# Winter Vacation (MT19) Problem Set

## 1 Choice under uncertainty

*An agent cares about the expected utility of her uncertain wealth, and the (von Neumann-Morgenstern) utility function she uses to evaluate a lottery is $u(w) = \log(w)$, where $w > 0$ is her wealth.*

### 1.1

*What is the coefficient of absolute risk aversion, $-\frac{u''(w)}{u'(w)}$, implied by these preferences? What is the coefficient of relative risk aversion, $-\frac{wu''(w)}{u'(w)}$? Are they increasing, decreasing, or constant in wealth?*

The coefficient of absolute risk aversion is computed as

$$\rho^a = -\frac{-\frac{1}{w^2}}{\frac{1}{w}} = \frac{1}{w},$$

and the coefficient of relative risk aversion is

$$\rho^r = -\frac{-w\frac{1}{w^2}}{\frac{1}{w}} = 1.$$

The coefficient of absolute risk aversion is decreasing in wealth – the agent will pay less to eliminate risk as she becomes richer – and the coefficient of relative risk aversion is constant. This should be obvious as when we look at models where agents have constant relative risk aversion preferences (i.e. $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$), we have logarithmic preferences as $\rho \to 1$.

### 1.2

*The agent with initial wealth $w$ faces uncertainty of the following form: either her wealth increases by a factor $\alpha$ ($\alpha > 1$), leaving her with final wealth $\alpha w$, with probability 0.5; or it falls, leaving her with only $\frac{w}{\alpha}$, with probability 0.5. How much would she be prepared to pay to avoid this uncertainty (compared to the initial situation with initial wealth $w$)?*

The agent faces

$$\begin{aligned} u(w) &= \begin{cases} \log(\alpha w) & \text{w.p. } \frac{1}{2}, \\ \log\left(\frac{w}{\alpha}\right) & \text{w.p. } \frac{1}{2}, \end{cases} \\ \implies \mathbb{E}\left[u(w)\right] &= \frac{1}{2}\log(\alpha w) + \frac{1}{2}\log\left(\frac{w}{\alpha}\right) \\ &= \log\left((\alpha w)^{1/2}\left(\frac{w}{\alpha}\right)^{1/2}\right) \\ &= \log w \end{aligned}$$

Thus, compared to the initial situation with wealth $w$ and utility $\log w$, the agent is willing to pay 0.

## 1.3

*Suppose this agent has initial wealth of \$10. She has the opportunity to enter a competition which pays \$19 if she wins (and nothing if she loses), where the probability of winning is $\frac{1}{3}$. The agent must pay \$2 up front to enter this competition. Should she enter the competition?*

We need to find the certainty equivalent for the agent, so let $a$ denote the amount of money the agent needs to have for sure in order to be indifferent to the lottery:

$$\begin{aligned} u(a) &= \mathbb{E}[u(w)] \\ \log a &= \frac{1}{3}\log(10+19-2) + \frac{2}{3}\log(10-2) \\ \log a &= \log 3 + \log 4 \\ a &= 12. \end{aligned}$$

The certainty equivalent, $a$, is 12, and so the agent should enter the competition, as $a$ is equal to the wealth from not entering the competition.

## 1.4

*Suppose instead that this agent has the opportunity to enter a competition which pays a benefit $b$ if she wins (and nothing if she loses), where the probability of winning is 0.5. The agent must pay a cost $c$ up front to enter this competition. Show that the agent must have initial wealth of*

$$w_0 = \frac{b-c}{b-2c}c$$

*if she is just willing to enter this competition.*

As before, we need to calculate the agent's certainty equivalent:

$$\begin{aligned} u(w_0) &= \frac{1}{2}u(w_0+b-c) + \frac{1}{2}u(w_0-c) \\ \log w_0 &= \frac{1}{2}\log(w_0+b-c) + \frac{1}{2}\log(w_0-c) \\ \log w_0 &= \frac{1}{2}\log(w_0+b-c)(w_0-c) \\ w_0 &= \left[w_0^2 + w_0 b - w_0 c - c w_0 - cb + c^2\right]^{1/2} \\ w_0^2 &= w_0^2 + w_0 b - 2c w_0 - cb + c^2 \\ cb - c^2 &= w_0 b - 2c w_0 \\ \therefore w_0 &= \frac{cb-c^2}{b-2c}, \end{aligned}$$

which completes the question.

