REVERSE MATHEMATICS AND UNIFORMITY IN PROOFS WITHOUT EXCLUDED MIDDLE

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Abstract. We prove that if an $\forall\exists$ sentence of a particular form is provable in a certain theory of higher order arithmetic without the law of the excluded middle, then it is uniformly provable in the weak classical theory RCA. Applying the contrapositive of this result, we give three examples where results of reverse mathematics can be used to demonstrate the nonexistence of proofs in certain intuitionistic systems.

1. Introduction

We study the proof theory of subsystems of higher-order arithmetic, with the goal of relating provability in intuitionistic systems with uniform provability in weak classical systems. In Section 2, we give the definition of the subsystem of intuitionistic higher-order arithmetic known as $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC}$. This subsystem is the fragment of constructive analysis consisting of Heyting arithmetic in all finite types with a restricted primitive recursion scheme and a choice scheme for arbitrary formulas. We prove the following theorem relating provability in $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC}$ with uniform provability in the weak classical system RCA. As the minimal ω-model of RCA contains only computable sets, this result can also be viewed as a link between intuitionistic provability and computable analysis.

Main Theorem. *If the system $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC}$ proves a statement of the form $\forall X\, \exists Y\, A(X, Y)$, where A is a formula of RCA and in the class Γ_1, then the uniformized statement*

$$\forall \langle x_n \mid n \in \mathbb{N}\rangle\, \exists \langle y_n \mid n \in \mathbb{N}\rangle \forall n\, A(x_n, y_n)$$

is provable in RCA.

The proof of the Main Theorem is presented in Section 3. It uses modified realizability, a well-known tool in proof theory. We have attempted to make that section accessible to a general reader who is familiar with mathematical logic but possibly not familiar with modified realizability.

In Section 4, we present several theorems of core mathematics that are provable in RCA but not in $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC}$. A reader who is willing to accept the Main Theorem should be able to skim Section 3 and proceed directly to this section.

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In Section 5, we prove and apply a variation of the Main Theorem based on the *Dialectica* interpretation of Gödel. This variation uses axiom systems with restricted induction, trading some extensionality strength for a Markov principle.

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2. Axiom systems

In this section, we present the axiom systems and schemes used in later sections. Our definitions make use of the following type notation. The type of a natural number is 0. The type of a function from objects of type ρ to objects of type τ is $\rho \to \tau$. For example, the type of a function from numbers to numbers is $0 \to 0$. As is typical in the literature, we will use the types 1 and $0 \to 0$ interchangeably, essentially identifying sets with their characteristic functions. We will often write superscripts on quantified variables to indicate their type.

The axiom systems we will use are all modifications of $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$. Here we are adopting the notation of Kohlenbach [6], in which $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$ is a restriction of Heyting arithmetic in all types. In particular, $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$ consists of $\mathsf{E\text{-}HA}^{\omega}$ (intuitionistic Heyting arithmetic in all finite types with the extensionality axioms described below) with induction restricted to quantifier-free formulas and primitive recursion restricted to type 0 functionals with parameters. This theory includes equality as a primitive relation only for type 0 objects (natural numbers). The added extensionality axioms consist of the scheme

$$\mathsf{E}\colon \forall x^{\rho} \forall y^{\rho} \forall z^{\rho \to \tau} \, (x =_{\rho} y \to z(x) =_{\tau} z(y)),$$

which can be used to define equality for higher types in terms of equality for lower types. $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$ also includes projection and substitution combinators (denoted by $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ in [6]), which allow terms to be defined using λ abstraction. For example, given $x \in \mathbb{N}$ and an argument list t, $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$ includes a term for $\lambda t.x$, the constant function with value x. We will use $\mathcal{L}(\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction})$ to denote the language of $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$. More details on $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction}$ are given by Kohlenbach [6].

