# Microeconomics 2 Supplementary Notes: Lagrangian Optimisation

By David H. Murakami

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## 1 Introduction

The Lagrangian Optimisation technique is a part of "constrained optimisation" in broader mathematics[1], and was developed by Joseph-Louis Lagrange (1736-1813). In short, it seeks to maximise (minimise) one function, the maximand (minimand), against another function, the constraint function. Immediately, it should be obvious to you that it has a large range of potential applications in economics. Many economic concepts actually drop out directly from the Lagrangian Optimisation technique!

## 2 Mathematical Prerequisites

In order to be accurate and efficient with using Lagrangian Optimisation, we must first go over a few mathematical prerequisites. The first is some basic arithmetic involving algebra. Be sure that you're confident with moving variables and values across each side of the equation. This sounds extremely basic and trivial, but it becomes very easy to make a mistake especially when you start moving powers from one side to the other. For example:

$$x^a = y$$

then

$$x = y^{\frac{1}{a}}$$

Things get difficult when we have to transfer powers that are fractions:

$$\begin{aligned} x^{\frac{1}{a}} &= y^{\frac{1}{b}} \\ \therefore x &= y^{\frac{a}{b}} \end{aligned}$$

[1]For more see Turkington (2006), Chiang & Wainwright (2005), and Bradley & Patton (2003).

The above example is obviously very simple. You will run into a lot more messier equations when you start working with production functions (e.g. $Y = AK^{\alpha}L^{1-\alpha}$) and utility functions (e.g. $U = x_1^{\alpha}x_2^{\beta}$).

Secondly, you must be confident with taking the derivative of a function. More specifically, in economics, we mostly deal with first-order partial derivatives.[2] If we assume that $f$ is a function of $x$, denoted as $f(x)$, then the first derivative of $f(x)$ differentiated with respect to $x$ is given as:

$$\frac{df(x)}{dx} = \frac{d}{dx}f(x)$$

or some of you may recall this notation from high school:

$$f'(x) = \frac{d}{dx}f(x)$$

where $f'(x)$ is the first derivative of $f(x)$ differentiated with respect to $x$ once.

With that in mind, remember the basic rules of derivatives:

$$\begin{aligned} f(x) &= x^a \\ f'(x) &= ax^{a-1} \end{aligned}$$

The above notation is for the case of functions of single variables (i.e. $x$ is the only variable of $f$). In economics we deal with multivariate equations, where $f$ could be a function of two or more variables. Let's assume that $f$ is a function of both $x_1$ and $x_2$:

$$f(x_1, x_2)$$

Then, the partial derivative of $f$ is the derivative of $f$ differentiated with respect to EITHER $x_1$ or $x_2$. In full notational form, consider $f$ differentiated with respect to $x_1$:

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{\partial}{\partial x_1}f(x_1, x_2)$$

which we sometimes shorthand to simply $f_1$. Likewise, if we differentiated $f(x_1, x_2)$ with respect to $x_2$ then we would refer to that partial derivative as simply $f_2$.

[2] Although, in higher levels of economics study, you must be confident with taking higher order derivatives in order to validate things like local maxima or minima and stability.

Remember, the partial derivative treats one of the variables as a variable of change (this is the one specified), and the other variables as constants (treat them like a normal number). So consider the following:

$$\begin{aligned} f(x_1, x_2) &= x_1^a + x_2^b \\ \therefore f_1 = \frac{\partial f(x_1, x_2)}{\partial x_1} &= a x_1^{a-1} \end{aligned}$$

Likewise:

$$f_2 = \frac{\partial f(x_1, x_2)}{\partial x_2} = b x_2^{b-1}$$

Hopefully that should have jogged your memory about these things. If these still seem vague, pull out your Year 12 calculus textbooks and/or your first year maths notes! Anyway, onto the Lagrange...

# 3 Setting up the Lagrange

The first thing we need to do when we are confronted with a potential constrained optimisation problem is to identify what it is that we need to optimise and whether or not we're being constrained by some other factor. This could be something that we often look at in microeconomics such as:

- Maximising utility against an income constraint;
- Minimising production costs against an output constraint, or;
- Maximising production volume against a cost constraint.

Once we've identified the equation that we need to optimise and its respective constraints, we can then set up our problem. I often like to write it down as the following, which is the notation used in Turkington (2006):

$$\begin{aligned} \text{Optimise} \quad & f(x_1, ..., x_n) \\ \text{subject to} \quad & g(x_1, ...x_n) = C \end{aligned}$$

where $f(x_1, ..., x_n)$ is the function called the objective function. The function $g(x_1, ..., x_2)$ is called the constraint function, and it equals a constant, $C$ (this could be a given level of income a consumer has, for example).

