

# Representations of reals in reverse mathematics

Jeffrey L. Hirst

Appalachian State University

December 20, 2005

## **Abstract**

Working in the framework of reverse mathematics, we consider representations of reals as rapidly converging Cauchy sequences, decimal expansions, and two sorts of Dedekind cuts. Converting single reals from one representation to another can always be carried out in  $\text{RCA}_0$ . However, the conversion process is not always uniform. Converting infinite sequences of reals in some representations to other representations requires the use of  $\text{WKL}_0$  or  $\text{ACA}_0$ .

Early in the study of computable analysis, several authors noted that

many representations of computable reals could be computably converted to other representations on a real by real basis [8], [7], [5]. Mostowski [4] observed that converting certain sequences of computable reals between representations was not a computable process. Because of the the significance of sequences of reals in computable analysis [6] and reverse mathematics [9], this is more than an idle curiosity.

We will analyze representations of reals using the techniques of reverse mathematics. The subsystems used in this paper are  $\text{RCA}_0$ ,  $\text{WKL}_0$ , and  $\text{ACA}_0$ . The systems differ in the available set comprehension axioms.  $\text{RCA}_0$  includes the recursive comprehension axiom, which essentially asserts the existence of relatively computable sets.  $\text{WKL}_0$  appends a weak version of König's lemma that says that infinite 0–1 trees have infinite paths.  $\text{ACA}_0$  adds a comprehension scheme for arithmetically definable sets. Simpson's book [9] is an excellent resource for complete details about these subsystems.

Section 1 introduces the various representations considered here and notions of equality between reals. Section 2 includes conversion results that can be proved in  $\text{RCA}_0$ , including conversions for single reals. Section 3 presents equivalence results showing the necessity of using stronger axiom systems for some conversions. That section ends with a table summarizing the results of

sections 2 and 3. Section 4 presents related results on sequences of irrational numbers and change of basis for expansions.

## 1 Representations of reals

We will consider four ways of representing reals and encoding these representations in  $\text{RCA}_0$ . The first is the usual rapidly converging Cauchy sequence used in reverse mathematics. A function  $\rho : \mathbb{N} \rightarrow \mathbb{Q}$  is a *rapidly converging Cauchy sequence* if  $\rho$  satisfies

$$\forall k \forall i |\rho(k) - \rho(k+i)| \leq 2^{-k}.$$

For our purposes, a decimal expansion is a special sort of rapidly converging Cauchy sequence in which  $\delta(j)$  gives the first  $j$  decimal places of the decimal representation of the real. Thus,  $\delta : \mathbb{N} \rightarrow \mathbb{Q}$  is a *decimal expansion* if  $\delta(0)$  is an integer or the special digit  $-0$ , and

$$\forall k \exists j \in \{0, \dots, 9\} (\delta(k+1) - \delta(k) = \text{sign}(\delta(0)) \cdot j \cdot 10^{-k-1}).$$

In this definition, decimal expansions terminating in either repeating nines or repeating zeros are allowed. We will treat these special cases in our discussion of equality. To make the signs work correctly, we must distinguish between  $-0$  and  $0$  as a digit. For example, the first digit of an element of the interval  $(-1, 0)$  will be  $-0$ . The first digit in a representation of  $0$  could be either  $-0$  or  $0$ .

The remaining two representations are forms of Dedekind cuts. Since  $\text{RCA}_0$  proves that the complement of any given subset of  $\mathbb{Q}$  exists, we can encode a cut by specifying just the elements of the lower set. To be precise, a set  $\lambda \subseteq \mathbb{Q}$  is a (lower) *Dedekind cut* if  $\emptyset \subsetneq \lambda \subsetneq \mathbb{Q}$  and

$$\forall s \in \mathbb{Q} \forall s' \in \mathbb{Q} ((s \in \lambda \wedge s' \notin \lambda) \rightarrow s < s').$$

This definition is exactly like that in section IV of Dedekind [1] in that cuts representing a rational number may or may not contain the rational. Many modern analysis texts specify the location of the rational in this case. We can append this requirement to the definition as follows. A set  $\sigma \subset \mathbb{Q}$  is an *open cut* if it is a Dedekind cut and  $\forall s \in \sigma \exists s' \in \sigma (s < s')$ . This completes our list of representations of reals: rapidly converging Cauchy sequences, decimal

expansions, Dedekind cuts, and open cuts.

In reverse mathematics, equality of sets is defined extensionally from equality on natural numbers. Similarly, equality of representations of reals requires definition. For example, following Simpson [9], if  $\rho$  and  $\tau$  are rapidly converging Cauchy sequences, then we say that  $\rho$  and  $\tau$  are equal (and write  $\rho = \tau$ ) if

$$\forall k \left( |\rho(k) - \tau(k)| \leq 2^{-k+1} \right).$$

Naïvely, we are saying that  $\rho = \tau$  if the sequences converge to the same real. Technically, we are abusing notation, since we may write  $\rho = \tau$  (as reals) even when  $\rho$  and  $\tau$  are not equal as sets.

Since a decimal expansion is a special sort of rapidly converging Cauchy sequence, equality of decimal expansions is defined as in the preceding paragraph. In  $\text{RCA}_0$  it is easy to prove that if  $\rho$  and  $\tau$  are decimal expansions, then  $\rho = \tau$  if and only if either  $\rho$  and  $\tau$  agree in every digit, or else (subject to renaming  $\rho$  and  $\tau$ ) there is a  $j$  such that  $\rho(i) = \tau(i)$  for  $i < j$ ,  $|\rho(j)| = |\tau(j)| + 10^j$ , and  $\rho(k) = 0$  and  $\tau(k) = 9$  for  $k > j$ . Of course, since decimal expansions are rapidly converging Cauchy sequences, equality

between reals in these two representations is defined.

