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Homomorphic Expansions for Knotted Trivalent Graphs

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Abstract

【保留】KTGs に対し a universal Vassiliev invariant が存在することは知られていた [MO97, CL07, Dan10]. KTGs において “edge unzip” という操作のみ準同型にならず、補正項が現れる。dotted Knotted Trivalent Graphs において Z^{old} が準同型となるように Z を 2通りで構成することが目的。

It has been known since old times [MO97, CL07, Dan10] that there exists a universal finite type invariant Z^{old} for Knotted Trivalent Graphs. While the behavior of Z^{old} under edge unzip is well understood, it is not plainly homomorphic as some “correction factors” appear.

In this paper we modify Z^{old} into a new expansion Z , defined on “dotted Knotted Trivalent Graphs”, which is homomorphic with respect to a large set of operations.

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Part I

Introduction

結び目理論とは位相幾何学の分野の一つであり、物理学とも関係する分野である。その中でも、結び目同士が異なるかどうかを区別する際に手段として使われるものとして結び目の不变量というものがある。

謝辞



Part II

Preliminaries

1 KTGs and Z^{old}

全ての辺は向きづけられているものとし、頂点は反時計回りに向きを与える。ループや円などの辺を許すこととする。All edges are oriented, and the vertices are given a counterclockwise orientation. Loops are allowed.

Definition 1.1. *Surface(曲面)* とは第2可算公理¹を満たす2次元多様体をいう。A surface is a 2-dimensional manifold that satisfies the second-countable axiom.

Definition 1.2. K を単体複体, $\sigma, \tau \in K$ を以下の条件を満たすとする。

- $\tau \not\preceq \sigma$,
- σ は K の最大の面単体で、他の最大面単体は τ を含まない。このような τ を *free face* という。

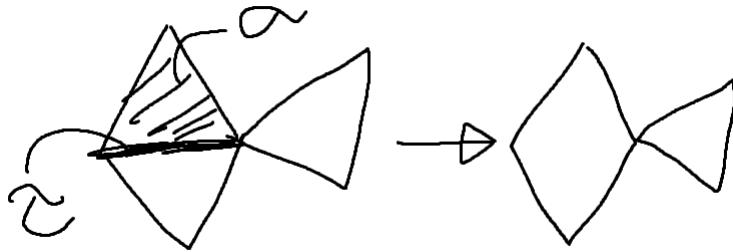
このとき, K の *collapse* とは, $\tau \preceq \gamma \preceq \sigma$ となる γ をすべて取り除くことをいう。Let K be a simplicial complex, and let $\sigma, \tau \in K$ satisfy the following conditions:

- $\tau \not\preceq \sigma$,
- σ is a maximal face simplex of K , and no other maximal face simplex contains τ . Such a τ is called a *free face*.

Then, a *collapse* of K is the removal of all γ such that $\tau \preceq \gamma \preceq \sigma$.

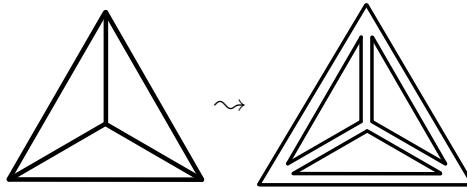


¹高々可算な開基を持つ。



Definition 1.3. 単体複体 K における *spine*(スパイン) とは, K の部分単体複体 K' であって, K を collapse して K' となるものをいう. A *spine* of a simplicial complex K is a sub-simplicial complex K' such that K collapses to K' .

Definition 1.4. graph Γ に対し *framed graph* (枠付きグラフ) Γ とは, Γ と, Γ を spine として曲面 Σ へ埋め込む²写像 $\Gamma \hookrightarrow \Sigma$ の組をいう. 特に Γ が Trivalent graph のとき, Γ を *framed trivalent graph*³ という. For a graph Γ , a *framed graph* Γ is a pair of Γ and an embedding $\Gamma \hookrightarrow \Sigma$ into a surface Σ such that Γ is a spine of Σ . In particular, when Γ is a trivalent graph, Γ is called a *framed trivalent graph*.



Definition 1.5. *Knotted trivalent graph (KTG)* を, framed trivalent graph γ から \mathbb{R}^3 への埋め込み, KTG の *skeleton* を trivalent graph Γ とする. (framed knots や links も含む)

A *Knotted Trivalent graph (KTG)* is an embedding of a framed trivalent graph γ into \mathbb{R}^3 . The *skeleton* of a KTG γ is the combinatorial object (trivalent graph Γ) behind it.

