_t^h(t), c_t^h(t+1)). \tag{1}$$

Production takes place in competitive firms with HOD1 production technology (CRTS), implying that they do not produce economic profits. Production in period t is given by

$$y(t) = F(K(t), L(t)) = K(t)^{\alpha} L(t)^{1-\alpha}.$$

Individuals are endowed with lifetime endowment of labour given by

$$l_t = [l_t^h(t), l_t^h(t+1)].$$

Total labour is given as

$$L(t) = \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t).$$

Aggregate labour of the young at time t is the first component of the RHS, and the aggregate labour of the old is the second component of the RHS. We also assume that $K(t)$ depreciates fully.

The economy has the following resource constraint:

$$y(t) = F(K(t), L(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1).$$

Members of generation t earn income in period t by offering all their labour endowment to firms at market wage, w_t, and use income to fuel consumption in period t, to fund borrowing and lending to other members of generation t, and for accumulation of private capital. The budget constraint for individual h when they're young is

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1), \tag{2}$$

where $a^h(t)$ are net asset holdings of individual h. $a^h(t) < 0$ implies net borrowing from other members of generation t. Individuals cannot borrow or lend across generations so

$$\sum_{h=1}^{N(t)} a^h(t) = 0.$$

The budget constraint for generation t individual in period $t+1$ is

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t a^h(t) + R_{t+1}^k k^h(t+1), \tag{3}$$

where R_t^k is the rental rate. Factor prices are determined by their marginal products due to competitive equilibrium:

$$\begin{aligned} w_t &= F_L(K(t), L(t)), \\ R_t^k &= F_K(K(t), L(t)). \end{aligned}$$

We can then combine the budget constraints of the young and old; from (2):

$$a^h(t) = w_t l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substitute this expression into (3) to get:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t w_t l_t^h(t) - R_t c_t^h(t) - R_t k^h(t+1) + R_{t+1}^k k^h(t+1),$$

collecting terms, we can yield an expression for $c_t^h(t)$:

$$c_t^h(t) = \frac{w_{t+1} l_t^h(t+1) - c_t^h(t+1)}{R_t} + w_t l_t^h(t) - k^h(t+1)\left[1 - \frac{R_{t+1}^k}{R_t}\right].$$

Since we assume that there are no arbitrage opportunities, the return on capital should equal the return on loans amongst members of a particular cohort, $R_t = R_{t+1}^k$. Thus the budget constraint becomes:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t} = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}. \tag{4}$$

A competitive equilibrium consists of a sequence of prices $\{w_t, R_t\}_{t=0}^{\infty}$, and quantities $\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty}$, such that each member h of each generation $t > 0$ maximises utility (1) subject to their lifetime budget constraint given by (4), and so that the equilibrium conditions

$$\begin{aligned} R_{t+1} &= R_t, \\ w_t &= F_L(K(t), L(t)), \\ R_t - 1 &= F_K(K(t), L(t)), \\ L(t) &= \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t), \end{aligned}$$

hold each period.

We attain optimal consumption by maximising utility (1) subject to (4) and yielding the consumption Euler equation. Given that our utility function is of the form

$$u(c_t^h(t), c_t^h(t+1)) = \ln c_t^h(t) + \ln c_t^h(t+1),$$

we differentiate the following equation wrt to consumption in period t:

$$\begin{aligned} &\arg\max_{c_t^h(t)} \left\{ \ln c_t^h(t) + \ln \left[R_t w_t l_t^h(t) - R_t c_t^h(t) + w_{t+1} l_t^h(t+1) \right] \right\} && (5) \\ &\implies 0 = \frac{1}{c_t^h(t)} - \frac{R_t}{c_t^h(t+1)} \\ &\quad\quad 1 = \frac{R_t c_t^h(t)}{c_t^h(t+1)}. && (6) \end{aligned}$$

We can then insert optimal consumption given by the Euler equation back into the budget constraint (4) to get:

$$2c_t^h(t) = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}.$$

We introduce a further simplifying assumption that the old do not supply their labour. i.e. $l_t^h(t+1) = 0$. Thus we have:

$$c_t^h(t) = \frac{1}{2} w_t l_t^h(t).$$

Now that we have consumption per period for an individual h when they are young, we want to pin down aggregate savings. We know that individuals only live for two periods – i.e. individual h of generation t will not live past $t+1$. Thus, they will not save in period $t+1$. To pin down aggregate savings we need the budget constraint of a young person given in (2):

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1),$$

then define savings for an individual h of generation t as:

$$s^h(t) = w_t l_t^h(t) - c_t^h(t) = a^h(t) + k^h(t+1).$$

Aggregating across an entire cohort gives:

$$\begin{aligned} S(t) = \sum_{h=1}^{N(t)} s^h(t) = \underbrace{\sum_{h=1}^{N(t)} a^h(t)}_{=0} + \sum_{h=1}^{N(t)} k^h(t+1) \\ \implies S(t) = K(t+1). \end{aligned}$$