Now we may turn to equality of cuts. Two Dedekind cuts are equal (as reals) if they differ in at most one element. Since open cuts are Dedekind cuts, this definition extends to comparisons between open cuts or between open cuts and other Dedekind cuts.  $\text{RCA}_0$  can prove that if two open cuts are equal (as reals) then they must agree on all elements, and so are equal as sets also.

Finally, suppose that  $\lambda$  is a Dedekind cut and  $\rho$  is a rapidly converging Cauchy sequence. We say that  $\lambda$  and  $\rho$  are equal (as reals) if

$$\forall k \forall s \forall s' ((s \in \lambda \wedge s' \notin \lambda) \rightarrow [s, s'] \cap [\rho(k) - 2^{-k}, \rho(k) + 2^{-k}] \neq \emptyset).$$

Intuitively, a rapidly converging Cauchy sequence can be viewed as specifying a real as a nested sequence of closed intervals, and similarly, a Dedekind cut can be viewed as specifying a real as the intersection of a set of closed intervals. If the intervals all overlap, then the two representations must correspond to the same real. It is also worth noting that the formula

$$[s, s'] \cap [\rho(k) - 2^{-k}, \rho(k) + 2^{-k}] \neq \emptyset$$

can be written as a comparison of rational endpoints,

$$\neg (\rho(k) + 2^{-k} < s \vee s' < \rho(k) - 2^{-k}) ,$$

which is a  $\Delta_0^0$  formula. Thus the formula encoding  $\lambda = \rho$  is  $\Pi_1^0$ , as are the formulas encoding equality between rapidly converging Cauchy sequences and equality between cuts.

We have defined four representations of real numbers, and have defined equality between any possible pair of representations. With this terminology, we can discuss conversions between representations.

## 2 Conversions in $\text{RCA}_0$

In this section, we will examine those situations where it is possible to convert a sequence of reals in one representation to a sequence in another representation while working within  $\text{RCA}_0$ . By the end of the section, we will be able to dispense with conversions of single reals. Conversions that require stronger axiom systems will be presented in the next section. In the statement of the following theorems, the notation  $(\text{RCA}_0)$  indicates that the result is provable in  $\text{RCA}_0$ .

**Theorem 1** ( $\text{RCA}_0$ ). *If  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  is a sequence of Dedekind cuts, then there is a sequence  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  of decimal expansions such that for each  $i \in \mathbb{N}$ ,  $\lambda_i = \delta_i$ .*

*Proof.* Suppose  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  is a sequence of Dedekind cuts. We will indicate how to compute  $\delta_i(j)$ , the  $j^{\text{th}}$  element of the  $i^{\text{th}}$  decimal expansion.

For  $j = 0$ , let  $z$  be the greatest integer in  $\lambda_i$ . Note that  $z$  exists because  $\lambda_i \neq \mathbb{Q}$  and the complement of  $\lambda_i$  is closed upward. If  $z \geq 0$ , then  $\delta_i(0) = z$ . If  $z < 0$ , then  $\delta_i(0) = z + 1$ , where  $-1 + 1$  is taken to be  $-0$ .

Suppose  $\delta_i(j)$  has been computed. If  $\delta_i(0) \geq 0$ , let  $d$  be the greatest element of  $K = \{k \cdot 10^{j+1} \mid k \in \{0, \dots, 9\}\}$  such that  $\delta_i(j) + d \in \lambda_i$ , and set  $\delta_i(j+1) = \delta_i(j) + d$ . If  $\delta_i(0) < 0$ , let  $d$  be the greatest element of  $K$  such that  $\delta_i(j) - d \notin \lambda_i$ , and set  $\delta_i(j+1) = \delta_i(j) - d$ .

The preceding computation shows that the proof of the existence of  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  can be carried out in  $\text{RCA}_0$ . The claim that  $\lambda_i = \delta_i$  for all  $i \in \mathbb{N}$  follows immediately from the definition of equality between Dedekind cuts and rapidly converging Cauchy sequences.  $\square$

Since every open cut is a Dedekind cut and every decimal expansion is a rapidly converging Cauchy sequence, Theorem 1 has the following corollary.

**Corollary 2** ( $\text{RCA}_0$ ). *If  $\langle \mu_i \rangle_{i \in \mathbb{N}}$  is a sequence of reals in a representation in*



the following list, then for any representation appearing lower in the list there is a sequence  $\langle \tau_i \rangle_{i \in \mathbb{N}}$  in that representation such that for all  $i \in \mathbb{N}$ ,  $\mu_i = \tau_i$ .

*open cuts,*

*Dedekind cuts,*

*decimal expansions,*

*rapidly converging Cauchy sequences.*

With one additional result, we can resolve the conversion problem for all single reals.

**Theorem 3** ( $\text{RCA}_0$ ). *Suppose  $\rho$  is a rapidly converging Cauchy sequence. Then there is an open cut  $\sigma$  such that  $\rho = \sigma$ .*

*Proof.* Let  $\rho$  be a rapidly converging Cauchy sequence. Either  $\rho$  represents a rational or it does not. (This assertion is not uniform.) If  $\rho$  represents the rational  $r$ , then let  $\lambda = \{q \in \mathbb{Q} \mid q < r\}$ . Otherwise,  $\rho$  is not equal to any rational. Consequently, for any  $q \in \mathbb{Q}$ , there is a  $k \in \mathbb{N}$  such that  $\rho(k) + 2^{-k} < q$  or  $\rho(k) - 2^{-k} > q$ . The open cut  $\lambda$  is constructed by excluding  $q$  when  $\rho(k) + 2^{-k} < q$  and including  $q$  when  $\rho(k) - 2^{-k} > q$ .  $\square$

Combining Theorem 3 and Corollary 2 for constant sequences yields the following corollary showing that for single reals all conversions can be carried out in  $\text{RCA}_0$ . The computability theoretic analog of this result was observed by Robinson [8], Myhill [5], and Rice [7].

