It is Taxing to be Coherent

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Introduction

- Models with no solution are incoherent; models with more than one solution are incomplete (Gourieroux, Laffont, and Monfort 1980) (GLM)
- New-Keynesian (NK) models with an active Taylor rule subject to an occasionally binding constraint are either incomplete (under support restrictions) or are incoherent (if shock is big enough) (Ascari and Mavroeidis 2022)
- Several ways to ensure uniqueness: i.e. price-level targeting (Holden 2022), passive Taylor rule with active fiscal policy, learning (Ascari, Mavroeidis, and McClung 2022).

This Paper

- Uses baseline NK model subject to the ZLB while maintaining full information rational expectations (FIRE)
- Finds that fiscal policy can restore model coherency and completeness
- Finds that simple Ricardian countercyclical fiscal policy needs to satisfy two properties to ensure coherency and completeness:
  1. sufficient countercyclicality
  2. high persistence

This Presentation

- Outlines coherency and completeness (CC) problem in NK models with the ZLB
- Sketch of methodology to check CC conditions in a linear DSGE model
- Overview of algorithm & challenges in checking for CC conditions in models with an endogenous state variable

Simple Example

- Consider a simple Fisherian model as in Aruoba, Cuba-Borda, and Schorfheide (2018):

$$1 = \mathbb{E}_t \left[M_{t,t+1} \frac{R_t}{\pi_{t+1}} \right] \tag{1}$$

$$R_t = \max \left\{ 1, r\pi^* \left(\frac{\pi_t}{\pi^*} \right)^{\phi_\pi} \right\}, \quad \phi_\pi > 1 \tag{2}$$

- Combine and log-linearise about non-stochastic steady state:

$$\mathbb{E}_t \hat{\pi}_{t+1} = \mathbb{E}_t \hat{M}_{t,t+1} + \max\{-\mu, \phi_\pi \hat{\pi}_t\} \tag{3}$$

Checking for Coherency and Completeness

- Checking coherency and completeness of a DGSE models is relatively straightforward.
  - Collect a vector of endogenous variables ($\mathbf{Y}_t$)
  - Collect a vector exogenous variables ($\mathbf{X}_t$)
  - Then rewrite the DSGE model in "generalised" matrix form
  - Using GLM Theorem check if the "general form" is invertible
- If the "general matrix form" is invertible, the model satisfies coherency and completeness.

Generalised Form and GLM Theorem

$$\underbrace{\overbrace{\overbrace{\mathbf{0} = \boldsymbol{A}_{s_t} \boldsymbol{Y}_t + \boldsymbol{C}_{s_t} \boldsymbol{X}_t}^{\text{Gourieroux, Laffont, and Monfort (1980)}} \qquad + \boldsymbol{B}_{s_t} \boldsymbol{Y}_{t+1|t} + \boldsymbol{D}_{s_t} \boldsymbol{X}_{t+1|t}}^{\text{Ascari and Mavroeidis (2022)}} + \boldsymbol{H}_{s_t} \boldsymbol{Y}_{t-1}}_{\text{This paper}}, \tag{4a}$$

$$s_t = \mathbb{1}_{\{\boldsymbol{a}^\top \boldsymbol{Y}_t + \boldsymbol{c}^\top \boldsymbol{X}_t + \boldsymbol{b}^\top \boldsymbol{Y}_{t+1|t} + \boldsymbol{d}^\top \boldsymbol{X}_{t+1|t} + \boldsymbol{h}^\top \boldsymbol{Y}_{t-1} > 0\}}. \tag{4b}$$

- Our contribution: Extend (4) with endogenous states using GLM Theorem.