$S(t)$ is dependent on wages in t and $t+1$ and returns to capital, given to us in our equilibrium conditions. Furthermore, we assumed that the elderly do not supply labour. Therefore, $K(t+1)$ depends on $L_t(t)$ and $L_{t+1}(t+1)$, parameters of $u(\cdot)$, α, and $K(t)$ – everything except for $K(t)$ are constants. Therefore, the law of motion of capital is given by some function of $K(t)$:

$$K(t+1) = G(K(t)).$$

Turning to the firm and production, we know that the firm produces output according to a Cobb-Douglas technology:

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}. \tag{7}$$

Taking the partial derivative of output wrt capital and labour yields the following factor prices:

$$\begin{aligned}
\frac{\partial Y(t)}{\partial K(t)} &= R_t = \alpha \left[\frac{K(t)}{L(t)}\right]^{\alpha-1} = \alpha k(t)^{\alpha-1}, \\
\frac{\partial Y(t)}{\partial L(t)} &= w_t = (1-\alpha) \left[\frac{K(t)}{L(t)}\right]^{\alpha} = (1-\alpha) k(t)^{\alpha}.
\end{aligned}$$

From our household FOCs, we have

$$c_t^h(t) = \frac{1}{2} w_t l_t^h(t) = \frac{1}{2}(1-\alpha) k(t)^{\alpha} l_t^h(t),$$

and aggregating across the cohort yields

$$C_t(t) = \frac{1}{2}(1-\alpha) K(t)^{\alpha} L(t)^{1-\alpha} = \frac{1}{2}(1-\alpha) Y(t).$$

The household FOCs and the above equation pins down aggregate savings:

$$S(t) = \frac{1}{2}(1-\alpha) Y(t),$$

and since $S(t) = K(t+1)$,

$$\implies K(t+1) = \frac{1}{2}(1-\alpha) Y(t).$$

Since labour is supply inelastically by the young, the law of motion of capital can be written as

$$K(t+1) = \frac{1}{2}(1-\alpha) K(t)^{\alpha}. \tag{8}$$

The steady state capital stock, $\bar{K}$, is given by the condition $\Delta K(t) = 0$[4]:

$$\Delta K(t) = 0 = K(t+1) - K(t) = \frac{1}{2}(1-\alpha) K(t)^{\alpha} - K(t),$$

[4]Market clearing in the steady state implies that

$$\begin{aligned}
y(t) &= K(t+1) + C_t(t) + C_t(t+1) \\
\implies K(t)^{\alpha} &= \frac{1}{2}(1-\alpha) K(t)^{\alpha} + \frac{1}{2}(1-\alpha) K(t)^{\alpha} + r_t K(t+1) \\
\bar{K}^{\alpha} &= \frac{1}{2}(1-\alpha) \bar{K}^{\alpha} + \frac{1}{2}(1-\alpha) \bar{K}^{\alpha} + \alpha \bar{K}^{\alpha-1} \bar{K}.
\end{aligned}$$

rearranging and solving for $\bar{K}$ yields

$$\begin{aligned}
0 &= \frac{1}{2}(1-\alpha)K(t)^{\alpha} - K(t) \\
K(t) &= \frac{1}{2}(1-\alpha)K(t)^{\alpha} \\
K(t)^{-\alpha}K(t) &= \frac{1-\alpha}{2} \\
\implies \bar{K} &= \left(\frac{1-\alpha}{2}\right)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

We can log linearise the law of motion for capital (8). First, take logs

$$\ln K(t+1) = \ln\left(\frac{1-\alpha}{2}\right) + \alpha \ln K(t),$$

and then taking a first order Taylor expansion gives:

$$\ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{1-\alpha}{2}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}).$$

In the steady state we know that

$$\ln \bar{K} = \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right),$$

which gives us:

$$\frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right) + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}),$$

simplifying this expressions yields:

$$\begin{aligned}
\frac{(K(t+1) - \bar{K})}{\bar{K}} &\approx \ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right) - \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right) + \alpha\frac{(K(t) - \bar{K})}{\bar{K}} \\
\hat{K}(t+1) &\approx \left[1 + \frac{\alpha - 1}{1-\alpha}\right] \ln\left(\frac{1-\alpha}{2}\right) + \alpha\hat{K}(t) \\
\hat{K}(t+1) &\approx \left[\frac{1-\alpha+\alpha-1}{1-\alpha}\right] \ln\left(\frac{1-\alpha}{2}\right) + \alpha\hat{K}(t) \\
\hat{K}(t+1) &\approx \alpha\hat{K}(t). \qquad (9)
\end{aligned}$$

where $\hat{K}$ is log deviation from the steady state value.

