Reverse Mathematics: Constructivism and Combinatorics

Jeff Hirst[1]
Appalachian State University
Boone, NC

These slides are available at: www.mathsci.appstate.edu/~jlh

September 7, 2012

ASU Mathematical Sciences Colloquium

[1] Jeff Hirst's research was partially supported by an OCSA (off-campus scholarly assignment) from Appalachian State University and a grant from the John Templeton Foundation. Any opinions expressed here are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

OCSA outcomes: Publications

- *Reverse mathematics and algebraic field extensions*, with François Dorais and Paul Shafer, submitted.
- *Disguising induction: Proofs of the pigeonhole principle for trees*, to appear in: Foundational Adventures: Essays in Honor of Harvey M. Friedman, (Neil Tennant, editor) Templeton Press (2012).
- *Reverse mathematics, trichotomy, and dichotomy*, with François Dorais and Paul Shafer, Journal of Logic and Analysis **4**:13, (2012) 1-14.
- *On Mathias generic sets*, with Peter Cholak and Damir Dzhafarov, pages 129-138 in: How the World Computes: Proceedings of the Turing Centenary Conference and 8th Conference on Computability in Europe, CiE 2012, LNCS 7318, (Cooper, Dawar, and Lowe, editors) Springer-Verlag (2012) ISBN: 978-3-642-30869-7.
- *More reverse mathematics of the Heine-Borel theorem*, with Jessica Miller, Journal of Logic and Analysis **4**:6, (2012) 1-10.
- *Hilbert versus Hindman*, Archive for Mathematical Logic **51**:1-2, (2012) 123-125.

OCSA outcomes: Presentations

- *Reverse mathematics and field extensions*, given at the Association for Symbolic Logic 2012 North American Annual Meeting on April 1, 2012.
- *Reverse mathematics and dichotomy*, given at the Joint Mathematics Meetings in Boston on January 6, 2012.
- *Reverse mathematics and persistent reals*, given in the Midwest Computability Seminar X at the University of Chicago on November 1, 2011.
- *Two familiar principles in disguise*, given in the Notre Dame Logic Seminar on October 27, 2011.
- *Two combinatorial proofs and some related questions* , given at the Reverse Mathematics Workshop at the University of Chicago on September 17, 2011.
- *Reverse Mathematics: Constructivism and Combinatorics*, given at Foundational Questions in the Mathematical Sciences, a meeting sponsored by the John Templeton Foundation at the International Academy Traunkirchen, Austria on July 8-12, 2011.

A weak form of Hindman's theorem

HIL: Suppose $f : \mathbb{N}^{<\mathbb{N}} \to k$ is a finite coloring of the finite subsets of the natural numbers. Then there is a an infinite sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ of distinct finite sets and a color $c < k$ such that for every finite set $F \subset \mathbb{N}$ we have $f(\cup_{i \in F} X_i) = c$.

HTU: Suppose $f : \mathbb{N}^{<\mathbb{N}} \to k$ is a finite coloring of the finite subsets of the natural numbers. Then there is a an infinite sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ of increasing finite sets and a color $c < k$ such that for every finite set $F \subset \mathbb{N}$ we have $f(\cup_{i \in F} X_i) = c$.

$X_i < X_j$ means $\max(X_i) < \min(X_j)$

Note: This material arose from discussions with Henry Towsner.

An example of reverse mathematics

Theorem

(RCA_0) *The following are equivalent:*

1. HIL :. *If* $f : \mathbb{N}^{<\mathbb{N}} \to k$ *then there is a color c and a sequence* $\langle X_i \rangle_{i \in \mathbb{N}}$ *of distinct sets such that all finite unions of sets in the sequence have the color c.*
2. $\mathsf{RT}(1)$: *If* $f : \mathbb{N} \to k$ *then there is a* $c < k$ *such that* $\{n \mid f(n) = c\}$ *is infinite.*

RCA_0 is a set of axioms that talk about natural numbers and sets of naturals numbers. The axioms include ordered semi-ring axioms for $\mathbb{N}$, induction for formulas with only one (number) quantifier, and a set-existence axiom for computable subsets of $\mathbb{N}$. Many theorems can be expressed in RCA_0, but only some can be proved in the system.

Theorem

(RCA_0) *The following are equivalent:*

1. HIL : *If* $f : \mathbb{N}^{<\mathbb{N}} \to k$ *then there is a color* c *and a sequence* $\langle X_i \rangle_{i \in \mathbb{N}}$ *of distinct sets such that all finite unions of sets in the sequence have the color* c*.*
2. $\mathsf{RT}(1)$: *If* $f : \mathbb{N} \to k$ *then there is a* $c < k$ *such that* $\{n \mid f(n) = c\}$ *is infinite.*

Sketch.

$(1) \to (2)$. Given $f : \mathbb{N} \to k$, define $g(X) = f(\max(X))$. Apply HIL. f is constant on the infinite set of maxima.

Theorem

(RCA_0) *The following are equivalent:*

1. HIL : *If* $f : \mathbb{N}^{<\mathbb{N}} \to k$ *then there is a color* c *and a sequence* $\langle X_i \rangle_{i \in \mathbb{N}}$ *of distinct sets such that all finite unions of sets in the sequence have the color* c*.*
2. $\mathrm{RT}(1)$: *If* $f : \mathbb{N} \to k$ *then there is a* $c < k$ *such that* $\{n \mid f(n) = c\}$ *is infinite.*

Sketch.

$(1) \to (2)$. Given $f : \mathbb{N} \to k$, define $g(X) = f(\max(X))$. Apply HIL. f is constant on the infinite set of maxima.