In Section 5, we will need to replace the extensionality scheme E with a weaker extensionality rule:

$$\mathsf{QF\text{-}ER}\colon \text{From } A_0 \to s =_{\rho} t \text{ deduce } A_0 \to r[s/x^{\rho}] =_{\tau} r[t/x^{\rho}],$$

where A_0 is quantifier free and $r[s/x^{\rho}]$ denotes the result of replacing the variable x of type ρ by the term s of type ρ in the term r of type τ. We will use $\widehat{\mathsf{WE{-}HA}}^{\omega}_{\restriction}$ to denote the axiom system with this weak form of extensionality.

We will also have use for the following axiom schemes, all appearing in [6].

- Induction for all formulas: For any formula A in $\mathcal{L}(\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction})$,

$$\mathsf{IA}\colon A(0) \to (\forall n(A(n) \to A(n+1)) \to \forall n A(n)).$$

- Independence of premise for $\exists$-free formulas: For x of any finite type, if A is $\exists$-free and does not contain x, then
$$\mathsf{IP}^{\omega}_{ef}\colon (A \to \exists x B(x)) \to \exists x(A \to B(x)).$$
- Independence of premise for universal premises: If A_0 is quantifier free, $\forall x$ represents a block of universal quantifiers, and y is of any type and is not free in $\forall x A_0(x)$, then
$$\mathsf{IP}^{\omega}_{\forall}\colon (\forall x A_0(x) \to \exists y B(y)) \to \exists y(\forall x A_0(x) \to B(y)).$$
- Markov principle: If A_0 is quantifier-free and $\exists x$ represents a block of existential quantifiers in any finite type, then
$$\mathsf{M}^{\omega}\colon \neg\neg\exists x A_0(x) \to \exists x A_0(x).$$
- Axiom of Choice: For any x and y of finite type,
$$\mathsf{AC}\colon \forall x \exists y\, A(x,y) \to \exists Y \forall x\, A(x, Y(x)).$$
- Restricted axiom of choice: For A_0 quantifier free,
$$\mathsf{QF\text{-}AC}^{1,0}\colon \forall x^1 \exists y^0\, A_0(x,y) \to \exists Y^{1\to 0} \forall x^1\, A_0(x, Y(x)).$$

The axiom system RCA^{ω}_0 consists of $\widehat{\mathsf{E}\text{--}\mathsf{HA}}^{\omega}_{\restriction}$ plus $\mathsf{QF\text{-}AC}^{1,0}$ and the law of the excluded middle. Restricting this theory to types 0 and 1 yields the system RCA_0. Appending the induction scheme IA to RCA^{ω}_0 yields RCA^{ω}, so the subscript 0 denotes restricted induction in these systems. Similarly, RCA_0 plus IA is denoted by RCA. Because these four systems include the law of the excluded middle, they also include all of classical predicate calculus.

The classical axiomatization of RCA_0, presented by Simpson [10], uses the set-based language L_2 with the membership relation symbol $\in$, rather than the langauge based on function application used in $\widehat{\mathsf{E}\text{--}\mathsf{HA}}^{\omega}_{\restriction}$. The system defined in the previous paragraph as RCA_0 is sometimes denoted RCA^2_0, to indicate it is a restriction of RCA^{ω}_0. As discussed by Kohlenbach [5], set-based RCA_0 and function-based RCA^2_0 are each included in a canonical definitional extension of the other. Throughout this paper, we use the functional variants of RCA_0 and RCA for convenience, knowing that our results apply equally to the traditionally axiomatized systems via this definitional extension.

3. Modified realizability

The most broadly applicable form of our theorem can be proved by an application of modified realizability, a technique introduced by Kreisel [7]. Excellent expositions on modified realizability are given by Kohlenbach [6] and Troelstra [11,12]. Indeed, minute modifications to stated results in these sources provide all the tools needed to easily derive our main result.

Modified realizability is a scheme for matching each formula A with a formula $t \,\mathsf{mr}\, A$, meaning "the sequence of terms t realizes A." The scheme is defined inductively as follows.