## 1.5

*Show that $w_0$ is decreasing in $b$ and increasing in $c$. Are these properties intuitively plausible?*

First, to show that $w_0$ is decreasing in $b$:

$$\begin{aligned}\frac{\partial w_0}{\partial b} &= \frac{c(b-2c)-(cb-c^2)1}{(b-2c)^2} \\ &= -\frac{c^2}{(b-2c)^2} < 0,\end{aligned}$$

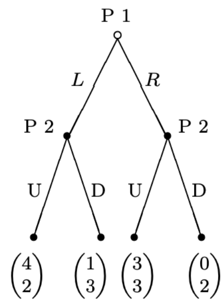
and that $w_0$ is increasing in $c$:

$$\begin{aligned}\frac{\partial w_0}{\partial c} &= \frac{(b-2c)(b-2c)-(cb-c^2)(-2)}{(b-2c)^2} \\ &= \frac{b^2-4bc+4c^2+2cb-2c^2}{(b-2c)^2} \\ &= \frac{b^2-2bc-2c^2}{(b-2c)^2} > 0.\end{aligned}$$

In words, the agent should be more willing to undertake lotteries with larger benefits from lower initial wealth. Conversely, our agent would require a larger initial wealth if the cost of playing the lottery is increasing. This is due to decreasing absolute risk aversion of the agent's utility function.

# 2 Bayesian-Nash equilibria and sequential games

*Consider the following game*




## 2.1

*Find a sub-game perfect Nash equilibrium of this game. Is it unique? Are there any other (pure or mixed) Nash equilibria?*

Begin by constructing the normal representation of this game:

Table 1: Normal form representation

| | | P2 | | | |
|---|---|---|---|---|---|
| | | $UU$ | $UD$ | $DU$ | $DD$ |
| P1 | $L$ | $\underline{4}$,2 | $\underline{4}$,2 | 1,$\underline{3}$ | $\underline{1}$,$\underline{3}$ |
| | $R$ | 3,$\underline{3}$ | 0,2 | $\underline{3}$,$\underline{3}$ | 0,2 |

There are two pure strategy Nash equilibria of this game are $(L, DD)$ and $(R, DU)$, with payoffs $(1, 3)$ and $(3, 3)$, respectively. Which of these can we rule out as non-credible threats? To start, recall the definition of a subgame: the game following a singleton information set, where all successor information sets are accessible only from that singleton. We have one singleton information set and two subgames:

- The subgame starting with $P2$ after $P1$ chooses $L$. The pure strategy is $(L, DD)$;
- The subgame starting with $P2$ after $P1$ chooses $R$. The unique pure strategy subgame perfect Nash equilibrium (SPNE) is $(R, DU)$.

Why is it unique? If $P1$ chooses $L$, $P2$ will always play $D$, and if $P1$ chooses $R$, $P2$ will always play $U$. $P1$ knows this and strictly prefers a payoff of 3 by choosing $R$, rather than a payoff of 1 by playing $L$.

Now, we need to find the mixed strategies. When $P1$ plays $L$, $P2$ is indifferent between $DU$ and $DD$. When $P1$ plays $R$, $P2$ is indifferent between $UU$ and $DU$. From the perspective of $P2$, $UD$ is strictly dominated and will never be played. $DU$ is weakly dominant and hence in equilibrium $P1$ does not mix. So, the two mixed strategies are:

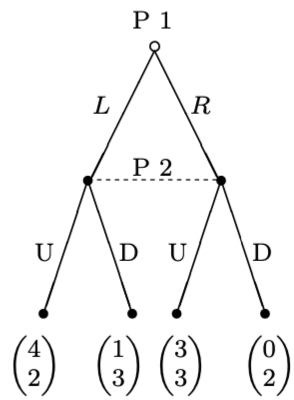
- $(L; DU \text{ w.p. } p, DD \text{ w.p. } 1 - p)$; and
- $(R; UU \text{ w.p. } p, DU \text{ w.p. } 1 - p)$.

