# Winter Vacation (MT19) Problem Set

## 1 Real business cycle models

*Does the real business cycle (RBC) model predict that real wages should be pro-cyclical or counter cyclical? How about unemployment? Why? What does the empirical evidence say about the direction and magnitude of the fluctuations in these variables in comparison to the model's predictions? What are the implications of labour market developments for interpreting the validity of the RBC framework?*

Consider the following basic RBC model, where the social planner wants to maximise

$$\mathbb{E}_t \left[ \sum_{i=0}^{\infty} \beta^i \left( U(C_{t+i}) - V(L_{t+i}) \right) \right], \tag{1}$$

where $C_t$ is consumption, $L_t$ is hours worked, and $\beta$ is the representative household's rate of time preference. The economy faces constraints described by

$$\begin{aligned} Y_t &= C_t + I_t = F(K_t, L_t), \\ K_{t+1} &= I_t + (1-\delta)K_t, \end{aligned}$$

where $F(K_t, L_t)$ is the production technology of output, $Y_t$, with constant returns to scale, $I_t$ is investment, and $\delta$ is the rate of depreciation of capital.We can simplify the problem by combining the constraints into one equation:

$$F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t. \tag{2}$$

With (1) and (2), we set up the Lagrangian,

$$\begin{aligned} Z = \mathbb{E}_t &\left[ \sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i})) \right] + \lambda_t \left( F(K_t, L_t) + (1-\delta)K_t - C_t - K_{t+1} \right) \\ &+ \beta \mathbb{E}_t \left[ \lambda_{t+1} \left( F(K_{t+1}, L_{t+1}) + (1-\delta)K_{t+1} - C_{t+1} - K_{t+2} \right) \right], \end{aligned}$$

and attain the following FOCs:

$$\begin{aligned} \frac{\partial Z}{\partial C_t} &: U'(C_t) - \lambda_t = 0, \\ \frac{\partial Z}{\partial C_{t+1}} &: \beta \mathbb{E}_t \left[ U'(C_{t+1}) \right] - \beta \mathbb{E}_t \left[ \lambda_{t+1} \right] = 0, \\ \frac{\partial Z}{\partial N_t} &: -V'(L_t) + \lambda_t F_L = 0, \\ \frac{\partial Z}{\partial N_{t+1}} &: -\beta \mathbb{E}_t \left[ V'(L_{t+1}) \right] + \beta \mathbb{E}_t \left[ \lambda_{t+1} F_L \right] = 0, \\ \frac{\partial Z}{\partial K_{t+1}} &: -\lambda_t + \beta \mathbb{E}_t \left[ \lambda_{t+1} \left( F_K + (1-\delta) \right) \right] = 0, \end{aligned}$$

Since markets are competitive, complete, and we have no externalities, we know by the second fundamental theorem of welfare economics that the social planner's allocation is equivalent to the competitive equilibrium. Thus we can first define the marginal product of capital as

$$R_{t+1} = F_K + 1 - \delta,$$

then the FOC for capital can be re-written as

$$\lambda_t = \beta \mathbb{E}_t \left[ \lambda_{t+1} R_{t+1} \right],$$

and this can be combined with the FOC for consumption to give the Keynes-Ramsey condition (the consumption Euler equation):

$$U'(C_t) = \beta \mathbb{E}_t [U'(C_{t+1}) R_{t+1}]. \tag{3}$$

Similarly, the marginal product of labour is equal to the real wage rate:

$$w_t = F_L,$$

and so the intertemporal Euler equation for labour supply is

$$\mathbb{E}_t \left[ V'(L_{t+1}) \right] = \mathbb{E}_t \left[ \lambda_t w_{t+1} \right],$$

and the intratemporal Euler equation for labour supply is

$$\lambda_t w_t = V'(L_t).$$

With the model defined, we can focus back to the question: The model predicts that wages are very pro-cyclical and that hours worked are weakly cyclical. This is due to productivity shocks entering $F(K_t, L_t)$ and directly affecting the marginal product of labour, raising wages (in the case of a positive productivity shock) temporarily. Households respond to a positive productivity shock and higher wages by raising present consumption and labour supply. The higher wages induce an income effect and substitution effect: Households do not have to work as much as they previously did to sustain consumption due to higher wages – however, the model predicts that this is small – but households choose to substitute their leisure time for more hours worked to take advantage of the higher wages to fund higher consumption. Essentially leading to a 'make hay while the sun shines' scenario.

