Theory of the Consumer and Firm

Intro Math for Economists (PEARL, Spring 2019)

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Introduction

- We have come quite a long way. We've developed optimisation skills – both unconstrained and constrained – and we've looked at the Envelope Theorem.
- It's now time to delve deeper into economic applications.
- We will look at neoclassical economic theory of the consumer and firm.

Mathematical Prerequisites: Binary Relations

Definition

A binary relation R is some relation or connection that may or may not hold between two elements of a set.

- i.e. R is a binary relation on set S if the statement xRy is either true or false $\forall\ x, y \in S$.
- e.g. Let S be the set of humans take R as 'is son of'.
- e.g. Take S as the set of real numbers and R as 'greater than or equal to' $\implies x \geq y$.

Binary Relations

Equivalence Relation:

Definition

A binary relation R on set S is an equivalence relation if it's reflexive, symmetric, and transitive.

1. Reflexive: $\forall\ x \in S,\ xRx$
2. Symmetric: $\forall\ x, y \in S,\ xRy \implies yRx$
3. Transitive: $\forall\ x, y, z \in S, xRy, yRz, \implies xRz$

- e.g. Let S be the set of real numbers, and R be 'is equal to'.
    - For real x, x is equal to itself (reflexive), if $x = y$ then $y = x$ (symmetric), and if $x = y$,$y = z \implies x = z$ (transitive).

Binary Relations

Preordering

Definition

A binary relation R on set S is preordering if it is reflexive and transitive.

- i.e. $\forall\ x \in S, xRx$, and for $x, y, z \in S$, if xRy, yRz then $\implies xRz$
- e.g. Suppose S is the set of all sets. Then for any set A, B, C, where $A \subset B$; if $A \subset B$, and $B \subset C$ then $\implies A \subset C$.

Binary Relations

Ordering

Definition

A binary relation R on S is ordering if it is a preordering and if xRy and yRx. This implies that $x = y$ in some sense.

- e.g. Suppose S is the set of real numbers, and $R = \geq$. Then if $x \geq y, y \geq x$ then $\implies x = y$.

Binary Relations

Completeness

Definition

Let R be preordering or ordering on set S. Then R is complete if for any $x, y \in S, xRy, yRx$. If $\exists$ two elements in S that do not meet this requirement then R is partial.

- e.g. Suppose S is the set of all sets. Then S would be partial as disjoint sets are not comparable under completeness.

Indices

Meaning of a^m where m is a positive integer:

$$a^m = \underbrace{a \times a \times \ldots \times a}_{m}$$

Also if $m = \frac{P}{Q}$, and P, Q are positive integers, then a is also positive.

Indices

- Consider $x^n = r \quad (1)$, where r is a positive number.
- Exactly n solutions to this equation – may be real or complex.
- But we assume one and only one positive root.

Indices

Definition

Any solution to $r(1)$ is called the n^{th} root of r. Assume one and only one solution that is a positive real number.

- Denote this root by $\sqrt[n]{r}$
- $\sqrt{81} \neq \pm 9$ under this definition. i.e. $\sqrt{81} = 9$ under this definition.

Indices

$$a^{P/Q} = \sqrt[Q]{a^P}$$

$$a^m \times a^n = a^{m+n}$$

$$(a^m)^n = a^{m \times n}$$

$$a^{-m} = \frac{1}{a^m}$$

Theory of the Consumer

- Consumer picks a commodity bundle:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad x_i \geq 0, \quad i = 1, ..., n$$

x_i is quantity of good i consumer over the period.

Definition

The set of all commodity bundles facing the consumer is called the consumption set X. $X \subset$ non-negative orthant $\mathbb{R}^n$. We usually assume that X is closed and convex.

Preference Preordering

Definition

On X we assume is defined a 'preference preordering'.

- Denote $\succeq$ as 'at least desired as'
- Assumptions placed on $\succeq$:
  1. Ordering: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$ then $\mathbf{x}$ is indifferent to $\mathbf{y}$, $\mathbf{x} \sim \mathbf{y}$. If $\mathbf{x} \succeq \mathbf{y}, \mathbf{y} \not\succeq \mathbf{x} \implies \mathbf{x} \succ \mathbf{y}$. i.e. $\mathbf{x}$ is preferred to $\mathbf{y}$.
  2. Completeness: $\forall\ \mathbf{x}, \mathbf{y} \in X$ either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$.
  3. Continuity: $\forall\ \mathbf{y} \in X$, the sets $\{\mathbf{x} : \mathbf{x} \succeq \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \preceq \mathbf{y}\}$ are closed sets.
  4. Strong Monotonicity: If $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \succeq \mathbf{y}$. i.e. one ice cream $<$ two ice creams.
  5. Local Non-Saturation: Given any $\mathbf{x}$ in X and any real number > 0, there exists some commodity bundle $\mathbf{y}$ in X in an neighbourhood of $\mathbf{x}$ such that $\mathbf{y} \succ \mathbf{x}$.

