The Envelope Theorem

Intro Math for Economists (PEARL, Spring 2019)

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Introduction

- We've covered a lot in a short time. So let's review.
- Unconstrained optimisation: We have a function which may be convex or concave, and we wish to find a critical value which is a minimum or maximum, respectively.
- Constrained optimisation: Basically like unconstrained optimisation, however we have to find a critical value subject to a set of constraints (e.g. budget constraint, production constraint, and so on).
- The envelope theorem is basically an extension of constrained optimisation. The notation may be difficult to understand at first, but if you can understand the main ideas, you will have an easy time in microeconomics.

Introduction

- Recall that when we conducted constrained optimisation (for Akane), we had endogenous variables (burgers and movies) and we had exogenous variables (budget and prices of burgers and movies).
- In a nutshell, exogenous variables are predetermined in our problems, and it's the endogenous variables that we are interested in solving.
- Natural question to ask when we conduct optimisation are the following:
- Firstly, how do the optimal values or equilibrium values x_i^* change when we change the value of one of the parameters or exogenous variables?
- Second, how does the optimal value of the objective function change when we change the value of one of the parameters or exogenous variables?

Mathematical Prerequisites: Total Derivatives

Suppose

$$y = f(x(u), u) = y(u)$$

If u changes, there are two effects on y:

1. A direct effect: $\frac{\partial y}{\partial u}$
2. An indirect effect through x.

The total derivative captures both these effects:

$$\frac{dy}{du} = \underbrace{\frac{\partial y}{\partial u}}_{\text{direct effect}} + \underbrace{\frac{\partial x}{\partial u}\frac{\partial y}{\partial x}}_{\text{indirect effect}}$$

Total Derivatives

More generally, if

$$y = f(\mathbf{x}(u), u), \quad \mathbf{x}(u) = \begin{pmatrix} x_1(u) \\ \vdots \\ x_n(u) \end{pmatrix},$$

Then

$$\frac{dy}{du} = \underbrace{\frac{\partial y}{\partial u}}_{\text{direct effect}} + \underbrace{\sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial y}{\partial x_i}}_{\text{indirect effect(s)}} \tag{1}$$

Partial Total Derivatives

Suppose

$$y = f(x_1(u,v), x_2(u,v), u, v) \tag{2}$$
$$\implies y = f(u,v)$$

We want to find the total effect of a change of u and y. When u changes we have:

1. A direct effect;
2. An indirect effect through x_1 and x_2.

What symbol should we use to capture the direct and indirect effects here? $\frac{\partial y}{\partial u}$ is usually reserved for the direct effect, and $\frac{dy}{du}$ is usually used for when y is a function of u alone. Chiang suggests the following for the total partial derivative:

$$\frac{\S y}{\S u}$$

Partial Total Derivatives

Let's just put aside the discussion about notation for now, and suppose we have (2). Then let's say that the partial total derivatives are given as follows:

$$\frac{dy}{du} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u}$$
$$\frac{dy}{dv} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial v} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial v} + \frac{\partial y}{\partial v}$$

Partial Total Derivatives

More generally, suppose

$$y = f(\mathbf{x}(\mathbf{a}), \mathbf{a}), \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

Where $\mathbf{x}$ and $\mathbf{a}$ are vectors. Each element of $\mathbf{x}$ is a function of $a_1, \ldots, a_m$.
Suppose a_i changes and it affects y directly and indirectly through $x_1, \ldots, x_n$:

$$\frac{dy}{da_i} = \underbrace{\frac{\partial y}{\partial a_i}}_{\text{direct effect}} + \underbrace{\sum_{j=1}^{n} \frac{\partial x_j}{\partial a_i} \frac{\partial y}{\partial x_j}}_{\text{indirect effect(s)}} \tag{3}$$

The Envelope Theorem

Consider the case where we optimise $f(\mathbf{x}, \mathbf{a})$, where $\mathbf{x}$ is the vector of endogenous variables and $\mathbf{a}$ is the vector of exogenous parameters.

For example, a firm has the following production function:

$$Q(L, K) = L^{1/2} + K^{1/2} \tag{4}$$

and it wishes to maximise total profits; given wage rate w, cost of capital r, and the price of output P.

$$\max \quad \pi = P(L^{1/2} + K^{1/2}) - wL - rK \tag{5}$$

$$\mathbf{x} = \begin{bmatrix} L \\ K \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} P \\ w \\ r \end{bmatrix}$$

The Envelope Theorem

Carrying out optimisation and attaining optimal point for (5) gives $\mathbf{x}^* = \mathbf{x}^*(\mathbf{a})$, where each $x_j^* = x_j^*(a_1, ..., a_m)$. We can designate a function to this optimality condition: $M(\mathbf{a})$.

$$M(\mathbf{a}) = \text{optimal } f(\mathbf{x}, \mathbf{a}) = f(\mathbf{x}^*(\mathbf{a}), \mathbf{a})$$

Definition

$M(\mathbf{a})$ is called the optimal value function – the optimal maximum value function or optimal minimum value function depending on the problem.

