2025年度 横浜国立大学 理工学部 数理科学EP 卒業研究

Homomorphic Expansions for Knotted Trivalent Graphs

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Abstract

【保留】KTGs に対し a universal Vassiliev invariant が存在することは知られていた [MO97, CL07, Dan10]．KTGs において "edge unzip" という操作のみ準同型にならず，補正項が現れる．dotted Knotted Trivalent Graphs において Z^{old} が準同型となるように Z を 2 通りで構成することが目的.

It has been known since old times [MO97, CL07, Dan10] that there exists a universal finite type invariant Z^{old} for Knotted Trivalent Graphs. While the behavior of Z^{old} under edge unzip is well understood, it is not plainly homomorphic as some "correction factors" appear.

In this paper we modify Z^{old} into a new expansion Z, defined on "dotted Knotted Trivalent Graphs", which is homomorphic with respect to a large set of operations.

Contents

1　Introduction

結び目理論とは位相幾何学の分野の一つであり，物理学とも関係する分野である．その中でも，結び目同士が異なるかどうかを区別する際に手段として使われるものとして結び目の不変量というものがある．

2　Acknowledgements

3 Preliminaries

3.1 KTGs and Z^{old}

If we consider graphs, all edges are oriented, and vertices be equipped with a cyclic orientation, and loops are allowed.

Definition 3.1. A *surface* is a compact, oriented 2-dimensional manifold with boundary that satisfies the second-countable axiom.

Definition 3.2. A *spine* of a simplicial complex Y is a subcomplex X of Y onto which Y collapses, where collapsing means successively removing finite number of pairs of a k-simplex Δ^k and a $(k+1)$-simplex Δ^{k+1} which is the unique $(k+1)$-simplex having Δ^k on its boundary.

Definition 3.3. For a graph Γ, a *framed graph* $\mathbf{\Gamma}$ is a pair (Γ, Σ) of 1-dimensional simplicial complex Γ and an embedding $\Gamma \hookrightarrow \Sigma$ of Γ into a surface Σ as a spine. In particular, when Γ is a trivalent graph, it is called a *framed trivalent graph*.

We regard two framed graphs (Γ, Σ) and (Γ, Σ') as equivalent if there exists an orientation-preserving diffeomorphism $h \colon \Sigma \to \Sigma'$ such that $h|_\Gamma = \mathrm{id}_\Gamma$. We consider framed graphs up to this equivalence. Under a fixed cyclic ordering of edges at each vertex, the framed graph is uniquely determined up to this equivalence.

Definition 3.4. For a framed trivalent graph $\mathbf{\Gamma} = (\Gamma, \Sigma)$, a *knotted trivalent graph (KTG)* γ is a triple (Γ, Σ, g) consisting of the framed graph Γ and an embedding $g \colon \Sigma \hookrightarrow \mathbb{R}^3$. The *skeleton* of a KTG γ is the trivalent graph Γ behind it.

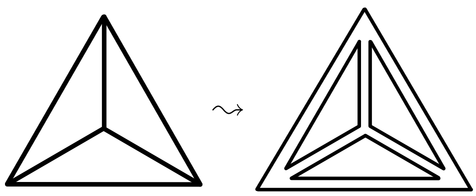


Figure 1: An image of a framed trivalent graph

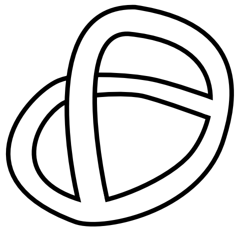


Figure 2: Example of a knotted trivalent graph

Definition 3.5. Two knotted trivalent graphs $\gamma_1 = (\Gamma, \Sigma, g)$ and $\gamma_2 = (\Gamma, \Sigma, h)$ are said to be *framed isotopic* if there exists a smooth embedding

$$\Phi \colon \Sigma \times I \to \mathbb{R}^3 \times I \quad (I = [0,1])$$

satisfying the following conditions:

(i) Φ is level-preserving, that is, for any $x \in \Sigma$ and $t \in I$, $\Phi(x,t) = (\varphi_t(x), t)$ for a smooth embedding $\varphi_t \colon \Sigma \hookrightarrow \mathbb{R}^3$,

(ii) The restriction $\Phi|_{\Gamma \times I}$ is an identity,

(iii) $\varphi_0 = g$ and $\varphi_1 = h$.

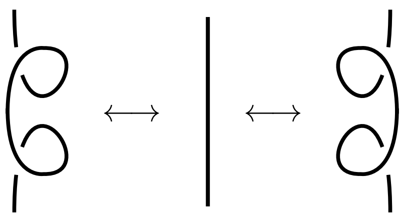
We identify KTGs whose are framed isotopic. For a trivalent graph Γ, we denote the vector space over $\mathbb{Q}$ generated by all linear combinations of KTGs with skeleton Γ by

$$\mathcal{K}(\Gamma) \coloneqq \left\{ \sum_{i=1}^{m} a_i \gamma_i \,\middle|\, m \in \mathbb{Z}_{>0}, a_i \in \mathbb{Q}, \gamma_i \text{ is a knotted trivalent graph.} \right\}.$$

Definition 3.6. For a graph Γ, a *chord diagram* D with support Γ is Γ together with an vertex-oriented uni-trivalent graph whose univalent vertices are on Γ; and the graph does not have any connected component homeomorphic to a circle. We call the uni-trivalent graph the chord graph of the diagram.

