RBC Model Smörgåsbord and Labour Markets

1 An analytic RBC model

Consider a version of the Real Business Cycle (RBC) model in which there is 100% capital depreciation and utility is log-linear in consumption and leisure.

1.1

Discuss why the competitive equilibrium of this economy is equivalent to the following social planning problem:

$$\max_{\{c_t, l_t\}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t (\ln c_t - \chi l_t),$$

subject to[1]

$$\begin{aligned} y_t &= c_t + k_t, \\ y_t &= A_t k_{t-1}^{\alpha} l_t^{1-\alpha}, \\ \ln A_t &= \rho \ln A_{t-1} + \epsilon_t. \end{aligned}$$

Derive the intertemporal Euler equations for consumption and labour supply, and the intratemporal Euler labour-consumption equation for this economy.

To compare the solution to the social planner's problem to the competitive equilibrium, we first set up a decentralised RBC model.

We first assume that households are the owners of capital, and rent the capital to firms; firms issue debt and equity to households in exchange. We could model the problem where households are the owners of firms, and that firms are the owners of capital. Both setups would lead to the same outcome, however, since eventually households would own the capital.

Households consume output, supply labour, and own capital. They earn a return for renting out the capital stock to firms each period of R_t. The household budget constraint is

$$C_t + K_{t+1} - (1-\delta)K_t = w_t L_t + R_t K_t + \Pi_t, \tag{1}$$

where C_t is consumption, K_t is period t capital stock, δ is the depreciation rate, w_t is the competitive market wage rate, L_t is labour supply, and Π_t are firm profits. The household takes the capital stock in period t as given, and its consumption and savings decisions directly influence the next period capital stock. We could include governments bonds here, but for simplicity these are omitted. We also assume that capital completely depreciates, so $\delta = 1$. Given its income and expenditure decisions, the

[1] Here I have adjusted the timing on capital to the end of period notation (and for the shock process too).

representative household's problem is

$$\begin{aligned} \max_{C_t, L_t, K_{t+1}} \quad & \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) + V(1 - L_{t+i})) \right], \\ \text{s.t. } & C_t + K_{t+1} = w_t L_t + R_t K_t + \Pi_t, \end{aligned} \tag{2}$$

where β is the household's subjective discount factor, and $U(\cdot)$ is the household's utility function which is concave and monotonically increasing in consumption, and $V(\cdot)$ is a function to account for the utility households receive for enjoying their leisure time.

Forming the Lagrangian for household problem yields

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) + V(1 - L_{t+i})) \right] \\ & + \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left(w_{t+i} L_{t+i} + R_{t+i} K_{t+i} + \Pi_{t+i} - C_{t+i} - K_{t+1+i} \right) \right]. \end{aligned} \tag{3}$$

Thus, our FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= U'(C_t) - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \lambda_t - \beta \mathbb{E}[\lambda_{t+1} R_{t+1}] = 0 \\ \frac{\partial \mathcal{L}}{\partial L_t} &= -V'(1 - L_t) + \lambda_t w_t = 0 \\ \frac{\partial \mathcal{L}}{\partial L_{t+1}} &= -\beta \mathbb{E}[V'(1 - L_{t+1})] + \beta \mathbb{E}[\lambda_{t+1} w_{t+1}] = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= w_t L_t + R_t K_t + \Pi_t - C_t - K_{t+1} = 0, \end{aligned}$$

and combining them yields

$$U'(C_t) = \beta \mathbb{E}_t \left[U'(C_{t+1}) R_{t+1} \right] \tag{4}$$

$$V'(1 - L_t) = U'(C_t) w_t \tag{5}$$

$$\mathbb{E}[V'(1 - L_{t+1})] = \mathbb{E}[\lambda_{t+1} w_{t+1}] \tag{6}$$

Where (4) is the household's consumption Euler equation, (5) is the household's intratemporial Euler equation between consumption and leisure, and (6) is the intertemporial labour relation. In order to yield expressions for R_{t+1} and w_t we need to solve the competitive firm's problem

$$\max_{K_t, L_t} \quad \Pi_t = F(K_t, L_t) - w_t L_t - R_t K_t,$$

where $F(K, L)$ is the representative firm's production function, which is homogenous of degree 1 and exhibits decreasing marginal returns to inputs. Differentiating for the firm's choices of capital and labour, and setting the derivatives equal to zero gives us our FOCs

$$\begin{aligned} \frac{\partial \Pi_t}{\partial K_t} &= F_K(K_t, L_t) - R_t = 0 \\ \frac{\partial \Pi_t}{\partial L_t} &= F_L(K_t, L_t) - w_t = 0, \end{aligned}$$

implying that

$$R_t^* = F_K(K, L) \tag{7}$$
$$w_t^* = F_L(K, L). \tag{8}$$

In words, the factors of production are paid their marginal product at the optimum. The competitive equilibrium is then a solution to (1), (4), (5), (7), and (8). There are five equations in five unknowns C, L, K, R, and w.

Now, we assess the social planner's problem. The social planner wants to maximise the representative household's utility subject to the economy's resource constraints. It faces the following problem:

$$\begin{aligned} \max_{C_t, L_t} \quad & \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_t) + V(1 - L_t)) \right] \\ \text{s.t.} \quad & K_{t+1} = Y_t - C_t \\ & Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \end{aligned}$$

where A_t follows the following process

$$\ln A_{t+1} = \rho A_t + \epsilon_{t+1}, \quad 0 < \rho < 1.$$

We can combine the constraints to the social planner's problem, and then form the Lagrangian

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) + V(1 - L_{t+i})) \right] \\ & + \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left(A_{t+i} K_{t+i}^{\alpha} L_{t+i}^{1-\alpha} - C_{t+i} - K_{t+1+i} \right) \right], \end{aligned}$$

which gives us the FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= U'(C_t) - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \lambda_t - \beta \mathbb{E}[\lambda_{t+1} \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha}] = 0 \\ \frac{\partial \mathcal{L}}{\partial L_t} &= -\chi + \lambda_t (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = 0 \\ \frac{\partial \mathcal{L}}{\partial L_{t+1}} &= -\beta\chi + \beta \mathbb{E}[\lambda_{t+1} (1-\alpha) A_{t+1} K_{t+1}^{\alpha} L_{t+1}^{-\alpha}] = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= A_t K_t^{\alpha} L_t^{1-\alpha} - C_t - K_{t+1} = 0. \end{aligned}$$

Combining the FOCs yields

$$U'(C_t) = \beta \mathbb{E}_t [U'(C_{t+1}) \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha}] \tag{9}$$
$$\chi = U'(C_t) \underbrace{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}_{F_L(K_t, L_t)}, \tag{10}$$
$$1 = \alpha\beta \frac{Y_t}{L_t} \mathbb{E} \left[\frac{L_{t+1}}{K_{t+1}} \right] \tag{11}$$

If we define the marginal value of an additional unit of capital next year as (recalling that we do not have depreciation):

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}} = \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha},$$

then we can rewrite (9) as

$$U'(C_t) = \beta \mathbb{E}_t[U'(C_{t+1}) R_{t+1}],$$

which is identical to the competitive economy consumption Euler equation. The intratemporial Euler equations are identical too.

We know that a social planner determines a Pareto optimum, defined to be some allocation such that there is no other allocation which some agents strictly prefer which does not make any agents worse off. We can conclude that under the assumptions given for the baseline RBC model, the following hold:

- First Welfare Theorem: A competitive equilibrium is Pareto optimal;
- Second Welfare Theorem: Any Pareto optimum can be supported as a competitive equilibrium with appropriate choice of endowments.

Necessary and sufficient conditions for the above are that we do not have externalities, markets are complete, and that there are no distortionary taxes.

1.2

Define the output to consumption ratio ϑ_t by $\vartheta_t = y_t/c_t$. Show that the Euler equation for consumption can then be written as a first order difference equation in terms of ϑ_t and $\mathbb{E}_t[\vartheta_{t+1}]$, and then calculate an expression for the steady state value of ϑ_t. Plot a graph of the ordinary difference equation with ϑ_t on the horizontal axis and $\mathbb{E}_t[\vartheta_{t+1}]$ on the vertical axis. Does your graph suggest that the ordinary difference equation for ϑ_t is stable?

The consumption to output ratio is defined as

$$\vartheta_t = \frac{Y_t}{C_t},$$

and our consumption Euler equation given the form of our utility function is

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[R_{t+1} \frac{1}{C_{t+1}} \right].$$

Given the results above, we can rewrite the Euler equation as

$$\begin{aligned} \frac{1}{C_t} &= \alpha\beta \mathbb{E}_t \left[\frac{Y_{t+1}}{K_{t+1}} \frac{1}{C_{t+1}} \right] \\ &= \alpha\beta \mathbb{E}_t \left[\frac{\vartheta_{t+1}}{K_{t+1}} \right], \end{aligned}$$

and then multiplying both the LHS and RHS by Y_t gives

$$\vartheta_t = \alpha\beta \mathbb{E}_t \left[\frac{\vartheta_{t+1}}{K_{t+1}} \right] Y_t. \tag{12}$$

In the steady state we can assume

$$\bar{\vartheta} = \alpha\beta\frac{\bar{\vartheta}}{\bar{K}},$$

which we can rewrite to get

$$\begin{aligned}
\bar{\vartheta}\bar{K} &= \alpha\beta\bar{\vartheta} \\
\bar{\vartheta}(\bar{Y} - \bar{C}) &= \alpha\beta\bar{\vartheta} \\
\bar{\vartheta}\bar{Y} - \bar{\vartheta}\bar{C} &= \alpha\beta\bar{\vartheta} \\
\bar{C} &= (1 - \alpha\beta)\bar{Y},
\end{aligned}$$

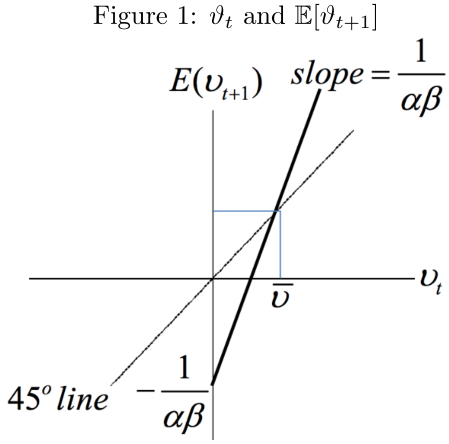
and dividing by the steady state level of consumption yields

$$\begin{aligned}
1 &= (1 - \alpha\beta)\frac{\bar{Y}}{\bar{C}} \\
\implies \bar{\vartheta} &= \frac{1}{1 - \alpha\beta}. \qquad (13)
\end{aligned}$$

1.2.1

If you think the ordinary difference equation is stable, then comment what factors affect the speed with which ϑ_t converges to its steady state value.