(b) Discuss how an increase in α affects the steady state capital stock and its law of motion when there is no population growth.[5]

[5] You may find it useful to take logs of the steady state equation and then differentiate with respect to α.

To see how an increase in α affects the steady state level of capital and its law of motion, begin by taking the derivative of $\bar{K}$ wrt α:

$$\begin{aligned}
\bar{K} &= \left[\frac{1-\alpha}{2}\right]^{\frac{1}{1-\alpha}} \\
\ln \bar{K} &= \frac{1}{1-\alpha} \ln\left(\frac{1-\alpha}{2}\right) \\
\frac{d \ln \bar{K}}{d\alpha} &= (-1)(1-\alpha)^{-2}(-1)\ln\left(\frac{1-\alpha}{2}\right) + \frac{1}{1-\alpha}\left(\frac{-1}{1-\alpha}\right) \\
&= \frac{1}{(1-\alpha)^2}\left[\ln\left(\frac{1-\alpha}{2}\right) - 1\right].
\end{aligned}$$

So, an increase in α leads to a decrease in the steady state level of capital. An increase in α implies a greater share of profits paid to capital at the expense of labour, leading to a fall in wage income, which leads to less savings and thus less capital. We can also see from equation (9) that an increase in α causes a stronger response in $\hat{K}(t+1)$ to changes in $\hat{K}(t)$. In other words, a change in capital in period t will lead to bigger changes in capital in period $t+1$ for larger values of α.

(c) Now introduce constant population growth so that $L_{t+1} = (1+\eta)L_t$. If K^1_{t+1} is the savings per young household made at time t then the aggregate capital stock per young household at time $t+1$ will be $(L_t K^1_{t+1})/L_{t+1}$. Derive the law of motion for capital in the economy with population growth. Solve for steady state and log-linearise the law of motion for capital. How does an increase in the population growth rate η affect these objects?

Population now grows at rate η so that

$$N(t+1) = (1+\eta)N(t).$$

Recall that aggregate capital in the model without population growth was simply the sum of savings by all individuals in the $N(t)$ cohort (since we assumed that aggregate borrowing/lending within the cohort summed to 0). Capital per young individual in period $t+1$ with population growth is now

$$k^h(t+1) = \frac{\sum_{h=1}^{N(t)} k^h(t+1)}{N(t+1)},$$

and since we know population grows at rate η we can rewrite this as:[6]

$$k^h(t+1)N(t) = \frac{K_t(t+1)}{(1+\eta)}.$$

We can use this definition to rewrite the law of motion of capital as

$$K(t+1) = \frac{(1-\alpha)}{2(1+\eta)} K(t)^{\alpha}. \tag{10}$$

[6] Note that this is aggregate capital, and hence we discount it by the factor $1+\eta$. If we were looking at aggregate capital belonging to a particular cohort,

We find the steady state of capital as before by setting $\Delta K(t) = K(t+1) - K(t) = 0$ and solving for $\bar{K}$:

$$0 = \frac{(1-\alpha)}{2(1+\eta)} K(t)^{\alpha} - K(t)$$
$$\implies \bar{K} = \left[\frac{(1-\alpha)}{2(1+\eta)}\right]^{\frac{1}{1-\alpha}}.$$

We can log linearise the law of motion of capital. Start by taking logs of (10):

$$\ln K(t+1) = \ln\left(\frac{(1-\alpha)}{2(1+\eta)}\right) + \alpha \ln K(t),$$

then take a first order Taylor expansion around the steady state:

$$\ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{(1-\alpha)}{2(1+\eta)}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}),$$

and since we know $\ln \bar{K} = \frac{1}{1-\alpha} \ln \frac{(1-\alpha)}{2(1+\eta)}$, substituting and rearranging gives us an expression for linearised log deviations from steady state:

$$\hat{K}(t+1) \approx \alpha \hat{K}(t).$$

So we can conclude that by including population growth in our OLG model, the steady state level of capital declines, however our transition dynamics for a shock to $K(t)$ remain unchanged relative to the model without population growth.

(d) How would labour and capital income taxes affect the steady state of the economy? To do this, assume that taxes are paid by the household and proportional to labour and capital income, then calculate how the taxes affect first order conditions. You should then be able to identify what changes in steady state. Explain the intuition behind your findings.