Empirically, the model's predictions are quite inaccurate. In the data, wages are only modestly pro-cyclical and hours worked are actually highly volatile, even more than output. There is evidence which suggests that wage cyclicality is biased downwards in the data, which may alleviate some of the criticisms of the RBC model. However, without modifying the model (e.g. Hansen's indivisible labour), we cannot address the discrepancy between the model and data when it comes to hours worked.

Jordi Gali has criticised the RBC model for failing to explain the labour market response to technology shocks. For example, Gali used VARs to show that in response to a positive productivity shock, hours worked actually declines in contrast to the model's predictions.

The primary issue with the RBC model is that it lacks any sort of propagation mechanism in order to explain what happens to the labour market following a productivity shock. Again, modifying the RBC model by adding things like variable labour utilisation, home production, and habit persistence, boosts the weak propagation mechanism of the model. An alternative, more drastic, approach is to fundamentally change the RBC model by adding nominal rigidities such as sticky prices (e.g. Calvo and Rotemberg pricing) and wages, and market imperfections (e.g. monopolistically competitive firms).

## 2 Ramsey model with government spending

*Consider a Ramsey type neoclassical growth model where the infinitely lived representative agent has preferences given by*

$$\int_0^\infty \exp(-\rho t)\left[\ln C_t + \ln G_t\right] dt,$$

*where $\rho$ is the discount factor, $C_t$ is private consumption, and $G_t$ is government spending. Production in this economy is given by a constant returns to scale production technology satisfying the Inada conditions. Households supply an exogenous amount of labour in every period, and government spending is financed by a lump-sum tax on households. There is no population growth, no technological change, and capital depreciates at the rate $\delta > 0$. Suppose the economy is initially at the steady state and subsequently the government increases spending to a permanently higher level. How will this permanent increase in government spending affect the new steady-state level of output?*

Begin by assuming competitive markets, which allows to write the representative household's problem as

$$\max_{\{C_t\}} \int_0^\infty \exp(-\rho t)[\ln C_t + \ln G_t]dt,$$

subject to

$$\dot{K}_t = w_t L_t + r_t K_t + \Pi_t - C_t - \delta K_t - T_t,$$

and so our current value Hamiltonian is

$$\mathcal{H} = \ln C_t + \ln G_t + \lambda_t\left[w_t L_t + (r_t - \delta)K_t + \Pi_t - C_t - T_t\right],$$

where labour is standardised to unity so $L_t = 1, \forall t$, $r_t$ is the return on capital, $\Pi_t$ is firms' profits (and are zero due to perfect competition), and $T_t$ are lump sum taxes which funds government expenditure. Government expenditure does not affect utility from private consumption; utility is additively separable between private consumption and government expenditure. Therefore, we can take the derivative of the Hamiltonian with respect to the control variable $C_t$ to yield our first FOC:

$$\frac{\partial \mathcal{H}}{\partial C_t} : \frac{1}{C_t} = \lambda_t, \tag{4}$$

and taking the derivative with respect to the state variable $K_t$ yields our second FOC:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial K_t} &: \lambda_t(r_t - \delta) = \rho\lambda_t - \dot{\lambda}_t \\ &\implies \dot{\lambda}_t = \lambda_t(\delta + \rho - r_t).\end{aligned} \tag{5}$$

