ECON4418: Macroeconomic Theory - Notes

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1 The Solow Growth Model

1.1 Some Basic Facts about Economic Growth

Over the past few centuries, standards of living in industrialised countries have reached levels almost unimaginable to our ancestors. Although comparisons are difficult, the best available evidence suggests that average real incomes today in the United States (US) and Western Europe are between 10 and 30 times larger than a century ago, and between 50 and 300 times larger than two centuries ago.

Moreover, worldwide growth is far from constant. Growth has been rising over most of modern history. Average growth rates in the industrialised countries were higher in the twentieth century than in the nineteenth, and higher in the nineteenth than in the eighteenth. Further, average incomes on the eve of the Industrial Revolution even in the wealthiest countries were not dramatically above subsistence level; this tells us that average growth over the millennia before the Industrial Revolution must have been very, very low.

One important exception to this general pattern of increasing growth is the productivity growth slowdown. Average annual growth in output per person in the US and other industrialised countries from the early 1970s to the mid-1990s was about a percentage point below its earlier level. The data since then suggest a rebound in productivity growth, at least in the US. How long the rebound will last and how widespread it will be is not yet clear.

This chapter focuses on the model that economists have traditionally used to study these issues, the Solow growth model. The Solow model is the starting point for almost all analyses of growth.

The principle conclusion of the Solow model is that the accumulation of physical capital cannot account for either the vast growth over time in output per person or the vast geographic differences in output per person. Specifically, suppose that capital accumulation affects output through the conventional channel that capital makes a direct contribution to production, for which it is paid its marginal product. Then the Solow model implies that the differences in real incomes that we are trying to understand are far too large to be accounted for by differences in real incomes as either exogenous and thus not explained by the model (in the case of technological progress, for example). Thus to address the central questions of growth theory, we must move beyond the Solow model.

We will first go through and extend the Solow model, by exploring the determinants of saving and investment. The Solow model has no optimisation in it; it takes the saving rate as exogenous and constant. We will modify this by making saving endogenous and potentially time-varying. In the first, saving and consumption decisions are made by a fixed set of infinitely lived households; in the second, the decisions are made by overlapping generations of households with finite horizons.

Relaxing the Solow model's assumption of a constant saving rate has three advantages:

1. It demonstrates that the Solow model's conclusions about the central questions of growth theory do not hinge on its assumption of a fixed saving rate;

2. It allows us to consider welfare issues. A model that directly specifies relations among aggregate variables provides no way of judging whether some outcomes are better or worse than others. The infinite-horizon and overlapping-generations models are built up from the behaviour of individuals, and can therefore be used to discuss welfare issues;

3. Infinite-horizon and overlapping generations models are used to study many issues in economics other than economic growth.

The main conclusion of this analysis is that endogenous technological progress is almost surely central to worldwide growth but probably has little to do with cross-country income differences.

1.2 Assumptions

1.2.1 Inputs and Outputs

The Solow model focuses on four variables: output (Y), capital (K), labour (L), and "knowledge" or the "effectiveness of labour" (A). At any time, the economy has some amounts of capital, labour, and knowledge, and these are combined with output. The production function takes the form:

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

where t denotes time. Notice that time does not enter the production function directly, but only through K, L and A. That is, output changes over time only if the inputs to production change. In particular, the amount of output obtained from given quantities of capital and labour rises over time–there is technological change progress–only if the amount of knowledge increases.

Notice also that A and L enter multiplicatively. AL is referred to as effective labour, and technological progress that enters in this fashion is known as labour augmenting or "Harrod neutral".

The central assumptions of the Solow model concern the properties of the production function and the evolution of the three inputs into production (capital, labour and knowledge) over time. We discuss each in turn.

1.2.2 Assumptions Concerning the Production Function

The model's critical assumption concerning the production function is that it has constant returns to scale (CRS) in its two arguments, capital and effective labour. That is, doubling the quantities of capital and effective labour (for example by doubling K and L with A fixed) doubles the amount produced. More generally, multiplying both arguments by any nonnegative constant c causes output to change by the same factor:

$$F(cK, cAL) = cF(K, AL), \quad \forall c \geq 0 \tag{2}$$

The assumption of constant returns can be thought of as a combination of two separate assumptions. The first is that the economy is big enough that the gains from specialisation have been exhausted. The second assumption is that inputs other than capital, labour and knowledge are relatively unimportant. In particular, the model neglects land and other natural resources.

The assumption of CRS allows us to work with the production function in intensive form. Setting $c = 1/AL$ in equation (2) yields:

$$F\left(\frac{K}{AL}, 1\right) = \frac{1}{AL}F(K, AL) \tag{3}$$

Here K/AL is the amount of capital per unit of effective labour, and $F(k, AL)/AL$ is Y/AL, output per unit of effective labour. Define $k = K/AL$, $y = Y/AL$ and $f(k) = F(k, 1)$. Then we can rewrite (3) as:

$$y = f(k) \tag{4}$$

That is, we can write output per unit of effective labour as a function of capital per unit of effective labour. These new variables, k and y, are not of interest in their own right. Rather, they are tools for learning about the

Figure 1: See Figure 1.1 of Romer

variables we are interested in. As we will see, the easiest way to analyse the model is to focus on the behaviour of k rather than to directly consider the behaviour of the two arguments of the production function, K and AL.

To see the intuition behind (4), think of dividing the economy into AL economies, each with 1 unit of effective labour and K/AL units of capital. Since the production function has CRS, reach of these small economies produces $1/AL$ as much as is produced in the large, undivided economy. Thus the amount of output per unit of effective labour depends on the quantity of capital per unit of effective labour, and not on the overall size of the economy. This is expressed mathematically above.

The intensive-form production function, $f(k)$, is assumed to satisfy $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$. Since $F(K, AL)$ equals $ALf(K/AL)$, it follows that the marginal product of capital, $\partial F(K, AL)/\partial K$, equals $ALf'(L/AL)(1/AL)$, which is just $f'(k)$. Thus the assumptions that $f'(k)$ is positive and $f''(k)$ is negative imply that the marginal product of capital is positive, but that it declines as capital per unit of effective labour rises. In addition, $f(\cdot)$ is assumed to satisfy the Inada conditions (Inada, 1964):

$$\lim_{k\to 0} f'(k) = \infty \tag{5}$$

$$\lim_{k\to\infty} f'(k) = 0 \tag{6}$$

These conditions (which are stronger than what is needed for the model) state that the marginal product of capital is very large when the capital stock is sufficiently small and that it becomes very small as the capital stock is sufficiently large; their role is to ensure that the path of the economy does not diverge.

A specific example of a production function is the Cobb-Douglas function:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1 \tag{7}$$

This production function is easy to analyse, and it appears to be a good first approximation to actual production functions. It is easy to check that the Cobb-Douglas function has CRS. Multiplying both inputs by c gives us:

$$\begin{aligned} F(cK, cAL) &= (cK)^{\alpha}(cAL)^{1-\alpha} \\ &= c^{\alpha}c^{1-\alpha}K^{\alpha}(AL)^{1-\alpha} \\ &= cF(K, AL) \end{aligned} \tag{8}$$

To find the intensive form of the production function, divide both inputs by AL; this yields:

$$\begin{aligned} f(k) &= F\left(\frac{K}{AL}, 1\right) \\ &= \left(\frac{K}{AL}\right)^{\alpha} \\ &= k^{\alpha} \end{aligned} \tag{9}$$

Equation (9) implies that $f'(k) = \alpha k^{\alpha-1}$. It is straightforward to check that this expression is positive, that it approaches infinity as k approaches zero, and that it approaches zero as k approaches infinity. Finally, $f''(k) = -(1-\alpha)\alpha k^{\alpha-2}$, which is negative.

1.2.3 The Evolution of the Inputs into Production

The remaining assumptions of the model concern how the stocks of labour, knowledge and capital change over time. The model is set in continuous time; that is, the variables of the model are defined at every point in time. The initial levels of capital, labour and knowledge are taken as given, and assumed to be strictly positive. Labour and knowledge grow at constant rates:

$$\dot{L}(t) = nL(t) \tag{10}$$

$$\dot{A}(t) = gA(t) \tag{11}$$

where n and g are exogenous parameters and where a dot over a variable denotes a derivative with respect to time (that is, $\dot{X}(t)$ is shorthand for $dX(t)/dt$). The growth rate of a variable refers to its proportional rate of change. That is, the growth rate of X refers to the quantity $\dot{X}(t)/X(t)$. Thus equation (10) implies that the growth rate of L is constant and equal to n, and (11) implies that A's growth rate is constant and equal to g.

A ket fact about growth rates is that the growth rate of a variable equals the rate of change of its natural log. That is, $\dot{X}(t)/X(t)$ equals $d\ln X(t)/dt$. To see this, note that since $\ln X$ is a function of X and X is a function of t, we can use the chain rule to write:

$$\begin{aligned}\frac{d\ln X(t)}{dt} &= \frac{d\ln X(t)}{dX(t)}\frac{dX(t)}{dt} \\ &= \frac{1}{X(t)}\dot{X}(t) \qquad (12)\end{aligned}$$

Applying the result that a variable's growth rate equals the rate of change of its log to (10) and (11) tells us that the rates of change of the logs of L and A are constant and that they equal n and g, respectively. Thus:

$$\ln L(t) = [\ln L(0)] + nt \tag{13}$$

$$\ln A(t) = [\ln A(0)] + gt \tag{14}$$

where $L(0)$ and $A(0)$ are the values of L and A at time 0. Exponentiating both sides of these equations gives us:

$$L(t) = L(0)e^{nt} \tag{15}$$

$$A(t) = A(0)e^{gt} \tag{16}$$

Thus, our assumption is that L and A each grow exponentially.

Output is divided between consumption and investment. The fraction of output devoted to investment, s, is exogenous and constant. One unit of output devoted to investment yields one unit of new capital. In addition, existing capital depreciates at rate δ. Thus:

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{17}$$

Although no restriction are placed on n, g and δ individually, their sum is assumed to be positive. This completes the description of the model.

Since this is the first model (of many) we will encounter, this is a good place for a general comment about modelling. The Solow model is grossly simplified in a host of ways. To give just a few examples, there is a only single good ("corn"); government is absent; fluctuations in employment are ignored; production is described by an aggregate production function with just three inputs; and the rates of saving, depreciation, population growth, and technological progress are constant. It is natural to think of these features of model as defects: the model omits many obvious features of the world, and surely some of those features are important to growth. But the purpose of a model is not to be realistic! After all, we already possess a model that is completely realistic–the world itself. The problem with that "model" is that it is too complicated to understand. A model's purpose is to provide insights about particular features of the world.

1.3 The Dynamics of the Model

In order to determine behaviour of the economy, we must analyse the behaviour of the third input, capital.