Figure 1: ϑ_t and $\mathbb{E}[\vartheta_{t+1}]$



The figure suggests that the ordinary difference equation is unstable (since in the steady state we should have $\vartheta_t = \vartheta_{t+1} = \bar{\vartheta}$).

1.2.2

If you think the ordinary difference equation is unstable, then comment on whether this is a sensible model.

The model is sensible despite the ODE being unstable. There is only one value of the output to consumption ratio which is sensible – its steady state value.

1.2.3

How does the Blanchard-Kahn condition matter for whether the ordinary difference equation is stable or unstable? Is the transversality condition respected or violated in this model?

The Blanchard-Kahn (BK) condition is that the number of unstable eigenvalues in the system of equations be equal to the number of control variables. There is a direct link between the instability of the ODE for ϑ_t and the BK condition in a linear rational expectations model. In our model we have 2 controls (consumption and labour supply) and one predetermined variable (capital). We therefore need two unstable roots and one stable root. In our case these are more complex because we have not linearised the model, but the intuition still goes through. The ODE we have derived is where the first unstable root shows up. The other would show up if we did a similar exercise starting the Euler equation for labour. The transversality condition plays a crucial role here, since any departure of ϑ_t from $\bar{\vartheta}$ would violate it. In other words, the unique rational expectations equilibrium satisfies $\vartheta_t = \bar{\vartheta}\ \forall t$.

For this model, the BK condition is satisfied and so the system is determinate and the transversality condition is not violated.

1.3

Assume that ϑ_t is equal to its steady state value. Show that in this case the solution of the model is given by:

$$
\begin{aligned}
l_t &= \frac{1-\alpha}{\chi(1-\alpha\beta)}, \\
y_t &= A_t k_t^{\alpha} \left(\frac{1-\alpha}{\chi(1-\alpha\beta)} \right)^{1-\alpha}, \\
c_t &= (1-\alpha\beta) y_t, \\
k_t &= y_t - c_t, \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t.
\end{aligned}
$$

A calibration of the model is $\alpha = 0.4$, $\chi = 2$, $\beta = 0.99$, $\rho = 0.95$, and $\sigma_\epsilon = 0.01$. Derive the numerical equations corresponding to output and calculate the steady state variables of the model. Simulate the economy for 50 periods and calculate the standard deviation of simulated output, consumption, and labour supply. Use impulse response functions to show how the endogenous variables in the economy react to a one standard deviation technology shock.

Steady state equations for consumption and ϑ were calculated in Section 1.2. From there, rearrange and solve to attain system of equations defining equilibrium.

Listing 1: **Simple RBC model in Dynare**

```

% Simple RBC Model
% David Murakami (Oxford)
% Macroeconomics (MPhil Economics)
% Very simple RBC model with full capital depreciation and 5
  equations

```

```
var
y $y$ (long_name='output')
k $k$ (long_name='capital stock')
h $h$ (long_name='labour supply')
A $A$ (long_name='technology')
c $c$ (long_name='consumption');

varexo eps $\epsilon$;

parameters alpha $\alpha$ (long_name='capital share')
beta $\beta$ (long_name='stochastic discount factor')
rho $\rho$ (long_name='technology shock persistence')
chi $chi$ (long_name='labour disutility parameter')
sigmaeps $\sigma_{\epsilon}$ (long_name='standard deviation of
    shock');

alpha = 0.40;
beta = 0.99;
rho = 0.95;
chi = 2;
sigmaeps = 0.01;

%%
model;
[name='Law of motion of capital']
k = y-c;

[name='Household labour supply']
h = (1-alpha)/(chi*(1-alpha*beta));

[name='Consumption']
c = (1-alpha*beta)*y;

[name='Production function']
y = A*k(-1)^(alpha)*h^(1-alpha);

[name='Shock process']
log(A) = rho*log(A(-1)) + eps;

end;

%%
initval;
A = 1;
k=0.1060660249;
h=0.4966887417;
```

```
c=0.1617774722;
y=0.2678434971;
end;

resid;

steady;

shocks;
var eps = sigmaeps^2;
end;

stoch_simul(order=1,irf=50);
```

Dynare (MATLAB) Output:

```
Residuals of the static equations:
Equation number 1 : 0 : Law of motion of capital
Equation number 2 : 0 : Household labour supply
Equation number 3 : 0 : Consumption
Equation number 4 : 0 : Production function
Equation number 5 : 0 : Shock process

STEADY-STATE RESULTS:

y                    0.267843
k                    0.106066
h                    0.496689
A                    1
c                    0.161777

MODEL SUMMARY

  Number of variables:          5
  Number of stochastic shocks:  1
  Number of state variables:    2
  Number of jumpers:            0
  Number of static variables:   3

MATRIX OF COVARIANCE OF EXOGENOUS SHOCKS
Variables            eps
eps             0.000100

POLICY AND TRANSITION FUNCTIONS
                y            k            h            A            c
Constant  0.267843     0.106066     0.496689     1.000000     0.161777
k(-1)     1.010101     0.400000            0            0     0.610101
```

A(−1)	0.254451	0.100763	0	0.950000	0.153689
eps	0.267843	0.106066	0	1.000000	0.161777

THEORETICAL MOMENTS

VARIABLE	MEAN	STD. DEV.	VARIANCE
y	0.2678	0.0140	0.0002
k	0.1061	0.0055	0.0000
h	0.4967	0.0000	0.0000
A	1.0000	0.0320	0.0010
c	0.1618	0.0084	0.0001

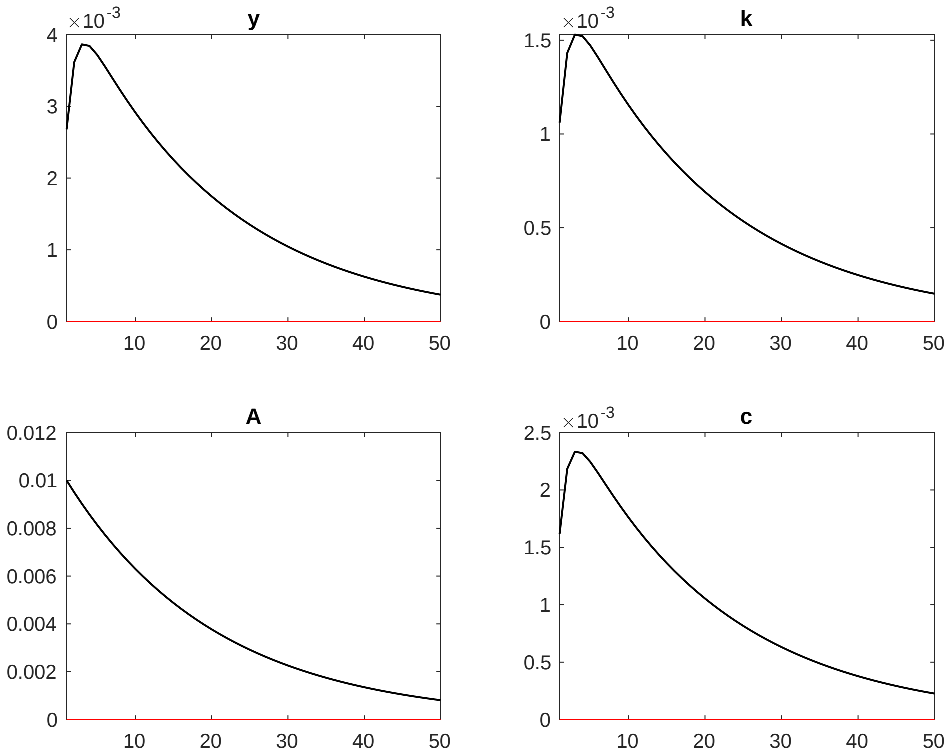
MATRIX OF CORRELATIONS

Variables	y	k	A	c
y	1.0000	1.0000	0.9908	1.0000
k	1.0000	1.0000	0.9908	1.0000
A	0.9908	0.9908	1.0000	0.9908
c	1.0000	1.0000	0.9908	1.0000

COEFFICIENTS OF AUTOCORRELATION

Order	1	2	3	4	5
y	0.9783	0.9407	0.8981	0.8550	0.8130
k	0.9783	0.9407	0.8981	0.8550	0.8130
A	0.9500	0.9025	0.8574	0.8145	0.7738
c	0.9783	0.9407	0.8981	0.8550	0.8130

Figure 2: Impulse Responses to Productivity Shock



2 Another analytic RBC model[2]

An economy is populated by an infinitely-lived representative agent with preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \ln C_t,$$

where C_t is consumption and ln *is the natural logarithm. β is a discount factor that satisfies $\beta \in (0, 1)$. There is no uncertainty or labour in this economy, and output Y_t is a linear function of K_t. Capital can be accumulated through a technology that has a constant elasticity of substitution form in existing capital and investment I_t. The constraints of the economy are therefore:*

$$\begin{aligned} Y_t &= C_t + I_t, \\ Y_t &= K_t, \\ K_{t+1} &= K_t^{\alpha} I_t^{\gamma}, \end{aligned}$$

with $\alpha > 0$, $\gamma > 0$, and $\alpha + \gamma < 1$.

2.1

Re-write the constraints of the economy as equations for consumption C_t and future capital stock K_{t+1} in terms of the current capital stock K_t and the investment rate $s_t = I_t/Y_t$.

First things first, substitute out Y_t from the constraints, and rearrange the first constraint to get

$$\begin{aligned} C_t &= K_t - I_t, \\ K_{t+1} &= K_t^{\alpha} I_t^{\gamma}. \end{aligned}$$

Then use the fact that consumption is nothing but income minus savings, and use the first equation to attain an expression for I_t in terms of C_t and K_t:

$$C_t = (1 - s_t) K_t, \tag{14}$$

and

$$\begin{aligned} K_{t+1} &= K_t^{\alpha} (K_t - C_t)^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha} C_t^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha} \left[(1 - s_t) K_t\right]^{\gamma} \\ &= K_t^{\alpha+\gamma} - K_t^{\alpha+\gamma} (1 - s_t)^{\gamma} \\ &= K_t^{\alpha+\gamma} \left[1 - 1 + s_t^{\gamma}\right] \\ &= K_t^{\alpha+\gamma} s_t^{\gamma}. \end{aligned} \tag{15}$$

2.2

Use the constraints expressed in terms of the current capital stock and the investment rate to derive the first order conditions of the social planner's problem. Interpret each condition briefly.

