It is Taxing to be Coherent

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Overview

Introduction

- Structural models with no solution are **incoherent**, and those with multiple solutions are **incomplete**.
- Ascari and Mavroeidis (2022) develop a method to study **coherency** and **completeness** (CC) conditions for linear dynamic forward-looking rational expectations models under an occasionally binding constraint.
- For a canonical New Keynesian (NK) model with a zero lower bound (ZLB), this method shows that the CC conditions are generally violated with an active Taylor Rule.
- Rational expectations require time-varying and correlated support restrictions on the distributions of the structural shocks. With appropriate restrictions, a very large number of equilibria can be supported.

This Paper

- Using the methodology of Ascari and Mavroeidis (2022) and Gourieroux, Laffont, and Monfort (1980) We show that simple fiscal policy rules can restore coherency and completeness to a NK model subject to the ZLB **and** active Taylor Rules.
- We maintain full information rational expectations (FIRE).

Literature Review

- Gourieroux, Laffont, and Monfort (1980) (GLM) derived conditions for coherency and completeness.
- Mavroeidis (2021) applied GLM's methodology to SVARs with occasionally binding constraints.
- Vast literature on New Keynesian models with ZLB constraint: Eggertsson and Woodford (2003), Fernández-Villaverde et al. (2015), and Aruoba, Cuba-Borda, and Schorfheide (2018), amongst many other notable contributions.

Example: Canonical NK Model

- Take a baseline NK model as found in, for example, Galí (2015):

$$\text{DISE: } \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right) + \epsilon_t, \tag{1}$$

$$\text{NKPC: } \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + u_t, \tag{2}$$

$$\text{TR: } i_t = \max\left\{-\mu, \phi_\pi \pi_t + \phi_y \hat{y}_t\right\}, \tag{3}$$

where $\mu = \log(r\pi^*)$.

Transformation of State-Space Models I

- Many solution methods of log-linearised models which feature the ZLB, such as Eggertsson and Woodford (2003), Guerrieri and Iacoviello (2015), Kulish, Morley, and Robinson (2017), Eggertsson and Singh (2019), and Holden (2022), can be verified for coherency in a simple manner.
- Transform the NK model into the following form:

$$\begin{aligned} \boldsymbol{A}_s \boldsymbol{Y}_t + \boldsymbol{B}_s \boldsymbol{Y}_{t+1|t} + \boldsymbol{C}_s \boldsymbol{X}_t + \boldsymbol{D}_s \boldsymbol{X}_{t+1|t} &= \boldsymbol{0}, \\ s_t = \mathbb{1}\{\boldsymbol{a}^\top \boldsymbol{Y}_t + \boldsymbol{b}^\top \boldsymbol{Y}_{t+1|t} + \boldsymbol{c}^\top \boldsymbol{X}_t + \boldsymbol{d}^\top &\boldsymbol{X}_{t+1|t}\}, \end{aligned} \tag{4}$$

where we let $\boldsymbol{Y}_t$ be a $n \times 1$ vector of endogenous variables, $\boldsymbol{X}_t$ be a $n_x \times 1$ vector of exogenous state variables, and $s_t \in \{0, 1\}$ be an indicator variable that is equal to 1 when some inequality constraint is slack and 0 otherwise. Appendix for k-state exogenous variables

Coherence and Completeness Requirements I

- All the determinants of $\boldsymbol{\mathcal{A}}_J$, $J \subset \{1, ..., k\}$, must share the same sign.
- Failure of this requirement implies that the model is generally incoherent, i.e., no MSV solution exists.
- Given the NK model (1)-(3), and suppose that, for simplicity, $k = 2$ and $u_t = \phi_y = 0$. Then $\boldsymbol{\mathcal{A}}_J$ are given by

