Banach's theorem in higher order reverse mathematics

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Abstract

In this paper, methods of second order and higher order reverse mathematics are applied to versions of a theorem of Banach that extends the Schroeder–Bernstein theorem. Some additional results address statements in higher order arithmetic formalizing the uncountability of the power set of the natural numbers. In general, the formalizations of higher order principles here have a Skolemized form asserting the existence of functionals that solve problems uniformly. This facilitates proofs of reversals in axiom systems with restricted choice.

1 Introduction

The Schroeder–Bernstein theorem is perhaps the best known result about cardinality. In full generality, it states that if A and B are sets, there is an injection $f\colon A \to B$, and there is an injection $g\colon B \to A$, then there is a bijection from A to B. Unfortunately, this theorem is not ideal for reverse mathematics analysis. If we add the assumption that $A, B \subseteq \mathbb{N}$, the result is computationally trivial: whenever $A, B \subseteq \mathbb{N}$ have the same cardinality, there is an $(A \oplus B)$-computable bijection between them.

In higher order reverse mathematics, we might consider the case where $A, B \subseteq 2^{\mathbb{N}}$ or $A, B \subseteq \mathbb{N}^{\mathbb{N}}$. In this setting, the Schroeder–Bernstein theorem is no longer trivial. However, because the theorem does not postulate any relationship between the bijection being constructed and the original two injections, obtaining reversals presents a challenge.

Our focus is a classical theorem of Banach [1] from 1924 more suited to reverse mathematical analysis. Banach argued this theorem captures the essence of proofs of the Schroeder–Bernstein theorem, such as the well known proof by Julius Kőnig.

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Theorem 1.1 (Banach)**.** If A and B are sets, $f\colon A \to B$ is an injection, and $g\colon B \to A$ is an injection, there are decompositions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$, $f(A_1) = B_1$, and $g(B_2) = A_2$.

Restating this in terms of the existence of a bijection gives a corollary that strengthens the Schroeder–Bernstein theorem, which we will also call Banach's Theorem.

Corollary 1.2. If f is an injection from a set A to a set B, and g is an injection from B to A, there is a bijection $h\colon A \to B$ such that, whenever $h(a) = b$, either $f(a) = b$ or $g(b) = a$.

A brief history of Banach's Theorem and the Schröder–Bernstein theorem is given by Remmel [12, Introduction]. An analysis of Banach's Theorem for subsets of $\mathbb{N}$, using subsystems of second order arithmetic, appears in Hirst's thesis [6, §3.2] and a related article [7]. That development uses symmetric marriage theorems to prove the following second order arithmetic results.

Theorem 1.3 ([7, Theorem 4.1])**.** RCA_0 proves the following are equivalent:

1. ACA_0.
2. (Countable Banach's Theorem) Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ be injections. Then there is a bijection $h\colon \mathbb{N} \to \mathbb{N}$ such that for all m and n, $h(n) = m$ implies either $f(n) = m$ or $g(m) = n$.

Theorem 1.4 ([7, Theorem 4.2])**.** RCA_0 proves the following are equivalent:

1. WKL_0.
2. (Bounded Countable Banach's Theorem) Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ be injections such that the ranges of f and g exist. Then there is a bijection $h\colon \mathbb{N} \to \mathbb{N}$ such that for all m and n, $h(n) = m$ implies either $f(n) = m$ or $g(m) = n$.

In this paper, we use methods from higher order reverse mathematics to study the *uniformity* of results like these. We are interested not only in the existence of the bijection h, but also whether there is a functional that can produce h uniformly from f and g. This question of uniformity is purely higher order, and cannot be expressed directly in second order reverse mathematics. To study this uniformity, we examine Skolemized versions of theorems. For example, instead of examining Banach's Theorem in a form such as

$$(\forall f, g, A, B)(\exists h)\, \Phi(f, g, A, B, h),$$

we consider the form

$$(\exists H)(\forall f, g, A, B)\, \Phi(f, g, A, B, H(f, g, A, B)).$$

Both versions of a theorem are of interest, of course, and the latter always follows from the former if we assume sufficient choice principles. We are interested in the Skolemized forms because they represent a particular kind of uniformity, and we typically do not assume enough choice to derive them directly from the un-Skolemized form. As discussed in Section 4, this is a different kind of uniformity than Weihrauch reducibility.

Section 2 begins with a survey of reverse mathematics results on countability. Sections 3 and 4 present a number of supporting lemmas to prepare for the analysis of Banach's theorem. Section 5 examines Theorems 1.3 and 1.4 from the viewpoint of Skolemized uniformity. Section 6 extends the study of Banach's Theorem to subsets of $2^{\mathbb{N}}$ and, more generally, subsets of compact metric spaces.

1.1 Formal theories

This work relies on several well studied systems of second order arithmetic and higher order arithmetic. Simpson [14] and Dzhafarov and Mummert [3] provide thorough references for reverse mathematics. Kohlenbach [9] provides a reference for higher order reverse mathematics. We follow Kohlenbach's definitions of higher order systems throughout this paper, noting any exceptions explicitly.

For the purposes of higher order reverse mathematics, we assume that our systems use the function based language of higher order arithmetic, rather than the set based language. Accordingly, $2^{\mathbb{N}}$ is used throughout this paper to denote the set of all functions from $\mathbb{N}$ to $\{0, 1\}$.

Many of our results will use fragments of the quantifier-free choice scheme. For types ρ and τ, we have the scheme

$$\mathsf{QF\text{-}AC}^{\rho,\tau}\colon (\forall x^\rho)(\exists y^\tau)A(x, y) \to (\exists Y^{\rho\to\tau})(\forall x^\rho)A(x, Y(x)),$$

where A is a quantifier free formula. Here A can have parameters of arbitrary type.