Applying a tax to labour and capital income will modify the individual's optimisation problem and the law of motion of capital. The budget constraint for individual h of generation t in period $t+1$ is now

$$c_t^h(t+1) = w_{t+1}(1-\tau_L) l_t^h(t+1) + R_t(1-\tau_K) a^h(t) + R_{t+1}(1-\tau_K) k^h(t+1),$$

where τ_L and τ_K are the tax rates on labour and capital, respectively.[7] From (2) we can get the following asset holdings of individual h:

$$a^h(t) = w_t(1-\tau_L) l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substituting this into the the adjusted budget constraint above gives the following (after rearranging and simplifying) lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t(1+\tau_K)} = w_t(1-\tau_L) l_t^h(t), \tag{11}$$

[7] In this setup of the model, taxes on borrowing and lending between individuals are taxed. Not applying the tax has significant consequences on the law of motion of capital and consumption Euler equations.

since we assumed that $l_t^h(t+1) = 0$ and that due to perfectly competitive markets, the interest on private borrowing and lending and the return on capital are the same so $R_t = R_{t+1}$. The individual's maximisation problem is therefore:

$$\arg\max_{c_t^h(t)} \left\{ \ln c_t^h(t) + \ln \left[R_t w_t (1-\tau_L) l_t^h(t) - (1-\tau_K) R_t c_t^h(t) \right] \right\},$$

yielding the following the Euler equation from the FOC:

$$\begin{aligned} 0 &= \frac{1}{c_t^h(t)} - \frac{(1-\tau_K)R_t}{c_t^h(t+1)} \\ 1 &= \frac{(1-\tau_K)R_t c_t^h(t)}{c_t^h(t+1)}. \end{aligned}$$

To derive the law of motion of capital, begin by substituting optimal consumption back into the individual's budget constraint:

$$\begin{aligned} 2c_t^h(t) &= w_t(1-\tau_L) l_t^h(t) \\ c_t^h(t) &= \frac{1}{2} w_t (1-\tau_L) l_t^h(t). \end{aligned}$$

Aggregating across $N(t)$ individuals gives:

$$C_t(t) = \frac{1}{2}(1-\alpha)(1-\tau_L) y(t),$$

and aggregate savings is given as:

$$\begin{aligned} S(t) &= (1-\tau_L)(1-\alpha)K(t)^\alpha - \frac{1}{2}(1-\tau_L)(1-\alpha)K(t)^\alpha. \\ &= \frac{1}{2}(1-\tau_L)(1-\alpha)K(t)^\alpha \end{aligned}$$

Since we know that $S(t) = K(t+1)$, and accounting for population growth, we can derive the law of motion for capital:

$$K(t+1) = \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^\alpha.$$

Then, to get the steady we set our condition as before, $\Delta K(t) = 0$:

$$\begin{aligned} 0 &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^\alpha - K(t) \\ K(t)^{1-\alpha} &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} \\ \implies \bar{K} &= \left[\frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} \right]^{\frac{1}{1-\alpha}}. \end{aligned} \tag{12}$$

The key intuition here is that taxes on labour reduces the steady state level of capital. Oddly, taxes on capital have no effect here. This is due to income and substitution effects that offset each other. The capital tax lowers the effective return on savings, which has a substitution effect – consumption when an individual becomes old becomes relatively more expensive, which tends to cause households to substitute towards consumption when they are young. Secondly, there is a wealth effect: the lower return on savings leads to lower lifetime income, which tends to cause households to reduce consumption in both periods. Because utility is logarithmic, the two effects on young consumption precisely cancel each other out, and as a result household saving – and thus capital – is unaffected.

(e) Suppose capital saved in period t does not depreciate fully after use in period $t+1$. Instead, assume that $1-\delta$ of the capital stock remains. How does this affect the transition dynamics of the system and how does it affect the steady state? By comparing what happens with depreciation to the effect of a capital tax in part (d), you should be able to answer this part of the question by direct reference to the first order conditions, deriving the implications for transition dynamics and steady state without further calculations.

Suppose capital does not fully depreciate after one period, and instead depreciates at rate δ, where $0<\delta<1$. The law of motion for capital with depreciation becomes

$$K(t+1) = \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^{\alpha} + (1-\delta)K(t).$$

Solving for steady state capital gives:

$$\begin{aligned} 0 &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^{\alpha} + (1-\delta)K(t) - K(t) \\ K(t)^{1-\alpha} - (1-\delta)K(t)^{1-\alpha} &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} \\ \implies \bar{K} &= \left[\frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{(1+\eta)\delta}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

To see how this slower depreciation rate affects transition dynamics, we can log linearise the law of motion of capital:

$$\hat{K}(t+1) \approx \left(\frac{\alpha}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{(1+\eta)\delta} \bar{K}^{\alpha-1} + (1-\delta)\right) \hat{K}(t).$$

Comparing our steady state capital with slower depreciation to our steady state with taxes and population growth (12), we can see a decline in the steady state level of capital, and a smoother transition dynamic for future capital in response to changes in capital today.

(f) *** **Bonus** *** Instead of assuming that the utility and technology functions are of the functional forms described above, can you think of any alternative functional forms which would lead to multiple steady states?