Proposition 3.7. *Two KTGs are framed isotopic if and only if their graph diagrams are related by a finite sequence of Reidemeister moves $R1', R2, R3$ and $R4$.*

$R1'$:



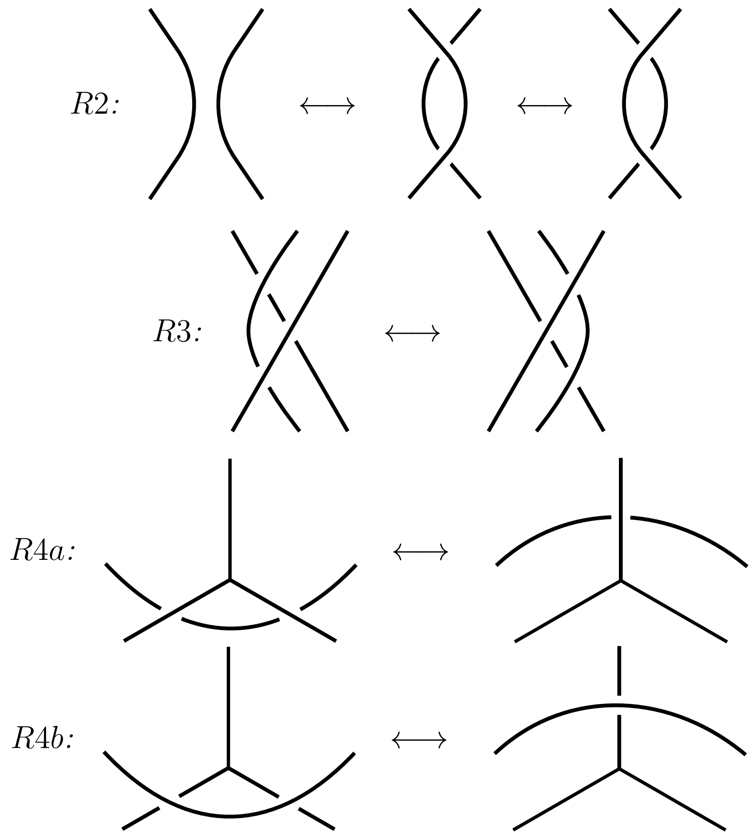


Figure 3: Reidemeister moves on knotted trivalent graphs

We omit the proof here. For details, please refer to [MO97, Theorem 1.4]. It is sufficient to show the invariance under extended Reidemeister moves for spatial graphs defined in [Yam87]. Note that we do not need the move in [Yam87] which changes the order of edges around a vertex, since we consider framed graphs with the blackboard framing.

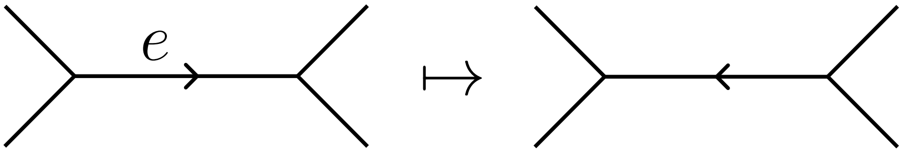
ここにつなぎの文が欲しい

There are four operations on KTGs:

4つの操作の説明が全部似たような文章になっている (defined in the same way 等) ため工夫したい

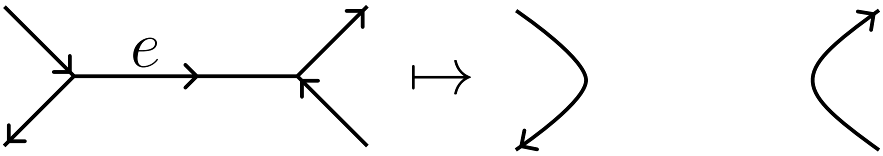
Definition 3.8. Let Γ be a trivalent graph and let e be an edge of Γ. We denote by $S_e(\Gamma)$ the graph obtained from Γ by reversing the orientation of the edge e. For a knotted trivalent graph $\gamma \in \mathcal{K}(\Gamma)$, the operation of *switch the orientation* of e is defined by reversing its orientation, and is denoted by S_e.

$$S_e\colon \mathcal{K}(\Gamma) \to \mathcal{K}(S_e(\Gamma))\,;\ \gamma \mapsto S_e(\gamma)$$



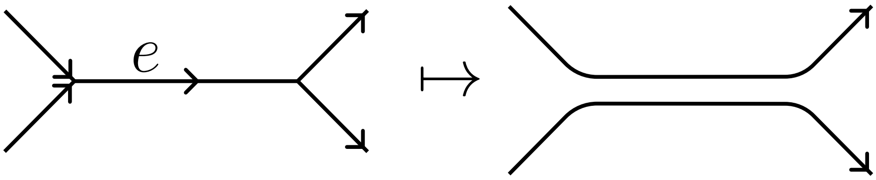
Definition 3.9. Let Γ be a trivalent graph and let e be an edge of Γ. We denote by $d_e(\Gamma)$ the graph obtained from Γ by removing e and th two vertices at the ends of e and smoothing the two resulting bivalent vertices into single edges. To do this, it is required that the orientations of the two edges connecting to e at either end match. For a knotted trivalent graph $\gamma \in \mathcal{K}(\Gamma)$, the operation of *delete* of e is defined in the same way, and is denoted by d_e.

$$d_e\colon \mathcal{K}(\Gamma) \to \mathcal{K}(d_e(\Gamma))\,;\, \gamma \mapsto d_e(\gamma)$$



Definition 3.10. Let Γ be a trivalent graph and let e be an edge of Γ. We denote by $u_e(\Gamma)$ the graph obtained from Γ by replacing e with two edges that are "very close to each other", and the two vertices at the ends of e will disappear. Again the edges at the vertex where e begins have to both be incoming, while the edges at the vertex where e ends must both be outgoing. For a knotted trivalent graph $\gamma \in \mathcal{K}(\Gamma)$, the operation of *unzip* of e is defined in the same way, and is denoted by u_e.