Definition 3.1. Let A be a formula in $\mathcal{L}(\widehat{\mathsf{E}\text{--}\mathsf{HA}}^{\omega}_{\restriction})$, and let x denote a possibly empty tuple of terms whose variables do not appear free in A. The formula $x \,\mathsf{mr}\, A$ is defined by:

(1) $x \,\mathsf{mr}\, A$ is A, if x is empty and A is a prime formula.
(2) $x, y \,\mathsf{mr}\, (A \wedge B)$ is $x \,\mathsf{mr}\, A \wedge y \,\mathsf{mr}\, B$.

(3) $z^0, x, y \mathsf{\ mr\ } (A \vee B)$ is $(z = 0 \to x \mathsf{\ mr\ } A) \wedge (z \neq 0 \to y \mathsf{\ mr\ } B)$.
(4) $y \mathsf{\ mr\ } (A \to B)$ is $\forall x(x \mathsf{\ mr\ } A \to yx \mathsf{\ mr\ } B)$.
(5) $x \mathsf{\ mr\ } (\forall y^\rho A(y))$ is $\forall y^\rho (xy \mathsf{\ mr\ } A(y))$.
(6) $z^\rho, x \mathsf{\ mr\ } (\exists y^\rho A(y))$ is $x \mathsf{\ mr\ } A(z)$.

The following soundness theorem for mr is almost identical to Theorem 5.8 of Kohlenbach [6]. This version varies only in the restriction of the primitive recursion scheme to type 0 operators. Consequently, the proof of this version is identical to the proof in [6], with the single observation that recursion on higher types is not utilized in that proof.

Theorem 3.2. *Let A be a formula in $\mathcal{L}(\widehat{\mathsf{E{-}HA}}^\omega_\restriction)$. If*

$$\widehat{\mathsf{E{-}HA}}^\omega_\restriction + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^\omega_{ef} \vdash A$$

then $\widehat{\mathsf{E{-}HA}}^\omega_\restriction + \mathsf{IA} \vdash t \mathsf{\ mr\ } A$, where t is a suitable tuple of terms of $\widehat{\mathsf{E{-}HA}}^\omega_\restriction$.

For prime formulas, A and $t \mathsf{\ mr\ } A$ are identical. However, the deduction of A from $t \mathsf{\ mr\ } A$ without the use of choice is only possible for some formulas. In the following, a formula is said to be *negative* if it is constructed from negated prime formulas by means of $\forall$, $\wedge$, $\to$, and $\perp$.

Definition 3.3. Γ_1 is the collection of formulas in $\mathcal{L}(\widehat{\mathsf{WE{-}HA}}^\omega_\restriction)$ defined inductively as follows.

(1) All prime formulas are elements of Γ_1.
(2) If A and B are in Γ_1, then so are $A \wedge B$, $A \vee B$, $\forall x A$, and $\exists x A$.
(3) If A is negative and $B \in \Gamma_1$, then $(\exists x A \to B) \in \Gamma_1$, where $\exists x$ may represent a block of existential quantifiers.

Lemma 5.20 of Kohlenbach [6] can also be proved in our restriction of $\mathsf{E\text{-}HA}^\omega$, yielding the following lemma.

Lemma 3.4. *If A is a formula of Γ_1, then $\widehat{\mathsf{E{-}HA}}^\omega_\restriction \vdash (t \mathsf{\ mr\ } A) \to A$.*

Proof. As noted in Kohlenbach [6], the lemma is proved by induction on the structure of formulas in Γ_1. No induction or primitive recursion on higher types is used in the argument, so $\widehat{\mathsf{E{-}HA}}^\omega_\restriction$ suffices. Motivation for the definition of Γ_1 is revealed in the proof of the clause for implications. The proof of that clause follows easily from Remark 3.4.4 of Troelstra [11], that if A is negative then $t \mathsf{\ mr\ } A$ and A are the same formula. $\square$

Applying Theorem 3.2 and Lemma 3.4, we now prove the following term extraction theorem, which is similar to the main theorem theorem on term extraction via mr (Theorem 5.13) of Kohlenbach [6]. Note that $\forall x \exists y A$ is in Γ_1 if and only if A is in Γ_1.