The Utility Function

Definition

A utility function, $U : X \to R$, is any function such that if $\mathbf{x} \succeq \mathbf{y}$, then $U(\mathbf{x}) \geq U(\mathbf{y})$ and if $\mathbf{x} \succ \mathbf{y}$ then $U(\mathbf{x}) > U(\mathbf{y})$.

- Note: We assume ordinality. i.e. If $U(\mathbf{x}) = 4, U(\mathbf{y}) = 2$ then it doesn't mean that $\mathbf{x}$ is twice as good as $\mathbf{y}$.
- Question: What conditions do we need to place on the preference preordering ($\succeq$) in order that $\exists$ a continuous utility function?

The Utility Function

Theorem

Suppose the preference preordering is complete, continuous, and strongly monotonic, then $\exists$ *a continuous utility function.*

The Consumer's Problem

- Let $\mathbf{P}$ be a vector of prices for goods i to n.

$$\mathbf{P} = \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}, \quad i = 1, ..., n$$

  and let Y be the consumer's income.
- The consumer's attainable set is then:

$$S = \{\mathbf{x} \in X | \mathbf{P}'\mathbf{x} \leq Y\}$$

- If we assume strong monotonicity and that goods can be consumed in any quantity, our problem can be:

$$\begin{aligned} \max \quad & U(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{P}'\mathbf{x} = Y \\ & \mathbf{x} \geq 0 \end{aligned}$$

Marshallian Demand Functions and Indirect Utility Function

- Utility navigation:

$$\begin{aligned} \max \quad & U(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{P}'\mathbf{x} = Y \end{aligned}$$

- The optimal point is attained at $\mathbf{x}^* = \mathbf{x}^*(\mathbf{P}, Y)$, the Marshallian demand function.
- The maximum value function for this problem: $V(\mathbf{P}, Y) = U(\mathbf{x}^*(\mathbf{P}, Y))$, known as the indirect utility function.

Marshallian Demand Functions and Indirect Utility Function

- Let
$$U(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$$
- Suppose we want to find the Marshallian demand function and indirect utility function. How should we go about doing this?
- Setup our problem and the Lagrangian function:

$$\begin{aligned} \max \quad & U(\mathbf{x}) = x_1^{\alpha} x_2^{\beta} \\ \text{s.t.} \quad & P_1 x_1 + P_2 x_2 = Y \end{aligned}$$

$$Z = x_1^{\alpha} x_2^{\beta} + \lambda (Y - P_1 x_1 - P_2 x_2) \tag{1}$$

Marshallian Demand Functions and Indirect Utility Function

- Get FOCs:

$$\frac{\partial Z}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} = \lambda P_1 \tag{2}$$

$$\frac{\partial Z}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} = \lambda P_2 \tag{3}$$

$$\frac{\partial Z}{\partial \lambda} = Y - P_1 x_1 - P_2 x_2 = 0 \tag{4}$$

- Start solving this problem by dividing (2) by (3). Should eventually get:

$$x_1^* = \frac{\alpha}{\alpha+\beta}\frac{Y}{P_1}, \quad x_2^* = \frac{\beta}{\beta+\alpha}\frac{Y}{P_2} \tag{5}$$

Marshallian Demand Functions and Indirect Utility Function

- Substitute the Marshallian demand functions, (5), into $U(\mathbf{x})$ to get the indirect utility function:

$$V(\mathbf{P}, Y) = A\frac{Y^{\alpha+\beta}}{P_1^{\alpha}P_2^{\beta}}$$

- Note: A is just a term to clean the expression.

Hicksian Demand Functions and Expenditure Function

- Suppose we flip the consumer's problem on its head, and we instead decide to minimise expenditure subject to some utility constraint.

$$\begin{aligned} \min \quad & \mathbf{P}'\mathbf{x} \\ \text{s.t.} \quad & U(\mathbf{x}) = \tilde{U} \end{aligned}$$

- The optimal point will occur at $\bar{\mathbf{x}}(\mathbf{P}, \tilde{U})$, the Hicksian demand functions.
- Minimum value function: $e(\mathbf{P}, \tilde{U}) = \mathbf{P}'\bar{\mathbf{x}}(\mathbf{P}, \tilde{U})$, the expenditure function.