The system $\mathsf{RCA}_0^\omega = \mathsf{E\text{-}PRA}^\omega + \mathsf{QF\text{-}AC}^{1,0}$ is a fragment of higher order arithmetic. It is axiomatized by a set of basic axioms along with induction for Σ^0_1 formulas and the choice scheme $\mathsf{QF\text{-}AC}^{1,0}$. The syntax has term-forming operations for λ abstraction and primitive recursion.

The system RCA_0^2 is a second order fragment of RCA_0^ω, with only types 0 and 1 for elements of $\mathbb{N}$ and functions $\mathbb{N} \to \mathbb{N}$, respectively. Formally, we have $\mathsf{RCA}_0^2 = \mathsf{E\text{-}PRA}^\omega + \mathsf{QF\text{-}AC}^{0,0}$. This system is equivalent to the set based system RCA_0 presented by Simpson [14], and we will henceforth denote RCA_0^2 by RCA_0 when no confusion is likely.

A sequence $\langle f_n : n \in \mathbb{N}\rangle$ is viewed as a map $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, so that $f_n(m) = f(\langle n, m\rangle)$, where $\langle\cdot, \cdot\rangle$ is a suitable pairing function.

Emulating Kohlenbach [9], we use parentheses around the name of a functional to denote the principle stating that the functional exists. For example, the principle $(\exists^2)$ asserts the existence of the functional $\exists^2$, defined below.

There are several ways to extend the comprehension axioms of second order arithmetic to the higher order setting. One particular functional (set) existence axiom for higher order arithmetic is $(\exists^2)$, defined by

$$(\exists^2)\colon (\exists\varphi^{1\to 0})(\forall f)(\varphi(f) = 0 \leftrightarrow (\exists n)[f(n) = 0]).$$

The functional $\varphi^{1\to 0}$ from this principle is itself called $\exists^2$. The system $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ implies the arithmetical comprehension scheme. Kohlenbach [9] showed that ACA_0^ω is conservative over ACA_0 for sentences in the language L_2.

Other functional existence principles correspond to ACA_0. Kohlenbach [9] presents two such functionals. One, μ_0, selects a zero of a function if such a zero exists:

$$(\mu_0)\colon (\exists\mu_0^2)(\forall f^1)[(\exists n^0)(f(n) =_0 0) \to f(\mu_0(f)) = 0].$$

Another returns the least zero of a function, in the fashion of Feferman [4, §2.3.3]:

$$(\mu)\colon (\exists\mu^2)(\forall f^1)((\exists n^0)(f(n) =_0 0) \to [f(\mu(f)) = 0 \land (\forall t < \mu(f))(f(t) \neq 0)]).$$

Proposition 1.5. The following are pairwise equivalent over RCA_0^ω: $(\exists^2)$, (μ_0), and (μ).

Proof. Kohlenbach [9, Proposition 3.9] proves the equivalence of $(\exists^2)$ and (μ_0). Because any μ satisfying (μ) also satisfies (μ_0), it suffices to show that RCA_0^ω proves thaxt (μ_0) implies (μ). Given a functional μ_0 as in the definition of (μ_0), $\mu(f)$ is the least $t \leq \mu_0(f)$ such that $f(t) = 0$. This functional is primitive recursive in μ_0 and thus exists by RCA_0^ω and (μ_0). □

2 Countability

One motivation for this research is the question: how difficult is it to prove $2^{\mathbb{N}}$ is uncountable? As usual, being uncountable simply means not being countable. There are many ways to express the principle that $2^{\mathbb{N}}$ is countable, with the following three being particularly natural:

- C_{enum}: there is a sequence $\langle f_n : n \in \mathbb{N}\rangle$ such that for all $g \in 2^{\mathbb{N}}$ there is an $n \in \mathbb{N}$ with $g = f_n$.
- C_{inj}: there is a functional $\Phi^{1\to 0}$ that is an injection from $2^{\mathbb{N}}$ to $\mathbb{N}$.
- C_{bij}: there is a functional $\Phi^{1\to 0}$ that is an bijection from $2^{\mathbb{N}}$ to $\mathbb{N}$.

The principles C_{inj} and C_{bij} cannot be stated in the language of second order arithmetic, but they can be stated in RCA_0^ω. When we say we assume C_{inj} or C_{bij}, this means we assume the existence of a functional with the property stated. Similarly, if we assume $\neg\mathsf{C}_{\text{inj}}$ or $\neg\mathsf{C}_{\text{bij}}$, this means we assume no functional has the property stated.

In context of set theory there is little reason to distinguish between C_{inj} and C_{bij}, because of the comprehension principles available. As discussed below, there are key distinctions between these principles in the context of theories of arithmetic with restricted comprehension principles.

Of course, C_{enum}, C_{bij}, and C_{inj} are classically false. There are two key questions: which systems are "strong enough" to disprove these false principles, and which are "weak enough" to be consistent with one or more of the principles. As is well known, Cantor's diagonalization proof allows us to disprove C_{enum} in very weak systems (compare Theorem II.4.9 of Simpson [14] showing $\mathbb{R}$ is uncountable in RCA_0).

Proposition 2.1. RCA_0^2 proves $\neg\mathsf{C}_{\text{enum}}$.