# 4 The Lagrangian Function

Now that we've set out our objective and constraint function, we can now set up the Lagrangian function. What this essentially does is convert the constraint problem into a unconstrained problem by introducing a new variable called the Lagrangian multiplier. We denote the Lagrangian function as $\mathcal{L}$, which follows:

$$\mathcal{L} = f(x_1, ..., x_n) + \lambda(C - g(x_1, ..., x_n))$$

where $\lambda$ is the Lagrangian multiplier. This later on becomes known as the "shadow price" in higher level microeconomics.[3]

As you can see, the Lagrangian function is set up by using both the objective function and the constraint function (after rearranging the terms to one side).

To see how this process works, let's work through an example...

# 5 Optimisation and First Order Conditions

This example is from Turkington (2006, pp. 154-155). Suppose we have the following problem:

$$\begin{array}{rl} \text{Maximise} & y = x_1^2 + 3x_1x_2 - 3x_2^2 \\ \text{subject to} & x_1 + 3x_2 = 6 \end{array}$$

Then, the Lagrangian function for this problem is:

$$\mathcal{L} = x_1^2 + 3x_1x_2 - 3x_2^2 + \lambda(6 - x_1 - 3x_2)$$

Now, in order to find the optimal values of $x_1$ and $x_2$, we need to take the partial derivatives of $\mathcal{L}$ to set up some first order conditions. This is essentially setting up the first derivative equal to zero in order to find a peak or a trough of the original non-linear equation. So let's differentiate $\mathcal{L}$ with respect to $x_1$, $x_2$, and $\lambda$, and set each of the derivatives to zero:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 + 3x_2 - \lambda = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 3x_1 - 6x_2 - 3\lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 6 - x_1 - 3x_2 = 0 \tag{3}$$

[3] Bonus participation marks if you can explain why ther is the case.

The above 3 equations are the First Order Conditions (FOCs) for this problem. From here, there's a myriad of ways to reach the optimal values of $x_1$, $x_2$, and $\lambda$. This is where your algebra skills come in. You are free to tackle the problem however you wish, but I'm going to start by looking at equation (1) and rearranging to get $\lambda$ in terms of $x_1$ and $x_2$:

$$\lambda = 2x_1 + 3x_2 \tag{4}$$

Then I'm going to sub $\lambda$ into equation (2) to get:

$$3x_1 - 6x_2 - 3(2x_1 + 3x_2) = 0$$

which when we expand the brackets results in:

$$\begin{aligned} -3x_1 - 15x_2 &= 0 \\ -15x_2 &= 3x_1 \\ x_1 &= -5x_2 \end{aligned} \tag{5}$$

Now, substitute $x_1$ into equation (3) to get the optimal value of $x_2$:

$$\begin{aligned} 6 - (-5x_2) - 3x_2 &= 0 \\ 6 + 5x_2 - 3x_2 &= 0 \\ 6 + 2x_2 &= 0 \\ 2x_2 &= -6 \\ \therefore x_2^* &= -3 \end{aligned}$$

with $x_2^*$ in hand, we know from equation (5) that $x_1^* = -5(-3) = 15$. Lastly, let's get the Lagrangian multiplier value, $\lambda$, from (1):

$$\begin{aligned} 2(15) + 3(-3) &= \lambda \\ \therefore \lambda^* &= 21 \end{aligned}$$

To summarise:

$$x_1^* = 15,\ x_2^* = -3,\ \lambda^* = 21$$

You can confirm that this is indeed the optimal by substituting $x_1^*$ and $x_2^*$ into the original constraint function:

$$\begin{aligned} x_1 + 3x_2 &= 6 \\ 15 + 3(-3) &= 6 \end{aligned}$$

Clearly, it holds.

## 6 Economic Application

So now let's have a look at the assignment question and why it is that $MRS = MRT$[4] leads to an optimal solution.

Akane's utility function, $U(B, M)$ is a function of Burgers, $B$, and Movies, $M$, and has the form $U = \sqrt{BM}$.[5] She has an income of \$96, and the price of Burgers are \$16 and the price of Movies are \$8. With this information, let's set up our problem:

$$\begin{aligned} \text{Maximise} \qquad & U(B, M) = \sqrt{BM} \\ \text{s.t.} \qquad & P_B B + P_M M = 96 \end{aligned}$$

Setting up the Lagrangian function for this problem gives us:

$$\mathcal{L} = \sqrt{BM} + \lambda(96 - P_B B - P_M M)$$

We then take first order partial derivatives, to attain our FOCs:

$$\frac{\partial \mathcal{L}}{\partial B} = \frac{1}{2} B^{-\frac{1}{2}} M^{\frac{1}{2}} - \lambda P_B = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial M} = \frac{1}{2} B^{\frac{1}{2}} M^{-\frac{1}{2}} - \lambda P_M = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 96 - P_B B - P_M M = 0 \tag{8}$$

