2025年度 横浜国立大学 理工学部 数理科学EP 卒業研究

Homomorphic Expansions for Knotted Trivalent Graphs

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Abstract

【保留】KTGs に対し a universal Vassiliev invariant が存在することは知られていた [MO97, CL07, Dan10]．KTGs において “edge unzip” という操作のみ準同型にならず，補正項が現れる．dotted Knotted Trivalent Graphs において Z^{old} が準同型となるように Z を 2 通りで構成することが目的.

It has been known since old times [MO97, CL07, Dan10] that there exists a universal finite type invariant Z^{old} for Knotted Trivalent Graphs. While the behavior of Z^{old} under edge unzip is well understood, it is not plainly homomorphic as some “correction factors” appear.

In this paper we modify Z^{old} into a new expansion Z, defined on “dotted Knotted Trivalent Graphs”, which is homomorphic with respect to a large set of operations.

Contents

1 Introduction

結び目理論とは位相幾何学の分野の一つであり，物理学とも関係する分野である．その中でも，結び目同士が異なるかどうかを区別する際に手段として使われるものとして結び目の不変量というものがある．

謝辞

2 Preliminaries

2.1 KTGs and Z^{old}

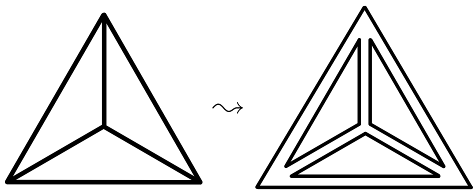
All edges are oriented, and the vertices are given a counterclockwise orientation. Loops are allowed.

Definition 2.1. A *surface* is a compact, oriented 2-dimensional manifold that satisfies the second-countable axiom.

Definition 2.2. A *spine* of a simplicial complex Y is a subcomplex X of Y onto which Y collapses, where collapsing means successively removing pairs of a k-simplex Δ^k and a $(k+1)$-simplex Δ^{k+1}, where Δ^{k+1} is the unique $(k+1)$-simplex having Δ^k on its boundary.

Definition 2.3. graph Γ に対し *framed graph* (枠付きグラフ) $\mathbf{\Gamma}$ とは，Γ と，Γ を spine として曲面 Σ へ埋め込む[1]写像 $\Gamma \hookrightarrow \Sigma$ の組をいう．特に Γ が Trivalent graph のとき，$\mathbf{\Gamma}$ を *framed trivalent graph*[2]という．
For a graph Γ, a *framed graph* is a 1-dimensional simplicial complex Γ together with an embedding $\Gamma \hookrightarrow \Sigma$ of Γ into a surface Σ as a spine. In particular, when Γ is a trivalent graph. it is denoted by $\mathbf{\Gamma} = (\Gamma, \Sigma)$ and is called a *framed trivalent graph.*

Moreover, for a framed trivalent graph $\mathbf{\Gamma} = (\Gamma, \Sigma)$, a *Knotted Trivalent graph (KTG)* is an embedding of a framed trivalent graph $\mathbf{\Gamma}$ into $\mathbb{R}^3$. The *skeleton* of a KTG γ is the combinatorial object (trivalent graph Γ) behind it.



Definition 2.4. *Knotted trivalent graph (KTG)* を，framed trivalent graph $\mathbf{\Gamma}$ から $\mathbb{R}^3$ への埋め込み，KTG の *skeleton* を trivalent graph Γ とする．(framed knots や links も含む)
A *Knotted Trivalent graph (KTG)* is an embedding of a framed trivalent graph $\mathbf{\Gamma}$ into $\mathbb{R}^3$. The *skeleton* of a KTG γ is the combinatorial object (trivalent graph Γ) behind it.

[1] グラフを 1 次元 CW 複体とみなす.
[2] 論文では thickened trivalent graph と書いてある.

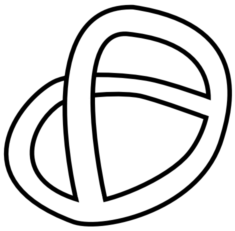


Figure 1: Knotted Trivalent Graph の例

KTGs において，skeleton が isotopy で移りあうものを同一視する．特に，framed knots や links は KTGs の特別な場合である．Trivalent graph Γ に対し，すべての KTG の集合を $\mathcal{K}(\Gamma)$ と書く．
We identify KTGs whose skeletons are isotopic. For a trivalent graph Γ, we denote the vector space over $\mathbb{Q}$ generated by all linear combinations of KTGs with skeleton Γ by

$$\mathcal{K}(\Gamma) \coloneqq \left\{ \sum_{i=1}^{m} a_i \gamma_i \,\middle|\, m \in \mathbb{N}, a_i \in \mathbb{Q}, \gamma_i \text{ is a knotted trivalent graph.} \right\}.$$

Definition 2.5. For a graph Γ, a *chord diagram D* with support Γ is Γ together with an vertex-oriented uni-trivalent graph whose univalent vertices are on Γ; and the graph does not have any connected component homeomorphic to a circle. We call the uni-trivalent graph the chord graph of the diagram.

Proposition 2.6. *KTGs の isotopy class と graph diagrams (交点の上下の情報を残した射影) で $R1', R2, R3, R4$ で移りあうものは 1 対 1 に対応する．*
Two KTGs are isotopic if and only if their graph diagrams are related by a finite sequence of Reidemeister moves $R1', R2, R3, R4$.