Outline of methodology

Theorem (GLM Theorem 1)

A system of piecewise-linear equations satisfies coherency and completeness conditions if its matrix form defined in (4)*, represented as* $F(\boldsymbol{Y}) = \lambda(\boldsymbol{X})$*, where* $F(\mathbf{Y}) = \sum_J \boldsymbol{\mathcal{A}}_J \mathbb{1}_{\boldsymbol{\mathcal{C}}_J} vec(\mathbf{Y})$*, is invertible; which requires that all determinants* $\mathcal{A}_J, J \subset \{1, ..., k\}$ *share the same sign.*

Verifying the Simple Example

- Simple Fisherian model (3) can be cast in "general form" as:

$$0 = \begin{cases} \left(-\phi_\pi \begin{pmatrix} \hat{\pi}_t^1 & \hat{\pi}_t^2 \end{pmatrix} + \begin{pmatrix} \hat{\pi}_{t+1}^1 & \hat{\pi}_{t+1}^2 \end{pmatrix} \boldsymbol{K}^\top + \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\exp(-r^L)}{r} & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{K}^\top \right) \boldsymbol{e}_i, \\ \left(-\phi_\pi \begin{pmatrix} \hat{\pi}_t^1 & \hat{\pi}_t^2 \end{pmatrix} + \begin{pmatrix} 0 & \mu \end{pmatrix} \boldsymbol{K}^\top + \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\exp(-r^L)}{r} & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{K}^\top \right) \boldsymbol{e}_i. \end{cases}$$

- Satisfies CC conditions if determinants of the following are the same sign:

$$\begin{aligned} \boldsymbol{\mathcal{A}}_{J_1} &= \boldsymbol{A}_1 \boldsymbol{I}_2 + \boldsymbol{B}_1 \boldsymbol{K}, \quad J_1 = \{1, 2\}, \\ \boldsymbol{\mathcal{A}}_{J_2} &= \boldsymbol{e}_1 \boldsymbol{e}_1^\top \boldsymbol{\mathcal{A}}_{J_4} + \boldsymbol{e}_2 \boldsymbol{e}_2^\top \boldsymbol{\mathcal{A}}_{J_1}, \quad J_2 = \{2\}, \\ \boldsymbol{\mathcal{A}}_{J_3} &= \boldsymbol{e}_2 \boldsymbol{e}_2^\top \boldsymbol{\mathcal{A}}_{J_4} + \boldsymbol{e}_1 \boldsymbol{e}_1^\top \boldsymbol{\mathcal{A}}_{J_1}, \quad J_3 = \{1\}, \\ \boldsymbol{\mathcal{A}}_{J_4} &= \boldsymbol{A}_0 \boldsymbol{I}_2 + \boldsymbol{B}_0 \boldsymbol{K}, \quad J_4 = \emptyset. \end{aligned} \tag{5}$$

Verifying the CC Conditions

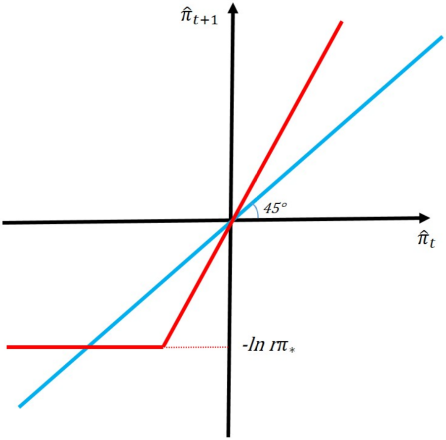
- Determinants are:

$$
\begin{aligned}
|\boldsymbol{\mathcal{A}}_{J_1}| &= (\phi_\pi - 1)(1 - p - q + \phi_\pi), \\
|\boldsymbol{\mathcal{A}}_{J_2}| &= p(1 - \phi_\pi) + q - 1, \\
|\boldsymbol{\mathcal{A}}_{J_3}| &= p - 1 + q(1 - \phi_\pi), \\
|\boldsymbol{\mathcal{A}}_{J_4}| &= p + q - 1.
\end{aligned} \tag{6}
$$

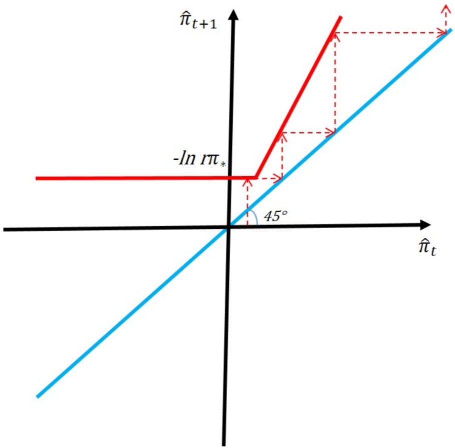
- Clearly, an active Taylor rule ($\phi_\pi > 1$) leads to a violation of the CC conditions.