Hicksian Demand Functions and Expenditure Function

- Let

$$U(\mathbf{x}) = x_1^{\alpha} x_2^{\beta}$$

- Suppose we are asked to find the Hicksian demand functions and the expenditure function. What shall we do?
- Yes, setup the problem and the Lagrangian:

$$\begin{aligned} \min \quad & P_1 x_1 + P_2 x_2 \\ \text{s.t.} \quad & x_1^{\alpha} x_2^{\beta} = \tilde{U} \end{aligned}$$

$$Z = P_1 x_1 + P_2 x_2 + \lambda(U^0 - x_1^{\alpha} x_2^{\beta}) \tag{6}$$

Hicksian Demand Functions and Expenditure Function

- Like always, get the FOCs:

$$\frac{\partial Z}{\partial x_1} = \alpha\lambda x_1^{\alpha-1} x_2^{\beta} = P_1 \tag{7}$$

$$\frac{\partial Z}{\partial x_2} = \beta\lambda x_1^{\alpha} x_2^{\beta-1} = P_2 \tag{8}$$

$$\frac{\partial Z}{\partial \lambda} = \tilde{U} - x_1^{\alpha} x_2^{\beta} = 0 \tag{9}$$

- Solve the above problem to get the Hicksian demand functions:

$$\bar{x}_1 = \tilde{U}^{\frac{1}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{P_2}{P_1}\right)^{\frac{\beta}{\alpha+\beta}}, \quad \bar{x}_2 = \tilde{U}^{\frac{1}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{P_1}{P_2}\right)^{\frac{\alpha}{\alpha+\beta}} \tag{10}$$

Hicksian Demand Functions and Expenditure Function

Begin by dividing (7) by (8):

$$\frac{P_1}{P_2} = \frac{\alpha}{\beta}\frac{x_2}{x_1},$$
$$\therefore x_2 = \frac{\beta}{\alpha}\frac{P_1 x_1}{P_2}$$

Then substitute this expression for x_2 into (9) to get:

$$\tilde{U} - x_1^{\alpha}\left(\frac{\beta}{\alpha}\right)^{\beta}\left(\frac{P_1}{P_2}\right)^{\beta} x_1^{\beta} = 0$$

$$\therefore x_1^{\alpha+\beta} = \tilde{U}\left(\frac{\alpha}{\beta}\right)^{\beta}\left(\frac{P_2}{P_1}\right)^{\beta} \implies \bar{x}_1 = \tilde{U}^{\frac{1}{\alpha+\beta}}\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{P_2}{P_1}\right)^{\frac{\beta}{\alpha+\beta}},$$

and by symmetry,

$$\bar{x}_2 = \tilde{U}^{\frac{1}{\alpha+\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{P_1}{P_2}\right)^{\frac{\alpha}{\alpha+\beta}}$$

Hicksian Demand Functions and Expenditure Function

- For the expenditure function, of the form $e = P_1\bar{x}_1 + P_2\bar{x}_2$.
- Easier to tackle this problem in bits. Begin by substituting $\bar{x}_1$ into $P_1\bar{x}_1$:

$$P_1\bar{x}_1 = K_1\tilde{U}^{\frac{1}{\alpha+\beta}}P_1\frac{P_2^{\frac{\beta}{\alpha+\beta}}}{P_1^{\frac{\beta}{\alpha+\beta}}}$$

but

$$\frac{P_1}{P_1^{\frac{\beta}{\alpha+\beta}}} = P_1^{\frac{\alpha+\beta-\beta}{\alpha+\beta}} = P_1^{\frac{\alpha}{\alpha+\beta}}$$

Hicksian Demand Functions and Expenditure Function

Therefore

$$P_1\bar{x}_1 = K_1 \tilde{U}^{\frac{1}{\alpha+\beta}} P_1^{\frac{\alpha}{\alpha+\beta}} P_2^{\frac{\beta}{\alpha+\beta}} = K_1(\tilde{U} P_1^{\alpha} P_2^{\beta})^{\frac{1}{\alpha+\beta}},$$

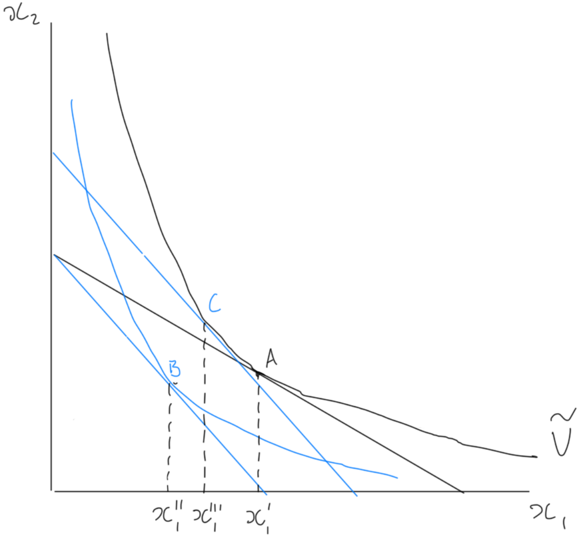
and by symmetry

$$P_2\bar{x}_2 = K_2(\tilde{U} P_2^{\beta} P_1^{\alpha})^{\frac{1}{\alpha+\beta}}.$$