**Corollary 4** ( $\text{RCA}_0$ ). *If  $\mu$  is a single real in any of the four representations, then there is a real  $\tau$  in each of the other representations such that  $\mu = \tau$ .*

*Proof.* Theorem 3 allows conversions from the bottom of the list in Corollary 2 to the top.  $\square$

In the next section we will see that the nonuniformity in the proof of Theorem 3 is unavoidable. Consequently, proving the analog of Corollary 4 for sequences of reals requires stronger axiom systems than  $\text{RCA}_0$ .

### 3 Conversions requiring $\text{WKL}_0$ and $\text{ACA}_0$

In this section we will show that conversions between some representations of reals require axioms beyond  $\text{RCA}_0$ . Our work will be simplified by the following technical lemma. This lemma extends a conservation result due to Kohlenbach [3, Proposition 3.1].

**Lemma 5** ( $\text{RCA}_0$ ). *The following are equivalent:*

1.  $\text{WKL}_0$ .
2. If  $\langle f_i \rangle_{i \in \mathbb{N}}$  and  $\langle g_i \rangle_{i \in \mathbb{N}}$  are sequences of functions with pairwise disjoint ranges, that is, such that  $\forall i \forall n \forall m (f_i(n) \neq g_i(m))$ , then there is a sequence of sets  $\langle X_i \rangle_{i \in \mathbb{N}}$  such that for each  $i$ ,  $\forall n (f_i(n) \in X_i \wedge g_i(n) \notin X_i)$ .
3. If  $\langle T_i \rangle_{i \in \mathbb{N}}$  is a sequence of infinite 0–1 trees, then there is a sequence  $\langle X_i \rangle_{i \in \mathbb{N}}$  such that for each  $i$ ,  $X_i$  is an infinite path through  $T_i$ .

*Proof.* Since the existence of a separating set for a single pair of functions implies  $\text{WKL}_0$  [9, Lemma IV.4.4], as does the existence of an infinite path through a single infinite 0–1 tree, it suffices to show that (2) and (3) follow from  $\text{WKL}_0$ .

Suppose  $\langle f_i \rangle_{i \in \mathbb{N}}$  and  $\langle g_i \rangle_{i \in \mathbb{N}}$  are sequences of functions with pairwise disjoint ranges, as in (2). Fix a bijection between  $\mathbb{N} \times \mathbb{N}$  and  $\mathbb{N}$ , and identify each ordered pair with its integer code. Define functions  $f$  and  $g$  by setting  $f(i, n) = (f_i(n), i)$  and  $g(i, n) = (g_i(n), i)$ . Since we are viewing ordered pairs as being interchangeable with their integer codes, we may think of  $f$  and  $g$  as functions from  $\mathbb{N}$  to  $\mathbb{N}$ . Note that if  $f(i, n) = g(j, m)$ , then  $j = i$  and  $f_i(n) = g_j(m) = g_i(m)$ , contradicting the claim that the ranges of  $f_i$  and  $g_i$  are disjoint. Thus  $f$  and  $g$  have disjoint ranges.  $\text{WKL}_0$  suffices to prove the

existence of a separating set  $X$  for  $f$  and  $g$  [9, Lemma IV.4.4]. For each  $i$ , let  $X_i = \{m \mid (m, i) \in X\}$ . Then for all  $n$ ,  $(f_i(n), i) \in X$  so  $f_i(n) \in X_i$ , and  $(g_i(n), i) \notin X$ , so  $g_i(n) \notin X_i$ . Thus  $\mathbf{WKL}_0$  proves (2) as desired.

Now we will use  $\mathbf{WKL}_0$  to prove (3). Let  $\langle T_i \rangle_{i \in \mathbb{N}}$  be a sequence of infinite 0–1 trees. Form a tree  $T$  of finite sequences of natural numbers as follows. For  $j \in \mathbb{N}$ , if for each  $i < j$  we are given a sequence  $\sigma_i$  in  $T_i$  of length  $j - i$ , then form the sequence

$$\sigma = (\sigma_0(0), (\sigma_0(1), \sigma_1(0)), \dots, (\sigma_0(j-1), \dots, \sigma_{j-1}(0))).$$

By identifying the inner finite sequences with their integer codes,  $\sigma$  can be viewed as a sequence of  $j$  natural numbers. Let  $T$  be the tree of all such sequences. Since each  $T_i$  is a 0–1 tree,  $\sigma(n)$  can take at most  $2^{n+1}$  possible values, so  $T$  is a bounded tree.  $\mathbf{WKL}_0$  suffices to prove the existence of an infinite path through  $T$  [9, Lemma IV.1.4]. Given a path  $X = (p_0, p_1, p_2, \dots)$  through  $T$ , for each  $i$  the sequence  $X_i = (p_i(i), p_{i+1}(i), p_{i+2}(i), \dots)$  is a path through  $T_i$ . This completes the proof of (3) from  $\mathbf{WKL}_0$ .  $\square$

Now we can turn to the theorems on converting representations. The next three theorems will enable us to completely analyze all possible conversions.