Illustrating the CC Violation

Coherent: $r\pi_* \geq 1$



Incoherent: $r\pi_* < 1$



Equilibria in Baseline NK model

- Baseline (log-linear) NK model as in Galí (2015):

$$\text{DISE: } \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - E_t \hat{\pi}_{t+1}) + \varepsilon_t, \tag{7}$$

$$\text{NKPC: } \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \hat{y}_t, \tag{8}$$

$$\text{TR: } \hat{i}_t = \max\{-\mu, \phi_\pi \hat{\pi}_t\}, \tag{9}$$

where $\mu = \log(r\pi^*)$

The Baseline NK model

- Assume that ε_t materialises and persists w.p. p and vanishes w.p $1-p$:

$$\varepsilon_t = \begin{cases} \frac{p}{\sigma}\hat{r}^T & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases} \tag{10}$$

- Model in transitory state is given by:

$$\hat{\pi}^T = \frac{\kappa}{1-p\beta}\hat{y}^T \quad AS, \tag{11a}$$

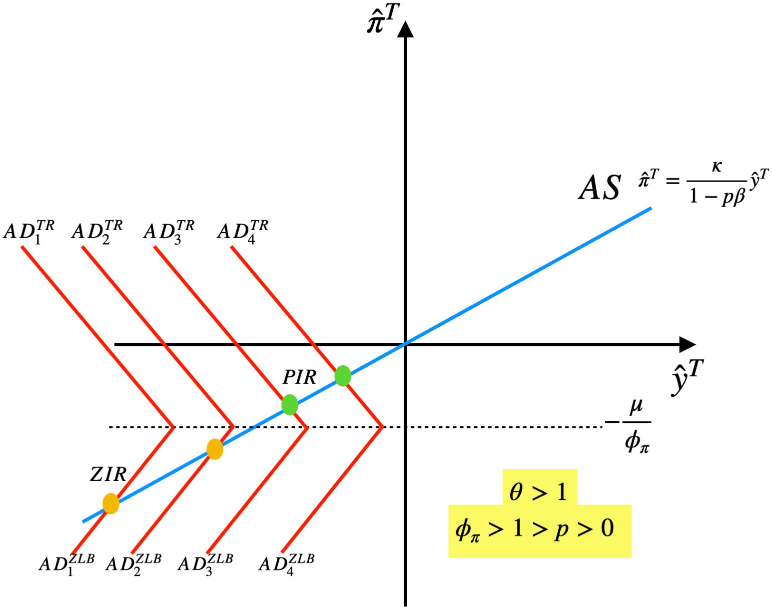
$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{p-\phi_\pi}\hat{y}^T - \frac{p}{p-\phi_\pi}\hat{r}^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T - \frac{\mu}{p} - \hat{r}^T & AD^{ZLB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{11b}$$

- Define slope ratio of AD^{ZLB} and AS:

$$\theta = \frac{\sigma(1-p)(1-p\beta)}{p\kappa} \tag{12}$$

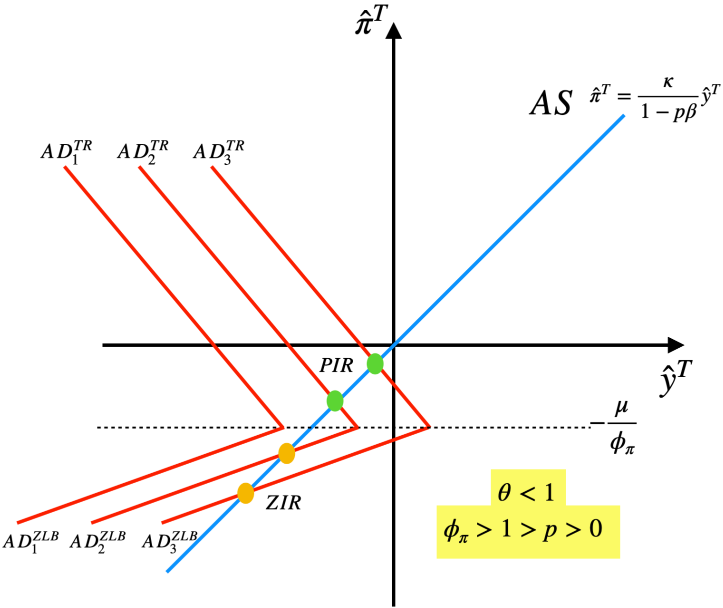
Illustrating the Baseline NK Model ($\theta > 1$)

