Linear Programming: Constrained Optimisation

Intro Math for Economists (PEARL, Spring 2019)

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Review of Unconstrained Optimisation

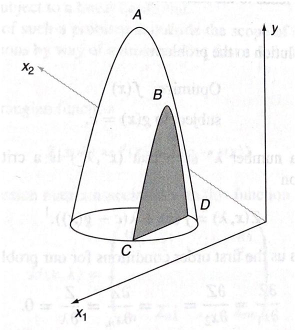
- The good news is that constrained optimisation is relatively easy and straightforward so long as you understood unconstrained optimisation.
- As such, let's use this chance to go over any questions or uncertainties you may have about unconstrained optimisation.

The Linear Programming Problem

- As mentioned in the introductory lecture, the classic economic linear programming problem can be generalised as:

$$
\begin{aligned}
\text{Optimise} \quad & Z = f(x_1, \ldots, x_n) \\
\text{subject to} \quad & g^1(x_1, \ldots, x_n) \leq = \geq c_1 \\
& \vdots \\
& g^m(x_1, \ldots, x_n) \leq = \geq c_m \\
& x_1 \geq 0, \ldots, x_n \geq 0
\end{aligned}
$$

The Linear Programming Problem



We only consider the points on the line segment CD. Such points, satisfying the constraints, form the set of feasible solutions. The constrained maximum is then B.

The Linear Programming Problem

- Unconstrained optimisation is common in economics.
- But more are constrained optimisation problems.
- Optimise (maximise/minimise) a function subject subject to an equality constraint.
- e.g. Maximise household utility subject to a budget constraint; minimise firm costs subject to production constraint; maximise profits subject to resource constraints; and so on.

The Lagrangian Multiplier Technique

- We know how to find maxima and minima when we face no constraints?
- How do we do this for constrained problems?
- One method: brute force algebra and calculus (substitute variables, cancel out, and then take derivatives). Another more elegant method: the Lagrangian multiplier technique.
- In a nutshell, Lagrange's method for constrained optimisation is to convert the problem into an unconstrained problem by introducing a new variable called the Lagrangian multiplier.

The Lagrangian Multiplier Technique

Definition

The Lagrangian function is:

$$Z(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda(c - g(\mathbf{x})) \tag{1}$$

where λ is the Lagrangian multiplier.
Suppose we can ensure that $g(\mathbf{x}) = c$. Then the last term in the Lagrangian function is zero for all λ and Z becomes the objective function $f(\mathbf{x})$. We could then optimise Z freely instead of optimising $f(\mathbf{x})$ subject to the equality constraint.

The Lagrangian Multiplier Technique

Ensuring that the constraint holds is easy. Consider

$$\frac{\partial Z}{\partial \lambda} = c - g(\mathbf{x})$$

If we set this derivative equal to zero, then $g(\mathbf{x}) = c$. We then get the following theorem:

Theorem

Suppose $\mathbf{x}^*$ is a solution to the problem:

$$\begin{aligned} \mathit{Optimise}\ & f(\mathbf{x}) \\ s.t.\ & g(\mathbf{x}) = c \end{aligned}$$

Then there is a number λ^* such that $(\mathbf{x}^*, \lambda^*)$ is a critical point of the Lagrangian function given in (1).

The Lagrangian Multiplier Technique

The previous theorem gives us the **first order conditions** for our problem, namely:

$$\frac{\partial Z}{\partial x_1} = \frac{\partial Z}{\partial x_2} = \ldots = \frac{\partial Z}{\partial x_n} = \frac{\partial Z}{\partial \lambda} = 0, \quad i = 1, \ldots, n.$$

Lagrangian Multiplier Example

- Let's work some examples. The first thing we need to do when we are confronted with a potential constrained optimisation problem is to identify what it is that we need to optimise and whether or not we're being constrained by some other factor.
- Once we've identified the equation that we need to optimise and its constraints, we can then set up our problem.
- Suppose we have the following problem:

$$\begin{aligned} \max \quad & y = x_1^2 + 3x_1x_2 - 3x_2^2 \\ \text{s.t.} \quad & x_1 + 3x_2 = 6 \end{aligned}$$

Lagrangian Multiplier Example

- The Lagrangian function for this problem is then:

$$Z = x_1^2 + 3x_1x_2 - 3x_2^2 + \lambda(6 - x_1 - 3x_2) \tag{2}$$

- We need optimal values of x_1 and x_2 (x_1^* and x_2^*).
- Need to take partial derivatives of Z w.r.t. to x_1, x_2 and λ and set up the first order conditions. This yields:

$$\frac{\partial Z}{\partial x_1} = 2x_1 + 3x_2 - \lambda = 0 \tag{3}$$

$$\frac{\partial Z}{\partial x_2} = 3x_1 - 6x_2 - 3\lambda = 0 \tag{4}$$

$$\frac{\partial Z}{\partial \lambda} = 6 - x_1 - 3x_2 = 0 \tag{5}$$

Lagrangian Multiplier Example

- Can solve this in many different ways. I'm going to start with (3) and rearranging it to get λ in terms of x_1 and x_2:

$$\lambda = 2x_1 + 3x_2 \tag{6}$$

- Then, I'm going to substitute λ from 6 into 4 to get:

$$3x_1 - 6x_2 - 3(2x_1 - 3x_2) = 0,$$

which we expand the brackets to get

$$x_1 = -5x_2 \tag{7}$$

Lagrangian Multiplier Example

- Sub x_1 from 7 into 5 to get the optimal value of x_2:

$$\begin{aligned} 6 - (5x_2) - 3x_2 &= 0 \\ 6 + 5x_2 - 3x_2 &= 0 \\ 6 + 2x_2 &= 0 \\ 2x_2 &= -6 \\ \therefore x_2^* &= -3 \end{aligned}$$

- With x_2^* in hand, we know from equation 7 that $x_1^* = -5(-3) = 15$. Lastly, let's get the Lagrangian multiplier value, λ, from 3:

$$\lambda^* = 2(15) + 3(-3) = 21$$

Lagrangian Multiplier Example

- Confirm the result by substituting the critical values x_1^*, x_2^* and λ^* into the original constraint function:

$$
\begin{aligned}
x_1 + 3x_2 &= 6 \\
15 + 3(-3) &= 6
\end{aligned}
$$

Clearly, it binds.