Proof. Given $\langle f_n : n \in \mathbb{N}\rangle$ witnessing C_{enum}, RCA_0^2 can construct the function g defined by $g(m) = 1 - f_m(m)$. Then $g \in 2^{\mathbb{N}}$, but g cannot be f_k for any $k \in \mathbb{N}$. □

The principles C_{inj} and C_{bij} have much more interesting behavior. Normann and Sanders [10] provide a detailed analysis of the negations of these principles, which they name NIN and NBI, respectively. (They formulate NIN and NBI for $\mathbb{R}$ but the results hold equally for $2^{\mathbb{N}}$.) Their Theorem 3.2 shows that the true principle NIN is not provable in the system $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ (which includes Π^1_∞ comprehension with parameters of type 1), and hence this system is consistent with C_{inj} [10, Theorem 3.26]. They also show that NIN is provable in $\mathsf{Z}_2^\Omega + \mathsf{QF\text{-}AC}^{0,1}$, which includes the functional $\exists^3$ in addition to Π^1_∞ comprehension. In the remainder of this section, we discuss some aspects of their results related to C_{inj} and C_{bij}.

A key issue in analyzing C_{inj} is that the range of an injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ may be hard to form with weak comprehension axioms. We will see that a similar issue arises in the study of Banach's theorem, as well, where the existence of the range of a functional becomes a key question. By contrast, it is relatively easy to disprove C_{bij} [10, Theorem 3.28].

Proposition 2.2 ($\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$)**.** There is no injection $\Phi\colon 2^{\mathbb{N}} \to \mathbb{N}$ for which the characteristic function for the range exists. In particular, C_{bij} is disprovable in $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

Proof. We will work in $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ and assume there is a bijection Φ from $2^{\mathbb{N}}$ to $\mathbb{N}$ with range $D = \{n : (\exists g)[\Phi(g) = n]\}$ given by a characteristic function. We will prove the principle C_{enum} by constructing a kind of left inverse of Φ, which will be a (possibly noninjective) enumeration of $2^{\mathbb{N}}$. Because RCA_0 proves $\neg\mathsf{C}_{\text{enum}}$, this gives a contradiction.

By assumption, for each $n \in D$ there is a $g \in 2^{\mathbb{N}}$ with $\Phi(g) = n$. Therefore, by $\mathsf{QF\text{-}AC}^{0,1}$, we may form a function f so that $(\forall n)[n \in D \to \Phi(f_n) = n]$. Then $\langle f_n : n \in \mathbb{N}\rangle$ is an enumeration of $2^{\mathbb{N}}$, so C_{enum} holds, a contradiction. □

We now explain how the lemma implies certain higher order formulations of the Schroeder–Bernstein theorem are nontrivial. Suppose, in the context of set theory, we wanted to try to use the Schroeder–Bernstein theorem to show $2^{\mathbb{N}}$ is countable. Because there is a trivial injection from $\mathbb{N}$ to $2^{\mathbb{N}}$, the other assumption in the Schroeder–Bernstein theorem is the existence of an injection from $2^{\mathbb{N}}$ to $\mathbb{N}$, that is, C_{inj}. The conclusion is the existence of a bijection, that is, C_{bij}. We can thus view the implication $\mathsf{C}_{\text{inj}} \to \mathsf{C}_{\text{bij}}$ as a specific formal instance of the Schroeder–Bernstein theorem. (Normann and Sanders [11] study a different formulation of the Schroeder–Bernstein theorem, which they call $\mathsf{CB}\mathbb{N}$.)

Corollary 2.3. The implication $\mathsf{C}_{\text{inj}} \to \mathsf{C}_{\text{bij}}$ is not provable in $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

Proof. $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ is consistent with C_{inj} but not C_{bij}. $\square$

Lemma 2.2 can also be used to obtain an upper bound on the strength required to disprove C_{inj}. Normann and Sanders prove a version of the following lemma using the principle $(\exists^3)$ as a formalization of Σ^1_1 comprehension with functional parameters.

Corollary 2.4 (see Normann and Sanders [10, Theorem 3.1])**.** C_{inj} is disprovable from $\mathsf{RCA}_0 + \mathsf{QF\text{-}AC}^{0,1}$ along with Σ^1_1 comprehension with parameters of type 2.

Proof. Assume Φ is a functional witnessing C_{inj}. Applying Σ^1_1 comprehension with parameter Φ, we can construct the range of Φ. We then obtain a contradiction from Proposition 2.2. $\square$

A final point of interest is that the classically false principle C_{inj}, although consistent with RCA_0^ω, has nontrivial set existence strength. Normann and Sanders discuss the contrapositive of the following proposition in the guise of a "trick" related to excluded middle [10, §3].

Proposition 2.5. C_{inj} implies $(\exists^2)$ over RCA_0^ω.

Proof. If Φ is an injection from $2^{\mathbb{N}}$ to $\mathbb{N}$ then Φ is discontinuous at every point (here we identify elements of $\mathbb{N}$ with constant functions from $\mathbb{N}$ to $\mathbb{N}$). The existence of a discontinuous functional implies $(\exists^2)$ by Proposition 3.7 of Kohlenbach [9]. $\square$

Thus, for example, there is no model of RCA_0^ω in which C_{inj} holds and every element of $2^{\mathbb{N}}$ is computable.

3 Bounding calculations of type 1 functions

This section contains several technical lemmas related to the range of a function $f\colon \mathbb{N} \to \mathbb{N}$. Each function of this type has a number of auxiliary functions related to its range. The most obvious is the characteristic function for the range. We write $\rho(f,g)$ as shorthand for the formula asserting that g is the characteristic function for the range of f. More formally,

$$\rho(f,g) \text{ is } (\forall n)[(\exists m)(f(m)=n) \leftrightarrow g(n)>0].$$

A bounding function can also be used to compute the range of f. We write $\beta(f,g)$ for the formula asserting that g is such a bounding function. Formally,

$$\beta(f,g) \text{ is } (\forall n)[(\exists m)(f(m)=n) \leftrightarrow (\exists t \leq g(n))(f(t)=n)].$$

The results below address the problem of converting between the characteristic function for the range of a function and a bounding function for the range, and the amount of uniformity present in the conversion. In the second order setting, principles asserting the existence of characteristic functions and the existence of bounding functions are interchangeable, as shown by the following two results.