Theorem 3.5. *Let $\forall x^\rho \exists y^\tau A(x, y)$ be a sentence in Γ_1 where ρ and τ are arbitrary types. If*

$$\widehat{\mathsf{E{-}HA}}^\omega_\restriction + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^\omega_{ef} \vdash \forall x^\rho \exists y^\tau A(x, y),$$

then $\mathsf{RCA}^\omega \vdash \forall x^\rho A(x, t(x))$, where t is a suitable term of $\widehat{\mathsf{E{-}HA}}^\omega_\restriction$.

Proof. Suppose $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef} \vdash \forall x^{\rho} \exists y^{\tau} A(x,y)$ where $A(x,y)$ is in Γ_1. By Theorem 3.2, there is a tuple t of terms of $\mathcal{L}(\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction})$ such that $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} \vdash t \mathrel{\mathsf{mr}} \forall x^{\rho} \exists y^{\tau} A(x,y)$. By clause (5) of Definition 3.1, $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} \vdash \forall x^{\rho}(t(x) \mathrel{\mathsf{mr}} \exists y^{\tau} A(x,y))$. By clause (6) of Definition 3.1, t must have the form t_0, t_1 and $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} \vdash \forall x^{\rho}[t_1(x) \mathrel{\mathsf{mr}} A(x, t_0(x))]$. Since $A(x,y)$ is in Γ_1, by Lemma 3.4, $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} \vdash \forall x^{\rho} A(x, t_0(x))$. Because RCA^{ω} is an extension of $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA}$, we see that $\mathsf{RCA}^{\omega} \vdash \forall x^{\rho} A(x, t_0(x))$. □

Our Main Theorem follows immediately from the next result, which is itself a direct consequence of the term extraction theorem.

Theorem 3.6. *Let $\forall x \exists y A(x,y)$ be a sentence in Γ_1. If*

$$\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef} \vdash \forall x \exists y \, A(x,y),$$

then

$$\mathsf{RCA}^{\omega} \vdash \forall \langle x_n \mid n \in \mathbb{N} \rangle \, \exists \langle y_n \mid n \in \mathbb{N} \rangle \, \forall n \, A(x_n, y_n).$$

Furthermore, if x and y are both type 1 (set) variables, and $\forall x \exists y A(x,y)$ is in $\mathcal{L}(\mathsf{RCA})$, then RCA^{ω} may be replaced by RCA in the implication.

Proof. Suppose $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef} \vdash \forall x^{\rho} \exists y^{\tau} A(x,y)$, and apply Theorem 3.5 to extract the term t such that $\mathsf{RCA}^{\omega} \vdash \forall x^{\rho} A(x, t(x))$. Working in RCA^{ω}, select a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$. Note that this sequence is actually a function of type $0 \to \rho$, so by lambda abstraction we can construct a function of type $0 \to \tau$ defined by $\lambda n.t(x_n)$. Taking $\langle y_n \mid n \in \mathbb{N} \rangle$ to be this sequence, we obtain $\forall n \, A(x_n, y_n)$. The final sentence of the theorem follows immediately from the fact that RCA^{ω} is a conservative extension of RCA for formulas in $\mathcal{L}(\mathsf{RCA})$. □

Afficionados of reverse mathematics may wonder whether the preceding theorem holds for systems with restricted induction. In Section 5, we prove that such a result can be obtained if the class of formulas is narrowed from Γ_1 to Γ_2.

4. Applications of the main theorem

In this section, we consider several theorems of core mathematics that are provable in RCA_0 but have uniformized versions that are not provable in RCA. In light of the Main Theorem, such results are not provable in $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$. Recall that RCA_0 is the subsystem of classical second-order arithmetic, containing the Δ^0_1 comprehension scheme and the arithmetical induction scheme restricted to Σ^1_0 formulas, typically used as a weak base system in the program of reverse mathematics. Simpson [10] gives a complete account of the parts of classical mathematics that can be developed in RCA_0. The system RCA appends IA to RCA_0. Where possible, we carry out proofs using restricted induction, as this will prove useful in the next section.

