Introduction to Neoclassical Growth Models

References

"The ABCs of RBCs: an introduction to dynamic macroeconomic models", McCandless, G., *Harvard University Press*, 2008.

1 General equilibrium model with constant relative risk aversion utility

Recall that we showed that the market interest rate in the GE model with heterogenous endowments is equal to the market interest rate in a GE model with a representative agent. The GE result assumed that utility was logarithmic in consumption. Will the market rate of interest be the same in GE models with and without heterogeneity if per-period preferences are of the CRRA form?

We wish to show if that the market rate of interest be the same as in GE models with and without heterogeneity if utility is of the CRRA form

$$U(C_t^i) = \frac{(C_t^i)^{1-\sigma}}{1-\sigma}.$$

To clean up the notation, for an ith individual, their utility maximisation problem is

$$\max_{\{C_{t+s}, A_{t+s+1}\}} \sum_{s=0}^{\infty} \beta^s \frac{C_{t+s}^{1-\sigma}}{1-\sigma},$$

subject to

$$A_{t+s+1} = R_{t+s} A_{t+s} + Y_{t+s} - C_{t+s}, \ \forall s \geq 0,$$

and the no-Ponzi condition:

$$\lim_{T\to\infty} \prod_{s=t}^{T} \frac{A_T}{R_s} \geq 0.$$

The Lagrangian for this problem can be written as the following

$$Z = \sum_{s=0}^{\infty} \beta^s U(C_{t+s}) + \sum_{s=0}^{\infty} \lambda_{t+s} \beta^s (Y_{t+s} + R_t A_{t+s} - A_{t+s+1} - C_{t+s}),$$

and the FOCs are

$$\frac{\partial Z}{\partial C_t} : U'(C_t) = \lambda_t \tag{1}$$

$$\frac{\partial Z}{\partial C_{t+1}} : \beta U'(C_{t+1}) = \beta \lambda_{t+1} \tag{2}$$

$$\frac{\partial Z}{\partial A_{t+1}} : \beta \lambda_{t+1} R_{t+1} - \lambda_t = 0. \tag{3}$$

Then substitute in the expressions for λ_t and λ_{t+1} from (1) and (2) into (3) to get the Keynes-Ramsey condition:

$$\beta U'(C_{t+1})R_{t+1} - U'(C_t) = 0$$
$$\implies R_{t+1} = \frac{U^{'}(C_t)}{\beta U'(C_{t+1})}.$$

In the case of log utility, $U(C_t) = \log C_t$, which then gives

$$R_{t+1} = \frac{1}{C_t} / \frac{\beta}{C_{t+1}} = \frac{1}{C_t} \frac{C_{t+1}}{\beta}$$
$$R_{t+1} = \frac{C_{t+1}}{\beta C_t}. \tag{4}$$

In the case of CRRA utility, $U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$, and $\frac{\partial}{\partial C_t} U(C_t) = U'(C_t) = \frac{1}{C_t^{-\sigma}}$, we have

$$R_{t+1} = \frac{1}{C_t^{\sigma}} / \frac{\beta}{C_{t+1}^{\sigma}} = \frac{1}{C_t^{\sigma}} \frac{C_{t+1}^{\sigma}}{\beta}$$
$$R_{t+1} = \frac{C_{t+1}^{\sigma}}{\beta C_t^{\sigma}}. \tag{5}$$

Now, for a slight numerical example, suppose $C_t = 3$, $\sigma = 0.5$, and $\beta = 0.95$, then

$$\text{log case: } R_{t+1} = \frac{3}{2.85} = 1.0526,$$
$$\text{CRRA case: } R_{t+1} = \frac{1.7321}{1.6455} = 1.0526.$$

For the heterogeneous case, the market level of interest is the same between log utility and CRRA utility. It's worth remembering that the two are equal if $\sigma = 1$.

Proof:

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma},$$

and applying L'Hopital's Rule

$$\lim_{\sigma \to 1} U(C_t) = \frac{0}{0}$$
$$\implies \lim_{\sigma \to 1} U(C_t) = \frac{\frac{\partial}{\partial \sigma} C_t^{1-\sigma}}{\frac{\partial}{\partial \sigma}(1-\sigma)} = \frac{C_t^{1-\sigma}(-1)\ln C_t}{-1}$$
$$= \ln C_t$$

■

Moving to a general equilibrium setup, there should be no difference between log or

CRRA preferences. Market rates are equal, so standard Walrasian assumptions hold:

$$\sum_{i=1}^{N} A_t^i = 0$$
$$\implies \sum_{i=1}^{N} Y_t^i = \sum_{i=1}^{N} C_t^i,$$

assuming that all households have identical CRRA preferences (via Gorman's Aggregation Theorem).

Theorem (Gorman's Aggregation Theorem): Consider an economy with finite number $N < \infty$ of commodities and set H of households. Suppose that the preference of house $i \in H$ can be represented by an indirect utility function of the form:

$$V^i(P, y^i) = a^i(P) + b(P)y^i,$$

then those preferences can be aggregated and represented by those of a representative agent with indirect utility:

$$V(P, y) = a(P) + b(P)y,$$

where $a(P) \equiv \int_{i \in H} a^i(P)\ di$ and $y \equiv \int_{i \in H} y^i\ di$ is aggregate income. Demand for good j (from Roy's Identity):

$$x_j^i(P, y^i) = \frac{-1}{b(P)} \frac{\partial a^i(P)}{\partial P_j} - \frac{1}{b(P)} \frac{\partial b(P)}{\partial P_j} y^i$$

implies linear Engel curves. We say that there exists a strong representative household if redistribution income or endowments across households does not affect the demand side. Gorman preferences are sufficient for a strong representative household.

2 GE model with alternative endowments

Consider a version of the simple GE endowment model from the first lecture in which there are two types of agents. Agents of Type 1 receive an endowment of $\bar{e}$ in even periods (starting in periods 0) and receive nothing in odd periods. Agents of Type 2 receive nothing in event periods (starting in period 0) and receive $\bar{e}$ in odd periods. There is borrowing and lending – in periods 0 an agent of Type 2 will have an incentive to borrow from an agent of Type 1 and repay them in periods 1. In period 1 the roles are reversed as an agent of Type 1 will want to borrow fro man agent of Type 2 and repay them in period 2.

This question asks you to work out the equilibrium rate of interest and consumption path of each type of agent in this economy. We assume that there is a continuum of unit mass of each type of agent, so that exactly half of all agents are of Type 1 and half of Type 2. Since there are infinitely many agents of each type in each continuum they take the market rate of interest as given and out of their control. Assume that the preferences of each type of agent are logarithmic in consumption and that the discount factor is β. Denote the market rate of interest in period t by r_t, and the consumption of Type 1 and Type 2 agents in period t by C_t^1 and C_t^2, respectively. Use the notation A_{t+1}^1 and A_{t+1}^2 to represent the borrowing of agents of Type 1 and Type 2 in period t that is repaid with interest in period $t+1$.