[2] This was a past exam question.

Substitute (14) into the objective function to get the Ramsey social planner's problem as

$$\max_{\{s_t, K_{t+1}\}} \sum_{s=0}^{\infty} \beta^s \ln(K_{t+s}(1 - s_{t+s})).$$

The Lagrangian is

$$\mathcal{L} = \ln(K_t(1 - s_t)) + \lambda_t \left[K_{t+1} - K_t^{\alpha+\gamma} s_t^{\gamma}\right],$$

and the FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial s_t} &= -\frac{K_t}{K_t(1 - s_t)} - \lambda_t \gamma K_t^{\alpha+\gamma} s_t^{\gamma-1} = 0 \\ \implies -\frac{1}{1 - s_t} &= \lambda_t \gamma K_t^{\alpha+\gamma} s_t^{\gamma-1}. \end{aligned} \tag{16}$$

The second FOC is actually a bit tricky. Notice that in the way the question is setup, K_t – the inherited level of capital – enters the social planner's objective function. But the planner can't pick K_t as it's already predetermined. Instead, the planner must pick K_{t+1}, so the Lagrangian for the FOC wrt K_{t+1} looks like this

$$\mathcal{L} = \ln(K_t(1 - s_t)) + \beta \ln(K_{t+1}(1 - s_{t+1})) + \lambda_t \left[K_{t+1} - K_t^{\alpha+\gamma} s_t^{\gamma}\right] + \beta \lambda_{t+1} \left[K_{t+2} - K_{t+1}^{\alpha+\gamma} s_{t+1}^{\gamma}\right],$$

which gives our second FOC,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \beta \frac{(1 - s_{t+1})}{K_{t+1}(1 - s_{t+1})} + \lambda_t - \beta \lambda_{t+1}(\alpha + \gamma) K_{t+1}^{\alpha+\gamma-1} s_{t+1}^{\gamma} = 0 \\ \implies \beta \lambda_{t+1}(\alpha + \gamma) K_{t+1}^{\alpha+\gamma-1} s_{t+1}^{\gamma} &= \beta \frac{1}{K_{t+1}} + \lambda_t. \end{aligned} \tag{17}$$

This question must've stung many on the exam...

2.3

Combine the two first order conditions and find the steady state investment rate in this economy. How does it vary with α and γ and why? (Hint: You may find it useful to work with a transformation of the investment rate $v_t = s_t/(\gamma(1 - s_t))$.)

Rearrange (16) to define v_t:

$$\frac{s_t}{\gamma(1 - s_t)} = -\lambda_t K_t^{\alpha+\gamma} s_t^{\gamma} = v_t.$$

Then, use the second FOC, (17), and substitute the value of λ_t into the expression for v_t:

$$\begin{aligned} v_t &= -\lambda_t K_t^{\alpha+\gamma} s_t^{\gamma} \\ &= -\left(\beta \lambda_{t+1}(\alpha + \gamma) K_{t+1}^{\alpha+\gamma-1} s_{t+1}^{\gamma} - \beta \frac{1}{K_{t+1}}\right) K_t^{\alpha+\gamma} s_t^{\gamma} \\ &= \left(\frac{1}{K_{t+1}} - \lambda_{t+1}(\alpha + \gamma) K_{t+1}^{\alpha+\gamma-1} s_{t+1}^{\gamma}\right) \beta K_t^{\alpha+\gamma} s_t^{\gamma}, \end{aligned}$$

and then do a bit of manipulation to get

$$
\begin{aligned}
&= \frac{K_{t+1}}{K_{t+1}} \left(\frac{1}{K_{t+1}} - \lambda_{t+1}(\alpha+\gamma) K_{t+1}^{\alpha+\gamma-1} s_{t+1}^{\gamma} \right) \beta K_t^{\alpha+\gamma} s_t^{\gamma} \\
&= \left(1 - (\alpha+\gamma)\lambda_{t+1} K_{t+1}^{\alpha+\gamma} s_{t+1}^{\gamma}\right) \beta \frac{K_t^{\alpha+\gamma} s_t^{\gamma}}{K_{t+1}} \\
&= \left(1 + (\alpha+\gamma) v_{t+1}\right) \beta \frac{K_t^{\alpha+\gamma} s_t^{\gamma}}{K_t^{\alpha+\gamma} s_t^{\gamma}} \\
v_t &= \beta + \beta(\alpha+\gamma) v_{t+1}. \qquad (18)
\end{aligned}
$$

In the steady state we have

$$
v = \frac{\beta}{1-\beta(\alpha+\gamma)},
$$

where we see that v is increasing in α and γ. Recall that α and γ are the share/weights of period t capital and investment, respectively, in the law of motion of capital. It stands to reason that if either of these shares increase, v, which is the ratio of the savings rate to the rate of consumption, should increase too. The higher shares induce more saving, on behalf of the social planner.

2.4

Derive and discuss the conditions under which there is a unique perfect foresight equilibrium with the investment rate at its steady state value. Your answer should include a diagram. How do parameter values affect the speed of convergence to the equilibrium?

We can rearrange (18) to get it into the following form:

$$
\mathbb{E}_t \mathbf{X}_{t+1} = \mathbf{A} \mathbf{X}_t,
$$

where in order to get unique convergence, we need the Blanchard-Kahn conditions to be met (number of eigenvalues of $\mathbf{A}$ outside of unit circle must be equal to the number of jump variables). So, write (18) as

$$
v_{t+1} = \frac{1}{\beta(\alpha+\gamma)} v_t - \frac{1}{\alpha+\gamma},
$$

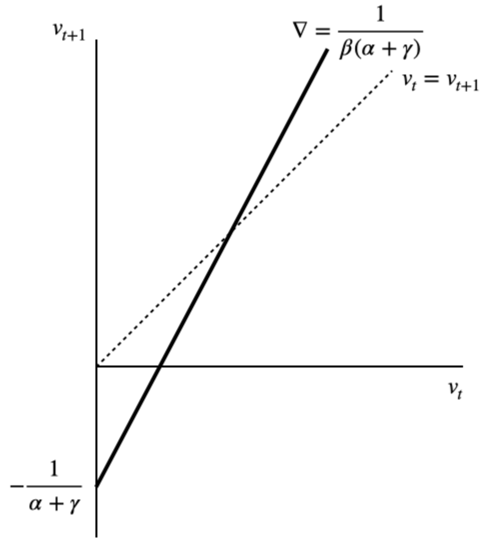
and plot it in (v_t, v_{t+1}) space, as below.

In the context of our simple model, we have a first order difference equation in v_t and v_{t+1}, and the Blanchard-Kahn condition requires that $\beta(\alpha+\gamma) < 1$ in absolute value. Intuitively, we need one of, or a combination of, the following conditions to be met:

- The household must have a low discount factor, which implies that the steady state gross interest rate is sufficiently high, leading to low consumption;
- Either or both of the weights of capital and investment are sufficiently low, which implies that it is relatively costly to transform current period capital and investments into future productive capital.

As for the steady state investment rate: we have $s = \frac{I}{Y}$, and so we can deduce that while the steady state value of the savings rate is affected by the parameters, the parameters do not affect the convergence to the unique equilibrium. The savings/investment rate is determined immediately and so there is no convergence to the equilibrium.

Figure 3: Plot of v_{t+1}



2.5

Returning to the equation for future capital stock derived in part 2.1, is it possible to have a constant investment rate but ever increasing capital in the model? Explain your answer and assess the realism of any restrictions applied to parameter values.

So, from (15), we previously had

$$K_{t+1} = K_t^{\alpha+\gamma} s_t^{\gamma}.$$

If we assume that $s_t = s,\ \forall t$, and take logs, we get

$$\ln K_{t+1} = (\alpha+\gamma)\ln K_t + \underbrace{\gamma \ln s}_{g}$$

$$g = \ln K_{t+1} - (\alpha+\gamma)\ln K_t,$$

and we can see that the capital stock diverges if $\alpha+\gamma$ is either greater than or less than 1. In particular, if $\alpha+\gamma<1$, then we will have an ever increasing capital stock. Previously, we required $|\beta(\alpha+\gamma)|<1$ in order to have a unique, perfect foresight equilibrium. So putting these inequalities together, we have

$$1 < |\alpha+\gamma| < \beta^{-1}.$$

Intuitively, we need high marginal product of capital and/or a good transformation technology to ensure that the capital stock never converges, but sufficient discounting that the investment rate converges. It is probably harder to satisfy the first rather than the second of these conditions.

3 Variable capacity utilisation[3]

A social planner wishes to maximise the welfare of a representative agent with preferences:

$$U_t = \sum_{s=0}^{\infty} \beta^s \left(\frac{c_{t+s}^{1-\sigma}}{1-\sigma} \right),$$

defined over a consumption stream $\{c_t\}$*.* $\sigma > 0$ *is the coefficient of relative risk aversion and* $\beta \in (0,1)$ *is the subjective discount factor. There is no uncertainty so there is no need to consider expectations. Output* y_t *depends on the capital stock* k_t *and the intensity* $h_t \in (0,1)$ *with which capital is utilised:*

$$y_t = (h_t k_t)^\alpha.$$

Capital depreciates more if it is used intensively, such that depreciation δ_t *is defined by the power function:*

$$\delta_t = \frac{1}{\omega} h_t^\omega,$$

with the parameter $\omega > 1$*.*

3.1

Write down the intertemporal budget constraint faced by the social planner, which defines k_{t+1} *as a function of* c_t*,* h_t*, and* k_t*.*

Using some identities, we can say that

$$y_t \equiv c_t + i_t,$$

where

$$i_t = k_{t+1} - k_t - (1-\delta_t)k_t.$$

So, we can write

$$\begin{aligned} y_t &\equiv c_t + i_t \\ \Leftrightarrow c_t + k_{t+1} &= (h_t k_t)^\alpha + k_t \left(1 - \frac{1}{\omega} h_t^\omega \right). \end{aligned} \tag{19}$$

3.2

Write down the maximisation problem of the social planner and derive the first order conditions for the optimal allocation. Briefly describe the economic intuition behind each first order condition.