Figure 2: See Figure 1.2

1.3.1 The Dynamics of k

Because the economy may be growing over time, it turns out to be much easier to focus on the capital stock per unit of effective labour, k, than on the unadjusted capital stock, K. Since $k = K/AL$, we can use the chain rule to find:

$$\begin{aligned} \dot{k}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[A(t)\dot{L}(t) + L(t)\dot{A}(t)] \\ &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{L}(t)}{L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{A}(t)}{A(t)} \end{aligned} \tag{18}$$

K/AL is simply k. From (10) and (11), $\dot{L}/L$ and $\dot{A}/A$ are n and g, respectively. $\dot{K}$ is given by (17). Substituting these facts into (18) yields:

$$\begin{aligned} \dot{k}(t) &= \frac{sY(t) - \delta K(t)}{A(t)L(t)} - k(t)n - k(t)g \\ &= s\frac{Y(t)}{A(t)L(t)} - \delta k(t) - nk(t) - gk(t) \end{aligned} \tag{19}$$

Finally, using the fact that Y/AL is given by $f(k)$, we have:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \tag{20}$$

Equation (20) is the key equation to the Solow model. It states that the rate of change of the capital stock per unit of effective labour is the difference between two terms. The first, $sf(k)$, is actual inve stment per unit of effect labour: output per unit of effective labour is $f(k)$, and the fraction of that output that is invested is s. The second term, $(n + g + \delta)k$, is break-even investment, the amount of investment that must be done just to keep k at its existing level. There are two reasons that some investment is needed to prevent k from falling. First, existing capital is depreciating. Second, the quantity of effective labour is growing.

Figure 3: See Figure 1.3

Figure 1.2 plots the two terms of the expression for $\dot{k}$ as functions of k. Break even investment, $(n + g + \delta)k$, is proportional to k. Actual investment, $sf(k)$, is a constant times output per unit of effective labour.

Since $f(0) = 0$, actual investment and break-even investment are equal at $k = 0$. The Inada conditions imply that at $k = 0$, $f'(k)$ is large, and thus that the $sf(k)$ line is steeper than the $(n + g + \delta)k$ line. Thus for small values of k, actual investment is larger than break-even investment. The Inada conditions also imply that $f'(k)$ falls toward zero as k becomes large. At some point, the slope of the actual investment line falls below the slope of the break-even investment line. With the $sf(k)$ line flatter than the $(n + g + \delta)k$ line, the two must eventually cross. Finally, the fact that $f''(k) < 0$ implies that the two lines intersect only once for $k > 0$. We let k^* denote the value of k where actual investment and break-even investment are equal.

Figure 1.3 summarises this information in the form of a phase diagram, which shows $\dot{k}$ as a function of k. If k is initially less than k^*, actual investment exceeds break-even investment, and so $\dot{k}$ is positive–that is, k is rising. Vice versa for when k exceeds k^*. Thus, regardless of where k starts, it converges to k^* and remains there.

1.3.2 The Balanced Growth Path

Since k converges to k^*, it is natural to ask how the variables of the model behave when k equals k^*. By assumption, labour and knowledge are growing at rates n and g, respectively. The capital stock, K, equals ALk; since k is constant at k^*, K is growing at rate $n+g$ (i.e. $\dot{K}/K$ equals $n+g$). With both capital and effective labour growing at rate $n+g$, the assumption of constant returns implies that output, Y, is also growing at that rate. Finally, capital per worker, K/L, and output per worker, Y/L, are growing at rate g. Thus, the Solow model implies that, regardless of its starting point, the economy converges to a balanced growth path or the steady state–a situation where each variable of the model is growing at a constant rate. On the balanced growth path, the growth rate of output per worker is determined solely by the rate of technological progress.

1.4 The Impact of Change in the Saving Rate

The parameter of the Solow model that policy is most likely to affect is the saving rate. The division of the government's purchases between consumption and investment goods, the division of its revenues between taxes and borrowing, and its tax treatments of saving and investment are all likely to affect the fraction of output that is invested. Thus it is natural to investigate the effects of a change in the saving rate.

For concreteness, we will consider a Solow economy that is in the steady-state, and suppose that there is a permanent increase in s. In addition to demonstrating the model's implications concerning the role of saving, this experiment will illustrate the model's properties when the economy is not in the steady state.

1.4.1 The Impact on Output

The increase is s shifts the actual investment line upward, and so k^* rises. This is shown in Figure 1.4. But k does not immediately jump to the new value of k^*. Initially, k is equal to the old value of k^*. At this level, actual investment now exceeds break-even investment–more resources are being devoted to investment than are needed to hold k constant–and so $\dot{k}$ is positive. Thus k begins to rise. It continues to rise until it reaches the new value of k^*, at which point it remains constant. These results are summarised in the first three panels of Figure 1.5. t_0 denotes the time of the increase in the saving rate. By assumption, s jumps up at time t_0 and remains constant thereafter. Since the jump in s causes actual investment to exceed

Figure 4: See Figure 1.4

Figure 5: See Figure 1.5

break-even investment by a strictly positive amount, $\dot{k}$ jumps from zero to a strictly positive constant. k rises gradually from the old value of k^* to the new value, and $\dot{k}$ falls gradually back to zero.

We are likely to be particularly interested in the behaviour of output per worker, Y/L. A permanent increase in the savings rate produces a temporary increase in the growth rate of output worker: k is rising for a time, but eventually it increases to the point where the additional saving is devoted entirely to maintaining the higher level of k.

The growth rate of output per worker, which is initially g, jumps upward at t_0 and then gradually returns to its initial level. Thus output per worker begins to rise above the path it was on and gradually settles into a higher path parallel to the first.

In sum, a change in the saving rate has a level effect but not a growth effect: it changes the economy's balanced growth path, and thus the level of output per worker at any point in time, but it does not affect the growth rate of output per worker on the balanced growth path. Indeed, in the Solow model only changes in the rate of technological progress have growth effects.

1.4.2 The Impact on Consumption

If we were to introduce households into the model, their welfare would depend not on output but on consumption: investment is simply an input into production in the future. Thus for many purposes we are likely to be more interested in the behaviour of consumption than in the behaviour of output.

Consumption per unit of effective labour equals output per unit of effective labour, $f(k)$, times the fraction of that output that is consumed, $1-s$. Thus, since s changes discontinuously at t_0 and k does not, initially consumption per unit of effective labour jumps downward. Consumption then rises gradually as k rises and s remains at its higher level. This is shown in the last panel of Figure 1.5.

Whether consumption eventually exceeds its level before the rise in s is not immediately clear. Let c^* denote consumption per unit of effective labour in the steady state. c^* equals output per unit of effective labour, $f(k^*)$, minus investment per unit of effective labour, $sf(k^*)$. In the steady state, actual investment equals break-even investment, $(n+g+\delta)k^*$. Thus:

$$c^* = f(k^*) - (n+g+\delta)k^* \tag{21}$$

k^* is determined by s and the other parameters of the model. We can therefore write $k^* = k^*(s,n,g,\delta)$. Thus (21) implies:

$$\frac{\partial c^*}{\partial s} = [f'(k^*(s,n,g,\delta)) - (n+g+\delta)]\frac{\partial k^*(s,n,g,\delta)}{\partial s} \tag{22}$$

We know that the increase in s raises k^*; that is, we know that $\partial k^*/\partial s$ is positive. Thus whether the increase raises or lowers consumption in the long run depends on whether $f'(k^*)$–the marginal product of capital–is more or less than $n+g+\delta$. Intuitively, when k rises, investment must rise by $n+g+\delta$ times the change in k for the increase to be sustained. If $f'(k^*)$ is less than $n+g+\delta$, then the additional output from the increased capital is not enough to maintain the capital stock at its higher level. In this case, consumption must fall to maintain the higher capital stock. If $f'(k^*)$ exceeds $n+g+\delta$, on the other hand, there is more than enough additional output to maintain k at its higher level, and so consumption rises.

Since consumption in the steady state equals output less break-even investment, c^* is the distance between $f(k)$ and $(n+g+\delta)k$ at $k=k^*$. If s is at the level that causes $f'(k^*)$ to just equal $n+g+\delta$, then $f(k)$ and

$(n+g+\delta)k$ loci are parallel at $k = k^*$. In this case, a marginal change in s has no effect on consumption in the long run, and consumption is at its maximum possible level among balanced growth paths. This value of k^* is known as the golden run level of the capital stock. We will refer to the golden rule again later.

1.5 Quantitative Implications

We are usually interested not just in a model's qualitative implications, but in its quantitative predictions. If, for example, the impact of a moderate increase in saving on growth remains large after several centuries, the result that the impact is temporary is of limited interest.

The approach we take is to consider approximations around the long run equilibrium.

1.5.1 The Effect on Output in the Long Run

The long run effect of a rise in saving on output is given by:

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s} \tag{23}$$

where $y^* = f(k^*)$ is the level of output per unit of effective labour in the steady state. Thus to find $\partial y^*/\partial s$, we need to find $\partial k^*/\partial s$. To do this, note that k^* is defined by the condition that $\dot{k} = 0$. Thus k^* satisfies:

$$sf(k^*(s,n,g,\delta)) = (n+g+\delta)k^*(s,n,g,\delta) \tag{24}$$

Equation (24) holds for all values of s. Thus the derivative of the two sides with respect to s are equal:

$$sf'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n+g+\delta)\frac{\partial k^*}{\partial s} \tag{25}$$

where the arguments of $k^*$are omitted simplicity. This can be rearranged to obtain:

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)} \tag{26}$$

Substituting (26) into (23) yields:

$$\frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta) - sf'(k^*)} \tag{27}$$

Two changes help in interpreting this expression. The first is to convert it to an elasticity by multiplying both sides by s/y^*. The second is to use the fact that $sf(k^*) = (n+g+\delta)k^*$ to substitute for s. Making these changes gives us:

$$\begin{aligned}
\frac{s}{y^*}\frac{\partial y^*}{\partial s} &= \frac{s}{f(k^*)}\frac{f'(k^*)f(k^*)}{(n+g+\delta) - sf'(k^*)} \\
&= \frac{(n+g+\delta)k^*f'(k^*)}{f(k^*)[(n+g+\delta) - (n+g+\delta)k^*f'(k^*)/f(k^*)]} \\
&= \frac{k^*\frac{f'(k^*)}{f(k^*)}}{1 - \left[\frac{k^*f'(k^*)}{f(k^*)}\right]}
\end{aligned} \tag{28}$$

$k^*f'(k^*)/f(k^*)$ is the elasticity of output with respect to capital at $k = k^*$. Denoting this by $\alpha_K(k^*)$, we have:

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} \tag{29}$$

Thus we have found a relatively simple expression for the elasticity of the steady state level of output with respect to the saving rate.

To think about the quantitative implications of (29), note that if markets are competitive and there are no externalities, capital earns its marginal product. Since output equals $ALf(k)$ and k equals K/AL, the marginal product of capital, $\partial Y/\partial K$, is $ALf'(k)[1/(AL)]$, or just $f'(k)$. Thus if capital earns its marginal product, the total amount earned by capital per unit of effective labour in the steady state is $k^*f'(k^*)$. The share of total income that goes to capital in the steady state is then $k^* f'(k^*)/f(k^*)$, or $\alpha_K(k^*)$.

Intuitively, a small of $\alpha_K(k^*)$ makes the impact of saving on output low for two reasons. First, it implies that the actual investment curve, $sf(k)$, bends fairly sharply. Second, a low value of $\alpha_K(k^*)$ means that the impact of a change in k^*on y^* is small.

1.5.2 The Speed of Convergence

In practice, we are interested not only in the eventual effects of some change, but also how rapidly those effects occur. Again, we can use approximations around the long run equilibrium to address this issue.