Figure: Transitory State of the New Keynesian Model ($\phi_\pi > 1$)



Illustrating the Baseline NK Model ($\theta < 1$)

Figure: Transitory State of the New Keynesian Model ($\phi_\pi > 1$)



Adding Fiscal Policy

- Now include fiscal policy; government spending is funded by output and lump-sum taxes

$$\tau_t + \tau_t^s Y_t = G_t$$

- Output tax offsets FP in NKPC:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \frac{c}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right) - g \Delta \widehat{g}_{t+1}, \tag{13}$$

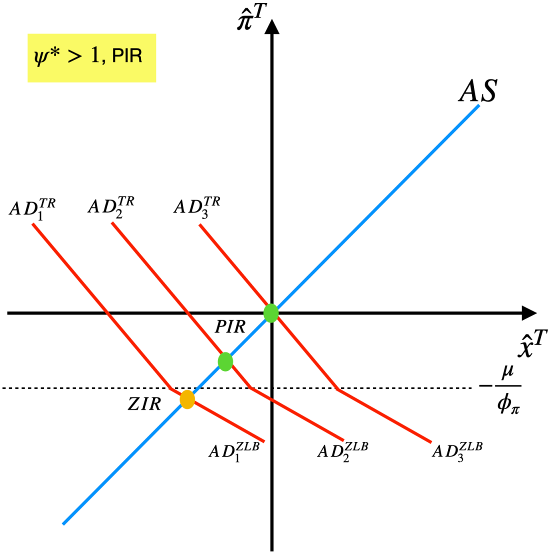
$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa_y \hat{x}_t - \frac{\epsilon}{\Phi}\left(\frac{1}{\epsilon}\hat{\tau}_t^s + \frac{\sigma}{c}\hat{g}_t\right), \tag{14}$$

$$\hat{i}_t = \max\{-\mu; \phi_\pi \hat{\pi}_t + \phi_y \hat{x}_t\}, \tag{15}$$

$$\Delta \widehat{g}_{t+1} = \psi_\pi \hat{\pi}_t + \psi_y \hat{x}_t \tag{16}$$

Illustrating the NK-FP Model

Figure: Transitory States under Active and Passive Fiscal Policy



Key Intuition

- Fiscal policy changes the slope of AD^{ZLB}
- Recall: AD^{ZLB}/AS slope ratio:

$$\theta = \frac{\sigma(1-p)(1-p\beta)}{\kappa_y c(p-\psi^*)} \tag{17}$$

- θ is determined by policy, ψ^*, and uncertainty, p

Endogenous State Solution

- Analytical solution with endogenous states is infeasible
- Model dynamics depend on all past realisations of the shock and endogenous state
- With no endogenous states, the support of $\boldsymbol{Y}$ is time-invariant

$$F(\boldsymbol{Y}) = \lambda(\boldsymbol{X}) \tag{18}$$

- With endogenous states

$$F(\boldsymbol{Y}_t) = \lambda(\boldsymbol{X}, \boldsymbol{Y}_{t-1}) \tag{19}$$

- To ensure uniqueness and existence, we use backward iterative algorithm
- Goal: find a path from $t = 0$ to some terminal T along which the mapping is invertible

Sketch of methodology and algorithm

NK Inertial FP Model

- So far, we have considered unit-root fiscal rules
- We extend analysis to inertial rules that do not feature a unit root $\implies$ endogenous state variables:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \frac{c}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right) - g\Delta\hat{g}_{t+1}, \tag{20}$$