Proposition 3.1 (RCA_0). For all $f\colon \mathbb{N} \to \mathbb{N}$, $(\exists g)\,\rho(f,g) \leftrightarrow (\exists h)\,\beta(f,h)$.

Proof. Working in RCA_0, suppose g is a characteristic function for the range of a function f. Then $h(n) = (\mu\, t)(g(n)=0 \vee f(t)=n)$ is the desired bounding function and exists by recursive comprehension.

Now suppose that h is a bounding function for the range of f. The characteristic function $g\colon \mathbb{N} \to \mathbb{N}$ can be defined by the formula

$$g(n) = \begin{cases} 1, & \text{if } (\exists t \leq h(n))(f(t)=n), \\ 0, & \text{otherwise,} \end{cases}$$

and hence g exists by recursive comprehension. □

The relationship between characteristic and bounding functions is uniform in the sense that RCA_0 proves the sequential extension of the previous result.

Proposition 3.2 (RCA_0). For every sequence $\langle f_i \rangle_{i\in\mathbb{N}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$, we have

$$(\exists \langle g_i \rangle_{i\in\mathbb{N}})(\forall n)\,\rho(f_n,g_n) \leftrightarrow (\exists \langle h_i \rangle_{i\in\mathbb{N}})(\forall n)\,\beta(f_n,h_n).$$

Proof. We will write $\langle f_i \rangle_{i\in\mathbb{N}}$ as a function of two variables, so $f(i,n) = f_i(n)$. Adapting the proof of the preceding result, write

$$h(i,n) = (\mu\, t)(g(i,n)=0 \vee f(i,t)=n)$$

and

$$g(i,n) = \begin{cases} 1, & \text{if } (\exists t \leq h(i,n))(f(i,t)=n), \\ 0, & \text{otherwise,} \end{cases}$$

to translate the sequences of auxiliary functions in RCA_0. □

Third order arithmetic can formalize “translating functionals” of type 2 to convert between characteristic functions and bounding functions for ranges. Principles asserting the existence of the translating functionals provide additional examples of Skolemized uniformity, distinct from the sequential uniformity often considered in second order settings.

As shown below, the existence of a translating functional from bounding functions to characteristic functions can be proved in RCA_0^ω. However, the reverse translation functional requires stronger assumptions. Thus the interchangeability of the two sorts of auxiliary functions witnessed in the second order setting by the previous two propositions does not extend to Skolemized functional formulations in third order arithmetic.

Proposition 3.3 (RCA_0^ω)**.** There is a functional $T_{\beta\to\rho}$ of type $1 \to 1$ that translates bounding functions into characteristic functions for ranges. That is,

$$(\exists T_{\beta\to\rho})(\forall f)(\forall g)[\beta(f,g) \to \rho(f, T_{\beta\to\rho}(f,g))].$$

Proof. Working in RCA_0^ω, by $\mathsf{QF\text{-}AC}^{1,0}$ there is a functional Y from $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$ to $\{0,1\}$ such that $Y(f,g,n) = 1$ if and only if $(\exists t \le g(n))[f(t) = n]$. Note that the defining formula is quantifier free because the bounded quantifier can be rewritten using a primitive recursive functional. The desired functional $T_{\beta\to\rho}(f,g)$ is then

$$T_{\beta\to\rho}(f,g) = \lambda n.Y(f,g,n). \qquad \square$$

Proposition 3.4 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. There is a functional $T_{\rho\to\beta}$ of type $1 \to 1$ that translates characteristic functions for ranges into bounding functions. That is:

$$(\forall f)(\forall g)[\rho(f,g) \to \beta(f, T_{\rho\to\beta}(f,g))].$$

Proof. To prove that (1) implies (2), assume $\mathsf{RCA}_0^\omega + (\exists^2)$. The base system RCA_0^ω suffices to prove the existence of the functional χ which takes the (type 1 code for the) pair (f,n) and maps it to the function $f_n\colon \mathbb{N} \to 2$ satisfying $f_n(t) = 0 \leftrightarrow f(t) = n$. By Proposition 3.9 of Kohlenbach [9], $(\exists^2)$ implies the existence of Feferman's μ functional satisfying the formula

$$(\forall f)[(\exists x)(f(x) = 0) \to f(\mu(f)) = 0].$$

The function $T_{\beta\to\rho}(f,g) = \mu(\chi(f,n))$ is the desired bounding function.

In the proof of the preceding implication, the principle $(\exists^2)$ is sufficiently strong that we can discard the given characteristic function and calculate the bounding function directly from f and $\exists_2$. We now show we can calculate $\exists_2$ from any translation functional $T_{\rho\to\beta}$ in the reverse direction.

To prove that (2) implies (1), suppose $T_{\rho\to\beta}$ is a translation functional as described in (2). We will show that this functional is not ε-δ continuous in the sense of Definition 3.5 of Kohlenbach [9].

We can view inputs f and g as a single sequence $\langle f(0), g(0), f(1), g(1) \dots \rangle$ and use the usual Baire space topology. The functional $T_{\rho\to\beta}$ is defined for every input of two type 1 arguments, including inputs f and g for which g is not a a characteristic function for the range of f. For example, let $f_1 \colon \mathbb{N} \to \mathbb{N}$ satisfy $f_1(n) = 1$ for all n. Let $g_2 \colon \mathbb{N} \to \mathbb{N}$ satisfy $g_2(0) = g_2(1) = 1$ and $g_2(n) = 0$ otherwise. Then g_2 is not a correct characteristic function for f_1. However, $T_{\rho\to\beta}(f_1, g_2) = h$ for some totally defined type 1 function h, and $h(0) = b$ for some value b.