2.1

Set up the optimisation problem of each agent in period 0, assuming that they start with no assets $A_0^1 = A_0^2 = 0$ and that they take the current and future path of the market interest rate as given. Solve for the consumption Euler equation of each type of agent and comment on the dynamics of consumption across time. What is the key economic mechanism driving consumption behaviour?

The maximisation problem for Type 1 households are

$$\max_{\{A_{t+s+1}^1, C_{t+s}^1\}} \sum_{s=0}^{\infty} \beta^s U(C_{t+s}^1),$$

subject to

$$A_{t+s+1}^1 = Y_{t+s}^1 + R_{t+s} A_{t+s}^1 - C_{t+s}^1,$$
$$Y_{t+s}^1 = \begin{cases} \bar{e} & \text{when } t+s \text{ is odd,} \\ 0 & \text{when } t+s \text{ is even.} \end{cases}$$

For Type 2 households:

$$\max_{\{A_{t+s+1}^2, C_{t+s}^2\}} \sum_{s=0}^{\infty} \beta^s U(C_{t+s}^2),$$

subject to

$$
\begin{aligned}
A^2_{t+s+1} &= Y^2_{t+s} + R_{t+s}A^2_{t+s} - C^2_{t+s}, \\
Y^2_{t+s} &= \begin{cases} \bar{e} & \text{when } t+s \text{ is even,} \\ 0 & \text{when } t+s \text{ is odd.} \end{cases}
\end{aligned}
$$

Since the forms are identical, we can solve and get FOCs as we usually do. For household i:

$$
Z = \sum_{s=0}^{\infty} \beta^s U(C^i_{t+s}) + \sum_{s=0}^{\infty} \beta^s \lambda_{t+s}(Y^i_{t+s} + R_{t+s}A^i_{t+s} - C^i_{t+s} - A^i_{t+s+1}),
$$

$$
\frac{\partial Z}{\partial C^i_t} : U'(C^i_t) = \lambda_t \tag{6}
$$

$$
\frac{\partial Z}{\partial C^i_{t+1}} : \beta U'(C^i_{t+1}) = \beta \lambda_{t+1} \tag{7}
$$

$$
\frac{\partial Z}{\partial A^i_{t+1}} : \beta \lambda_{t+1} R_{t+1} = -\lambda_t. \tag{8}
$$

Again, using (6) and (7) to get the Keynes-Ramsey condition from (8):

$$
\begin{aligned}
\beta U'(C^i_{t+1})R_{t+1} - U'(C^i_t) = 0 \\
\implies U'(C^i_t) = \beta U'(C^i_{t+1})R_{t+1}.
\end{aligned}
$$

And since consumption is logarithmic, we have

$$
\begin{aligned}
\frac{1}{C^i_t} &= \frac{\beta R_{t+1}}{C^i_{t+1}} \\
\implies C^i_{t+1} &= \beta R_{t+1} C^i_t.
\end{aligned} \tag{9}
$$

The key mechanism here is consumption smoothing. Future consumption is growing if $\beta R_{t+1} > 1$, and shrinking $\beta R_{t+1} < 1$. Of course, if $\beta R_{t+1} = 1$ then consumption between period t and $t+1$ is constant.

2.2

State the market clearing conditions and use them and the consumption Euler equation of each type to derive the market interest rate. What does this imply for the dynamics of consumption for each type?

We derived the consumption Euler equations for each household type in the previous question. It's worth stating the market clearing conditions:

$$
\begin{aligned}
C^1_{t+s} + C^2_{t+s} = Y^1_{t+s} + Y^2_{t+s} \leftrightarrow \\
C^1_t + C^2_t = \bar{e}, \\
C^1_{t+1} + C^2_{t+1} = \bar{e}.
\end{aligned}
$$

So we can say

$$C_t^1 + C_t^2 = C_{t+1}^1 + C_{t+1}^2,$$

and substituting in our expression from (9) yields

$$\begin{aligned} C_t^1 + C_t^2 &= \beta R_{t+1}(C_t^1 + C_t^2) \\ \implies 1 &= \beta R_{t+1} \\ \implies \bar{R} &= \frac{1}{\beta}. \end{aligned}$$

So what do we have? For this particular setup, the market interest rate is constant, the gross interest rate is greater than unity (since $\beta < 1$), and it is decreasing in β, the subjective discount factor. Furthermore, if we substitute $\bar{R}$ back into (9), we get:

$$C_{t+1}^i = C_t^i.$$

Thus, for the special case of log utility, consumption in each period, for a particular individual, is constant.

Finally, it is worth noting that even though the form of utility is the same for Type 1 and Type 2 households, nothing guarantees that $C_t^1 = C_t^2$ nor can we say that $C_{t+s}^1 = C_{t+s}^2$. What is equal, if we look at our FOCs, is that the marginal rates of consumption are equal for the different types.

2.3

Show that the present value budget constraints of each agent in period 0 take the following forms:

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{C_t^1}{R^t} &= \sum_{t=0}^{\infty} \frac{\bar{e}}{R^{2t}}, \\ \sum_{t=0}^{\infty} \frac{C_t^2}{R^t} &= \sum_{t=0}^{\infty} \frac{\bar{e}}{R^{2t+1}}, \end{aligned}$$

where $R = (1 + r)$ is the [gross] market interest rate derived in part 2. Explain carefully how you use the transversality condition and the value of initial assets in your calculation.

Begin by rolling forward the budget constraint. For Type 1 households this is:

$$A_{t+1}^1 = Y_t^1 + RA_t^1 - C_t^1, \tag{10}$$

$$A_{t+2}^1 = Y_{t+1}^1 + RA_{t+1}^1 - C_{t+1}^1, \tag{11}$$

$$A_{t+3}^1 = Y_{t+2}^1 + RA_{t+2}^1 - C_{t+2}^1, \tag{12}$$

an assuming we start at $t = 0$, doing recursive substitution yields the following

$$\begin{aligned}
A_1^1 &= RA_0^1 + Y_0^1 - C_0^1, \\
A_2^1 &= RA_1^1 + Y_1^1 - C_1^1 \\
&= R\left(Y_0^1 + RA_0^1 - C_0^1\right) + Y_1^1 - C_1^1, \\
A_3^1 &= RA_2^1 + Y_2^1 - C_2^1 \\
&= R\left(R\left(Y_0^1 + RA_0^1 - C_0^1\right) + Y_1^1 - C_1^1\right) + Y_2^1 - C_2^1, \\
&\vdots
\end{aligned}$$

and eventually we would get

$$\lim_{t\to\infty} \frac{A_t^1}{R^t} = A_0^1 + \frac{1}{R}\sum_{t=0}^{\infty} \frac{Y_t^1 - C_t^1}{R^t}. \tag{13}$$