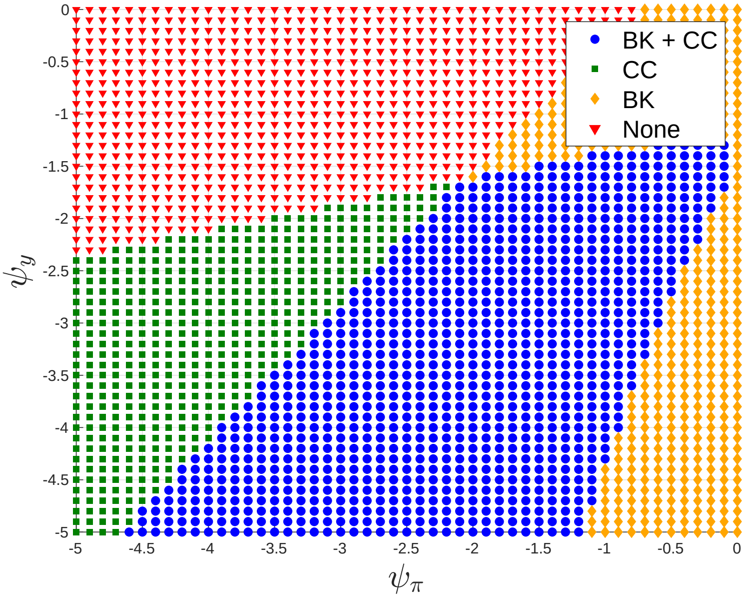
$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa_y \hat{x}_t - \kappa_g \hat{g}_t, \tag{21}$$

$$\hat{i}_t = \max\{-\mu; \phi_\pi \hat{\pi}_t + \phi_y \hat{x}_t\} \tag{22}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \psi_\pi \hat{\pi}_t + \psi_y \hat{x}_t \tag{23}$$

Derivations and intuition

Blanchard-Kahn and CC Regions



Conclusion

- Ricardian fiscal policy can restore uniqueness in presence of ZLB
- FP needs to satisfy two properties: (i) sufficiently countercyclical, (ii) persistent
- We develop operational algorithm to check uniqueness conditions with an endogenous state

Sketch of Methodology (Forward Looking) I

- Assume that shocks are two-state stationary Markovian

$$\boldsymbol{K} = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \tag{24}$$

- Along an MSV solution, we have

$$\mathbb{E}[\boldsymbol{Y}_{t+1}|\boldsymbol{Y}_t = \mathbf{Y}\boldsymbol{e}_i] = \mathbb{E}_t[\boldsymbol{Y}_{t+1}|\boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{Y}\boldsymbol{K}^\top \boldsymbol{e}_i \tag{25}$$

- This allows to rewrite the general form (with no endogenous states, i.e. $\boldsymbol{h} = \mathbf{0}$) as

$$\begin{aligned} \mathbf{0} &= (\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}^\top)\boldsymbol{e}_i, \\ s_i &= \mathbb{1}\{(\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top)\boldsymbol{e}_i > 0\},\ i = 1, ..., k. \end{aligned} \tag{26}$$

- This can be rewritten as

$$F(\mathbf{Y}) = \sum_J \boldsymbol{\mathcal{A}}_J \mathbb{1}_{\boldsymbol{\mathcal{C}}_J} \mathrm{vec}(\mathbf{Y}), \tag{27}$$

  where $\boldsymbol{\mathcal{C}}_J = \{\mathbf{Y} : \mathbf{Y} \in \mathbb{R}^{n\times k}, s_i = \mathbb{1}_{\{i\in J\}}\}$

