Self-balancing and Velocity Control of Two-Wheeled Mobile Robot Based on LQR Sliding Mode

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***Abstract*—This paper introduces a control method based on LQR sliding mode for the self-balancing and velocity control of the two-wheeled mobile robots(WMR) with external disturbance. The WMR motion mainly consists of the swing of the pendulum and the rotation of the wheel without taking turning movement into consideration. Firstly, a dynamic model of the two-wheeled mobile robots(WMR) is constructed, in which the WMR motor voltage is used as the input of the Newton method, and the non-complete sliding-free external disturbance is considered. Then, in order to realize the real-time control of the swing angle and speed, the dynamic control based on the LQR sliding mode control method is derived. On the theoretical level, the correctness of the proposed controller is proved by the Lyapunov method. The numerical simulation results also illustrate the effectiveness of the proposed control method.**

***Index Terms*—two-wheeled mobile robots, LQR sliding mode, Newton method, Lyapunov method**

I. Introduction

In the past few years, a vast majority of studies on Wheeled Mobile Robots(WMR) have been attracted lots of attention due to its small size, portable, high flexible etc so that it can be applied into traditional industry, military, search and rescue work, daily patrol and many aspects of our life. There are also many researches on balancing control and tracking control of this underactuated system. As the continuous innovation and improvement of the control theory technology, modern control methods such as backstepping [3], fuzzy control, neural network control [4]- [5], [9], ADRC [7],and adaptive control [12]etc have been applied to the control system of two-wheeled self-balancing mobile robots. Felix Grasser and others from the Swiss Federal Institute of Technology (EPFL) proposed a concept of two-wheeled robot named JOE [1], they applied the pole placement method into the control system designing. In [2], a sliding-mode control method is proposed for two-wheeled mobile robots in polar coordinates. On the one hand, most of them proposed their control scheme without considering external disturbances which can not be ignored. On the other hand,although these intelligent methods also achieve the control performance we want get,their robustness and anti-jamming performance are not very good. And recently, in [8], the author tackled the control problem when the WMR moved on the slope,but he ignored the external disturbance influence. For uncertain mobile robots, a new adaptive control scheme is designed in [11] to achieve path tracking while considering slipping,skidding and external disturbances as well. However, they only considered the kinematics when designing the controller.

With the continuous advancement of technology, the theory of sliding mode control has gradually improved.Many scholars applied this technology into a variety of control systems duo to its anti-interference performance, strong robustness and unaffected by parameter changing. In this paper, a novel LQR sliding mode control method is proposed for the two-wheeled mobile robots with external disturbance to achieve balance control and velocity control. Different from previous researches, the external disturbance the forces generated by the car body swinging are considered in the process of modeling. And then, combining LQR with sliding mode control, the wheeled mobile robot is unaffected by the effects of limited external disturbances on the whole and has less time to reach the balance point.

Considering the quality of system dynamics of the WMR, we give some variables clear definition in TABLE I:

II. System Modeling via Newton Methods

The WMR can move forward or backward around the central axis of two wheels and rotate around the vertical axis. To combine the WMR's behavior and its input, we need a math model of the system. Thus, we need to model the WMR with the common methods, like Newton methods classical mechanical analysis or Lagrange method based on energy analysis. This paper build the mathematical dynamic model of the WMR subjected to external disturbance via Newton methods.

