Neoclassical Growth Models Part II: The OLG and Ramsey Model

References

"The ABCs of RBCs: an introduction to dynamic macroeconomic models", McCandless, G., *Harvard University Press*, 2008.

"Advanced Macroeconomics", Romer, D., *McGraw Hill Education*, 4th edition, 2012.

1 OLG with Population Growth

Consider a standard two period OLG model with log utility and Cobb-Douglas production technology $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Capital depreciates completely after a period and households only work when young (they inelastically supply labour normalised to 1 when young).

1.1

Assume that there is no population growth, so that K_{t+1}^1 is the savings per young household made at time t and the aggregate capital stock per young household at $t+1$. Write down the optimisation problem of the household when it is young and solve for the optimal consumption and savings decisions it makes, taking the wage rate w_t and return to capital r_{t+1} as given. Write down the problem of the perfectly competitive firm and show that market clearing implies a law of motion for capital of the form:

$$K_{t+1} = \frac{1}{2}(1-\alpha)K_t^{\alpha}.$$

Solve for the steady state capital stock and log-linearise the law of motion for capital.

The basic OLG is described as follows. Let there be an infinite sequence of time, $t = 0, 1, 2, ..., \infty$. Each generation born in period t is referred to as generation t. There are $N(t)$ members of generation t, and people live for 2 periods, generation t is young in t and old in $t+1$. Generation t does not exist in period $t+2$.

A member h of generation t has utility

$$u_t^h(c_t^h(t), c_t^h(t+1)). \tag{1}$$

Production takes place in competitive firms with HOD1 production technology (CRTS), implying that they do not produce economic profits. Production in period t is given by

$$y(t) = F(K(t), L(t)) = K(t)^{\alpha} L(t)^{1-\alpha}.$$

Individuals are endowed with lifetime endowment of labour given by

$$l_t = [l_t^h(t), l_t^h(t+1)].$$

Total labour is given as

$$L(t) = \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t).$$

Aggregate labour of the young at time t is the first component of the RHS, and the aggregate labour of the old is the second component of the RHS. We also assume that $K(t)$ depreciates fully.

The economy has the following resource constraint:

$$y(t) = F(K(t), L(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1).$$

Members of generation t earn income in period t by offering all their labour endowment to firms at market wage, w_t, and use income to fuel consumption in period t, to fund borrowing and lending to other members of generation t, and for accumulation of private capital. The budget constraint for individual h when they're young is

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1), \tag{2}$$

where $a^h(t)$ are net asset holdings of individual h. $a^h(t) < 0$ implies net borrowing from other members of generation t. Individuals cannot borrow or lend across generations so

$$\sum_{h=1}^{N(t)} a^h(t) = 0.$$

The budget constraint for generation t individual in period $t+1$ is

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t a^h(t) + R_{t+1} k^h(t+1). \tag{3}$$

Factor prices are determined by their marginal products due to competitive equilibrium:

$$\begin{aligned} w_t &= F_L(K(t), L(t)) \\ R_t - 1 &= F_K(K(t), L(t)). \end{aligned}$$

We can combine the budget constraints of the young and old. From (2):

$$a^h(t) = w_t l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substitute this expression into (3) to get:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t w_t l_t^h(t) - R_t c_t^h(t) - R_t k^h(t+1) + R_{t+1} k^h(t+1),$$

collecting terms, we can yield an expression for $c_t^h(t)$:

$$c_t^h(t) = \frac{w_{t+1} l_t^h(t+1) - c_t^h(t+1)}{R_t} + w_t l_t^h(t) - k^h(t+1) \left[1 - \frac{R_{t+1}}{R_t}\right].$$

Since we assume that there are no arbitrage opportunities, the return on capital should equal the return on loans amongst members of a particular cohort, $r_t = r_{t+1}$. Thus the budget constraint becomes:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t} = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}. \tag{4}$$

A competitive equilibrium consists of a sequence of prices $\{w_t, R_t\}_{t=0}^{\infty}$, and quantities $\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty}$, such that each member h of each generation $t > 0$ maximises utility (1) subject to their lifetime budget constraint given by (4), and so that the equilibrium conditions

$$\begin{aligned}
R_{t+1} &= R_t, \\
w_t &= F_L(K(t), L(t)), \\
R_t - 1 &= F_K(K(t), L(t)), \\
L(t) &= \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t),
\end{aligned}$$

hold each period.

We attain optimal consumption by maximising utility (1) subject to (4) and yielding the consumption Euler equation. Given that our utility function is of the form

$$u(c_t^h(t), c_t^h(t+1)) = \ln c_t^h(t) + \ln c_t^h(t+1),$$

we differentiate the following equation wrt to consumption in period t:

$$\begin{aligned}
&\arg\max_{c_t^h(t)} \left\{\ln c_t^h(t) + \ln\left[R_t w_t l_t^h(t) - R_t c_t^h(t) + w_{t+1} l_t^h(t+1)\right]\right\} && (5) \\
&\implies 0 = \frac{1}{c_t^h(t)} - \frac{R_t}{c_t^h(t+1)} \\
&\qquad 1 = \frac{R_t c_t^h(t)}{c_t^h(t+1)}. && (6)
\end{aligned}$$

We can then insert optimal consumption given by the Euler equation back into the budget constraint (4) to get:

$$2c_t^h(t) = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}.$$

We introduce a further simplifying assumption that the old do not supply their labour. i.e. $l_t^h(t+1) = 0$. Thus we have:

$$c_t^h(t) = \frac{1}{2} w_t l_t^h(t).$$

Now that we have consumption per period for an individual h when they are young, we want to pin down aggregate savings. We know that individuals only live for two periods – i.e. individual h of generation t will not live past $t+1$. Thus, they will not save in period $t+1$. To pin down aggregate savings we need the budget constraint of a young person given in (2):