Suppose by way of contradiction that $T_{\rho\to\beta}$ is ε-δ continuous. Then for every pair (f, g) in some neighborhood N of (f_1, g_2) we must have $T_{\rho\to\beta}(f, g)(0) = b$. Let $f_2 \colon \mathbb{N} \to 2$ be a function that is 1 for every $t \le b$, outputs a sufficient number of ones so that (f_2, g_2) is in the neighborhood N, and is eventually constantly zero. Then g_2 is a correct characteristic function for f_2, so $\rho(f_2, g_2)$ holds. However, $T_{\rho\to\beta}(f_2, g_2)(0) = b$. Thus $T_{\rho\to\beta}(f_2, g_2)$ is not a bounding function for f_2 because $(\exists t)[f_2(t) = 0]$ but $(\forall t \le T_{\rho\to\beta}(f_2, g_2)(0))[f_2(t) \ne 0]$. Thus $\beta(f_2, T_{\rho\to\beta}(f_2, g_2))$ fails. This contradicts the implication given in item (2) of the proposition. Thus $T_{\rho\to\beta}$ must not be ε-δ continuous. By Proposition 3.7 of Kohlenbach [9], $(\exists^2)$ follows. □

Let R be the Weihrauch problem taking a type 1 function as an input and yielding output consisting of the characteristic function of the range of the input. Let B be the Weihrauch problem that outputs bounding functions as described above. Ideas from the proof of Proposition 3.1 can be adapted to show that R and B are weakly Weihrauch equivalent, and strongly Weihrauch incomparable. Summarizing, analyses based on sequential second order statements, Skolemized higher order statements, and Weihrauch reducibility yield different results. This indicates that there are three distinct notions of uniformity considered here.

4 Realizers for omniscience principles

The principle $(\exists^2)$ is closely related to a certain formulation of the limited principle of omniscience. The Weihrauch problem LPO asks for a realizer that determines whether an infinite sequence of natural numbers contains a zero. Indeed, the definition of $(\exists^2)$ could be rewritten as

$$(\exists^2)\colon (\exists R_{\mathsf{LPO}})(\forall f^1)[R_{\mathsf{LPO}}(f) = 0 \leftrightarrow (\exists n)(f(n) = 0)]$$

to emphasize $(\exists^2)$ asserts the existence of a realizer for this problem,

The Weihrauch problem LLPO, related to the lesser limited principle of omniscience, asks for a realizer to identify a parity (even or odd) on which a sequence of numbers is zero, assuming either that all even positions are zero or all odd positions are zero. We will use a principle asserting the existence of a realizer for LLPO:

$$(\mathsf{LLPO})\colon (\exists R_{\mathsf{LLPO}} \leq 1)(\forall f^1)([(\forall n)(f(2n) = 0) \vee (\forall n)(f(2n+1) = 0)] \\ \rightarrow (\forall n)[f(2n + R_{\mathsf{LLPO}}(f)) = 0]).$$

Often it is more convenient to work with an equivalent form that asks for the parity of the first location where a sequence is zero, if there is such a location:

$$(\mathsf{LLPOmin})\colon (\exists R_{\mathsf{LLPOmin}})(\forall f^1)(\forall n) \\ [f(n) = 0 \rightarrow R_{\mathsf{LLPOmin}}(f) \equiv_{\text{mod } 2} (\mu\, t \leq n)(f(t) = 0)].$$

For example, suppose $f = \langle 1, 0, 1, 0, 0 \ldots \rangle$ denotes the infinite sequence consisting of $1, 0, 1$ followed by all zeros. Then $R_{\mathsf{LPO}}(f) = 1$ because the sequence contains a 0; $R_{\mathsf{LLPO}}(f) = 1$ because $f(2n+1) = 0$ for all n; and $R_{\mathsf{LLPOmin}}(f) = 1$ because the first zero occurs in position 1, which is odd. For the sequence $g = \langle 1, 1, 0, 0, \ldots \rangle$, $R_{\mathsf{LPO}}(g) = 1$; $R_{\mathsf{LLPOmin}} = 0$ because the first zero occurs at position 2, which is even; and the value of R_{LLPO} is not determined by its defining formula.

One motivation of $\mathsf{LLPOmin}$ is that its value is determined for every sequence that includes a zero. The next proposition shows that LLPO and $\mathsf{LLPOmin}$ are equivalent for our purposes. For Weihrauch problems P and Q expressible in the language of RCA_0^ω, we say that RCA_0^ω proves $\mathsf{P} \leq_{\text{sW}} \mathsf{Q}$ if RCA_0^ω proves there are functionals $\varphi, \psi\colon \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ such that, for every realizer R_Q of Q, the functional $\psi \circ R_\mathsf{Q} \circ \varphi$ is a realizer of P.

Proposition 4.1. RCA_0^ω proves that the problems LLPO and $\mathsf{LLPOmin}$ are strongly Weihrauch equivalent, and that the principles (LLPO) and ($\mathsf{LLPOmin}$) are equivalent.

Proof. First, assume R is a realizer for $\mathsf{LLPOmin}$. Define a preprocessing function $h\colon \mathbb{N} \rightarrow \mathbb{N}$ such that $h(n) = 0$ if $n \neq 0$ and $h(0) = 1$. Define a postprocessing function $w(n) = 1 - (n \bmod 2)$. Then $S = w \circ R \circ h$ is a realizer for LLPO.

To see this, assume g is an instance of LLPO. If g is identically zero, then whichever value in $\{0, 1\}$ is produced by S is acceptable. If g is not identically zero, then $h \circ g$ is zero on exactly the inputs where g is nonzero. Thus $R(h \circ g)$ is the parity of the first location where g is nonzero, and $w \circ R(h \circ g)$ is the parity for which g is always zero. This shows $\mathsf{LLPO} \leq_{\text{sW}} \mathsf{LLPOmin}$.