**Theorem 6** ( $\text{RCA}_0$ ). *The following are equivalent:*

1.  $\text{WKL}_0$ .
2. *If  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  is a sequence of rapidly converging Cauchy sequences then there is a sequence  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  of decimal expansions such that for each  $i \in \mathbb{N}$ ,  $\rho_i = \delta_i$ .*

*Proof.* To prove that (1) implies (2), assume  $\text{WKL}_0$  and let  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  be a sequence of rapidly converging Cauchy sequences. For each  $\rho_i$ , construct a tree  $T_i$  as follows. Put a sequence  $\delta$  in  $T_i$  if  $\delta$  is an initial segment of a decimal expansion and for each  $j < \text{lh}(\delta)$ ,  $\rho_i(j) - 2^{-j+1} \leq \delta(j) \leq \rho_i(j) + 2^{-j+1}$ . For each  $k$ , each initial segment of the sequence consisting of the first  $k$  digits of the decimal expansion of  $\rho_i(k)$  satisfies these conditions, so  $T_i$  is an infinite tree. If  $\delta_i$  is an infinite path through  $T_i$ , then from the definition of equality for rapidly converging Cauchy sequences,  $\rho_i = \delta_i$ .  $\text{RCA}_0$  suffices to prove that the sequence  $\langle T_i \rangle_{i \in \mathbb{N}}$  exists, and by Lemma 5,  $\text{WKL}_0$  proves the sequence  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  exists.

To prove the reversal, it suffices to use  $\text{RCA}_0$  and (2) to separate the ranges of disjoint functions [9, Lemma IV.4.4]. The computable analysis counterexample corresponding to this implication appears as part of Theorem

4 of [4]. Suppose  $f$  and  $g$  are injections such that  $\forall n \forall m (f(n) \neq g(m))$ . Define a sequence of rapidly converging Cauchy sequences as follows. For each  $i$  and  $j$ , let

$$\rho_i(j) = \begin{cases} 1 & \text{if } \forall k < j (f(k) \neq i \wedge g(k) \neq i), \\ 1 + 2^{-k} & \text{if } k < j \wedge f(k) = i, \text{ and} \\ 1 - 2^{-k} & \text{if } k < j \wedge g(k) = i. \end{cases}$$

By the recursive comprehension axiom,  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  exists. Apply (2) to obtain a sequence of decimal expansions  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  such that for all  $i \in \mathbb{N}$ ,  $\rho_i = \delta_i$ . Note that if  $f(k) = i$  then  $\delta_i(0) = 1$ , and if  $g(k) = i$  then  $\delta_i(0) = 0$ . Thus the function  $\chi(i) = \delta_i(0)$  is a characteristic function for a separating set for the ranges of  $f$  and  $g$ .  $\square$

**Theorem 7 (RCA<sub>0</sub>).** *The following are equivalent:*

1.  $\text{WKL}_0$ .
2. *If  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  is a sequence of decimal expansions then there is a sequence  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  of Dedekind cuts such that for each  $i \in \mathbb{N}$ ,  $\delta_i = \lambda_i$ .*

*Proof.* To prove that (1) implies (2), assume  $\text{WKL}_0$  and let  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  be a sequence of decimal expansions. Fix an enumeration of  $\mathbb{Q}$ . Note that the sign of a decimal expansion  $\delta_i$  can be determined from  $\delta_i(0)$ . For each  $\delta_k$

define a pair of functions  $f_k$  and  $g_k$  as follows. If  $\delta_k$  is greater than 0 or equal to 0, let  $f_k(m) = q$ , where  $q$  is the first element of  $\mathbb{Q}$  that is not in  $[f_k(m)] \cup [g_k(m)]$  (the ranges of  $f_k$  and  $g_k$  on values less than  $m$ ) that satisfies  $q < \delta_k(m)$ . Let  $g_k(m) = q$  where  $q$  is the first element of  $\mathbb{Q}$  that is not in  $[f_k(m+1)] \cup [g_k(m)]$  that satisfies  $q > \delta_k(m) + 10^{-m}$ . If  $\delta_k$  is less than 0 or equal to  $-0$ , let  $f_k(m) = q$  where  $q$  is the first element of  $\mathbb{Q}$  that is not in  $[f_k(m)] \cup [g_k(m)]$  that satisfies  $q < \delta_k(m) - 10^{-m}$ . Let  $g_k(m) = q$  where  $q$  is the first element of  $\mathbb{Q}$  that is not in  $[f_k(m+1)] \cup [g_k(m)]$  that satisfies  $q > \delta_k(m)$ .  $\text{RCA}_0$  suffices to prove the existence of the sequences  $\langle f_k \rangle_{k \in \mathbb{N}}$  and  $\langle g_k \rangle_{k \in \mathbb{N}}$ . By Lemma 5,  $\text{WKL}_0$  proves the existence of a sequence  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  such that for each  $k$ ,  $\lambda_k$  contains the range of  $f_k$  and is disjoint from the range of  $g_k$ .

We will show that  $\lambda_k$  is a Dedekind cut and  $\delta_k = \lambda_k$ . Suppose that  $\delta_k(0)$  is greater than 0 or equal to 0. If  $q \in \mathbb{Q}$  and  $q < \delta_k$ , then for some  $m$ ,  $q < \delta_k(m)$ . Since  $\delta_k$  is an increasing function, for some  $n > m$ ,  $f_k(n) = q$ , so  $q \in \lambda_k$ . If  $q \in \mathbb{Q}$  and  $q > \delta_k$ , then for some  $m$ ,  $q > \delta_k(m) + 10^{-m}$ . Since  $\delta_k(j) + 10^{-j}$  is a decreasing function in  $j$ , for some  $n > m$ ,  $g_k(n) = q$  and so  $q \notin \lambda_k$ . Thus  $\lambda_k$  is a Dedekind cut equal to  $\delta_k$ . Since  $\lambda_k$  is a separating set, if  $\delta_k$  is a rational then  $\delta_k$  may or may not be an element of  $\lambda_k$ . Thus,

we have not shown that  $\lambda_k$  is an open cut. The proof that  $\lambda_k$  is the desired Dedekind cut when  $\delta_k(0)$  is negative or  $-0$  is similar.