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1),$$

then define savings for an individual h of generation t as:

$$s^h(t) = w_t l_t^h(t) - c_t^h(t) = a^h(t) + k^h(t+1).$$

Aggregating across an entire cohort gives:

$$\begin{aligned} S(t) &= \sum_{h=1}^{N(t)} s^h(t) = \underbrace{\sum_{h=1}^{N(t)} a^h(t)}_{=0} + \sum_{h=1}^{N(t)} k^h(t+1) \\ \implies S(t) &= K(t+1). \end{aligned}$$

$S(t)$ is dependent on wages in t and $t+1$ and returns to capital, given to us in our equilibrium conditions. Furthermore, we assumed that the elderly do not supply labour. Therefore, $K(t+1)$ depends on $L_t(t)$ and $L_{t+1}(t+1)$, parameters of $u(\cdot)$, α, and $K(t)$ – everything except for $K(t)$ are constants. Therefore, the law of motion of capital is given by some function of $K(t)$:

$$K(t+1) = G(K(t)).$$

Turning to the firm and production, we know that the firm produces output according to a Cobb-Douglas technology:

$$Y(t) = K(t)^\alpha L(t)^{1-\alpha}. \tag{7}$$

Taking the partial derivative of output wrt capital and labour yields the following factor prices:

$$\begin{aligned} \frac{\partial Y(t)}{\partial K(t)} &= R_t = \alpha \left[\frac{K(t)}{L(t)}\right]^{\alpha-1} = \alpha k(t)^{\alpha-1}, \\ \frac{\partial Y(t)}{\partial L(t)} &= w_t = (1-\alpha)\left[\frac{K(t)}{L(t)}\right]^{\alpha} = (1-\alpha)k(t)^\alpha. \end{aligned}$$

From our household FOCs, we have

$$c_t^h(t) = \frac{1}{2} w_t l_t^h(t) = \frac{1}{2}(1-\alpha)k(t)^\alpha l_t^h(t),$$

and aggregating across the cohort yields

$$C_t(t) = \frac{1}{2}(1-\alpha)K(t)^\alpha L(t)^{1-\alpha} = \frac{1}{2}(1-\alpha)Y(t).$$

The household FOCs and the above equation pins down aggregate savings:

$$S(t) = \frac{1}{2}(1-\alpha)Y(t),$$

and since $S(t) = K(t+1)$,

$$\implies K(t+1) = \frac{1}{2}(1-\alpha)Y(t).$$

Since labour is supply inelastically by the young, the law of motion of capital can be written as

$$K(t+1) = \frac{1}{2}(1-\alpha)K(t)^{\alpha}. \tag{8}$$

The steady state capital stock, $\bar{K}$, is given by the condition $\Delta K(t) = 0$[1]:

$$\Delta K(t) = 0 = K(t+1) - K(t) = \frac{1}{2}(1-\alpha)K(t)^{\alpha} - K(t),$$

rearranging and solving for $\bar{K}$ yields

$$\begin{aligned}
0 &= \frac{1}{2}(1-\alpha)K(t)^{\alpha} - K(t) \\
K(t) &= \frac{1}{2}(1-\alpha)K(t)^{\alpha} \\
K(t)^{-\alpha}K(t) &= \frac{1-\alpha}{2} \\
\implies \bar{K} &= \left(\frac{1-\alpha}{2}\right)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

We can log linearise the law of motion for capital (8). First, take logs

$$\ln K(t+1) = \ln\left(\frac{1-\alpha}{2}\right) + \alpha \ln K(t),$$

and then taking a first order Taylor expansion gives:

$$\ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{1-\alpha}{2}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}).$$

In the steady state we know that

$$\ln \bar{K} = \frac{1}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right),$$

which gives us:

$$\frac{1}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right) + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}),$$

[1] Market clearing in the steady state implies that

$$\begin{aligned}
y(t) &= K(t+1) + C_t(t) + C_t(t+1) \\
\implies K(t)^{\alpha} &= \frac{1}{2}(1-\alpha)K(t)^{\alpha} + \frac{1}{2}(1-\alpha)K(t)^{\alpha} + r_t K(t+1) \\
\bar{K}^{\alpha} &= \frac{1}{2}(1-\alpha)\bar{K}^{\alpha} + \frac{1}{2}(1-\alpha)\bar{K}^{\alpha} + \alpha\bar{K}^{\alpha-1}\bar{K}.
\end{aligned}$$

simplifying this expressions yields:

$$
\begin{aligned}
\frac{(K(t+1)-\bar{K})}{\bar{K}} &\approx \ln\left(\frac{1-\alpha}{2}\right) + \frac{\alpha}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right) - \frac{1}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right) + \alpha\frac{(K(t)-\bar{K})}{\bar{K}} \\
\tilde{K}(t+1) &\approx \left[1+\frac{\alpha-1}{1-\alpha}\right]\ln\left(\frac{1-\alpha}{2}\right) + \alpha\tilde{K}(t) \\
\tilde{K}(t+1) &\approx \left[\frac{1-\alpha+\alpha-1}{1-\alpha}\right]\ln\left(\frac{1-\alpha}{2}\right) + \alpha\tilde{K}(t) \\
\tilde{K}(t+1) &\approx \alpha\tilde{K}(t). \qquad (9)
\end{aligned}
$$

where $\tilde{K}$ is log deviation from the steady state value.

1.2

Discuss how an increase in α affects the steady state capital stock and its law of motion when there is no population growth.