Differentiating (4) with respect to time yields

$$\dot{\lambda}_t = -\frac{1}{C_t^2}\dot{C}_t,$$

and we can combine this with (5) to yield

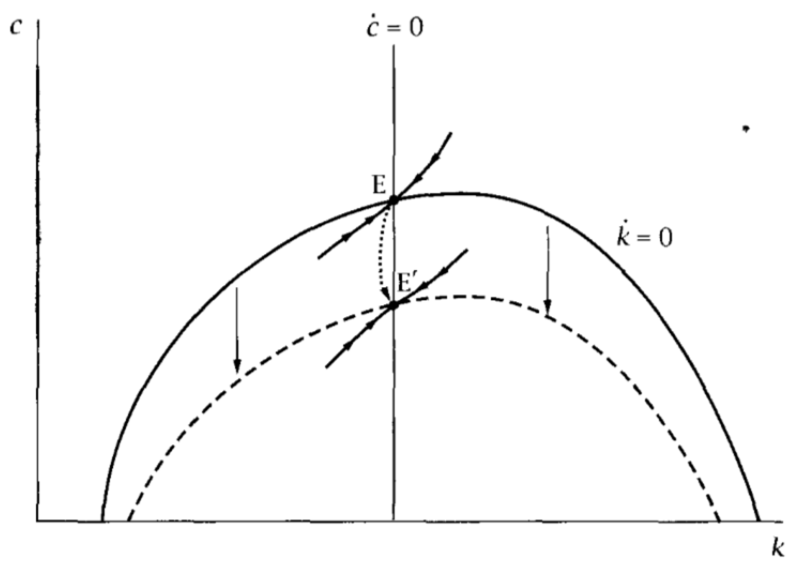
$$
\begin{aligned}
-\frac{1}{C_t^2}\dot{C}_t &= \frac{1}{C_t}(\delta + \rho - r_t) \\
\implies \dot{C}_t &= C_t(r_t - \delta - \rho).
\end{aligned}
$$

Set $\dot{C}_t = 0$ and our law of motion equation $\dot{K}_t = 0$, and we have two equations describing equilibrium dynamics:

$$
\begin{aligned}
\dot{C}_t &= C_t(r_t - \delta - \rho) = 0, \\
\dot{K}_t &= w_t L_t + (r_t - \delta)K_t - C_t - T_t = 0.
\end{aligned}
$$

These dynamics are identical to an economy where government expenditures do not enter the household's utility at all. Thus, any increases in $G_t$ will simply be funded by an increase in $T_t$ and a downward shift in the locus of equilibria dictated by $\dot{K}_t = 0$. That is, the economy will undergo a crowding out affect and household private consumption will fall instantly by the same amount as $G_t$ increases.

Figure 1: The effects of a permanent increase in government expenditure




Note also that here we assumed that the increase in $G_t$ is unanticipated – households do not have an opportunity to smooth their consumption moving from the initial equilibrium $E$ to $E'$ as shown in Figure 1.

# 3 Asset pricing puzzles

*Traditional asset pricing models typically load all uncertainty on the supply side of the economy. For example, in Lucas-tree type models, agents are exposed to endowment shocks, while in production economies total factor productivity is most often the sole driver of exogenous variations. With reference to the fundamental asset pricing equation, explain how the introduction of demand shocks (in the form of stochastic changes to agents' rate of time preference) can help fix asset pricing puzzles.*

For simplicity, let's assume a two-period setup where we wish to value the payoff $A_{t+1}$ at time $t$. Buying an asset today will yield its price plus a dividend, that is

$$A_{t+1} = P_{t+1} + D_{t+1}.$$

Then assume that utility is CRRA or logarithmic, and that total sum utility from consumption in period $t$ and $t+1$ is

$$U(C_t) + \beta \mathbb{E}_t \left[ U(C_{t+1}) \right],$$

and further assume that an agent can or sell $A_{t+1}$ freely at a price $P_t$. The optimal purchase volume of $A_{t+1}$ is given by the following problem

$$\max_{\{b\}} \; U(C_t) + \beta \mathbb{E}_t \left[ U(C_{t+1}) \right],$$

subject to

$$\begin{aligned} C_t &= \alpha_t - P_t b, \\ C_{t+1} &= \alpha_{t+1} + A_{t+1} b, \end{aligned}$$

where $\alpha_t$ is the original consumption level (if the agent bought none of the asset) and $b$ is the amount of the asset the agent chooses to buy. Substituting the constraints into the objective function, differentiating with respect to $b$, and setting the derivative equal to 0 yields

$$\begin{aligned} &\frac{\partial}{\partial b} \left\{ U(\alpha_t - P_t b) + \beta \mathbb{E}_t \left[ U(\alpha_{t+1} + A_{t+1} b) \right] \right\} \\ 0 &= -U'(C_t) P_t + \beta \mathbb{E}_t \left[ U'(C_{t+1}) A_{t+1} \right] \\ \therefore P_t &= \beta \mathbb{E}_t \left[ \frac{U'(C_{t+1})}{U'(C_t)} A_{t+1} \right]. \end{aligned} \tag{6}$$