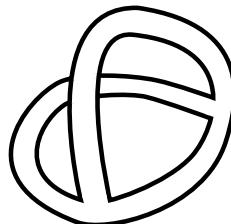


Figure 1: Knotted Trivalent Graph の例

²グラフを 1 次元 CW 複体とみなす.

³論文では thickened trivalent graph と書いてある.

KTGsにおいて, skeletonがisotopyで移りあうものを同一視する. 特に, framed knotsやlinksはKTGsの特別な場合である. Trivalent graph Γ に対し, すべてのKTGの集合を $\mathcal{K}(\Gamma)$ と書く. We identify KTGs whose skeletons are isotopic. For a trivalent graph Γ , we denote the vector space over \mathbb{Q} generated by all linear combinations of KTGs with skeleton Γ by $\mathcal{K}(\Gamma)$.

$$\mathcal{K}(\Gamma) := \left\{ \sum_{i=1}^m a_i \gamma_i \mid m \in \mathbb{N}, a_i \in \mathbb{Q}, \gamma_i \in \mathcal{K}(\Gamma) \right\}$$

Proposition 1.6. *Framed linksと, framed linkのdiagramsで3つのReidemeister変形 $R1', R2, R3$ の操作で移りあうものを同一視したものは1対1に対応する. Two framed links are isotopic if and only if their diagrams are related by a finite sequence of Reidemeister moves $R1', R2, R3$.*

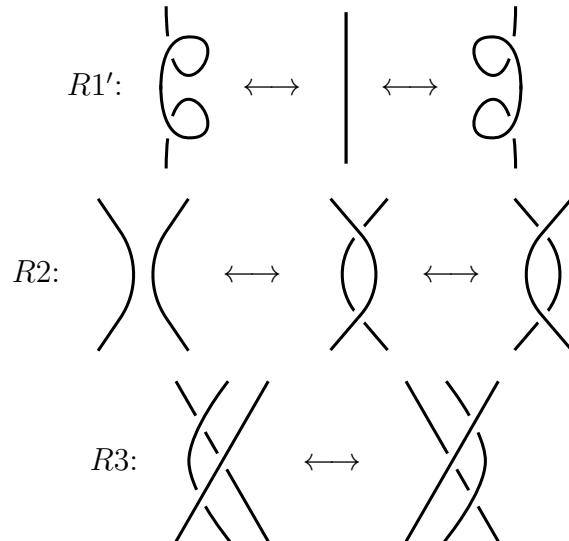


Figure 2: Framed knotsにおける3種類のReidemeister変形

Proof. [Oht02] P15 Theorem 1.8 □

Proposition 1.7. *KTGsのisotopy classとgraph diagrams(交点の上下の情報を持った射影)で $R1', R2, R3, R4$ で移りあうものは1対1に対応する. Two KTGs are isotopic if and only if their graph diagrams are related by a finite sequence of Reidemeister moves $R1', R2, R3, R4$.*

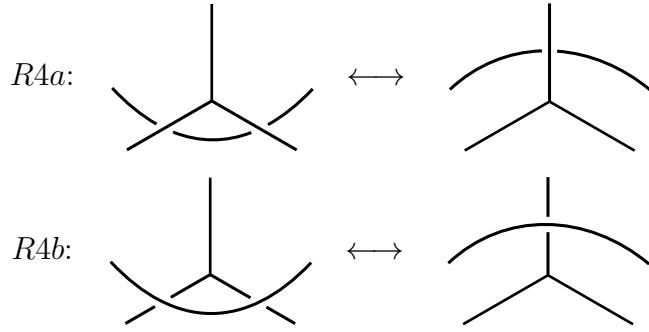


Figure 3: Framed knots における 3 種類の Reidemeister 変形

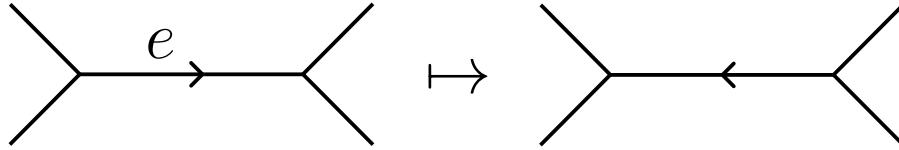
Proof.