To see how an increase in α affects the steady state level of capital and its law of motion, begin by taking the derivative of $\bar{K}$ wrt α:

$$
\begin{aligned}
\bar{K} &= \left[\frac{1-\alpha}{2}\right]^{\frac{1}{1-\alpha}} \\
\ln\bar{K} &= \frac{1}{1-\alpha}\ln\left(\frac{1-\alpha}{2}\right) \\
\frac{d\ln\bar{K}}{d\alpha} &= (-1)(1-\alpha)^{-2}(-1)\ln\left(\frac{1-\alpha}{2}\right) + \frac{1}{1-\alpha}\left(\frac{-1}{1-\alpha}\right) \\
&= \frac{1}{(1-\alpha)^2}\left[\ln\left(\frac{1-\alpha}{2}\right) - 1\right].
\end{aligned}
$$

So, an increase in α leads to a decrease in the steady state level of capital. An increase in α implies a greater share of profits paid to capital at the expense of labour, leading to a fall in wage income, which leads to less savings and thus less capital. We can also see from equation (9) that an increase in α causes a stronger response in $\tilde{K}(t+1)$ to changes in $\tilde{K}(t)$. In other words, a change in capital in period t will lead to bigger changes in capital in period $t+1$ for larger values of α.

1.3

Now introduce constant population growth so that $L_{t+1} = (1+\eta)L_t$. If K^1_{t+1} is the savings per young household made at time t then the aggregate capital stock per young household at $t+1$ will be $(L_t K^1_{t+1})/L_{t+1}$. Derive the law of motion for capital in the economy with population growth. Solve for steady state and log linearise the law of motion for capital. How does an increase in the population growth rate η affect these objects?

Population now grows at rate η so that

$$N(t+1) = (1+\eta)N(t).$$

Recall that aggregate capital in the model without population growth was simply the sum of savings by all individuals in the $N(t)$ cohort (since we assumed that aggregate borrowing/lending within the cohort summed to 0). Capital per young individual in period $t+1$ with population growth is now

$$k^h(t+1) = \frac{\sum_{h=1}^{N(t)} k^h(t+1)}{N(t+1)},$$

and since we know population grows at rate η we can rewrite this as:[2]

$$k^h(t+1)N(t) = \frac{K_t(t+1)}{(1+\eta)}.$$

We can use this definition to rewrite the law of motion of capital as

$$K(t+1) = \frac{(1-\alpha)}{2(1+\eta)}K(t)^{\alpha}. \tag{10}$$

We find the steady state of capital as before by setting $\Delta K(t) = K(t+1) - K(t) = 0$ and solving for $\bar{K}$:

$$\begin{aligned} 0 &= \frac{(1-\alpha)}{2(1+\eta)}K(t)^{\alpha} - K(t) \\ \implies \bar{K} &= \left[\frac{(1-\alpha)}{2(1+\eta)}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

We can log linearise the law of motion of capital. Start by taking logs of (10):

$$\ln K(t+1) = \ln\left(\frac{(1-\alpha)}{2(1+\eta)}\right) + \alpha \ln K(t),$$

then take a first order Taylor expansion around the steady state:

$$\ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{(1-\alpha)}{2(1+\eta)}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}),$$

and since we know $\ln \bar{K} = \frac{1}{1-\alpha} \ln \frac{(1-\alpha)}{2(1+\eta)}$, substituting and rearranging gives us an expression for linearised log deviations from steady state:

$$\tilde{K}(t+1) \approx \alpha \tilde{K}(t).$$

So we can conclude that by including population growth in our OLG model, the steady state level of capital declines, however our transition dynamics for a shock to $K(t)$ remain unchanged relative to the model without population growth.

[2]Note that this is aggregate capital, and hence we discount it by the factor $1+\eta$. If we were looking at aggregate capital belonging to a particular cohort,

1.4

How would labour and capital income taxes affect the steady state of the economy? To do this, assume that taxes are paid by the household and proportional to labour and capital income, then calculate how the taxes affect FOCs. You should then be able to identify what changes in steady state. Explain the intuition behind your findings.

Applying a tax to labour and capital income will modify the individual's optimisation problem and the law of motion of capital. The budget constraint for individual h of generation t in period $t+1$ is now

$$c_t^h(t+1) = w_{t+1}(1-\tau_L)l_t^h(t+1) + R_t(1-\tau_K)a^h(t) + R_{t+1}(1-\tau_K)k^h(t+1),$$

where τ_L and τ_K are the tax rates on labour and capital, respectively.[3] From (2) we can get the following asset holdings of individual h:

$$a^h(t) = w_t(1-\tau_L)l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substituting this into the the adjusted budget constraint above gives the following (after rearranging and simplifying) lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t(1+\tau_K)} = w_t(1-\tau_L)l_t^h(t), \tag{11}$$

since we assumed that $l_t^h(t+1) = 0$ and that due to perfectly competitive markets, the interest on private borrowing and lending and the return on capital are the same so $R_t = R_{t+1}$. The individual's maximisation problem is therefore:

$$\arg\max_{c_t^h(t)} \left\{ \ln c_t^h(t) + \ln\left[R_t w_t(1-\tau_L)l_t^h(t) - (1-\tau_K)R_t c_t^h(t)\right] \right\},$$

yielding the following the Euler equation from the FOC:

$$\begin{aligned} 0 &= \frac{1}{c_t^h(t)} - \frac{(1-\tau_K)R_t}{c_t^h(t+1)} \\ 1 &= \frac{(1-\tau_K)R_t c_t^h(t)}{c_t^h(t+1)}. \end{aligned}$$

To derive the law of motion of capital, begin by substituting optimal consumption back into the individual's budget constraint:

$$\begin{aligned} 2c_t^h(t) &= w_t(1-\tau_L)l_t^h(t) \\ \therefore c_t^h(t) &= \frac{1}{2}w_t(1-\tau_L)l_t^h(t). \end{aligned}$$