The terminology in the following theorem is well known; we give formal definitions as needed later in the section.

Theorem 4.1. *Each of the following statements is provable in RCA_0 but not provable in $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$.*

(1) *Every 2×2 matrix has a Jordan decomposition.*

(2) *Every quickly converging Cauchy sequence of rational numbers can be converted to a Dedekind cut representing the same real number.*
(3) *Every enumerated filter on a countable poset can be extended to an unbounded enumerated filter.*

We show that each statement (4.1.1)–(4.1.3) is provable in RCA_0 but not provable in $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$ by noting that each statement is in Γ_1 and showing that the uniformization of each statement implies a strong comprehension axiom over RCA_0. Because these strong comprehension axioms are not provable even with the added inductive strength of RCA, we may apply the Main Theorem to obtain the desired results. The stronger subsystems include WKL_0 and ACA_0, which are discussed thoroughly by Simpson [10]. WKL_0 appends a weak form of König's lemma to the axioms of RCA_0, while ACA_0 appends comprehension for sets defined by arithmetical formulas. We remark that there are many other statements that are provable in RCA_0 but not $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$; we have chosen these three to illustrate the what we believe to be the ubiquity of this phenomenon in various branches of core mathematics.

We begin with statement (4.1.1). We consider only finite square matrices whose entries are complex numbers represented by quickly converging Cauchy sequences. In RCA_0, we say that a matrix M *has a Jordan decomposition* if there are matrices (U, J) such that $M = UJU^{-1}$ and J is a matrix consisting of Jordan blocks. We call J the *Jordan canonical form* of M. The fundamental definitions and theorems regarding the Jordan canonical form are presented by Halmos [2, Section 58]. Careful formalization of (4.1.1) shows that the statement is in Γ_1.

Lemma 4.2. RCA_0 *proves that every* 2×2 *matrix has a Jordan decomposition.*

Proof. Let M be a 2×2 matrix. RCA_0 proves that the eigenvalues of M exist and that for each eigenvalue there is an eigenvector. (Compare Exercise II.4.11 of Simpson [10], which notes that the basics of linear algebra, including fundamental properties of Gaussian elimination, are provable in RCA_0.) If the eigenvalues of M are distinct, then the Jordan decomposition is trivial to compute from the eigenvalues and eigenvectors. If there is a unique eigenvalue and there are two linearly independent eigenvectors then the Jordan decomposition is similarly trivial to compute.

Suppose that M has a unique eigenvalue λ but not two linearly independent eigenvectors. Let u be any eigenvector and let $\{u, v\}$ be a basis. It follows that $(M - \lambda I)v = au + bv$ is nonzero. Now $(M - \lambda I)(au + bv) = b(M - \lambda I)v$, because u is an eigenvector of M with eigenvalue λ. This shows $(M - \lambda I)$ has eigenvalue b, which can only happen if $b = 0$, that is, if $(M - \lambda I)v$ is a scalar multiple of u. Thus $\{u, v\}$ is a chain of generalized eigenvectors of M; the Jordan decomposition can be computed directly from this chain. $\square$

It is not difficult to see that the previous proof makes use of the law of the excluded middle.

Remark 4.3. Proofs similar to that of Lemma 4.2 can be used to show that for each standard natural number n the principle that every $n \times n$ matrix has a Jordan decomposition is provable in RCA_0. We do not know whether the principle that every finite matrix has a Jordan decomposition is provable in RCA_0.

The next lemma is foreshadowed by previous research. It is well known that the function that sends a matrix to its Jordan decomposition is discontinuous. Kohlenbach [5] has shown that, in the extension RCA_0^ω of RCA_0 to all finite types, the existence of a higher-type object encoding a non-sequentially-continuous real-valued function is equivalent to the corresponding higher-type extension of ACA_0.

