Some Integration Tricks

1 The Gaussian integral

Consider the following integral:

$$G = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx.$$

Here G is not a function of two functions – i.e., we don't have a factor of x multiplying the exponential term, so we can't use integration by parts or substitution here.

We can however square G, and call the second integral some dummy variable, say, y:

$$\begin{aligned} G^2 &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dxdy. \end{aligned}$$

Now, change the coordinates to polar coordinates:

$$\begin{aligned} &(x, y) \to (r\cos(\theta), r\sin(\theta)) \\ \implies &dxdy = rdrd\theta. \end{aligned}$$

Then, the integral becomes:

$$G^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) rdrd\theta.$$

Now we have a factor of r sitting next to the exponential, so we can make the substitution:

$$\begin{aligned} u &= \frac{r^2}{2} \\ \implies rdr &= du, \end{aligned}$$

so:

$$\begin{aligned} G^2 &= \int_0^{2\pi}\int_0^{\infty} \exp(-u)dud\theta \\ &= 2\pi\left[-\exp(-u)\right]_0^{\infty} \\ &= 2\pi, \end{aligned}$$

and so:

$$\begin{aligned} G &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \sqrt{2\pi}. \end{aligned}$$

2 Exponentials of quadratic functions

Consider an integral of the following form:

$$\int_{-\infty}^{\infty} \exp\left\{-(ax^2 + bx + c)\right\} dx,$$

where $a > 0$ can be converted into the form of a Gaussian integral by completing the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$
$$\implies \int_{-\infty}^{\infty} \exp\left\{-(ax^2 + bx + c)\right\} dx = \int_{-\infty}^{\infty} \exp\left\{-a\left(a + \frac{b}{2a}\right)^2 - c + \frac{b^2}{4a}\right\} dx,$$

and then make a substitution:

$$u = \sqrt{2a}\left(x - \frac{b}{2a}\right)$$
$$\implies dr = \frac{1}{\sqrt{2a}} du,$$

and so:

$$\begin{aligned}\implies \int_{-\infty}^{\infty} \exp\left\{-(ax^2 + bx + c)\right\} dx &= \frac{1}{\sqrt{2a}} \exp\left(\frac{b^2}{4a} - c\right) \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{1}{\sqrt{2a}} \exp\left(\frac{b^2}{4a} - c\right) G \\ &= \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} - c\right).\end{aligned}$$

3 Differentiation under the integral (Leibniz's Integral Rule)

This technique involves moving the derivative of an integral under the integral sign, as implied by its name. The general rule can be found in most textbooks or the internet. Considering the specific case where the integration limits are constants, the rule can be written as:

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha) dx = \int_a^b \frac{\partial f(x, \alpha)}{\partial x} dx.$$

How do you use it? Let's say we want to evaluate the integral:

$$I_2 = \int_{-\infty}^{\infty} x^2 \exp\left(-ax^2\right) dx,$$

where $a > 0$ is a constant. Generalise the problem as:

$$\begin{aligned}
I(\alpha) &= \int_{-\infty}^{\infty} \exp\left(\alpha x^2\right) dx \\
\implies I'(\alpha) &= \frac{d}{d\alpha} \int_{-\infty}^{\infty} \exp\left(\alpha x^2\right) dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial \alpha} \left[\exp\left(\alpha x^2\right)\right] dx \\
&= \int_{-\infty}^{\infty} x^2 \exp\left(\alpha x^2\right) dx.
\end{aligned}$$

We see that $I_2 = I'(-a)$. The integral $I(\alpha)$ looks familiar – it is the previous integral we evaluated if $\alpha < 0$, $b = c = 0$:

$$\begin{aligned}
I(\alpha) &= \sqrt{\frac{\pi}{-\alpha}} \\
\implies I'(\alpha) &= \frac{\sqrt{\pi}}{2(-\alpha)^{\frac{3}{2}}},
\end{aligned}$$

and so we see that:

$$\begin{aligned}
I_2 &= I'(-a) \\
&= \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}.
\end{aligned}$$

Integrals of the form:

$$\int_{-\infty}^{\infty} x^{2a} \exp\left(-ax^2\right) dx,$$

can be found by taking n derivatives of $I(\alpha)$ with respect to α and setting $\alpha = -a$, i.e.:

$$\begin{aligned}
I_{2n} &= \int_{-\infty}^{\infty} x^{2n} \exp\left(-ax^2\right) dx \\
&= \frac{(2n-1)!!}{(2a)^n} \sqrt{\frac{\pi}{a}},
\end{aligned}$$

where $n!! = n(n-2)!!$, $0!! = 1 = 1!!$ is the double factorial operator.

For the same integral with odd powers of x in the integrand the result is 0 due to the integrand being an odd function of x and the limits being symmetric about 0.