Aggregating across $N(t)$ individuals gives:

$$C_t(t) = \frac{1}{2}(1-\alpha)(1-\tau_L)y(t),$$

[3] In this setup of the model, taxes on borrowing and lending between individuals are taxed. Not applying the tax has significant consequences on the law of motion of capital and consumption Euler equations.

and aggregate savings is given as:

$$\begin{aligned}S(t) &= (1-\tau_L)(1-\alpha)K(t)^\alpha - \frac{1}{2}(1-\tau_L)(1-\alpha)K(t)^\alpha. \\ &= \frac{1}{2}(1-\tau_L)(1-\alpha)K(t)^\alpha\end{aligned}$$

Since we know that $S(t) = K(t+1)$, and accounting for population growth, we can derive the law of motion for capital:

$$K(t+1) = \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^\alpha.$$

Then, to get the steady we set our condition as before, $\Delta K(t) = 0$:

$$\begin{aligned}0 &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^\alpha - K(t) \\ K(t)^{1-\alpha} &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} \\ \implies \bar{K} &= \left[\frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta}\right]^{\frac{1}{1-\alpha}}. \qquad (12)\end{aligned}$$

The key intuition here is that taxes on labour reduces the steady state level of capital. Oddly, taxes on capital have no effect here. This is due to income and substitution effects that offset each other. The capital tax lowers the effective return on savings, which has a substitution effect – consumption when an individual becomes old becomes relatively more expensive, which tends to cause households to substitute towards consumption when they are young. Secondly, there is a wealth effect: the lower return on savings leads to lower lifetime income, which tends to cause households to reduce consumption in both periods. Because utility is logarithmic, the two effects on young consumption precisely cancel each other out, and as a result household saving – and thus capital – is unaffected.

1.5

Suppose capital saved in period t does not fully depreciate after use in period $t+1$. Instead, assume that $1-\delta$ of the capital stock remains. How does this affect the transition dynamics of the system and how does it affect the steady state? By comparing what happens with depreciation to the effect of a capital tax in part 4 of this question, you should be able to answer this part of the question by direct reference to the FOCs, deriving the implications for transition dynamics and steady state without further calculations.

Suppose capital does not fully depreciate after one period, and instead depreciates at rate δ, where $0 < \delta < 1$. The law of motion for capital with depreciation becomes

$$K(t+1) = \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^\alpha + (1-\delta)K(t).$$

Solving for steady state capital gives:

$$\begin{aligned} 0 &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} K(t)^{\alpha} + (1-\delta)K(t) - K(t) \\ K(t)^{1-\alpha} - (1-\delta)K(t)^{1-\alpha} &= \frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{1+\eta} \\ \implies \bar{K} &= \left[\frac{1}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{(1+\eta)\delta} \right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

To see how this slower depreciation rate affects transition dynamics, we can log linearise the law of motion of capital:

$$\tilde{K}(t+1) \approx \left(\frac{\alpha}{2} \cdot \frac{(1-\alpha)(1-\tau_L)}{(1+\eta)\delta} \bar{K}^{\alpha-1} + (1-\delta) \right) \tilde{K}(t).$$

Comparing our steady state capital with slower depreciation to our steady state with taxes and population growth (12), we can see a decline in the steady state level of capital, and a smoother transition dynamic for future capital in response to changes in capital today.

1.6

[Advanced] Instead of assuming that the utility and technology functions are of the functional forms described above, can you think of any alternative functional forms which lead to multiple steady states?

An ill-behaved OLG model is one in which the steady state of capital may not have one unique level. Potential candidates for production and utility functions which may lead to multiple steady states could be quadratic, cubic, or other such polynomial functions. For example we could use a CES production function

$$Y_t = \left[\alpha K(t)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) L_t^{\frac{\epsilon-1}{\epsilon}} \right],$$

where $\epsilon > 0$ is the elasticity of substitution between capital and labour.

2 OLG with social security

Consider again the standard OLG model with log preferences and Cobb-Douglas production technology. Assume that population obeys the law of motion $N(t+1) = (1+\eta)N(t)$. The object of the problem is to show how a pay-as-you-go (PAYG) social security system compares with a fully funded (FF) social security system. In a PAYG system, the young are taxed a fixed amount T when young, and the old in the same period receive the transfer $(1+\eta)T$. The old get more per person than the young lose because the population is growing – there are always more young people than old people. In a FF system, the young pay the amount T when young, which the government invests in capital. Then, when old, they receive the amount $R_{t+1}T$ where R_{t+1} is the return on capital between period t and $t+1$.

2.1

Compare the effects of each of these systems on the equilibrium capital stock. How do they affect the steady state?

Begin by looking at the household budget constraints for a generation t individual in period t and $t+1$, given by (2) and (3):

$$\begin{aligned} c_t^h(t) + k^h(t+1) &= w_t l_t^h(t) - a^h(t), \\ c_t^h(t+1) &= \underbrace{w_{t+1} l_t^h(t+1)}_{=0} + R_t a^h(t) + R_{t+1} k^h(t+1). \end{aligned}$$

Aggregating and then accounting for taxes gives

$$\begin{aligned} C_t(t) + K(t+1) &= w_t L(t) - T, \\ C_t(t+1) &= R_{t+1} K(t+1) + T(1+\eta), \\ \implies C_t(t) + \frac{C_t(t+1)}{R_{t+1}} &= w_t L(t) + \left(\frac{1+\eta-R_{t+1}}{R_{t+1}} \right) T. \end{aligned} \tag{13}$$