Therefore the expenditure function is:

$$e(\mathbf{P}, \tilde{U}) = K(\tilde{U} P_1^{\alpha} P_2^{\beta})^{\frac{1}{\alpha+\beta}}$$

Marshallian and Hicksian Demand Functions



Marshallian and Hicksian Demand Functions

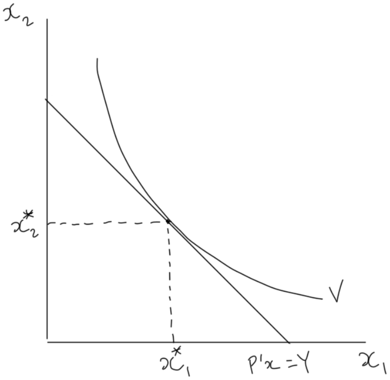
- Suppose $P_1 \uparrow$
- x_1' to x_1'' is given by the Marshallian demand function.
- But Hicksian demand function insists on keeping $\tilde{U}$ constant. Therefore, shift the budget constraint out parallel.
- Movement from x_1' to x_1''' given by Hicksian demand function – not observable, purely theoretical.

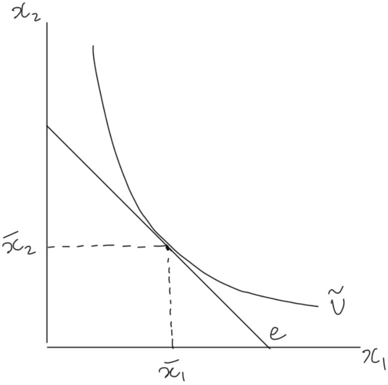
Consistency Properties of Marshallian and Hicksian Demand Functions

$$\begin{aligned} \max \quad & U(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{P}'\mathbf{x} = Y \\ \mathbf{x}^* &= \mathbf{x}^*(\mathbf{P}, Y) \\ V(\mathbf{P}, Y) &= U(\mathbf{x}^*(\mathbf{P}, Y)) \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{P}'\mathbf{x} \\ \text{s.t.} \quad & U(\mathbf{x}) = \tilde{U} \\ \bar{\mathbf{x}} &= \bar{\mathbf{x}}(\mathbf{P}, \tilde{U}) \\ e(\mathbf{P}, \tilde{U}) &= \mathbf{P}'\bar{\mathbf{x}}(\mathbf{P}, \tilde{U}) \end{aligned}$$

Consistency Properties of Marshallian and Hicksian Demand Functions





Consistency Properties of Marshallian and Hicksian Demand Functions

- Suppose $Y = e$. Then, that implies $V = U, \mathbf{x}^* = \bar{\mathbf{x}}$:

$$V(\mathbf{P}, e(\mathbf{P}, \tilde{U}) = \tilde{U}$$
$$\mathbf{x}^*(\mathbf{P}, e(\mathbf{P}, \tilde{U})) = \bar{\mathbf{x}}(\mathbf{P}, \tilde{U})$$

- Suppose we set $\tilde{U} = V$. Then,

$$e(\mathbf{P}, V(\mathbf{P}, Y)) = Y$$
$$\bar{\mathbf{x}}(\mathbf{P}, V(\mathbf{P}, Y)) = \mathbf{x}^*(\mathbf{P}, Y)$$

Consistency Properties of Marshallian and Hicksian Demand Functions

- Consider our previous problem where $U(\mathbf{x}) = x_1^\alpha x_2^\beta$. Then

$$V(\mathbf{P}, Y) = \frac{AY^{\alpha+\beta}}{P_1^\alpha P_2^\beta}$$

- To find the expenditure function we use the consistency properties that $V(\mathbf{P}, e) = \tilde{U}$.

$$\implies \frac{Ae^{\alpha+\beta}}{P_1^\alpha P_2^\beta} = \tilde{U}, \quad \therefore e^{\alpha+\beta} = \frac{\tilde{U} P_1^\alpha P_2^\beta}{A},$$

$$\therefore e = \left(\frac{\tilde{U} P_1^\alpha P_2^\beta}{A}\right)^{\frac{1}{\alpha+\beta}}$$

We can also show that $A = K$ from the previous problem.