Lemma 4.4. *The following principle implies* ACA_0 *over* RCA_0 (*and hence over* RCA). *For every sequence* $\langle M_i \mid i \in \mathbb{N}\rangle$ *of* 2×2 *real matrices, such that each matrix* M_i *has only real eigenvalues, there are sequences* $\langle U_i \mid i \in \mathbb{N}\rangle$ *and* $\langle J_i \mid i \in \mathbb{N}\rangle$ *such that* (U_i, J_i) *is a Jordan decomposition of* M_i *for all* $i \in \mathbb{N}$.

Proof. We first demonstrate a concrete example of the discontinuity of the Jordan form. For any real z, let $M(z)$ denote the matrix

$$M(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

The matrix $M(0)$ is the identity matrix, and so is its Jordan canonical form. If $z \neq 0$ then $M(z)$ has the following Jordan decomposition:

$$M(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}^{-1}.$$

The crucial fact is that the entry in the upper-right-hand corner of the Jordan canonical form of $M(z)$ is 0 if $z = 0$ and 1 if $z \neq 0$.

Let h be an arbitrary function from $\mathbb{N}$ to $\mathbb{N}$. We will assume the principle of the theorem and show that the range of h exists. It is well known that RCA_0 can construct a function $n \mapsto z_n$ that assigns each n a quickly converging Cauchy sequence z_n such that, for all n, $z_n = 0$ if and only n is not in the range of h. Form a sequence of matrices $\langle M(z_n) \mid n \in \mathbb{N}\rangle$; according to the principle, there is an associated sequence of Jordan canonical forms. The upper-right-hand entry of each of these canonical forms is either 0 or 1, and it is possible to effectively decide between these two cases. Thus, in RCA_0, we may form the range of h using the sequence of Jordan canonical forms as a parameter. □

We now turn to statement (4.1.2). Recall that the standard formalization of the real numbers in RCA_0, as described by Simpson [10], makes use of quickly converging Cauchy sequences of rationals. Alternative formalizations of the real numbers may be considered, however. We define a *Dedekind cut* to be a subset Y of the rational numbers such that not every rational number is in Y and if $p \in Y$ and $q < p$ then $q \in Y$. We say that a Dedekind cut Y is *equivalent* to a quickly converging Cauchy sequence $\langle a_i \mid i \in \mathbb{N}\rangle$ if any only if the equivalence

$$q \in Y \Leftrightarrow q \leq \lim_{i\to\infty} a_i$$

holds for every rational number q. Formalization of (4.1.2) shows that it is in Γ_1.

Hirst [3] has proved the following results that relate Cauchy sequences with Dedekind cuts. Together with the Main Theorem, these results show that statement (4.1.2) is provable in RCA_0 but not $\widehat{\mathsf{E\text{-}HA}}{}^\omega_\restriction + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^\omega_{ef}$.

Lemma 4.5 (Hirst [3, Corollary 4])**.** *The following is provable in* RCA_0. *For any quickly converging Cauchy sequence* x *there is an equivalent Dedekind cut.*

Lemma 4.6 (Hirst [3, Corollary 9]). *The following principle is equivalent to* WKL_0 *over* RCA_0 (*and hence over* RCA). *For each sequence* $\langle X_i \mid i \in \mathbb{N}\rangle$ *of quickly converging Cauchy sequences there is a sequence* $\langle Y_i \mid i \in \mathbb{N}\rangle$ *of Dedekind cuts such that* X_i *is equivalent to* Y_i *for each* $i \in \mathbb{N}$.

Statement (4.1.3), which is our final application of the Main Theorem, is related to countable posets. In RCA_0, we define a *countable poset* to be a set $P \subseteq \mathbb{N}$ with a coded binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. A function $f\colon \mathbb{N} \to P$ is called an *enumerated filter* if for every $i, j \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $f(k) \preceq f(i)$ and $f(k) \preceq f(j)$, and for every $q \in P$ if there is an $i \in \mathbb{N}$ such that $f(i) \preceq q$ then there is a $k \in \mathbb{N}$ such that $f(k) = q$. An enumerated filter is called *unbounded* if there is no $q \in P$ such that $q \prec f(i)$ for all $i \in \mathbb{N}$. An enumerated filter f *extends* a filter g if the range of g (viewed as a function) is a subset of the range of f. If we modify the usual definition of an enumerated filter to include an auxiliary function $h : \mathbb{N}^2 \to \mathbb{N}$ such that for all i and j, $f(h(i,j)) \preceq f(i)$ and $f(h(i,j)) \preceq f(j)$, then (4.1.3) is in Γ_1.