Substituting (13) into our utility function, we maximise the following

$$\arg\max_{\{C_t\}} \ \ln C_t + \ln C_{t+1},$$

and yield the familiar consumption Euler equation:

$$\begin{aligned} \frac{1}{C_t(t)} - \frac{R_{t+1}}{C_t(t+1)} &= 0 \\ \implies C_t(t+1) &= C_t(t) R_{t+1}, \end{aligned}$$

which is the same as in (6). Therefore we substitute the value for $C_t(t+1)$ from our Euler equation back into our budget constraint (13) to get

$$\begin{aligned} C_t(t) + \frac{C_t(t) R_{t+1}}{R_{t+1}} &= w_t L(t) + \left[\frac{1+\eta-R_{t+1}}{R_{t+1}} \right] T \\ 2C_t(t) &= w_t L(t) + \left[\frac{1+\eta-R_{t+1}}{R_{t+1}} \right] T \\ \implies C_t(t) &= \frac{1}{2} \left(w_t L(t) + \left[\frac{1+\eta-R_{t+1}}{R_{t+1}} \right] T \right). \end{aligned}$$

Similarly, we can do the same substitution for $C_t(t)$ to yield

$$\begin{aligned}\frac{C_t(t+1)}{R_{t+1}} + \frac{C_t(t+1)}{R_{t+1}} &= w_t L(t) + \left[\frac{1+\eta-R_{t+1}}{R_{t+1}}\right] T \\ \implies C_t(t+1) &= \frac{1}{2}\left(R_{t+1} w_t L(t) + \left[1+\eta-R_{t+1}\right] T\right).\end{aligned}$$

To get the law of motion of capital, as before, rewrite $C_t(t)$ in terms of the production function

$$\begin{aligned}C_t(t) &= \frac{1}{2}\left((1-\alpha)\left[\frac{K(t)}{L(t)}\right]^{\alpha} L(t) + \left[\frac{1+\eta-R_{t+1}}{R_{t+1}}\right] T\right) \\ &= \frac{1}{2}(1-\alpha)y(t) + \frac{1}{2}\left[\frac{1+\eta-R_{t+1}}{R_{t+1}}\right] T, \\ \implies S(t) = K(t+1) &= y(t) - C_t(t) \quad \underbrace{+R_{t+1}T}_{\text{Pension benefit}} \\ \therefore K(t+1) &= \frac{1}{2}(1-\alpha)y(t) - \frac{1}{2}\left[\frac{1+\eta-R_{t+1}}{R_{t+1}}\right] + R_{t+1}T \\ &= \frac{1}{2}(1-\alpha)y(t) - T\left[\frac{1+\eta-R_{t+1}}{2R_{t+1}} + R_{t+1}\right] \\ K(t+1) &= \frac{(1-\alpha)y(t)}{2} - \frac{1}{2}\left[\frac{1+\eta+R_{t+1}}{R_{t+1}}\right] T. \qquad (14)\end{aligned}$$

We can also rewrite (14) strictly in terms of $K(t)$ and $K(t+1)$, first note that labour and capital are paid their marginal products

$$\begin{aligned}w_t &= (1-\alpha)\left(\frac{K(t)}{L(t)}\right)^{\alpha}, \\ R_t &= \alpha\left(\frac{K(t)}{L(t)}\right)^{\alpha-1} + 1,\end{aligned}$$

and labour $L(t)$ is normalised to unity, and note that $K(t+1) = \frac{K(t+1)}{1+\eta}$ since we have to account for population growth, which gives us

$$\begin{aligned}K(t+1) &= \frac{1}{2}\frac{(1-\alpha)}{(1+\eta)} K(t)^{\alpha} - \frac{1}{2(1+\eta)}\left[\frac{2+\eta+\alpha K(t+1)^{\alpha-1}}{\alpha K(t+1)^{\alpha-1}+1}\right] T \\ &= \frac{1-\alpha}{2(1+\eta)} K(t)^{\alpha} - \frac{1}{2(1+\eta)}\left[\frac{(2+\eta) + \frac{\alpha}{K(t+1)^{1-\alpha}}}{1 + \frac{\alpha}{K(t+1)^{1-\alpha}}}\right] T \\ &= \frac{1-\alpha}{2(1+\eta)} K(t)^{\alpha} - \frac{T}{2(1+\eta)}\left[\frac{(2+\eta)K(t+1)^{1-\alpha}+\alpha}{K(t+1)^{1-\alpha}+\alpha}\right],\end{aligned}$$

and in the steady state this is

$$\bar{K} = \frac{1-\alpha}{2(1+\eta)}\bar{K}^{\alpha} - \frac{T}{2(1+\eta)}\left[\frac{(2+\eta)\bar{K}^{1-\alpha}+\alpha}{\bar{K}^{1-\alpha}+\alpha}\right].$$

Therefore, for any $T > 0$, $K(t+1)$ is lower for a given $K(t)$ compared to the baseline case without social security, which implies that the steady state K is also lower. Now, let's look at the FF system. The budget constraints are

$$C_t(t) + K(t+1) = w_t L(t) - T,$$
$$C_t(t+1) = R_{t+1} K(t+1) + R_{t+1} T,$$

which becomes

$$C_t(t) + \frac{C_t(t+1)}{R_{t+1}} = w_t L(t),$$

which is exactly what we had in (4) in the baseline model. So consumption in the FF case will be the same as the baseline case. Savings, on the other hand is given by

$$S(t) = K(t+1) = \frac{1}{2} w_t L(t) - T,$$

which is lower than in the baseline case. However, as we now have government savings $K_G(t+1) = T$, total savings is given by $\frac{1}{2} w_t L(t)$, which is also the same as in the baseline case. Thus, all equilibrium dynamics will be identical to baseline. In both cases, the social security scheme reduces the need for households to save. However, while in the FF case those reduced private savings are replaced with government savings so that the capital stock is unaffected; in the PAYG case, private savings are not replaced and thus the capital stock is lower.

2.2

Discuss which system you believe is better.