Applications of the Envelope Theorem: Roy's Identity

- Now we still start looking at some applications of the Envelope Theorem. Let's first start with Roy's Identity.

Theorem

Let $\mathbf{x}^*(\mathbf{P}, Y)$ *be the Marshallian demand functions and* $V(\mathbf{P}, Y)$ *be the indirect utility function. Then*

$$x_i^*(\mathbf{P}, Y) = -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial Y}$$

Applications of the Envelope Theorem: Roy's Identity

Proof.

Start with $Z = U(\mathbf{x}) + \lambda(Y - \mathbf{P}'\mathbf{x})$. By the Envelope Theorem:

$$\frac{\partial V}{\partial P_1} = \left.\frac{\partial Z}{\partial P_1}\right|_{\mathbf{x}^*,\lambda^*} = -\lambda^* x_i^*$$

Also

$$\frac{\partial V}{\partial Y} = \left.\frac{\partial Z}{\partial Y}\right|_{\mathbf{x}^*,\lambda^*} = \lambda^*$$

$$\therefore x_i^* = -\frac{\partial V}{\partial P_i} \Big/ \frac{\partial V}{\partial Y}$$

□

Applications of the Envelope Theorem: Shephard's Lemma

Theorem

If $\bar{x}_i = (\mathbf{P}, \tilde{U})$ is the Hicksian demand function for good i and $e(\mathbf{P}, \tilde{U})$ is the expenditure function, then:

$$\bar{x}_i = \frac{\partial e}{\partial P_i}$$

Applications of the Envelope Theorem: Shephard's Lemma

Proof.

e is the minimum value function of $\mathbf{P}'\mathbf{x}$ subject to $U(\mathbf{x}) = \tilde{U}$, where the Lagrange is:

$$Z = \mathbf{P}'\mathbf{x} + \lambda(\tilde{U} - U(\mathbf{x}))$$

By the Envelope Theorem:

$$\frac{\partial e}{\partial P_i} = \left.\frac{\partial Z}{\partial P_i}\right|_{\bar{x},\bar{\lambda}} = \bar{x}_i$$

Note

$$\frac{\partial e}{\partial \tilde{U}} = \left.\frac{\partial Z}{\partial \tilde{U}}\right|_{\bar{x},\bar{\lambda}} = \bar{\lambda}$$

where $\bar{\lambda}$ is the shadow price. □

Applications of the Envelope Theorem: Slutsky's Equation

- Using the properties we've just learnt, we can derive the famous Slutsky's Equation.
- From the consistency properties:

$$\bar{x}_i(\mathbf{P}, \tilde{U}) = x_i^*(\mathbf{P}, e(\mathbf{P}, \tilde{U}))$$

Take $\tilde{U} = V = \max \ U^*$

$$\bar{x}_i(\mathbf{P}, U^*) = x_i^*(\mathbf{P}, e(\mathbf{P}, U^*))$$

- Differentiate w.r.t. P_i

$$\left.\frac{\partial \bar{x}_i}{\partial P_i}\right|_{P_i, U^*} = \frac{\partial x_i^*}{\partial P_i} + \frac{\partial x_i^*}{\partial Y}\frac{\partial e(\mathbf{P}, U^*)}{\partial P_i}$$

Applications of the Envelope Theorem: Slutsky's Equation

- But from Shephard's Lemma and the consistency properties:

$$\frac{\partial e(\mathbf{P}, U^*)}{\partial P_i} = \bar{x}_i(\mathbf{P}, U^*) = \bar{x}_i(\mathbf{P}, V(\mathbf{P}, \tilde{U})) = x_i^*$$

Definition

Slutsky's Equation

$$\frac{\partial x_i^*}{\partial P_i} = \underbrace{\frac{\partial \bar{x}_i}{\partial P_i}}_{\text{Substitution Effect}} - \underbrace{\frac{x_i^* \partial x_i^*}{\partial Y}}_{\text{Income Effect}}$$

Symmetry of the Substitution Effect

- By Shephard's Lemma:

$$\frac{\partial \bar{x}_i}{\partial P_j} = \frac{\partial}{\partial P_j}\frac{\partial e}{\partial P_i}$$

and by Young's Theorem

$$= \frac{\partial^2 e}{\partial P_j \partial P_i} = \frac{\partial^2 e}{\partial P_i \partial P_j} = \frac{\partial \bar{x}_j}{\partial P_i}$$

- i.e. The substitution effect of ΔP_j on good i is equal to ΔP_i on good j.

$$\frac{\partial \bar{x}_i}{\partial P_j} = \left\| \frac{\partial x_i^*}{\partial P_j} + \frac{x_j^* \partial x_i^*}{\partial Y} = \frac{\partial x_j^*}{\partial P_i} + \frac{x_i^* \partial x_j^*}{\partial Y} \right\| = \frac{\partial \bar{x}_j}{\partial P_i} \tag{11}$$