This consumption-capital asset pricing model is for a simple two-period case, but it can be extended for an infinite horizon case. To see the relevance of changes to agents' rate of time preference, begin by rewriting (6) and assume that utility of logarithmic:

$$P_t = \beta \mathbb{E}_t \left[ \frac{C_t}{C_{t+1}} A_{t+1} \right],$$

then let $g^c$ denote consumption growth between period $t$ and $t+1$, and we know that $\beta = R^{-1}$, so we can derive an expression for the risk free rate, $R$:

$$R = \mathbb{E}_t \left[ \frac{A_{t+1}}{(1+g^c)} \right] = \frac{1}{\beta}.$$

Asset pricing puzzles, such as the equity premium puzzle, highlight the inability of neoclassical economic models to reconcile appropriate risk aversion of agents and returns on assets with different risk profiles. For example, Mehra and Prescott found that neoclassical models cannot explain the discrepancy between returns on 90-day T-bills and US stocks for reasonable calibrations of risk aversion. Their finding was that in order to explain the equity premium, agents would require a relative risk aversion parameter of over 20. In the simple example above, by writing (6) with log utility (and implicitly assuming a coefficient of relative risk aversion of unity), we can explain the equity premium as being closely tied to agents' rate of time preference.

# 4 The AK model

*Robert Solow, in the paper 'Perspectives on Growth Theory' (Journal of Economic Perspectives, 1994), criticises the AK model of endogenous growth by stating:*
*'This version of the endogenous-growth model is very unrobust. It cannot survive without exactly constant returns to capital. But you would have to believe in the tooth fair to expect that kind of luck.'*
*Explain the basic growth mechanism in the AK model, and assess whether Solow is being unfair.*

Begin by deriving the model: Households maximise their present discounted value of utility

$$\int_0^\infty \exp(-\rho t)U(C_t)dt,$$

As in the Ramsey model, the optimality condition is from the consumption Euler equation:

$$\dot{C}_t = C_t(r_t - \rho - \delta),$$

when we assume logarithmic utility. Firm output is given by the following production technology

$$Y_t = f(K_t, A) = AK_t,$$

where $A$ is exogenous technology and the $K_t$ is the capital stock. Thus, firms maximise profits

$$AK_t - rK_t,$$

and therefore the rental rate on capital is equal to the marginal product of capital

$$r_t = A.$$

Thus, the law of motion for capital is

$$\dot{K}_t = AK_t - C_t - \delta K_t.$$

The model does not have a steady state but it has a balanced growth path:

$$\begin{aligned} \frac{\dot{C}_t}{C_t} &= A - \rho - \delta = g, \\ \frac{\dot{K}_t}{K_t} &= A - \frac{C_t}{K_t} - \delta = g \\ \implies \frac{C_t}{K_t} &= A - \delta - g. \end{aligned}$$

Fundamental to the AK model is that $K$ embodies both physical and human capital, and thus firms' production technology is a special case of the Cobb-Douglas production function $Y_t = AK_t^\alpha L_t^{1-\alpha}$ where $\alpha = 1$. Hence, the model relies on constant returns to scale production. The fact that the return on capital is now a constant, $A$, eliminates any potential for there being transition dynamics.

In assessing Solow's critique of the AK model, we first need to recognise its strengths. The AK model can first account for persistent and positive growth rates of output (we could also show positive growth rates for output per capita) – something that we observe empirically, and a feature which neoclassical models cannot address. Where the AK model struggles is with its inability to explain convergence, which is another important empirical observation. Neoclassical models such as the Solow-Swan and Ramsey model predict that economies with lower GDP and capital per-capita levels undergo rapid growth initially, before plateauing out as the economy reaches its steady state. The main mechanism for this convergence dynamic is the fact that production in neoclassical model has constant returns to scale in capital and labour together. In other words, the further an economy is away from its steady state, the faster it grows as it has a relatively high marginal product of capital.

Now consider the case of two geographically proximate economies with similar characteristics (e.g. states in the US), such as similar population growth, savings, and depreciation rates. Suppose that one economy had lower GDP and capital per-capita than the other. Under the AK model and constant returns to scale, the two economies would never converge to one another as they grow at exogenous rates. Whereas under a neoclassical model, the two economies would tend toward their steady states – which is something that we observe in the data.

So is Solow being fair in his critique of AK models? Perhaps. AK models can explain long run growth, but can't say much for convergence. However, neoclassical models can explain convergence but do not have a convincing story for long run technological growth. As I previously mentioned, the fact that the AK model lumps physical and human capital together under a catch-all capital term is the model's primary weakness.