The Lagrangian is

$$\mathcal{L} = \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[c_t + k_{t+1} - (h_t k_t)^\alpha - k_t \left(1 - \frac{1}{\omega} h_t^\omega \right) \right] + \beta \lambda_{t+1} \left[c_{t+1} + k_{t+2} - (h_{t+1} k_{t+1})^\alpha - k_{t+1} \left(1 - \frac{1}{\omega} h_{t+1}^\omega \right) \right],$$

[3] Another past exam question.

and the FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_t} &= c_t^{-\sigma} + \lambda_t = 0,\\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_t - \beta\alpha\lambda_{t+1}(h_{t+1}k_{t+1})^{\alpha-1}h_{t+1} - \beta\lambda_{t+1}\left(1 - \frac{1}{\omega}h_{t+1}^{\omega}\right) = 0,\\ \frac{\partial \mathcal{L}}{\partial h_t} &= -\alpha\lambda_t(h_tk_t)^{\alpha-1}k_t + \lambda_t k_t h_t^{\omega-1} = 0.\end{aligned}$$

Clean these up a bit to get:

$$c_t^{-\sigma} = \lambda_t, \tag{20}$$

$$\lambda_t = \beta\lambda_{t+1}\left[\alpha h_{t+1}^{\alpha}k_{t+1}^{\alpha-1} + 1 - \frac{1}{\omega}h_{t+1}^{\omega}\right], \tag{21}$$

$$\alpha h_t^{\alpha-1}k_t^{\alpha} = k_t h_t^{\omega-1}. \tag{22}$$

The first two FOCs give us the consumption Euler equation (or Keynes-Ramsey condition):

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left(\alpha h_{t+1}^{\alpha}k_{t+1}^{\alpha-1} + 1 - \frac{1}{\omega}h_{t+1}^{\omega}\right),$$

which is a ratio of the marginal utilities of consumption in periods t and $t+1$. More precisely, we can see from (21) that the marginal utility of consumption today is equal to a product of the discounted marginal utility tomorrow and a future resource return – i.e., decrease consumption today by Δ, at a loss of $u'(c_t)\Delta$ in utility; invest to get return$\times\Delta$ tomorrow; that investment is worth $\beta u'(c_t)\times$return$\times\Delta$ in terms of utility today; and, along the optimal path, an agent must be indifferent between these two options.

3.3

By suitable algebraic manipulation of the first order conditions obtained in part 2, show that the optimal allocation satisfies an intertemporal Euler equation of the form

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}\left[\left(1 - \frac{1}{\omega}\right)h_{t+1}^{\omega} + 1\right].$$

What condition on ω and β needs to be satisfied in this equation to ensure that the long run (steady state) level of capital utilisation h is in the interval $[0,1)$ for all $\omega > 1$. Interpret your answer.

From (20) and (21), we have

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}\left[\alpha h_{t+1}^{\alpha}k_{t+1}^{\alpha-1} - \frac{1}{\omega}h_{t+1}^{\omega} + 1\right],$$

and then use (22) to write this expression as

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}\left[\alpha h_{t+1}^{\alpha}k_{t+1}^{\alpha-1} - \frac{1}{\omega}h_{t+1}^{\omega} + 1\right]\\ &= \beta c_{t+1}^{-\sigma}\left[h_{t+1}^{\omega} - \frac{1}{\omega}h_{t+1}^{\omega} + 1\right]\\ &= \beta c_{t+1}^{-\sigma}\left[\left(1 - \frac{1}{\omega}\right)h_{t+1}^{\omega} + 1\right],\end{aligned}$$

and note we used some trickery as the FOC implies $\alpha h_{t+1}^{\alpha-1} k_{t+1}^{\alpha-1} = h_t^{\omega-1}$ so it stands to reason that we can also write $\alpha h_{t+1}^{\alpha} k_{t+1}^{\alpha-1} = h_{t+1}^{\omega}$.

Evaluating the above equation at the steady state gives

$$\begin{aligned} 1 &= \beta \left[\left(1 - \frac{1}{\omega}\right) h^\omega + 1 \right] \\ \frac{1-\beta}{\beta\left(1-\frac{1}{\omega}\right)} &= h^\omega, \end{aligned}$$

or

$$h^\omega = \frac{\frac{1}{\beta} - 1}{1 - \frac{1}{\omega}}.$$

In order to keep h in the interval $[0, 1)$, we need h^ω to also be in the interval $[0, 1)$ since $\omega > 1$. This condition is met so long as the numerator is not too large, and/or the denominator is not too small. This means that β cannot be too small (need a sufficiently high discount factor), and/or ω is not too small.

Intuitively, too small a discount factor implies that households are over-incentivised to save their income, and too small a value for ω implies a rapid rate of depreciation. In either of these circumstances, an efficient allocation would imply a very high usage of capital – perhaps beyond 100% capacity.

3.4

Log-linearise the first order conditions in part (2) to obtain the intratemporal condition between $\hat{h}_t$ and $\hat{k}_t$, and an intertemporal condition between $\hat{c}_{t+1}$, $\hat{c}_t$, and $\hat{h}_{t+1}$.

First, start with consumption Euler equation/Keynes-Ramsey condition:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} \left[\left(1 - \frac{1}{\omega}\right) h_{t+1}^\omega + 1 \right] \\ c_t &= \beta^\sigma c_{t+1} \left[\left(1 - \frac{1}{\omega}\right) h_{t+1}^\omega + 1 \right]^\sigma, \end{aligned}$$

but note that the 1 inside the brackets is going to be a pain to deal with.

Use the "Taylor approximation method":[4]

$$\begin{aligned} c_t = {} & \underbrace{\beta^\sigma \bar{c} \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma}_{f(\bar{c},\bar{h})} + \underbrace{\beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma}_{f_c(\bar{c},\bar{h})} (c_{t+1} - \bar{c}) \\ & + \underbrace{\sigma \beta^\sigma \bar{c} \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^{\sigma-1} \omega \left(1 - \frac{1}{\omega}\right) \bar{h}^{\omega-1}}_{f_h(\bar{c},\bar{h})} (h_{t+1} - \bar{h}), \end{aligned}$$

[4] It turns out that, according to the answer key, one could assume that $(1 - \frac{1}{\omega})\bar{h}^\omega + 1 = \beta^{-1}$. This may be useful for an easier log-linearisation.

set $\bar{c} = f(\bar{c}, \bar{h})$, to write

$$\begin{aligned} c_t - \bar{c} &= \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma (c_{t+1} - \bar{c}) \\ &+ \sigma \beta^\sigma \bar{c} \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^{\sigma - 1} \omega \left(1 - \frac{1}{\omega}\right) \bar{h}^{\omega - 1} (h_{t+1} - \bar{h}), \end{aligned}$$

and then divide through by $\bar{c}$ (and do a bit of manipulation with $\bar{h}$):

$$\begin{aligned} \frac{c_t - \bar{c}}{\bar{c}} &= \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma \frac{c_{t+1} - \bar{c}}{\bar{c}} \\ &+ \sigma \beta^\sigma \bar{c} \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^{\sigma - 1} \omega \left(1 - \frac{1}{\omega}\right) h^{\omega - 1} \frac{h_{t+1} - \bar{h}}{\bar{c}} \frac{\bar{h}}{\bar{h}}, \\ \hat{c}_t &\approx \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma \hat{c}_{t+1} + \sigma \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^{\sigma - 1} \omega \left(1 - \frac{1}{\omega}\right) \bar{h}^\omega \hat{h}_{t+1}. \end{aligned} \tag{23}$$

The third FOC should be a lot easier to deal with:

$$\begin{aligned} \alpha h_t^{\alpha - 1} k_t^\alpha &= k_t h_t^{\omega - 1} \\ \Leftrightarrow k_t^{1 - \alpha} &= \alpha h_t^{\alpha - \omega}, \end{aligned}$$

then take logs:

$$(1 - \alpha) \ln k_t = \ln \alpha + (\alpha - \omega) \ln h_t,$$

and use the fact that $\ln X_t = \ln \bar{X} + \frac{X_t - \bar{X}}{\bar{X}}$:

$$(1 - \alpha) \left[\ln \bar{k} + \frac{k_t - \bar{k}}{\bar{k}} \right] = (\alpha - \omega) \left[\ln \bar{h} + \frac{h_t - \bar{h}}{\bar{h}} \right].$$

Assume that in the steady state we have $(1 - \alpha) \ln \bar{k} = (\alpha - \omega) \ln \bar{h}$, to then get the log linearisation,

$$\hat{k}_t = \frac{\alpha - \omega}{1 - \alpha} \hat{h}_t. \tag{24}$$

3.5

Suppose the social planner inherits a sufficiently large capital stock so the optimal allocation requires that the capital stock next period will be above its long run value, i.e. $\hat{k}_{t+1} > 0$. Will consumption rise or fall between period t and period $t + 1$? What is the intuition for this feature of the optimal allocation? Is capital used more or less intensively?

Combine (23) and (24) to see that

$$\hat{c}_t \approx \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^\sigma \hat{c}_{t+1} + \underbrace{\sigma \beta^\sigma \left[\left(1 - \frac{1}{\omega}\right) \bar{h}^\omega + 1 \right]^{\sigma - 1} \omega \left(1 - \frac{1}{\omega}\right) \bar{h}^\omega \frac{1 - \alpha}{\alpha - \omega}}_{<0} \hat{k}_{t+1},$$

because $\alpha < \omega$. Thus, if $\hat{k}_{t+1} > 0$, then it stands to reason that consumption would fall between periods t and $t + 1$ – i.e., consumption, above trend, will decline. This makes intuitive sense. Too much capital implies that the optimal allocation of consumption would be to consume away upon the inheritance of excess capital, and then slowly decay afterward.

From the intratemporal Euler equation, there is an inverse relationship between the level of capital and its intensity. Thus, as capital declines, the intensive use of capital rises.

4 Public and private capital[5]

Consider an economy in which consumption goods can be produced using the following production technology:

$$y_t = K_t^\alpha,$$

where y_t is output, K_t is capital input and $\alpha \in (0, 1)$. Capital input is a combination of private capital k_t and public capital κ, which is provided free of charge by the government:

$$K_t = k_t^\gamma \kappa_t^\theta, \ \gamma, \theta > 0.$$

A social planner maximises the expected present discounted utility of a representative consumer

$$\max_{\{c_t, k_{t+1}\}} \ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

by choosing consumption c_t and future private capital k_{t+1}. The social planner does not choose public capital, which is instead set exogenously by the political process. The utility function satisfies $u' > 0, u'' < 0$. There is unit marginal rate of transformation between output, consumption goods, private capital and public capital so the law of motion for private capital stock is

$$c_t + k_{t+1} + \kappa_{t+1} = y_t + k_t + \kappa_t$$

with neither private nor public capital subject to depreciation.