The Envelope Theorem

- Recall our questions at the start of this lecture:

1. How do the optimal values or equilibrium values x_i^* change when we change the value of one of the parameters or exogenous variables?
   - i.e. We want $\frac{\partial x_j^*}{\partial a_i}$.
2. Second, how does the optimal value of the objective function change when we change the value of one of the parameters or exogenous variables?
   - This question is answered by the Envelope Theorem.
   - $M(\mathbf{a}) = f(\mathbf{x}^*(\mathbf{a}), \mathbf{a})$: if $\mathbf{a}$ is changes there will be a direct effect and an indirect effect.

The Envelope Theorem

Definition

The Envelope Theorem states

$$\frac{dM}{da_i} = \left.\frac{\partial f(x_j, a)}{\partial a_i}\right|_{\mathbf{x}^*}$$

The Envelope Theorem

Proof.

Suppose $M(\mathbf{a}) = f(\mathbf{x}^*(\mathbf{a}), \mathbf{a})$

$$\frac{dM}{da_i} = \frac{\partial f(\mathbf{x}^*, \mathbf{a})}{\partial \mathbf{a}_i} + \sum_{j=1}^{n} \frac{\partial f(\mathbf{x}^*, \mathbf{a})}{\partial x_j} \frac{\partial x_j^*}{\partial a_i} \tag{6}$$

and by FOCs:

$$\frac{\partial f(\mathbf{x}^*(\mathbf{a}), \mathbf{a})}{\partial x_j^*} = \left.\frac{\partial f(\mathbf{x}, \mathbf{a})}{\partial x_j}\right|_{\mathbf{x}^*} = 0 \tag{7}$$

Therefore, the partial total derivative is

$$\frac{dM}{da_i} = \frac{\partial f(\mathbf{x}^*, \mathbf{a})}{\partial a_i} = \left.\frac{\partial f(\mathbf{x}, \mathbf{a})}{\partial a_i}\right|_{\mathbf{x}^*} \tag{8}$$

Applying the Envelope Theorem

Consider the example problem in (4) and (5). Suppose we want to know how optimal profit changes when there is a change in the wage rate.

Two ways to go about this:

1. Traditional way: Find optimal values $L^*(P, w, r)$ and $K^*(P, w, r)$, substitute into π to get max value function $\pi^8(P, w, r)$, and then differentiate π^* w.r.t. w. Troublesome.
2. Envelope way: Take the partial derivative of (5) w.r.t. w

$$\frac{d\pi}{dw} = \left.\frac{\partial \pi}{\partial w}\right|_{L^*, K^*} = -L^*$$

Much more efficient.

Applying the Envelope Theorem

The last example was actually one of unconstrained optimisation. Let's look at a classical (constrained) programming problem. Parameters can enter through the objective function and the constraint function or parameters we have isolated on the RHS of constraints. For example:

$$\begin{aligned} \max \quad U &= x_1^{\alpha} x_2^{\beta} \\ \text{s.t. } P_1 x_1 + P_2 x_2 &= Y \end{aligned} \tag{9}$$

We have the following parameters: α, β from the objective function; P_1, P_2 from the constraint function; and Y from the RHS of the constraint.

Applying the Envelope Theorem

Recall that constructing a Lagrangian for the following problem:

$$\begin{aligned} \text{optimise } & f(\mathbf{x}, \mathbf{a}) \\ \text{s.t. } & g^1(\mathbf{x}, \mathbf{a}) = 0 \\ & \vdots \\ & g^m(\mathbf{x}, \mathbf{a}) = 0 \end{aligned}$$

would give us $Z = f(\mathbf{x}, \mathbf{a}) - \sum_{k=1}^{m} \lambda_k g^k(\mathbf{x}, \mathbf{a})$, and the following FOCs:

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{k=1}^{m} \lambda_k \frac{\partial g^k}{\partial x_j} = 0, \quad \text{at } \mathbf{x}^*, \lambda^*, \quad j = 1, ..., n$$

$$\frac{\partial Z}{\partial \lambda_k} = -g^k(\mathbf{x}, \mathbf{a}) = 0, \quad \text{at } \mathbf{x}^*, \lambda^*, \quad k = 1, ..., m.$$

The Envelope Theorem (again)

Repeating our previous definition, the Envelope Theorem states

$$\frac{dM}{da_i} = \left.\frac{\partial Z}{\partial a_i}\right|_{\mathbf{x}^*,\lambda^*}$$

The Envelope Theorem (again)

Proof.