For simplicity, we focus on the behaviour of k rather than y. Our goal is thus to determine how rapidly k approaches k^*. We know that $\dot{k}$ is determined by k: recalled that the key equations of the model is $\dot{k} = sf(k) - (n+g+\delta)k$. Thus we can write $\dot{k} = \dot{k}(k)$. When k equals k^*, $\dot{k}$ is zero. A first order Taylor series approximation of $\dot{k}(k)$ around $k = k^*$ is:

$$\dot{k} \backsimeq \left[\left. \frac{\partial \dot{k}(k)}{\partial k} \right|_{k=k^*} \right] (k - k^*) \tag{30}$$

That is, $\dot{k}$ is approximately equal to the product of difference between k and k^* and the derivative of $\dot{k}$ with respect to k at $k = k^*$. Let λ denote $-\partial \dot{k}(k)/\partial k|_{k=k^*}$. With this definition, (30) becomes:

$$\dot{k}(t) \backsimeq -\lambda[k(t) - k^*] \tag{31}$$

Equation (31) implies that in the vicinity of the steady state, k moves toward k^* at a speed approximately proportional to its distance from k^*. That is, the growth rate of $k(t) - k^*$ is approximately constant and equal to $-\lambda$. This implies:

$$k(t) \backsimeq k^* + e^{\lambda t}[k(0) - k^*] \tag{32}$$

This is because of the fact that the system is stable (that is, that k converges to k^*) and that we are linearising the equation for $\dot{k}$ around $k = k^*$.

We must find λ; this is where the specifics of the model enter the analysis. Differentiating (20) for $\dot{k}$ with respect to k and evaluating the resulting expression at $k = k^*$ yields:

$$\begin{aligned}
\lambda &\equiv -\left.\frac{\partial \dot{k}(k)}{\partial k}\right|_{k=k^*} = -[sf'(k^*) - (n+g+\delta)] \\
&= (n+g+\delta) - sf'(k^*) \\
&= (n+g+\delta) - \frac{(n+g+\delta)k^* f'(k^*)}{f(k^*)} \\
&= [1-\alpha_K(k^*)](n+g+\delta) \qquad (33)
\end{aligned}$$

Thus, k converges to its balanced growth path value at rate $[1-\alpha_K(k^*)](n+g+\delta)$. In addition one can show hat y approaches y^* at the same rate that k approaches k^*. That is, $y(t) - y^* \backsimeq e^{-\lambda t}[y(0) - y^*]$.

1.6 The Solow Model and the Central Questions of Growth Theory

The Solow model identifies two possible sources of variation–either over time or across parts of the world–in output per worker: differences in capital per worker (K/L) and differences in the effectiveness of labour (A). We have seen, however, that only growth in the effectiveness of labour can lead to permanent growth in output per worker, and that for reasonable cases the impact of changes in capital per worker on output per worker is modest.

As a result, only differences in the effectiveness of labour have any reasonable hope of accounting for the vast differences in wealth across time and space. Specifically, the central conclusion of the Solow model is that if the returns that capital commands in the market are a rough guide to its contributions in output, then variations in the accumulation of physical capital do not account for a significant part of either worldwide economic growth or cross country income differences.

There are two ways to see that the Solow model implies that differences in capital accumulation cannot account for large differences in incomes, one direct and the other indirect. Suppose X was the factor difference in output per worker between two economies on the basis of differences in capital per worker. If output per worker differs by a factor X, the difference in log output per worker between the two economies is $\ln X$. Since the elasticity of output per worker with respect to capital per worker is α_K, log capital per worker must differ by $(\ln X)/\alpha_K$. That is, capital per worker differs by a factor of $e^{(\ln X)/\alpha_K}$, or X^{1/α_K}.

Output per worker in the major industrialised countries today is on the order of 10 times larger than it was 100 years ago, and 10 times larger than it is in poor countries today. Thus we would like to account for values of X in the vicinity of 10. Our analysis implies that doing this on the basis of differences in capital requires a difference of a factor of $10^{1/\alpha_K}$ in capital per worker. For $\alpha_K = \frac{1}{3}$, this is a factor of 1000. Even if the capital's share is one half, which is well above what data on capital income suggest, one still needs a difference of a factor of 100. There is no evidence of such differences in capital stocks. Capital-output ratios are roughly constant over time. Thus the capital stock per worker in industrialised countries is roughly 10 times larger than it was 100 years ago, not 100 or 1000 times larger. Similarly, although capital output ratios vary somewhat across countries, the variation is not great. For example, the capital output ratio appears to be 2 to 3 times larger in industrialised countries than in poor countries; thus capital per worker is "only" about 20 to 30 times larger. In sum, differences in capital per worker are far smaller than those needed to account for the differences in output per worker that we are trying to understand.

Another, more indirect way, of seeing that the model cannot account for large variations in output per worker on the basis of differences in capital per worker is to notice that the required differences in capital imply enormous differences in the rate of return on capital (Lucas, 1990). If markets are competitive, the rate of return on capital equals its marginal product, $f'(k)$, minus depreciation, δ. Suppose that the production function is Cobb-Douglas, which in intensive form is $f(k) = k^{\alpha}$. With this production function, the elasticity of output with respect to capital is simply α. The marginal product of capital is:

$$\begin{aligned} f'(k) &= \alpha k^{\alpha-1} \\ &= \alpha y^{\frac{\alpha-1}{\alpha}} \end{aligned} \tag{34}$$

which implies that the elasticity of the marginal product of capital with respect to output is $-(1-\alpha)/\alpha$. If $\alpha = \frac{1}{3}$, a tenfold difference in output per worker arising from differences in capital per worker thus implies a hundredfold difference in the marginal product of capital. And since the return to capital is $f'(k) - \delta$, the difference in rates of return is even larger. Again, there is no evidence of such differences in rates of return. Direct measurement of returns on financial assets, for example, suggests only moderate variation over time and across countries. If rates of return were larger by factor of 10 or 100 in poor countries than in rich countries, there would be immense incentives to invest in poor countries.

The other potential source of variation in output per worker in the Solow model is the effectiveness of labour. Attributing differences in standards of living to differences in the effectiveness of labour does not require huge differences in capital or in rates of return. Unfortunately, however, the Solow model has little to say about the effectiveness of labour. Most obviously, the growth of the effectiveness of labour is exogenous: the model takes as given the behaviour of the variable it identifies as the driving force of growth.

More fundamentally, the model does not identify what the "effectiveness of labour" is; it is just a catchall for factors other than labour and capital that affect output. Thus saying that differences in income are due to differences in the effectiveness of labour is no different than saying that they are not due to differences in capital per worker. We must therefore further define this notion before we proceed.

Possible interpretations of A are the education and skills of the labour force, the strength of property rights, the quality of infrastructure, cultural attitudes toward entrepreneurship and work, and so on. Or A may reflect a combination of forces. For any proposed view of what A represents, one would again have to address the questions of how it affects output, how it evolves over time, and why it differs across parts of the world.

The other possibility is that the importance of capital is understated in the Solow model. If capital encompasses more than just physical capital, or if physical capital has positive externalities, then the private return on physical capital is not an accurate guide to capital's importance in production. In this case, the calculations we have done may be misleading.

1.7 Empirical Applications

1.7.1 Growth Accounting

In many situations we are interested in the proximate determinants of growth. That is, we often want to know how much of growth over some period is due to increases in various factors of production, and how much stems from other forces. Growth accounting, which was pioneered by Abramovitz (1956) and Solow (1957), provides a way of tackling this subject. To see how growth accounting works, consider again the production function $Y(t) = F(K(t), A(t)L(t))$. This implies:

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)}\dot{K}(t) + \frac{\partial Y(t)}{\partial L(t)}\dot{L}(t) + \frac{\partial Y(t)}{\partial A(t)}\dot{A}(t) \tag{35}$$

where $\partial Y/\partial L$ and $\partial Y/\partial A$ denote $[\partial Y/\partial(AL)]A$ and $[\partial Y/\partial(AL)]L$, respectively. Dividing both sides by $Y(t)$ and rewriting the terms on the right hand side yields:

$$\begin{aligned}\frac{\dot{Y}(t)}{Y(t)} &= \frac{K(t)}{Y(t)}\frac{\partial Y(t)}{\partial K(t)}\frac{\dot{K}(t)}{K(t)} + \frac{L(t)}{Y(t)}\frac{\partial Y(t)}{\partial L(t)}\frac{\dot{L}(t)}{L(t)} + \frac{A(t)}{Y(t)}\frac{\partial Y(t)}{\partial A(t)}\frac{\dot{A}(t)}{A(t)} \\ &\equiv \alpha_K(t)\frac{\dot{K}(t)}{K(t)} + \alpha_L(t)\frac{\dot{L}(t)}{L(t)} + R(t) \end{aligned} \tag{36}$$

Subtracting $\dot{L}(t)/L(t)$ from both sides and using the fact that $\alpha_K(t) + \alpha_L(t) = 1$ gives us an expression for the growth rate of output per worker:

$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t) \tag{37}$$

The growth rates of Y, K and L are straightforward to measure. And we know that if capital earns its marginal product, α_K can be measured using data on the share of income that goes to capital. $R(t)$ can then be measured as a residual–and hence is termed as the "Solow residual". The Solow residual is sometimes interpreted as a measure of the contribution of technological progress. As the derivation shows, however, it reflects all sources of growth other than the contribution of capital accumulation via its private return.

Growth accounting only examines the immediate determinants of growth: it asks how much factor accumulation, improvements in the quality of inputs, and so on contribute to growth while ignoring the deeper issue of what causes the changes in those determinants.

2 The Ramsey-Cass-Koopmans Infinite-Horizon Model

We will now investigate a model that resembles the Solow model but in which the dynamics of economic aggregates are determined by decisions at the microeconomic level. The Ramsey-Cass-Koopmans Infinite-Horizon Model (the Ramsey model) is conceptually simple. Competitive firms rent capital and hire labour labour to produce and sell output, and a fixed number of infinitely lived households supply labour, hold capital, consume and save.

2.1 Assumptions

2.1.1 Firms

There are a large number of identical firms. Each has access to the production function $Y = F(K, AL)$, which satisfies the same assumptions as the Solow model. The firms hire workers and rent capital in competitive factor markets and sell their output in a competitive output market. Firms take A as given; as in the Solow model, A grows exogenously at rate g. The firms maximise profits. They are owned by the households, so any profits they earn accrue to the households.

2.1.2 Households

There are a large number of identical households. The size of each household grows at rate n. Each member of the household supplies 1 unit of labour at every point in time. In addition, the household rents whatever capital it owns to firms. It has initial capital holdings of $K(0)/H$, where $K(0)$ is the initial amount of capital in the economy and H is the number of households. For simplicity, here we assume there is no depreciation. The household divides its income at each point in time between consumption and saving so as to maximise its lifetime utility. The household's utility function takes the form:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt \tag{38}$$

$C(t)$ is the consumption of each member of the household at any time t. $u(\cdot)$ is the instantaneous utility function, which gives each member's utility at a given date. $L(t)$ is the total population of the economy; $L(t)/H$ is therefore the number of members of the household. Thus $u(C(t))L(t)/H$ is the household's total instantaneous utility at t. Finally, ρ is the discount rate; the greater is ρ, the less the household values future consumption relative to current consumption.

The instantaneous utility function takes the form:

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}, \quad \theta > 0, \quad \rho - n - (1-\theta)g > 0 \tag{39}$$

This functional form is needed for the economy to converge to a balanced growth path. It is known as constant relative risk aversion (CRRA) utility. The reason for the name is that the coefficient of relative risk aversion for this utility function is θ, and thus is independent of C.

Since there is no uncertainty in this model, the household's attitude toward risk is not directly relevant. But θ also determines the willingness to shift consumption between different periods. When θ is smaller, marginal utility falls more slowly as consumption rises, and so the household is more willing to allow its consumption to vary over time. Specifically, one can show that the inter temporal elasticity of substitution between consumption at any two points in time is $1/\theta$.