Make sure you polish your algebra before you attempt to this! We can make life a bit easier for ourselves by moving the two price terms in equations (6) and (7) to the right hand side (RHS), which then gives us:

$$\begin{aligned} \frac{1}{2} B^{-\frac{1}{2}} M^{\frac{1}{2}} &= \lambda P_B \\ \frac{1}{2} B^{\frac{1}{2}} M^{-\frac{1}{2}} &= \lambda P_M \end{aligned}$$

If we take a closer look at the components on the LHS, we actually have Akane's marginal utility of Burgers, $MU_B$, and her marginal utility of Movies, $MU_M$. Recall that the marginal utility of good $x_i$ is the partial derivative of the utility function derived with respect to good $x_i$.

$$MU_i = \frac{\partial U(\mathbf{x})}{\partial x_i}$$

[4] The optimal point in a consumer optimisation problem occurs where the marginal rate of substitution (MRS) is equal to the marginal rate of transformation (MRT).

[5] I'm changing Pizza to Burgers in ther case, as it is less confusing when we use $P_i$ for the price of good $i$.

and the utility function for Akane, $U(B, M)$, was inputted directly into the Lagrangian function! So that means we have:

$$\begin{aligned} MU_B &= \lambda P_B \\ MU_M &= \lambda P_M \end{aligned}$$

This should look much more familiar to you now. We can go one step further by dividing these two equations to yield the optimal point of the consumer's utility maximisation problem (and to cancel out the two $\lambda$ terms):

$$\frac{MU_B}{MU_M} = \frac{P_B}{P_M}$$

which we know, in economic jargon, as:

$$MRS = MRT$$

So, it must be the case, that if we find the optimal values of $B$ and $M$ for this Lagrangian problem, that the condition of $MRS = MRT$ will hold! Economic theory will tell us at that point the consumer is maximising her utility subject to their budget constraint. Mathematically, we know at that point the FOCs will give us an optimal solution. We've just made an important realisation! So, let's find $B^*$ and $M^*$:

$$\frac{\frac{1}{2}B^{-\frac{1}{2}}M^{\frac{1}{2}}}{\frac{1}{2}B^{\frac{1}{2}}M^{-\frac{1}{2}}} = \frac{P_B}{P_M}$$

Rearrange this mess to get either $B$ or $M$ in terms of the other variable. I got:

$$\begin{aligned} \frac{\frac{1}{2}M^{\frac{1}{2}}M^{\frac{1}{2}}}{\frac{1}{2}B^{\frac{1}{2}}B^{\frac{1}{2}}} &= \frac{P_B}{P_M} \\ \frac{M}{B} &= \frac{16}{8} \\ M &= 2B \end{aligned}$$

Substitute this into equation (8):

$$\begin{aligned} 96 - 16B - 8(2M) &= 0 \\ 96 - 16B - 16B &= 0 \\ 32B &= 96 \\ \therefore B^* &= 3 \end{aligned}$$

Substitute this into the above expression for $M$ to get $M^* = 2(3) = 6$. We can also find the shadow price for this problem using either equation (6) or (7). I'm going to use (6):

$$\begin{aligned}
\frac{1}{2}B^{-\frac{1}{2}}M^{\frac{1}{2}} - \lambda P_B &= 0 \\
\frac{1}{2}(3)^{-\frac{1}{2}}(6)^{\frac{1}{2}} - \lambda(16) &= 0 \\
\frac{1}{2}\left(\frac{6}{3}\right)^{\frac{1}{2}} &= 16\lambda \\
\therefore \lambda^* &= 8\sqrt{2} \approx 11.3
\end{aligned}$$

To summarise, the optimal bundle for Akane is:

$$B^* = 3,\ M^* = 6,\ \lambda^* = 8\sqrt{2}$$

At this point, Akane's indifference curve is tangent to her budget constraint!

Hopefully these notes helped you in understanding the Lagrangian optimisation process. As I said before, it has a large range of potential uses in economics, which I'm sure you will come across in further study.

## References


[1] Bradley, T. and Patton, P., *Essential Mathematics for Economics and Business* (John Wiley & Sons, West Sussex, Second Edition, 2003), pp. 378-390.

[2] Chiang, A., C. and Wainwright, K., *Fundamental Methods of Mathematical Economics* (McGraw-Hill Irwin, New York, Fourth Edition, 2005), pp. 347-353.

[3] Turkington, D., *Mathematical Tools for Economics* (Wiley-Blackwell Publishing, Oxford, 2006), pp. 154-155.