PEARL Introductory Macroeconomics: Additional Notes on the Production Function

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1 Introduction

These are some supplementary notes to the lecture notes provided in-class by Professor Fujiwara, and will hopefully answer frequent questions I have received from students. I want to try and emphasis a few key points raised in the lectures: a) why do we need to look at production functions; b) what can production functions tell us about firm decisions, and c) what will be the importance of the production function in your future studies.

To start, let's try and formally define a production function. We can look at something like this:

$$Y = F(K, L)$$

Which very simply says that output, Y, is a function of two inputs: capital, K, and labour, L. This is the most simple concept of a production function. It says inputs of capital and labour **produce** the output, Y. i.e., K and L go into F, and out comes Y. We haven't even talked about how K and L make Y, just that they simply do.

So, why should we care about production functions? Previously in the lecture, we discussed what causes differences in wealth/income amongst different countries. Some students stated factors, such as:

- the quality of institutions such as competitive markets, property rights, political stability, openness to trade, and open financial markets; and
- geographic location and endowment of natural resources.

The above factors are called ultimate factors, and most people agreed that they tend to be important in the determination of a nation's wealth. Other students pointed out that proximate factors, such as natural disasters and war, were also important in wealth determination, and many people also agreed that this seemed reasonable to assume.

In addition, we also discussed that investment or increases in a nation's capital stock – things like factories, roads, airports, assets, computers, trucks, and so on, as well increases in productivity/innovation – could also lead to increases in GDP and thus an increase in wealth and income.

Then, it seems intuitive that ultimate and proximate factors, investment, and productivity are all seemingly related in this GDP puzzle, and that if we want to understand why – and more importantly **how** – some countries are more wealthy than others, we need a framework to process these things. This is where the production function comes in. It will be your first step into the much wider world of economic modelling.

Using the production function, we can understand how increases in the aforementioned endowments lead to increases in output and income. We can answer questions such as:

- If productivity increases by 2%, by how much will GDP increase?
- If we increase labour through immigration, what are the broader economic ramifications?
- If we lose production resources (capital) due to a natural disaster, how much do we expect GDP to fall by?

Of course, there are limits as to what we can analyse with a simple production function. However, in your future studies in macroeconomics, you will learn how to add more functionality to your models, giving you further insights in how economies operate. But, for now, let's stick to a simple model based on a simple production function.

2 The Cobb-Douglas Production Function

By far, the most common type of production function you will encounter in macroeconomics, even as you proceed onto further studies and research, is the Cobb-Douglas Production Function (CDPF). The CDPF is based on the Cobb-Douglas functional form, proposed by economists Charles Cobb and Paul Douglas in the early 20th century. Economists favour the CDPF because it has good analytical properties and is statistically robust. The CDPF that we will focus on is of the following form:

$$Y = AK^{\alpha}L^{1-\alpha}, \qquad 0 < \alpha < 1$$

Where A is our productivity variable, α is the capital share of output, and $1-\alpha$ is the labour share of output. Before we further explore the implications of these variables, let's first go over some assumptions and properties of the CDPF.

Assumption 1: The CDPF features constant returns to scale

This assumption implies that if we, say, double inputs (L and K), then we will double output. We can prove this with some simple algebra:

$$\begin{aligned} F(K,L) &= Y = AK^{\alpha}L^{1-\alpha} \\ 2 * F(K,L) &= F(2K,2L) = A(2K)^{\alpha}(2L)^{1-\alpha} \\ &= 2^{\alpha+1-\alpha}AK^{\alpha}L^{1-\alpha} \\ &= 2AK^{\alpha}L^{1-\alpha} \\ &= 2Y \quad \therefore QED \end{aligned}$$

The mechanism for this is in our assumption that $\alpha + (1-\alpha) = 1$. If we altered this assumption so that the powers of K and L did not sum up to 1, then we would have either increasing returns to scale or decreasing returns to scale. You can test this yourself if you wish.

Assumption 2: The CDPF marginal product of capital or labour exhibits diminishing returns

We have covered marginal products before, so this shouldn't be new material. Recall that the marginal product is the additional output you can generate by an incremental increase in an input factor. Here we have two inputs (K and L), so we can look at two marginal products – one with respect to capital, and one with respect to labour. What we want to show is that if we increase one factor input, yes, output will increase, however it will increase at a slower and slower rate as you keep increasing that input factor.