Three additional features of the instantaneous utility function are worth mentioning. First, $C^{1-\theta}$ is increasing in C if $\theta < 1$ but decreasing if $\theta > 1$; dividing $C^{1-\theta}$ by $1-\theta$ thus ensures that the marginal utility of consumption is positive regardless of the value of θ. Second, in the special case of $\theta \to 1$, the instantaneous utility function simplifies to $\ln C$; this is often useful case to consider. Third, the assumption that $\rho - n - (1-\theta)g > 0$ ensures that lifetime utility does not diverge: if this were not the case, the household can attain infinite lifetime utility, and its maximisation problem does not have well defined solution.

2.2 The Behaviour of Households and Firms

2.2.1 Firms

Firms' behaviour is relatively simple. At each point in time they employ the stocks of capital and labour, them them their marginal products, and sell the resulting output. Because the production function has CRS and the economy is competitive, firms earn zero profits.

Because we assume no depreciation, and because markets are competitive, we assume that the real rate of return on capital equals its earnings per unit time. Thus the real interest rate at time t is:

$$r(t) = f'(k(t)) \tag{40}$$

Labour's marginal product is $\partial F(K, AL)/\partial L$, which equals $A\partial F(K, AL)/\partial AL$. In terms of $f(\cdot)$, the intensive form of the production function, this is $A[f(k) - kf'(k)]$. Thus the real wage at t is:

$$W(t) = A(t)[f(k(t)) - k(t)f'(k(t))] \tag{41}$$

The wage per unit of effective labour is therefore:

$$w(t) = f(k(t)) - k(t)f'(k(t)) \tag{42}$$

2.2.2 Household's Budget Constraint

The representative household takes the paths of r and w as given. Its budget constraint is that the present value of its lifetime consumption cannot exceed its initial wealth plus the present value of its lifetime labour income. We need to account for the fact that r may vary over time. To do this, define:

$$R(t) = \int_{\tau=0}^{t} r(\tau)d\tau \tag{43}$$

One unit of the output good invested at time 0 yields $e^{R(t)}$ units of the good at t; equivalently, the value of 1 unit of output at time t in terms of output at time 0 is $e^{-R(t)}$. For example, if r is constant at some level $\bar{r}$, $R(t)$ is simply $\bar{r}t$ and the present value of 1 unit of output at t is $e^{-\bar{r}t}$. More generally, $e^{R(t)}$ shows the effects of continuously compounding interest over the period [0,t].

Since the household has $L(t)/H$ members, its labour income at t is $W(t)L(t)/H$, and its consumption expenditures are $C(t)L(t)/H$. Its initial wealth is $1/H$ of total wealth at time 0, or $K(0)/H$. The household's budget constraint is therefore:

$$\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} dt \tag{44}$$

In general, it is not possible to find the integrals in this expression. Fortunately, we can express the budget constraint in terms of the limiting behaviour of the household's capital holdings; and it is usually possible to describe the limiting behaviour of the economy. Begin by bringing all terms in (44) over to the same side and combine the two integrals; this gives:

$$\frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H} dt \geq 0 \tag{45}$$

We can write the integral from $t = 0$ to $t = \infty$ as a limit. Thus (45) is equivalent to:

$$\lim_{s\to\infty} \left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H} dt\right] \geq 0 \tag{46}$$

Now note that the household's capital holdings at time s are:

$$\frac{K(s)}{H} = e^{R(s)} \frac{K(0)}{H} + \int_{t=0}^{s} e^{R(s)-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H} dt \tag{47}$$

To understand (47), observe that $e^{R(s)}K(0)/H$ is the contribution of the household's initial wealth to its wealth at s. The household's saving at t

is $[W(t) - C(t)]L(t)/H$ (which may be negative); $e^{R(s)-R(t)}$ shows how the value of that saving changes from t to s.

The expression in (47) is $e^{R(s)}$ times the expression in brackets in (46). Thus we can write the budget constraint as simply:

$$\lim_{s\to\infty} e^{-R(s)}\frac{K(s)}{H} \geq 0 \tag{48}$$

Expressed in this form, the budget constraint states that the present value of the household's asset holdings cannot be negative in the limit. Equation (48) is also known as the "no Ponzi game condition". A Ponzi game is a scheme in which someone issues debt and rolls it over forever. That is, the issuer always obtains the funds to pay off debt when it comes due by issuing new debt. Such a scheme allows the issuer to have a present value of lifetime consumption that exceeds the present value of his or her lifetime resources. We rule out these schemes.

2.2.3 Household's Maximisation Problem

The representative household wants to maximise its lifetime utility subject to its budget constraint. As in the Solow model, it is easier to work with variables normalised by the quantity of effective labour. To do this, we need to express both the objective function and the budget constraint in terms of consumption and labour income per unit of effective labour.

Begin by defining $c(t)$ as consumption per unit of effective labour. Thus, $C(t)$, consumption per worker, equals $A(t)c(t)$. The household's instantaneous utility (39) is therefore:

$$\begin{aligned}\frac{C(t)^{1-\theta}}{1-\theta} &= \frac{[A(t)c(t)]^{1-\theta}}{1-\theta}\\ &= \frac{[A(0)e^{gt}]^{1-\theta}c(t)^{1-\theta}}{1-\theta}\\ &= A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta} \qquad (49)\end{aligned}$$

Substituting (49) and the fact that $L(t) = L(0)e^{nt}$ into the household's objective function, (38)-(39), yields:

$$\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \left[A(0)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \right] \frac{L(0)e^{nt}}{H} dt \\
&= A(0)^{1-\theta} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-\rho t} e^{(1-\theta)gt} e^{nt} \frac{c(t)^{1-\theta}}{1-\theta} dt \\
&= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt
\end{aligned} \tag{50}$$

Here, $B = A(0)^{1-\theta} L(0)/H$ and $\beta = \rho - n - (1-\theta)g$. From (39), β is assumed to be positive.

Now consider the budget constraint, (44). The household's total consumption at t, $C(t)L(t)/H$, equals consumption per unit of effective labour, $c(t)$, times the household's quantity if effective labour, $A(t)L(t)/H$. Similarly, its total labour income at t equals the wage per unit of effective labour, $w(t)$, times $A(t)L(t)/H$. And its initial capital holdings are capital per unit of effective labour at time 0, $k(0)$, times $A(0)L(0)/H$. Thus we can rewrite (44) as:

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) \frac{A(t)L(t)}{H} dt \leq k(0) \frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt \tag{51}$$

$A(t)L(t)$ equals $A(0)L(0)e^{(n+g)t}$. Substituting this fact into (51) and dividing both sides by $A(0)L(0)/H$ yields:

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt \tag{52}$$

Finally, because $K(s)$ is proportional to $k(s)e^{(n+g)s}$, we can rewrite the no Ponzi game condition as:

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) \geq 0 \tag{53}$$

2.2.4 Household Behaviour

The household's problem is to choose the path of $c(t)$ to maximise lifetime utility (50), subject to the budget constraint (52). Although this involves choosing c at each instant of time (rather than choosing a finite set of variables, as in standard maximisation problems), conventional maximisation techniques can be used. Since the marginal utility of consumption is always positive, the household satisfies its budget constraint with equality. We can therefore use the objective function (50) and the budget constraint (52) to set up the Lagrangian:

$$\begin{aligned}\mathcal{L} &= B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt \qquad (54)\\ &\quad +\lambda\left[k(0)+\int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}w(t)dt-\int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)dt\right]\end{aligned}$$

The household chooses c at each point in time; that is, it chooses infinitely many $c(t)$'s. The first order condition for an individual $c(t)$ is:

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)}e^{(n+g)t} \qquad (55)$$

The household's behaviour is characterised by (55) and the budget constraint (52).

To see what (55) implies for the behaviour of consumption, first take logs of both sides:

$$\begin{aligned}\ln B-\beta t-\theta\ln c(t) &= \ln\lambda - R(t)+(n+g)t\\ &= \ln\lambda-\int_{\tau=0}^{t} r(\tau)d\tau+(n+g)t \qquad (56)\end{aligned}$$

where the second line uses the definition of $R(t)$ as $\int_{\tau=0}^{t} r(\tau)d\tau$. Now note that since the two sides are equal for every t, the derivatives of the two sides with respect to t must be the same. This condition is:

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{r(t)-n-g-\beta}{\theta} \\ &= \frac{r(t)-\rho-\theta g}{\theta} \qquad (57)\end{aligned}$$

where the second line uses the definition of β as $\rho - n - (1-\theta)g$.

To interpret (57), note that since $C(t)$ equals $c(t)A(t)$, the growth rate of C is given by:

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{c}(t)}{c(t)} \\ &= g + \frac{r(t)-\rho-\theta g}{\theta} \\ &= \frac{r(t)-\rho}{\theta} \qquad (58)\end{aligned}$$

where the second line uses (57). This condition states that consumption per worker is rising if the real return exceeds the rate at which the household discounts future consumption, and is falling if the reverse holds. The smaller is θ–the less marginal utility changes as consumption changes–the larger are the changes in consumption in response to differences between the real interest rate and the discount rate.

Equation (57) is known as the Euler equation for this maximisation problem. A more intuitive way of thinking about it is to consider the household's consumption at two consecutive moments in time. Specifically, imagine the household reducing c at some date t by a small amount, $\triangle c$, investing this additional saving for a short period of time $\triangle t$, and then consuming the proceeds at time $t + \triangle t$; assume when it does this, the household leaves consumption and capital holdings at all times other than t and $t + \triangle t$ unchanged. If the household is optimising, the marginal impact of this change on lifetime utility must be zero. If the impact is strictly positive, the household can marginally raise its lifetime utility by making the change. And if

Figure 6: See Figure 2.1

Figure 7: See Figure 2.2

the impact is strictly negative, the household can raise its lifetime utility by making the opposite change.

2.3 The Dynamics of the Economy

2.3.1 The Dynamics of c

Since all households are the same, equation (57) describes the evolution of c not just for a single household but for the economy as a whole. Since $r(t) = f'(k(t))$, we can rewrite (57) as:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} \tag{59}$$

Thus $\dot{c}$ is zero when $f'(k)$ equals $\rho + \theta g$. Let k^* denote this level of k. When k exceeds k^*, $f'(k)$ is less than $\rho + \theta g$, and so $\dot{c}$ is negative; when k is less than k^*, $\dot{c}$ is positive. This information is summarised in Figure 2.1. The arrows show the direction of motion of c.

2.3.2 The Dynamics of k

As in the Solow model, $\dot{k}$ equals actual investment minus break even investment. Since we are assuming there is no depreciation, break even investment is $(n+g)k$. Actual investment is output minus consumption, $f(k)-c$. Thus:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t) \tag{60}$$

Figure 2.2 summarises the dynamics of k.

Figure 8: See Figure 2.3

Figure 9: See Figure 2.4

2.3.3 The Phase Diagram

Figure 2.3 combines the information of Figures 2.1 and 2.2. Figure 2.3 is drawn with k^* less than the golden rule level of k.

This is because k^* was defined earlier as $f'(k^*) = \rho + \theta g$, and that the golden rule of k is defined by $f'(k_{GR}) = n + g$. Since $f''(k)$ is negative, k^* is less than k_{GR} if and only if $\rho + \theta g$ is greater than $n + g$. This is the same as $\rho - n - (1 - \theta)g > 0$, which we assumed to hold so that lifetime utility does not diverge. Thus k^* is to the left of the peak of the $\dot{k} = 0$ curve.