Lagrangian Multiplier Economic Application

- Now let's look at an application of Lagrangian optimisation to economics.
- In microeconomics we say that a consumer is behaving optimally if their marginal rate of substitution (MRS) is equal to their marginal rate of transformation (MRT).
- Suppose Akane's utility function, $U(B, M)$, is a function of burgers, B, and movies, M, and has the form $U = \sqrt{BM}$. She has an income of \$96, and the price of burgers are \$16 and the price of movies are \$8. With this, we can setup the problem

$$\begin{aligned} \max \quad & U(B, M) = \sqrt{BM} \\ \text{s.t.} \quad & P_B B + P_M M = 96 \end{aligned}$$

Lagrangian Multiplier Economic Example

- Setting up the Lagrangian function for this problem gives us:

$$Z = \sqrt{BM} + \lambda(96 - P_B B - P_M M) \tag{8}$$

- Taking first order partial derivatives gives us our FOCs:

$$\frac{\partial Z}{\partial B} = \frac{1}{2}B^{-1/2}M^{1/2} - \lambda P_B = 0 \tag{9}$$

$$\frac{\partial Z}{\partial M} = \frac{1}{2}B^{1/2}M^{-1/2} - \lambda P_M = 0 \tag{10}$$

$$\frac{\partial Z}{\partial \lambda} = 96 - P_B B - P_M M = 0 \tag{11}$$

Lagrangian Multiplier Economic Example

- Again, there are many ways to go about finding B^*, M^* and λ^*. For starters, I'm going to make things easier by moving the two price terms in 9 and 10 to the right hand side (RHS):

$$\frac{1}{2}B^{-1/2}M^{1/2} = \lambda P_B \tag{12}$$

$$\frac{1}{2}B^{1/2}M^{-1/2} = \lambda P_M \tag{13}$$

- If we look at the LHS, we actually have Akane's marginal utility of burgers, MU_B, and her marginal utility of movies, MU_M.
- Recall that the marginal utility of good x_i is the partial derivative of the utility function derived w.r.t. good x_i:

$$MU_i = \frac{\partial U(\mathbf{x})}{\partial x_i}$$

Lagrangian Multiplier Economic Example

- The utility function for Akane, $U(B, M)$, was inputted directly into the Lagrangian function! So that means we have:

$$MU_B = \lambda P_B \tag{14}$$
$$MU_M = \lambda P_M \tag{15}$$

- We can actually go further and divide the terms to yield the optimal point of Akane's utility maximisation problem (and to cancel out the two λ terms):

$$\frac{MU_B}{MU_M} = \frac{P_B}{P_M} \tag{16}$$
$$\text{i.e. } MRS = MRT \tag{17}$$

Lagrangian Multiplier Economic Example

- So, it must be the case that if we find the optimal values of B and M for Akane's Lagrangian problem, that the condition of $MRS = MRT$ will hold!
- Economic theory tells us at that point Akane is maximising her utility subject to her budget constraint.
- Mathematically, we know at that point the FOCs will give us an optimal solution. THIS IS IMPORTANT!

Lagrangian Multiplier Economic Example

- Let's find B^* and M^*:

$$\frac{\frac{1}{2}B^{-1/2}M^{1/2}}{\frac{1}{2}B^{1/2}M^{-1/2}} = \frac{P_B}{P_M} \tag{18}$$

- Rearrange this mess:

$$\begin{aligned} \frac{\frac{1}{2}M^{1/2}M^{1/2}}{\frac{1}{2}B^{1/2}B^{1/2}} &= \frac{P_B}{P_M} \\ \frac{M}{B} &= \frac{16}{8} \\ M &= 2B \end{aligned} \tag{19}$$

- Substitute 19 into 11, which gives us:

$$B^* = 3 \tag{20}$$

Lagrangian Multiplier Economic Example

- Then sub 20 into 19 to get $M^* = 2(3) = 6$.
- We can also find the shadow price for Akane's problem using either equation 9 or 10. I'm going to use 9:

$$\frac{1}{2}B^{-1/2}M^{1/2} - \lambda P_B = 0$$

$$\frac{1}{2}(3)^{-1/2}(6)^{1/2} - \lambda(16) = 0$$

$$\frac{1}{2}(\frac{6}{3})^{1/2} = 16\lambda$$

$$\therefore \lambda^* = 8\sqrt{2} \approx 11.3$$

- To summarise, the optimal bundle (tangent point) for Akane is:

$$B^* = 3, M^* = 6, \lambda^* = 8\sqrt{2}$$

Concluding Remarks

- That's essentially it for constrained optimisation.
- You actually have the tools now to solve many future optimisation problems in economics.
- The equations will get harder (for example, you will look at dynamic problems), but the essential logic remains the same:
- How can we find the maximum/minimum point of a function when we are bound by constraints?
- As always, we skipped a lot of material to get this lecture concluded. Please refer to the readings (especially Turkington) for more information.