Mummert has proved the following two lemmas about extending filters to unbounded filters (see Lempp and Mummert [8] and the remarks after Lemma 4.1.1 of Mummert [9]). These lemmas show that (4.1.3) is provable in RCA_0 but not $\widehat{\mathsf{E{-}HA}}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$.

Lemma 4.7 (Lempp and Mummert [8, Theorem 3.5]). RCA_0 *proves that any enumerated filter on a countable poset can be extended to an unbounded enumerated filter.*

Lemma 4.8 (Lempp and Mummert [8, Theorem 3.6]). *The following statement implies* ACA_0 *over* RCA_0 (*and hence over* RCA)*: Given a sequence* $\langle P_i \mid i \in \mathbb{N}\rangle$ *of countable posets and a sequence* $\langle f_i \mid i \in \mathbb{N}\rangle$ *such that* f_i *is an enumerated filter on* P_i *for each* $i \in \mathbb{N}$*, there is a sequence* $\langle g_i \mid i \in \mathbb{N}\rangle$ *such that, for each* $i \in \mathbb{N}$*,* g_i *is an unbounded enumerated filter on* P_i *extending* f_i.

We close this section by noting that the proof-theoretic results of Section 3 are proved by finitistic methods. Consequently, inituitionists might accept arguments like those presented here to establish the non-provability of certain theorems from intuitionistic systems.

5. The *Dialectica* interpretation

By replacing the application of modified realizability with an application of Gödel's *Dialectica* interpretation, we can reformulate the results of Section 3 in axiom systems with restricted induction. This has some appeal, as reverse mathematics is traditionally carried out using systems with restricted induction. Perhaps more importantly, the reformulation allows us to append a Markov principle to the initial intuitionistic axiom system. These benefits have a cost: we must weaken the extentionality axioms of our systems, and the new version of our main theorem will only hold for a smaller class of formulas.

Extended discussions of Gödel's *Dialectica* interpretation are given by Avigad and Feferman [1], Kohlenbach [6], and Troelstra [11]. The interpretation assigns to each formula A a formula A^D of the form $\exists x \forall y\, A_D$, where A_D is quantifier free and each quantifier may represent a block of quantifiers of the same kind. The blocks of quantifiers in A^D may include variables of any finite type. We follow Avigad and

Feferman [1] in defining the *Dialectica* interpretation inductively via the following six clauses, in which $A^D = \exists x \forall y\, A_D$ and $B^D = \exists u \forall v\, B_D$.

(1) If A an atomic formula then x and y are both empty and $A^D = A_D = A$.
(2) $(A \wedge B)^D = \exists x \exists u \forall y \forall v\, (A_D \wedge B_D)$.
(3) $(A \vee B)^D = \exists z \exists x \exists u \forall y \forall v\, ((z = 0 \wedge A_D) \vee (z = 1 \wedge B_D))$.
(4) $(\forall z\, A(z))^D = \exists X \forall z \forall y\, A_D(X(z), y, z)$.
(5) $(\exists z\, A(z))^D = \exists z \exists x \forall y\, A_D(x, y, z)$.
(6) $(A \to B)^D = \exists U \exists Y \forall x \forall v\, (A_D(x, Y(x, v)) \to B_D(U(x), v))$.

A negated formula $\neg A$ is treated as an abbreviation of $A \to \bot$.

As noted by Kohlenbach [6, Section 8.3], the proof of Theorem 8.6 of [6] can be easily modified to yield the following analog of the soundness theorem, Theorem 3.2.

Theorem 5.1. *Let A be a formula in $\mathcal{L}(\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction})$. If*

$$\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction} + \mathsf{AC} + \mathsf{IP}^{\omega}_{\forall} + \mathsf{M}^{\omega} \vdash \forall x \exists y A(x, y),$$

then $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction} \vdash \forall x A^D(x, t(x))$, where t is a suitable term of $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction}$.