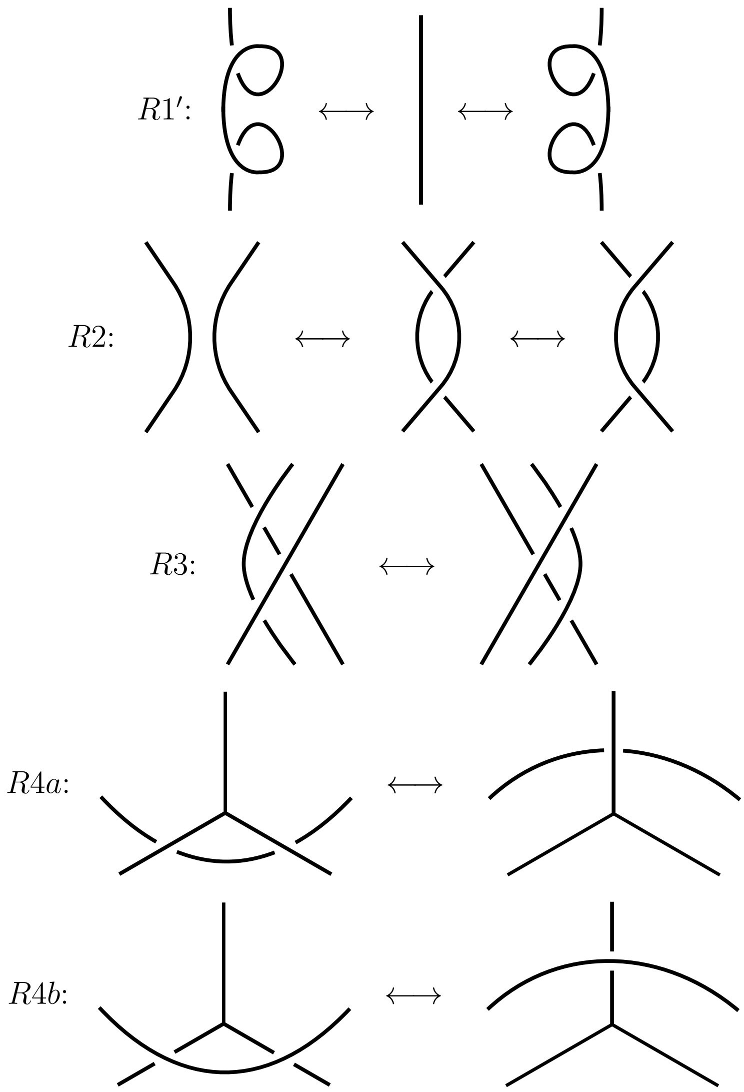


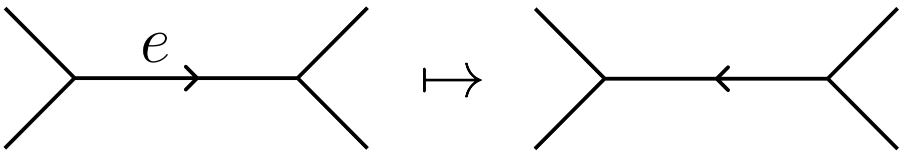
Figure 2: Knotted Trivalent Graph における 4 種類の Reidemeister 変形

証明は省略する．詳しくは [MO97, Theorem 1.4] を参照されたい．ここでは [Yam87] で定義された空間グラフのための拡張された Reidemeister moves のもとで不変であることを示せば十分であると述べられている．注意として，blackboard framing を用いているため，[Yam87] において定義されている頂点周りの edge の順序を変える移動は必要としない．
We omit the proof here. For details, please refer to [MO97, Theorem 1.4]. It is sufficient to show the invariance under extended Reidemeister moves for spatial graphs defined in [Yam87]. Note that we do not need the move in [Yam87] which changes the order of edges around a vertex, since we consider framed graphs with the blackboard framing.

There are four operations on KTGs:

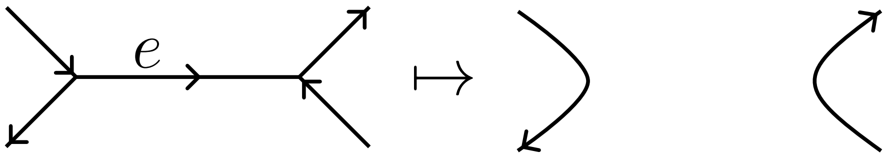
Definition 2.7. KTG を $\gamma \in \mathcal{K}(\Gamma)$ とし，Γ の edge を e とする．e の *switch the orientation* を，向きを変えるものとして定め，$S_e(\gamma)$ と書く．
Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and e be an edge of Γ. The *switch the orientation* of e is defined as reversing its orientation, and is denoted by $S_e(\gamma)$.

$$S_e \colon \mathcal{K}(\Gamma) \to \mathcal{K}(S_e(\Gamma))\,;\, \gamma \mapsto S_e(\gamma)$$



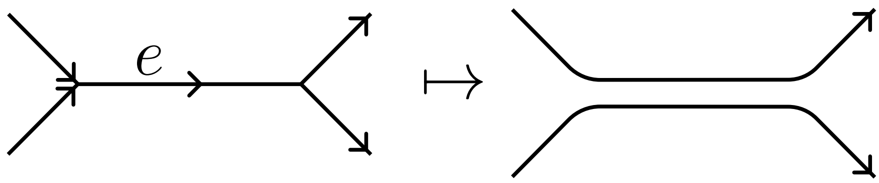
Definition 2.8. Γ の edge であって，両端に接続されている edge の向きが一致している e を *delete* するとは，e を削除し，三価性を保つように e の両端の頂点を削除することをいう．
Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and let e be an edge of γ. *Delete* of e is defined as removing e and the two vertices at the ends of e also cease to exist to preserve the trivalence. To do this, it is required that the orientations of the two edges connecting to e at either end match.

$$d_e \colon \mathcal{K}(\Gamma) \to \mathcal{K}(d_e(\Gamma))\,;\, \gamma \mapsto d_e(\gamma)$$



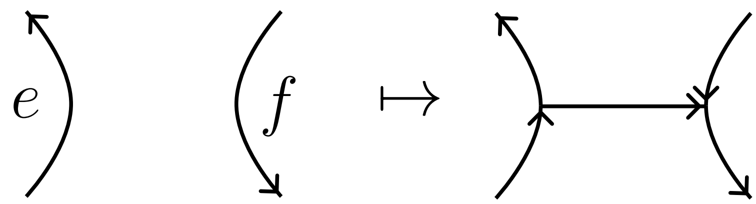
Definition 2.9. Γ の edge e を *unzip* するとは，e を “限りなく近い”2 つの edges に分け，端点をなくすことをいう．端点をなくしたとき，edge の向きが合っていることが必要である．同様の議論で framed graph $\mathbf{\Gamma}$ に対し，unzip を定義できる．
Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and let e be an edge of γ. *Unzip* the edge e is replacing it by two edges that are “very close to each other”. Thw two vertices at the ends of e will disappear. Again the edges at the vertex where e begins have to both be incoming, while the edges at the vertex where e ends must both be outgoing.

$$u_e \colon \mathcal{K}(\Gamma) \to \mathcal{K}(u_e(\Gamma))\,;\, \gamma \mapsto u_e(\gamma)$$



Definition 2.10. 2 つの trivalent graph とその edge のペア $(\Gamma, e), (\Gamma', f)$ の *connected sum* $\Gamma \#_{e,f} \Gamma'$ とは e, f をつなぐ edge を新たに作ること．well-defined であるために，新たな edge の向きは Γ から Γ' への向きとし，KTGs においてはねじれを許さず，辺を付ける場合は e, f の右側に付けるとする．(2 次元では自由に動かせないため左右が重要)
Let $(\Gamma, e), (\Gamma', f)$ be two pairs of trivalent graphs and their edges. The *connected sum* $\Gamma \#_{e,f} \Gamma'$ is obtained by joining e and f by a new edge. For this to be well-defined, we also need to specify the direction of the new edge, the cyclic orientations at each new vertex, and in the case of KTGs, the framing on the new edge. To compress notatio, let us declare that the new edge be oriented from Γ towards Γ', have no twists, and, using the blackboard framing, be attached to the right side of e and f.

$$\#_{e,f} \colon \mathcal{K}(\Gamma) \times \mathcal{K}(\Gamma') \to \mathcal{K}(\Gamma \#_{e,f} \Gamma')$$



KTGs の finite type invariants は，links におけるものと同様に定義する．同じ skeleton の KTGs の形式和を許し，得られたベクトル空間を特異点の解消によってフィルター分けする．
We define finite type invariants of KTGs in the same way as for links. In detail, we allow linear combinations of KTGs with the same skeleton, and filter the resulting vector space by the resolution of singular points.

Definition 2.11. *n-singular KTG* とは，n この特異点を持つ trivalent graph の $\mathbb{R}^3$ へのはめ込み．各特異点は横断的な 2 重点か，“F” と書かれた線上の点である．
An *n-singular KTG* is a trivalent graph immersed in $\mathbb{R}^3$ with n singular points: each singular point is a transverse double point or a point on an edge marked with an “F”.