□

There are four operations on KTGs:

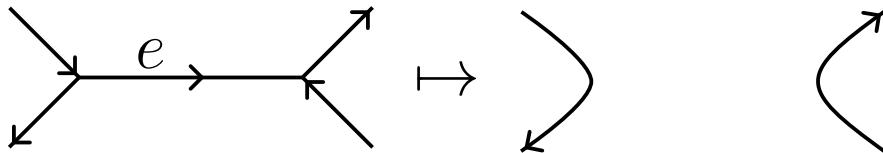
Definition 1.8. KTG を $\gamma \in \mathcal{K}(\Gamma)$ とし, Γ の edge を e とする. e の *switch the orientation* を, 向きを変えるものとして定め, $S_e(\gamma)$ と書く. Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and e be an edge of Γ . The *switch the orientation* of e is defined as reversing its orientation, and is denoted by $S_e(\gamma)$.

$$S_e: \mathcal{K}(\Gamma) \rightarrow \mathcal{K}(S_e(\Gamma)); \gamma \mapsto S_e(\gamma)$$



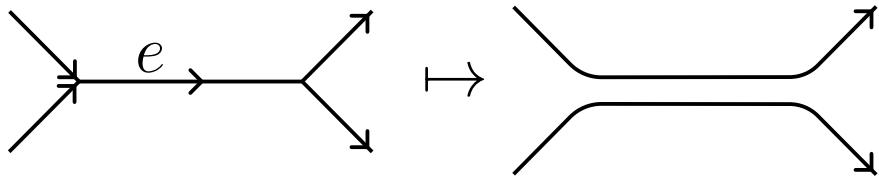
Definition 1.9. Γ の edge であって, 両端に接続されている edge の向きが一致している e を *delete* するとは, e を削除し, 三価性を保つように e の両端の頂点を削除することをいう. Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and let e be an edge of γ . *Delete* of e is defined as removing e and the two vertices at the ends of e also cease to exist to preserve the trivalence. To do this, it is required that the orientations of the two edges connecting to e at either end match.

$$d_e: \mathcal{K}(\Gamma) \rightarrow \mathcal{K}(d_e(\Gamma)); \gamma \mapsto d_e(\gamma)$$



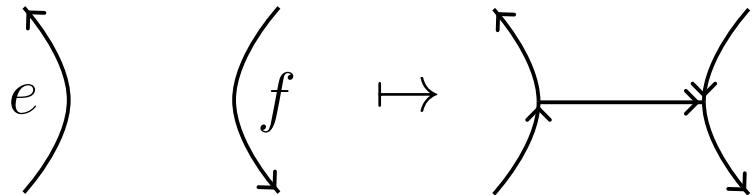
Definition 1.10. Γ の edge e を *unzip* するとは, e を “限りなく近い” 2つの edges に分け, 端点をなくすことをいう. 端点をなくしたとき, edge の向きが合っていることが必要である. 同様の議論で framed graph Γ に対し, unzip を定義できる. Let Γ be a trivalent graph and let $\gamma \in \mathcal{K}(\Gamma)$ be a KTG, and let e be an edge of γ . Unzip the edge e is replacing it by two edges that are “very close to each other”. The two vertices at the ends of e will disappear. Again the edges at the vertex where e begins have to both be incoming, while the edges at the vertex where e ends must both be outgoing.

$$u_e : \mathcal{K}(\Gamma) \rightarrow \mathcal{K}(u_e(\Gamma)) ; \gamma \mapsto u_e(\gamma)$$



Definition 1.11. 2つの trivalent graph とその edge のペア $(\Gamma, e), (\Gamma', f)$ の connected sum $\Gamma \#_{e,f} \Gamma'$ とは e, f をつなぐ edge を新たに作ること. well-defined であるために, 新たな edge の向きは Γ から Γ' への向きとし, KTGs においてはねじれを許さず, 辺を付ける場合は e, f の右側に付けるとする. (2次元では自由に動かせないため左右が重要) Let $(\Gamma, e), (\Gamma', f)$ be two pairs of trivalent graphs and their edges. The connected sum $\Gamma \#_{e,f} \Gamma'$ is obtained by joining e and f by a new edge. For this to be well-defined, we also need to specify the direction of the new edge, the cyclic orientations at each new vertex, and in the case of KTGs, the framing on the new edge. To compress notation, let us declare that the new edge be oriented from Γ towards Γ' , have no twists, and, using the blackboard framing, be attached to the right side of e and f .