While we might be tempted to conclude that the PAYG system is worse since it reduces the steady state level of capital and output, it may not actually be welfare reducing. In particular, define the golden rule level of capital K_{GR} as the level that maximises total steady state consumption. Since total steady state consumption is simply output net of capital replacement, $K^\alpha - \eta K$, we have

$$K_{GR} = \left(\frac{\alpha}{\eta}\right)^{\frac{1}{1-\alpha}}.$$

The steady state level of the capital under the FF system, meanwhile, is:

$$K = \left[\frac{1-\alpha}{2(1+\eta)}\right]^{\frac{1}{1-\alpha}}.$$

If $K > K_{GR}$, then the steady state level of capital is inefficiently high. In this case, employing a PAYG system that reduces the steady state level of capital may actually enhance welfare. This can be seen directly from the consolidated lifetime budget constraint of the PAYG scheme (13): taking derivatives of the RHS wrt T and evaluating at $T = 0$ (i.e. the baseline case) yields at steady state

$$\frac{\eta - r}{1 + r}.$$

If $\eta > r$, then increasing T marginally from zero will increase lifetime wealth for all agents, allowing them to consume more. It can be verified that the condition $\eta > r$ is precisely the condition that $K > K_{GR}$.

3 Ramsey with Government Spending

Consider an extension of the Ramsey model where we include government expenditures that are financed by lump sum taxes. Government expenditures $\{g_t\}_{t=0}^{\infty}$ *will be treated as exogenous and we will assume that they don't enter utility. Household preferences are given by*

$$\int_0^\infty \exp(-\rho t) \ln c_t \; dt.$$

Suppose the economy is initially in steady state with $g = 0$*. Then at time* t^**,* g *jumps unexpectedly to* $g^* > 0$*. Aside from the introduction of government expenditures, everything is as in the lecture notes for the Ramsey model without technological progress.*

3.1

Derive the set of equations that define the equilibrium behaviour of consumption c_t *and capital* k_t*.*

The Hamiltonian to our problem is

$$\mathcal{H}(c_t, k_t) = \ln(c_t)\exp(\rho t) + \lambda_t[w_t l + (r_t - \delta)k_t + \pi_t - c_t - t_t],$$

where labour is standardised to unity, r_t is the return on capital, δ is the capital depreciation rate, π_t are firm profits, and t_t are government lump sum taxes which funds government consumption. Taking the derivative of the Hamiltonian with respect to the control variable, c_t, setting it equal to 0 gives the consumption FOC:

$$\frac{\partial \mathcal{H}}{\partial c_t} = \frac{1}{c_t}\exp(\rho t) - \lambda = 0, \tag{15}$$

and for the state variable, k_t:

$$\frac{\partial \mathcal{H}}{\partial k_t} = \lambda_t(r_t - \delta) + \dot{\lambda}_t = 0. \tag{16}$$

From (15) we know $\frac{1}{c_t}\exp(-\rho t) = \lambda_t$, or

$$u'(c_t)\exp(-\rho t) = \lambda_t,$$

and differentiating this expression wrt to t gives:

$$\begin{aligned} \dot{\lambda}_t &= u''(c_t)\dot{c}_t \exp(-\rho t) - \rho u'(c_t)\exp(-\rho t) \\ \implies \dot{c}_t &= \frac{\dot{\lambda}_t}{u''(c_t)\exp(-\rho t)} + \frac{\rho u'(c_t)}{u''(c_t)}. \end{aligned}$$

Substituting our value of $\dot{\lambda}_t$ from (16) gives the following:

$$\dot{c}_t = \frac{\rho u'(c_t)}{u''(c_t)} - \frac{\lambda_t(r_t - \delta)}{u''(c_t)\exp(-\rho t)},$$

and substituting the value of λ from our FOC gives

$$\dot{c} = \frac{u'(c_t)[\rho - r_t + \delta]}{u''(c_t)},$$

and since $u(c_t) = \log c_t$, our expression for $\dot{c}_t$ becomes

$$\dot{c}_t = c_t(r_t - \delta - \rho).$$

Our law of motion for capital is

$$\dot{k}_t = w_t l + (r_t - \delta)k_t + \pi_t - c_t - t_t,$$

and due to perfect competition we can assume that

$$y_t = w_t l + r_t k_t + \pi_t = f(k_t, l),$$

which gives us

$$\dot{k}_t = f(k_t, l) - c_t - t_t - \delta k_t.$$

This gives us two equations which characterise equilibrium and the transition dynamics of consumption and capital:

$$\dot{c}_t = c_t(r_t - \delta - \rho) = 0, \tag{17}$$

$$\dot{k}_t = f(k_t, l) - c_t - t_t - \delta k_t = 0. \tag{18}$$

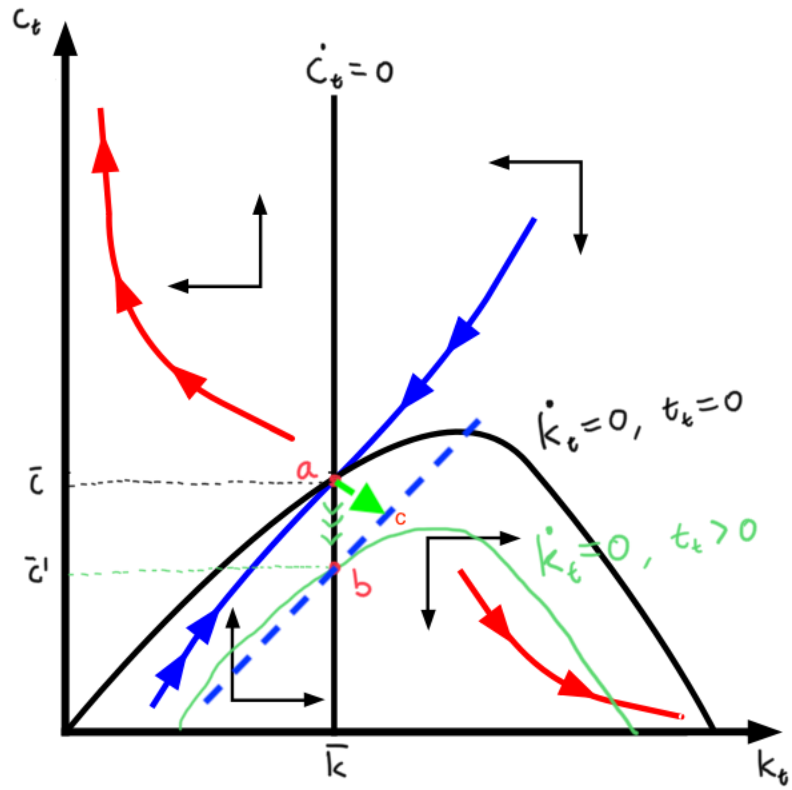
3.2

How does the increase in government spending affect the steady state of the system?

An increase in government spending is simply an increase in t_t in (18) as the government's expenditure does not enter into the household's maximisation problem. The transition dynamics are described below.

3.3

Use a phase diagram to show how the economy will evolve following the increase in g.



Following an increase in government spending, g, the government funds its higher expenditure by increasing lump sum taxes t_t. This causes a downward shift in the $\dot{k} = 0$ relation, resulting in the economy 'jumping' down in the steady state level of consumption (from point a to b). Steady state level of capital remains unchanged.

3.4

Now suppose that at time t^* government expenditures do not increase immediately, but instead agents learn at time t^* that government expenditures will increase from 0 to $g^* > 0$ at time $t^{**} > t^*$. Use a phase diagram to show the transition path following the announcement at t^*.

The new steady state following the increase in g is precisely the same as before, however because households dislike large jumps in consumption and utility, and because the policy is anticipated, the dynamics are different as households now smooth their consumption. Households will start at point a, and then upon the policy announcement, will smooth their consumption by saving and moving to point c, before slowly moving toward b. Even after the policy kicks in, the household will enjoy higher consumption levels because they are slowly moving from c to b by consuming out of their prior excess savings.

4 Ramsey with different household objectives

Suppose we have a Ramsey economy with population growth $L_t = L_0 \exp(\eta t)$ so $\dot{L}/L_t = \eta$, where η is the rate of population growth. The object of this question is to compare how population growth affects the behaviour of the economy under two different specifications of the household objective function.

- *Case a): Objective is the utility of each household member:*

$$\int_0^\infty U\left(\frac{C_t}{L_t}\right)\exp(-\rho t)\ dt.$$

- *Case b): Objective is total household utility:*

$$\int_0^\infty L_t U\left(\frac{C_t}{L_t}\right)\exp(-\rho t)\ dt.$$

Aside from population growth and the objective of the household, everything is as in the lecture notes for the Ramsey model without technological progress.[4] *You can assume that $U(\cdot)$ has the CRRA form.*

4.1

Derive the systems of equations that defines the equilibrium behaviour of consumption and capital in case a).

Using the current value Hamiltonian method, we have

$$\mathcal{H} = \frac{\left(\frac{C_t}{L_t}\right)^{1-\sigma}}{1-\sigma} + \lambda_t(w_t L_t + (r_t - \delta)K_t - C_t),$$

and the following FOCs

$$\mathcal{H}_{C_t}: \left(\frac{C_t}{L_t}\right)^{-\sigma}\frac{1}{L_t} = \lambda_t, \tag{19}$$

$$\mathcal{H}_{K_t}: \lambda_t(r_t - \delta) = \rho\lambda_t - \dot{\lambda}, \tag{20}$$

$$\implies \dot{\lambda} = \lambda_t(\delta + \rho - r_t),$$

and taking the derivative with respect to t for (19) gives

$$\dot{\lambda} = -\left(\frac{C_t}{L_t}\right)^{-\sigma}\frac{1}{L_t^2}\dot{L} - \sigma\left(\frac{C_t}{L_t}\right)^{-\sigma-1}\left(\frac{L_t\dot{C} - C_t\dot{L}}{L_t^2}\right)\frac{1}{L_t},$$

[4] *Be careful though. When there is no population growth we can talk interchangeably about aggregate consumption and consumption per household (assuming that the constant population size is normalised to unity), as in the lecture notes. When the population is growing as in this question, it will no longer be the case.*

and then substituting in $\dot{\lambda}$ from (20) gives

$$\lambda_t(\delta+\rho-r_t) = -\left(\frac{C_t}{L_t}\right)^{-\sigma}\frac{1}{L_t^2}\dot{L} - \sigma\left(\frac{C_t}{L_t}\right)^{-\sigma-1}\left(\frac{L_t\dot{C}-C_t\dot{L}}{L_t^2}\right)\frac{1}{L_t},$$

and then substitute in λ_t from (19)

$$\left(\frac{C_t}{L_t}\right)^{-\sigma}\frac{1}{L_t}(\delta+\rho-r_t) = -\left(\frac{C_t}{L_t}\right)^{-\sigma}\frac{1}{L_t^2}\dot{L} - \sigma\left(\frac{C_t}{L_t}\right)^{-\sigma-1}\left(\frac{L_t\dot{C}-C_t\dot{L}}{L_t^2}\right)\frac{1}{L_t},$$

and then simplify

$$\begin{aligned}
\frac{1}{L_t}(\delta+\rho-r_t) &= -\frac{1}{L_t^2}\dot{L} - \sigma\left(\frac{L_t}{C_t}\right)\left(\frac{L_t\dot{C}-C_t\dot{L}}{L_t^2}\right)\frac{1}{L_t} \\
\delta+\rho-r_t &= -\frac{\dot{L}}{L_t} - \frac{\sigma}{C_t}\frac{L_t\dot{C}-C_t\dot{L}}{L_t} \\
&= -\eta - \frac{\sigma L_t\dot{C}}{C_tL_t} + \frac{\sigma C_t\dot{L}}{C_tL_t} \\
\delta+\rho-r_t &= -\eta - \sigma\frac{\dot{C}}{C_t} + \sigma\eta.
\end{aligned}$$