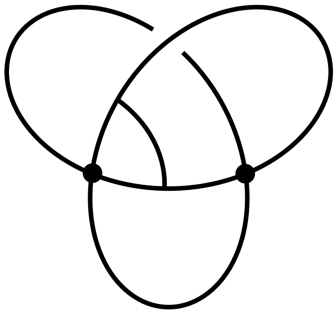
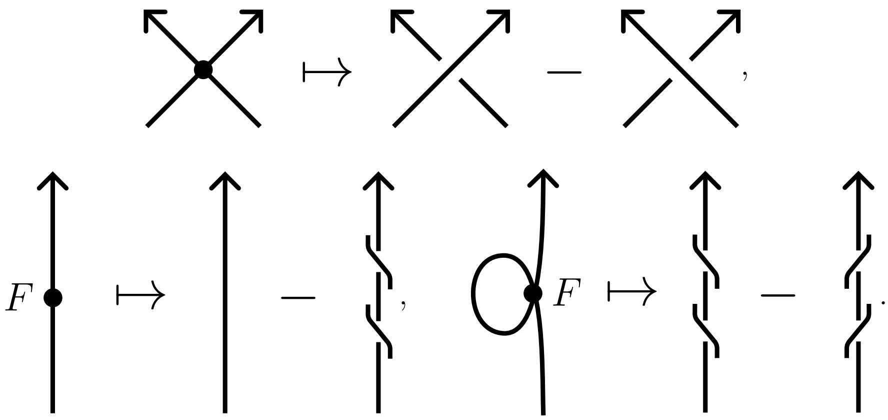


Figure 3: Example of an n-singular KTG.

$n \geq 1$ に対し以下のようなベクトル空間を考える．For $n \geq 1$, we consider the following vector space:

$$\mathcal{F}'_n(\Gamma) \coloneqq \left\{ \sum_{i=1}^{m} a_i \gamma_i \;\middle|\; \begin{array}{l} m \in \mathbb{N}, a_i \in \mathbb{Q}, \gamma'_i\text{: KTG which has} \\ \quad\text{at least } n \text{ double point with skeleton } \Gamma \end{array} \right\}$$

全ての特異点を解消する写像 $\rho\colon \mathcal{F}'_*(\Gamma) \to \mathcal{F}_0(\Gamma)$ を以下のように定める: We define a map $\rho\colon \mathcal{F}'_*(\Gamma) \to \mathcal{F}_0(\Gamma)$ that resolves all double points as follows:



$\mathcal{F}'_n(\Gamma)$ $(n \geq 1)$ に対し，$\mathcal{F}_n(\Gamma)$ を $\mathcal{F}_n(\Gamma) \coloneqq \rho(\mathcal{F}'_n(\Gamma))$ と定義すると，明らかに $\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma)$ であり，
For each $\mathcal{F}'_n(\Gamma)$ $(n \geq 1)$, we define $\mathcal{F}_n(\Gamma) \coloneqq \rho(\mathcal{F}'_n(\Gamma))$. Then, obviously, $\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma)$, and we obtain a following filtration:

$$\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma) \supset \mathcal{F}_1(\Gamma) \supset \mathcal{F}_2(\Gamma) \supset \mathcal{F}_3(\Gamma) \cdots .$$

という filtration が得られる．この filtration において隣合う 2 つのベクトル空間から得られる商ベクトル空間を $\mathcal{A}_n(\Gamma) \coloneqq \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma)$ とし，associated graded space を以下のように定義する:

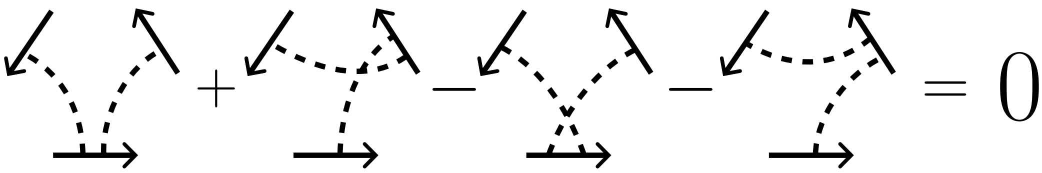
In this filtration, we denote the quotient vector space obtained from two adjacent vector spaces by $\mathcal{A}_n(\Gamma) \coloneqq \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma)$, and define the associated graded space as follows:

$$\mathcal{A}(\Gamma) \coloneqq \bigoplus_{n=0}^{\infty} \mathcal{A}_n(\Gamma) \left(= \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma) \right)$$

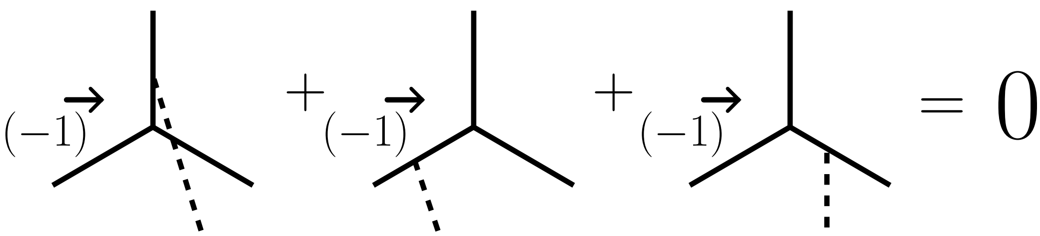
次数 n の chord diagram を基底とする $\mathbb{Q}$ 上のベクトル空間を $\mathcal{D}_n(\Gamma)$ と書き，$\mathcal{D}(\Gamma) \coloneqq \bigoplus_n \mathcal{D}_n(\Gamma)$ とする．Chord diagram における chords を縮めて double points に対応させることで，$\mathcal{D}(\Gamma)$ から $\mathcal{A}(\Gamma)$ への自然な全射 π が存在する．
We denote by $\mathcal{D}_n(\Gamma)$ the vector space over $\mathbb{Q}$ generated by chord diagrams of order n, and set $\mathcal{D}(\Gamma) \coloneqq \bigoplus_n \mathcal{D}_n(\Gamma)$. There is a natural surjection π from $\mathcal{D}(\Gamma)$ to $\mathcal{A}(\Gamma)$ by associating the chords in a chord diagram to double points.

We call the following relations the $4T$ and VI relations in $\mathcal{D}(\Gamma)$:

- $(4T)$ Four term relation



- (VI) Vertex invariance relation



図に描かれていない部分には graph があるが，それらは全て同じでなければならない．4T では反時計回りの向きを与える (これ必要？)．VI において，$(-1)^{\rightarrow}$ は，chord の付いた edge が外向きなら-1, 内向きなら 1 をかける (つまり式は 8 つある).
In both pictures, there may be other chords in the parts of the graph not shown, but they have to be the same throughout. In $4T$, all skeleton parts (solid lines) are oriented counterclockwise. In VI, the sign $(-1)^{\rightarrow}$ is -1 if the edge the chord is ending on is oriented to be outgoing from the vertex, and $+1$ if it is incoming (thus there are 8 versions of the relation).

Theorem 2.12. *The relations* $4T$ *and* VI *are contained in* $\ker \pi$*.*

この定理の証明は Appendix A で与える．
The proof of this theorem is given in Appendix A.