A. Analysis of the Left Wheel

The force analysis of the left wheel is shown in Fig.1. According to Newton's second law, the analysis of the dy-

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TABLE I

Variable	*definition*
M_p	The mass of the WMR's pendulum
M_ω	The mass of the WMR's right and left wheel
l	Distance between the z axis and centroid of the chassis
D	Distance between right wheel and left wheel
R	Radius of the wheel
x_c	Displacement of the wheels
$\dot{x_c}$	Velocity of the wheels
T_R, T_L	The driven-torque of left and right wheel
J_ω	Moment of inertia of the wheels rotating around the z axis
J_δ	Moment of inertia of the chassis with respect to the y axis
J_p	Moment of inertia of the chassis with respect to z axis
k_m	The motor torque constant
k_e	The motor back electromotive force constant
θ	Pitch angle of the WMR's pendulum
δ	Yaw angle
U_L, U_R	The driven-voltage of motor

namic characteristic of left wheel around X-axis is described as following:

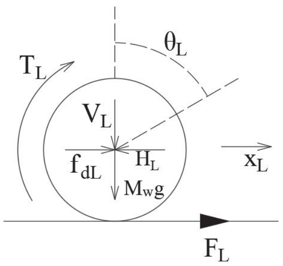

Fig. 1. Force Analysis of Left-Wheel

$$\left\{ \begin{array}{l} \ddot{x}_L M_w = F_L - H_L + f_{dL} \\ \ddot{\theta}_L J_w = T_L - F_L R \end{array} \right. \tag{1}$$

where H_L is the force between the left wheel and pendulum, F_L is the friction force between the left wheel and floor.

B. Analysis of the Right Wheel

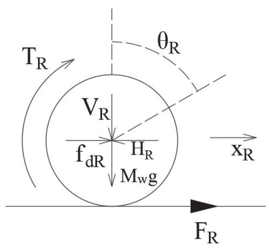

Fig. 2. Force Analysis of Right-Wheel

The force analysis of the right wheel is shown in Fig.2. According to Newtonian second law, the analysis of the dynamic characteristic of right wheel on X-axis is described as following:

$$\left\{ \begin{array}{l} \ddot{x}_R M_w = F_R - H_R + f_{dR} \\ \ddot{\theta}_R J_w = T_R - F_R R \end{array} \right. \tag{2}$$

where H_R is the force between the right wheel and pendulum, F_R is the friction force between the right wheel and floor.

Eliminate the friction between the left wheel and the ground F_L ,the friction between the right wheel and the ground F_R, we can get:

$$\left\{ \begin{array}{l} H_L = f_{dL} + \frac{T_L}{R} - \ddot{x}_L M_w - \frac{\ddot{x}_L}{R^2} J_w \\ H_R = f_{dR} + \frac{T_R}{R} - \ddot{x}_R M_w - \frac{\ddot{x}_R}{R^2} J_w \end{array} \right. \tag{3}$$

When the WMR turns, it has the following relationship:

$$\ddot{\delta} J_\delta = (H_L - H_R) \frac{D}{2} \tag{4}$$

According to $\ddot{x}_R = R\dot{x}_R, \ddot{x}_L = R\dot{x}_L$ and substituting (4) into (3),we can get:

$$\begin{aligned} \ddot{\delta} = & \frac{D}{\left(2J_\delta R + M_w D^2 R + J_w \frac{D^2}{R}\right)} (T_L - T_R) \\ & + \frac{DR(f_{dL} - f_{dR})}{\left(2J_\delta R + M_w D^2 R + J_w \frac{D^2}{R}\right)} \end{aligned} \tag{5}$$

C. Analysis of the WMR's Pendulum

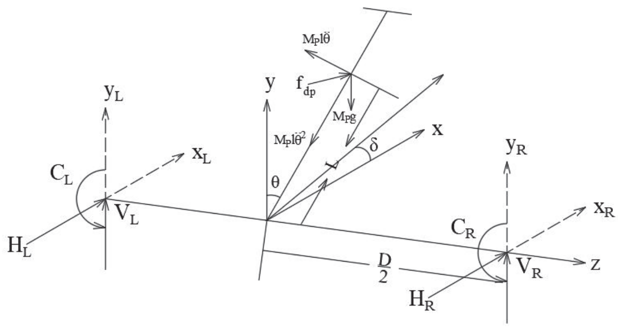

Fig. 3. Force analysis of the WMR's pendulum

The force analysis of the WMR's pendulum is shown in Fig. 3, and the dynamic characteristic of the WMR's pendulum can be analysised according to the Newton's second law.