## 2.2

*Suppose that player 2 cannot observe player 1's choice. Draw the new extensive form of the game. What is the set of Nash equilibria?*

The extensive form of the game is now:

Figure 1: Imperfect information extensive form



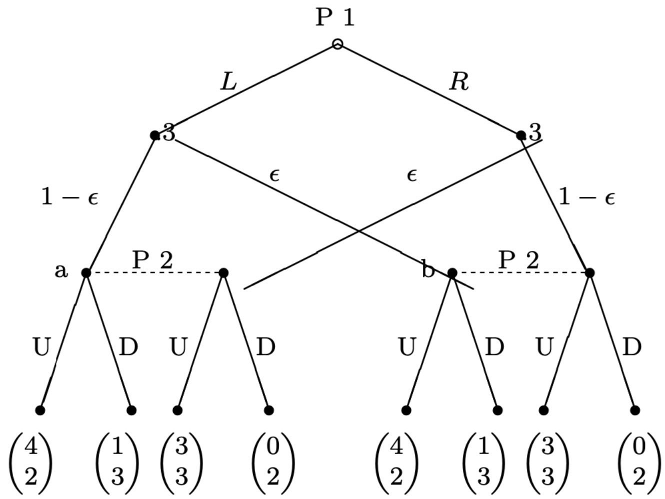

Here, for $P1$, $L$ is the dominant strategy, so the unique Nash equilibrium is $(L, D)$ with a payoff of $(1, 3)$.

## 2.3

*Suppose now that, with probability $1-\epsilon$, player 2 observes the actual choice of player 1; but with probability $\epsilon$, player 2 is mistaken and observes the opposite choice. The probability $\epsilon$ is small and is common knowledge to both players. Draw the new extensive form. Find the unique pure weak perfect Bayesian equilibrium. Comment.*

The extensive form of the game is now:

Figure 2: Imperfect information extensive form




There are two information sets for $P2$ – what they see/do after:

- $P2$ observes $L$ and it was played, or $R$ was played but $P2$ believes that $L$ was observed; or
- $P2$ observes $R$ and was played, or that $L$ was played but $P2$ believes that $R$ was observed.

Denote by $\alpha$, $P2$'s belief on being at node $a$ in the left information set, and $\beta$ the belief that she is on node $b$ in the right information set. In the left information set, the sequentially rational action for $P2$ is to choose $U$ whenever:

$$\begin{aligned} 2\alpha + 3(1-\alpha) &\geq 3\alpha + 2(1-\alpha) \\ 3 - \alpha &\geq \alpha + 2 \\ \frac{1}{2} &\geq \alpha. \end{aligned} \tag{1}$$

In the right information set, the sequentially rational action for $P2$ is to choose $U$ whenever:

$$\begin{aligned} 2\beta + 3(1-\beta) &\geq 3\beta + 2(1-\beta) \\ 3 - \beta &\geq \beta + 2 \\ \frac{1}{2} &\geq \beta. \end{aligned} \tag{2}$$

If $P2$ believes that $P1$ chooses $L$ with probability 1, then $\alpha = 1$ and $\beta = 1$ would be $P2$'s consistent beliefs. In other words, $P2$ believes that $P1$ chose $L$ no matter what, so she could be mistaken with probability $\epsilon$ that she's in the right information set, or that she's correct with probability $1 - \epsilon$ that she's in the left information set. $P2$ would always pick

$D$ as it strictly dominates $U$, given her beliefs, and so $P1$'s best response is to actually choose $L$. If $P1$ does not choose $L$, then she will end up with a payoff of 0 instead of 1 (as $P2$ will always play $D$). We thus have a weak perfect Bayesian Nash equilibrium (WPBE): $\{(L, DD), \alpha = 1, \beta = 1\}$.