The no-Ponzi condition would imply that the LHS of (13) would be equal to 0. In other words, we do not allow the household to keep borrowing into the next period indefinitely – or that borrowing increases at a rate higher than the rate of interest. If we assume that $A_0^1 = 0$, then we can eliminate it from the RHS of (13). After splitting the final term on the RHS, we are left with

$$\sum_{t=0}^{\infty} \frac{Y_t^1}{R^t} = \sum_{t=0}^{\infty} \frac{C_t^1}{R^t}.$$

By symmetry, we can do a similar exercise for the Type 2 household:

$$\sum_{t=0}^{\infty} \frac{Y_t^2}{R^t} = \sum_{t=0}^{\infty} \frac{C_t^2}{R^t},$$

and we can generalise for the two types:

$$\sum_{t=0}^{\infty} \frac{\bar{e}}{R^t} = \sum_{t=0}^{\infty} \frac{C_t^i}{R^t}.$$

We are almost ready to answer the question. Now, consider the alternating nature of the two household types. We know that with log utility, $C_t^1 = C_{t+1}^1 = \cdots = \bar{C}^1$ and, by symmetry, $C_t^2 = \bar{C}^2$ $\forall t$. Further, we know that Type 1 households receive endowment $\bar{e}$ in even periods ($t = 0, 2, 4, ...$), so we can rewrite their present value budget constraint as

$$\sum_{t=0}^{\infty} \frac{\bar{e}}{R^{2t}} = \sum_{t=0}^{\infty} \frac{\bar{C}^1}{R^t}, \tag{14}$$

and we know that Type 2 households receive an endowment in odd periods ($t = 1, 3, 5, ...$), so their budget constraint can be written as

$$\sum_{t=0}^{\infty} \frac{\bar{e}}{R^{2t+1}} = \sum_{t=0}^{\infty} \frac{\bar{C}^2}{R^t}. \tag{15}$$

This completes our answer.

2.4

Substitute out for C_t^1 and C_t^2 in the present value budget constraints by using the consumption Euler equations derived in part 1. Obtain the consumption allocation of each type of agent in period 0 and check that markets clear. Interpret any asymmetry in the allocation for period 0.

We actually partially did this in part 3. From (14) and (15) we can pull out $\bar{e}$ and $\bar{C}^i$ from the summation terms as they are constants:

$$\begin{aligned}
\bar{e}\sum_{t=0}^{\infty} R^{-2t} &= \bar{C}^1 \sum_{t=0}^{\infty} R^{-t} \\
\implies \bar{e}\frac{1}{1-R^{-2}} &= \bar{C}^1 \frac{1}{1-R^{-1}},
\end{aligned}$$

and

$$\begin{aligned}
\bar{e}\sum_{t=0}^{\infty} \frac{1}{R^{2t+1}} &= \bar{C}^2 \sum_{t=0}^{\infty} \frac{1}{R^t} \\
\frac{\bar{e}}{R}\sum_{t=0}^{\infty} \frac{1}{R^{2t}} &= \sum_{t=0}^{\infty} \frac{\bar{C}^2}{R^t} \\
\implies \frac{\bar{e}}{R}\frac{1}{1-R^{-2}} &= \bar{C}^2 \frac{1}{1-R^{-1}}.
\end{aligned}$$

We now need to check that markets clear: $\bar{C}^1 + \bar{C}^2 = \bar{e}$.[1] So we need to attain expressions for $\bar{C}^1$ and $\bar{C}^2$ (this is going to be rough):

$$\begin{aligned}
\bar{C}^1 &= \frac{\bar{e}(1-R^{-1})}{1-R^{-2}} \\
&= \frac{\bar{e}-\bar{e}R^{-1}}{1-R^{-2}}
\end{aligned}$$

[1] Note that we could swap $\bar{C}^i$ for C_0^i here. The results are the same.

multiply numerator and denominator on the RHS by R

$$
\begin{aligned}
&= \frac{\bar{e}R - \bar{e}}{R - R^{-1}} \\
&= \bar{e}\frac{R-1}{R-\frac{1}{R}} \\
&= \bar{e}\frac{R-\frac{R}{R}}{\frac{R^2}{R}-\frac{1}{R}} \\
&= \bar{e}\frac{\frac{R^2}{R}-\frac{R}{R}}{\frac{R^2}{R}-\frac{1}{R}} \\
&= \bar{e}\frac{R^2-R}{R}\Big/\frac{R^2-1}{R} \\
&= \bar{e}\frac{(1+r)^2-(1+r)}{1+r}\Big/\frac{(1+r)^2-1}{1+r} \\
&= \bar{e}\frac{1+2r+r^2-1-r}{1+r}\Big/\frac{1+2r+r^2-1}{1+r} \\
&= \bar{e}\frac{2r+r^2-r}{1+r}\Big/\frac{2r+r^2}{1+r} \\
&= \bar{e}\frac{2r+r^2-r}{2r+r^2} \\
&= \bar{e}\frac{2+r-1}{2+r} \\
\therefore \bar{C}^1 &= \bar{e}\left(\frac{1+r}{2+r}\right),
\end{aligned}
$$

and for $\bar{C}^2$:

$$
\begin{aligned}
\bar{C}^2 &= \frac{\bar{e}}{R}\frac{1-R^{-1}}{1-R^{-2}} \\
&= \bar{e}\frac{1-R^{-1}}{R-R^{-1}} \\
&= \bar{e}\frac{R-1}{R}\Big/\frac{R^2-1}{R} \\
&= \bar{e}\frac{1+r-1}{1+r}\Big/\frac{1+2r+r^2-1}{1+r} \\
&= \bar{e}\frac{r}{1+r}\Big/\frac{2r+r^2}{1+r} \\
&= \bar{e}\frac{r}{2r+r^2} \\
\therefore \bar{C}^2 &= \bar{e}\left(\frac{1}{2+r}\right).
\end{aligned}
$$

Now to check market clearing:

$$
\begin{aligned}
\bar{C}^1 + \bar{C}^2 &= \bar{e} \\
\bar{e}\left(\frac{1+r}{2+r}\right) + \bar{e}\left(\frac{1}{2+r}\right) &= \bar{e} \\
\bar{e}\left(\frac{2+r}{2+r}\right) &= \bar{e},
\end{aligned}
$$

therefore, we have verified that markets do indeed clear. What about the timing and asymmetry between the two types of households?

Well, given that $r > 0$ ($R > 1$), then we can presume that $C_0^1 > C_0^2$. Since markets start in period 0, Type 1 households start with endowments and thus start with more wealth than Type 2 households. This is a wealth effect, and market equilibrium tilts in the favour of Type 1 households.

Note also that since $R = \frac{1}{\beta}$, we can rewrite the consumption allocations as

$$
\begin{aligned}
\bar{C}^1 &= \bar{e}\frac{1}{1+\beta} \\
\bar{C}^2 &= \bar{e}\frac{\beta}{1+\beta},
\end{aligned}
$$

from which the role of the discount factor becomes clear. The more impatient that consumers are (lower β), the higher is the consumption of Type 1 agent relative to the Type 2. It makes sense as having an endowment in period 0 is very valuable if agents are impatient.