$$\begin{aligned} M_p \ddot{x}_p = & M_p \left(\ddot{x}_c - l\sin\theta\dot{\theta}^2 + l\cos\theta\ddot{\theta}\right) \\ & = (H_L + H_R) + f - M_p l\ddot{\theta}\cos\theta - M_p l\dot{\theta}^2 \sin\theta \\ \left(M_p + 2M_w + \tfrac{2J_w}{R^2}\right) & \ddot{x}_c + 2M_p l\cos\theta\ddot{\theta} = \tfrac{T_L + T_R}{R} + f \end{aligned} \tag{6}$$

where x_p is the displacement of the centroid of the WMR's pendulum, $x_p = x_c + l\sin\theta$, $f = f_{dp} + f_{dL} + f_{dR}$.

Along the horizontal direction of WMR's pendulum, we can analyze

$$\begin{aligned} & M_p \ddot{x}_p \cos\theta = M_p \left(\ddot{x}_c - l\sin\theta\dot{\theta}^2 + l\cos\theta\ddot{\theta}\right)\cos\theta \\ & = M_p \ddot{x}_c \cos\theta - M_p l\sin\theta\cos\theta\dot{\theta}^2 + M_p l\cos^2\theta\ddot{\theta} \\ & = M_p g\sin\theta + (H_L + H_R)\cos\theta - (V_L + V_R)\sin\theta - M_p l\ddot{\theta} \end{aligned} \tag{7}$$

According to the law of rotation, we can get

$$J_p \ddot{\theta} = (V_L + V_R)\, L\sin\theta - (H_L + H_R)\, L\cos\theta - (T_L + T_R) \tag{8}$$

united (9) and (10), remove $(V_L + V_R)Lsin\theta$ and $(H_L + H_R)Lcos\theta$, we can get:

$$\begin{aligned} &M_p l\cos\theta\ddot{x}_c + \left(J_p + M_p l^2\cos^2\theta + M_p l^2\right)\ddot{\theta} \\ &= M_p l^2 \sin\theta\cos\theta\dot{\theta}^2 + M_p gL\sin\theta - (T_L + T_R) \end{aligned} \tag{9}$$

united (6)-(8) and (11), the state variable about θ and x_c can be obtained:

$$\begin{aligned} \ddot{x}_c = &\frac{\left[\frac{M_p l^2 + M_p l^2\cos^2\theta + J_p}{R} + 2M_p l\cos\theta\right](T_R + T_L)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &- \frac{-2{M_p}^2 l^3\sin\theta\cos^2\theta\dot{\theta}^2 - 2{M_p}^2 g l^2\sin\theta\cos\theta}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &+ \frac{(f_{dp} + f_{dR} + f_{dL})\left(M_p l^2 + M_p l^2\cos^2\theta + J_p\right)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \end{aligned} \tag{10}$$

$$\begin{aligned} \ddot{\theta} = &\frac{-\left[\frac{M_p l\cos\theta}{R} + M_p + 2M_w + \frac{2J_w}{R^2}\right](T_R + T_L)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &+ \frac{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)\left(M_p l^2\sin\theta\cos\theta\dot{\theta}^2 + M_p g l\sin\theta\right)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &- \frac{M_p l\cos\theta(f_{dp} + f_{dR} + f_{dL})}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \end{aligned} \tag{11}$$

where $a = 2{M_p}^2 l^2\cos^2\theta$.

According to the relationship between motor voltage and the WMR's torque:

$$\begin{cases} T_L = K_m\left(U_L - K_e\dot{\theta}\right) \\ T_R = K_m\left(U_R - K_e\dot{\theta}\right) \end{cases} \tag{12}$$

where U_L and U_R represents the control voltage of the left and right wheels, K_m represents the motor torque constant, and K_e represents the motor back electromotive force constant.