It remains to show that (2) implies  $\text{WKL}_0$ . As in the preceding theorem, we will use (2) to separate the ranges of disjoint functions. Let  $f$  and  $g$  be injections such that for all  $m$  and  $n$ ,  $f(m) \neq g(n)$ . For the following, let  $[d]^n$  denote a string of  $n$  copies of the digit  $d$ . Define a sequence of decimal expansions,  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  by setting

$$\delta_k(n) = \begin{cases} .[1]^t[2]^{n-t} & \text{if } t < n \wedge g(t) = k, \\ .[1]^t[0]^{n-t} & \text{if } t < n \wedge f(t) = k, \text{ and} \\ .[1]^n & \text{otherwise.} \end{cases}$$

Let  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  be a sequence of Dedekind cuts such that for each  $i \in \mathbb{N}$ ,  $\delta_i = \lambda_i$ . Then the set  $S = \{i \mid \frac{1}{9} \in \lambda_i\}$  contains every element of the range of  $f$  and no elements of the range of  $g$ .  $\square$

**Theorem 8** ( $\text{RCA}_0$ ). *The following are equivalent:*

1.  $\text{ACA}_0$ .
2. *If  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  is a sequence of Dedekind cuts, then there is a sequence  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  of open cuts such that for each  $i \in \mathbb{N}$ ,  $\lambda_i = \sigma_i$ .*



*Proof.* First, assume (1) and let  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  be a sequence of Dedekind cuts. For each  $i \in \mathbb{N}$ , if  $\exists q \in \lambda_i \forall q' \in \lambda_i (q' \leq q)$ , then let  $\sigma_i = \lambda_i - \{q\}$ . Otherwise, let  $\sigma_i = \lambda_i$ .  $\text{ACA}_0$  proves that the sequence  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  exists, and the omission of maxima guarantees that each  $\sigma_i$  is an open cut.

To prove the converse, we will use (2) to find the range of an injection [9, Lemma III.1.3]. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be an injection. Define the sequence  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  of Dedekind cuts by putting  $q \in \mathbb{Q}$  in  $\lambda_i$  if and only if  $q \leq 0$  or

$$q > 0 \wedge (\exists t < 1/q)(f(t) = i).$$

$\text{RCA}_0$  suffices to prove that the sequence  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  exists and that each  $\lambda_i$  is a Dedekind cut. (Indeed, each  $\lambda_i$  is a closed lower Dedekind cut for some rational.) By (2), there is a sequence of open cuts  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  satisfying  $\sigma_i = \lambda_i$  for each  $i \in \mathbb{N}$ . Since  $\exists t(f(t) = k)$  if and only if  $0 \in \sigma_k$ , recursive comprehension proves that the range of  $f$  exists.  $\square$

The remaining analysis of the conversions of the representations of sequences of reals consists of two easy corollaries to the preceding theorems.

**Corollary 9** ( $\text{RCA}_0$ ). *The following are equivalent:*

1.  $\text{WKL}_0$ .

2. If  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  is a sequence of rapidly converging Cauchy sequences, then there is a sequence  $\langle \lambda_i \rangle_{i \in \mathbb{N}}$  of Dedekind cuts such that for all  $i \in \mathbb{N}$ ,  $\rho_i = \lambda_i$ .

*Proof.* To prove that (1) implies (2), concatenate Theorem 6 and Theorem 7. Since every decimal expansion is a rapidly converging Cauchy sequence, (2) above implies (2) of Theorem 7, so  $\text{WKL}_0$  follows by Theorem 7.  $\square$

**Corollary 10** ( $\text{RCA}_0$ ). *The following are equivalent:*

1.  $\text{ACA}_0$ .
2. If  $\langle \delta_i \rangle_{i \in \mathbb{N}}$  is a sequence of decimal expansions, then there is a sequence of open cuts  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  such that for all  $i \in \mathbb{N}$ ,  $\delta_i = \sigma_i$ .
3. If  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  is a sequence of rapidly converging Cauchy sequences, then there is a sequence of open cuts  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  such that for all  $i \in \mathbb{N}$ ,  $\rho_i = \sigma_i$ .

*Proof.* Since  $\text{ACA}_0$  implies  $\text{WKL}_0$ , the proof of (3) from (1) follows from a concatenation of Theorem 6, Theorem 7, and Theorem 8. Since every decimal expansion is a rapidly converging Cauchy sequence, (2) is a special case of (3). It remains to show that (2) implies (1). By Theorem 1,  $\text{RCA}_0$  proves that every sequence of Dedekind cuts can be converted to a sequence

of decimal expansions, so (2) above implies (2) of Theorem 8, and  $\text{ACA}_0$  follows by Theorem 8. (Theorem 6 of [4] includes a computable analysis counterexample corresponding to a direct proof of (1) from (2).)  $\square$

We summarize the results of the preceding two sections in the following table. Each table entry corresponds to a conversion from a sequence of the row type to a sequence of the column type. Row and column labels are:

$\rho$ : rapidly converging Cauchy sequence,

$\delta$ : decimal expansion,

$\lambda$ : Dedekind cut, and

$\sigma$ : open cut.