Conversely, suppose S is a realizer for LLPO. Define a preprocessing function $J\colon \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ such that, for $f \in \mathbb{N}^\mathbb{N}$,

$$J(f)(n) = \begin{cases} 1, & \text{if } (\exists m < n)(f(m) = 0), \\ f(n), & \text{otherwise.} \end{cases}$$

Thus $J(f)$ and f agree through the first zero of f, but afterwards $J(f)$ takes only the value 1. Hence there is at most one input k for which $h(J(f))(k)$ is nonzero, and if there is such a k then it is the least input for which $f(k) = 0$. This means that $h(J(f))$ is in the domain of LLPO, and $S(h(J(f)))$ produces the parity of k. Thus $S \circ (h \circ J)$ is a realizer for $\mathsf{LLPOmin}$. We have shown $\mathsf{LLPOmin} \leq_{\mathrm{sW}} \mathsf{LLPO}$.

The preprocessing and postprocessing functionals in this argument can all be formed in RCA_0^ω, which can verify the correctness of the argument. None of the postprocessing functions require access to the original instance of a problem. Hence RCA_0^ω proves that LLPO or $\mathsf{LLPOmin}$ are strongly Weihrauch equivalent.

Concatenation of the preprocessing and postprocessing functionals with any LLPO realizer yields an $\mathsf{LLPOmin}$ realizer, so the principle (LLPO) implies the principle ($\mathsf{LLPOmin}$). The converse follows in a similar fashion. $\square$

Results of Weihrauch analysis include $\mathsf{LLPO} <_W \mathsf{LPO}$ and the parallelized form $\widehat{\mathsf{LLPO}} <_W \widehat{\mathsf{LPO}}$. See Weihrauch [15, §4] and Brattka and Gherardi [2, Theorem 7.13] for proofs. Consequently, the following result may intially be surprising. The underlying difference is that Weihrauch reducibility requires a single reduction that works for all realizers; the argument below breaks into cases depending on the behavior of the realizer.

Proposition 4.2 (RCA_0^ω)**.** (LLPO) implies ($\exists^2$).

Proof. Working in RCA_0^ω, by Proposition 4.1, it is sufficient to assume the existence of $R = R_{\mathsf{LLPOmin}}$ as provided by ($\mathsf{LLPOmin}$), and prove ($\exists^2$) holds.

Let $f = \langle 1, 1, 1 \ldots \rangle$ be the infinite sequence of ones. Our goal is to show that R is sequentially discontinuous at f. We will construct a sequence $\langle g_n \rangle$ such that $\lim_{n\to\infty} g_n = f$ and for each n, $R(g_n)$ disagrees with $R(f)$. In particular, if $R(f) = 1$, we want $R(g_n) = 0$ for all n, so we define g_n as a sequence of $2+2n$ ones followed by all zeros. On the other hand, if $R(g_n) = 1$, we want $R(g_n) = 1$ for all n, so we define g_n as a sequence of $1 + 2n$ ones followed by all zeros. Summarizing, for each n and m we have

$$g_n(m) = \begin{cases} 1, & \text{if } m < 1 + R(f) + 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Note that RCA_0^ω proves the existence of the sequence $\langle g_n \rangle$, that $\lim_{n\to\infty} g_n = f$, and that for all n, $R(g_n) \neq R(f)$. Thus R is sequentially discontinuous and ($\exists^2$) follows by Proposition 3.7 of Kohlenbach [9] (see Proposition 6.1).

The proof of Kohlenbach's proposition is based on the proof of Lemma 1 of Grilliot [5]. We append that argument here to give a direct derivation of ($\exists^2$) from ($\mathsf{LLPOmin}$). Let the function f and the sequence $\langle g_n \rangle$ be defined as above. RCA_0^ω suffices to prove the existence of the operator $J \colon \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ defined for $h \colon \mathbb{N} \to \mathbb{N}$ and $j \in \mathbb{N}$ by

$$J(h)(j) = \begin{cases} 1, & \text{if } (\forall x \leq j)[h(x) \neq 0], \\ g_i, & \text{if } i \leq j \wedge i = (\mu t)[h(t) = 0]. \end{cases}$$

Note that $J(h) = f$ if $(\forall x)[h(x) \neq 0]$. On the other hand, if i is the least value for which $h(i) = 0$, then $J(h) = g_i$. Consequently, for all $h \colon \mathbb{N} \to \mathbb{N}$, $R(J(h)) = R(f)$ if and only if $(\forall x)[h(x) \neq 0]$. Thus $R_{\mathsf{LPO}}(h) = 1 - |R(J(h)) - R(f)|$, so the existence of R_{LPO} follows by RCA_0^ω. □

Theorem 4.3 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. (LLPO).

Proof. To show that (1) implies (2), as noted in Proposition 1.5, Proposition 3.9 of Kohlenbach [9] shows that the principle $(\exists^2)$ proves the existence of Feferman's μ functional which satisfies:

$$(\forall f)[(\exists x)(f(x) = 0) \to [f(\mu(f)) = 0 \wedge (\forall t < \mu(f))(f(t) \neq 0)]]$$

The remainder function $\mathsf{rm}(n, 2)$ yielding the remainder of dividing n by 2 is primitive recursive. Thus RCA_0^ω proves the existence of the composition functional $\mathsf{rm}(\mu(f), 2)$, which satisfies the definition of (LLPO).

The converse was proved as Proposition 4.2 above. □

For an alternative proof of the forward implication of Theorem 4.3, we can use a formalized Weihrauch reducibility result. The next two results illustrate this process. The following proposition converts formal Weihrauch reducibility to proofs of implications of Skolemized functional existence principles.