To show diminishing returns, we begin by attaining equations for our marginal products. Recall that the marginal product of capital (MPK) or labour (MPL) is the first-order partial derivative of output derived with respect to either capital or labour. Thus, the marginal product of capital is:

$$\begin{aligned} MPK = \frac{\partial Y}{\partial K} &= \alpha A K^{\alpha-1} L^{1-\alpha} \\ &= \alpha \frac{AL^{1-\alpha}}{K^{1-\alpha}} \\ &= \alpha A \left(\frac{L}{K}\right)^{1-\alpha} \\ &= \alpha \frac{Y}{K} \end{aligned}$$

and by symmetry, the marginal product of labour is:

$$\begin{aligned} MPL = \frac{\partial Y}{\partial L} &= (1-\alpha) A K^{\alpha} L^{1-\alpha-1} \\ &= (1-\alpha) \frac{AK^{\alpha}}{L^{\alpha}} \\ &= (1-\alpha) A \left(\frac{K}{L}\right)^{\alpha} \\ &= (1-\alpha) \frac{Y}{L} \end{aligned}$$

If the first-order partial derivative of output – the marginal product – defines the rate of change of output with respect to a change in either capital or labour, then the second-order partial derivative of output shows the rate of change of the marginal product with respect to a change in either capital or labour. Thus, to prove that our assumption holds, we take the second-order partial derivative of output/the first-order partial derivative of either MPK and MPL:

$$\begin{aligned} \frac{\partial_2 Y}{\partial K^2} = \frac{\partial MPK}{\partial K} &= (\alpha - 1)\alpha A K^{\alpha-1-1} L^{1-\alpha} \\ \implies \frac{\partial MPK}{\partial K} &< 0 \quad \therefore QED \end{aligned}$$

Similarly for MPL:

$$\begin{aligned} \frac{\partial_2 Y}{\partial L^2} = \frac{\partial MPL}{\partial L} &= -\alpha(1-\alpha) A K^{\alpha} L^{-\alpha-1} \\ \implies \frac{\partial MPL}{\partial L} &< 0 \quad \therefore QED \end{aligned}$$

Recall from your calculus classes that if the second-order derivative of a function is negative, then the function will increase/decrease at a decreasing rate. This completes our proof for Assumption 2.

2.1 The Capital Share in Output

We can do a quick mathematical proof that α is the capital share in output. To do so we first assume that markets are competitive. Under such an assumption, the marginal product of capital should be equal to the rental rate (interest rate, R) on capital.

$$MPK = R$$

We derived this result in class both mathematically and intuitively, so I won't repeat that material here. To keep things simple though, there is an opportunity cost in undertaking investment in capital – both for those that demand and supply capital. On the demand side, instead of investing in say, a factory, you could instead invest in the stock market – the opportunity cost of undertaking the investment in the factory. Likewise, on the supply side, instead of providing funds or lending to entrepreneurs and firms, one could simply take their income and save it in a bank account (to earn interest; this may not be applicable to Japan, however) or take the funds and invest in some bonds or shares. Again, there is an opportunity cost to accumulating capital. Thus we assume that in competitive markets, firms will accumulate capital to the level that the marginal product of capital is equal to the rental rate on capital. Anything less, then the firm would leave untapped profits on the table; anything more, and the firm would be able to increase profits by reallocating funds away from capital accumulation.

Therefore, throughout the economy, the total amount of income paid out to capital is $MPK \times K$: its marginal product multiplied by its total stock. Then, dividing this by output allows us to derive capital's share in output. We can mathematically show this:

$$\begin{aligned}\frac{MPK \times K}{Y} &= \frac{\alpha \frac{Y}{K} K}{Y} \\ &= \frac{\alpha A K^{\alpha-1} L^{1-\alpha} K}{A K^{\alpha} L^{1-\alpha}} \\ &= \frac{\alpha A K^{\alpha-1+1} L^{1-\alpha}}{A K^{\alpha} L^{1-\alpha}} \\ &= \frac{\alpha A K^{\alpha} L^{1-\alpha}}{A K^{\alpha} L^{1-\alpha}} \\ &= \alpha \quad \therefore QED\end{aligned}$$

Which economists have estimated to be approximately equal to $\frac{1}{3}$.

As an additional exercise, try considering the case for labour. Begin with the result we derived in class:

$$MPL = W$$

where W is the wage rate paid out to labour.

2.2 Output per Worker

This is a minor caveat, but is something good to be aware of. In many macroeconomics textbooks and academic papers, people may express output in per worker terms. That is, the amount of output each worker can produce by having a set amount of capital each. We can derive the CDPF in per worker terms:

$$
\begin{aligned}
\frac{Y}{L} &= \frac{F(K,L)}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} \\
y = F(k,1) &= \frac{AK^{\alpha}L^{1-\alpha}}{L} \\
&= AK^{\alpha}L^{1-\alpha}L^{-1} \\
&= AK^{\alpha}L^{-\alpha} \\
&= \frac{AK^{\alpha}}{L^{\alpha}} \\
&= A\left(\frac{K}{L}\right)^{\alpha} \\
y &= Ak^{\alpha}
\end{aligned}
$$

where lower case variables, y and k, denote per worker output and capital, respectively. Graphically, we can draw this as:

Figure 1: **Output per Worker**