$4T$, VI が kernel に含まれることは分かったが，これ以上の relations が存在 “しない” ことを示すのは困難である．これを示すには，universal finite type invariant $\mathbb{Q}$KTG $\to \mathcal{A}$ を構成するのが最善である (ここでは定義しないが，後で一般の文脈で定義する)．これは，T.Le, H.Murakami, J. Murakami, T.Ohtsuki の結果をもとに，また Drinfeld の associator の理論を用いて [KO, CD],[BN] での Kontsevich integral を拡張する形で [MO97] で初めて得られた．
Although it is easy to see that these relations are contained in kernel, showing that there are no more is difficult, and is best achieved by constructing a universal finite type invariant $\mathbb{Q}$KTG $\to \mathcal{A}$ (we do not define universal finite type invariants here, but will do so later in the general context).

KTGs の各 operation は $\mathcal{A}$ 上の operation を誘導する．($\mathcal{A}$ は $\mathcal{K}(\Gamma)$ の associated graded space である．)
Each operation on KTGs induces an operation on $\mathcal{A}$ (the associated graded space of $\mathcal{K}(\Gamma)$).

- orientation switch

図

- edge delete

図

- edge unzip

図

- connected sum well-defined である．Introduction to Vassiliev knot invariants(Chmutov) の Lemma4.2.9

図

Theorem 2.13. *Any KTG can be obtained from the trivially embedded tetrahedron and the twisted tetrahedron by a finite sequence of the four operations defined above.*

Proof. aa

図

□

Theorem 2.14. *Any n-singular KTG can be obtained from the trivially embedded tetrahedron, twisted tetrahedron and singular twisted tetrahedron using the four operations.*

Proof. Same as Theorem 2.13. □

2.2 Algebraic structures and expansions

$\mathcal{K}$ において，orientation switch, edge delete, edge unzip, connected sum を linear に拡張し，$\mathbb{Q}$ 係数の形式和を許すように拡張することで，$\mathcal{K}$ は vector space となる．
By linearly extending the operations of orientation switch, edge delete, edge unzip, and connected sum on $\mathcal{K}$ to allow linear combinations with coefficients in $\mathbb{Q}$, $\mathcal{K}$ becomes a vector space.

Definition 2.15. Γ を KTG とする．$\mathcal{K}(\Gamma)$ において，係数の和が 0 となるような形式和全体から生成される集合を $\mathcal{I}(\Gamma)$ と書き，$\mathcal{I} \coloneqq \bigoplus_{\Gamma'} \mathcal{I}(\Gamma')$ とする．
Let Γ be a KTG. Let $\mathcal{I}(\Gamma)$ be the sub-structure made out of all such combinations in which the sum of coefficients is 0, and let $\mathcal{I} \coloneqq \bigoplus_{\Gamma'} \mathcal{I}(\Gamma')$.

Example 2.16. Let γ_1, γ_2 and γ_3 be KTGs with skeleton Γ. Then, $\gamma_1 - \gamma_2, \gamma_1 - \frac{1}{2}\gamma - \frac{1}{2}\gamma_3 \in \mathcal{I}(\Gamma)$.

Definition 2.17. $\mathcal{I}^m$ を，$\mathcal{I}$ の元を少なくとも m 個含むようなものから任意の演算の合成で得られる元が生成する $\mathcal{K}$ の部分空間とする．つまり，
Let $\mathcal{I}^m$ be the set of all outputs of arbitrary compositions of the operations in $\mathcal{K}$ that have at least m inputs in $\mathcal{I}$. In other words,

$$\mathcal{I}^m \coloneqq \left\{ \gamma \in \mathcal{K} \;\middle|\; \begin{array}{c} \text{There exist } n,\ f\colon \prod_{i=1}^{n} \mathcal{K} \to \mathcal{K},\ x_1, \ldots, x_n \in \mathcal{K} \\ \text{such that } \gamma = f(x_1, \ldots, x_n), \#\{i \mid x_i \in \mathcal{I}\} \geq m. \end{array} \right\}.$$

さらに，$\mathcal{I}^m(\Gamma) \coloneqq \mathcal{I}^m \cap \mathcal{I}(\Gamma)$ とする．Moreover, we define $\mathcal{I}^m(\Gamma) \coloneqq \mathcal{I}^m \cap \mathcal{I}(\Gamma)$.

ここで，$\mathcal{I}^m$ は明らかに filtration の構造をもつ．Clearly, $\mathcal{I}^m$ has a filtration structure.

Lemma 2.18. $\mathcal{I}(\Gamma) = \{\sum_i c_i(\gamma_i - \gamma'_i) \mid \gamma_i, \gamma'_i$*: generators of* $\mathcal{K}(\Gamma), c_i \in \mathbb{Q}\}$*.*

Proof. ($\supset$) Each coefficient of $c_i(\gamma_i - \gamma'_i)$ is 0, so the sum of coefficients is also 0.
($\subset$) For any element of $\mathcal{I}$, it can be written as $\sum_{i=1}^n c_i\gamma_i$. Since $\sum_{i=1}^n c_i = 0$, we have $c_n = -\sum_{i=1}^{n-1} c_i$. Thus

$$\sum_{i=1}^{n} c_i\gamma_i = c_1\gamma_1 + c_2\gamma_2 + \cdots + \left(-\sum_{i=1}^{n-1} c_i\right)\gamma_n = \sum_{i=1}^{n-1} c_i(\gamma_i - \gamma_n).$$

□

Theorem 2.19. $\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ *for all* $n \geq 0$ *and skeleton* Γ*.*

Proof. (i) $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$

($\supset$) 任意の $\mathcal{F}_1(\Gamma)$ の元は少なくとも 1 つの double point を持つような 1-singular KTG の交点の正負の差を元に持つため，$\mathcal{F}_1(\Gamma) \subset \mathcal{I}(\Gamma)$.
Since any element $\gamma \in \mathcal{F}_1(\Gamma)$ has at least one double point, so there exist $\gamma_+, \gamma_- \in \mathcal{K}(\Gamma)$ such that $\gamma = \gamma_+ - \gamma_-$, thus $\mathcal{F}_1(\Gamma) \subset \mathcal{I}(\Gamma)$.
($\subset$) 任意の $\mathcal{I}(\Gamma)$ の元は Lemma 2.18 より $\sum_i c_i(\gamma_i - \gamma'_i)$ と書ける．$\mathcal{F}_1(\Gamma)$ において，同じ skeleton を持つ任意の 2 つの KTG は crossing change により移りあうため，$\gamma_i - \gamma'_i$ を 1 点における正負の交差の差 $\tilde{\gamma}_i - \tilde{\gamma}'_i$ となるようにできる．よって

$$\sum_i c_i(\gamma_i - \gamma'_i) = \sum_i c_i(\tilde{\gamma}_i - \tilde{\gamma}'_i) \in \mathcal{F}_1(\Gamma).$$