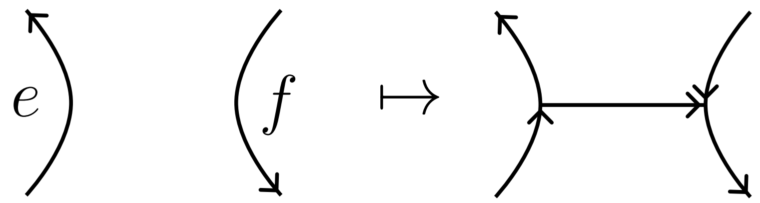
$$u_e\colon \mathcal{K}(\Gamma) \to \mathcal{K}(u_e(\Gamma))\,;\, \gamma \mapsto u_e(\gamma)$$



Definition 3.11. Let $(\Gamma, e), (\Gamma', f)$ be two pairs of trivalent graph and their edge. We denote by $\Gamma \#_{e,f} \Gamma'$ the graph obtained by joining e and f by a new edge. For this to be well-defined, the new edge be oriented from Γ towards Γ', and we also need to specify the direction of the new edge, the cyclic orientations at each new vertex. Let $\gamma \in \mathcal{K}(\Gamma)$ and $\gamma' \in \mathcal{K}(\Gamma')$. For two pairs

$(\gamma, e), (\gamma', f)$, the operation of *connected sum* is defined in the same way, and is denoted by $\#_{e,f}$. To compress notation, let us declare that the new edge be oriented from γ towards γ', have no twists, and, using the blackboard framing, be attached to the right side of e and f.

$$\#_{e,f} \colon \mathcal{K}(\Gamma) \times \mathcal{K}(\Gamma') \to \mathcal{K}(\Gamma \#_{e,f} \Gamma')$$



We define finite type invariants of KTGs in the same way as for links. In detail, we filter the resulting vector space by the resolution of "singular points".

Definition 3.12. An *n-singular KTG* is a trivalent graph immersed in $\mathbb{R}^3$ with n singular points: each singular point is a transverse double point or a point on an edge marked with an "F". ←こいつは何者か？

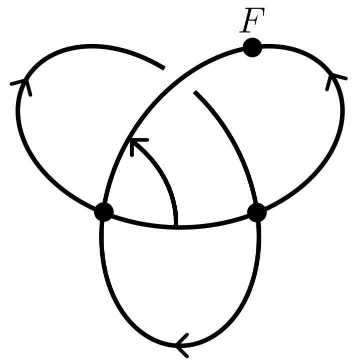
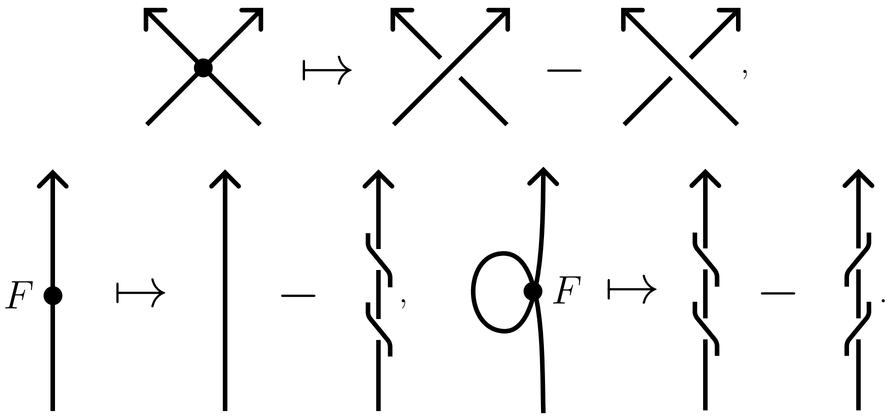


Figure 4: Example of an n-singular KTG.

For $n \geq 0$, we consider the following vector space:

$$\mathcal{F}'_n(\Gamma) := \left\{ \sum_{i=1}^{m} a_i \gamma_i \;\middle|\; \begin{array}{l} m \in \mathbb{Z}_{>0}, a_i \in \mathbb{Q}, \gamma'_i \colon \text{KTG which has} \\ \quad \text{at least } n \text{ double point with skeleton } \Gamma \end{array} \right\}$$

We define a map $\rho \colon \mathcal{F}'_*(\Gamma) \to \mathcal{F}_0(\Gamma)$ that resolves all singular points as follows:



ひねりの説明未完 For each $\mathcal{F}'_n(\Gamma)$ $(n \geq 0)$, we define $\mathcal{F}_n(\Gamma) \coloneqq \rho(\mathcal{F}'_n(\Gamma))$. Then, obviously, $\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma)$, and we obtain the following filtration:

$$\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma) \supset \mathcal{F}_1(\Gamma) \supset \mathcal{F}_2(\Gamma) \supset \mathcal{F}_3(\Gamma) \cdots .$$

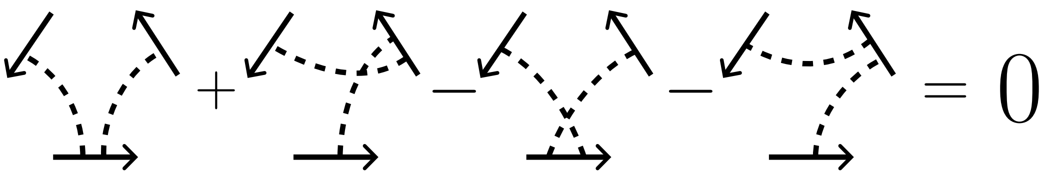
In this filtration, we denote the quotient vector space obtained from two adjacent vector spaces by $\mathcal{A}_n(\Gamma) \coloneqq \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma)$, and define the associated graded space as follows:

$$\mathcal{A}(\Gamma) \coloneqq \bigoplus_{n=0}^{\infty} \mathcal{A}_n(\Gamma) \left(= \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma) \right)$$

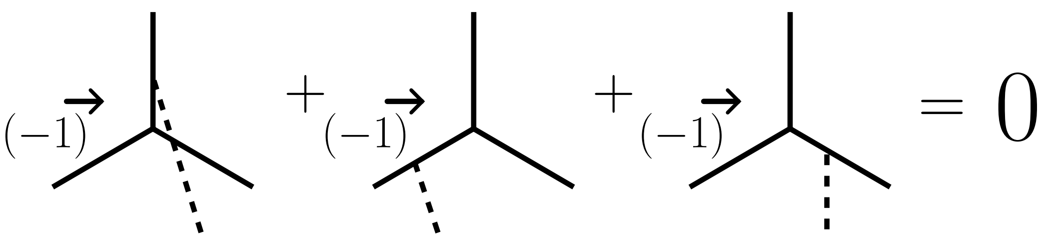
We denote by $\mathcal{D}_n(\Gamma)$ the vector space over $\mathbb{Q}$ generated by chord diagrams of order n, and set $\mathcal{D}(\Gamma) \coloneqq \bigoplus_n \mathcal{D}_n(\Gamma)$. There is a natural surjection π from $\mathcal{D}(\Gamma)$ to $\mathcal{A}(\Gamma)$ by associating the chords in a chord diagram to double points.