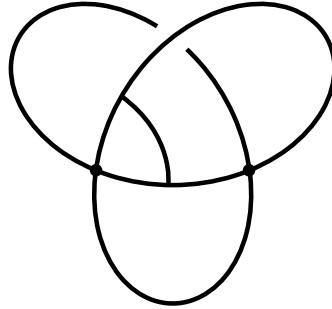
$$\#_{e,f} : \mathcal{K}(\Gamma) \times \mathcal{K}(\Gamma') \rightarrow \mathcal{K}(\Gamma \#_{e,f} \Gamma')$$



KTGs の finite type invariants は, links におけるものと同様に定義する. 同じ skeleton の KTGs の形式和を許し, 得られたベクトル空間を特異点の解

消によってフィルター分けする。We define finite type invariants of KTGs in the same way as for links. In detail, we allow linear combinations of KTGs with the same skeleton, and filter the resulting vector space by the resolution of singular points.

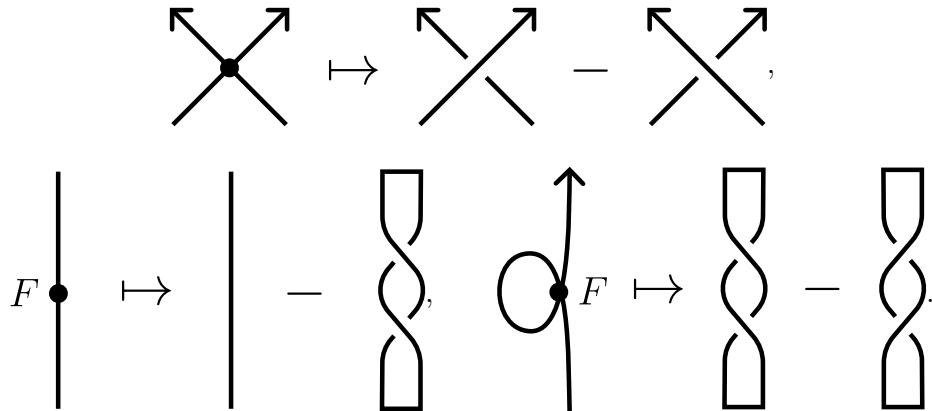
Definition 1.12. *n-singular KTG* とは, n この特異点を持つ trivalent graph の \mathbb{R}^3 へのはめ込み. 各特異点は横断的な 2 重点か, “F”と書かれた線上の点である. An *n-singular KTG* is a trivalent graph immersed in \mathbb{R}^3 with n singular points: each singular point is a transverse double point or a point on an edge marked with an “F”.



$n \geq 1$ に対し以下のようなベクトル空間を考える. For $n \geq 1$, we consider the following vector space:

$$\mathcal{F}'_n(\Gamma) := \left\{ \sum_{i=1}^m a_i \gamma'_i \mid m \in \mathbb{N}, a_i \in \mathbb{Q}, \gamma'_i: \text{KTG which has at least } n \text{ double point with skeleton } \Gamma \right\}$$

全ての特異点を解消する写像 $\rho: \mathcal{F}'_*(\Gamma) \rightarrow \mathcal{F}_0(\Gamma)$ を以下のように定める: We define a map $\rho: \mathcal{F}'_*(\Gamma) \rightarrow \mathcal{F}_0(\Gamma)$ that resolves all double points as follows:



$\mathcal{F}'_n(\Gamma)$ ($n \geq 1$) に対し, $\mathcal{F}_n(\Gamma)$ を $\mathcal{F}_n(\Gamma) := \rho(\mathcal{F}'_n(\Gamma))$ と定義すると, 明らかに $\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma)$ であり, For each $\mathcal{F}'_n(\Gamma)$ ($n \geq 1$), we define $\mathcal{F}_n(\Gamma) :=$

$\rho(\mathcal{F}'_n(\Gamma))$. Then, obviously, $\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma)$, and we obtain a following filtration:

$$\mathcal{K}(\Gamma) = \mathcal{F}_0(\Gamma) \supset \mathcal{F}_1(\Gamma) \supset \mathcal{F}_2(\Gamma) \supset \mathcal{F}_3(\Gamma) \cdots.$$