With the above deduction, substituting (14) into (5), (12), (13), the dynamic equations of WMR can be obtained

$$\begin{aligned} \ddot{x}_c = &\frac{K_m\left[\frac{M_p l^2 + M_p l^2\cos^2\theta + J_p}{R} + 2M_p l\cos\theta\right](U_R + U_L)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &- \frac{2K_m K_e\dot{\theta}\left[\frac{M_p l^2 + M_p l^2\cos^2\theta + J_p}{R} + 2M_p l\cos\theta\right]}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &+ \frac{-2{M_p}^2 l^2\sin\theta\cos\theta\left(l\cos\theta\dot{\theta}^2 + g\right) + fb}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \end{aligned} \tag{13}$$

$$\begin{aligned} \ddot{\theta} = &\frac{-K_m\left[\frac{M_p l\cos\theta}{R} + M_p + 2M_w + \frac{2J_w}{R^2}\right](T_R + T_L)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &+ \frac{+2K_m K_e\dot{\theta}\left[\frac{M_p l\cos\theta}{R} + M_p + 2M_w + \frac{2J_w}{R^2}\right]}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \\ &+ \frac{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)M_p l\sin\theta\left(l\cos\theta\dot{\theta}^2 + g\right) - M_p l f\cos\theta}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(M_p l^2 + M_p l^2\cos^2\theta + J_p) - a} \end{aligned} \tag{14}$$

where $a = 2{M_p}^2 l^2\cos^2\theta, f = f_{dp} + f_{dR} + f_{dL}, b = M_p l^2 + M_p l^2\cos^2\theta + J_p$.

D. Linearize the system of WMR

In order to better design controller to make the WMR stable, we will linearize the equation we get above near the equilibrium point and in a small angle. We set $\sin\theta \approx \theta, \cos\theta \approx 1, \dot{\theta}^2 \approx 0$, the model of WMR, denoted by the above equation, can be simplified as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{v}_c \\ \dot{\theta} \\ \dot{\omega} \\ \dot{\delta} \\ \ddot{\delta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & a_{11} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{22} & a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ v_c \\ \theta \\ \omega \\ \delta \\ \dot{\delta} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ b_{11} & b_{11} \\ 0 & 0 \\ b_{21} & b_{21} \\ 0 & 0 \\ b_{31} & b_{31} \end{bmatrix} \begin{bmatrix} U_L + U_R \\ U_L - U_R \end{bmatrix} + \begin{bmatrix} 0 \\ D_1 \\ 0 \\ D_2 \\ 0 \\ D_3 \end{bmatrix} \end{aligned} \tag{15}$$

where we definite the state variable $x = \begin{bmatrix} x_c & v_c & \theta & \omega & \delta & \dot{\delta} \end{bmatrix}^T$, and

$$\begin{cases} a_{11} = \frac{-2K_m K_e\left[\frac{2M_p l^2 + J_p}{R} + 2M_p l\right]}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ a_{12} = \frac{-2{M_p}^2 l^2 g}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ a_{21} = \frac{2K_m K_e\left[\frac{M_p l}{R} + M_p + 2M_w + \frac{2J_w}{R^2}\right]}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ a_{22} = \frac{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)M_p g l\theta}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ b_{11} = \frac{K_m\left[\frac{M_p l^2 + M_p l^2 + J_p}{R} + 2M_p l\right](U_R + U_L)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ b_{21} = \frac{-K_m\left[\frac{M_p l}{R} + M_p + 2M_w + \frac{2J_w}{R^2}\right]}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ b_{31} = \frac{K_m D}{\left(2J_\delta R + M_w D^2 R + J_w\frac{D^2}{R}\right)} \\ D_1 = \frac{(f_{dp} + f_{dR} + f_{dL})\left(2M_p l^2 + J_p\right)}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ D_2 = \frac{-M_p l(f_{dp} + f_{dR} + f_{dL})}{\left(M_p + 2M_w + \frac{2J_w}{R^2}\right)(2M_p l^2 + J_p) - 2{M_p}^2 l^2} \\ D_3 = \frac{DR(f_{dL} - f_{dR})}{\left(2J_\delta R + M_w D^2 R + J_w\frac{D^2}{R}\right)} \end{cases} \tag{16}$$