Sketch of Methodology (Forward Looking) II

- (27) must be invertible for the system to have a unique solution.
- This is the case if determinants of $\boldsymbol{\mathcal{A}}_J$ have the same sign:

$$\begin{aligned}
\boldsymbol{\mathcal{A}}_{J_1} &= \boldsymbol{A}_1 \boldsymbol{I}_2 + \boldsymbol{B}_1 \boldsymbol{K}, \quad J_1 = \{1, 2\}, \\
\boldsymbol{\mathcal{A}}_{J_2} &= \boldsymbol{e}_1 \boldsymbol{e}_1^\top \boldsymbol{\mathcal{A}}_{J_4} + \boldsymbol{e}_2 \boldsymbol{e}_2^\top \boldsymbol{\mathcal{A}}_{J_1}, \quad J_2 = \{2\}, \\
\boldsymbol{\mathcal{A}}_{J_3} &= \boldsymbol{e}_2 \boldsymbol{e}_2^\top \boldsymbol{\mathcal{A}}_{J_4} + \boldsymbol{e}_1 \boldsymbol{e}_1^\top \boldsymbol{\mathcal{A}}_{J_1}, \quad J_3 = \{1\}, \\
\boldsymbol{\mathcal{A}}_{J_4} &= \boldsymbol{A}_0 \boldsymbol{I}_2 + \boldsymbol{B}_0 \boldsymbol{K}, \quad J_4 = \emptyset.
\end{aligned} \tag{28}$$

Back

Methodology with an Endogenous State I

- Consider the simple case of a single endogenous state. For a date T whereby $t \geq T$, an MSV solution $f(y_{t-1}, \boldsymbol{X}_t)$ can be written as

$$\mathbf{Y}_t = \boldsymbol{G} y_{t-1} + \boldsymbol{Z},$$

- Again, assume $k = 2$
- General form:

$$\begin{aligned}\mathbf{0} = &\left(\boldsymbol{A}_{s_{t,i}} \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{G} \boldsymbol{e}_i\right) y_{t-1} \\ &+ \left(\boldsymbol{A}_{s_{t,i}} \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{Z} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}} \mathbf{X} + \boldsymbol{D}_{s_{t,i}} \mathbf{X} \boldsymbol{K}^\top\right) \boldsymbol{e}_i,\end{aligned} \tag{29}$$

  for all $i = 1, ..., k$.
- For a given regime J corresponding to the k states and their transitions, a slackness condition for the constraint $s_{t,i}$ is determined which gives a system of $2nk$ polynomial equations in the $2nk$ unknowns $\boldsymbol{G}$ and $\boldsymbol{Z}$ by equating the coefficients on y_{t-1} and the constant terms to zero, respectively.

Methodology with an Endogenous State II

- As these conditions are polynomial and not piecewise linear in $\boldsymbol{G}$ and $\boldsymbol{Z}$, the algorithm and theorem of Gourieroux, Laffont, and Monfort (1980) is no longer suitable to check coherency.
- Build a "brute force" algorithm which essentially goes through all possible 2^k J regime configurations to check if there are any feasible solutions that satisfy the inequality constraints.
- We know that at T, the solution to the model takes the following form

$$\boldsymbol{Y}_T = \boldsymbol{G}_{J_0} y_{T-1} + \boldsymbol{Z}_{J_0},$$

where $J_0 \in J$ defines the configuration of regimes in T. $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can be solved for from (29):

$$\mathbf{0} = \boldsymbol{A}_{s_{t,i}} \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{G} \boldsymbol{e}_i, \tag{30}$$

$$\mathbf{0} = \left(\boldsymbol{A}_{s_{t,1}} \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{Z} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}} \mathbf{X} + \boldsymbol{D}_{s_{t,i}} \mathbf{X} \boldsymbol{K}^\top \right) \boldsymbol{e}_i, \tag{31}$$

$\forall i = 1, ..., k.$

Methodology with an Endogenous State III

- $\boldsymbol{Y}_T$ is a function of $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$. Thus, $\boldsymbol{Y}_T$ is known and we can solve for $\boldsymbol{Y}_{T-1}$ from

$$\begin{aligned}\mathbf{0} = &\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i}\boldsymbol{G}_{J_0}\boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Y}_{T-1}\boldsymbol{e}_i\right)\\ &+ \left(\boldsymbol{B}_{s_{T-1},i}\boldsymbol{Z}_{J_0}\boldsymbol{K}^\top + \boldsymbol{C}_{s_{T-1},i}\mathbf{X} + \boldsymbol{D}_{s_{T-1},i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i}y_{T-2}.\end{aligned}$$