Proposition 4.4. If P and Q are problems and (P) and (Q) are the associated Skolemized functional existence principles, then RCA_0^ω proves

$$P \leq_{\mathsf{W}} \mathsf{Q} \to ((\mathsf{Q}) \to (\mathsf{P})).$$

Proof. Working in RCA_0^ω, suppose $P \leq_{\mathsf{W}} \mathsf{Q}$ and (Q) hold. Let φ and ψ be the functionals witnessing $P \leq_{\mathsf{W}} \mathsf{Q}$ and let R_{Q} witness (Q). RCA_0^ω proves the existence of the composition $\psi(R_{\mathsf{Q}}(\varphi(x)), x)$, which can be directly shown to realize the principle (P). □

Next, we prove the formalized Weihrauch reducibility result corresponding to the forward implication of Theorem 4.3.

Proposition 4.5. RCA_0^ω proves $\mathsf{LLPO} \leq_{\mathrm{sW}} \mathsf{LPO}$.

Proof. We again work with $\mathsf{LLPOmin}$ in place of LLPO. Let $\varphi \colon \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the preprocessing functional defined by:

$$\varphi(h)(n) = \begin{cases} 1, & \text{if } (\forall t \leq n)(h(t) > 0), \\ 1, & \text{if } (\exists t \leq n)(h(t) = 0) \text{ and the least such } t \text{ is odd}, \\ 0, & \text{if } (\exists t \leq n)(h(t) = 0) \text{ and the least such } t \text{ is even}. \end{cases}$$

Note that $\varphi(h)$ is the sequence of all ones except when the first zero in the range of h occurs in an even location. Define the postprocessing functional $\psi(h,n) = n$. If R_{LPO} is a realizer for LPO, then $\psi(h, R_{\mathsf{LPO}}(\varphi(h))$ is a realizer for LLPO. Because ψ makes no use of h, this shows that $\mathsf{LLPOmin} \leq_{\mathrm{sW}} \mathsf{LPO}$. Applying Proposition 4.1, we see that $\mathsf{LLPO} \leq_{\mathrm{sW}} \mathsf{LPO}$. $\square$

As mentioned before, the forward implication of Theorem 4.3 follows immediately from Proposition 4.5 and Proposition 4.4. The next result uses Theorem 4.3 to give a short proof of one direction of Proposition 3.4 of Kohlenbach [8], showing that a uniform version of weak König's lemma is equivalent to $(\exists^2)$. This equivalence is also included in Proposition 3.9 of Kohlenbach [9]. This equivalence will be helpful in the analysis of Banach's Theorem for $\mathbb{N}$ in the next section. (For a discussion of uniform WWKL see Theorem 3.2 of Sakamoto and Yamazaki [13].)

Proposition 4.6 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. (WKL) There is a functional $\mathsf{WKL}\colon \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ such that if T is a code for an infinite tree in $2^\mathbb{N}$, then $\mathsf{WKL}(T)$ is an infinite path in T.

Proof. As noted in the proof of Proposition 3.4 of Kohlenbach [8], the proof that (1) implies (2) follows from the fact that, given the functional $\exists^2$, primitive recursion can define a functional which selects an infinite branch of an infinite binary tree. For a short proof of the converse, it suffices to show that (WKL) implies ($\mathsf{LLPOmin}$). Consider an instance $f\colon \mathbb{N} \to 2$ for $\mathsf{LLPOmin}$. Let $\langle 1\rangle_n$ denote the sequence of n ones. Define the 0-1 tree T_f by:

- Only sequences of the form $0^\frown\langle 1\rangle_n$ and $1^\frown\langle 1\rangle_n$ are in T_f,
- $0^\frown\langle 1\rangle_n \in T_f$ if and only if either $(\forall t \leq n)[f(t) \neq 0]$ or $(\mu t \leq n)[f(t) = 0]$ is even, and
- $1^\frown\langle 1\rangle_n \in T_f$ if and only if either $(\forall t \leq n)[f(t) \neq 0]$ or $(\mu t \leq n)[f(t) = 0]$ is odd.

RCA_0^ω can prove the existence of a functional φ that maps each f to T_f. For each f, the first element of $\mathsf{WKL}(\varphi(f))$ is an $\mathsf{LLPOmin}$ solution for f. $\square$

5 Banach's Theorem on $\mathbb{N}$

This section reformulates Theorems 1.3 and 1.4 as higher order functional existence statements. In particular, Theorem 5.8 shows that, in the Skolemized higher order setting, the bounded version is equivalent to the unbounded version. This collapse mimics that of the uniform principle (WKL). Our discussion begins with the formulation of the bounded principle and its proof from (WKL).

Definition 5.1. A bounded Banach functional $\mathsf{bB}_{\mathbb{N}}$ on $\mathbb{N}$ is defined as follows. For injective functions $f_0 \colon \mathbb{N} \to \mathbb{N}$ and $f_1 \colon \mathbb{N} \to \mathbb{N}$ with bounding functions b_0 and b_1, $\mathsf{bB}_{\mathbb{N}}(f_0, f_1, b_0, b_1)$ is a bijective function $h \colon \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $h(m) = n$ implies $f_0(m) = n$ or $f_1(n) = m$. As usual, the parenthesized expression $(\mathsf{bB}_{\mathbb{N}})$ denotes the principle asserting the existence of a bounded Banach functional for $\mathbb{N}$.

Proposition 5.2 (RCA_0^ω)**.** (WKL) implies $(\mathsf{bB}_{\mathbb{N}})$.