という filtration が得られる. この filtration において隣合う 2 つのベクトル空間から得られる商ベクトル空間を $\mathcal{A}_n(\Gamma) := \mathcal{F}_n/\mathcal{F}_{n+1}$ とし, associated graded space を以下のように定義する: In this filtration, we denote the quotient vector space obtained from two adjacent vector spaces by $\mathcal{A}_n(\Gamma) := \mathcal{F}_n/\mathcal{F}_{n+1}$, and define the associated graded space as follows:

$$\mathcal{A}(\Gamma) := \bigoplus_{n=0}^{\infty} \mathcal{A}_n(\Gamma) \left(= \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\Gamma)/\mathcal{F}_{n+1}(\Gamma) \right)$$

$\mathcal{A}(\Gamma)$ は chord diagram を用いて表すことができる. $\mathcal{A}(\Gamma)$ can be represented using chord diagrams.

Definition 1.13. skeleton graph Γ 上の n 次の *chord diagram* とは, Γ の辺上の $2n$ この点のペアからなる組み合わせ的なものであり, 辺の向きを保つ同相写像で移りあうものを同一視する. 特に, 次数 n の chord diagram を基底とする \mathbb{Q} 上のベクトル空間を $\mathcal{D}_n(\Gamma)$ と書く. A *chord diagram* of order n on a skeleton graph Γ is a combinatorial object consisting of a pairing of $2n$ points on the edges of Γ , up to orientation-preserving homeomorphisms of the edges. In particular, we denote by $\mathcal{D}_n(\Gamma)$ the vector space over \mathbb{Q} generated by chord diagrams of order n .

Proposition 1.14. $\pi: \mathcal{D}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$ is well-defined and surjective.

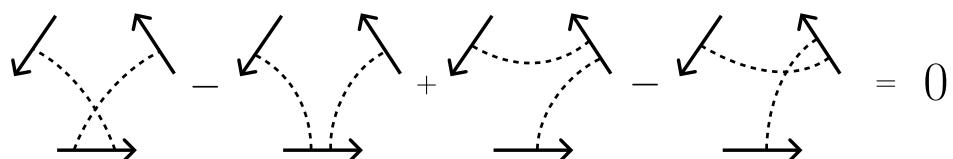
Proof. この論文だけ $\mathcal{A}(\Gamma)$ の構成法が違うので一旦パス.

(結局は $\{\text{Chord diagrams}\}/(\text{VI}, \text{4T}) \cong \mathcal{A}$ のはず)

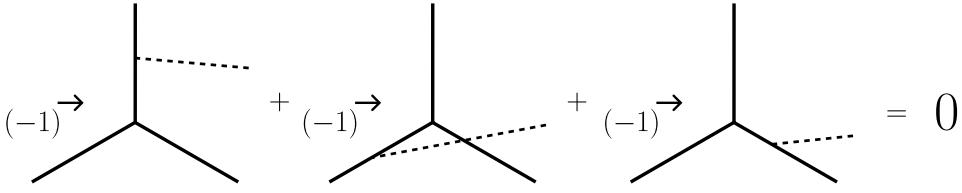
□

上記の写像の kernel に含まれる 2 つの関係式がある. There are two relations contained in the kernel of the above map π .

- (4T) Four term relation



- (VI) Vertex invariance relation



図に描かれていない部分には graph があるが、それらは全て同じでなければならぬ。4T では反時計回りの向きを与える（これ必要？）。VIにおいて、 $(-1)^\rightarrow$ は、chord の付いた edge が外向きなら -1、内向きなら 1 をかける（つまり式は 8 つある）。In both pictures, there may be other chords in the parts of the graph not shown, but they have to be the same throughout. In 4T, all skeleton parts (solid lines) are oriented counterclockwise. In VI, the sign $(-1)^\rightarrow$ is -1 if the edge the chord is ending on is oriented to be outgoing from the vertex, and $+1$ if it is incoming (thus there are 8 versions of the relation).

$4T, VI$ の relations が存在することは分かったが、これ以上の relation が存在“しない”ことを示すのは困難である。これを示すには、universal finite type invariant $\mathbb{Q}KTG \rightarrow \mathcal{A}$ を構成するのが最善である（ここでは定義しないが、後で一般の文脈で定義する）。これは、T.Le, H.Murakami, J. Murakami, T.Ohtsuki の結果をもとに、また Drinfeld の associator の理論を用いて [KO, CD],[BN] での Kontsevich integral を拡張する形で [MO97] で初めて得られた。Although it is easy to see that these relations are present, showing that there are no more is difficult, and is best achieved by constructing a universal finite type invariant $\mathbb{Q}KTG \rightarrow \mathcal{A}$ (we do not define universal finite type invariants here, but will do so later in the general context).