2.3.4 The Initial Value of c

This issue is addressed in Figure 2.4. For concreteness, $k(0)$ is assumed to be less than k^*. The figure shows the trajectory of c and k for various assumptions concerning the initial level of c.

2.3.5 The Saddle Path

Although this discussion has been in terms of a single value of k, the idea is general. For any positive initial level of k, there is a unique initial level of c that is consistent with household's intertemporal optimisation. The function this initial c as a function k is known as the saddle path. It is shown in Figure 2.5. For any starting value for k, the initial c must be the value on the saddle path. The economy then moves along the saddle path to Point E.

Figure 10: See Figure 2.5

2.4 Welfare

A natural question is whether the equilibrium of this economy represents a desirable outcome. The answer to this question is simple. The first welfare theorem from microeconomics tells us that if markets are competitive and complete and there are no externalities (and if the number of agents is finite), then the decentralised equilibrium is Pareto-efficient–that is, it is impossible to make anyone better off without making someone else worse off. Since the conditions of the first welfare theorem hold in our model, the equilibrium must be Pareto efficient.

To see this more clearly, consider the problem facing a social planner who can dictate the division of output between consumption and investment at each date and who wants to maximise the lifetime utility of a representative household. This problem is identical to that of an individual household except that, rather than taking the paths of w and r as given, the planner takes into account the fact these are determined by the path of k, which is in turn determined by (60).

The intuitive argument involving consumption at consecutive moments used to derive (57) or (59) applies to the social planner as well: reducing c by $\triangle c$ at time t and investing the proceeds allows the planner to increase c at time $t + \triangle t$ by $e^{f'(k(t))\triangle t} e^{-(n+g)\triangle t} \triangle c$. Thus $c(t)$ along the path chosen by the planner must satisfy (59). And since equation (60) giving the evolution of k reflects technology, not preferences, the social planner must obey it as well. Finally, as with the households' optimisation problem, paths that result in negative capital stocks, and paths that cause consumption to approach zero can be ruled out on the grounds that they do not maximise households' utility.

In short, the solution the social planner's problem is for the initial value of c to be given by the value on the saddle path, and for c and k to then move along the saddle path. That is, the competitive equilibrium maximises the welfare of the representative household.

2.5 The Balanced Growth Path

2.5.1 Properties of the Balanced Growth Path

The behaviour of the economy once it has converged to Point E is identical to that of the Solow economy on the balanced growth path. Capital, output and consumption per unit of effective labour are constant. Since y and c are constant, the saving rate, $(y - c)/y$ is also constant. The total capital stock, total output, and total consumption grow at rate $n + g$. And capital per worker, output per worker, and consumption per worker grow at rate g.

Thus the central implications of the Solow model concerning the driving forces of economic growth do not hinge on its assumption of a constant saving rate. Even when saving is endogenous, growth in the effectiveness of labour remains the only source of persistence growth in output per worker.

One can show, just like we did in the previous section, that unless differences in capital per worker and rates of return to capital are enormous, differences in output per worker across companies will not be significant.

2.5.2 The Social Optimum and the Golden Rule Level of Capital

The only difference between the balanced growth paths of the Solow and Ramsey models is that a balanced growth path with a capital stock above the golden rule level is not possible in the Ramsey model. In the Solow model, a sufficiently high saving rate causes the economy to reach a balanced growth path with the property that there are feasible alternatives that involve higher consumption at every moment. In the Ramsey model, in contrast, saving is derived from the behaviour of households whose utility depends on their consumption, and there are no externalities. As a result, it cannot be an equilibrium for the economy to follow a path where higher consumption can be attained in every period; if the economy were on such a path, households would reduce their saving and take advantage of this opportunity.

This can be seen in Figure 2.5. If the initial capital stock exceeds the golden rule level (remember the golden rule level is the peak of the $\dot{k} = 0$ line), initial consumption is above the level needed to keep k constant; thus $\dot{k}$ is negative. k gradually approaches k^*, which is below the golden rule level.

Finally, the fact that k^* is less than the golden rule capital stock implies that the economy does not converge to the balanced growth path that yields the maximum sustainable level of c. The intuition for this result is clearest in the case of g equal to zero. In this case, k^* is defined by:

$$f'(k^*) = \rho \tag{61}$$

$$f'(k_{GR}) = n \tag{62}$$

and our assumption that $\rho - n - (1-\theta)g > 0$ simplifies to:

$$\rho > n \tag{63}$$

Since k^* is less than k_{GR}, an increase in saving starting at $k = k^*$ would cause consumption per worker to eventually rise above its previous level and remain there. But because households value present consumption more than future consumption, the benefit of the eventual permanent increase in consumption is bounded. At some point–specifically, when k exceeds k^*–the tradeoff between the temporary short term sacrifice and the permanent long term gain is sufficiently unfavourable that accepting it reduces rather than raises lifetime utility. Thus k converges to a value below the golden rule level. Because k^* is the optimal level of k for the economy to converge to, it is known as the modified golden rule capital stock.

2.6 The Effects of a Fall in the Discount Rate

Consider a Ramsey model economy that is in the steady state, and suppose that there is a fall in ρ, the discount rate. Because ρ is the parameter

Figure 11: See Figure 2.6

governing household's preferences between current and future consumption, this change is the closest analogue in this model to a rise in the saving rate in the Solow Model.

For our purposes, we assume that the change is unexpected, so households are unable to change their behaviour prior to the shock.

2.6.1 Qualitative Effects

Since the evolution of k is determined by technology rather than preferences, ρ enters the equation for $\dot{c}$ but not the one for $\dot{k}$. Thus only the $\dot{c} = 0$ locus is affected. Thus the $\dot{c} = 0$ line shifts to the right, resulting in an increase in k^*. This is shown in Figure 2.6.

At the time of the change in ρ, the value of the k is given by the history of the economy, and it cannot change discontinuously. In particular, k at the time of the change equals the value of k^* on the old balanced growth path. In contrast, c–the rate at which households are consuming–can jump at the time of the shock.

Given our analysis of the dynamics of the economy, it is clear what occurs: at the instant of the change, c jumps down so that the economy is on the new saddle path. Thereafter, c and k rise gradually to their new balanced growth path values; these are higher than their values on the original balanced growth path.

Thus the effects of a fall in the discount rate are similar to the effects of a rise in the saving rate in the Solow model with a capital stock below the golden rule level. In both cases, k rises gradually to a new higher level, and in both c initially falls but then rises to a level above the one it started at. Thus, just as with the permanent rise in the saving rate in the Solow model, the permanent fall in the discount rate produces temporary increases in the growth rates of capital per worker and output per worker. The only difference between the two experiments is that, in the case of a fall in ρ, in general the fraction of output that is saved is not constant during the adjustment process.

2.6.2 The Rate of Adjustment and the Slope of the Saddle Path

Equations (59) and (60) describe $\dot{c}(t)$ and $\dot{k}(t)$ as functions of $k(t)$ and $c(t)$. A fruitful way to analyse their quantitative implications for the dynamics of the economy is to replace these nonlinear equations with linear approximations around the balanced growth path. Thus we begin by taking first order Taylor approximations around $k = k^*$ and $c = c^*$. That is:

$$\dot{c} \backsimeq \frac{\partial \dot{c}}{\partial k}[k - k^*] + \frac{\partial \dot{c}}{\partial c}[c - c^*] \tag{64}$$

$$\dot{k} \backsimeq \frac{\partial \dot{k}}{\partial k}[k - k^*] + \frac{\partial \dot{k}}{\partial c}[c - c^*] \tag{65}$$

where the above partial derivatives are all evaluated at $k = k^*, c = c^*$. It helps to define $\tilde{c} = c - c^*$ and $\tilde{k} = k - k^*$. Since c^* and k^* are both constant, $\dot{\tilde{c}} = \dot{c}$ and $\dot{\tilde{k}} = \dot{k}$. We can therefore rewrite the above system of equations as:

$$\dot{\tilde{c}} = \frac{\partial \dot{c}}{\partial k}\tilde{k} + \frac{\partial \dot{c}}{\partial c}\tilde{c} \tag{66}$$

$$\dot{\tilde{k}} = \frac{\partial \dot{c}}{\partial k}\tilde{k} + \frac{\partial \dot{c}}{\partial c}\tilde{c} \tag{67}$$

In matrix form:

$$\begin{bmatrix} \dot{\tilde{c}} \\ \dot{\tilde{k}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial c} \\ \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial c} \end{bmatrix}_{k=k^*, c=c^*} \begin{bmatrix} \tilde{k} \\ \tilde{c} \end{bmatrix} \tag{68}$$

Figure 12: See Figure 2.7

From:

$$\frac{\partial \dot{c}}{\partial k} = \frac{f''(k^*)c^*}{\theta} \tag{69}$$

$$\frac{\partial \dot{c}}{\partial c} = \frac{f'(k^*) - \rho - \theta g}{\theta} = 0 \tag{70}$$

$$\frac{\partial \dot{k}}{\partial k} = f'(k^*) - (n+g) \tag{71}$$

$$\frac{\partial \dot{k}}{\partial c} = -1 \tag{72}$$

It then follows:

$$\dot{\tilde{c}} = \frac{f''(k^*)c^*}{\theta}\tilde{k} \tag{73}$$

and:

$$\begin{aligned} \dot{\tilde{k}} &= [f'(k^*) - (n+g)\tilde{k} - \tilde{c} \\ &= [(\rho + \theta g) - (n+g)]\tilde{k} - \tilde{c} \\ &= \beta\tilde{k} - \tilde{c} \end{aligned} \tag{74}$$

Looking growth rates:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} \backsimeq \frac{f''(k^*)c^*}{\theta}\frac{\tilde{k}}{\tilde{c}} \tag{75}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} \simeq \beta - \frac{\tilde{c}}{\tilde{k}} \tag{76}$$

Equations (75) and (76) imply that the growth rates of $\tilde{c}$ and $\tilde{k}$ depend only on the ratio of $\tilde{c}$ and $\tilde{k}$.

Figure 2.7 shows the line along which the economy converges smoothly to (k^*,c^*); it is labelled AA. This is the saddle path of the linearised system. The figure also shows the line along which the economy moves directly away from (k^*,c^*); it is labelled BB. If the initial values of $c(0)$ and $k(0)$ lay along this line, (75) and (76) would imply that $\tilde{c}$ and $\tilde{k}$ would grow steadily.

2.7 The Effects of Government Purchases

Thus far, we have left government out of our model. Yet modern economies devote their resources not just to investment and private consumption but also to public uses. In the US, for example, about 20 percent of total output is purchased by the government; in many other countries the figure is considerably higher. It is thus natural to extend our model to include a government sector.

2.7.1 Adding Government to the Model

Assume that the government buys output at rate $G(t)$ per unit of effective labour per unit time. Government purchases are assumed not to affect utility from private consumption; this can occur if the government devotes the goods to some activity that does not affect utility at all, or if utility equals the sum of utility from private consumption and utility from government provided goods. Purchases are assumed not affect future output; that is, they are devoted purely to public consumption and not public investment. The purchases are financed by lump-sum taxes of amount $G(t)$ per unit of effective labour per unit time; thus the government always runs a balanced budget. We will consider deficits later.

Investment is now the difference between output and the sum of private consumption and government purchases. Thus the equation of motion for k, (60), becomes:

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t) \tag{77}$$

A higher value of G shifts the $\dot{k} = 0$ locus down: the more goods that are purchased by government, the fewer that can be purchased privately if k is to be held constant.