2.5

What is the consumption allocation across types in period 1? How would you feel about this if you were a consumer of Type 2? What key economic mechanism does this demonstrate?

Due to perfection consumption smoothing (and the ability to borrow and lend), consumption patterns in period 1 are the same as period 0. i.e. Type 1 households continue to have higher consumption than those of Type 2. This occurs because there is a commitment to repay debts that were taken on in period 0. If agents were able to renege on debts in period 1, then those of Type 2 would want to re-optimise. It's an illustration of the time inconsistency of the optimal plan that was decided at time 0.

3 GE model with alternating labour productivity

We now develop the setup of the model in the preceding section. Instead of agents having endowments that alternate between periods, we assume that the productivity of agents alternates between periods. In even periods (starting in period 0) agents of Type 1have a productivity of $\bar{\theta}$, which means that for every unit of time they spend working they produce $\bar{\theta}$ of the consumption good. In odd periods the agents of Type 1 have zero productivity and so produce nothing irrespective of how much time they spend working. Agents of Type 2 have the opposite labour productivity, produce $\bar{\theta}$ per unit time in odd periods and nothing per unit time in even periods. If labour supply is exogenous then this environment is isomorphic to that in the previous question with $\bar{\theta} = \bar{e}$ so we know what will happen. To make things more interesting, you are asked to calculate the equilibrium if agents endogenously choose their labour supply. It should be obvious that agents will never supply labour in periods during in which they have no productivity, but how much labour will they supply when they are productive? This question asks you to calculate GE consumption and labour supply of each type of agent in periods when they are productive and periods when they are not. It's a tough question so give it your best shot.

The optimisation problem of an agent of Type i at time 0 is as follows:

$$\max_{\{C^i_{t+s}, L^i_{t+s}, A^i_{t+s+1}\}} \sum_{s=0}^{\infty} \beta^s \left[\log C^i_{t+s} + \log(1 - L^i_{t+s})\right]$$

subject to

$$A^i_{t+s+1} = R_t A^i_{t+s} + \theta^i_{t+s} L^i_{t+s} - C^i_{t+s},$$

where

$$\theta^1_{t+s} = \begin{cases} \bar{\theta} & \text{when } s \text{ is even,} \\ 0 & \text{when } s \text{ is odd,} \end{cases}$$

and

$$\theta^2_{t+s} = \begin{cases} \bar{\theta} & \text{when } s \text{ is odd,} \\ 0 & \text{when } s \text{ is even,} \end{cases}$$

and not that total possible labour supply is normalised to 1 so $L^i_t \in [0, 1)$.

3.1

Derive the first order conditions of an agent of Type i. You should obtain an intertemporal Euler equation for consumption that holds in every period and an intratemporal condition linking the marginal utility of consumption to the marginal disutility that applies in periods when the agent has positive labour productivity. In periods where an agent has no productivity the intratemporal condition will be replaced by the corner solution $L^i_t = 0$.

We can tackle this problem in different ways (direct substitution, Lagrangian, or dynamic programming). Let's setup the Lagrange as

$$Z = \sum_{s=0}^{\infty} \beta^s \left[\log C^i_{t+s} + \log(1 - L^i_{t+s})\right] + \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} \left(R_t A^i_{t+s} + \theta^i_{t+s} L^i_{t+s} - C^i_{t+s} - A^i_{t+s+1}\right),$$

which will give us the following FOCs:

$$\frac{\partial Z}{\partial C_t^i} : \frac{1}{C_t^i} - \lambda_t = 0 \tag{16}$$

$$\frac{\partial Z}{\partial C_{t+1}^i} : \beta \frac{1}{C_{t+1}^i} - \beta \lambda_{t+1} = 0 \tag{17}$$

$$\frac{\partial Z}{\partial L_t^i} : -\frac{1}{1 - L_t^i} + \lambda_t \theta_t^i = 0 \tag{18}$$

$$\frac{\partial Z}{\partial A_{t+1}^i} : -\lambda_t + \beta \lambda_{t+1} R_{t+1} = 0. \tag{19}$$

Using our FOCs we can first get the Keynes-Ramsey condition by combining (16), (17), and (19):

$$\lambda_t = \frac{1}{C_t^i} \implies \frac{1}{C_t^i} = \beta \lambda_{t+1} R_{t+1},$$

$$\implies \frac{1}{C_t^i} = \beta R_{t+1} \frac{1}{C_{t+1}^i}$$

$$\therefore C_{t+1}^i = \beta R_{t+1} C_t^i. \tag{20}$$

The intertemporal condition is derived from (16) and (18):

$$\lambda_t \theta_t^i = \frac{1}{1 - L_t^i}$$

$$\implies \frac{\theta_t^i}{C_t^i} = \frac{1}{1 - L_t^i}. \tag{21}$$

3.2

Introduce the notation $\tilde{R}_t = (1 + r_0)(1 + r_1)...(1 + r_{t+1})$ for $t > 0$ and assume no initial assets and suitable transversality condition. Show that the Euler equation for consumption and the budget constraint can be written in the following forms:

$$C_t^i = \beta^t \tilde{R}_t C_0^i \tag{22}$$

$$\sum_{t=0}^{\infty} \frac{\theta_t^i L_t^i}{\tilde{R}_t} = \frac{C_0^i}{1 - \beta}. \tag{23}$$

Begin with (20) and assume $t = 0$:

$$C_1^i = \beta R_1 C_0^i,$$

and then roll one period forward:

$$C_2^i = \beta R_2 C_1^i,$$

now substitute C_1^i in the above equation to get

$$C_2^i = \beta R_2 \beta R_1 C_0^i$$
$$= \beta^2 R_1 R_2 C_0^i,$$

which we can generalise for any t as

$$C_t^i = \beta^t \tilde{R}_t C_0^i,$$

which is (22). Now, look at the budget constrain at period 0:

$$A_1^i = R_0 A_0^i + \theta_0^i L_0^i - C_0^i,$$

and then roll one period forward:

$$\begin{aligned} A_2^i &= R_1 A_1^i + \theta_1^i L_1^i - C_1^i \\ &= R_1 \left[R_0 A_0^i + \theta_0^i L_0^i - C_0^i \right] + \theta_1 L_1^i - C_1^i, \end{aligned}$$

and again:

$$\begin{aligned} A_3^i &= R_2 A_2^i + \theta_2^i L_2^i - C_2^i \\ &= R_2 \left[R_1 \left[R_0 A_0^i + \theta_0^i L_0^i - C_0^i \right] + \theta_1 L_1^i - C_1^i \right] + \theta_2^i L_2^i - C_2^i, \end{aligned}$$

which we can write as

$$\lim_{T \to \infty} \frac{A_T^i}{\tilde{R}_T} = A_0^i + \sum_{t=0}^{\infty} \frac{\theta_t^i L_t^i}{\tilde{R}_t} - \sum_{t=0}^{\infty} \frac{C_t^i}{\tilde{R}_t}.$$

Now, recall that initial asset holdings are 0, so $A_0^i = 0$, and consider that the transversality condition states that at the limit, there should be no borrowing or lending beyond the final period, T, so $A_T = 0$. We then get

$$\sum_{t=0}^{\infty} \frac{\theta_t^i L_t^i}{\tilde{R}_t} = \sum_{t=0}^{\infty} \frac{C_t^i}{\tilde{R}_t},$$

and we can simply substitute in (22) to get

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{\theta_t^i L_t^i}{\tilde{R}_t} &= \sum_{t=0}^{\infty} \frac{\beta^t \tilde{R}_t C_0^i}{\tilde{R}_t} \\ &= C_0^i \sum_{t=0}^{\infty} \beta^t \\ &= \frac{C_0^i}{1-\beta}, \end{aligned}$$

which is simply (23).