An ill-behaved OLG model is one in which the steady state of capital may not have one unique level. Potential candidates for production and utility functions which may lead to multiple steady states could be quadratic, cubic, or other such polynomial functions. For example we could use a CES production function

$$Y_t = \left[\alpha K(t)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)L_t^{\frac{\epsilon-1}{\epsilon}}\right],$$

where $\epsilon > 0$ is the elasticity of substitution between capital and labour.

9 Hall's random walk theory of consumption

Consider the standard intertemporal model of consumption with infinite horizon, rational expectations consumers, and perfect capital markets. Consumption is denoted c_t, income transfers are y_t, the real interest rate is $R = 1 + r$ and is fixed, and initial financial assets are A_0.

(a) Write down the lifetime budget constraint for this model taking $t = 0$ as the first period. What further assumptions are needed for Hall's random walk result for consumption to hold? Derive the result mathematically and provide economic intuition for the result (include in your answer economic intuition for the roles played by each of the assumptions that you have identified as necessary for the result).

The lifetime budget constraint is:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \frac{c_t}{R^t} = RA_0 + \mathbb{E}_t \sum_{t=0}^{\infty} \frac{y_t}{R^t}.$$

Further assumptions needed are quadratic utility and $\beta = 1/R$. The Euler equation is:

$$u'(c_t) = \frac{\beta}{R}\mathbb{E}_t u'(c_{t+1}).$$

If $u(c_t)$ is quadratic then $u'(c_t)$ is linear, for example it could be $\alpha - \gamma c_t$, then the Euler equation becomes:

$$\alpha - \gamma c_t = \alpha - \gamma \mathbb{E}_t c_{t+1},$$

which simplifies to

$$c_t = \mathbb{E}_t c_{t+1} \Leftrightarrow c_{t+1} = c_t + \epsilon_t,$$

with $\mathbb{E}_t \epsilon_t = 0$.

Intuition: the representative household desires to smooth (because of concave utility) and flat (given that $\beta = 1/R$) consumption path. Linear marginal utility is key to the result.

(b) Assume that in any period income is either y^L or y^H each with probability 0.5, with $y^L < y^H$ and $(y^L + y^H)/2 = y^*$. Explain why **before** y_0 (income in the first period) is known, the intertemporal budget constraint can be written as:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{c_t}{R^t} \right] = RA_0 + \frac{R}{r} y^*.$$

Hint: Use the fact that:

$$\sum_{i=0}^{\infty} ab^i = \frac{a}{1-b}, \quad \text{iff } |b| < 1.$$

Start by writing:

$$\mathbb{E}_t y_t = \frac{y^L + y^H}{2} = y^*.$$

Then using the hint provided, you can write

$$\mathbb{E}_t \sum_{t=0}^{\infty} \frac{y_t}{R^t} = \sum_{t=0}^{\infty} \frac{y^*}{R^t} = \frac{R}{r} y^*.$$

Thus, the lifetime budget constraint can be written as required after you substitute in the above expression.

(c) Suppose households choose c_0 after learning whether y_0 is y^L or y^H, but still not knowing hwat income is going to be thereafter. How much would households consume if they received $y_0 = y^L$? What if $y_0 = y^H$? In your answers assume that all the further assumptions in part (a) hold, and make use of the random walk result for consumption.

Start by noting that once y_0 is known, we can write the present value of expected income as:

$$y_0 + \sum_{t=1}^{\infty} \frac{y^*}{R^t} = y_0 + \frac{1}{R} \sum_{t=0}^{\infty} \frac{y^*}{R^t} = y_0 + \frac{y^*}{r}.$$

(Carefully note the indexes used on the summation operators.)

From the first part of this question we know that

$$c_t = \mathbb{E}_t c_{t+1} \implies c_0 = \mathbb{E} c_1 = \mathbb{E} c_2 = ...,$$

then the LHS of the budget constraint becomes

$$\mathbb{E}_t \sum_{t=0}^{\infty} \frac{c_t}{R^t} = \frac{R}{r} c_0.$$

Doing a bit of algebra we can then write

$$\frac{R}{r} c_0 = RA_0 + y_0 + \frac{y^*}{r},$$

or

$$c_0 = rA_0 + \frac{r}{R} y_0 + \frac{y^*}{R}.$$

Substituting y^L or y^H for y_0 gives the desired expressions.

10 The RBC model

Consider a basic RBC model, where the social planner wants to maximise

$$\mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(L_{t+i})) \right],$$

where C_t is consumption, L_t is hours worked, and β is the representative household's rate of time preference (discount factor). The economy faces constraints described by:

$$\begin{aligned} Y_t &= C_t + I_t, \\ Y_t &= F(K_t, L_t), \\ K_{t+1} &= I_t + (1-\delta)K_t, \end{aligned}$$

where $F(K_t, L_t)$ is the production technology of output, Y_t with constant returns to scale, I_t is investment, and δ is the rate of depreciation of capital. We can simplify the problem by combining the constraints into one equation:

$$F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t.$$

(a) Does the real business cycle (RBC) model predict that real wages should be procyclical or countercyclical? How about employment? Why?