$$\begin{aligned}
\boldsymbol{\mathcal{A}}_{J_1} &= \boldsymbol{A}_1 \boldsymbol{I}_2 + \boldsymbol{B}_1 \boldsymbol{K}, \quad J_1 = \{1, 2\}, \\
\boldsymbol{\mathcal{A}}_{J_2} &= \boldsymbol{e}_1 \boldsymbol{e}_1^\top (\boldsymbol{A}_0 \boldsymbol{I}_2 + \boldsymbol{B}_0 \boldsymbol{K}) + \boldsymbol{e}_2 \boldsymbol{e}_2^\top (\boldsymbol{A}_1 \boldsymbol{I}_2 + \boldsymbol{B}_1 \boldsymbol{K}), \quad J_2 = \{2\}, \\
\boldsymbol{\mathcal{A}}_{J_3} &= \boldsymbol{e}_2 \boldsymbol{e}_2^\top (\boldsymbol{A}_0 \boldsymbol{I}_2 + \boldsymbol{B}_0 \boldsymbol{K}) + \boldsymbol{e}_1 \boldsymbol{e}_1^\top (\boldsymbol{A}_1 \boldsymbol{I}_2 + \boldsymbol{B}_1 \boldsymbol{K}), \quad J_3 = \{1\}, \\
\boldsymbol{\mathcal{A}}_{J_4} &= \boldsymbol{A}_0 \boldsymbol{I}_2 + \boldsymbol{B}_0 \boldsymbol{K}, \quad J_4 = \emptyset.
\end{aligned} \tag{5}$$

- The relevant matrices are given by

$$\boldsymbol{A}_0 = \begin{pmatrix} 1 & -\kappa \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_1 = \begin{pmatrix} 1 & -\kappa \\ \frac{\phi_\pi}{\sigma} & 1 \end{pmatrix}, \quad \boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{pmatrix} -\beta & 0 \\ -\frac{1}{\sigma} & -1 \end{pmatrix}.$$

Coherence and Completeness Requirements II

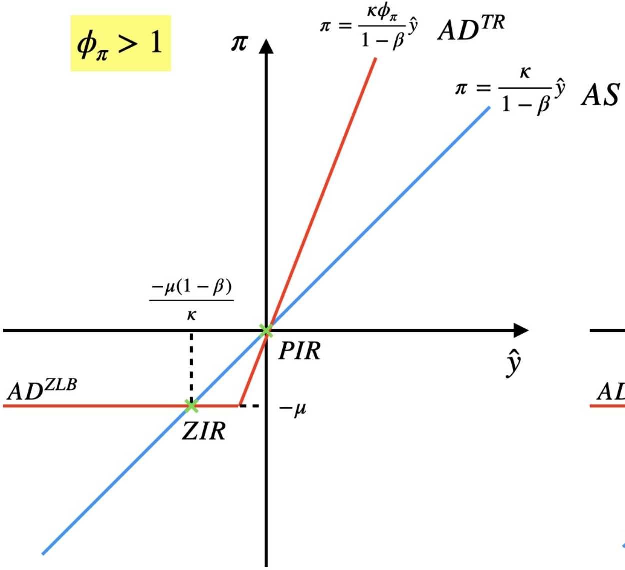
- Observe that, assuming $p = q = 1$, the determinants of $\boldsymbol{\mathcal{A}}_{J_1}$ and $\boldsymbol{\mathcal{A}}_{J_4}$ are:

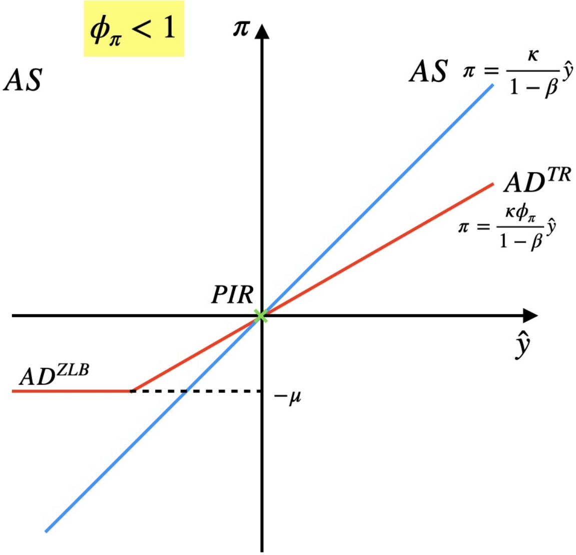
$$|\boldsymbol{\mathcal{A}}_{J_1}| = \begin{vmatrix} 1-\beta & -\kappa \\ \frac{\phi_\pi - 1}{\sigma} & 0 \end{vmatrix} = \frac{\kappa(\phi_\pi - 1)}{\sigma} > 0, \tag{6}$$

$$|\boldsymbol{\mathcal{A}}_{J_4}| = \begin{vmatrix} 1-\beta & -\kappa \\ -\frac{1}{\sigma} & 0 \end{vmatrix} = -\frac{\kappa}{\sigma} < 0. \tag{7}$$