By assumption, household preferences are unchanged. Since the Euler equation is derived from household's preferences without imposing their lifetime budget constraint, this condition continues to hold as before. The taxes that finance the government's purchases affect households' budget constraint however. Specifically, (52) becomes:

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} [w(t) - G(t)] e^{(n+g)t} dt \tag{78}$$

2.7.2 The Effects of Permanent and Temporary Changes in Government Purchases

To see the implications of the model, suppose that the economy is on a balanced growth path with $G(t)$ constant at some level G_L, and that there is an unexpected, permanent increase in G to G_H. We know that this causes a downward shift in the $\dot{k} = 0$ locus by the amount of the increase in G. Since government purchases do not affect the Euler equation, the $\dot{c} = 0$ locus is unaffected.

Through the no Ponzi game condition and positive capital stock assumption, we know that c must jump so that the economy is on its new saddle path. In this case, the adjustment takes a simple form: c falls by the amount of the increase in G, and the economy is immediately on its new balanced growth path. Intuitively, the permanent increases in government purchases and taxes reduces households' lifetime wealth. And because the increases in purchases and taxes are permanent, there is no scope for households to raise their utility by adjusting the time pattern of their consumption. Thus the fall in consumption is equal to the full amount of the increase in government purchases, and the capital stock and the real interest rate are unaffected.

An older approach to modelling consumption behaviour would be to assume that consumption depends only on current disposable income and that it moves less than one-for-one with disposable income. As a result, the rise in government purchases crowds out investment, and so the capital stock starts

Figure 13: See Figure 2.8

Figure 14: See Figure 2.9

to fall and the real interest rate starts to rise. Our analysis shows that those results rest critically on the assumption that households follow mechanical rules: with intertemporal optimisation, a permanent increase in government purchases does not cause crowding out. These results are shown in Figure 2.8.

A more complicated case is provided by the unanticipated increase in G that is expected to be temporary. For simplicity, assume that the terminal date for government spending changes is known with certainty. See Figure 2.9 for a range of changes to government spending. In the case of temporary increases of government purchases, we see changes to not only c but also the capital stock k and the real interest rate, r.

Note that once again allowing for forward-looking behaviour yields insights we would not get from the older approach of assuming that consumption depends only current disposable income. The idea that households do not look ahead and put some weight on the likely future path of government purchases and taxes is implausible.

3 The Diamond Overlapping-Generations Model

3.1 Assumptions

We now turn to the Diamond overlapping-generations model. The central difference between the Diamond model and the Ramsey model is that there is turnover in the population: new individuals are continually being born, and old individuals are continually dying.

With turnover, it turns out to be simpler to assume that time is discrete rather than continuous. To further simplify the analysis, the model assumes that each individual lives only two periods. It is the general assumption of turnover in the population, however, and not the specific assumptions of discrete time and two period lifetimes, that is crucial to the model's results.

The individual born at t has a CRRA utility function defined as:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t}^{1-\theta}}{1-\theta}, \quad \theta > 0, \quad \rho > -1 \tag{79}$$

L_t individuals are born in period t. As before, population grows at rate n; thus $L_t = (1+n)L_{t-1}$. Each individual supplies 1 unit of labour when he or she is young and divides the resulting labour income between first period consumption and saving. In the second periods, the individual simply consumes the saving and any interest he or she earns from period 1. As before, the above functional form is needed for balanced growth. Because lifetimes are finite, we no longer have to assume $\rho > n+(1-\theta)g$ to ensure that lifetime utility does not diverge. If $\rho > 0$ then individuals place greater weight on first periods than second period consumption; if $\rho < 0$ then the reverse holds.

Production is described by the same assumptions as before. There are many firms, each with production function $Y_t = F(K_t, A_tL_t)$. The production functions are assumed to exhibit CRS and satisfy the Inada conditions. A again grows at exogenous rate g, so that $A_t = (1+g)A_{t-1}$. Markets are competitive, so firms earn zero profits and labour and capital are paid their marginal products. The real interest rate is therefore:

$$r_t = f'(k_t) \tag{80}$$

and real wages are:

$$w_t = f(k_t) - k_t f'(k_t) \tag{81}$$

3.2 Household Behaviour

The second period consumption of an individual born at t is:

$$C_{2t+1} = (1 + r_{t+1})(w_t A_t - C_{1t}) \tag{82}$$

Dividing both sides of this expression by $1 + r_{t+1}$ and bringing C_{1t} over to the left hand side yields the individual's budget constraint:

$$C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = A_t w_t \tag{83}$$

This condition states that the present value of lifetime consumption equals initial wealth (which is zero) plus the value of lifetime labour income (which is $w_t A_t$).

The individual maximises (79) subject to (83). Setting up the Lagrangian:

$$\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1-\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda \left[A_t w_t - \left(C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} \right) \right] \tag{84}$$

The first order conditions for C_{1t} and C_{2t+1} are:

$$C_{1t}^{-\theta} = \lambda \tag{85}$$

$$\frac{1}{1+\rho} C_{2t+1}^{-\theta} = \frac{1}{1 + r_{t+1}} \lambda \tag{86}$$

This can be rearranged to:

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1 + r_{t+1}}{1 + \rho} \right)^{\frac{1}{\theta}} \tag{87}$$

The above expression (the Euler equation here) is similar to the Euler equation in the Ramsey model. It implies that whether an individual's consumption is increasing or decreasing over time depends on whether the real rate of return is greater or less than the discount rate. θ again determines how much individuals' consumption varies in response to differences between r and ρ. Remember that θ is the coefficient of relative risk aversion, and the inverse of this is the intertemporal elasticity of substitution.

We can use the Euler equation and the budget constraint to express C_{1t} in terms of labour income and the real interest rate. Specifically, multiplying both sides of (87) by C_{1t} and substituting into (83) gives:

$$C_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} C_{1t} = A_t w_t \tag{88}$$

This implies:

$$C_{1t} = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \tag{89}$$

Equation (89) shows that the interest rate determines the fraction of income the individual consumes in the first period. If we let $s(r)$ denote the fraction of income saved, (89) implies:

$$s(r) = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \tag{90}$$

We can therefore simplify (89) as:

$$C_{1t} = [1 - s(r_{t+1})] A_t w_t \tag{91}$$

Equation (90) implies that young individuals' saving is increasing r if and only if $(1+r)^{\frac{1-\theta}{\theta}}$ is increasing in r. The derivative with respect to r is:

$$\left((1+r)^{\frac{1-\theta}{\theta}}\right) dr = \frac{1-\theta}{\theta}(1+r)^{\frac{1-2\theta}{\theta}} \tag{92}$$

Thus s is increasing in r if θ is less than 1, and decreasing if θ is greater than 1. Intuitively, a rise in r has both an income and substitution effect. The fact that the tradeoff between consumption in the two periods has become more favourable for second period consumption tends to increase saving (the substitution effect), but the fact that given amount of saving yields more second period consumption tends to decrease saving (the income effect).

Substitution effects dominate for low values of θ. When individuals have strong preferences for similar levels of consumption in the two periods (when θ) is high, the income effect dominates. If $\theta = 1$, as in the case of logarithmic utility, young individuals' saving rate is independent of r.

3.3 The Dynamics of the Economy

3.3.1 The Equation Motion of k

As in the Ramsey model, we can aggregate individuals' behaviour to characterise the dynamics of the economy. Consider the total capital stock in the economy in period $t+1$:

$$K_{t+1} = s(r_{t+1})L_t A_t w_t \tag{93}$$

Note that because saving in period t depends on labour income that period and on the return on capital that savers expect the next period, it is w in period t and r in period $t+1$ that enter the expression for the capital stock in period $t+1$. Dividing both sides of (93) by $L_{t+1}A_{t+1}$ gives us an expression for capital per unit of effective labour:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t \tag{94}$$

We can substitute for r_{t+1} and w_t to obtain:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)] \tag{95}$$

Figure 15: See Figure 2.11

3.3.2 The Evolution of k

Equation (95) implicitly defines k_{t+1} as a function of k_t. It therefore determines how k evolves over time given its initial value. A value of k_t such that $k_{t+1} = k_t$ satisfies (95) is a steady state value of k: once k reaches that value, it remains there.

To explore the behaviour of k, we need to move away from the general case and look at an example of logarithmic utility and Cobb-Douglas production.

3.3.3 Logarithmic Utility and Cobb-Douglas Production

When $\theta = 1$, the fraction of labour income saved is $1/(2+\rho)$. And when production is Cobb-Douglas, $f(k) = k^\alpha$ and $f'(k) = \alpha k^{\alpha-1}$. Equation (95) therefore becomes:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{(2+\rho)} (1-\alpha) k_t^\alpha \tag{96}$$

Plotting the dynamics of k such as in Figure 2.11, we can see that k will converge to k^* (we already ruled out that k_t cannot equal zero or any negative values). The properties of the economy once it has converged to the steady state is the same as in the Solow model and Ramsey model.: the saving rate is constant, output per worker is growing at rate g, the capital output ratio is constant and so on.

To see how the economy responds to shocks, consider our usual example of a fall in the discount rate, ρ, when the economy is initially in its steady state. The fall in the discount rate causes the young to save a greater fraction of their labour income. Thus the k_{t+1} function shifts up. This is depicted in Figure 2.12. Thus the effect of a fall in the discount rate in the Diamond model in the case we are considering are similar to its effects in the Ramsey model, and to the effects of a rise in the saving rate in the Solow model.

The change shifts the paths over time of output and capital per worker permanently up, but it leads only to temporary increases in the growth rates of these variables.

3.3.4 The Speed of Convergence

Once again, we may be interested in the model's quantitative as well as qualitative implications. In the special case we are considering, we can solved for the steady state values of k and y. Equation (96) gives k_{t+1} as a function of k_t. The economy is in the steady state when k_{t+1} and k_t are equal. That is, k^* is defined by:

$$k^* = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)k^{*\alpha} \tag{97}$$

Solving this expression for k^* yields:

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}} \tag{98}$$

Since y equals k^α, this implies:

$$y^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}} \tag{99}$$

This expression shows that the model's parameters affect output per unit of effective labour in the steady state.

We can also find how quickly the economy converges to the balanced growth path. To do this, we again linearise around the steady state. That is, we replace the equation of motion for k, (96), with a first order Taylor approximation around $k = k^*$. We know that when k_t equals k^*, k_{t+1} also equals k^*. Thus:

$$k_{t+1} \simeq k^* + \left(\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}\right)(k_t - k^*) \tag{100}$$

Let λ denote dk_{t+1}/dk_t evaluated at $k_t = k^*$. With this definition, we can write:

$$k_t - k^* \simeq \lambda^t (k_0 - k^*) \tag{101}$$

where k_0 is the initial value of k.

The convergence to the steady state is determined by λ. If λ is between 0 and 1, the system converges smoothly. If λ is between -1 and 0, there are damped oscillations toward k^*. If λ is greater than 1, the system explodes. Finally, if λ is less than -1, there are explosive oscillations. To find λ, we return to (96):

$$\begin{aligned}
\lambda \equiv \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} &= \alpha \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} k^{*\alpha-1} \\
&= \alpha \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{(\alpha-1)}{(1-\alpha)}} \\
&= \alpha
\end{aligned} \tag{102}$$

λ is simply α, capital's share of output.