The Integrability Problem

- Question: Suppose we are given functions, $\mathbf{x}(\mathbf{P}, Y)$. What conditions are needed to be placed on these functions such that they can be considered as Marshallian demand functions?
  - Answer: The Slutsky symmetry conditions (11).
- Question: Suppose are given a set of Marshallian demand functions. Is it possible to work out the utility function that gave rise to them?
  - Answer: Don't know. Has not been solved (as far as I am aware).
- Question: Suppose we have an indirect utility function, $V(\mathbf{P}, Y)$. Can we obtain the utility function which gave rise to V?
  - Answer: In general, we don't know.

Theory of Production

- Suppose we have a representative competitive firm with one output and n inputs.
- Let y be the quantity of output, $\mathbf{x}$ be the quantity of inputs and let a production be represented by

$$\begin{bmatrix} y \\ -\mathbf{x} \end{bmatrix}_{(n+1)\times 1}$$

Definition

The production set Y is the set of all possible technical feasible solutions. The input requirements set $V(y)$.

$$V(y) = \{\mathbf{x} \in \mathbb{R}^n | \begin{bmatrix} y \\ -\mathbf{x} \end{bmatrix} \in Y\}$$

Theory of Production

Assumptions of $V(y)$:

1. Monotonicity: If $\mathbf{x} \in V(y)$ and $\mathbf{x}' \geq \mathbf{x}$, then $\mathbf{x} \in V(y)$.
2. Convexity: $V(y)$ is a convex set.
3. Regularity: $V(y)$ is closed and non empty.

Definition

If $V(y)$ is regular, convex, and monotonic then for a given y, the boundary of $V(y)$ gives the isoquant.

Note:

- The isoquant gives efficient production.
- If $V(y)$ does not satisfy the above conditions, then the boundary of $V(y)$ need not be the isoquant.

Theory of Production

Definition

A production function $f(\mathbf{x})$ is defined this way: $f(\mathbf{x}) = \{y \in \mathbb{R}^n | y$ is the max output produced by $-\mathbf{x}$ such that $y - \mathbf{x} \in Y\}$.

Notation:

$$y = f(\mathbf{x})$$

i.e. Maximum level of output that can be produced using inputs $\mathbf{x}$.

Homogeneity

- We often assume production functions are homogenous.

Definition

$y = f(\mathbf{x})$ is homogeneous of degree r if $f(\lambda \mathbf{x}) = \lambda^r f(\mathbf{x})$, where λ is a real number.

- e.g. If $f(x_1, x_2) = x_1^2 x_2^3$, then

$$\begin{aligned} f(\lambda x_1, \lambda x_2) &= (\lambda x_1)^2 (\lambda x_2)^3 = \lambda^{2+3} x_1^2 x_2^3 \\ &= \lambda^5 f(x_1, x_2) \end{aligned}$$

we say that $f(x_1, x_2)$ is homogenous of degree 5.

Euler's Theorem

Definition

If $f(x)$ is homogenous of degree r then

$$\frac{x_1 \partial f}{\partial x_1} + \ldots + \frac{x_n \partial f}{\partial x_n} = rf(\mathbf{x})$$

- e.g. If $f_1 = 2x_1x_2^3$ and $f_2 = 3x_1^2x_2^2$. Then

$$x_1 f_1 + x_2 f_2 = 2x_1^2x_2^3 + 3x_1x_2^3 = 5f(x_1, x_2)$$

The Technical Rate of Substitution

Definition

Changes to inputs x_1 and x_2 in order to maintain same level of output.

i.e.

$$dy = \frac{df}{dx_1}dx_1 + \frac{df}{dx_2}dx_2 = 0$$

$$\implies \frac{dx_2}{dx_1} = -\frac{\partial f}{\partial x_1} \Big/ \frac{\partial f}{\partial x_2}$$

Returns to Scale

- We say that technology exhibits constant returns to scale if: $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}) \ \forall \ \lambda > 0$.
- Increasing returns to scale if: $f(\lambda \mathbf{x}) > \lambda f(\mathbf{x}) \ \forall \ \lambda > 1$.
- Decreasing returns to scale if: $f(\lambda \mathbf{x}) < \lambda f(\mathbf{x}) \ \forall \ \lambda > 1$.
- e.g. Suppose $y = f(\mathbf{x}) = Ax_1^\alpha x_2^\beta$. Then $f(\lambda \mathbf{x}) = \lambda^{\alpha+\beta} Ax_1^\alpha x_2^\beta$
    - $\alpha + \beta = 1 \implies$ CRTS.
    - $\alpha + \beta > 1 \implies$ IRTS.
    - $\alpha + \beta < 1 \implies$ DRTS.