- For every t the determinants relevant for CC conditions are given by

$$|\mathcal{A}_{J_0 J_1}| = \prod_i^k \det\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i}\boldsymbol{G}_{J_0}\boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top\right).$$

Methodology with an Endogenous State IV

- If $k = 2$, the determinants can be rewritten as

$$
\begin{aligned}
|\mathcal{A}_{J_0\{1,2\}}| &= \det\left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_1 \boldsymbol{g}^\top\right) \det\left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_2 \boldsymbol{g}^\top\right), \\
|\mathcal{A}_{J_0\{2\}}| &= \det\left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_1 \boldsymbol{g}^\top\right) \det\left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_2 \boldsymbol{g}^\top\right), \\
|\mathcal{A}_{J_0\{1\}}| &= \det\left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_1 \boldsymbol{g}^\top\right) \det\left(\boldsymbol{A}_1 + \boldsymbol{B}_1 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_2 \boldsymbol{g}^\top\right), \\
|\mathcal{A}_{J_0\{\emptyset\}}| &= \det\left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_1 \boldsymbol{g}^\top\right) \det\left(\boldsymbol{A}_0 + \boldsymbol{B}_0 \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_2 \boldsymbol{g}^\top\right).
\end{aligned}
$$

- If the model satisfies CC, the solution is given by

$$
\begin{aligned}
\boldsymbol{Y}_{T-1} \boldsymbol{e}_i = &-\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i} \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top\right)^{-1} \\
&\left[\left(\boldsymbol{B}_{s_{T-1},i} \boldsymbol{Z}_{J_0} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{T-1},i} \mathbf{X} + \boldsymbol{D}_{s_{T-1},i} \mathbf{X} \boldsymbol{K}^\top\right) \boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i} y_{T-2}\right],
\end{aligned}
$$

$\forall i = 1, ..., k.$

Methodology with an Endogenous State V

- Iterating the solution backwards implies that all the determinants $\mathcal{A}_{J_0,\dots,J_{T-t}}$ must have the same sign. The recursive solution will be given by

$$\boldsymbol{Y}_t = \boldsymbol{G}_{J_0,\dots,J_{T-t}} y_{t-1} + \boldsymbol{Z}_{J_0,\dots,J_{T-t}},$$

where $\boldsymbol{G}_{J_0,\dots,J_{T-t}}$ and $\boldsymbol{Z}_{J_0\dots,J_{T-t}}$ can be computed recursively using

$$\begin{aligned}\boldsymbol{Z}_{J_0\dots,J_{T-t},i} = -\left(\boldsymbol{A}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G}_{J_0,\dots,J_{T-t-1}} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top\right)^{-1} \\ \left(\boldsymbol{B}_{s_{t,i}} \boldsymbol{Z}_{J_0,\dots,J_{T-t-1}} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}} \mathbf{X} + \boldsymbol{D}_{s_{t,i}} \mathbf{X} \boldsymbol{K}^\top\right) \boldsymbol{e}_i,\end{aligned} \tag{32}$$

$$\boldsymbol{G}_{J_0,\dots,J_{T-t},i} = -\left(\boldsymbol{A}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G}_{J_0,\dots,J_{T-t-1}} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top\right)^{-1} \boldsymbol{h}_{s_{t,i}}. \tag{33}$$

- The recursive solution from terminal T solves the model backwards to $t = 1$ and implies up to $2^{(T-1)k}$ solution paths.
- Given some initial condition, y_0, and conditional on satisfaction of CC conditions, the recursive solution is unique. If the CC conditions are not satisfied, there can be either no or multiple solutions. Back