4.1

Derive and interpret the first order conditions of the social planner's problem. Assume that public capital is constant in all periods so $\kappa_t = \kappa, \ \forall t > 0$.

Rewrite the "law of motion" (it's more a budget constraint) using the definitions that we are given:

$$c_t + k_{t+1} + \kappa = \left(\kappa^\theta k_t^\gamma\right)^\alpha + k_t + \kappa,$$

which allows us to write the Lagrangian as

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[u(c_{t+s}) + \lambda_{t+s}\left(c_{t+s} + k_{t+1+s} - \kappa^\theta k_{t+s}^\gamma - k_{t+s}\right)\right].$$

Thus, the FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) + \lambda_t = 0, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_t - \beta \mathbb{E}_t \left[\lambda_{t+1}\left(\gamma\alpha\kappa^{\theta\alpha} k_{t+1}^{\gamma\alpha-1} + 1\right)\right] = 0. \tag{26}$$

[5] Another past exam question.

4.2

Solve for the steady state level of private capital.

Combining the FOCs, we get

$$
\begin{aligned}
u'(c_t) &= -\beta \mathbb{E}_t \left[\lambda_{t+1} \left(\gamma\alpha\kappa^{\theta\alpha} k_{t+1}^{\gamma\alpha-1} + 1 \right) \right] \\
u'(c_t) &= \beta \mathbb{E}_t \left[u'(c_{t+1}) \left(\gamma\alpha\kappa^{\theta\alpha} k_{t+1}^{\gamma\alpha-1} + 1 \right) \right],
\end{aligned}
$$

and in the steady state we have $c_t = c_{t+1} = \bar{c}$ and $k_{t+1} = \bar{k}$:

$$
\begin{aligned}
1 &= \beta\gamma\alpha\kappa^{\theta\alpha} \bar{k}^{\gamma\alpha-1} + \beta \\
\frac{1-\beta}{\beta\gamma\alpha\kappa^{\theta\alpha}} &= \bar{k}^{\gamma\alpha-1} \\
\bar{k} &= \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{1}{\gamma\alpha-1}}. \qquad (27)
\end{aligned}
$$

4.3

Show that the steady state level of private capital is increasing in κ if and only if $\alpha\gamma < 1$. Interpret this condition.

From the above, we have

$$
\begin{aligned}
\frac{d\bar{k}}{d\kappa} &= \frac{1}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{1}{\gamma\alpha-1}-1} \left(\theta\alpha \frac{1-\beta^{-1}}{\gamma\alpha} \kappa^{-\theta\alpha-1} \right) \\
&= \frac{1}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{2-\gamma\alpha}{\gamma\alpha-1}} \left(\theta\alpha \frac{1-\beta^{-1}}{\gamma\alpha\kappa^{1+\theta\alpha}} \right),
\end{aligned}
$$

or, with a bit of cleaning up:

$$
\begin{aligned}
&= \frac{1}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{2-\gamma\alpha}{\gamma\alpha-1}} \left(\frac{1-\beta^{-1}}{\gamma\alpha\kappa^{\theta\alpha}} \right) \frac{\theta\alpha}{\kappa} \\
&= -\frac{1}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{2-\gamma\alpha}{\gamma\alpha-1}} \left(\frac{1-\beta^{-1}}{\gamma\alpha\kappa^{\theta\alpha}} \right) \frac{\theta\alpha}{\kappa} \\
&= -\frac{\theta\alpha}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{2-\gamma\alpha}{\gamma\alpha-1}+1} \frac{1}{\kappa} \\
&= -\frac{\theta\alpha}{\gamma\alpha-1} \left(\frac{\beta^{-1}-1}{\gamma\alpha\kappa^{\theta\alpha}} \right)^{\frac{1}{\gamma\alpha-1}} \frac{1}{\kappa} \\
&= \frac{\theta\alpha}{1-\gamma\alpha} \frac{\bar{k}}{\kappa}, \qquad (28)
\end{aligned}
$$

where it's evident that $\bar{k}$ is increasing in κ if and only $\alpha\gamma < 1$ (and if $\beta < 1$).

We could also use total differentiation. After combining the FOCs and evaluating at the steady state, we have

$$\begin{aligned}
0 &= \beta\gamma\kappa^{\theta\alpha}\bar{k}^{\gamma\alpha-1} + \beta - 1 \\
&= (\gamma\alpha - 1)\beta\gamma\kappa^{\theta\alpha}\bar{k}^{\gamma\alpha-2}d\bar{k} + \theta\alpha\beta\gamma\kappa^{\theta\alpha-1}\bar{k}^{\gamma\alpha-1}d\kappa \\
(1-\gamma\alpha)\beta\gamma\kappa^{\theta\alpha}\bar{k}^{\gamma\alpha-2}\frac{d\bar{k}}{d\kappa} &= \theta\alpha\beta\gamma\kappa^{\theta\alpha-1}\bar{k}^{\gamma\alpha-1} \\
\frac{d\bar{k}}{d\kappa} &= \frac{\theta\alpha\beta\gamma\kappa^{\theta\alpha-1}\bar{k}^{\gamma\alpha-1}}{(1-\gamma\alpha)\beta\gamma\kappa^{\theta\alpha}\bar{k}^{\gamma\alpha-2}} \\
&= \frac{\theta\alpha}{(1-\gamma\alpha)}\frac{\bar{k}}{\kappa}.
\end{aligned}$$

The interpretation for this is that $\bar{k}$ increases with κ so long as there is diminishing returns to k_t.

4.4

Under what condition(s) is steady state output increasing in κ?

Steady state output is

$$\bar{y} = \bar{k}^{\gamma\alpha}\kappa^{\theta\alpha}.$$

So, differentiate this with respect to κ (noting that $\bar{k}$ changes with respect to changes in κ)

$$\frac{d\bar{y}}{d\kappa} = \gamma\alpha\bar{k}^{\gamma\alpha-1}\kappa^{\theta\alpha}\frac{d\bar{k}}{d\kappa} + \theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1},$$

and substitute in our result from (28):

$$\begin{aligned}
\frac{d\bar{y}}{d\kappa} &= \gamma\alpha\bar{k}^{\gamma\alpha-1}\kappa^{\theta\alpha}\frac{\theta\alpha}{(1-\gamma\alpha)}\frac{\bar{k}}{\kappa} + \theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1} \\
&= \gamma\alpha\frac{\theta\alpha}{(1-\gamma\alpha)}\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1} + \theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1} \\
&= \theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1}\left(\frac{\gamma\alpha}{1-\gamma\alpha} + 1\right) \\
&= \theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1}\left(\frac{1}{1-\gamma\alpha}\right). \qquad (29)
\end{aligned}$$

So, the condition for $\bar{y}$ to be increasing in κ is that $\gamma\alpha < 1$, since $\theta\alpha\bar{k}^{\gamma\alpha}\kappa^{\theta\alpha-1} > 0$.

4.5

Assume that $\alpha\gamma < 1$ and the economy is in steady state for $\kappa_t = \kappa$ for $t = 0$ when politicians unexpectedly announce a policy to increase the future level of public capital. The increase is permanent that such that $\kappa_t = \kappa' > \kappa$, $\forall t > 1$. Answer the following:

4.5.1

Describe the impact of the new policy on consumption, investment, private capital and output. Pay particular attention to the difference between short-run and long-run effects.

Let's first discuss the long term effects. We see that so long as $\gamma\alpha < 1$, then $\bar{y}$ and $\bar{k}$ increases. This in turn will lead to consumption and investment rising in the long run too.

In the short run, output is fixed because the level of private capital is predetermined and the new investment has not yet increased the level of public capital. From the resource constraint, it follows that $c_t + k_{t+1}$ must decrease when κ_{t+1} increases. With the marginal value of private capital increasing due to the extra public capital, it must be that the presented discounted value of future utilities rises, which via an Envelope Theorem argument implies that $u'(c_t)$ rises and consumption c_t initially falls. The effect on investment $k_{t+1} - k_t$ is ambiguous, depending on the strength of wealth effects.

4.5.2

Recently, a number of governments have increased public investment in an attempt to stimulate economic activity. Given your answer above, explain how the short-run effects of increased investment in public capital depends on how soon the investment becomes productive.

The answer above states that output may decline in $t + 1$ if the fall in private capital is sufficient to offset the benefits of higher public capital. if it took even longer for the public investments to become productive, then the situation could be even worse, with a protracted period in which output is below the level it would otherwise have been. If public investments become productive quickly, then this is much less problematic.

5 A representative household model

A representative household consists of a continuum of ex ante identical workers indexed by $j \in [0, 1]$. We begin by analysing a static model in which the consumption of worker of type j is denoted c^j and the hours worked by a worker of type j is denoted n^j. A worker can choose between working a fixed number of hours 1 and not working at all, so all adjustment is at the extensive margin. If a worker chooses to work then they receive wage W and incur disutility of working $v(1) > 0$. If a worker chooses not to work then they receive no wage and incur no disutility. Assume that the wage is above the reservation wage so the problem is not degenerate. The problem of the representative household is to choose consumption c^j and n^j subject to their budget constraint. With preferences over consumption assumed to be logarithmic, the problem of the representative household can be written as:

$$\max_{c^j, n^j} \int_0^1 \left[\log c^j - v(1)n^j\right] dj,$$

subject to

$$\int_0^1 c^j \, dj \leq W \int_0^1 n^j \, dj.$$

5.1

Derive the first order conditions and solve for c^j. Discuss the intuition for how c^j varies as a function of W and $v(1)$.

This is a maximisation problem, so set up the Lagrangian by bringing everything to the RHS of the $\leq$ inequality constraint:

$$\max_{c^j, n^j} \left\{ \int_0^1 \left[\log c^j - v(1)n^j\right] dj + \lambda \left[W \int_0^1 n^j \, dj - \int_0^1 c^j \, dj \right] \right\}.$$

The FOCs are -

$$\frac{1}{c^j} - \lambda = 0,$$
$$-v(1) + \lambda W = 0.$$

Combine these to get

$$c^j = \frac{W}{v(1)}.$$

Thus as the wage, W, rises, consumption will rise. Likewise, as the disutility of working shrinks, consumption will rise. Also note that each worker's consumption is identical.