Neoclassical Problems of the Firm

- No distinction between the short run and long run
- Profit maximisation: maximise profit for given set of parameters, namely, price of out, P, and price of inputs, $\mathbf{w}$:

$$\max \;\; Pf(\mathbf{x}) - \mathbf{w}'\mathbf{x} \rightarrow \mathbf{x}^* = \mathbf{x}^*(P, \mathbf{w})$$

  where $\mathbf{x}{*}(P, \mathbf{w})$ is the firm's demand for inputs.
- Maximum value function:

$$\pi(P, \mathbf{w}) = Pf(\mathbf{x}^*(P, \mathbf{w}) - \mathbf{w}'\mathbf{x}^*(P, \mathbf{w})$$

Neoclassical Problems of the Firm

- Cost minimisation problem:

$$\begin{aligned} \min \quad & \mathbf{w}'\mathbf{x} \\ \text{s.t.} \quad & f(\mathbf{x}) = y \end{aligned}$$

  where the optimal point occurs at $\bar{\mathbf{x}} = \bar{\mathbf{x}}(y, \mathbf{w})$: the conditional demand function for inputs.
- Minimum value function: $C(\mathbf{w}, y) = \mathbf{w}'\bar{\mathbf{x}}(y, \mathbf{w})$
- Combining the two problems:

$$\max \quad Py - C(\mathbf{w}, y)$$

  where the optimal point occurs at $y^*(\mathbf{w}, P)$, the supply function.
- Minimum value function: $\pi(P, \mathbf{w}) = Py^*(\mathbf{w}, P) - C(\mathbf{w}, y^*(\mathbf{w}, P))$.

Neoclassical Problems of the Firm

- e.g. Suppose $y = Ax_1^{\alpha}x_2^{\beta}$, and we have to find the conditional demand functions and the cost function.
- Set up problem and the Lagrangian function:

$$\min \quad w_1x_1 + w_2x_2 \tag{12}$$
$$\text{s.t.} \quad Ax_1^{\alpha}x_2^{\beta} = y \tag{13}$$

$$Z = w_1x_1 + w_2x_2 + \lambda(y - Ax_1^{\alpha}x_2^{\beta})$$

- We could solve this problem by attain FOCs, and then solving:

$$\frac{\partial Z}{\partial x_1} = 0, \quad \frac{\partial Z}{\partial x_2} = 0, \quad \frac{\partial Z}{\partial \lambda} = 0$$

- But let's try something different (hopefully your algebra skills are good).

Neoclassical Problems of the Firm

Start with (13)

$$x_2^{\beta} = yA^{-1}x_1^{-\alpha}$$
$$\therefore x_2 = y^{\frac{1}{\beta}}A^{-\frac{1}{\beta}}x_1^{-\frac{\alpha}{\beta}}$$

Then sub x_2 into (12):

$$w_1x_1 + w_2y^{\frac{1}{\beta}}A^{-\frac{1}{\beta}}x_1^{-\frac{\alpha}{\beta}}$$

take the partial differential w.r.t. x_1:

$$w_1 - \frac{\alpha}{\beta}w_2y^{\frac{1}{\beta}}A^{-\frac{1}{\beta}}x_1^{\frac{-\alpha+\beta}{\beta}} = 0$$

Neoclassical Problems of the Firm

cont...

$$x_1^{\frac{-\alpha+\beta}{\beta}} = \frac{\beta w_1}{\alpha w_2} y^{-\frac{1}{\beta}} A^{\frac{1}{\beta}}$$

$$\therefore \bar{x}_1 = \left(\frac{\beta w_1}{\alpha w_2}\right)^{-\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}}$$

$$\bar{x}_1 = \left(\frac{\alpha}{\beta}\frac{w_2}{w_1}\right)^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} \tag{14}$$

and by symmetry:

$$\bar{x}_2 = \left(\frac{\beta}{\alpha}\frac{w_1}{w_2}\right)^{\frac{\alpha}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} \tag{15}$$

Neoclassical Problems of the Firm

This is going to be messy. Basically, sub (14) and (15) into (12):

$$\begin{aligned}
C(\mathbf{w}, y) &= \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \\
&\quad + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \\
&= A^{-\frac{1}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}}
\end{aligned}$$

Applications of the Envelope Theorem: Shephard's Lemma (again)

Theorem

Let $C(\mathbf{w}, y)$ be the differentiable cost function and $\bar{x}_i(\mathbf{w}, y)$ be the firm's conditional demand function for input i. Then

$$\bar{x}_i(\mathbf{w}, y) = \frac{\partial C(\mathbf{w}, y)}{\partial w_i}, \quad i = 1, ..., n.$$

Applications of the Envelope Theorem: Shephard's Lemma (again)

Proof.