We know that r_t is the marginal product of capital, $f_K(K_t, L_t)$, and we can rearrange our terms to get

$$\sigma\frac{\dot{C}}{C_t} = f_K(K_t, L_t) - \delta - \rho - (1-\sigma)\eta,$$

and then we can write things in per-capita terms so that $c_t = \frac{C_t}{L_t}$ and $\frac{\dot{c}}{c_t} = \frac{\dot{C}}{C_t} - \frac{\dot{L}}{L_t} = \frac{\dot{C}}{C_t} - \eta$. We assumed that the production function is HOD1, and therefore the marginal product is also HOD1.

$$\sigma\frac{\dot{c}}{c_t} = f_K(k_t) - \delta - \rho - \eta. \tag{21}$$

Next, we rewrite the law of motion of capital in per capita terms to get our second equation which describes equilibrium dynamics

$$\begin{aligned}
\frac{\dot{K}}{L_t} &= \frac{w_tL_r + (r_t-\delta)K_t - C_t}{L_t} - \eta k_t \\
&= w_t + (r_t-\delta)k_t - c_t - \eta k_t \\
&= f_L + k_tf_K - \delta k_t - c_t - \eta k_t \\
\therefore \dot{k} &= f(k_t) - c_t - (\delta+\eta)k_t, \qquad (22)
\end{aligned}$$

where we use Euler's theorem $f_L\frac{L_t}{L_t} + f_K\frac{K_t}{L_t} = f\left(\frac{K_t}{L_t}\right)$ to simplify our expression, and this yields our second equation which defines equilibrium dynamics.

4.2

Derive the system of equations that defines the equilibrium behaviour of consumption and capital in case b).

This is a bit easier. First, write down our present value Hamiltonian as

$$\mathcal{H} = \frac{L_t \left(\frac{C_t}{L_t}\right)^{1-\sigma}}{1-\sigma} + \lambda_t(w_t L_t + (r_t - \delta)K_t - C_t),$$

and get our FOCs

$$\mathcal{H}_{C_t} : \left(\frac{C_t}{L_t}\right)^{-\sigma} = \lambda_t, \tag{23}$$

$$\mathcal{H}_{K_t} : \lambda_t(r_t - \delta) = \rho\lambda_t - \dot{\lambda}, \tag{24}$$

$$\implies \dot{\lambda} = \lambda_t(\delta + \rho - r_t),$$

and then differentiate (23) wrt t to get

$$\dot{\lambda} = -\sigma \left(\frac{C_t}{L_t}\right)^{-\sigma-1} \left(\frac{\dot{C}L_t - C_t\dot{L}}{L_t^2}\right),$$

and use our values of $\dot{\lambda}$ and λ_t from our FOCs to get

$$\begin{aligned}
\lambda_t(\delta + \rho - r_t) &= -\sigma \left(\frac{C_t}{L_t}\right)^{-\sigma-1} \left(\frac{\dot{C}L_t - C_t\dot{L}}{L_t^2}\right) \\
\left(\frac{C_t}{L_t}\right)^{-\sigma} (\delta + \rho - r_t) &= -\sigma \left(\frac{C_t}{L_t}\right)^{-\sigma-1} \left(\frac{\dot{C}L_t - C_t\dot{L}}{L_t^2}\right) \\
\delta + \rho - r_t &= -\sigma \frac{L_t}{C_t} \left(\frac{\dot{C}L_t - C_t\dot{L}}{L_t^2}\right) \\
\delta + \rho - r_t &= -\sigma \frac{\dot{C}}{C_t} + \frac{\sigma \dot{L}}{L_t} \\
\delta + \rho - r_t &= \sigma\eta - \sigma \frac{\dot{C}}{C_t},
\end{aligned}$$

which we can rearrange as

$$\sigma \frac{\dot{C}}{C_t} = f_K(K_t, L_t) - \delta - \rho + \sigma\eta.$$

And then, once again, rewrite everything in per capita terms to get

$$\sigma \frac{\dot{c}}{c_t} = f_K(k_t) - \delta - \rho, \tag{25}$$

where, as before, we use the fact that

$$\frac{\dot{c}}{c_t} = \frac{\dot{C}}{C_t} - \frac{\dot{L}}{L_t} = \frac{\dot{C}}{C_t} - \eta.$$

The second equation defining equilibrium dynamics is given by the law of motion of capital in per capita terms, which is the same as part a):

$$\begin{aligned}\frac{\dot{K}}{L_t} &= \frac{w_t L_r + (r_t - \delta)K_t - C_t}{L_t} - \eta k_t \\ &= w_t + (r_t - \delta)k_t - c_t - \eta k_t \\ &= f_L + k_t f_K - \delta k_t - c_t - \eta k_t \\ \therefore \dot{k} &= f(k_t) - c_t - (\delta + \eta)k_t. \end{aligned} \tag{26}$$

4.3

How does the steady-state level of per-capita consumption differ between case a) and case b)? Explain.

In case a) the steady state level of k_t will be lower than in case b). It follows that the steady state consumption per capital will also be lower as long as $\rho > \eta$. To see this, recall that in steady state

$$c = f(k) - (\delta + \eta)k,$$

and the steady state level of c will be increase in k if $f_K(k) > \delta + \eta$ which arises when $\rho > \eta$ (use the steady state condition of f_K to see this).