If $P2$ believes that $P1$ chooses $R$ with probability 1, then with a similar analogous argument to the above, $P2$'s consistent beliefs would be $\alpha = 0$ and $\beta = 0$, and she will always play $U$. Knowing this, $P1$ will not play $R$, and will play $L$. Thus, we do not have a WPBE here.

Finally, notice that the SPNE found in the previous question does not survive here.

# 3 Consumer theory

## 3.1

*Prove Roy's Identity:*

$$x_i(\mathbf{P}, w) = -\frac{\partial V(\mathbf{P}, w)/\partial P_i}{\partial V(\mathbf{P}, w)/\partial w},$$

*where $V(\mathbf{P}, w)$ is a household's indirect utility function, and $x_i(\mathbf{P}, w)$ is their Walrasian/Marshallian demand for good $i$ when they have wealth $w$ and a face of vector of prices $\mathbf{P}$.*

The consumer problem is as follows:

$$\max \; u(\mathbf{x}),$$

subject to

$$\mathbf{P}^\top \mathbf{x} = w,$$

where $u(\mathbf{x})$ is a continuous and concave utility function representing continuous and convex preferences, $\mathbf{x}$ is a vector of $k$ consumption goods such that $\{x_i\}_{i=1}^k \geq 0 \; \forall i$ , $\mathbf{P}$ is a vector of prices for the $k$ consumption goods, and we assume insatiability and that markets clear in a Walrasian fashion such that the consumer expends all their wealth on consumption goods. The Lagrangian of the consumer problem is:

$$Z = u(\mathbf{x}) + \lambda(w - \mathbf{P}^\top \mathbf{x}),$$

and by the Envelope Theorem:

$$\frac{\partial V(\mathbf{P}, w)}{\partial P_i} = \left.\frac{\partial Z}{\partial P_i}\right|_{\mathbf{x}^*, \lambda^*} = -\lambda^* x_i^*,$$

also,

$$\frac{\partial V(\mathbf{P}, w)}{\partial w} = \left.\frac{\partial Z}{\partial w}\right|_{\mathbf{x}^*, \lambda^*} = \lambda^*,$$

and hence it follows that:

$$x_i^*(\mathbf{P}, w) = -\frac{\partial V(\mathbf{P}, w)/\partial P_i}{\partial V(\mathbf{P}, w)/\partial w}. \tag{3}$$

This completes the proof of Roy's Identity.

## 3.2

*Assume that the household's expenditure function takes the Gorman Polar Form:*

$$e(\mathbf{P}, \tilde{U}) = a(\mathbf{P}) + \tilde{U} b(\mathbf{P}). \tag{4}$$

*Using part 3.1 or otherwise, show that the response of demand to wealth is independent of the level of wealth.*

We know

$$e(\mathbf{P}, \tilde{U}) = e(\mathbf{P}, V(\mathbf{P}, w)) = w,$$

and hence we can write (4) as

$$\begin{aligned} a(\mathbf{P}) + V(\mathbf{P}, w) b(\mathbf{P}) &= w \\ \therefore V(\mathbf{P}, w) &= \frac{w - a(\mathbf{P})}{b(\mathbf{P})}. \end{aligned} \tag{5}$$

Then, using Roy's Identity, the Marshallian demand for good $i$ is

$$\begin{aligned} x_i^*(\mathbf{P}, w) &= -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial w}, \\ \frac{\partial V}{\partial P_i} &= \frac{\partial}{\partial P_i}\left[ w b(\mathbf{P})^{-1} - \frac{a(\mathbf{P})}{b(\mathbf{P})} \right] \\ &= -w b(\mathbf{P})^{-2} b'(\mathbf{P}) - \frac{a'(\mathbf{P}) b(\mathbf{P}) - a(\mathbf{P}) b'(\mathbf{P})}{b(\mathbf{P})^2} \\ &= -\frac{w b'(\mathbf{P})}{b(\mathbf{P})^2} - \frac{a'(\mathbf{P}) b(\mathbf{P}) - a(\mathbf{P}) b'(\mathbf{P})}{b(\mathbf{P})^2} \\ &= \frac{a(\mathbf{P}) b'(\mathbf{P}) - w b'(\mathbf{P}) - a'(\mathbf{P}) b(\mathbf{P})}{b(\mathbf{P})^2}, \text{ and,} \\ \frac{\partial V}{\partial w} &= \frac{1}{b(\mathbf{P})}, \end{aligned}$$