3.3

We now derive the market equilibrium when t is an even period. What will total output be in period t as a function of consumption, given that only agents of Type 1 will be working? Now introduce market clearing in period t such that total output in period t is equal to total consumption $C_t^1 + C_t^2$ in period t. Use the Euler equation from (22) to re-write the *market clearing condition in terms of consumption at time 0 (hint: 0 is an even number) and hence show that the market interest rate in even periods must satisfy* $\tilde{R}_t = \beta^{-t}$.

From our FOCs, we have the intratemporal condition (18):

$$\frac{\theta_t^i}{C_t^i} = \frac{1}{1 - L_t^i},$$

and if we're looking at Type 1 households during even periods, this condition becomes:

$$\begin{aligned} \frac{\bar{\theta}}{C_t^1} &= \frac{1}{1 - L_t^1} \\ \implies L_t &= \frac{\bar{\theta} - C_t^1}{\bar{\theta}}. \end{aligned}$$

Since $\bar{\theta}$ is a productivity term, total output will be the product of $\bar{\theta}$ and L_t^1, and simply $\bar{\theta} - C_t^1$. We know that market clearing requires total output to be equal to total consumption, which we write as

$$\bar{\theta} - C_t^1 = C_t^1 + C_t^2.$$

We can collect terms

$$\bar{\theta} = 2C_t^1 + C_t^2$$

and substitute in our Keynes-Ramsey condition (22):

$$\begin{aligned} \bar{\theta} &= 2\beta^t \tilde{R}_t C_0^1 + \beta^t \tilde{R}_t C_0^2 \\ &= \beta^t \tilde{R}_t (2C_0^1 + C_0^2). \end{aligned}$$

But we note that in even periods, such as $t = 0$, $\bar{\theta} = 2C_0^1 + C_0^2$, and so

$$\begin{aligned} 2C_0^1 + C_0^2 &= \beta^t \tilde{R}_t (2C_0^1 + C_0^2) \\ \therefore \tilde{R}_t &= \frac{1}{\beta^t}, \ \forall t = 0, 2, 4, ... \end{aligned}$$

3.4

Continuing with the analysis when t *is an even period, solve the present discounted value budget constraint (23) of a agent in Type 1 at time 0 to show that in an even period*

$$\begin{aligned} C_t^1 &= \frac{\bar{\theta}}{2 + \beta} \\ L_t^1 &= \frac{1 + \beta}{2 + \beta}. \end{aligned}$$

Use what you know about total output and market clearing in period 0 to show that the consumption and labour supply of Type 2 agent in an even period is:

$$\begin{aligned} C_t^2 &= \frac{\beta \bar{\theta}}{2 + \beta} \\ L_t^2 &= 0. \end{aligned}$$

We know from (23) that

$$\sum_{t=0}^{\infty} \frac{\theta_t^i L_t^i}{\tilde{R}_t} = \frac{C_0^i}{1-\beta},$$

and that for a Type 1 agent in even periods, this can be written as

$$\sum_{t=0}^{\infty} \frac{\bar{\theta} L_t^1}{\tilde{R}_t} = \frac{C_0^1}{1-\beta},$$

and from market clearing conditions we know that total output $\bar{\theta} L_t^1 = \bar{\theta} - C_t^1$, so

$$\sum_{t=0}^{\infty} \frac{\bar{\theta} - C_t^1}{\tilde{R}_t} = \frac{C_0^1}{1-\beta},$$

and we know that $C_t^1 = \beta^t \tilde{R}_t C_0^1$, so

$$\begin{aligned}
\sum_{t=0}^{\infty} \frac{\bar{\theta} - \beta^t \tilde{R}_t C_0^1}{\tilde{R}_t} &= \frac{C_0^1}{1-\beta} \\
\sum_{t=0}^{\infty} \left(\frac{\bar{\theta}}{\tilde{R}_t} - \frac{\beta^t \tilde{R}_t C_0^1}{\tilde{R}_t} \right) &= \frac{C_0^1}{1-\beta} \\
\sum_{t=0}^{\infty} \left(\frac{\bar{\theta}}{\tilde{R}_t} - \beta^t C_0^1 \right) &= \frac{C_0^1}{1-\beta},
\end{aligned}$$

and we know that $\tilde{R}_t = \beta^t$, so

$$\begin{aligned}
\sum_{t=0}^{\infty} \left(\beta^t \bar{\theta} - \beta^t C_0^1 \right) &= \frac{C_0^1}{1-\beta} \\
\sum_{t=0}^{\infty} \beta^t \left(\bar{\theta} - C_0^1 \right) &= \frac{C_0^1}{1-\beta}.
\end{aligned}$$

It may seem that we're stuck, however recall that we're only looking at even periods, so we can use a trick and index β on the LHS by $2t$:

$$\sum_{t=0}^{\infty} \beta^{2t} \left(\bar{\theta} - C_0^1 \right) = \frac{C_0^1}{1-\beta},$$

which allows us to simplify the summation term

$$\begin{aligned}
\frac{\bar{\theta} - C_0^1}{1-\beta^2} &= \frac{C_0^1}{1-\beta} \\
\frac{\bar{\theta}}{1-\beta^2} &= \frac{C_0^1}{1-\beta} + \frac{C_0^1}{1-\beta^2} \\
\bar{\theta} &= \frac{C_0^1(1-\beta^2)}{1-\beta} + C_0^1,
\end{aligned}$$

and doing some tedious algebra gives

$$
\begin{aligned}
C_0^1 &= \frac{\bar{\theta}}{\frac{1-\beta^2}{1-\beta} + 1} \\
C_0^1 &= \frac{\bar{\theta}}{\frac{1-\beta^2}{1-\beta} + \frac{1-\beta}{1-\beta}} \\
C_0^1 &= \frac{\bar{\theta}}{\frac{2-\beta^2-\beta}{1-\beta}} \\
C_0^1 &= \frac{\bar{\theta}}{\frac{(2+\beta)(1-\beta)}{(1-\beta)}} \\
C_0^1 &= \frac{\bar{\theta}}{2+\beta}. \qquad (24)
\end{aligned}
$$