5.2

Your solution to the representative household's problem will require some proportion $\phi = \int_0^1 n^j dj$ of workers to work. Rewrite the budget constraint in terms of ϕ and use your answer to the previous part to solve for ϕ. How does ϕ depend on W and $v(1)$? Why?

Rewrite the Lagrangian using the definition of ϕ, and by noting that consumption is identical amongst all households:

$$\max_{c, \phi} \left\{ \log c - v(1)\phi + \lambda \left[W\phi - c \right] \right\},$$

and for the sake of convenience, assume that by Walras' Law and non-satiation, $W\phi = c$ holds with equality. The FOCs are

$$\begin{aligned}\frac{1}{c} - \lambda &= 0, \\ -v(1) + \lambda W &= 0,\end{aligned}$$

which we can rearrange to get

$$\begin{aligned}v(1) &= \frac{W}{c} \\ v(1)c &= W \\ v(1)W\phi &= W \\ \phi &= \frac{1}{v(1)}.\end{aligned}$$

The FOCs state that the proportion of workers is simply the inverse of the disutility of labour. ϕ is independent of the wage, W, because of the separable log-linear specification of the utility function, which means W only has a wealth and not a substitution effect.

5.3

Discuss the similarities and differences between your results and the Hansen-Rogerson lotteries model of aggregate labour supply. Do you feel more or less positive towards lotteries after seeing the representative household model?

I'm just going to quote Martin Ellison's notes on this, as his answer is perfectly succinct: The representative household model discussed is equivalent to the Hansen-Rogerson lotteries model. Household members have equal consumption and the household as a whole is indifferent between which members work. The equal consumption feature mimics the perfect insurance in the lotteries model – here households are self-insuring their members. Household members being indifferent about who works is equivalent to who works being decided by lottery. This should then make you feel more positive about the lotteries model. In effect, we rationalise the lotteries by saying there is a representative household instead of a representative consumer. The representative household provides the insurance and still does not overcome the indeterminacy of who works, in which case things might as well be decided by a lottery.

5.4

Now analyse a dynamic version of the representative household model in which time is continuous but finite, i.e. $t \in [0, 1]$. The household is assumed to discount utility at the market rate of interest r. the dynamic problem of the representative household is:

$$\max_{\{c^j, n^j\}} \int_0^1 \int_0^1 \exp(-rt) \left[\log c_t^j - v(1) n_t^j\right] dj dt,$$

subject to

$$\int_0^1 \exp(-rt) \left[\int_0^1 c_t^j dj\right] dt \leq \int_0^1 \exp(-rt) \left[W \int_0^1 n_t^j dj\right] dt.$$

Solve the dynamic problem for c_t^j *and the proportion of workers working in any given period,* $\phi_t = \int_0^1 n_t^j dj$. *Interpret your results and state whether you remain more or less positive about the lotteries model.*

We can proceed with setting up our Lagrangian as before – there is no need to make a Hamiltonian here:

$$\max_{\{c^j, n^j\}} \left\{ \int_0^1 \int_0^1 \exp(-rt) \left[\log c_t^j - v(1) n_t^j \right] dj dt + \lambda \left(\int_0^1 \exp(-rt) \left[W \int_0^1 n_t^j dj \right] dt - \int_0^1 \exp(-rt) \left[\int_0^1 c_t^j dj \right] dt \right) \right\},$$

and then differentiate with respect to c^j and n^j to get our FOCs:

$$\int_0^1 \exp(-rt) \frac{1}{c_t^j} dt - \lambda \int_0^1 \exp(-rt) dt = 0$$

$$\frac{1}{c_t^j} = \lambda,$$

$$-\int_0^1 \exp(-rt) v(1) dt + \lambda W \int_0^1 \exp(-rt) = 0$$

$$v(1) = \lambda W,$$

which is exactly what we had in 5.2. λ_t is independent of t, so c_t^j is independent of both t and j.

If $\phi_t = \int_0^1 n_t^j dj$, then our problem is (after noting that c_t^j is independent on both t and j):

$$\max_{\{c^j, n^j\}} \left\{ \int_0^1 \exp(-rt) \left[\log c - v(1) \phi_t \right] dt + \lambda \left(W \int_0^1 \exp(-rt) \phi_t dt - c \int_0^1 \exp(-rt) dt \right) \right\},$$

and the FOC wrt ϕ_t is

$$\phi_t = \frac{1}{v(1)},$$

which is what we had previously.

We therefore see that the dynamic model gives the same allocation as the static model in the first part of this question. The equivalence between a representative household model and the lottery is still valid in a dynamic setting.

5.5

Now introduce human capital by assuming that the productivity of a worker of type j *is increasing in the experience of a worker of that type, so that human capital can be defined as:*

$$h_t^j = \int_0^1 n_s^j ds.$$

Human capital is rewarded such that if it is below the threshold level $\tilde{h}$ *then the worker is classified as unskilled and receives wage* $W = 1$. *If human capital is above the threshold level* $\tilde{h}$ *then the worker is classified as skilled and receives a wage* $W = H > 1$. *Without doing any further calculations, explain how would the representative household would decide who works in this case. Do any similarities with the Hansen-Rogerson lottery model remain? Can you design a lottery that leads to the same aggregate outcome as the representative household problem?*

I'll quote Martin Ellison again:

If workers can accumulate human capital and become skilled, then it makes sense for the members of the representative household to specialise into being workers and non-workers. Once a household member is skilled (or on their way to being skilled) then they have higher productivity (or higher expected future productivity) so they should work rather than a household member who is not skilled. This means that the standard Hansen-Rogerson lottery is no longer equivalent to the problem of the representative household.

Indeed, the problem of who works in the representative household is determinate once some workers have accumulated experience. It would not be optimal for the representative household to randomly choose members to work each period.

However, an indeterminacy still remains in the representative household problem because there is no mechanism to decide which members become workers and which not at period 0 when the household is "born" and no members have any work experience. A once-and-for-all lottery could work here whereby the household member is assigned randomly to be a worker or not at time 0. Optimality requires that $\phi_t = 1/v(1)$ are workers but does not determine which members become workers.

5.6

A worker is said to be frontloading if they work in the first period of their lifetime then take vacation. They are backloading if they take a vacation first and then work. In the human capital model will there be an incentive to frontload or backload work? How does this compare to the model without human capital?

Workers want to backload their work. There is a benefit to doing this in the model with human capital because it enables workers to delay (and hence discount more) the disutility they receive from working. To understand why this is a solution, consider the disutility of work associated with backload:

$$\bar{v} = v(1) \int_s^1 \exp(-rt)dt,$$

where s is the date they start working. Starting from this allocation, consider a perturbation in which the household supplies some work earlier and takes some leisure later, but keeps the disutility of work fixed at $\bar{v}$. Because of discounting, such a shift allows the household to work less total time over the integral $[0, 1]$ (i.e., $\int_0^1 n_t dt$ would be smaller), but involves a smaller proportion of its time as a high skilled worker. That would lower the present value of income associated with a given disutility work and so is suboptimal.

6 Macroeconomic co-movement

This question is about the relationship between aggregate consumption and aggregate hours worked. Recall from the lecture notes that the intertemporal optimisation predicts a positive relationship between aggregate consumption and leisure (assuming both consumption and leisure are normal goods). This implies a negative relationship between aggregate consumption and hours worked, which is not what we observed in the data. To rectify this, consider a model of a representative household with a continuum of ex ante identical workers. Workers can either work for wage W *with disutility of work* $v(1)$ *or not work and have no wage and disutility* $v(0) < v(1)$. *We assume* $\sigma > 1$ *so consumption and leisure are substitutes. The problem of the representative household is to decide the proportion* e_t *of workers who work, and the consumption levels* C_t^e *and* C_t^u *of working and non-working members respectively. Preferences are non-separable between consumption and the disutility of working so the problem of the representative household is*:

$$\max_{C_t^e, C_t^u, e_t} \left[e_t \frac{(C_t^e)^{1-\sigma} v(1)}{1-\sigma} + (1-e_t) \frac{(C_t^u)^{1-\sigma} v(0}{1-\sigma} \right],$$

subject to

$$e_t C_t^e + (1-e_t) C_t^u = R_t K + W_t e_t.$$

The budget constraint states that consumption must be equal to capital income (assuming K *is owned by the household but fixed) plus labour income.*

6.1

Show that the first order conditions of the representative household problem are:

$$C_t^e - C_t^u = \left(\frac{\sigma - 1}{\sigma} \right) W_t, \tag{30}$$

$$\frac{C_t^e}{C_t^u} = \left(\frac{v(1)}{v(0)} \right)^{\frac{1}{\sigma}}. \tag{31}$$

The FOCs from our problem are

$$\frac{\partial \mathcal{L}}{\partial e_t} = \frac{(C_t^e)^{1-\sigma} v(1)}{1-\sigma} - \frac{(C_t^u)^{1-\sigma} v(0)}{1-\sigma} + \lambda w_t - \lambda C_t^e - \lambda C_t^u = 0 \tag{32}$$

$$\frac{\partial \mathcal{L}}{\partial C_t^e} = e_t (C_t^e)^{-\sigma} v(1) - \lambda e_t = 0 \tag{33}$$

$$\frac{\partial \mathcal{L}}{\partial C_t^u} = (1-e_t)(C_t^u)^{-\sigma} v(0) - (1-e_t)\lambda = 0, \tag{34}$$

Substituting and rearranging from (33) and (34):

$$(C_t^e)^{-\sigma} v(1) = \lambda,$$
$$(C_t^u)^{-\sigma} v(0) = \lambda,$$

which implies

$$
\begin{aligned}
(C_t^e)^{-\sigma} v(1) &= (C_t^u)^{-\sigma} v(0) \\
\left[\frac{C_t^e}{C_t^u}\right]^{-\sigma} &= \frac{v(0)}{v(1)} \\
\frac{C_t^e}{C_t^u} &= \left[\frac{v(0)}{v(1)}\right]^{-\frac{1}{\sigma}} \\
\frac{C_t^e}{C_t^u} &= \left[\frac{v(1)}{v(0)}\right]^{\frac{1}{\sigma}}.
\end{aligned}
$$