(b) What does the empirical evidence say about the direction and magnitude of the fluctuations in these variables in comparison with the model's predictions?

(c) What are the implications of labour market developments for interpreting the validity of the RBC framework?

With the objective function and simplified constraint, we set up the Lagrangian,

$$\begin{aligned} Z = \mathbb{E}_t &\left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i})) \right] + \lambda_t \left(F(K_t, L_t) + (1-\delta)K_t - C_t - K_{t+1} \right) \\ &+ \beta \mathbb{E}_t \left[\lambda_{t+1} \left(F(K_{t+1}, L_{t+1}) + (1-\delta)K_{t+1} - C_{t+1} - K_{t+2} \right) \right], \end{aligned}$$

and attain the following FOCs:

$$\begin{aligned} \frac{\partial Z}{\partial C_t} &: U'(C_t) - \lambda_t = 0, \\ \frac{\partial Z}{\partial C_{t+1}} &: \beta \mathbb{E}_t \left[U'(C_{t+1}) \right] - \beta \mathbb{E}_t \left[\lambda_{t+1} \right] = 0, \\ \frac{\partial Z}{\partial N_t} &: -V'(L_t) + \lambda_t F_L = 0, \\ \frac{\partial Z}{\partial N_{t+1}} &: -\beta \mathbb{E}_t \left[V'(L_{t+1}) \right] + \beta \mathbb{E}_t \left[\lambda_{t+1} F_L \right] = 0, \\ \frac{\partial Z}{\partial K_{t+1}} &: -\lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} \left(F_K + (1-\delta) \right) \right] = 0, \end{aligned}$$

Since markets are competitive, complete, and we have no externalities, we know by the second fundamental theorem of welfare economics that the social planner's allocation is equivalent to the competitive equilibrium. Thus we can first define the marginal product of capital as

$$R_{t+1} = F_K + 1 - \delta,$$

then the FOC for capital can be re-written as

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} R_{t+1} \right],$$

and this can be combined with the FOC for consumption to give the Keynes-Ramsey condition (the consumption Euler equation):

$$U'(C_t) = \beta \mathbb{E}_t [U'(C_{t+1}) R_{t+1}]. \tag{13}$$

Similarly, the marginal product of labour is equal to the real wage rate:

$$w_t = F_L,$$

and so the intertemporal Euler equation for labour supply is

$$\mathbb{E}_t \left[V'(L_{t+1}) \right] = \mathbb{E}_t \left[\lambda_t w_{t+1} \right],$$

and the intratemporal Euler equation for labour supply is

$$\lambda_t w_t = V'(L_t).$$

With the model defined, we can focus back to the question: The model predicts that wages are very pro-cyclical and that hours worked are weakly cyclical. This is due to productivity shocks entering $F(K_t, L_t)$ and directly affecting the marginal product of labour, raising wages (in the case of a positive productivity shock) temporarily. Households respond to a positive productivity shock and higher wages by raising present consumption and labour supply. The higher wages induce an income effect and substitution effect: Households do not have to work as much as they previously did to sustain consumption due to higher wages – however, the model predicts that this is small – but households choose to substitute their leisure time for more hours worked to take advantage of the higher wages to fund higher consumption. Essentially leading to a 'make hay while the sun shines' scenario.

Empirically, the model's predictions are quite inaccurate. In the data, wages are only modestly pro-cyclical and hours worked are actually highly volatile, even more than output. There is evidence which suggests that wage cyclicality is biased downwards in the data, which may alleviate some of the criticisms of the RBC model. However, without modifying the model (e.g. Hansen's indivisible labour), we cannot address the discrepancy between the model and data when it comes to hours worked.

Jordi Gali has criticised the RBC model for failing to explain the labour market response to technology shocks. For example, Gali used VARs to show that in response to a positive productivity shock, hours worked actually declines in contrast to the model's predictions. The primary issue with the RBC model is that it lacks any sort of propagation mechanism in order to explain what happens to the labour market following a productivity shock. Again, modifying the RBC model by adding things like variable labour utilisation, home production, and habit persistence, boosts the weak propagation mechanism of the model. An alternative, more drastic, approach is to fundamentally change the RBC model by adding nominal rigidities such as sticky prices (e.g. Calvo and Rotemberg pricing) and wages, and market imperfections (e.g. monopolistically competitive firms).