- We observe that with an active Taylor Rule, $\phi_\pi > 1$, the model is generally incoherent as thoroughly shown by Ascari and Mavroeidis (2022).

Illustrating the Baseline NK Model





Adding a Fiscal Authority

- Now, we augment the baseline NK model by adding in a fiscal authority which implements simple lump-sum taxes that funds its spending:

$$(1-\tau_t^c)P_t C_t + B_t = (1-\tau_t^w)W_t L_t + R_{t-1}B_{t-1}.$$

- The NK model can then be written in log-linearised form as:

$$\text{DISE: } \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{c}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1}) - g\mathbb{E}_t \Delta\hat{g}_{t+1} + \epsilon_t, \tag{8}$$

$$\text{NKPC: } \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + u_t, \tag{9}$$

$$\text{TR: } i_t = \max\{-\mu, \phi_\pi \pi_t + \phi_y \hat{y}_t\}, \tag{10}$$

where $c = \bar{C}/\bar{Y}$ is the steady-state consumption-output ratio, $g = \bar{G}/\bar{Y}$ is the steady-state government expenditure-output ratio, and $\Delta\hat{g}_t$ is the (log-linearised) difference in government expenditure between periods t and $t-1$.

"Knife-Edge" Fiscal Rule

- The "knife-edge" fiscal rule:

$$\mathbb{E}_t \Delta \hat{g}_{t+1} = \mathbb{1}\{i_t = -\mu\}(\psi_\pi \pi_t + \psi_y \hat{y}_t). \tag{11}$$

  The CC conditions are satisfied so long as:

$$\begin{aligned} \psi_\pi &= \frac{\bar{C}}{\bar{G}\sigma}\phi_\pi, \\ \psi_y &= \frac{\bar{C}}{\bar{G}\sigma}\phi_y, \end{aligned} \tag{12}$$

  which also allows (10) to follow an active Taylor Rule ($\phi_\pi > 1$).
- (12) implies that government spending increases in inflation.
- It is straightforward to see that since the model is now linear, it is generally coherent and complete.
- This fiscal policy rule also applies to models where monetary policy is strictly inflation targeting, whereby if $\phi_y = 0$ then $\psi_y = 0$.

"Knife-Edge" Fiscal Rule

Proposition ("Knife-Edge" Fiscal Rule)

A baseline New Keynesian model with fiscal policy that consists of government spending and lump-sum taxes as defined in (8)-(10), is generally coherent and complete when the sensitivity parameters of the fiscal instrument, ψ_π and ψ_y, allow fiscal policy to replicate monetary policy at the ZLB, as in (12).

Inflation and Output Gap Targeting Fiscal Rule

- Contemporaneous inflation (and output gap) targeting:

$$\mathbb{E}_t \Delta \hat{g}_{t+1} = \psi_\pi \pi_t. \tag{13}$$

Proposition (Inflation and Output Gap Targeting Fiscal Rule)

A baseline New Keynesian model with fiscal policy that consists of government spending and lump-sum taxes as defined in (8)-(10) and (13), is generally coherent and complete when the absolute value of the degree of reaction of the fiscal instrument, $|\psi_\pi|$, is sufficiently large.