Start with our Lagrangian:

$$Z = f(\mathbf{x}, \mathbf{a}) - \sum_{k=1}^{m} \lambda_k g^k(\mathbf{x}, \mathbf{a})$$

$$Z(\mathbf{x}^*(\mathbf{a}), \lambda^*(\mathbf{a}), \mathbf{a}) = f(\mathbf{x}^*(\mathbf{a}), \mathbf{a}) - \sum_{k=1}^{m} \lambda_k g^k(\mathbf{x}^*, \mathbf{a})$$

But $g^k(\mathbf{x}, \mathbf{a}) = 0$ at $\mathbf{x}^*, \lambda^*$, therefore $g^k(\mathbf{x}^*, \mathbf{a}) = 0$.

$$\therefore Z(\mathbf{x}^*(\mathbf{a}), \lambda^*(\mathbf{a}), \mathbf{a}) = Z^*(\mathbf{a}) = f(\mathbf{x}^*(\mathbf{a})) = M(\mathbf{a})$$

□

The Envelope Theorem (again)

Proof.

cont...

$$\frac{dM(\mathbf{a})}{da_i} = \underbrace{\frac{\partial Z^*}{\partial a_i}}_{\text{direct}} + \underbrace{\sum_{j=1}^{n} \frac{\partial Z^*}{\partial x_j^*} \frac{\partial x_j^*}{\partial a_i}}_{\text{indirect}} + \underbrace{\sum_{k=1}^{m} \frac{\partial Z^*}{\partial \lambda_k^*} \frac{\partial \lambda_k^*}{\partial a_i}}_{\text{indirect}}$$

But we know

$$\frac{dZ^*}{dx_j} = \left.\frac{\partial Z}{\partial x_j}\right|_{\mathbf{x}^*,\lambda^*} = 0 = \frac{dZ^*}{d\lambda_i^*} = \left.\frac{\partial Z}{\partial \lambda_i}\right|_{\mathbf{x}^*,\lambda^*}, \text{ and}$$

$$\frac{\partial Z^*}{\partial a_i} = \left.\frac{\partial Z}{\partial a_i}\right|_{\mathbf{x}^*,\lambda^*} \text{ QED}$$

□

Applying the Envelope Theorem

Let's look at (9) again:

$$\begin{aligned} \max \quad & U = x_1^{\alpha} x_2^{\beta} \\ \text{s.t.} \quad & P_1 x_1 + P_2 x_2 = Y \end{aligned}$$

Suppose we want to know how max utility changes when P_1 increases.

Applying the Envelope Theorem

By the Envelope Theorem, we can solve this by forming the Lagrangian function and then partially differentiating at the optimal point:

$$\frac{dU^*}{dP_1} = \left.\frac{\partial Z}{\partial P_1}\right|_{\mathbf{x}^*,\lambda^*}$$

So

$$Z = x_1^\alpha x_2^\beta + \lambda(Y - P_1 x_1 - P_2 x_2)$$
$$\therefore \frac{dU^*}{dP_1} = \left.\frac{\partial Z}{\partial P_1}\right|_{\mathbf{x}^*,\lambda^*} = -\lambda x_1\big|_{\mathbf{x}^*,\lambda^*} = -\lambda^* x_1^*$$

Applying the Envelope Theorem: Shadow Prices

- Let's look at another example of the application of the envelope theorem, and at a concept that is quite important in economics: shadow prices.
- Suppose we have the following problem:

$$\text{optimise } f(\mathbf{x}, \mathbf{a})$$
$$\text{s.t. } g^k(\mathbf{x}, \mathbf{a}) = 0$$

Isolate the parameters onto the RHS of the constraint:

$$h^k(\mathbf{x}, \mathbf{c}) = b_k, \quad k = 1, ..., m$$

Formulate Lagrangian:

$$Z = f(\mathbf{x}, \mathbf{a}) + \sum_{k=1}^{m} \lambda_k (b_k - h^k(\mathbf{x}, \mathbf{c}))$$

Applying the Envelope Theorem: Shadow Prices

- We are interested in how the optimal value function changes when b_k changes, say, $\frac{dM}{db_k}$.
- By the Envelope Theorem:

$$\frac{dM}{db_k} = \left.\frac{\partial Z}{\partial b_k}\right|_{\mathbf{x}^*,\lambda^*} = \lambda_k^*$$

- What does this mean? Suppose our problem is a maximisation problem and we allow b_k to increase by one unit.
- How much is that increase in b_k worth to a person confronted by this problem? $\uparrow b_k \implies k$th constraint is less binding $\implies \uparrow M$.
- At most, one unit increase in b_k would be worth the increase in M.
- But that increase in M is equivalent to λ_k^*. i.e. λ^* tells us how much an extra unit of b_k is worth.
- This is called the shadow price

Concluding Remarks

- The Envelope Theorem has a wide range of applications in economics.
- Again, we skipped a lot of material in the interest of time. So be sure to read up for more information.
- If you were able to follow this lecture closely, then microeconomics will be a breeze.
- Next lecture we will look at further applications of the Envelope Theorem.