The rate of convergence differs from that in the Solow model. The reason is that although the saving of the young is a constant fraction of their income and their income is a constant fraction of total income, the dissaving of the old is not a constant fraction of total income. The dissaving of the old as a fraction of output is $k_t/f(k_t)$. The fact that there are diminishing returns to capital implies that this ratio is increasing in k. Since this term enters negatively into saving, it follows that total saving as a fraction of output is a decreasing function of k. Thus total saving as a fraction of output is above its steady state value when $k < k^*$, and is below when $k > k^*$. As a result, convergence is more rapid than in the Solow model.

The Diamond model does not better than the Solow or Ramsey models at answering our basic questions about growth. Because of the Inada conditions, k_{t+1} must be less than k_t. Specifically, since the saving of the

Figure 16: See Figure 2.15

young cannot exceed the economy's total output, k_{t+1} cannot be greater than $f(k_t)/[(1+n)(1+g)]$. And because the marginal product of capital reaches zero as k becomes large, this must eventually be less than k_t. The fact that k_{t+1} is eventually less than k_t implies that unbounded growth of k is not possible. Thus, once again, growth in the effectiveness of labour is the only potential source of long run growth in output per worker. Because of the possibility of multiple k^*'s, the model does imply that otherwise identical economies can converge to different steady states simply because of differences in their initial conditions. But, as in the Solow and Ramsey models, we can account for quantitatively large differences in output per worker in this way only by positing immense differences in capital per worker and in rates of return.

3.4 Government in the Diamond Model

As in the Ramsey model, it is natural to ask what happens in the Diamond model if we introduce a government that makes purchases and levies taxes. For simplicity, we focus on the case of logarithmic utility and Cobb-Douglas production.

Let G_t denote the government's purchases of goods per unit of effective labour in period t. Assume it finances those purchases by lump-sum taxes on the young. When the government finances its purchases entirely with taxes, workers' after-tax income in period t is $(1-\alpha)k_t^\alpha - G_t$ rather than $(1-\alpha)k_t^\alpha$. The equation of motion for k, (96), therefore becomes:

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{(2+\rho)}[(1-\alpha)k_t^\alpha - G_t] \tag{103}$$

A higher G_t therefore reduces k_{t+1} for a given k_t. We can see the effects of this in Figure 2.15.

Thus, unlike the Ramsey model, higher government purchases lead to a lower capital stock and a higher real interest rate. Intuitively, since individuals live for two periods, they reduce their first period consumption less than one for one with the increase in G. But since taxes are levied only in the first period of life, tis means that their saving falls. As usual, the economy movies smoothly from the initial steady state to the one.

Temporary (which are known to be so) increases in government expenditure, however, do not change the capital stock once G returns back to its original levels.

4 Consumption

This chapter focuses on households' consumption choices. Consumption and investment are important to both growth and fluctuations. With regard to fluctuations, consumption and investment make up the majority of the demand for goods. Thus to understand how such forces affect aggregate output, we must study how consumption and investment are determined.

There are two other reasons for studying consumption and investment. First, they introduce some important issues involving financial markets. Financial markets affect the macroeconomy mainly through their impact on consumption and investment. In addition, consumption and investment have important feedback effects on financial markets. Second, much of the most insightful empirical work in macroeconomics in recent decades has been concerned with consumption and investment.

4.1 Consumption under Certainty: The Permanent Income Hypothesis

4.1.1 Assumptions

Although we have already looked at consumption in the Solow, Ramsey and Diamond models, here we start with a simple case. Consider an individual who lives for T periods whose lifetime utility is:

$$U = \sum_{t=1}^{T} u(C_t), \quad u'(C_t) > 0, \quad u''(C_t) < 0 \tag{104}$$

where $u(C_t)$ is the instantaneous utility function and C_t is consumption in period t. The individual has initial wealth of A_0 and labour incomes of $Y_1, Y_2, ..., Y_T$ in the T periods of his or her life. The individual can borrow and save at an exogenous interest rate, subject only to the constraint that any outstanding debt be repaid at the end of his or her life. For simplicity, this interest rate is set to 0. Thus the individual's budget constraint is:

$$\sum_{t=1}^{T} C_t \leq A_0 + \sum_{t=1}^{T} Y_t \tag{105}$$

4.1.2 Behaviour

Since the marginal utility of consumption is always positive, the individual satisfies the budget constraint with equality. The Lagrangian for his or her problem is therefore:

$$\mathcal{L} = \sum_{t=1}^{T} u(C_t) + \lambda \left(A_0 + \sum_{t=1}^{T} Y_t - \sum_{t=1}^{T} C_t \right) \tag{106}$$

The first order condition for C_t is:

$$u'(C_t) = \lambda \tag{107}$$

Since the above condition holds for every periods, the marginal utility of consumption is constant. And since the level of consumption uniquely determines its marginal utility, this means that consumption must be constant. Thus $C_1 = C_2 = ... = C_T$. Substituting this fact into the budget constraint yields:

$$C_t = \frac{1}{T} \left(A_0 + \sum_{\tau=1}^{T} Y_\tau \right), \quad \forall t \tag{108}$$

The term in parenthesis is the individual's total lifetime resources.

4.1.3 Implications

This analysis implies that the individual's consumption in a given period is determined not by income that period, but by income over his or her entire lifetime. In the terminology of Friedman (1957), the right hand side of (108) is permanent income, and the difference between current and permanent income is transitory income.

To see the importance of the distinction between permanent and transitory income, consider the effect of a windfall gain of amount Z in the first period of life. Although this windfall raises current income by Z, it raises permanent income by Z/T. One implication is that a temporary tax cut may have little impact on consumption, if the individual's horizon is fairly long.

Our analysis also implies that although the time pattern of income is not important to consumption, it is critical to saving. The individual's saving in period t is the difference between income and consumption:

$$\begin{aligned} S_t &= Y_t - C_t \\ &= \left(Y_t - \frac{1}{T} \sum_{\tau=1}^{T} Y_\tau \right) - \frac{1}{T} A_0 \end{aligned} \tag{109}$$

where the second line uses (108) to substitute for C_t. Thus saving is high when income is high relative to its average–that is, when transitionary income is high. Similarly, when current income is less than permanent income, saving is negative. Thus the individual uses saving and borrowing to smooth the path of consumption. This is the key idea of the permanent income hypothesis of Modigliani and Brumberg (1954) and Friedman (1957).

4.1.4 What is Saving?

At a more general level, the basic idea of the permanent income hypothesis is a simple insight about saving: saving is future consumption. As long as an individual does not sae just for the sake of saving, he or she saves to consume in the future.

This observation suggests that many common statements may be incorrect. For example, it is often asserted that poor individuals save a smaller fraction of their incomes than the wealthy do because their incomes are little above the level needed to provide a minimal standard of living. But this claim overlooks the fact that individuals who have trouble obtaining even a low standard of living today may also have trouble obtaining that standard in the future. Thus their saving is likely to be determined by the time pattern of their income, just as it is for the wealthy.

4.1.5 Empirical Application: Understanding Estimated Consumption Functions

The traditional Keynesian consumption function posits that consumption is determined by current determined disposable income. Keynes (1936) argued that the aggregate consumption was positively related to aggregate income. He claimed further that higher incomes would lead to a higher proportion of income being devoted to saving.

The importance of the consumption function to Keynes' analysis of fluctuations led many researchers to estimate the relationship between consumption and current income. Contrary to Keynes' claims, these studies did not demonstrate a consistent, stable relationship. Across households at a point in time, the relationship is indeed of the type that Keynes postulated; see for example panel (a) in Figure 8.1. But within a country over time, aggregate consumption is essentially proportional to aggregate income; that is; one sees a relationship like that in panel (b) in Figure 8.1.

As Friedman demonstrates, the permanent income hypothesis provides a straightforward explanation of all these findings. Suppose consumption is in fact determined by permanent income: $C = Y^P$. Current income equals the sum of permanent and transitory income: $Y = Y^P + Y^T$. And since transitory income reflects departures of current income from permanent income, in most samples it has a mean near zero and is roughly uncorrelated with permanent income.

The permanent income hypothesis predicts that the key determinant of the slope of an estimated consumption function is the relative variation in permanent and transitory income:

$$\hat{\beta}_Y = \frac{\mathrm{Var}(Y^P)}{\mathrm{Var}(Y^P) + \mathrm{Var}(Y^T)}$$

Intuitively, an increase in current income is associated with an increase in consumption only to the extent that it reflects an increase in permanent income. When the variation permanent income is much greater than the variation in transitory income, almost all differences in current income reflect differences in permanent income; thus consumption rises nearly one for one with current income. But when the variation in permanent income is small relative to the variation in transitory income, little of the variation in current income comes from variation in permanent income, and so consumption rises little with current income.

4.2 Consumption under Uncertainty: The Random Walk Hypothesis

4.2.1 Individual Behaviour

We now extend our analysis to account for uncertainty. Suppose there is uncertainty about the individual's labour income each period. Continue to assume that the interest rate and the discount rate are zero. In addition, suppose that the instantaneous utility function, $u(\centerdot)$, is quadratic. Thus the individual maximises:

$$E[U] = E\left[\sum_{t=1}^{T}\left(C_t - \frac{a}{2}C_t^2\right)\right], \quad a > 0 \tag{110}$$

However, we assume the same budget as before:

$$\sum_{t=1}^{T} C_t \leq A_0 + \sum_{t=1}^{T} Y_t$$

To describe the individual's behaviour, we use our usual Euler equation approach. Assuming optimising behaviour on behalf of the individual, consider a reduction in C_1 of dC from the value the individual initially chose and an equal increase in consumption at some future date from the value he or she would have chosen. If the individual is optimising, a marginal change of this type does not affect expected utility. Let $E_1[\centerdot]$ denote expected utility conditional on information known in period 1. Thus:

$$1 - aC_1 = E_1[1 - aC_1], \quad t = 2, 3, ..., T. \tag{111}$$

This is because marginal utility of consumption in period 1 is $1 - aC_1$, and so the change in consumption has a utility cost of $(1 - aC_1)dC$. But there's also a future utility gain of $E_1[1 - aC_1]dC$.

Since $E_1[1 - aC_1]$ equals $1 - aE_1[C_1]$, this implies:

$$C_1 = E_1[C_t],] \quad t = 2, 3, ..., T. \tag{112}$$

The individual knows that his or her lifetime consumption will satisfy the budget constraint (105) with equality. Thus the expectations of the two sides of the constraint must equal:

$$\sum_{t=1}^{T} E_1[C_t] = A_0 + \sum_{t=1}^{T} E_1[Y_t] \tag{113}$$

Equation (112) implies that the left hand side of (113) is TC_1. Substituting this into (113) and dividing by T yields:

$$C_1 = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} E_1[Y_t]\right) \tag{114}$$

That is, the individual consumes $1/T$ of his or her expected lifetime resources.

4.2.2 Implications

Equation (112) implies that the expectation as of period 1 of C_2 equals C_1. More generally, we imply that each period, expected next period consumption equals current consumption. This implies that changes in consumption are unexpected shocks. By the definition of expectations, we can write:

$$C_t = E_{t-1}[C_t] + e_t \tag{115}$$

where e_t is a variable whose expectation as of period $t-1$ is zero. Thus, since $E_{t-1}[C_t] = C_{t-1}$, we have:

$$C_t = C_{t-1} + e_t \tag{116}$$

This is Hall's famous result that the permanent income hypothesis implies that consumption follows a random walk (Hall, 1978). The intuition for this result is straightforward: if consumption is expected to change, the individual can do a better job of smoothing consumption. Suppose for example that the current marginal utility of consumption is greater than the expected future marginal utility of consumption, and thus that the individual is better raising current consumption. Thus the individual adjusts his or her current consumption to the point where consumption is not expected to change.