$$\begin{aligned} \therefore -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial w} &= -\left[ \frac{a(\mathbf{P}) b'(\mathbf{P}) - w b'(\mathbf{P}) - a'(\mathbf{P}) b(\mathbf{P})}{b(\mathbf{P})^2} \right] b(\mathbf{P}) \\ &= \frac{-b(\mathbf{P}) a(\mathbf{P}) b'(\mathbf{P}) + b(\mathbf{P}) w b'(\mathbf{P}) + a'(\mathbf{P}) b(\mathbf{P})^2}{b(\mathbf{P})^2} \\ &= a'(\mathbf{P}) + \frac{w b'(\mathbf{P}) - a(\mathbf{P}) b'(\mathbf{P})}{b(\mathbf{P})} \\ \implies x_i^*(\mathbf{P}, w) &= \frac{\partial a(\mathbf{P})}{\partial P_i} + \frac{\partial b(\mathbf{P})}{\partial P_i} \left[ \frac{w - a(\mathbf{P})}{b(\mathbf{P})} \right]. \end{aligned} \tag{6}$$

Then, we can show

$$\frac{\partial x_i}{\partial w} = \frac{\partial b(\mathbf{P})}{\partial P_i} \frac{1}{b(\mathbf{P})}, \tag{7}$$

therefore proving that the response of demand to wealth is independent of the level of wealth.

### 3.3

*Consider the special case of (4) where:*

$$\begin{aligned} a(\mathbf{P}) &= \sum_j P_j \gamma_j, \\ b(\mathbf{P}) &= \prod_j \left( \frac{P_j}{\beta_j} \right)^{\beta_j}. \end{aligned}$$

*Show that the Walrasian demand for good $i$ in this case is:*

$$x_i(\mathbf{P}, w) = \gamma_i + \frac{\beta_i}{P_i}\left(w - \sum_j P_j \gamma_j\right). \tag{8}$$

*Give an economic interpretation of the parameters $\beta_i$ and $\gamma_i$.*

Start by writing our indirect utility function

$$V(\mathbf{P}, w) = \frac{w - a(\mathbf{P})}{b(\mathbf{P})} = \frac{w - \sum_j P_j \gamma_j}{\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}},$$

and first find $\frac{\partial V}{\partial P_i}$:

$$\begin{aligned}
\frac{\partial V}{\partial P_i} &= \frac{-\gamma_i \left[\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}\right] - \left[w - \sum_j P_j \gamma_j\right] \frac{\beta_i}{P_i} \prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}}{\left[\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}\right]^2}, \\
\frac{\partial V}{\partial w} &= \frac{1}{\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}}, \\
\therefore x_i(\mathbf{P}, w) &= -\left\{\frac{-\gamma_i \left[\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}\right] - \left[w - \sum_j P_j \gamma_j\right] \frac{\beta_i}{P_i} \prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}}{\left[\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}\right]^2}\right\} \prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j} \\
&= \gamma_i + \frac{\beta_i}{P_i}\left(w - \sum_j P_j \gamma_j\right). \qquad (9)
\end{aligned}$$

The parameters $\beta_i$ and $\gamma_i$ are the marginal propensity to consume out of "supernumerary" income (i.e., income at the margin), and the "subsistence" level of consumption, respectively.