We call the following relations the $4T$ and VI relations in $\mathcal{D}(\Gamma)$:

- $(4T)$ Four term relation



- (VI) Vertex invariance relation



In both pictures, there may be other chords in the parts of the graph not shown, but they have to be the same throughout. In $4T$, all skeleton parts (solid lines) are oriented counterclockwise. In VI, the sign $(-1)^{\rightarrow}$ is -1 if the edge the chord is ending on is oriented to be outgoing from the vertex, and $+1$ if it is incoming (thus there are 8 versions of the relation).

Theorem 3.13. *The relations* $4T$ *and* VI *are contained in* $\ker \pi$*.*

The proof of this theorem is given in Appendix A.

$4T$, VI が kernel に含まれることは分かったが，これ以上の relations が存在 “しない” ことを示すのは困難である．これを示すには，universal finite type invariant $\mathbb{Q}\mathrm{KTG} \to \mathcal{A}$ を構成するのが最善である (ここでは定義しないが，後で一般の文脈で定義する)．これは，T.Le, H.Murakami, J. Murakami, T.Ohtsuki の結果をもとに，また Drinfeld の associator の理論を用いて [KO, CD],[BN] での Kontsevich integral を拡張する形で [MO97] で初めて得られた．
Although it is easy to see that these relations are contained in kernel, showing that there are no more is difficult, and is best achieved by constructing a universal finite type invariant $\mathbb{Q}\mathrm{KTG} \to \mathcal{A}$ (we do not define universal finite type invariants here, but will do so later in the general context).

Each operation on KTGs induces an operation on $\mathcal{A}$ (the associated graded space of $\mathcal{K}(\Gamma)$).

- orientation switch

図

- edge delete

図

- edge unzip

図

- connected sum well-defined である．Introduction to Vassiliev knot invariants(Chmutov) の Lemma4.2.9

図

Theorem 3.14. *Any KTG can be obtained from the trivially embedded tetrahedron and the twisted tetrahedron by a finite sequence of the four operations defined above.* 引用文献書け

Proof. aa

図

□

Theorem 3.15. *Any n-singular KTG can be obtained from the trivially embedded tetrahedron, twisted tetrahedron and singular twisted tetrahedron using the four operations.*

Proof. Same as Theorem 3.14. □

3.2 Algebraic structures and expansions

By linearly extending the operations of orientation switch, edge delete, edge unzip and connected sum on $\mathcal{K}$ to allow linear combinations with coefficients in $\mathbb{Q}$, $\mathcal{K}$ becomes a vector space.

Definition 3.16. Let Γ be a KTG. Let $\mathcal{I}(\Gamma)$ be the sub-structure made out of all such combinations in which the sum of coefficients is 0, and let $\mathcal{I} := \bigoplus_{\Gamma'} \mathcal{I}(\Gamma')$.

Example 3.17. Let γ_1, γ_2 and γ_3 be KTGs with skeleton Γ. Then, $\gamma_1 - \gamma_2, \gamma_1 - \frac{1}{2}\gamma - \frac{1}{2}\gamma_3 \in \mathcal{I}(\Gamma)$.

Definition 3.18. Let $\mathcal{I}^m$ be the set of all outputs of arbitrary compositions of the operations in $\mathcal{K}$ that have at least m inputs in $\mathcal{I}$. In other words,

$$\mathcal{I}^m := \left\{ \gamma \in \mathcal{K} \;\middle|\; \begin{array}{c} \text{There exist } n,\ f\colon \prod_{i=1}^{n} \mathcal{K} \to \mathcal{K},\ x_1, \ldots, x_n \in \mathcal{K} \\ \text{such that } \gamma = f(x_1, \ldots, x_n), \#\{i \mid x_i \in \mathcal{I}\} \geq m. \end{array} \right\}.$$

Moreover, we define $\mathcal{I}^m(\Gamma) := \mathcal{I}^m \cap \mathcal{I}(\Gamma)$.

Clearly, $\mathcal{I}^m$ has a filtration structure.

Lemma 3.19. $\mathcal{I}(\Gamma) = \{\sum_i c_i(\gamma_i - \gamma_i') \mid \gamma_i, \gamma_i'\colon$ *generators of* $\mathcal{K}(\Gamma), c_i \in \mathbb{Q}\}$.

Proof. ($\supset$) Each coefficient of $c_i(\gamma_i - \gamma_i')$ is 0, so the sum of coefficients is also 0.
($\subset$) For any element of $\mathcal{I}$, it can be written as $\sum_{i=1}^{n} c_i\gamma_i$. Since $\sum_{i=1}^{n} c_i = 0$, we have $c_n = -\sum_{i=1}^{n-1} c_i$. Thus

$$\sum_{i=1}^{n} c_i\gamma_i = c_1\gamma_1 + c_2\gamma_2 + \cdots + \left(-\sum_{i=1}^{n-1} c_i\right)\gamma_n = \sum_{i=1}^{n-1} c_i(\gamma_i - \gamma_n).$$

□

Theorem 3.20. *$\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ for all $n \geq 0$ and skeleton Γ.*

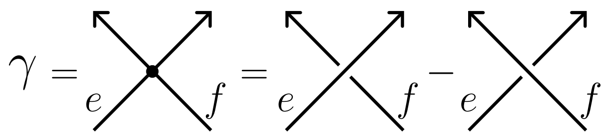
Proof. (i) $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$

($\supset$) Since any element $\gamma \in \mathcal{F}_1(\Gamma)$ has at least one double point, so there exist $\gamma_+, \gamma_- \in \mathcal{K}(\Gamma)$ such that $\gamma = \gamma_+ - \gamma_-$, thus $\mathcal{F}_1(\Gamma) \subset \mathcal{I}(\Gamma)$.
($\subset$) Any element of $\mathcal{I}(\Gamma)$ can be written as $\sum_i c_i(\gamma_i - \gamma_i')$ by Lemma 3.19. In $\mathcal{F}_1(\Gamma)$, any two KTGs with the same skeleton can be related by crossing changes, so we can make $\gamma_i - \gamma_i'$ into the difference of positive and negative crossings at one point $\tilde{\gamma}_i - \tilde{\gamma}_i'$. Therefore,