$C(\mathbf{w}, y)$ is the minimum value function of

$$\begin{aligned} \min \quad & \mathbf{w}'\mathbf{x} \\ \text{s.t.} \quad & f(\mathbf{x}) = y \end{aligned}$$

$$\therefore Z = \mathbf{w}'\mathbf{x} + \lambda(y - f(\mathbf{x}))$$

By the Envelope Theorem:

$$\frac{\partial C}{\partial w_i} = \left.\frac{\partial Z}{\partial w_i}\right|_{\bar{\mathbf{x}}, \bar{\lambda}} = \bar{x}_i$$

$$\frac{\partial C}{\partial y} = \left.\frac{\partial Z}{\partial y}\right|_{\bar{\mathbf{x}}, \bar{\lambda}} = \bar{\lambda}$$

Properties of Cost Functions

- $C(\mathbf{w}, y)$ is increasing in factor prices, $\mathbf{w}$:

$$\frac{\partial C}{\partial w_i} > 0$$

But by Shephard's Lemma:

$$\frac{\partial C}{\partial w_i} = \bar{x}_i$$

$$\therefore \Longrightarrow \bar{x} > 0$$

- $C(\mathbf{w}, y)$ is homogeneous of degree 1 in $\mathbf{w}$. Therefore, $\frac{\partial C}{\partial w_i}$ is homogeneous of degree 0 in $\mathbf{w}$; which implies that $\bar{x}_i(\mathbf{w}, y)$ is homogenous of degree 0 in $\mathbf{w}$.
- $C(\mathbf{w}, y)$ is concave in $\mathbf{w}$.

Duality of the Firm Problem

- Consider the production function, $y = f(\mathbf{x})$, and the cost function, $C(\mathbf{w}, y)$.
- These are often referred to as duals – in the sense that they contain the same amount of economic information (this should be familiar).
- i.e. From the production function we can derive the cost function, and from the cost function we can derive the production function.
- Economically, levels of input that minimises the costs of producing a certain level of output, regardless of input prices, must represent efficient production.
- Conditional production functions do this.

Duality of the Firm Problem

Suppose

$$C(\mathbf{w}, y) = y w_1^{\alpha} w_2^{1-\alpha}$$

Find the production function. By Shephard's Lemma, the conditional demand functions are:

$$\bar{x}_1 = \frac{\partial C}{\partial w_1} = \alpha y w_1^{\alpha-1} w_2^{1-\alpha} = \alpha y \left(\frac{w_2}{w_1}\right)^{1-\alpha} \tag{16}$$

$$\bar{x}_2 = \frac{\partial C}{\partial w_2} = (1-\alpha) y w_1^{\alpha} w_2^{-\alpha} = (1-\alpha) y \left(\frac{w_2}{w_1}\right)^{-\alpha} \tag{17}$$

From (16) and (17), solve for y. After cleaning up, you should get :

$$y = \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}} x_1^{\alpha} x_2^{1-\alpha} = K x_1^{\alpha} x_2^{1-\alpha}$$

Applications of the Envelope Theorem: Hotelling's Lemma

Definition

Let y^* be the firm's supply function and x_i^* be the firm's demand function for input i. Let π^* be the profit function, then the supply function is

$$y^* = \frac{\partial \pi^*}{\partial P}$$
$$x_i^* = -\frac{\partial \pi^*}{\partial w_i}$$

Applications of the Envelope Theorem: Hotelling's Lemma

Proof.

$\pi^*(P, \mathbf{w})$ is the max value function of the problem

$$\begin{aligned} \max \quad & Py - \mathbf{w}'\mathbf{x} \\ \text{s.t.} \quad & y = f(\mathbf{x}) \end{aligned}$$

$$Z = Py - \mathbf{w}'\mathbf{x} + \lambda(y - f(\mathbf{x}))$$

By the Envelope Theorem

$$\frac{\partial \pi^*}{\partial P} = \left.\frac{\partial Z}{\partial P}\right|_{\mathbf{x}^*, y^*, \lambda^*} = y^*$$

$$\frac{\partial \pi^*}{\partial w_i} = \left.\frac{\partial Z}{\partial w_i}\right|_{\mathbf{x}^*, y^*, \lambda^*} = x_i^*$$

Concluding Remarks

- Well, that's it!
- As always, we did rush through things. You probably noticed that we skipped a lot of detail in order to derive the main results of the envelope theorem and its applications to economics.
- Hopefully we have some time left over to discuss simple linear regression models...