Put this expression for Type 1 initial consumption into the intratemporal condition:

$$
\begin{aligned}
\frac{\bar{\theta}}{C_t^1} &= \frac{1}{1-L_t^1} \\
\implies \frac{\bar{\theta}}{\frac{\bar{\theta}}{2+\beta}} &= \frac{1}{1-L_0^1} \\
2+\beta &= \frac{1}{1-L_0^1} \\
1-L_0^1 &= \frac{1}{2+\beta} \\
L_0^1 &= 1 - \frac{1}{2+\beta},
\end{aligned}
$$

and again applying some algebraic manipulation we get

$$
\begin{aligned}
L_0^1 &= \frac{2+\beta}{2+\beta} - \frac{1}{2+\beta} \\
\therefore L_0^1 &= \frac{1+\beta}{2+\beta}. \qquad (25)
\end{aligned}
$$

Now, since we're in an even period, only Type 1 households work. We just showed that in even periods total output is $\bar{\theta}L_t^1$, and since markets clear we must have total output equalling total consumption. i.e. $\bar{\theta}L_t^1 = C_t^1 + C_t^2$. We showed that from (18) that total output can be expressed as $\bar{\theta}L_t^1 = \bar{\theta} - C_t^1$, thus we showed that $\bar{\theta} - C_t^1 = C_t^1 + C_t^2$, and so

$$\bar{\theta} = 2C_t^1 + C_t^2.$$

Now substitute (24) into the above equation to get

$$
\begin{aligned}
\bar{\theta} &= 2\left(\frac{\bar{\theta}}{2+\beta}\right) + C_0^2 \\
C_0^2 &= \bar{\theta} - \frac{2\bar{\theta}}{2+\beta} \\
&= \frac{\bar{\theta}(2+\beta)}{2+\beta} - \frac{2\bar{\theta}}{2+\beta} \\
&= \frac{2\bar{\theta} + \bar{\theta}\beta - 2\bar{\theta}}{2+\beta} \\
\therefore C_0^2 &= \frac{\bar{\theta}\beta}{2+\beta},
\end{aligned} \tag{26}
$$

which is what we were asked to show. Finally, to complete our answer, we know that in even periods, Type 2 households have zero productivity, and so $L_t^2 = 0 \; \forall$ even t.

3.5

Now turn attention to what happens when t is an odd period. You will need to follow similar steps to those in part 3 and 4 but your answers will be different. Start by writing total output in period t as a function of consumption, given that now only agents of Type 2 will be working. Then introduce market clearing in period t and use the Euler equation for consumption from (22) to rewrite the market clearing condition in terms of consumption at 0 (which is still an even number even though t is odd). You should be able to show that the market interest rate in even periods satisfies

$$
\tilde{R}_t = \frac{2+\beta}{1+2\beta}\beta^{-t}.
$$

Again, we look to our FOCs. From (18), we know that

$$
\begin{aligned}
\frac{\theta_t^i}{C_t^i} &= \frac{1}{1-L_t^i} \\
\implies \theta^i L_t &= \theta^i - C_t^i,
\end{aligned}
$$

and in odd periods this is

$$
\bar{\theta} L_t^2 = \bar{\theta} - C_t^2,
$$

and market clearing requires that this expression for total output be equal to total consumption

$$
\bar{\theta} - C_t^2 = C_t^1 + C_t^2.
$$

From our Keynes-Ramsey condition (20) we have

$$
C_{t+1}^i = \beta R_{t+1} C_t^i,
$$

and rolling this back to period 0 gives

$$C_1^i = \beta R_1 C_0^i,$$

which we can do recursive substitution with, as we did before, to get

$$C_t^i = \beta^t \tilde{R}_t C_0^i,$$

and then substituting this into our market clearing condition $\bar{\theta} - C_t^2 = C_t^1 + C_t^2$ gives

$$\begin{aligned}
\bar{\theta} - \beta^t \tilde{R}_t C_0^2 &= \beta^t \tilde{R}_t C_0^1 + \beta^t \tilde{R}_t C_0^2 \\
\bar{\theta} &= \beta^t \tilde{R}_t C_0^1 + 2\beta^t \tilde{R}_t C_0^2 \\
\implies \bar{\theta} &= \beta^t \tilde{R}_t (C_0^1 + 2C_0^1).
\end{aligned}$$

Now, instead of saying that $\bar{\theta} = C_t^1 + 2C_t^2$, let's instead use our expressions for consumption we found previously in (24) and (26):

$$\begin{aligned}
C_0^1 &= \frac{\bar{\theta}}{2+\beta}, \\
C_0^2 &= \frac{\bar{\theta}\beta}{2+\beta},
\end{aligned}$$

and so

$$\begin{aligned}
C_0^1 + 2C_0^2 &= \frac{\bar{\theta}}{2+\beta} + 2\frac{\bar{\theta}\beta}{2+\beta} \\
&= \bar{\theta}\left(\frac{1+2\beta}{2+\beta}\right),
\end{aligned}$$

and so we have

$$\begin{aligned}
\bar{\theta} &= \beta^t \tilde{R}_t \bar{\theta}\left(\frac{1+2\beta}{2+\beta}\right) \\
\therefore \tilde{R}_t &= \beta^{-t}\frac{2+\beta}{1+2\beta}.
\end{aligned}$$

3.6

Show that the consumption and labour supply of agents in an odd period t are:

$$\begin{aligned}
C_t^1 &= \frac{\bar{\theta}}{1+2\beta},\ L_t^1 = 0, \\
C_t^2 &= \frac{\beta\bar{\theta}}{1+2\beta},\ L_t^2 = \frac{1+\beta}{1+2\beta}.
\end{aligned}$$

Once again, begin with the result we got after doing recursive substitution on our Keynes-Ramsey condition (22):

$$C_t^i = \beta^t \tilde{R}_t C_0^i,$$

and so for Type 1 households, after substituting in our values for C_0^1 and $\tilde{R}_t$ for odd periods, we have

$$
\begin{aligned}
C_t^1 &= \beta^t \left[\beta^{-t} \frac{2+\beta}{1+2\beta}\right] \left[\frac{\bar{\theta}}{2+\beta}\right] \\
&= \frac{2+\beta}{1+2\beta} \frac{\bar{\theta}}{2+\beta} \\
&= \frac{\bar{\theta}(2+\beta)}{2+\beta+4\beta+2\beta^2} \\
&= \frac{\bar{\theta}(2+\beta)}{2+5\beta+2\beta^2} \\
&= \frac{\bar{\theta}(2+\beta)}{(1+2\beta)(2+\beta)} \\
\therefore C_t^1 &= \frac{\bar{\theta}}{1+2\beta}.
\end{aligned}
$$