III. DECOUPLING DESIGN OF WMR

Since the pitch angle and the yaw angle of the WMR are not related to each other, the decoupling matrix T can be used to convert the torque input on the left U_L and right wheels U_R into the voltage input in the speed U_θ and in the yaw direction U_δ. There, follow the above method, we can decouple the original sixth-order system into one fourth-order speed subsystem and a second-order yaw subsystem. The state space equation of the WMR can be transformed into the following form:

$$\begin{aligned}\begin{bmatrix} \dot{x}_c \\ \dot{v}_c \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{12} & a_{11} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{22} & a_{21} & 0 \end{bmatrix} \begin{bmatrix} x_c \\ v_c \\ \theta \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 2b_{11} \\ 0 \\ 2b_{21} \end{bmatrix} \begin{bmatrix} U_\theta \\ U_\delta \end{bmatrix} + \begin{bmatrix} 0 \\ D_1 \\ 0 \\ D_2 \end{bmatrix} \\ &= A_{11}x_{11} + B_{11}U_1 \end{aligned} \tag{17}$$

$$\begin{aligned}\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ 2b_{31} \end{bmatrix} U_\delta + \begin{bmatrix} 0 \\ D_3 \end{bmatrix} \\ &= A_{22}x_{22} + B_{22}U_2 \end{aligned} \tag{18}$$

where the decoupling unit of the system is

$$\begin{bmatrix} U_L \\ U_R \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} U_\theta \\ U_\delta \end{bmatrix} \tag{19}$$

The value of WMR is related variable designed in the paper are given in TABLE II.

TABLE II

Variable	*Value*	*Variable*	*Value*
M_p	16.33Kg	J_r	$0.00623Kg \cdot m^2$
M_ω	1.13Kg	J_p	$0.8039Kg \cdot m^2$
l	0.086m	J_p	$0.190646Kg \cdot m^2$
D	0.4m	k_m	$0.0508N{\cdot}m/V$
k_e	0.5732	R	0.105m

Substituting it into the state space equation we have obtained, we can get

$$\begin{aligned}\begin{bmatrix} \dot{x}_c \\ \dot{v}_c \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -15.0457 & -1.0236 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 41.9119 & 0.1789 \end{bmatrix} \begin{bmatrix} x_c \\ v_c \\ \theta \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0.1607 \\ 0 \\ -0.3121 \end{bmatrix} \begin{bmatrix} U_\theta \\ U_\delta \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1607\,(f_{dp}+f_{dR}+f_{dL}) \\ 0 \\ -0.3121\,(f_{dp}+f_{dR}+f_{dL}) \end{bmatrix} \end{aligned} \tag{20}$$

IV. THE CONTROLLER DESIGNING OF WMR

In the process of designing the controller, this paper only considers the decoupled fourth-order velocity subsystem.

$$\begin{aligned}\dot{x}=Ax+Bu+D=\begin{bmatrix} \dot{x}_c \\ \dot{v}_c \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -15.0457 & -1.0236 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 41.9119 & 0.1789 \end{bmatrix} \begin{bmatrix} x_c \\ v_c \\ \theta \\ \omega \end{bmatrix} \\ + \begin{bmatrix} 0 \\ 0.1607 \\ 0 \\ -0.3121 \end{bmatrix} \begin{bmatrix} U_\theta \\ U_\delta \end{bmatrix} &+ \begin{bmatrix} 0 \\ 0.1607\,(f_{dp}+f_{dR}+f_{dL}) \\ 0 \\ -0.3121\,(f_{dp}+f_{dR}+f_{dL}) \end{bmatrix} \end{aligned} \tag{21}$$

A. Linear Quadratic Regulator (LQR)

Without considering the external disturbances, the state equation of the system can be written as

$$\dot{x} = Ax + Bu \tag{22}$$

LQR optimal regulator design is finding the appropriate state feedback control law $u = -Kx$ in the system to minimize the following objective functions of a linear system.