Although RCA^{ω}_0 is a conservative extension of $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction}$, we still need to convert A^D back to A. Unfortunately, RCA^{ω}_0 can only prove $A^D \to A$ for certain formulas. The class Γ_2, as found in (for example) Definition 8.10 of Kohlenbach [6], is a subset of these formulas.

Definition 5.2. Γ_2 is the collection of formulas in $\mathcal{L}(\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction})$ defined inductively as follows.

(1) All prime formulas are elements of Γ_2.
(2) If A and B are in Γ_2, then so are $A \wedge B$, $A \vee B$, $\forall x A$, and $\exists x A$.
(3) If A is purely universal and $B \in \Gamma_2$, then $(\exists x A \to B) \in \Gamma_2$, where $\exists x$ may represent a block of existential quantifiers.

Kohlenbach [6, Lemma 8.11] establishes the following analog of Lemma 3.4.

Lemma 5.3. *Let A be a formula of Γ_2. Then $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction} \vdash A^D \to A$.*

We can adapt our proof of Theorem 3.5 to obtain the following term extraction theorem.

Theorem 5.4. *Let $\forall x^\rho \exists y^\tau A(x, y)$ be a sentence in Γ_2 with arbitrary types ρ and τ. If $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction} + \mathsf{AC} + \mathsf{IP}^{\omega}_{\forall} + \mathsf{M}^{\omega} \vdash \forall x^\rho \exists y^\tau A(x, y)$, then $\mathsf{RCA}^{\omega}_0 \vdash \forall x^\rho A(x, t(x))$, where t is a suitable term of $\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction}$.*

Because the proof of Theorem 3.6 makes no use of induction, it also serves to prove the *Dialectica* version of our main theorem.

Theorem 5.5. *Let $\forall x \exists y A(x, y)$ be a sentence in Γ_2. If*

$$\widehat{\mathsf{WE-HA}}^{\omega}_{\restriction} + \mathsf{AC} + \mathsf{IP}^{\omega}_{\forall} + \mathsf{M}^{\omega} \vdash \forall x \exists y\, A(x, y),$$

then

$$\mathsf{RCA}^{\omega}_0 \vdash \forall \langle x_n \mid n \in \mathbb{N} \rangle\, \exists \langle y_n \mid n \in \mathbb{N} \rangle \forall n\, A(x_n, y_n).$$

Furthermore, if x and y are both type 1 (set) variables, and $\forall x \exists y A(x, y)$ is in $\mathcal{L}(\mathsf{RCA}_0)$, then RCA^{ω}_0 may be replaced by RCA_0 in the implication.

Clearly, this *Dialectica* version of the main theorem is less broadly applicable due to the restriction on the class of formulas allowed. While Γ_2 may not be the largest class of formulas for which an analog of Theorem 5.5 can be obtained, any class substituted for Γ_2 must omit a substantial collection of formulas. For example, any class replacing Γ_2 must omit some formulas of the form $\forall x \exists y A(x, y)$ where $A(x, y)$ is classically equivalent to a Π^0_2 formula. This observation follows immediately from an examination of the collection scheme arguments of Kohlenbach [4].

In practice, requiring formulas to be in Γ_2 may not be excessively restrictive. Examination of the statements in Theorem 4.1 shows that the hypotheses in their implications are purely universal, and each of them is in Γ_2. Thus an application of Theorem 5.5 shows that Theorem 4.1 holds with $\widehat{\mathsf{E{-}HA}}{}^{\omega}_{\restriction} + \mathsf{IA} + \mathsf{AC} + \mathsf{IP}^{\omega}_{ef}$ replaced by $\widehat{\mathsf{WE{-}HA}}{}^{\omega}_{\restriction} + \mathsf{AC} + \mathsf{IP}^{\omega}_{\forall} + \mathsf{M}^{\omega}$.

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