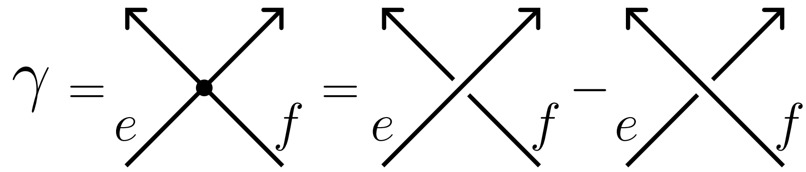
Any element of $\mathcal{I}(\Gamma)$ can be written as $\sum_i c_i(\gamma_i - \gamma'_i)$ by Lemma 2.18. In $\mathcal{F}_1(\Gamma)$, any two KTGs with the same skeleton can be related by crossing changes, so we can make $\gamma_i - \gamma'_i$ into the difference of positive and negative crossings at one point $\tilde{\gamma}_i - \tilde{\gamma}'_i$. Therefore,

$$\sum_i c_i(\gamma_i - \gamma'_i) = \sum_i c_i(\tilde{\gamma}_i - \tilde{\gamma}'_i) \in \mathcal{F}_1(\Gamma).$$

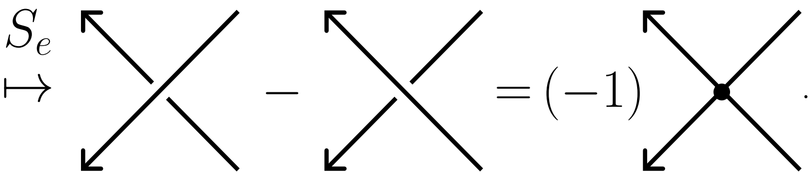
(ii) $\mathcal{I}^n(\Gamma) \subset \mathcal{F}_n(\Gamma)$

By $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$, any element $\gamma \in \mathcal{I}^n(\Gamma)$ is generated by at least n elements of $\mathcal{F}_1(\Gamma)$. It is enough to check that the four operations for an edge with double points preserve number of double points.

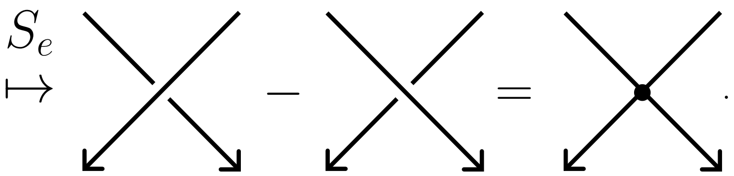
• orientation switch of an edge with a double point,



If e does not connect to f,

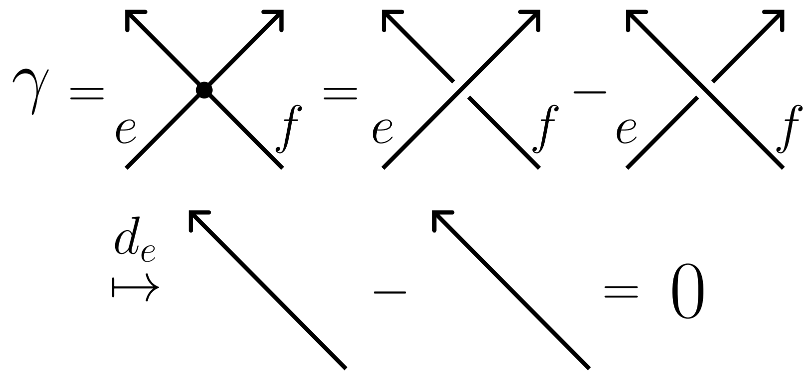


If e connects to f,



In both cases, the number of double points does not change.

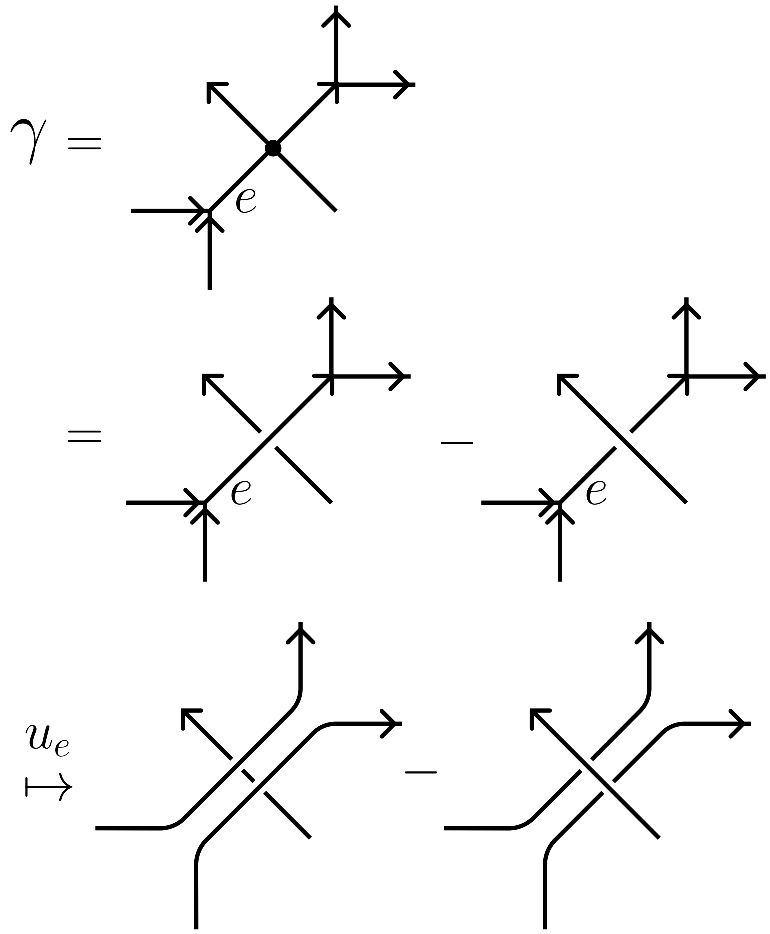
• edge delete,

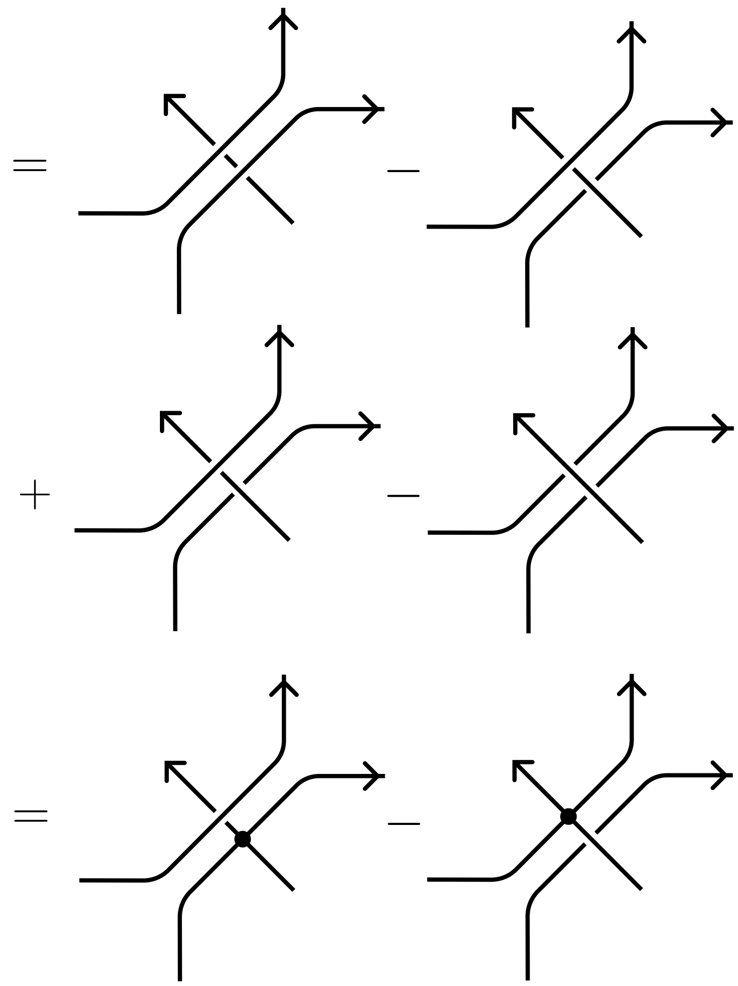


Both cases, if e connects to f or not, $d_e(\Gamma) = 0$. 0 is included in $\mathcal{F}_n$ for all $n \geq 0$, so the number of double points does not change.