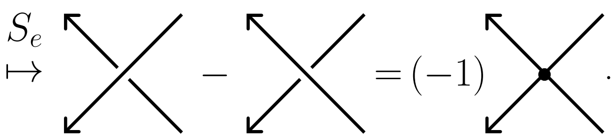
$$\sum_i c_i(\gamma_i - \gamma_i') = \sum_i c_i(\tilde{\gamma}_i - \tilde{\gamma}_i') \in \mathcal{F}_1(\Gamma).$$

(ii) $\mathcal{I}^n(\Gamma) \subset \mathcal{F}_n(\Gamma)$

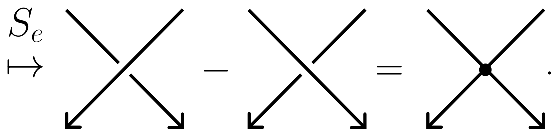
By $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$, any element $\gamma \in \mathcal{I}^n(\Gamma)$ is generated by at least n elements of $\mathcal{F}_1(\Gamma)$. It is enough to check that the four operations for an edge with double points preserve number of double points. In the case of orientation switch,



If e does not connect to f,

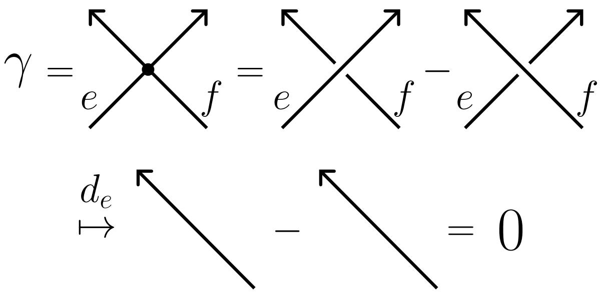


If e connects to f,



In both cases, the number of double points does not change.

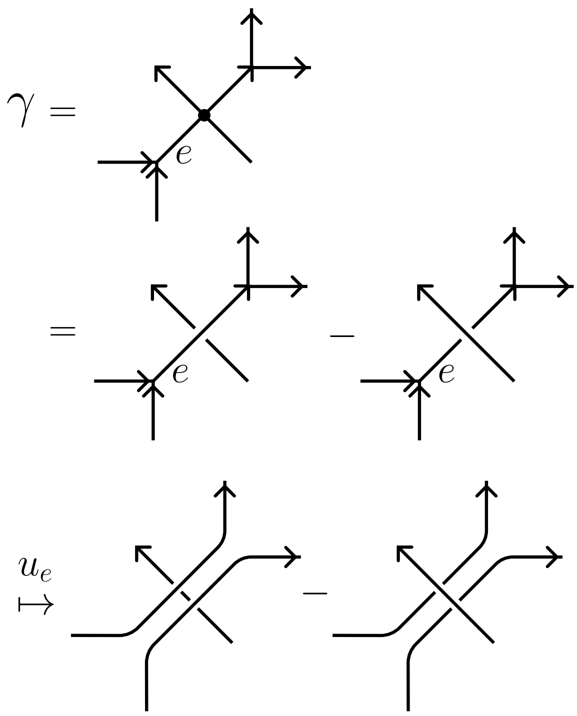
In the case of edge delete,

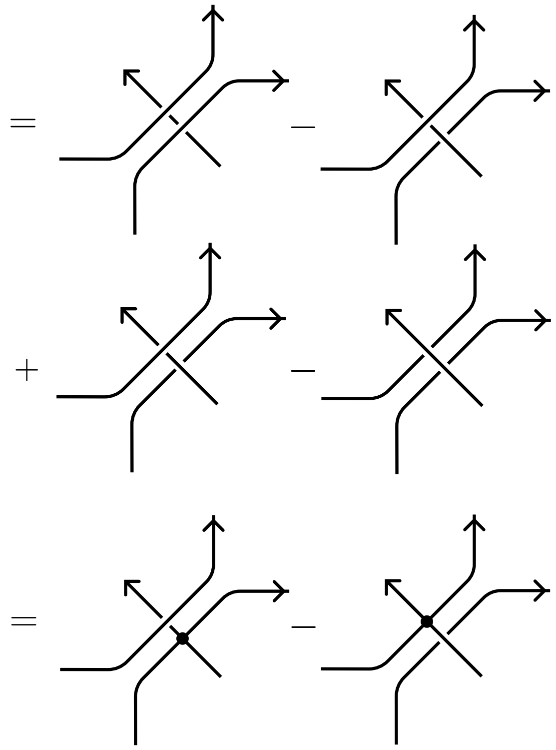


Both cases, if e connects to f or not, $d_e(\Gamma) = 0$. 0 is included in $\mathcal{F}_n$ for all $n \geq 0$, so the number of double points does not change.

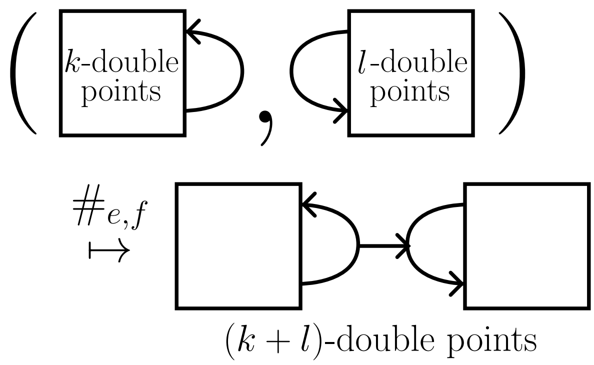
In the case of edge unzip,

We prove with a technical method by adding and subtracting same diagrams as follows:





In the case of connected sum,



(iii) $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$ Since any n-singular KTG can be obtained from n pieces of 1-singular KTGs and the four operations by Theorem 3.15, $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$.