Note that this analysis shows that quadratic utility is the source of certainty equivalence behaviour: if utility is not quadratic, marginal utility is not linear, and this results in consumption in period t not equalling expected consumption in future periods.

4.3 Empirical Application: Two Tests of the Random Walk Hypothesis

Hall's random walk result ran strongly counter to existing views about consumption. The traditional view of consumption over the business cycle implies that when output declines, consumption declines but is expected to recover; thus it implies that there are predictable movements in consumption. Hall's extension of the permanent income hypothesis, in contrast, predicts that when output declines unexpectedly, consumption only declines by the amount of the fall in permanent income; as a result; it is not expected to recover.

4.3.1 Campbell and Mankiw's Test Using Aggregate Data

Campbell and Mankiw (1989) used an instrumental variable approach to test Hall's hypothesis against a specific alternative. The alternative they consider is that some fraction of consumers simply spend their current income, and the remainder behave according to Hall's theory. They find that lagged changes in income have almost no predictive power for future changes. This suggests that Hall's failure to find predictive power of lagged income movements for consumption is not strong evidence against the traditional view of consumption.

Thus the Campbell and Mankiw estimates suggest quantitatively large and statistically significant departures from the predictions of the random walk model: consumption appears to increase by about 50 cents in response to an anticipated one dollar increase in income, and the null hypothesis of no effect is strongly rejected. At the same time, the estimates are far below 1. Thus the results also suggest that the permanent income hypothesis is important to understanding consumption.

4.4 The Interest Rate and Saving

An important issue concerning consumption involves its response to rates of return. For example, many economists believe by offering tax incentives on interest income, it would encourage more individuals to save. However, if individuals are unresponsive to rates of return, tax breaks are meaningless.

4.4.1 The Interest Rate and Consumption Growth

Returning back to consumption under certainty, we now allow for a nonzero interest rate, and a nonzero discount rate. This largely repeats material from the Diamond model, so our analysis will be brief here. The budget constraint for an individual is given by:

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t} C_t \leq A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t} Y_t \tag{117}$$

where r is the real interest rate, and the constraint satisfies the conditions we established earlier; that the present value of lifetime consumption cannot exceed initial wealth plus the present value of lifetime labour income.

By factoring in a nonzero subjective discount factor, ρ, we now express the individual's utility function (a CRRA utility function) as:

$$U = \sum_{t=1}^{T} \frac{1}{(1+\rho)^t} \frac{C_t^{1-\theta}}{1-\theta} \tag{118}$$

Now consider our usual experiment of a decrease in consumption in some period, period t, accompanied by an increase in consumption in the next period by $1+r$ times the amount of the decrease. Optimisation requires that marginal change of this type has no effect on lifetime utility. Since marginal utilities in periods t and $t+1$ are $C_t^{-\theta}/(1+\rho)^t$ and $C_{t+1}^{-\theta}/(1+\rho)^{t+1}$, this condition is:

$$\frac{1}{(1+\rho)^t} C_t^{-\theta} = (1+r) \frac{1}{(1+\rho)^{t+1}} C_{t+1}^{-\theta} \tag{119}$$

We can rearrange this to obtain the Euler equation for this problem:

$$\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\rho} \right)^{\frac{1}{\theta}} \tag{120}$$

This analysis implies that once we account for the possibility that the real interest rate and the discount rate are not equal, consumption need not be

Figure 17: See Figure 8.2

a random walk: consumption is rising over time if r exceeds ρ and falling if r is less than ρ.

We can model this problem such as in Figure 8.2.

5 Investment

This chapter investigates the demand for investment. We begin by presenting a baseline model of investment where firms face a perfectly elastic supply of capital goods and can adjust their capital stocks costlessly. We will see hat this model, even though it is a natural one to consider, provides little insight into actual investment. It implies, for example, that discrete changes in the economic environment produces infinite rates of investment or disinvestment.

We then develop the q theory model of investment. The model's key assumption is that firms face costs of adjusting their capital stocks. As a result, the model avoids the unreasonable implications of the baseline case and provides a useful framework for analysing the effects that expectations and current conditions have on investment.

5.1 Investment and the Cost of Capital

5.1.1 The Desired Capital Stock

Consider a firm that can rent capital at a price of r_K. The firms' profits at a point in time are given by $\pi(K, X_1, X_2, ..., X_n) - r_K K$, where K is the amount of capital the firm rents and the X's are variables that it takes as given. In the case of a perfectly competitive firm, for example, the X's include the price of the firm's product and the costs of other inputs. $\pi(\centerdot)$ is assumed to account for whatever optimisation the firm can do on dimensions other than its choice of K. For a competitive firm, for example, $\pi(\centerdot) - r_K K$ gives the firm's profits at the profit maximising choices of inputs other than capital given K and the X's. We assume that $\pi_K > 0$ and $\pi_{KK} < 0$.

The first order condition for profit maximising choice of K is:

$$\pi_K(\centerdot) = r_K \tag{121}$$

That is, the firm rents capital up to the point where its marginal revenue product equals its rental price.

We can show that K is decreasing in r_K. A similar analysis can be used to find the effects of changes in the X's on K.

5.1.2 The User Cost of Capital

More capital is not rented by is owned by the firms that use it. Thus there is no clear empirical counterpart of r_K. This difficulty has given rise to a large literature on the user cost of capital.

Consider a firm that owns a unit of capital. Suppose the real market price of the capital at timet is $p_K(t)$, and consider the firm's choice between selling the capital and continuing to use it. Keeping the capital has three costs to the firm. First, the firm foregoes the interest it would receive if it sold the capital and saved the proceeds. This has a real cost of $r(t)p_K(t)$ per unit time, where $r(t)$ is the real interest rate. Second, the capital is depreciating. This has a cost of $\delta p_K(t)$ per unit time. And third, the price of the capital may be changing. This increases the cost of it using the capital if the price is falling (since the firm obtains less if it waits to sell the capital(and decreases the cost if the price is rising. This has a unit cost of $-\dot{p}_K(t)$ per unit time. Putting the three components together yields the user cost of capital:

$$\begin{aligned} r_K(t) &= r(t)p_K(t) + \delta p_K(t) - \dot{p}_K(t) \\ &= \left[r(t) + \delta - \frac{\dot{p}_K(t)}{p_K(t)} \right] p_K(t) \end{aligned} \tag{122}$$

This analysis ignores taxes. In practice, however, the tax treatments of investment and of capital income have large effects on the user cost of capital. Accounting for an investment tax credit may yield the following expression:

$$r_K(t) = \left[r(t) + \delta - \frac{\dot{p}_K(t)}{p_K(t)} \right] (1 - f\tau) p_K(t) \tag{123}$$

Here, $f\tau$ denotes the fraction of the marginal corporate tax rate. Thus investment tax credits reduce the user cost of capital, and hence increases firms' desired capital stocks.

5.1.3 Difficulties with the Baseline Model

This simple model of investment has at least two major failings as a description of actual behaviour. The first concerns the impact of changes in the exogenous variables. Our model concerns firms' demand for capital, and it implies that firms' desired capital stocks are smooth functions of the exogenous variables. As a result, a discrete change in an exogenous variable leads to a discrete change in the desired capital stock. Suppose, for example, that the Federal Reserve reduces interest rates by a discrete amount. This discretely reduces the cost of capital, r_K. This in turn means that the capital stock that satisfies (121) ruses discretely.

The problem is that since the rate of change of the capital stock equals investment minus depreciation, a discrete change in the capital stock requires an infinite rate of investment. For the economy as a whole, however, investment is limited by the economy's output; thus aggregate investment cannot be infinite.

The second problem with the model is that it does not identify any mechanism through which expectations affect investment demand. The model implies that firms equate the current marginal revenue product of capital with its current user cost, without regard to what they expect future marginal revenue products or user costs to be.

5.2 A Model of Investment with Adjustment Costs

We now turn to a model of investment with adjustment costs. For concreteness, the adjustment costs are assumed to be internal.; it is straightforward, however, to reinterpret the model as one of external adjustment costs. The model is known as the q theory model of investment.

5.2.1 Assumptions

Consider an industry with N identical firms. A representative firm's real profits at time t, are proportional to its capital stock, $\kappa(t)$, and decreasing in the industry wide capital stock, $K(t)$; thus they take the form $\pi(K(t))\kappa(t)$, where $\pi'(.) < 0$. The assumption of proportional profits are reasonable if the production function exhibits CRS, markets are competitive and the supply of all factors other than capital is perfectly elastic.

The key assumption of this model is that firms face costs of adjusting their capital stocks. The adjustment costs are a convex function of the rate of change in the firm's capital stock, $\dot{\kappa}$. Specifically, the adjustment costs, $C(\dot{\kappa})$, satisfy $C(0) = 0$, $C'(0) = 0$ and $C''(.) > 0$. This implies that it is costly for the firm to increase or decrease its capital stock.

The purchase price of capital goods is constant and equal to 1; thus there are no external adjustment costs. Finally, for simplicity, the depreciation rate is assumed to be zero. It follows that $\dot{\kappa}(t) = I(t)$, where I is the firm's investment.

This implies that the firm's profits at a point in time are $\pi(K)\kappa - I - C(I)$. The firm maximises the present value of these profits:

$$\Pi = \int_{t=0}^{\infty} e^{-rt}[\pi(K(t))\kappa(t) - I(t) - C(I(t))]dt \tag{124}$$

where we assume for simplicity that the real interest rate is constant. Each firm takes the path of the industry wide capital stock, K, as given.

5.2.2 A Discrete Time Version of the Firm's Problem

In discrete time the firm's objective function is:

$$\tilde{\Pi} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}[\pi(K_t)\kappa_t - I_t - C(I_t)] \tag{125}$$

Where assume that $\kappa_t = \kappa_{t-1} + I_t$ for all t. We can think of the firm as choosing investment and capital stock each period subject to the constraint $\kappa_t = \kappa_{t-1} + I_t$ for each t. Since there are infinitely many periods, there are infinitely many constraints. The Lagrangian for the firm's maximisation problem is therefore:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^r} [\pi(K_t)\kappa_t - I_t - C(I_t)] + \sum_{t=0}^{\infty} \lambda_t(\kappa_{t-1} + I_t - \kappa_t) \tag{126}$$

5.3 Tobin's q

Our analysis of the firm's maximisation problem implies that q summarises all information about the future that is relevant to a firm's investment decision. q shows how an additional dollar of capital affects the present value of profits. Thus the firm wants to increase its capital stock if q is high and reduce it if q is low; the firm does not need to know anything about the future other than the information that is summarised in q in order to make this decision.

We know that q is the present discounted value of the future marginal revenue products of a unit of capital. In the continuous time case, we can therefore express q as:

$$q(t) = \int_{\tau=t}^{\infty} e^{-r(\tau-t)} \pi(K(\tau)) d\tau \tag{127}$$

There is another interpretation of q. A unit increase in the firm's capital stock inreases the present vaue of the firm's profits by q, and thus raises the value of the firm by q. Thus q is the market value of a unit of capital. If there is a market for shares in firms, for example, the total value of a firm with one more unit of capital than another firm exceeds the value of the other by q. And since we have assumed that the purchase price of capital is fixed at 1, q is also the ratio of the market value of a unit of capital to its replacement cost.

The ratio of the market value to the replacement cost of capital is known as Tobin's q. Our analysis implies what is relevant to investment is marginal q– the ratio of the market value of a marginal unit of capital to its replacement cost.