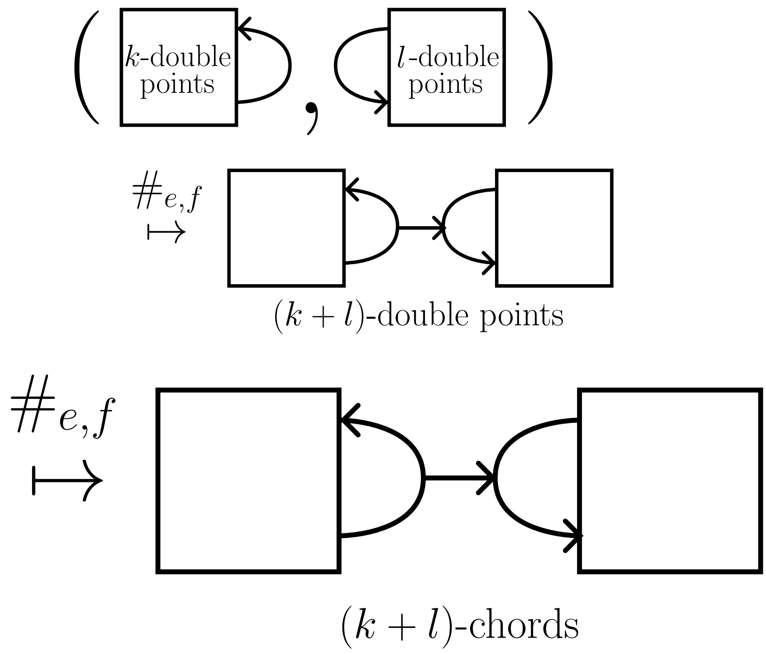
• edge unzip,

We prove with a technical method by adding and subtracting same diagrams as follows:





- connected sum.



(iii) $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$ Since any n-singular KTG can be obtained from n pieces of 1-singular KTGs and the four operations by Theorem 2.14, $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$.

Therefore, $\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ for all $n \geq 0$ and skeleton Γ. $\square$

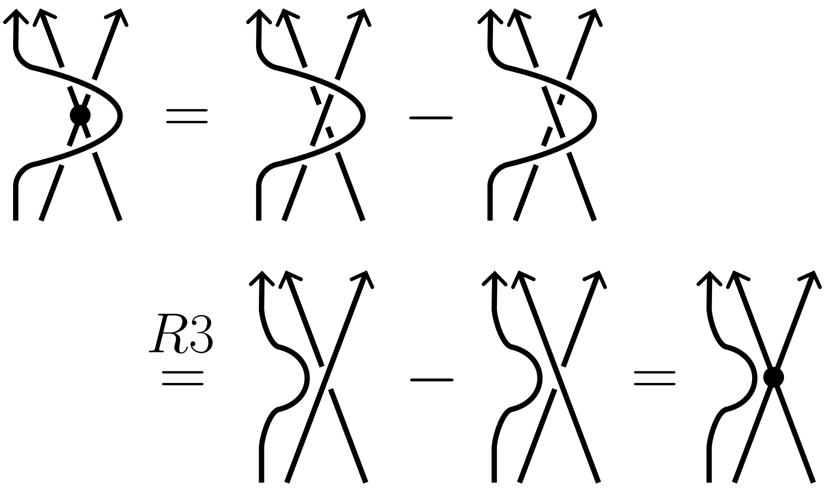
Definition 2.20. Let Γ, $\mathcal{K}(\Gamma)$ be a skeleton and the set of KTGs with skeleton Γ. An *expansion* Z for $\mathcal{K}(\Gamma)$ is a map $Z\colon \mathcal{K}(\Gamma) \to \hat{\mathcal{A}}(\Gamma) = \prod_{n=0}^{\infty} \mathcal{A}_n(\Gamma)$ such that

(i) If $\gamma \in \mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$, then $Z(\gamma) \in \prod_{n \geq m} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$,

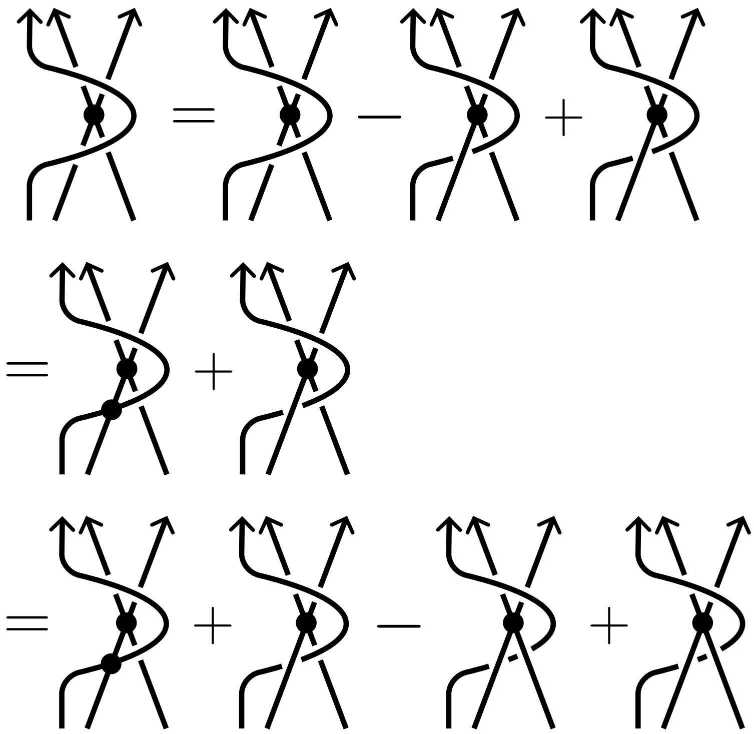
(ii) $\operatorname{gr} Z\colon\ \operatorname{gr}\mathcal{K}(-) \to \operatorname{gr}\operatorname{proj}\mathcal{K}(-)$ is the identity map, where $\operatorname{proj}\mathcal{K}(\Gamma) \coloneqq \bigoplus_{n=0}^{\infty} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$.