Then from (32)

$$
\frac{1}{1-\sigma}\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] = \lambda C_t^e - \lambda C_t^u - \lambda w_t,
$$

we then substitute values for λ from (33) and (34)

$$
\frac{1}{1-\sigma}\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] = (C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0) - \lambda w_t
$$

$$
\begin{aligned}
\frac{1}{(1-\sigma)}\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] - \left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= -\lambda w_t \\
\left[\frac{1}{1-\sigma} - 1\right]\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= -\lambda w_t \\
\left[\frac{1}{1-\sigma} - \frac{1-\sigma}{1-\sigma}\right]\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= -\lambda w_t \\
\left[\frac{\sigma}{1-\sigma}\right]\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= -\lambda w_t \\
\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= \left[\frac{1-\sigma}{\sigma}\right](-\lambda w_t) \\
\left[(C_t^e)^{1-\sigma} v(1) - (C_t^u)^{1-\sigma} v(0)\right] &= \left[\frac{\sigma-1}{\sigma}\right]\lambda w_t \\
\left[\frac{C_t^e}{(C_t^e)^{\sigma}} v(1) - \frac{C_t^u}{(C_t^u)^{\sigma}} v(0)\right] &= \left[\frac{\sigma-1}{\sigma}\right]\lambda w_t \qquad (35)
\end{aligned}
$$

and then substitute

$$
\begin{aligned}
\left[\frac{C_t^e}{C_t^u}\right]^{-\sigma} &= \frac{v(0)}{v(1)} \\
\Leftrightarrow (C_t^e)^{-\sigma} &= (C_t^u)^{-\sigma}\frac{v(0)}{v(1)},
\end{aligned}
$$

into (35):

$$\begin{aligned}\left[\frac{C_t^e}{(C_t^u)^\sigma}v(1)\frac{v(0)}{v(1)} - \frac{C_t^u}{(C_t^u)^\sigma}v(0)\right] &= \left[\frac{\sigma-1}{\sigma}\right]\lambda w_t \\ \left[\frac{C_t^e}{(C_t^u)^\sigma}v(0) - \frac{C_t^u}{(C_t^u)^\sigma}v(0)\right] &= \left[\frac{\sigma-1}{\sigma}\right]\lambda w_t \\ (C_t^u)^{-\sigma}v(0)\left[C_t^e - C_t^u\right] &= \left[\frac{\sigma-1}{\sigma}\right]\lambda w_t\end{aligned}$$

then substitute in the value for λ from (34) to yield

$$\left[C_t^e - C_t^u\right] = \left[\frac{\sigma-1}{\sigma}\right] w_t.$$

6.2

Define aggregate consumption and aggregate hours worked as:

$$C_t = e_t C_t^e + (1-e_t)C_t^u, \tag{36}$$

$$N_t = e_t. \tag{37}$$

Assumed the existence of a steady state $(\bar{C}^u, \bar{C}^e, \bar{e}, \bar{W})$. Log-linearise equations (30), (31), (36), and (37) to show that there is a relationship between deviations in aggregate consumption, wage, and hours worked from steady state:

$$\tilde{C}_t = \tilde{W}_t + \frac{1-\omega}{1+(\bar{e}^{-1}-1)\omega}\tilde{N}_t, \tag{38}$$

where $\omega = \bar{C}^u/\bar{C}^e$.

Linearising the four equations gives:

$$\begin{aligned}\tilde{W}_t &= \frac{\bar{C}^e}{\bar{C}^e - \bar{C}^u}\tilde{C}_t^e - \frac{\bar{C}^u}{\bar{C}^e - \bar{C}^u}\tilde{C}_t^u \\ \tilde{C}_t^e &= \tilde{C}_t^u \\ \tilde{C}_t &= \frac{\bar{e}\bar{C}^e}{\bar{C}}(\tilde{e}_t + \tilde{C}_t^e) - \frac{\bar{C}^u}{\bar{C}}\tilde{C}_t^u - \frac{\bar{e}\bar{C}^u}{\bar{C}}(\tilde{e}_t + \tilde{C}_t^u) \\ \tilde{N}_t &= \tilde{e}_t.\end{aligned}$$

6.3

We now add production to the economy by defining a firm's production function $Y_t = K^\alpha N_t^{1-\alpha}$ and assume perfect factor markets so that labour is paid at its marginal product and $W_t = (1-\alpha)\frac{Y_t}{N_t}$.[6] *Assuming the existence of a steady state in $\bar{Y}$ and $\bar{N}$, log-linearise these two equations to obtain a relationship between wage $\tilde{W}_t$ and hours worked $\tilde{N}_t$. Substitute this in (38). Interpret the resulting equation in detail. What are the effects at work in the co-movement of aggregate consumption and aggregate hours worked?*

[6] *Note that we have no described investment and the evolution of capital so there is no reason to believe that $Y_t = C_t$ in equilibrium.*

First, note that K is fixed, so log linearising the two equations gives:

$$\tilde{Y}_t = (1-\alpha)\tilde{N}_t \tag{39}$$

$$\tilde{W}_t = \tilde{Y}_t - \tilde{N}_t. \tag{40}$$

Substituting (39) into (40) yields

$$\tilde{W}_t = -\alpha \tilde{N}_t.$$

Substituting this into the equation for log deviations of consumption from steady state gives:

$$\tilde{C}_t = \left[\frac{1-\omega}{1+(\bar{e}^{-1}-1)\omega} - \alpha \right] \tilde{N}_t.$$

Evidently, increases to capital lead to increases in consumption, and increases in working hours lead to increases in consumption too. This is because consumption and leisure are substitutes linked by $\sigma > 1$, which implies that as leisure increases, consumption decreases, and that has leisure decreases (work increasing), consumption increases.

7 To let shirk or not to let shirk

This question asks about a simplified version of the Shapiro-Stiglitz shirking model covered in the lecture notes. On the worker side, the model is identical except we assume there are no unemployment benefits.

7.1

Simplify the derivation in the lecture notes to obtain the no-shirking conditions without unemployment benefits:

$$w \geq e + \frac{e}{q}(a + b + r). \tag{41}$$

Just to recap the simple Shapiro-Stiglitz model presented in class: There are N workers with utility depending on wages, w, and effort, e. For simplicity, we assume that the workers are risk-neutral:

$$U(w, e) = w - e,$$

and effort takes two values

$$e = \begin{cases} e & \text{no shirking,} \\ 0 & \text{shirking.} \end{cases}$$

A shirking workers has zero productivity, so the firm must ensure no shirking. Firm cannot constantly monitor individual worker's effort, so an inspection randomly occurs per unit time with a probability of q. If a worker is caught shirking, they are fired. We also assume that b represents the probability that a worker will randomly leave the firm per period too.

Let r be the worker's discount rate and let their discounted utility stream for being unemployed be V_U. Thus, we have the following utility streams: V_E^S for shirking if currently employed; V_E^N for not shirking if currently employed. Thus, if we treat the streams as an asset, we have

$$\begin{aligned} rV_E^S &= \underbrace{w}_{\text{payoff}} + \underbrace{(b+q)}_{\text{probability of capital loss}} \underbrace{(V_U - V_E^S)}_{\text{capital loss if realised}}, \\ rV_E^N &= \underbrace{w - e}_{\text{payoff}} + \underbrace{b}_{\text{probability of capital loss}} \underbrace{(V_U - V_E^N)}_{\text{capital loss if realised}}. \end{aligned}$$

We can rearrange these expression in order to get our no-shirking condition (NSC):

$$\begin{aligned} rV_E^S + (b+q)V_E^S &= w + (b+q)V_U \\ V_E^S &= \frac{w + (b+q)V_U}{r + b + q}, \end{aligned}$$

and

$$\begin{aligned} rV_E^N + bV_E^N &= w - e + bV_U \\ V_E^N &= \frac{w - e + bV_U}{r + b}. \end{aligned}$$

Then the firm must ensure that the utility stream of not shirking must be at least as good as the utility stream from shirking:

$$\begin{aligned} V_E^N &\geq V_E^S \\ \Leftrightarrow \frac{w - e + bV_U}{r + b} &\geq \frac{w + (b+q)V_U}{r + b + q}, \end{aligned}$$

and with a bit cleaning up:

$$
\begin{aligned}
w - e + bV_U - \frac{(r+b)}{r+b+q} w &\geq \frac{(r+b)(b+q)}{r+b+q} V_U \\
w\left(1 - \frac{r+b}{r+b+q}\right) &\geq \frac{(r+b)(b+q)}{r+b+q} V_U + e - bV_U \\
w\left(\frac{r+b+q-r-b}{r+b+q}\right) &\geq e + V_U \left[\frac{(r+b)(b+q)}{r+b+q} - \frac{b(r+b+q)}{r+b+q}\right] \\
w\left(\frac{q}{r+b+q}\right) &\geq e + V_U \left[\frac{(r+b)(b+q) - b(r+b+q)}{r+b+q}\right] \\
w &\geq \frac{r+b+q}{q} e + rV_U.
\end{aligned}
$$

We also have

$$rV_U = y + a(V_E^N - V_U),$$

and if substitute this into our NSC (while assuming that unemployment benefits, $y = 0$), we get

$$w \geq \frac{r+b+q}{q} e + a(V_E^N - V_U),$$

or (after a bunch of tedious algebra):

$$w \geq e + \frac{e}{q}(a+b+r).$$

7.2

We are interested in whether it makes sense for a firm to monitor its workers and pay wages sufficiently high to prevent shirking. The problem was trivial in the lecture notes because shirking workers were assumed to be completely unproductive. To make the problem interest we therefore assume that even workers who shirk have some productivity. In particular, we assume that shirking workers have productivity 1 and non-shirking workers have productivity $x > 1$, so the firm receives revenue 1 if its worker shirks and revenue x if its worker does not shirk. Monitoring is assumed to have unit cost, so the expected cost of monitoring with probability q is simply q. The expected profit of a firm monitoring with probability q and paying wage w is therefore:

$$\pi = R - w - q, \tag{42}$$

where $R = x$ if the no-shirking condition (41) is satisfied by q and w, and $R = 1$ if the no-shirking condition (41) is violated. Suppose the firm decides not to bother satisfying the non-shirking condition and simply lives with shirking workers.[7] *What monitoring probability will the firm set if they maximise expected profit? What wage will it pay? What will be its profits?*

If the firm decides not to satisfy the NSC, then they set $q = 0$ and $w = 0$ and the profit is $\pi = 1$ with certainty. The assumption that the worker has to take a job if offered at any wage (even if $w = 0$) is important. If the worker could turn down the job then they would prefer to stay in the pool of unemployed workers and wait for a better paid job.