Doing the same for Type 2 households gives

$$
\begin{aligned}
C_t^2 &= \beta^t \left[\beta^{-t} \frac{2+\beta}{1+2\beta}\right] \left[\frac{\bar{\theta}\beta}{2+\beta}\right] \\
&= \frac{2+\beta}{1+2\beta} \frac{\bar{\theta}\beta}{2+\beta} \\
&= \frac{2\bar{\theta}\beta+\bar{\theta}\beta^2}{(1+2\beta)(2+\beta)},
\end{aligned}
$$

using from above, and so

$$
\begin{aligned}
C_t^2 &= \frac{\bar{\theta}\beta(2+\beta)}{(1+2\beta)(2+\beta)} \\
\therefore C_t^2 &= \frac{\bar{\theta}\beta}{1+2\beta}.
\end{aligned}
$$

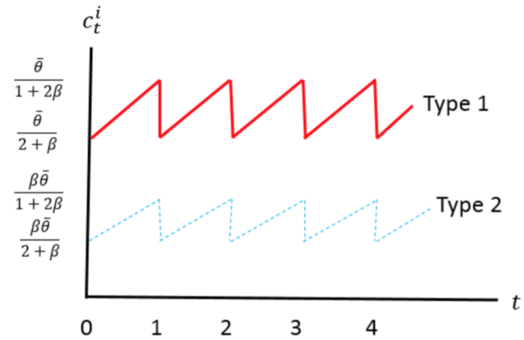
For the labour supply decisions, we know that in odd periods $L_t^1 = 0$, so we need to find Type 2's labour supply. From our FOCs, the intratemporal condition for Type 2 households can be written as

$$\bar{\theta}L_t^2 = \bar{\theta} - C_t^2,$$

and doing substitution gives

$$
\begin{aligned}
\bar{\theta}L_t^2 &= \bar{\theta} - \frac{\bar{\theta}\beta}{1+2\beta} \\
L_t^2 &= 1 - \frac{\beta}{1+2\beta} \\
&= \frac{1+2\beta-\beta}{1+2\beta} \\
\therefore L_t^2 &= \frac{1+\beta}{1+2\beta},
\end{aligned}
$$

Figure 2: Consumption



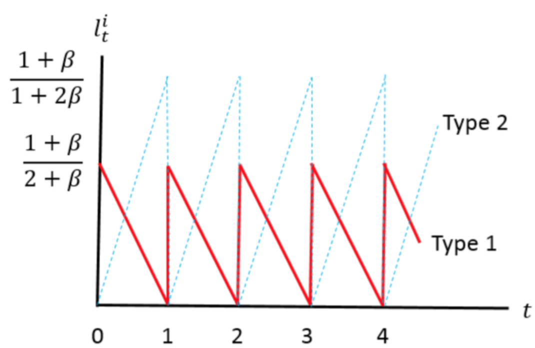
thus completing our answer.

3.7

Draw a plot of consumption and labour supply of each type of agent over time. Comment on what is happening, paying particular attention to the cases when $\beta \to 0$ and $\beta \to 1$.

Let's look at Professor Ellison's graphs:

Figure 1: Labour supply



We note two key results: First, the labour supply of Type 2 agents in odd periods is higher than the labour supply of Type 1 agents in even periods because

$$\frac{1+\beta}{1+2\beta} > \frac{1+\beta}{2+\beta}.$$

Secondly, consumption of Type 1 agents is always higher than that of Type 2 agents.

Why? Recall that Type 1 agents have an advantage in being productive in period 0 when commitments are made. They are in control as they a first-mover advantage, and there's a wealth effect which we discovered previously. The result is that Type 1 agents work less in even periods than Type 2 agents in odd periods. The effect is so strong that enough is produced in odd periods such that the consumption of both types of agents rises.

When $\beta \to 0$ we get the extreme case where Type 2 agents consume nothing in even period and work all possible hours in odd periods (in exchange for some consumption in odd periods). This makes sense because Type 2 agents with no productivity in period 0 are in a very weak position. Meanwhile, when $\beta \to \infty$, the asymmetry in allocations across individuals and time disappears. Again, this makes sense as if agents patient forever, there is nothing special about getting a first mover advantage in period 0.

4 A Lucas tree economy

Instead of an endowment economy with trade in bonds, the object of this exercise is to examine the determination of stock prices. To this end, suppose there is one asset in the economy, which we will call a tree, and that each agent i at time t holds a share s_t^i of the tree. The tree produces fruit (dividends) d_t every period, which is returned to households for consumption in proportion to their share in the tree. Every period, agents can buy and sell shares of the tree on a stock market which clears in the usual Walrasian fashion. The per-period utility of agents is logarithmic $U(C_t^i) = \log C_t^i$ and the discount factor is β.

Denote the price of a tree in period t as p_t so that the price of a share s_t^i is $p_t s_t^i$. The sequence of dividends $\{d_t\}$ is an exogenous given series that is known in advance.

4.1

Set up the optimisation problem faced by a household of Type i in period t, where a household of Type i is distinguished by its holding of shares s_t^i.

Our problem is as follows:

$$\max_{\{C_t^i, s_{t+1}^i\}} \sum_{t=0}^{\infty} \beta^t \log C_t^i,$$

subject to the following budget constraint:

$$s_{t+1}^i p_t = s_t^i p_t + s_t^i d_t - C_t^i.$$

4.2

Derive the first order condition associated with the problem of a Type i household.

First, set up the Lagrange for this social planner's problem:

$$Z = \sum_{t=0}^{\infty} \beta^t \log C_t^i + \sum_{t=0}^{\infty} \beta^t \lambda_t \left(s_t^i p_t + s_t^i d_t - C_t^i - s_{t+1}^i p_t \right),$$

which gives us the following FOCs

$$\begin{aligned}
Z_{C_t^i} :& \frac{1}{C_t^i} - \lambda_t = 0, \\
Z_{C_{t+1}^i} :& \frac{\beta}{C_{t+1}^i} - \beta\lambda_{t+1} = 0, \\
Z_{s_{t+1}^i} :& \beta\lambda_{t+1} p_{t+1} + \beta\lambda_{t+1} d_{t+1} - \lambda_t p_t = 0.
\end{aligned}$$

From which we can infer and derive our Keynes-Ramsey condition (consumption Euler equation):

$$\begin{aligned}
\lambda_t &= \frac{1}{C_t^i}, \\
\lambda_{t+1} &= \frac{1}{C_{t+1}^i}, \\
\implies p_t \frac{1}{C_t^i} &= \frac{\beta}{C_{t+1}^i}(p_{t+1} + d_{t+1}) \\
C_{t+1}^i &= \beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] C_t^i.
\end{aligned}$$

4.3

Define an equilibrium in this setup and show that the initial price of a tree p_t satisfies:[2]

$$p_t = \frac{\beta}{1-\beta} d_t.$$

Why does an increase in future dividends have no impact on the price of the tree?