Inertial NK-FP Model I

- The MSV solution is given by

$$\boldsymbol{Y}_T = \boldsymbol{G}_{J_0} y_{T-1} + \boldsymbol{Z}_{J_0},$$

- where $J_0 \in J$ defines the configuration of regimes in T
- $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can be solved for from

$$\mathbf{0} = \boldsymbol{A}_{s_{t,i}} \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{G} \boldsymbol{e}_i, \tag{34}$$

$$\mathbf{0} = \left(\boldsymbol{A}_{s_{t,1}} \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{Z} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}} \mathbf{X} + \boldsymbol{D}_{s_{t,i}} \mathbf{X} \boldsymbol{K}^\top \right) \boldsymbol{e}_i, \tag{35}$$

$\forall i = 1, ..., k.$

- This yields $\boldsymbol{Y}_T$, we then can solve for $\boldsymbol{Y}_{T-1}$

$$\begin{aligned}\mathbf{0} = &\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i} \boldsymbol{G}_{J_0} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Y}_{T-1} \boldsymbol{e}_i \right) \\ &+ \left(\boldsymbol{B}_{s_{T-1},i} \boldsymbol{Z}_{J_0} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{T-1},i} \mathbf{X} + \boldsymbol{D}_{s_{T-1},i} \mathbf{X} \boldsymbol{K}^\top \right) \boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i} y_{T-2}.\end{aligned} \tag{36}$$

Inertial NK-FP Model II

- NK-IFP model:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \frac{c}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^n\right), \tag{37a}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa_y \hat{x}_t - \kappa_g \widehat{GR}_t, \tag{37b}$$

$$\hat{i}_t = \max\{-\mu; \phi_\pi \hat{\pi}_t + \phi_y \hat{x}_t\}, \tag{37c}$$

$$\widehat{GR}_t = \rho_\tau \widehat{GR}_{t-1} + \psi_\pi \hat{\pi}_t + \psi_y \hat{x}_t, \tag{37d}$$

with

$$\hat{r}_t^n = -\frac{\sigma}{c}(g \mathbb{E}_t \Delta \widehat{GR}_{t+1} + \varepsilon_t).$$

- Can be evaluated about two absorbing states, either PIR or ZIR.
  - PIR: $\{\hat{x}, \hat{\pi}, \hat{i}, \widehat{GR}\} = \{0, 0, 0, 0\}$
  - ZIR $\implies \hat{i} = -\mu$:

$$\{\hat{x}, \hat{\pi}, \hat{i}, \widehat{GR}\} = \left\{-\frac{(1-\beta)(1-\rho_\tau) + \kappa_g \psi_\pi}{\kappa_y(1-\rho_\tau) - \kappa_g \psi_y}\mu, -\mu, -\mu, \frac{\psi_y \hat{x} + \psi_\pi \hat{\pi}}{1-\rho_\tau}\right\}$$

Inertial NK-FP Model III

- Under certain fiscal policy rules, the above ZIR equilibrium is not consistent with the constraint on the TR and, thus, the ZIR equilibrium is ruled out.
- Specifically, we require that $\hat{x}$ be sufficiently large such that the ZLB constraint on $\hat{i}_t$ is not binding:

$$-\mu < -\phi_\pi \mu + \phi_y \hat{x} \implies -\frac{\mu(1-\phi_\pi)}{\phi_y} < \hat{x}, \quad \phi_y \neq 0$$

- This implies

$$\frac{1-\phi_\pi}{\phi_y} > \frac{(1-\beta)(1-\rho_\tau) + \kappa_g \psi_\pi}{\kappa_y(1-\rho_\tau) - \kappa_g \psi_y},$$

  with the LHS being negative under conventional restrictions on TR coefficients.
- As ρ_τ tends to unity, we get

$$\lim_{\rho_\tau \to 1} \frac{(1-\beta)(1-\rho_\tau) + \kappa_g \psi_\pi}{\kappa_y(1-\rho_\tau) - \kappa_g \psi_y} = -\frac{\psi_\pi}{\psi_y} < \frac{1-\phi_\pi}{\phi_y},$$

  which holds under countercyclical fiscal policy, $\psi_y < 0$, $\psi_\pi < 0$ with ψ_π sufficiently large in absolute value.

Inertial NK-FP Model IV

- Thus, under countercyclical fiscal policy, if ρ_τ is sufficiently large there cannot exist a ZIR absorbing state.

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