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{23}$$

where $K = R^{-1}B^TP$, P is the solution of Riccati differential equation, and is the parameter matrix to be designed. Q is a semi-positive symmetric matrix, which is the weighting matrix of the state variable x. R is a positive definite symmetric matrix, which is the weighting matrix for the control input variable u.

The state feedback matrix of the system under control can solved through using the command functionin $K = lqr(A, B, Q, R)$ in Matlab. If we choose $Q = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $R = 1$,we can get $K = \begin{bmatrix} -10.0000 & -25.1082 & -340.2245 & -58.1937 \end{bmatrix}$.

B. LQR Sliding Mode Method

In the case of considering the external disturbance, under the action of LQR control, the original equation of state can be written as $\dot{x} = (A - BK)x + D$, where D is the external disturbance, and it is bounded $D \leq \gamma$.

The system under LQR control is only vulnerable to external disturbance, and may even makes the system unstable. Since sliding mode control technology is insensitive to external disturbance and parameter changes, this paper a new control technology that is combining LQR control and sliding mode control to reduce the impact of external disturbance on the system.

Select the sliding surface for the subsystem:

$$s = (x - x(0)) - A_k \int_0^t x d\tau \tag{24}$$

where the initial value $s(0) = 0$.

This paper designs the sliding mode control law based on the LQR control law to suppress the external disturbance towards the system and make the system stable, the comprehensive control law of the system is:

$$C_\theta = -Kx - k_1 s - \eta sgn(s) \tag{25}$$

where $k1$ and η is positive constant.
Proof:
First, define a Lyapunov function

$$V = \frac{1}{2}s^2 \tag{26}$$

$$
\begin{aligned}
\dot{V} &= s\dot{s} = s^{T}(\dot{x} - A_k x) \\
&= s(Ax + Bu + D - A_k x) \\
&= s(Ax + B\left(-Kx - k_1 s - \eta \operatorname{sgn}(s)\right) + D{-}A_k x) \\
&= s(B(-k_1 s - \eta \operatorname{sgn}(s) + D) \\
&= -k_1 B s^2 - \eta B\left|s\right| + s^T D \\
&\leq -k_1 B s^2 - \eta B\left|s\right| + \gamma\left|s\right| \\
&= -k_1 B s^2 - (\eta B - \gamma)\left|s\right| \leq 0
\end{aligned} \tag{27}
$$

If and only if $(\eta B - \gamma) > 0$, the system can be stable. In order to minimize the system chatter caused by sliding mode control,we choose the $sat(s)$ function instead of $sgn(s)$. where

$$
sat(s) = \begin{cases} 1 & s > \delta \\ ks & |s| \leq \delta \\ -1 & s < \delta \end{cases} \qquad s = \frac{1}{\Delta}, \Delta = 0.1
$$

V. Simulation

A. The Control Effect with LQR Methods

The designing controller with LQR methods was used to control the linear model of WMR in the position of near zero angle. Given an initial state $x = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$, and the control effect with this controller we designed is shown in Fig.4.

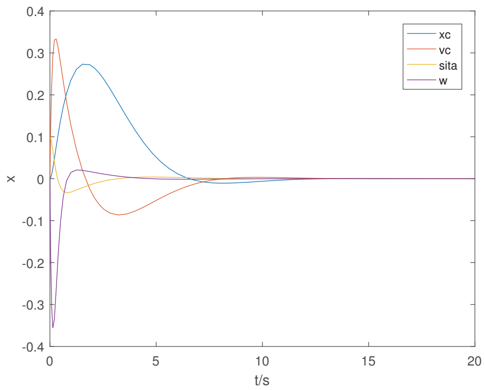

Fig. 4. The control effect with LQR methods without external disturbance

In Fig.4, the control effect with this controller we designed for the linear model of WMR without external disturbance is good. It has a small overshoot with its control and the displacement, velocity, swing angle, the angular speed of WMR can reach to near zero point in about 11s. But In Fig.5, we can see overshoot, and displacement, velocity, swing angle and the angular speed of WMR are also more or less affected by external disturbance. Thus the system dynamic performance of WMR need be further improved.