The conversion results are either provable in  $\text{RCA}_0$  (as shown in §2), or equivalent to the designated subsystem (as shown in this section).

from \ to	$\rho$	$\delta$	$\lambda$	$\sigma$
$\rho$	$\text{RCA}_0$	$\text{WKL}_0$	$\text{WKL}_0$	$\text{ACA}_0$
$\delta$	$\text{RCA}_0$	$\text{RCA}_0$	$\text{WKL}_0$	$\text{ACA}_0$
$\lambda$	$\text{RCA}_0$	$\text{RCA}_0$	$\text{RCA}_0$	$\text{ACA}_0$
$\sigma$	$\text{RCA}_0$	$\text{RCA}_0$	$\text{RCA}_0$	$\text{RCA}_0$

## 4 Related results

As noted in the reversal of Theorem 8, conversions from Dedekind cuts to open cuts require  $\text{ACA}_0$ , even for sequences consisting only of rationals. On the other hand, conversions of purely irrational sequences can be carried out in  $\text{RCA}_0$ , as shown by the following theorem and corollary.

**Theorem 11** ( $\text{RCA}_0$ ). *If  $\langle \rho_i \rangle_{i \in \mathbb{N}}$  is a sequence of rapidly converging Cauchy sequences each of which converges to an irrational number, then there is a sequence  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  of open cuts such that for all  $i \in \mathbb{N}$ ,  $\sigma_i = \rho_i$ .*

*Proof.* Given the sequence  $\langle \rho_i \rangle_{i \in \mathbb{N}}$ , determine if  $q \in \mathbb{Q}$  is in  $\sigma_k$  as follows. Since  $\rho_k$  is irrational,  $\rho_k \neq q$ . Find  $n$  so large that  $\rho_k(n) - 2^{-n} > q$  or  $\rho_k(n) + 2^{-n} < q$ . If the first inequality holds, include  $q$  in  $\sigma_k$ . If the second holds then exclude  $q$  from  $\sigma_k$ .  $\text{RCA}_0$  suffices to prove that  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  exists, each  $\sigma_i$  is an open cut, and for each  $i$ ,  $\sigma_i = \rho_i$ .  $\square$

**Corollary 12** ( $\text{RCA}_0$ ). *Any sequence of irrationals in any of the four representations can be converted to a sequence in any other representation.*

*Proof.* Immediate from Corollary 2 and Theorem 11.  $\square$

In general, separating rationals and irrationals requires  $\text{ACA}_0$  as shown by the following theorem and corollary.

**Theorem 13.** ( $\text{RCA}_0$ ) *The following are equivalent:*

1.  $\text{ACA}_0$ .
2. *If  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  is a sequence of open cuts then the set  $\{i \in \mathbb{N} \mid \sigma_i \in \mathbb{Q}\}$  exists.*

*Proof.* First, assume (1) and suppose  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  is a sequence of open cuts. Note that  $\sigma_i \in \mathbb{Q}$  if and only if

$$\exists q \in \mathbb{Q} \forall q' \in \mathbb{Q} (q' \notin \sigma_i \rightarrow q \leq q').$$

Since each rational can be encoded by a natural number, this formula is arithmetical. Thus, the desired set exists by arithmetical comprehension.

To prove that (2) implies (1), assume  $\text{RCA}_0$  and let  $f$  be an injection.

Include  $q$  in  $\sigma_i$  if and only if

- $\exists k (q < -2^{-k} \wedge \forall t \leq k (f(t) \neq i))$ , or
- $\exists t (f(t) = i \wedge q < -2^{-t}/\pi)$ .

$\text{RCA}_0$  suffices to prove that  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  exists, that each  $\sigma_i$  is an open cut, and that  $\sigma_i = 0$  if  $i \notin \text{Range}(f)$  and  $\sigma_i$  is irrational otherwise. The complement

of  $\{i \in \mathbb{N} \mid \sigma_i \in \mathbb{Q}\}$  is the range of  $f$ , so an application of [9, Lemma III.1.3] yields  $\text{ACA}_0$ .  $\square$

**Corollary 14** ( $\text{RCA}_0$ ). *For any of the four representations of reals, the following are equivalent.*

1.  $\text{ACA}_0$ .
2. *If  $\langle \tau_i \rangle_{i \in \mathbb{N}}$  is a sequence of reals in the specified representation, then the set  $\{i \in \mathbb{N} \mid \tau_i \in \mathbb{Q}\}$  exists.*

*Proof.* To prove that (1) implies (2), assume  $\text{ACA}_0$  and let  $\langle \tau_i \rangle_{i \in \mathbb{N}}$  be a sequence of reals. Apply results from Section 3 to convert  $\langle \tau_i \rangle_{i \in \mathbb{N}}$  to open cuts. An application of Theorem 13 yields the desired set.

To prove the converse, assume  $\text{RCA}_0$  and suppose (2) holds. By Corollary 2,  $\text{RCA}_0$  proves that (2) above implies (2) of Theorem 13.  $\text{ACA}_0$  follows from Theorem 13.  $\square$

In Theorems 3 and 5 of [4], Mostowski analyzed change of basis for sequences of decimal expansions in a computable analysis setting. Theorems 16 and 17 give the reverse mathematical analogs of his results. The following terminology is useful in the proofs. A *base  $b$  expansion* is defined in the same manner as a decimal expansion, using  $b$  in place of 10 and integers less

than  $b$  as digits. A base  $b$  expansion is *terminating* if there is some point after which every digit is zero or every digit is  $b - 1$ . Applying the definition of equality between rapidly converging Cauchy sequences, this means that a terminating base  $b$  expansion is always equal to (but not necessarily the same as) an expansion ending in zeros. The next lemma shows that termination may or may not be conserved under change of basis. For natural numbers  $a$  and  $b$ , we will use the notation  $a \mid b$  to denote “ $a$  divides  $b$ ” and  $a \nmid b$  to denote “ $a$  does not divide  $b$ .”