## 3.4

*Show that the income elasticity of demand in this case equals:*

$$\eta_i = \frac{\partial \log x_i}{\partial \log w} = \frac{\beta_i}{\theta_i}, \tag{10}$$

*where $\theta_i = \frac{P_i x_i}{w}$ is the budget share of good $i$. How does this income elasticity behave as income grows indefinitely? Is this behaviour empirically plausible?*

From (8), we have:

$$x_i(\mathbf{P}, w) = \gamma_i + \frac{\beta_i}{P_i} \prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j} \left[\frac{w - \sum_j P_j \gamma_j}{\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}}\right] = a'(P_i) + b'(P_i)\left[\frac{w - a(\mathbf{P})}{b(\mathbf{P})}\right],$$

and recall when we do log-linearisation that

$$d \log x = \frac{dx}{x},$$

and so:

$$\begin{aligned}
\frac{\partial \log x_i}{\partial \log w} &= \frac{\partial x_i}{\partial w} \frac{w}{x_i} \\
&= \frac{b'(P_i)}{b(\mathbf{P})} \frac{w}{x_i} \\
&= \frac{\frac{\beta_i}{P_i} \prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}}{\prod_j \left(\frac{P_j}{\beta_j}\right)^{\beta_j}} \frac{w}{x_i} \\
&= \frac{\beta_i}{P_i} \frac{w}{x_i} \\
&\Leftrightarrow \frac{\beta_i}{\frac{P_i x_i}{w}} \\
\therefore \eta_i &= \frac{\beta_i}{\theta_i}. \qquad (11)
\end{aligned}$$

As income $w$ grows without bound, this approaches a limit of $\beta_i$ and so the income elasticity approaches a limit of one. In other words, tastes a asymptotically homothetic. This is contrary to so much empirical evidence (e.g. Engel's Law).

### 3.5

*Show that the own-price elasticity of demand in this case equals:*

$$\frac{\partial \log x_i}{\partial \log P_i} = -\frac{\beta_i}{P_i x_i} \left( w - \sum_{j \neq i} P_j \gamma_j \right).$$

*Show that, for small values of the budget share $\theta_i$, this elasticity is approximately proportional to (minus) the income elasticity of demand in (10), where the factor proportionality is independent of $i$. Is this outcome empirically plausible?*

Similar to the previous question, we have

$$\frac{\partial \log x_i}{\partial \log P_i} = \frac{\partial x_i}{\partial P_i} \frac{P_i}{x_i},$$

so we have:

$$
\begin{aligned}
x_i &= \gamma_i + \frac{\beta_i}{P_i}\left(w - \sum_j P_j \gamma_j\right), \\
\implies \frac{\partial x_i}{\partial P_i} &= -\frac{\beta_i w}{P_i^2} - \left[-\frac{\beta_i}{P_i^2}\sum_j (P_j\gamma_j) + \frac{\beta_i}{P_i}\gamma_i\right] \\
&= -\frac{\beta_i w}{P_i^2} + \frac{\beta_i}{P_i^2}\sum_j (P_j\gamma_j) - \frac{\beta_i}{P_i}\gamma_i \\
&= \frac{\beta_i}{P_i}\left[\frac{1}{P_i}\sum_j (P_j\gamma_j) - \frac{w}{P_i} - \gamma_i\right], \\
\implies \frac{\partial x_i}{\partial P_i}\frac{P_i}{x_i} &= \frac{\beta_i}{x_i}\left[\frac{1}{P_i}\sum_j (P_j\gamma_j) - \frac{w}{P_i} - \gamma_i\right] \\
&= -\frac{\beta_i}{P_i x_i}\left[P_i\gamma_i - \sum_j (P_j\gamma_j) + w\right] \\
&\Leftrightarrow -\frac{\beta_i}{P_i x_i}\left(w - \sum_{j \neq i} P_j\gamma_j\right),
\end{aligned}
$$

completing our proof. This result is “Pigou’s/Deaton’s Law”. It holds for any demand function generated by additively separable preferences: the LES is a member of the subclass (due to Pollak) that is nested by both the Gorman Polar Form and additive separability. This outcome is very counter-factual: it would be strange if income and price elasticities lined up so neatly as this (with the ratio of the price and income elasticity the same for all goods), and the data reject it. It is particularly implausible that the approximation is better for more disaggregated data.