KTGs の各 operation は \mathcal{A} 上の operation を誘導する。 $(\mathcal{A}$ は $\mathcal{K}(\Gamma)$ の associated graded space である。) Each operation on KTGs induces an operation on \mathcal{A} (the associated graded space of $\mathcal{K}(\Gamma)$)。

- orientation switch
- edge delete
- edge unzip
- connected sum well-defined である。Introduction to Vassiliev knot invariants(Chmutov) の Lemma4.2.9

Theorem 1.15. *Any KTG can be obtained from the trivially embedded tetrahedron and the twisted tetrahedron by a finite sequence of the four operations defined above.*

2 Algebraic structures and expansions

\mathcal{K} において, orientation switch, edge delete, edge unzip, connected sum を linear に拡張し, \mathbb{Q} 係数の形式和を許すように拡張することで, \mathcal{K} は vector space となる. By linearly extending the operations of orientation switch, edge delete, edge unzip, and connected sum on \mathcal{K} to allow linear combinations with coefficients in \mathbb{Q} , \mathcal{K} becomes a vector space.

Definition 2.1. Γ を KTG とする. $\mathcal{K}(\Gamma)$ において, 係数の和が 0 となるような形式和全体から生成される集合を $\mathcal{I}(\Gamma)$ と書き, $\mathcal{I} := \bigoplus_{\Gamma'} \mathcal{I}(\Gamma')$ とする. Let Γ be a KTG. Let $\mathcal{I}(\Gamma)$ be the sub-structure made out of all such combinations in which the sum of coefficients is 0, and let $\mathcal{I} := \bigoplus_{\Gamma'} \mathcal{I}(\Gamma')$.

Example 2.2. Let γ_1, γ_2 and γ_3 be KTGs with skeleton Γ . Then, $\gamma_1 - \gamma_2, \gamma_1 - \frac{1}{2}\gamma - \frac{1}{2}\gamma_3 \in \mathcal{I}(\Gamma)$.

Definition 2.3. \mathcal{I}^m を, \mathcal{I} の元を少なくとも m 個含むようなものから任意の演算の合成で得られる元が生成する \mathcal{K} の部分空間とする. つまり, Let \mathcal{I}^m be the set of all outputs of arbitrary compositions of the operations in \mathcal{K} that have at least m inputs in \mathcal{I} . In other words,

$$\mathcal{I}^m := \left\{ \gamma \in \mathcal{K} \mid \begin{array}{l} \text{There exist } n, f: \prod_{i=1}^n \mathcal{K} \rightarrow \mathcal{K}, x_1, \dots, x_n \in \mathcal{K} \\ \text{such that } \gamma = f(x_1, \dots, x_n), \#\{i \mid x_i \in \mathcal{I}\} \geq m \end{array} \right\}.$$

さらに, $\mathcal{I}^m(\Gamma) := \mathcal{I}^m \cap \mathcal{I}(\Gamma)$ とする. Moreover, we define $\mathcal{I}^m(\Gamma) := \mathcal{I}^m \cap \mathcal{I}(\Gamma)$.

ここで, \mathcal{I}^m は明らかに filtration の構造をもつ. Clearly, \mathcal{I}^m has a filtration structure.

Lemma 2.4. $\mathcal{I}(\Gamma) = \{\sum_i c_i (\gamma_{1i} - \gamma_{2i}) \mid \gamma_{1i}, \gamma_{2i}: \text{generators of } \mathcal{K}(\Gamma), c_i \in \mathbb{Q}\}$.

Proof. (\supset) This is obvious.

(\subset) For any element of \mathcal{I} , it can be written as $\sum_{i=1}^n c_i \gamma_i$. Since $\sum_{i=1}^n c_i = 0$, we have $c_n = -\sum_{i=1}^{n-1} c_i$. Thus

$$\sum_{i=1}^n c_i \gamma_i = c_1 \gamma_1 + c_2 \gamma_2 + \cdots + \left(-\sum_{i=1}^{n-1} c_i \right) \gamma_n = \sum_{i=1}^{n-1} c_i (\gamma_i - \gamma_n).$$

□

Theorem 2.5. $\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ for all $n \geq 0$ and skeleton Γ .