B. The Control Effect with LQR Sliding Methods

In Fig.6, the control effect of the system with external disturbance through LQR sliding methods is good. The swing angle, angular speed, displacement, velocity of the WMR can reach to balance point in a shorter time and basically

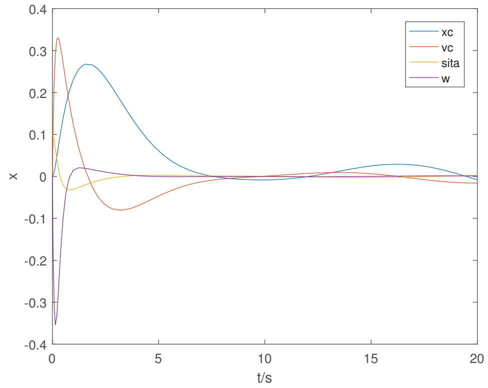

Fig. 5. The control effect with LQR methods existing external disturbance

immune to external disturbance. The fly in the ointment is that the system's overshoot has increased slightly, but the overall control effect is improved and its anti-interference performance is very good.

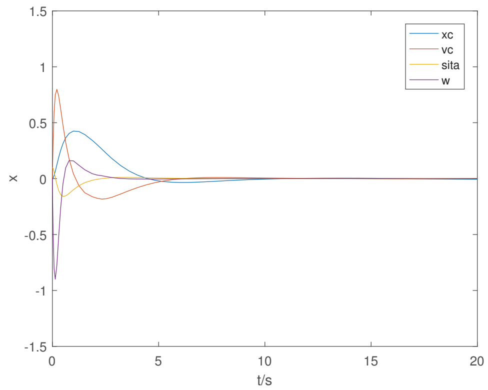

Fig. 6. The control effect with LQR sliding methods existing external disturbance

Given the initial condition $x = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$, and the input of LQR sliding controller is shown as Fig.7. We can notice that it can reach to zero point in a shorter time with nearly no chatter.

VI. Conclusion

This article models the WMR with external disturbance through Newton methods firstly, and then designs the LQR sliding mode controller to improve the robustness of the system. Comparing the the results with LQR method and the results using the LQR sliding mode controller, we can notice that the latter has better robustness. The swing angle, angular speed, displacement, velocity of the WMR can reach to the balance point in a shorter time and basically immune to external interference compared to the former method. And in the process of designing controller, we replace the ideal sliding mode of symbol function by saturated function to reduce the system chattering. The fly in the ointment is that the system's

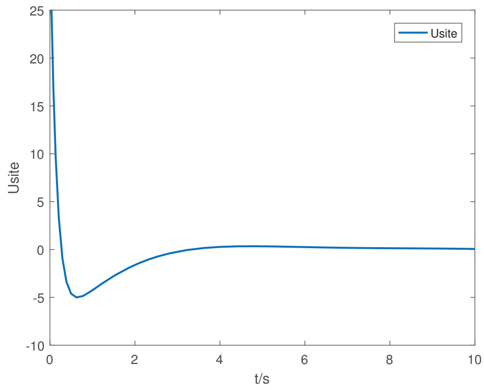

Fig. 7. Input of LQR sliding mode controller

overshoot has increased slightly, but not affect the control effect, and improve system robustness. In future work, I will try to reduce the system's overshoot as much as possible while enhancing system robustness and improving adjustment time.

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