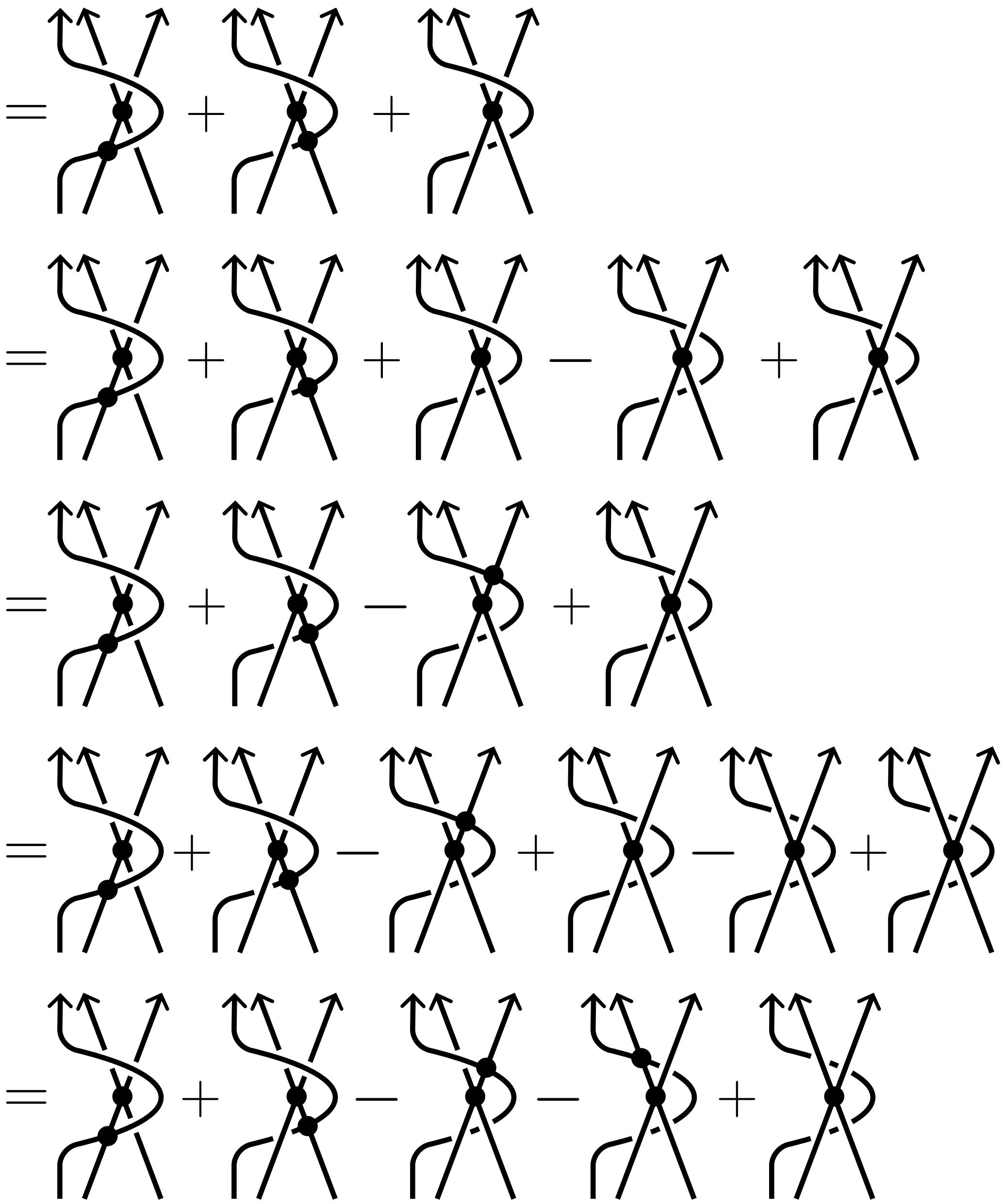
Appendix A: Proof of Theorem 2.12

In $\mathcal{A}(\Gamma) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma)$, there are no double points, but for the readers to understand the proof easily, we draw double points in the following figures. Any chord can move freely as shown below:

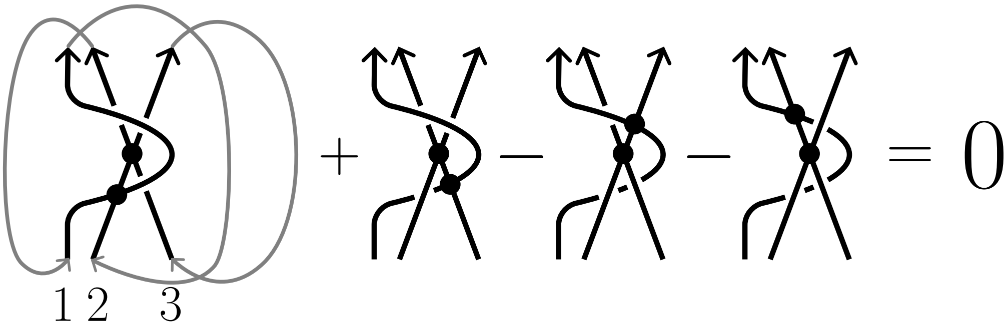


Now, we derive $4T$ relation by adding and subtracting same diagrams as follows:

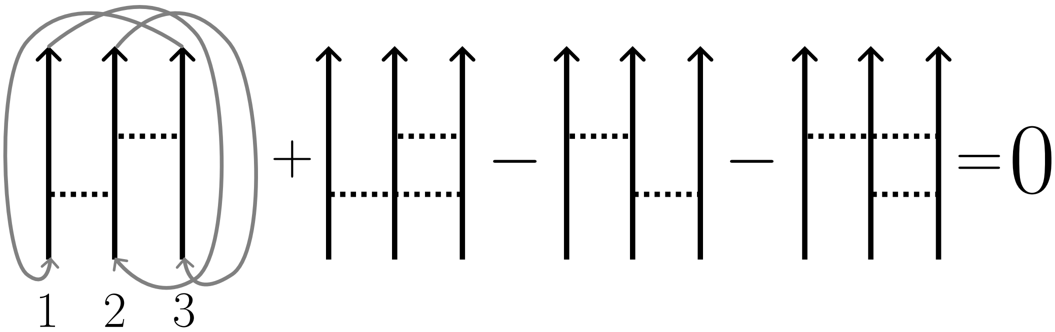




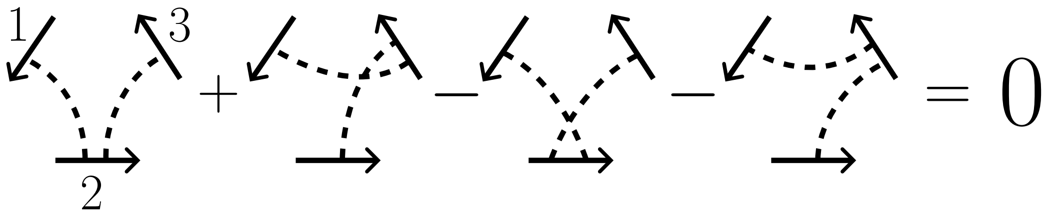
Thus, we have a new equation with two double points and connect ordered edges:

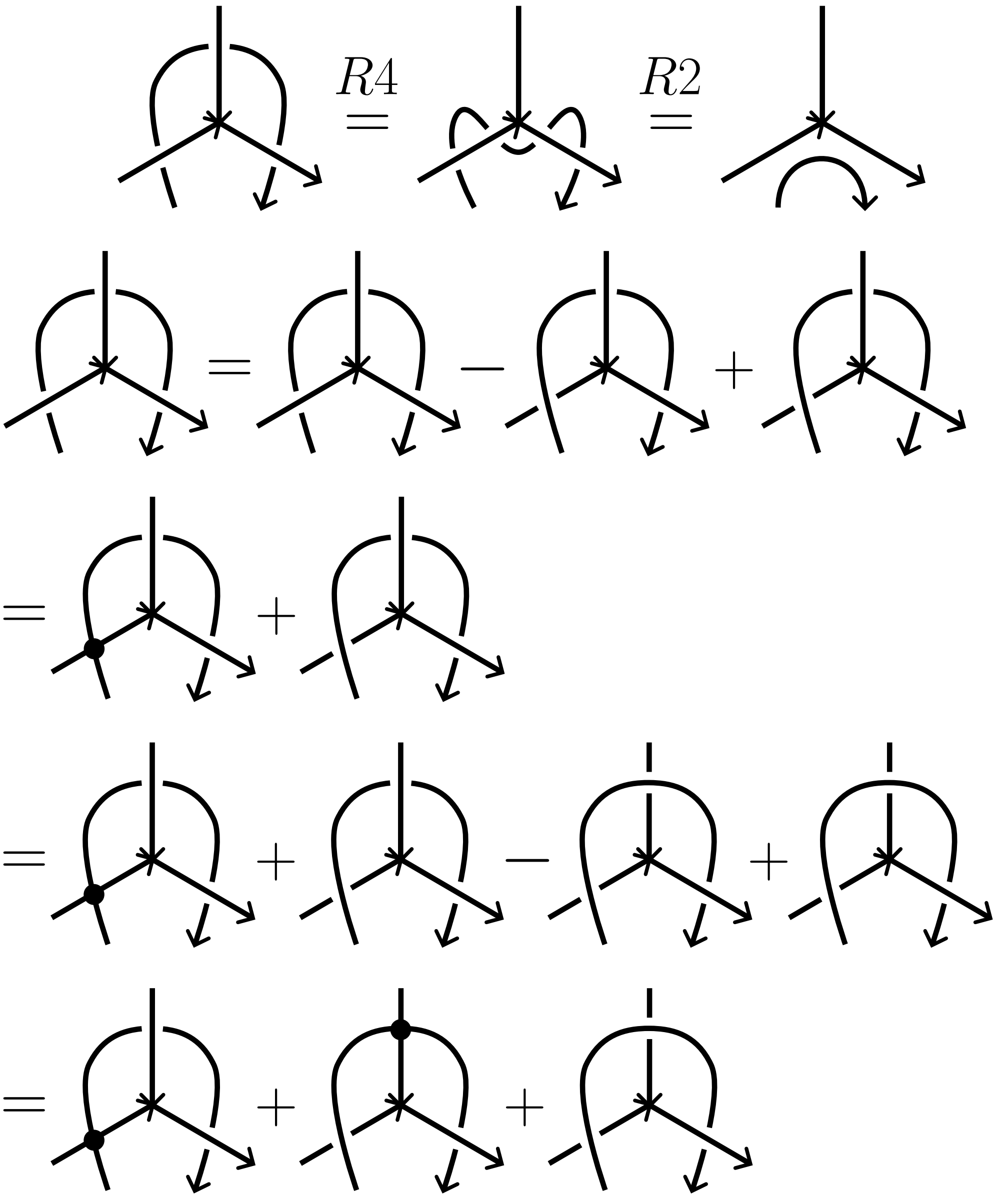


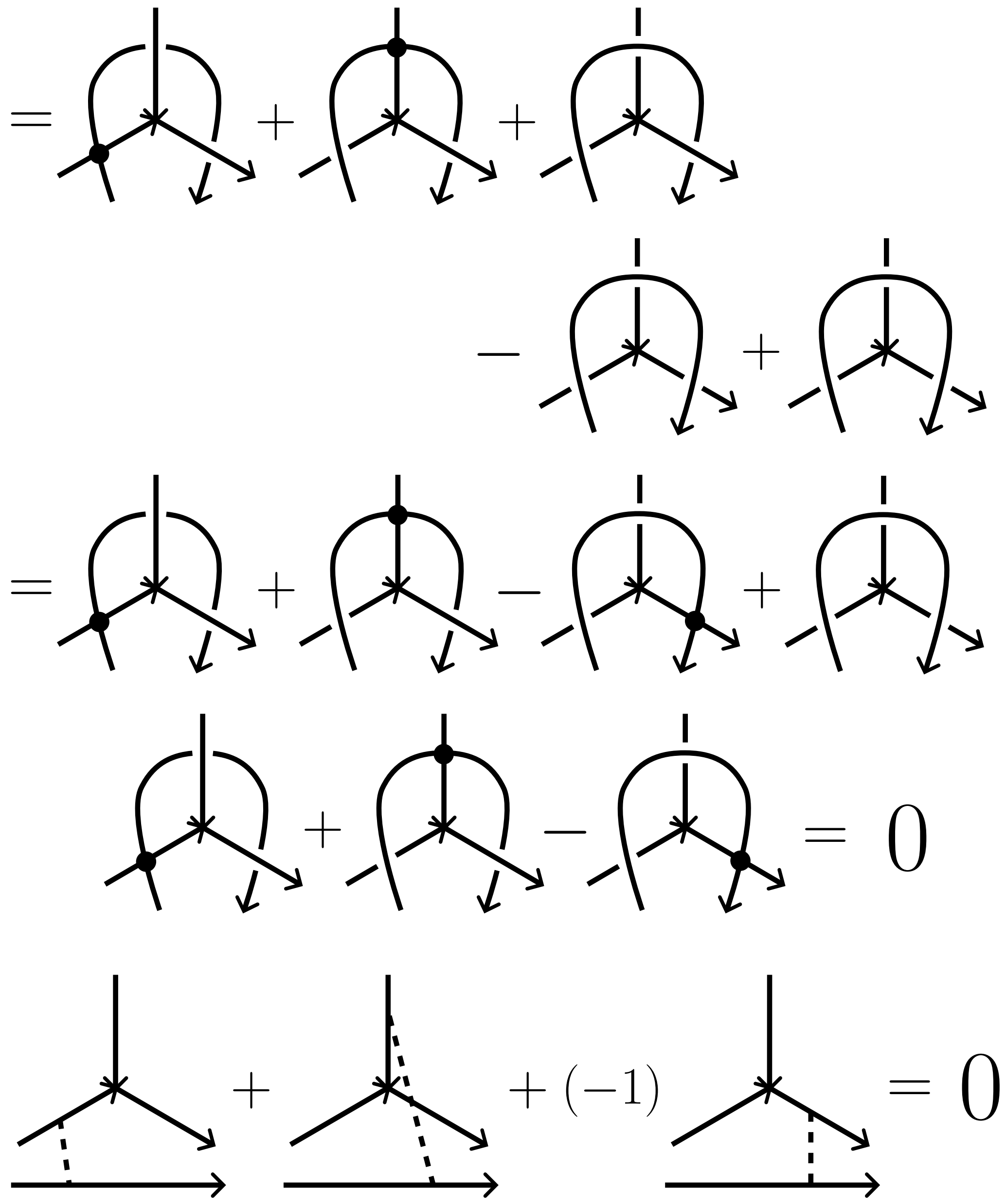
Then we straighten the three edges.



Finally, we obtain the $4T$ relation:




R4
R2


= + +
− +
= + − +
+ − = 0
+ + (−1) = 0

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