11 An analytic RBC model

An economy is populated by an infinitely-lived representative agent with preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \ln C_t,$$

where C_t is consumption and ln is the natural logarithm. β is a discount factor that satisfies $\beta \in (0, 1)$. There is no uncertainty or labour in this economy, and output Y_t is a linear function of capital K_t. Capital can be accumulated through a technology that has a constant elasticity of substitution form in existing capital and investment I_t. The constraints of the economy are therefore:

$$\begin{aligned} C_t + I_t &= Y_t, \\ Y_t &= K_t, \\ K_{t+1} &= K_t^{\alpha} I_t^{\gamma}, \end{aligned}$$

with $\alpha > 0$, $\gamma > 0$, and $\alpha + \gamma < 1$.

(a) Re-write the constraints of the economy as equations for consumption C_t and future capital stock K_t in terms of the current capital stock K_t and the investment rate $s_t = I_t / Y_t$.

First things first, substitute out Y_t from the constraints, and rearrange the first constraint to get

$$\begin{aligned} C_t &= K_t - I_t, \\ K_{t+1} &= K_t^{\alpha} I_t^{\gamma}. \end{aligned}$$

Then use the fact that consumption is nothing but income minus savings, and use the first equation to attain an expression for I_t in terms of C_t and K_t:

$$C_t = (1 - s_t) K_t, \tag{14}$$

and

$$\begin{aligned} K_{t+1} &= K_t^{\alpha} (K_t - C_t)^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha} C_t^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha} \left[(1 - s_t) K_t\right]^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha+\gamma} (1 - s_t)^{\gamma} \\ &= K_t^{\alpha+\gamma} \left[1 - 1 + s_t^{\gamma}\right] \\ &= K_t^{\alpha+\gamma} s_t^{\gamma}. \end{aligned} \tag{15}$$

(b) Use the constraints expressed in terms of current capital stock and the investment rate to derive the first-order conditions of the social planner's problem. Interpret each condition briefly.

Substitute (14) into the objective function to get the Ramsey social planner's problem as

$$\max_{\{s_t, K_{t+1}\}} \sum_{s=0}^{\infty} \beta^s \ln(K_{t+s}(1 - s_{t+s})).$$

The Lagrangian is

$$\mathcal{L} = \ln(K_t(1-s_t)) + \lambda_t\left[K_{t+1} - K_t^{\alpha+\gamma}s_t^{\gamma}\right],$$

and the FOCs are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial s_t} &= -\frac{K_t}{K_t(1-s_t)} - \lambda_t\gamma K_t^{\alpha+\gamma}s_t^{\gamma-1} = 0\\ \implies -\frac{1}{1-s_t} &= \lambda_t\gamma K_t^{\alpha+\gamma}s_t^{\gamma-1}. \end{aligned} \tag{16}$$

The second FOC is actually a bit tricky. Notice that in the way the question is setup, K_t – the inherited level of capital – enters the social planner's objective function. But the planner can't pick K_t as it's already predetermined. Instead, the planner must pick K_{t+1}, so the Lagrangian for the FOC wrt K_{t+1} looks like this

$$\mathcal{L} = \ln(K_t(1-s_t)) + \beta\ln(K_{t+1}(1-s_{t+1})) + \lambda_t\left[K_{t+1} - K_t^{\alpha+\gamma}s_t^{\gamma}\right] + \beta\lambda_{t+1}\left[K_{t+2} - K_{t+1}^{\alpha+\gamma}s_{t+1}^{\gamma}\right],$$

which gives our second FOC,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial K_{t+1}} &= \beta\frac{(1-s_{t+1})}{K_{t+1}(1-s_{t+1})} + \lambda_t - \beta\lambda_{t+1}(\alpha+\gamma)K_{t+1}^{\alpha+\gamma-1}s_{t+1}^{\gamma} = 0\\ \implies \beta\lambda_{t+1}(\alpha+\gamma)K_{t+1}^{\alpha+\gamma-1}s_{t+1}^{\gamma} &= \beta\frac{1}{K_{t+1}} + \lambda_t. \end{aligned} \tag{17}$$

(c) Combine the two first-order conditions and find the steady-state investment rate in this economy. How does it vary with α and γ and why? Hint: You may find it useful to work with a transformation of the investment rate $\nu_t = s_t/[\gamma(1-s_t)]$.