Therefore, $\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ for all $n \geq 0$ and skeleton Γ. $\square$

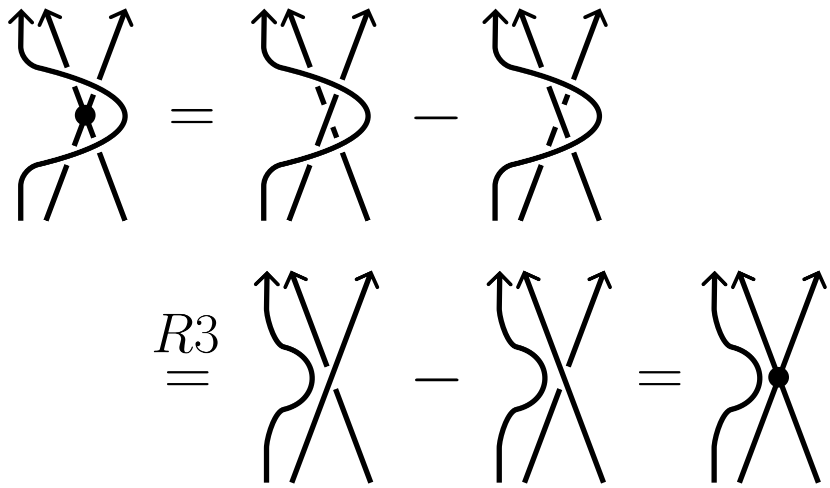
Definition 3.21. Let Γ, $\mathcal{K}(\Gamma)$ be a skeleton and the set of KTGs with skeleton Γ. An *expansion* Z for $\mathcal{K}(\Gamma)$ is a map $Z\colon \mathcal{K}(\Gamma) \to \hat{\mathcal{A}}(\Gamma) = \prod_{n=0}^{\infty} \mathcal{A}_n(\Gamma)$ such that

(i) If $\gamma \in \mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$, then $Z(\gamma) \in \prod_{n\geq m} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$,

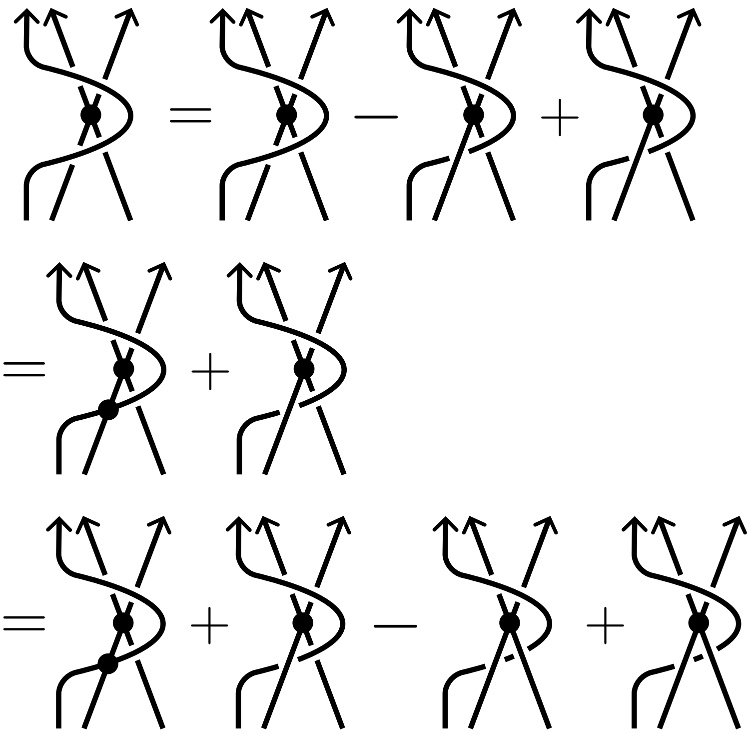
(ii) $\operatorname{gr} Z\colon \operatorname{gr}\mathcal{K}(-) \to \operatorname{gr}\operatorname{proj}\mathcal{K}(-)$ is the identity map, where $\operatorname{proj}\mathcal{K}(\Gamma) \coloneqq \bigoplus_{n=0}^{\infty} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$.

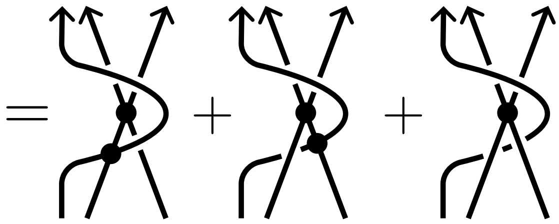
Appendix A: Proof of Theorem 3.13

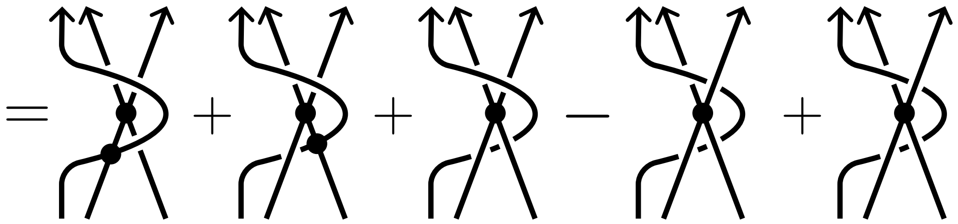
In $\mathcal{A}(\Gamma) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma)$, there are no double points, but for the readers to understand the proof easily, we draw double points in the following figures. Any chord can move freely as shown below:

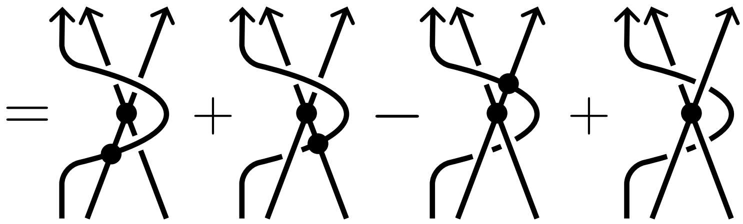


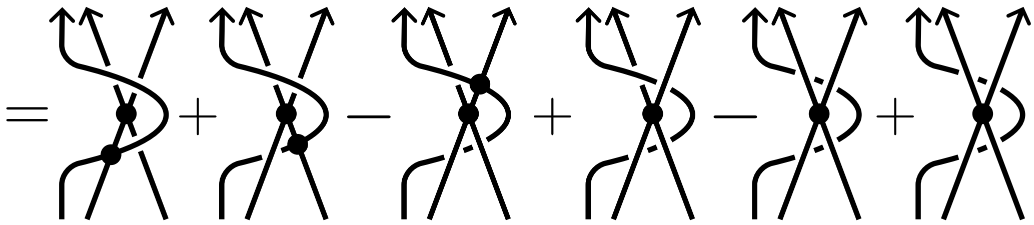
Now, we derive $4T$ relation by adding and subtracting same diagrams as follows:

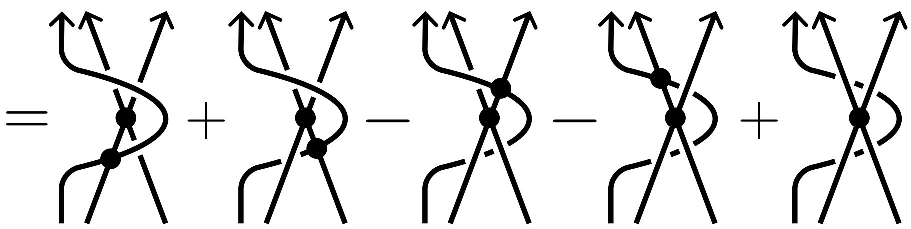




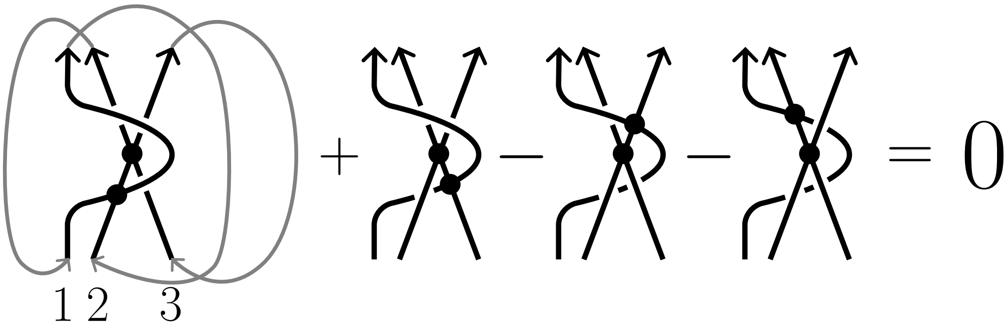




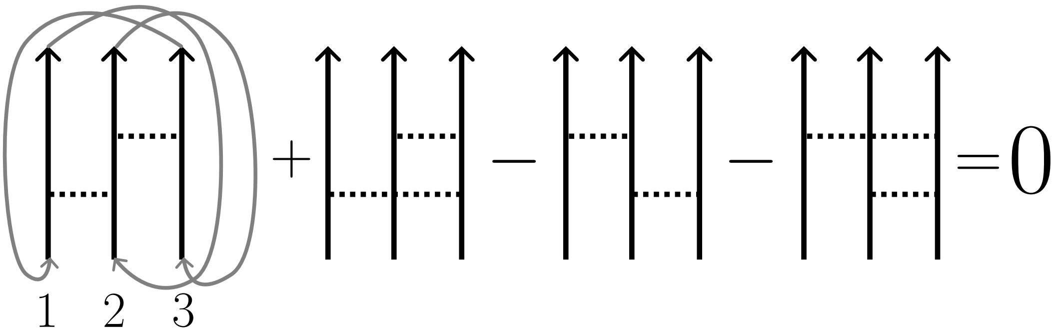




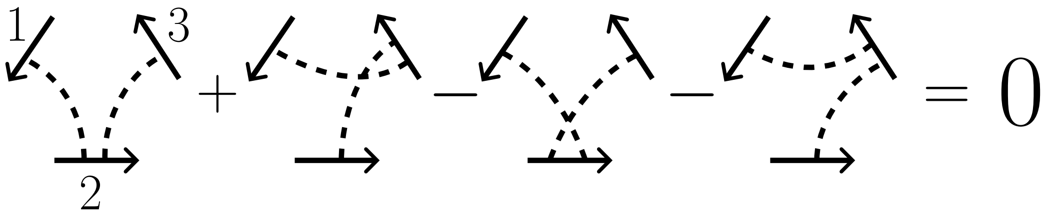
Thus, we have a new equation with two double points and connect ordered edges:

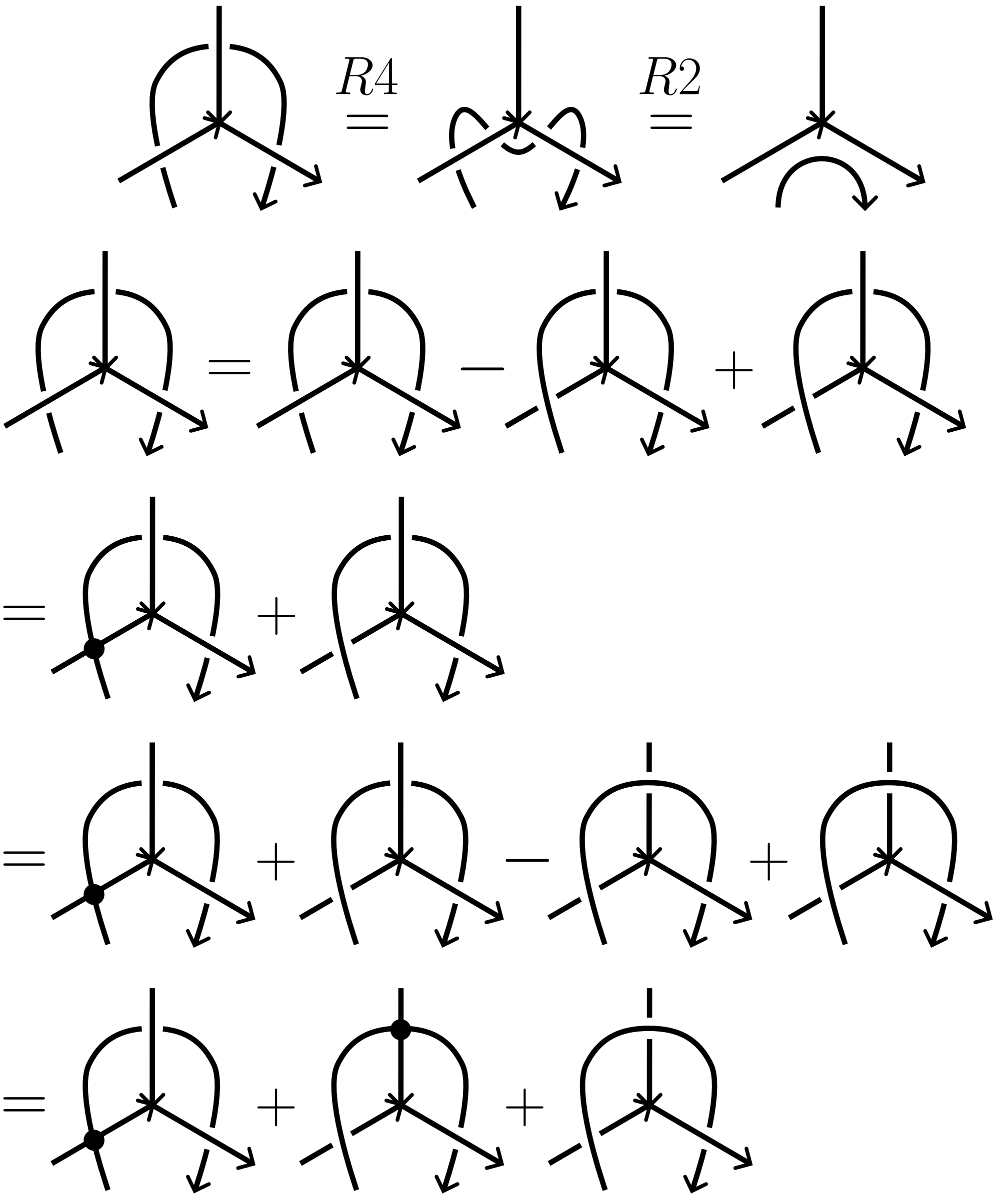


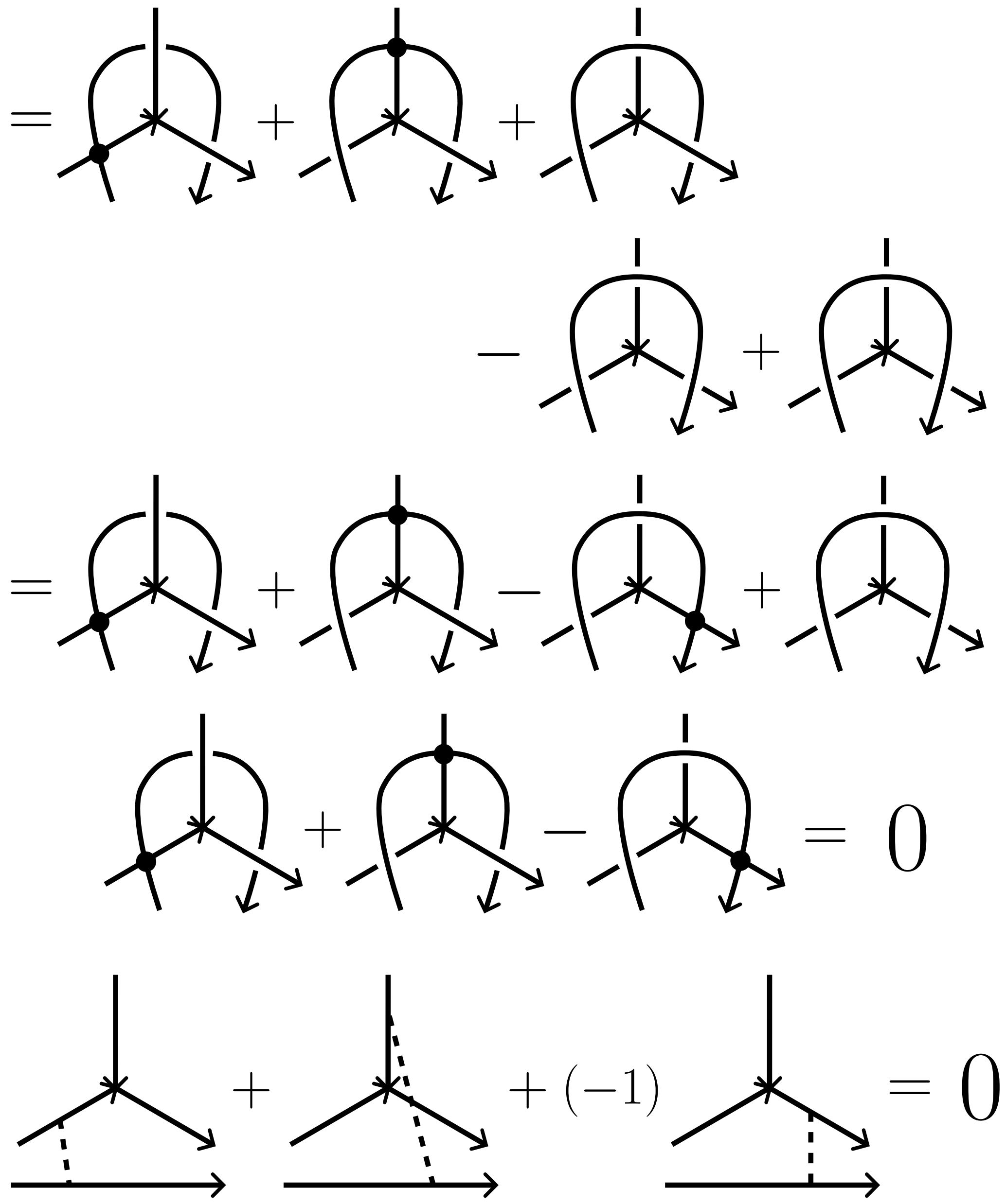
Then we straighten the three edges.



Finally, we obtain the $4T$ relation:




R4
R2



Bibtexのエクスポートはゴミなので自分で書け

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