A competitive equilibrium is a sequence of prices and allocations $\{p_t, d_t, C_t^i, S_{t+1}^i\}$ such that $\{C_t^i, s_{t+1}^i\}$ solves the household problem given prices $\{p_t, d_t\}$, and there is market clearing such that

$$\begin{aligned}
\sum_{i=1}^N C_t^i &= d_t, \\
\sum_{i=1}^N s_{t+1}^i &= 1.
\end{aligned}$$

Aggregating and applying market clearing to the consumption Euler equation gives us

$$\begin{aligned}
\sum_{i=1}^N C_{t+1}^i &= \beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] \sum_{i=1}^N C_t^i \\
\leftrightarrow d_{t+1} &= \beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] d_t \\
\implies p_t &= \beta \left[p_{t+1} + d_{t+1} \right] \frac{d_t}{d_{t+1}}.
\end{aligned}$$

[2] *You will need to iterate the Euler equation to achieve this.*

Solving forward gives

$$
\begin{aligned}
p_t &= \beta \left[p_{t+1} + d_{t+1} \right] \frac{d_t}{d_{t+1}} \\
&= \beta \left[\underbrace{\left(\beta \left[p_{t+2} + d_{t+2} \right] \frac{d_{t+1}}{d_{t+2}} \right)}_{p_{t+1}} + d_{t+1} \right] \frac{d_t}{d_{t+1}} \\
&= \beta \left[\underbrace{\left(\beta \left[\underbrace{\beta \left[p_{t+3} + d_{t+3} \right] \frac{d_{t+2}}{d_{t+3}}}_{p_{t+2}} + d_{t+2} \right] \frac{d_{t+1}}{d_{t+2}} \right)}_{p_{t+1}} + d_{t+1} \right] \frac{d_t}{d_{t+1}} \\
&= \beta d_t + \beta^2 d_t + \beta^3 d_t + p_{t+3} \frac{d_t}{d_{t+3}} + \ldots \\
p_t &= \frac{\beta}{1-\beta} d_t.
\end{aligned}
$$

Future dividends have no impact on the current price because future dividends are completely discounted. To see why this is the case, consider an increase in d_{t+1}. This will have two effects. Firstly, it increases the price of the share, secondly it increases future discount rates. In the simple logarithmic model these effects exactly cancel out to leave the share price only depending on the current dividend. Yes, it's good when future dividends go up but if future discounting also goes up then there will be no overall impact.

4.4

Now assume that utility is CRRA

$$
U(C_t^i) = \frac{(C_t^i)^{1-\sigma}}{1-\sigma},
$$

where $\sigma > 0$ and $\sigma \neq 1$. Derive an expression for the price of the tree at time t as a function of the stream of future dividends and discuss how this price is affected by changes in future dividends. In particular, explain if and when the current price of the tree decreases if future dividends are increased.[3]

Solve for FOCs given CRRA preferences

$$
Z = \sum_{t=0}^{\infty} \beta^t \frac{(C_t^i)^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left(s_t^i p_t + s_t^i d_t - C_t^i - s_{t+1}^i p_t \right)
$$

[3]For example, in response to knowledge that fertiliser will be applied in the future.

$$
\begin{aligned}
Z_{C_t^i} :& (C_t^i)^{-\sigma} - \lambda_t = 0, \\
Z_{C_{t+1}^i} :& \beta (C_{t+1}^i)^{-\sigma} - \beta \lambda_{t+1} = 0, \\
Z_{s_{t+1}^i} :& \beta \lambda_{t+1} p_{t+1} + \beta \lambda_{t+1} d_{t+1} - \lambda_t p_t = 0.
\end{aligned}
$$

From which we can infer and drive our Keynes-Ramsey condition (consumption Euler equation):

$$
\begin{aligned}
\lambda_t &= (C_t^i)^{-\sigma}, \\
\lambda_{t+1} &= (C_{t+1}^i)^{-\sigma}, \\
\implies p_t (C_t^i)^{-\sigma} &= \beta (C_{t+1}^i)^{-\sigma} (p_{t+1} + d_{t+1}) \\
(C_{t+1}^i)^{\sigma} &= \beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] (C_t^i)^{\sigma}. \\
C_{t+1}^i &= \left(\beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] \right)^{\frac{1}{\sigma}} C_t^i,
\end{aligned}
$$

and then applying market clearing conditions and aggregating, the competitive general equilibrium gives us

$$
\begin{aligned}
\sum_{i=1}^{N} C_{t+1}^i &= \left(\beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] \right)^{\frac{1}{\sigma}} \sum_{i=1}^{N} C_t^i \\
\leftrightarrow d_{t+1} &= \left(\beta \left[\frac{p_{t+1} + d_{t+1}}{p_t} \right] \right)^{\frac{1}{\sigma}} d_t \\
\implies p_t &= \beta \left[p_{t+1} + d_{t+1} \right] \left(\frac{d_t}{d_{t+1}} \right)^{\sigma}.
\end{aligned}
$$

Then, solving forward gives

$$
\begin{aligned}
p_t &= \beta \left[p_{t+1} + d_{t+1} \right] \left(\frac{d_t}{d_{t+1}} \right)^{\sigma} \\
&= \beta \left[\left(\underbrace{\beta \left[p_{t+2} + d_{t+2} \right] \left(\frac{d_{t+1}}{d_{t+2}} \right)^{\sigma}}_{p_{t+1}} \right) + d_{t+1} \right] \left(\frac{d_t}{d_{t+1}} \right)^{\sigma} \\
&= \beta \left[\underbrace{\left(\beta \left[\underbrace{\beta \left[p_{t+3} + d_{t+3} \right] \left(\frac{d_{t+2}}{d_{t+3}} \right)^{\sigma}}_{p_{t+2}} + d_{t+2} \right] \left(\frac{d_{t+1}}{d_{t+2}} \right)^{\sigma} \right)}_{p_{t+1}} + d_{t+1} \right] \left(\frac{d_t}{d_{t+1}} \right)^{\sigma} \\
&= \sum_{j=1}^{\infty} \beta^j d_{t+j} \left(\frac{d_t}{d_{t+j}} \right)^{\sigma}.
\end{aligned}
$$

The effect of a change in future dividends on the current price is

$$\left.\frac{\partial p_t}{\partial d_{t+j}}\right|_{\sigma>1} = \beta^j (1-\sigma)\left(\frac{d_t}{d_{t+j}}\right)^{\sigma} < 0.$$

So for the price of a tree to decrease with an increase in future dividends, you need σ to be more than 1. Implicitly the idea is that, once this is the case, the consumption smoothing motive is so high than an agent facing higher wealth in the future would like to sell some of their share in the tree today to consume more in the current period. Since all agents will be doing the same thing, the price of the tree must drop sufficiently that all households still want to hold shares and markets are able to clear. If $\sigma < 1$ then the consumption smoothing motive is low and households want to sell their shares less now, hence the price does not drop.

4.5

Does this economy behave as a representative agent economy? Explain.

Yes, see the explanation of Gorman aggregation in the first question.