Rearrange (16) to define v_t:

$$\frac{s_t}{\gamma(1-s_t)} = -\lambda_t K_t^{\alpha+\gamma}s_t^{\gamma} = v_t.$$

Then, use the second FOC, (17), and substitute the value of λ_t into the expression for v_t:

$$\begin{aligned}v_t &= -\lambda_t K_t^{\alpha+\gamma}s_t^{\gamma}\\ &= -\left(\beta\lambda_{t+1}(\alpha+\gamma)K_{t+1}^{\alpha+\gamma-1}s_{t+1}^{\gamma} - \beta\frac{1}{K_{t+1}}\right)K_t^{\alpha+\gamma}s_t^{\gamma}\\ &= \left(\frac{1}{K_{t+1}} - \lambda_{t+1}(\alpha+\gamma)K_{t+1}^{\alpha+\gamma-1}s_{t+1}^{\gamma}\right)\beta K_t^{\alpha+\gamma}s_t^{\gamma},\end{aligned}$$

and then do a bit of manipulation to get

$$\begin{aligned}&= \frac{K_{t+1}}{K_{t+1}}\left(\frac{1}{K_{t+1}} - \lambda_{t+1}(\alpha+\gamma)K_{t+1}^{\alpha+\gamma-1}s_{t+1}^{\gamma}\right)\beta K_t^{\alpha+\gamma}s_t^{\gamma}\\ &= \left(1-(\alpha+\gamma)\lambda_{t+1}K_{t+1}^{\alpha+\gamma}s_{t+1}^{\gamma}\right)\beta\frac{K_t^{\alpha+\gamma}s_t^{\gamma}}{K_{t+1}}\\ &= \left(1+(\alpha+\gamma)v_{t+1}\right)\beta\frac{K_t^{\alpha+\gamma}s_t^{\gamma}}{K_t^{\alpha+\gamma}s_t^{\gamma}}\\ v_t &= \beta + \beta(\alpha+\gamma)v_{t+1}.\end{aligned} \tag{18}$$

In the steady state we have

$$v = \frac{\beta}{1-\beta(\alpha+\gamma)},$$

where we see that v is increasing in α and γ. Recall that α and γ are the share/weights of period t capital and investment, respectively, in the law of motion of capital. It stands to reason that if either of these shares increase, v, which is the ratio of the savings rate to the rate of consumption, should increase too. The higher shares induce more saving, on behalf of the social planner.

(d) Derive and discuss the conditions under which there is a unique perfect foresight equilibrium with the investment rate at its steady-state value. Your answer should include a diagram. How do parameter values affect the speed of convergence to the equilibrium?

We can rearrange (18) to get it into the following form:

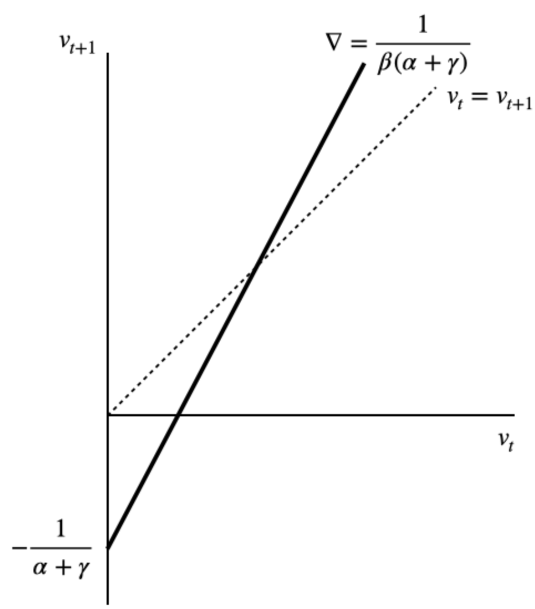
$$\mathbb{E}_t \mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t,$$

where in order to get unique convergence, we need the Blanchard-Kahn conditions to be met (number of eigenvalues of $\mathbf{A}$ outside of unit circle must be equal to the number of jump variables). So, write (18) as

$$v_{t+1} = \frac{1}{\beta(\alpha+\gamma)} v_t - \frac{1}{\alpha+\gamma},$$

and plot it in (v_t, v_{t+1}) space, as below.

Figure 4: Plot of v_{t+1}



(e) Returning to the equation for future capital stock derived in part (a), is it possible to have a constant investment rate but ever-increasing capital in the model? Explain your answer and assess the realism of any restrictions applied to parameter values.

So, from (15), we previously had

$$K_{t+1} = K_t^{\alpha+\gamma} s_t^{\gamma}.$$

If we assume that $s_t = s, \ \forall t$, and take logs, we get

$$\ln K_{t+1} = (\alpha + \gamma) \ln K_t + \underbrace{\gamma \ln s}_{g}$$
$$g = \ln K_{t+1} - (\alpha + \gamma) \ln K_t,$$

and we can see that the capital stock diverges if $\alpha + \gamma$ is either greater than or less than 1. In particular, if $\alpha + \gamma < 1$, then we will have an ever increasing capital stock. Previously, we required $|\beta(\alpha + \gamma)| < 1$ in order to have a unique, perfect foresight equilibrium. So putting these inequalities together, we have

$$1 < |\alpha + \gamma| < \beta^{-1}.$$

Intuitively, we need high marginal product of capital and/or a good transformation technology to ensure that the capital stock never converges, but sufficient discounting that the investment rate converges. It is probably harder to satisfy the first rather than the second of these conditions.