$(2) \to (1)$. Given $f : \mathbb{N}^{<\mathbb{N}} \to k$, define $g(n) = f([0, n])$. Apply $\mathrm{RT}(1)$ to find $n_0, n_1, \ldots$ monochromatic. Let $X_i = [0, n_i]$. $\square$

Why bother?

Based on Tait's analysis of Hilbert's program, Simpson [5] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for Π^0_1 sentences.

$\mathsf{WKL}_0 + \mathsf{RT}(1)$ is conservative over PRA for Π^0_2 formulas.

Since $\mathsf{WKL}_0 + \mathsf{RT}(1)$ proves $\mathsf{RCA}_0 + \mathsf{HIL}$, we know HIL is finitistically reducible.

Why bother?

Based on Tait's analysis of Hilbert's program, Simpson [5] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for Π^0_1 sentences.

$\mathsf{WKL}_0 + \mathsf{RT}(1)$ is conservative over PRA for Π^0_2 formulas.

Since $\mathsf{WKL}_0 + \mathsf{RT}(1)$ proves $\mathsf{RCA}_0 + \mathsf{HIL}$, we know HIL is finitistically reducible.

$\mathsf{RCA}_0 + \mathsf{HTU}$ proves ACA_0 [1], so $\mathsf{RCA}_0 + \mathsf{HTU}$ proves Π^0_1 formulas that PRA can't.

The consistency of PRA is a Π^0_1 formula.
HTU is not finitistically redicible.

Dichotomy is provable in RCA_0,

Theorem

(RCA_0) *If α is a real number, then $\alpha \geqslant 0$ or $\alpha \leqslant 0$.*

but sequential dichotomy is not...

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0 *(Infinite 0-1 trees have infinite paths.)*
2. *If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of reals, then there is a set I such that for all i, $i \in I$ implies $\alpha_i \geqslant 0$ and $i \notin I$ implies $\alpha_i \leqslant 0$.*

Note: This material is joint work with François Dorais and Paul Shafer [2].

Since RCA_0 proves that sequential dichotomy implies WKL_0, RCA_0 cannot prove sequential dichotomy.

By a result of Hirst and Mummert [4], since RCA_0 cannot prove sequential dichotomy, $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathsf{ef}}$ does not prove dichotomy.

($\mathsf{E\text{-}HA}^\omega$ is an axiomatization of constructive analysis, AC is a choice scheme, and $\mathsf{IP}^\omega_{\mathsf{ef}}$ is an independence of premise scheme for $\exists$-free formulas.)

The result from [4] is not a biconditional, but a *computable restriction* of sequential dichotomy can indicate a candidate for a *constructive* restriction of dichotomy.

Definition: A real α is persistent if

- $\forall s(\alpha(s) \geqslant 0 \to \exists t(t > s \wedge \alpha(t) \geqslant 0))$
  ... the expansion of α has no last non-negative rational and
- $\forall s(\alpha(s) \leqslant 0 \to \exists t(t > s \wedge \alpha(t) \leqslant 0))$
  ... the expansion of α has no last non-positive rational.

Definition: A real α is persistent if

- $\forall s(\alpha(s) \geqslant 0 \to \exists t(t > s \wedge \alpha(t) \geqslant 0))$
  ... the expansion of α has no last non-negative rational and
- $\forall s(\alpha(s) \leqslant 0 \to \exists t(t > s \wedge \alpha(t) \leqslant 0))$
  ... the expansion of α has no last non-positive rational.

Theorem: (RCA_0) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \leqslant 0$ and $i \notin I$ implies $\alpha_i \geqslant 0$.

Theorem: $(\widehat{\text{E-HA}}^{\omega}_{\restriction})$ If α is a persistent real, then $\alpha \geqslant 0$ or $\alpha \leqslant 0$.

Moral: Reverse math can assist in formulating constructive results.

Bibliography

[1] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, Logic and combinatorics (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 125–156. MR891245 (88d:03113).

[2] François Dorais, Jeffry L. Hirst, and Paul Shafer, *Reverse mathematics, trichotomy, and dichotomy*, J. Log. Anal. **4** (2012), no. 13, 1-14, DOI 10.4115/jla.2012.4.13.

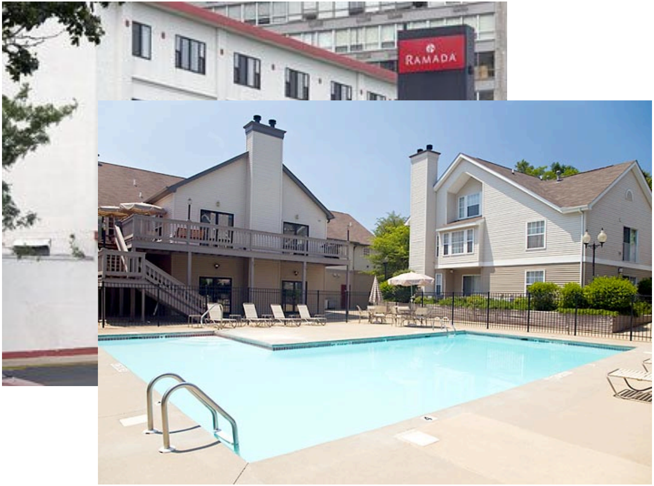
[3] Jeffry L. Hirst, *Hilbert versus Hindman*, Archive for Mathematical Logic **51** (2012), no. 1-2, 123–125, DOI 10.1007/s00153-011-0257-4.

[4] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162. DOI 10.1215/00294527-1306163 MR2794648.

[5] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.

Obligatory snapshots. . .

Obligatory snapshots. . .



Obligatory snapshots. . .

Obligatory snapshots. . .

Obligatory snapshots. . .

Obligatory snapshots. . .