Proof. (i) $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$

(\supset) 任意の $\mathcal{F}_1(\Gamma)$ の元は少なくとも 1 つの double point を持つような 1-singular KTG の交点の正負の差を元に持つため, $\mathcal{F}_1(\Gamma) \subset \mathcal{I}(\Gamma)$.

Since any element $\gamma \in \mathcal{F}_1(\Gamma)$ has at least one double point, so there exist $\gamma_+, \gamma_- \in \mathcal{K}(\Gamma)$ such that $\gamma = \gamma_+ - \gamma_-$, thus $\mathcal{F}_1(\Gamma) \subset \mathcal{I}(\Gamma)$.

(\subset) 任意の $\mathcal{I}(\Gamma)$ の元は Lemma 2.4 より $\sum_i c_i(\gamma_i - \gamma'_i)$ と書ける. $\mathcal{F}_1(\Gamma)$ において, 同じ skeleton を持つ任意の 2 つの KTG は crossing change により移りあうため, $\gamma_i - \gamma'_i$ を 1 点における正負の交差の差 $\tilde{\gamma}_i - \tilde{\gamma}'_i$ なるようにできる. よって

$$\sum_i c_i(\gamma_i - \gamma'_i) = \sum_i c_i(\tilde{\gamma}_i - \tilde{\gamma}'_i) \in \mathcal{F}_1(\Gamma).$$

Any element of $\mathcal{I}(\Gamma)$ can be written as $\sum_i c_i(\gamma_i - \gamma'_i)$ by Lemma 2.4. In $\mathcal{F}_1(\Gamma)$, any two KTGs with the same skeleton can be related by crossing changes, so we can make $\gamma_i - \gamma'_i$ into the difference of positive and negative crossings at one point $\tilde{\gamma}_i - \tilde{\gamma}'_i$. Therefore,

$$\sum_i c_i(\gamma_i - \gamma'_i) = \sum_i c_i(\tilde{\gamma}_i - \tilde{\gamma}'_i) \in \mathcal{F}_1(\Gamma).$$

(ii) $\mathcal{I}^n(\Gamma) \subset \mathcal{F}_n(\Gamma)$

By $\mathcal{I}(\Gamma) = \mathcal{F}_1(\Gamma)$, any element $\gamma \in \mathcal{I}^n(\Gamma)$ is generated by at least n elements of $\mathcal{F}_1(\Gamma)$. we check that the four operations preserve number of double points.

- orientation switch of an edge with double point. 定義より, switch



する edge につく chord の数だけ -1 倍するため, double point の数は変わらない.

- edge delete. edge に chord が接続されている場合, 定義より dia-



gram は 0 になるため, double point の数は変わらない. edge に接続されていない場合, double point の数は変わらない.

- connected sum.



- edge unzip. Unzip を行う edge に chord が接続されている場合, chord は 2 つに分かれるため, double point の数は変わらない.

(iii) $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$

Lemma 2.6. *Any n -singular KTG can be obtained from the trivially embedded tetrahedron, twisted tetrahedron and singular twisted tetrahedron using the four operations.*

Proof. Same as Theorem 1.15. □

Since any n -singular KTG can be obtained from n pieces of 1-singular KTGs by Lemma 2.6 and the four operations, $\mathcal{F}_n(\Gamma) \subset \mathcal{I}^n(\Gamma)$.

Therefore, $\mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$ for all $n \geq 0$ and skeleton Γ . □

Definition 2.7. Let $\Gamma, \mathcal{K}(\Gamma)$ be a skeleton and the set of KTGs with skeleton Γ . An *expansion* Z for $\mathcal{K}(\Gamma)$ is a map $Z: \mathcal{K}(\Gamma) \rightarrow \hat{\mathcal{A}}(\Gamma) = \prod_{n=0}^{\infty} \mathcal{A}_n(\Gamma)$ such that if $\gamma \in \mathcal{I}^n(\Gamma) = \mathcal{F}_n(\Gamma)$, then $Z(\gamma) \in \prod_{n \geq m} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$ and $\text{gr } Z: \text{gr } \mathcal{K}(-) \rightarrow \text{gr } \text{proj } \mathcal{K}(-)$ is the identity map, where $\text{proj } \mathcal{K}(\Gamma) := \bigoplus_{n=0}^{\infty} \mathcal{I}^n(\Gamma)/\mathcal{I}^{n+1}(\Gamma)$.

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