[7] *Assume that workers cannot turn down a job even if the wage is really bad. They can only decide whether to shirk or not shirk.*

7.3

Suppose the firm now decides to set the monitoring probability and wage to induce its workers not to shirk. Set up an optimisation problem for the firm to determine the optimal monitoring probability q^* and optimal wage w^* to prevent shirking. Discuss how and why these variables depend on the primitives of the model, name $a, e, a, b,$ and r.

Conditional on the firm needing to induce no shirking, the problem of the firm is:

$$\max_{w,q} \ R - w - q,$$

subject to

$$w = e + \frac{e}{q}(a + b + r).$$

The FOC wrt q implies

$$q^* = \sqrt{e(a + b + r)},$$

and the wage satisfies

$$w^* = e + \sqrt{e(a + b + r)}.$$

We see that the monitoring probability and optimal wage are increasing in $e(a + b + r)$. These reflect NSC that requires the firm to create the correct incentives for workers not to shirk. To take just one of these, suppose there is an increase in a the probability of an unemployed worker finding a job. This increases the incentive for a worker to shirk and needs to be offset by the firm. The increased incentives not to shirk are created by increasing the wage paid and increasing the monitoring probability. If e increases, then the wage increases by more. This is because it is the most cost-effective way for the firm to induce no shirking.

7.4

Show that there will be no shirking in the economy iff:

$$x - e - 2\sqrt{e(a + b + r)} > 1.$$

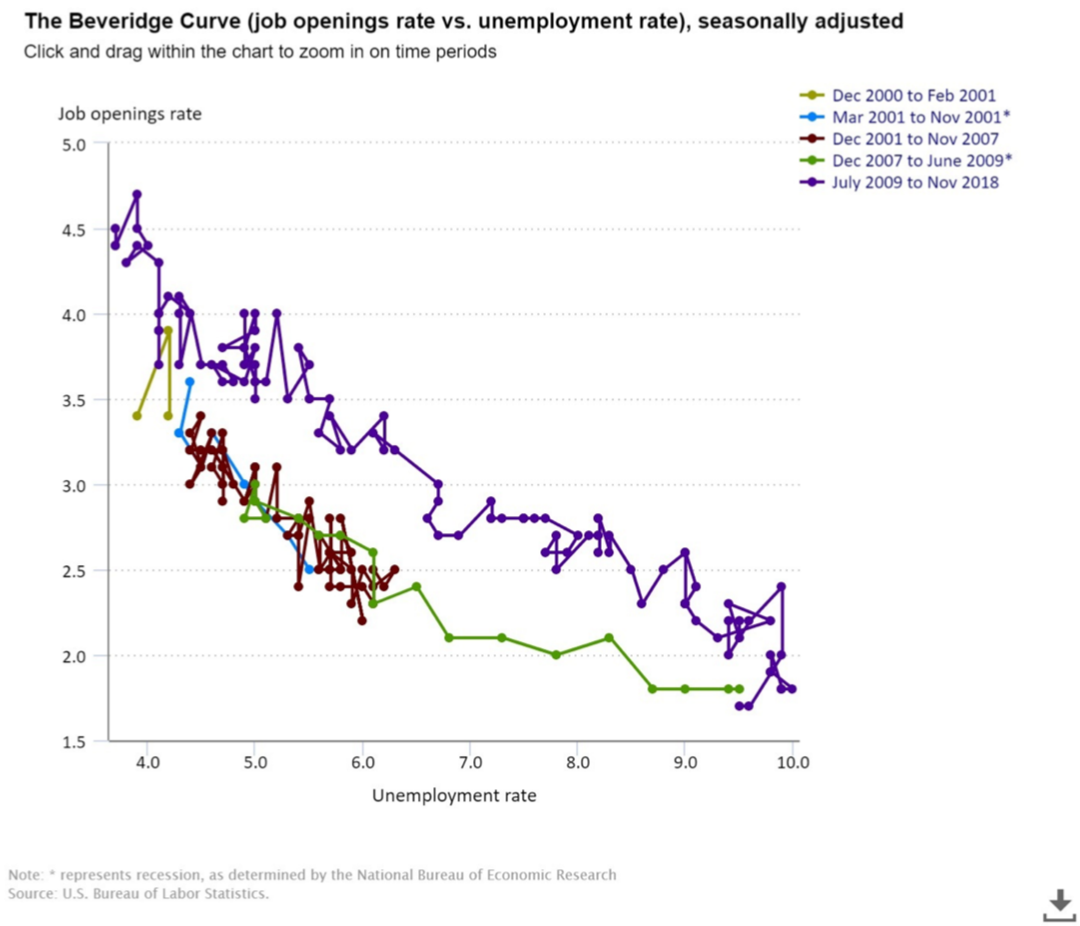
Firms will choose to induce no-shirking if the profit from doing so (i.e., the profit in part 7.3) exceeds the profit from not doing so (i.e., the profit in part 7.2). In other words, firms induce shirking if and only if:

$$R - e - 2\sqrt{e(a + b + r)} > 1.$$

8 The Beveridge curve in recent years

The figure below plots vacancies against unemployment in the US for 2000-2018. Interpret the different pattern observed in terms of the Diamon-Mortensen-Pissaridies search and match model studied in the lectures. Carefully describe any relationships you use.

Figure 4: US Vacancies and Unemployment (2000-2018)



Search and match models have become common as a means of understanding the macroeconomics of the labour market. One reason for this is a number of empirical studies which have examined the behaviour of labour markets over the business cycle. These studies reveal a number of different phenomena such as:

- Even in recessions, large numbers of firms have unfilled vacancies and in booms some firms are laying off workers.
- In every period, there are large gross market flows; movements in job creation and job destruction. The change in unemployment – which most of us see when we read the news – reflects net flows

only (i.e., job destruction less job creation) and so is only part of the overall labour market story.

- Job creation is slightly pro-cyclical but job destruction is strongly counter-cyclical with big spikes in recessions. In other words, big increases in unemployment are caused by occasional large periods of job destruction. The fact that job creation and job destruction have different cyclical properties suggests that labour market allocations are not well coordinated – it takes several periods for the unemployed to find vacancies.

To understand changes in the unemployment rate, we need to consider worker flows. Let's define the entry rate into unemployment as δ_t, where

$$\delta_t = \frac{\text{flow of employed workers becoming unemployed during period } t}{\text{employed at the beginning of period } t}.$$

In other words, δ_t is the average probability of a worker becoming unemployed in period t. Next, let's define the exit rate from unemployment, p_t, as:

$$p_t = \frac{\text{flow of unemployed workers becoming employed during period } t}{\text{unemployed at the beginning of period } t},$$

and so p_t is the average probability of an unemployed person becoming employed in period t. Thus, we have changes in unemployed workers, u_t, defined as

$$\begin{aligned} u_t - u_{t-1} &= \delta_t(1 - u_{t-1}) - p_t u_{t-1} \\ \Leftrightarrow \Delta u_t &= \text{gross entry flow} - \text{gross exit flow.} \end{aligned}$$

Then, we define the matching function as the following:

$$m(v_t, u_t),$$

where v_t is the number of job vacancies, and n_t is the number of employed workers and can be defined as

$$u_t = 1 - (1 - \delta)n_{t-1},$$

where $\delta_t = \delta_{t+1} = \delta$ for all t. It should also be noted that $m(v_t, n_t)$ typically is a constant returns to scale (CRS) technology (see, for example "The Beveridge Curve" by Blanchard et al. (1989)). The assumption of a CRS matching function implies that a single number, the ratio of vacancies to unemployment, summarises the tightness of the labour market. Define $\theta_t = v_t/u_t$ and note that CRS implies:

$$\frac{m(v_t, u_t)}{v_t} = m\left(1, \frac{1}{v_t/u_t}\right) = q\left(\frac{v_t}{u_t}\right) = q(\theta_t),$$

which is the matching rate for vacancies (probability of filling a vacancy), and:

$$\frac{m(v_t, u_t)}{u_t} = m\left(\frac{v_t}{u_t}, 1\right) = p\left(\frac{v_t}{u_t}\right) = p(\theta_t),$$

which is the matching rate for the unemployed (probability of finding a job).

Our assumption that $m(v_t, u_t)$ exhibits CRS and that is increasing in both arguments imply that $m(\theta_t)$ is increasing in θ_t, but that the increase is less than proportional. Thus, when the labour market is tighter (when $\theta_t \uparrow$), the job finding rate is higher and the vacancy filling rate is lower.

Next, we turn to some simple labour force dynamics. We assume that total labour, $\bar{L}$, is given by

$$\bar{L} = n_t + u_t,$$

and therefore the employment and unemployment rates, n_t^r and u_t^r, sum to unity:

$$1 = n_t^r + u_t^r.$$

What about employment in the long run? Each period, a fraction, δ, of workers lose (exogenously) their job, and each period a fraction, $q(\theta_t)$, of vacancies are filled. Let us denote the law of motion of employment as

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t,$$

where the first term on the RHS – jobs that survive separation – is a stock, and the second term – new jobs created – is a flow. Note, also, that we can write the new jobs created term as

$$q(\theta_t)v_t = p(\theta_t)u_t.$$

So we can rewrite the law of motion of employment as

$$\begin{aligned} n_t &= (1-\delta)n_{t-1} + p(\theta_t)u_t \\ \implies n &= (1-\delta)n + p(\theta)u \\ \delta n &= p(\theta)u, \end{aligned}$$

and if we normalise the labour force, we can write this as

$$\begin{aligned} \delta n^r &= p(\theta)u^r \\ \Leftrightarrow \delta(1-u^r) &= p(\theta)u^r \\ u^r &= \frac{\delta}{\delta + p(\theta)}, \end{aligned} \tag{43}$$

which is the long run rate of unemployment, and it is determined by the job separation rate, δ, and the job finding rate, $p(\theta)$. Note that fiscal policy and labour market institutions are critical as they both affect $p(\theta)$ in this model. If the model had endogenous job separation, then δ would also be affected by policy and institutions. Plotting (43) in (u^r, v^r) space gives us what is known as the Beveridge curve, and is empirically shown in Figure 4.

The data seems to suggest that over time the BC has shifted up over time – i.e., for every vacancy rate, the associated unemployment rate has trended upwards. This is consistent with shocks to $p(\theta)$, which have lowered the probability of a worker finding a job.

Figure 5: The Beveridge Curve