Optimal Monetary Policy Fiscal Rule

- The case with discretionary optimal policy (OP):
- As in Nakata (2018) and Nakata and Schmidt (2019), the OP condition, when i_t is not constrained, reads as:

$$\alpha_y \hat{y}_t + \kappa \pi_t = 0. \tag{14}$$

- When the ZLB is non-binding, the model is given by condition (14), together with the NKPC (9), and the government spending rule given by:

$$\mathbb{E}_t \Delta \hat{g}_{t+1} = \psi_\pi \pi_t + \psi_y \hat{y}_t. \tag{15}$$

- When the ZLB is binding, the model is given by the following set of equations:

$$\begin{aligned} \hat{y}_t &= \mathbb{E}_t \hat{y}_{t+1} - \frac{c}{\sigma}\left(-\mu - \mathbb{E}_t \hat{\pi}_{t+1}\right) - g\mathbb{E}_t \Delta \hat{g}_{t+1} + \varepsilon_t, \\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + u_t. \end{aligned} \tag{16}$$

Optimal Monetary Policy Fiscal Rule

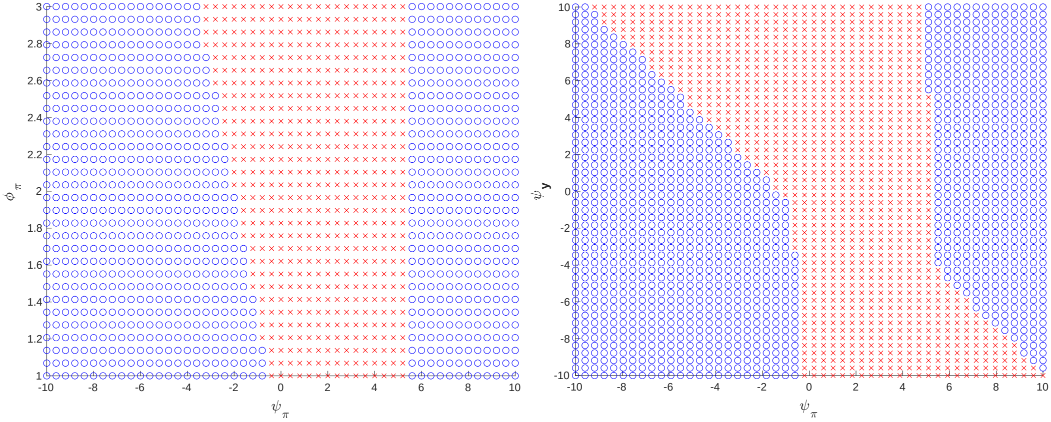
- After putting the model in canonical form (4), and numerically validating the CC conditions, we then have the following proposition:

Proposition (Optimal Monetary Policy under Discretion Fiscal Rule)

A baseline New-Keynesian model with fiscal policy, that consists of government spending and lump-sum taxes as defined in (9), (14), and (16) is generally coherent and complete when the degree of reaction of the fiscal instrument, ψ_π, is sufficiently large.

CC Regions

Figure: CC regions under different fiscal feedback rules



(A) $\Delta \hat{g}_{t+1} = \psi_\pi \pi_t$

(B) $\Delta \hat{g}_{t+1} = \psi_\pi \pi_t + \psi_y \hat{y}_t$

Note: Blue region(s) denote areas where coherency and completeness conditions are generally satisfied.

Concluding Remarks

- In this paper we have considered the coherency and completeness conditions of a baseline New Keynesian model, that is generally incoherent due to the presence of an occasionally binding constraint (ZLB) on the conventional monetary policy instrument.
- We find that the model satisfies the CC conditions if the fiscal authority uses an appropriate fiscal rule.
- Firstly, we show that if a policy authority has an instrument in the dynamic IS curve, the linear form of an otherwise piece-wise linear model can be restored.
- Secondly, we find that, under mild assumptions, there exists appropriate fiscal policy (tax regimes) that can render the model coherent and complete.