**Lemma 15** ( $\text{RCA}_0$ ). *For all  $b$  and  $c$ , there is an  $n$  such that  $c \mid b^n$  if and only if every real with a terminating base  $c$  expansion has a terminating base  $b$  expansion. In particular, if for all  $n$  we have  $c \nmid b^n$ , then the base  $b$  expansion of  $1/c$  is nonterminating.*

*Proof.* Suppose that for some  $t$  and  $n$ ,  $tc = b^n$ . Let  $\sigma$  be a terminating base  $c$  expansion. We may assume that  $\sigma$  terminates in zeros, so for some  $j$ ,

$$\sigma = \sigma(0) + \text{sign}(\sigma(0)) \sum_{i=1}^j \frac{\sigma_i}{c^i}$$

where  $0 \leq \sigma_i \leq c - 1$  for each  $i \leq j$ . Since  $\frac{\sigma_i}{c^i} = \frac{t^i \sigma_i}{t^i c^i} = \frac{t^i \sigma_i}{b^{ni}}$ , we have

$$\sigma = \sigma(0) + \text{sign}(\sigma(0)) \sum_{i=1}^j \frac{t^i \sigma_i}{b^{ni}},$$

so  $\sigma$  can be expressed as a terminating base  $b$  expansion.

To prove the converse, suppose that for every value of  $n$ ,  $c \nmid b^n$ . Suppose by way of contradiction that  $1/c$  has a terminating base  $b$  expansion. Then we may write

$$\frac{1}{c} = \sum_{i=1}^j \frac{\beta_i}{b^i} = \frac{t}{b^j}$$

for some  $t \in \mathbb{N}$ . Thus  $ct = b^j$ , contradicting our divisibility assumption.

Thus,  $1/c$  has no terminating base  $b$  expansion.  $\square$

**Theorem 16 (RCA<sub>0</sub>).** *If  $c \mid b^n$  for some  $n$ , then for every sequence  $\langle \beta_i \rangle_{i \in \mathbb{N}}$  of base  $b$  expansions there is a sequence  $\langle \gamma_i \rangle_{i \in \mathbb{N}}$  of base  $c$  expansions such that for all  $i \in \mathbb{N}$ ,  $\beta_i = \gamma_i$ .*

*Proof.* This argument is essentially a formalization of the proof of Theorem 3 of [4]. Suppose  $c \mid b^n$ . By Lemma 15, whenever  $\gamma = \beta$  where  $\gamma$  is a base  $c$  expansion and  $\beta$  is a base  $b$  expansion, if  $\gamma$  terminates then so does  $\beta$ .

Consider a single base  $b$  expansion; call it  $\beta$ . As usual, let  $\beta(k)$  denote



the result of truncating  $\beta$  after the first  $k$  digits to the right of the decimal point. Let  $(\beta(k))_c$  denote the base  $c$  expansion of  $\beta(k)$ . Suppose by way of contradiction that there is a  $j$  such that for all  $k$ ,  $(\beta(k))_c$  and  $(\beta(k) + b^{-k})_c$  disagree somewhere in the first  $j$  digits. In this case there are two base  $c$  expansions  $\gamma_0$  and  $\gamma_1$  such that  $\beta = \gamma_0 = \gamma_1$  and  $\gamma_0$  and  $\gamma_1$  disagree somewhere in the first  $j$  digits. This implies that  $\gamma_0$  and  $\gamma_1$  must be terminating. Let  $\gamma$  denote the element of  $\{\gamma_0, \gamma_1\}$  that terminates in zeros. Since  $\beta = \gamma$  and  $\gamma$  terminates,  $\beta$  must terminate also, and we may assume that  $\beta$  ends in zeros. Choose  $m$  so large that  $m > j$  and for all  $k > m$ ,  $\beta(k) = \beta(m)$  and  $\gamma(k) = \gamma(m)$ . Choose  $p > m$  such that for all  $k > p$ ,  $b^{-k} < c^{-m-1}$ . Thus when  $k > p$ ,  $(\beta(k))_c = \gamma(m)$ ,  $(\beta(k) + b^{-k})_c < \gamma(m) + c^{-m-1}$ , and  $(\beta(k))_c$  must agree with  $(\beta(k) + b^{-k})_c$  on the first  $j$  digits, contradicting our assumption. Thus for every  $j$  there is a  $k$  such that  $(\beta(k))_c$  and  $(\beta(k) + b^{-k})_c$  agree on the first  $j$  digits. Furthermore, for any  $m$  greater than such  $k$ ,  $(\beta(m))_c$  and  $(\beta(k))_c$  agree on the first  $j$  digits.

Now we can present the algorithm for converting  $\langle \beta_i \rangle_{i \in \mathbb{N}}$  to  $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ . For any  $i$  and  $j$ , find a  $k$  so large that  $(\beta_i(k))_c$  and  $(\beta_i(k) + b^{-k})_c$  agree on the first  $j$  digits. Let  $\gamma_i(j)$  consist of those  $j$  digits.  $\text{RCA}_0$  suffices to prove that  $\langle \gamma_i \rangle_{i \in \mathbb{N}}$  exists and is a sequence of base  $c$  expansions, and that  $\beta_i = \gamma_i$  for all

$i \in \mathbb{N}$ .

□

**Theorem 17** ( $\text{RCA}_0$ ). *If for all  $n$  we have  $c \nmid b^n$ , then the following are equivalent:*

1.  $\text{WKL}_0$ .
2. *For every sequence  $\langle \beta_i \rangle_{i \in \mathbb{N}}$  of base  $b$  expansions there is a sequence of base  $c$  expansions such that for all  $i \in \mathbb{N}$ ,  $\beta_i = \gamma_i$ .*

*Proof.* Suppose that for all  $n$ ,  $c$  does not divide  $b^n$ . Since base  $b$  expansions are rapidly converging Cauchy sequences, and 10 can be replaced by  $c$  in the proof of Theorem 6, the statement that (1) implies (2) can be proved by adapting the proof of Theorem 6.