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Appendix: Transformation of State-Space Models I

- Coherency of the system (4) requires that there exists some function $f(\cdot)$ such that the minimum state variable (MSV) solution can be represented as $\boldsymbol{Y}_t = f(\boldsymbol{X}_t)$.
- Assume that the exogenous states $\boldsymbol{X}_t$ are k-state stationary first-order Markov processes with transition kernel $\boldsymbol{K}$.[3]
- Stack the possible states of $\boldsymbol{X}_t$ for states $i = 1, .., k$ into a $n_x \times k$ matrix $\mathbf{X}$. Then, let $\boldsymbol{e}_i$ denote the i-th column of the $k \times k$ identity matrix $\boldsymbol{I}_k$, such that $\mathbf{X}\boldsymbol{e}_i$, the i-th column of $\mathbf{X}$, is the i-th state of $\boldsymbol{X}_t$.[4]
- Then define $\mathbf{Y}$ as an $n \times k$ matrix whose i-th column, $\mathbf{Y}\boldsymbol{e}_i$, corresponds to $\boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i$ along a MSV solution.

Appendix: Transformation of State-Space Models II

- Thus, along a MSV solution we have:

$$\mathbb{E}[\boldsymbol{Y}_{t+1}|\boldsymbol{Y}_t = \mathbf{Y}\boldsymbol{e}_i] = \mathbb{E}_t[\boldsymbol{Y}_{t+1}|\boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i]\mathbf{Y}\boldsymbol{K}^\top\boldsymbol{e}_i.$$

  Substituting this into (4), $\mathbf{Y}$ must satisfy the following in order for the CC conditions to be satisfied:

$$\begin{aligned} \mathbf{0} &= (\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}^\top)\boldsymbol{e}_i, \\ s_i &= \mathbb{1}\{(\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top)\boldsymbol{e}_i > 0\},\ i = 1, \ldots, k. \end{aligned} \tag{17}$$

- This system of equations relates $\mathbf{Y}$ to $\mathbf{X}$, and can be expressed as $F(\mathbf{Y}) = \lambda(\mathbf{X})$, where $\lambda(\cdot)$ is some function of $\mathbf{X}$, and $F(\cdot)$ is a piecewise linear continuous function of $\mathbf{Y}$.

Appendix: Transformation of State-Space Models III

- If $J \subset \{1, ..., k\}$, then the piecewise linear function $F(\mathbf{Y})$ can then be expressed as:

$$F(\mathbf{Y}) = \sum_{J} \boldsymbol{\mathcal{A}}_J \mathbb{1}_{\boldsymbol{\mathcal{C}}_J} \mathrm{vec}(\mathbf{Y}). \tag{18}$$

  where $\boldsymbol{\mathcal{C}}_J = \{\mathbf{Y} : \mathbf{Y} \in \mathbb{R}^{n \times k}, s_i = \mathbb{1}_{\{i \in J\}}\}$.

- In words: $\boldsymbol{\mathcal{A}}_J$ and $\boldsymbol{\mathcal{C}}_J$ are such that if $F(\mathbf{Y})$ in (18) is invertible, then the linear system is coherent and complete – i.e., that there exists a unique MSV solution, as stipulated in Gourieroux, Laffont, and Monfort (1980). Back to CC Conditions

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[3]For a 2 state transition kernel, $\boldsymbol{K}$ is given by:

$$\boldsymbol{K} = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

where p and q are transition probabilities.

[4]The elements of the transition kernel $\boldsymbol{K}$ are $\boldsymbol{K}_{ij} = \Pr(\boldsymbol{X}_{t+1} = \mathbf{X}\boldsymbol{e}_j | \boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i)$ and hence, $\mathbb{E}_t[\boldsymbol{X}_{t+1} | \boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{X}\boldsymbol{K}^\top \boldsymbol{e}_i$.

Appendix: Calibration

Table: Parameter Values and Description

σ	2	Coefficient of relative risk-aversion
ζ	0.67	Frisch elasticity of labour supply
β	0.99	Discount factor
τ^c	0.2	Steady-state level of fiscal instrument
κ	0.23	NKPC coefficient
Ψ^c	0.25	Coefficient on fiscal instrument
θ	0.75	Calvo probability of changing price
ε	10	Elasticity of substitution between goods
c	0.75	Fraction of consumption in output
g	0.25	Fraction of government spending in output
α_y	0.2	Output gap weight in Social Planner loss function
ϕ_π	1.5	Weight on inflation, Taylor Rule
ϕ_y	0.2	Weight on output gap, Taylor Rule