The proof of the converse is essentially a formalization of the construction in Theorem 5 of [4]. By Lemma 15, let  $\beta$  be the nonterminating base  $b$  expansion of  $1/c$ . Since  $\beta$  does not terminate, after any given point in the expansion a digit greater than 0 must occur and a digit less than  $b - 1$  must occur. For any  $k$ , let  $n_k > k$  be the first location to the right of the  $k^{\text{th}}$

decimal place in  $\beta$  that has a value less than  $b - 1$  and define

$$\beta_k^\uparrow(j) = \begin{cases} \beta(j) & \text{for } j < n_k, \\ b - 1 & \text{for } j = n_k, \\ 0 & \text{for } j > n_k. \end{cases}$$

Note that  $\beta_k^\uparrow > 1/c$ . Similarly, when  $n_k > k$  is the first location to the right of the  $k^{\text{th}}$  decimal place in  $\beta$  that has a value greater than 0, define

$$\beta_k^\downarrow(j) = \begin{cases} \beta(j) & \text{for } j < n_k, \\ 0 & \text{for } j \geq n_k. \end{cases}$$

Note that  $\beta_k^\downarrow < 1/c$ . Let  $f$  and  $g$  be functions with disjoint ranges, and define

$\langle \beta_i \rangle_{i \in \mathbb{N}}$  by

$$\beta_i(j) = \begin{cases} \beta_t^\uparrow(j) & \text{if } t \leq j \text{ and } f(t) = i, \\ \beta_t^\downarrow(j) & \text{if } t \leq j \text{ and } g(t) = i, \\ \beta(j) & \text{otherwise.} \end{cases}$$

Apply (2) to find a sequence  $\langle \gamma_i \rangle_{i \in \mathbb{N}}$  of base  $c$  expansions such that  $\beta_i = \gamma_i$  for all  $i \in \mathbb{N}$ . The set  $S = \{i \mid \gamma_i(1) \geq 1/c\}$  is a separating set for the ranges of  $f$  and  $g$ . □

We close by observing that many of the reversals of the results on sequences can be converted to arguments in constructive analysis for negative statements about single reals. As an example, consider the proof of (2) implies (1) in Theorem 17. To increase the concreteness of the discussion, suppose  $b = 2$  and  $c = 10$ . Thus  $\beta$  is the base 2 expansion of  $1/10$ , that is,  $\beta = .000\overline{1100}$  in standard base 2 notation. Let  $P$  denote a formal theory that is assumed to be consistent and that has proofs that can be Gödel numbered. (A reasonable choice would be Peano arithmetic.) Let  $S$  denote a statement whose status is completely open. That is,  $S$  might or might not be provable in  $P$  and  $\neg S$  might or might not be provable in  $P$ . (At the moment,  $S$  could be the Goldbach conjecture.) Define  $\beta_0$  by setting

$$\beta_0(j) = \begin{cases} \beta_t^\uparrow(j) & \text{if some } t \leq j \text{ encodes a proof of } S \text{ in } P, \\ \beta_t^\downarrow(j) & \text{if some } t \leq j \text{ encodes a proof of } \neg S \text{ in } P, \\ \beta(j) & \text{otherwise.} \end{cases}$$

$\beta_0$  is a constructive base 2 expansion. Note that if  $\gamma = \beta_0$  and  $\gamma$  is a base 10 expansion, then  $\gamma(1) \geq 1/10$  implies there is no proof of the negation of  $S$  in  $P$  and  $\gamma(1) < 1/10$  implies there is no proof of  $S$  in  $P$ . Since we lack sufficient information about the provability of  $S$  to determine the value of  $\gamma(1)$ , there is

no constructive base 10 expansion that is equal to  $\beta_0$ . A constructivist might summarize by saying that some base 2 expansions cannot be converted to base 10 expansions. For more on constructive representations of reals, see [10].

## Bibliography

- [1] Richard Dedekind, *Essays on the theory of numbers. I:Continuity and irrational numbers. II:The nature and meaning of numbers*, Authorized translation by Wooster Woodruff Beman, Dover Publications Inc., New York, 1963. MR0159773 (28 #2989)
- [2] Jeffrey L. Hirst, *Minima of initial segments of infinite sequences of reals*, MLQ Math. Log. Q. **50** (2004), 47–50. MR2029605 (2005d:03107)
- [3] Ulrich Kohlenbach, *On uniform weak König’s lemma*, Ann. Pure Appl. Logic **114** (2002), 103–116, Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999). MR1879410 (2002k:03104)
- [4] A. Mostowski, *On computable sequences*, Fund. Math. **44** (1957), 37–51. MR0091242 (19,934a)
- [5] John Myhill, *Criteria of constructibility for real numbers*, J. Symbolic Logic **18** (1953), 7–10. MR0054549 (14,938e)
- [6] Marian B. Pour-El and J. Ian Richards, *Computability in analysis and physics*, Per-

- spectives in Mathematical Logic, Springer-Verlag, Berlin, 1989, ISBN 3-540-50035-9.  
MR1005942 (90k:03062)
- [7] H. G. Rice, *Recursive real numbers*, Proc. Amer. Math. Soc. **5** (1954), 784–791.  
MR0063328 (16,104a)
- [8] Raphael M. Robinson, *Review of: Rekursive Funktionen by Rózsa Péter*, J. Symb. Logic **16** (1951), 280–282.
- [9] Stephen G. Simpson, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999, ISBN 3-540-64882-8. MR1723993 (2001i:03126)
- [10] A. S. Troelstra, *Intuitionistic extensions of the reals*, Nieuw Arch. Wisk. (3) **28** (1980), 63–113. MR568437 (82a:03060)