Proof. We work in RCA_0^ω. For bounded injections $\vec{f} = \langle f_0, f_1, b_0, b_1 \rangle$, we will describe the computation of a related tree $T_{\vec{f}}$ so that any infinite path through $T_{\vec{f}}$ defines an injection h satisfying Banach's theorem. If p is an infinite path through $T_{\vec{f}}$, the bijection h will be defined by:

$$h(n) = \begin{cases} f_0(n), & \text{if } p(n) = 0, \\ (\mu\, t \leq b_1(n))(f_1(t) = n), & \text{if } p(n) = 1. \end{cases}$$

A finite sequence $\sigma \in 2^{<\mathbb{N}}$ is included in $T_{\vec{f}}$ if it satisfies the following four conditions, (i)–(iv), each ensuring an aspect of the back-and-forth construction of the bijection h.

First, if there is an $m < \text{length}(\sigma)$ which is not in the range of f_1, we ensure that $h(m) = f_0(m)$.

(i) If $m < \text{length}(\sigma)$ and $(\forall t \leq b_1(m))[f_1(t) \neq m]$ then $\sigma(m) = 0$.

Next, if m is $f_1(t)$ for some t and t is not in the range of f_0, we set $h(m) = t$ in the following fashion.

(ii) If $m < \text{length}(\sigma)$, there is a $t \leq b_1(m)$ such that $f_1(t) = m$, and $(\forall s \leq b_0(t))[f_0(s) \neq t]$, then $\sigma(m) = 1$.

The next clause ensures that h is injective.

(iii) If $m, n < \text{length}(\sigma)$, $\sigma(m) = 0$, and $\sigma(n) = 1$, then $f_1(f_0(m)) \neq n$.

This final clause ensures that h is surjective.

(iv) If $m, n < \text{length}(\sigma)$, $\sigma(m) = 0$, and $\sigma(n) = 1$, then $f_1(f_0(n)) \neq m$.

The sequences satisfying the clauses are closed under initial segments, so $T_{\vec{f}}$ is a tree. The second order proof of the bounded Banach theorem in WKL_0 (Theorem 1.4) shows that $T_{\vec{f}}$ is infinite. The construction of $T_{\vec{f}}$ terminates for arbitrary choices of $\vec{f}$, even if f_0 and f_1 are not injections or if b_0 or b_1 gives incorrect bounding information. Thus RCA_0^ω proves the existence of a functional mapping $\vec{f}$ to $T_{\vec{f}}$. Whenever $T_{\vec{f}}$ is infinite, $\mathsf{WKL}(\vec{f})$ yields an infinite path. Concatenating these functionals with the one computing the bijection h as described at the beginning of the proof yields the desired Banach functional. $\square$

The preceding proposition differs from the second order analog (Theorem 1.3) in the formulation of the bounding functions. The original second order version was formulated with characteristic functions for the ranges of the injections. However, in the calculation of $T_{\vec{f}}$, the use of an incorrect characteristic function could result in an unbounded nonterminating search, causing the functional mapping $\vec{f}$ to $T_{\vec{f}}$ to be undefined on some inputs. This difficulty could be circumvented by using the fact that the uniform principle (WKL) implies $(\exists^2)$, but the argument presented here uses a single application of (WKL).

Our proof of the unbounded version of Banach's Theorem for $\mathbb{N}$ from $(\exists^2)$ uses a proposition relating $(\exists^2)$ to the existence of bounding functions.

Definition 5.3. The functional $\mathsf{b}\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ maps any function $f\colon \mathbb{N} \to \mathbb{N}$ to a bounding function $\mathsf{b}(f)$ for f. In the notation of section §5, for all f we have $\beta(f, \mathsf{b}(f))$. As usual, the parenthesized expression (b) denotes the principle asserting the existence of a bounding functional.

Lemma 5.4 (RCA_0^ω)**.** $(\exists^2)$ implies (b).

Proof. RCA_0^ω proves the existence of a functional $\mathsf{z}\colon \mathbb{N}^\mathbb{N} \times \mathbb{N} \to 2^\mathbb{N}$ where $\mathsf{z}(f, n) = g$ satisfies $g(m) = 0$ if and only if $f(m) = n$. By Proposition 1.5, $(\exists^2)$ proves the existence of Kohlenbach's μ_0. By composition and λ abstraction, there is a functional $\mathsf{b}(f)$ mapping f to the bounding function $\mu_0(\mathsf{z}(f, n))$. □

The next definition formulates an unbounded form of Banach's theorem on $\mathbb{N}$. Using the principle (b), the unbounded form can be derived from the bounded form.

Definition 5.5. A Banach functional on $\mathbb{N}$, denoted $\mathsf{B}_\mathbb{N}$, is defined as follows. For injective functions $f_0\colon \mathbb{N} \to \mathbb{N}$ and $f_1\colon \mathbb{N} \to \mathbb{N}$, $\mathsf{B}_\mathbb{N}(f_0, f_1)$ is a bijective function $h\colon \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $h(m) = n$ implies $f_0(m) = n$ or $f_1(n) = m$. As usual, the parenthesized expression $(\mathsf{B}_\mathbb{N})$ denotes the principle asserting the existence of a Banach functional for $\mathbb{N}$.

Proposition 5.6 (RCA_0^ω)**.** $(\exists^2)$ implies $(\mathsf{B}_\mathbb{N})$.

Proof. Working in RCA_0^ω, assume $(\exists^2)$. By Proposition 4.6, we have (WKL), so by Proposition 5.2, we have $(\mathsf{bB}_\mathbb{N})$. By Lemma 5.4, we have the functional b mapping functions to associated bounding functions. The composition functional $\mathsf{bB}(f_0, f_1, \mathsf{b}(f_0), \mathsf{b}(f_1))$ satisfies $(\mathsf{B}_\mathbb{N})$. □

The next proposition essentially shows that the principle $(\exists^2)$ can be deduced from the restricted form of Banach's theorem for $\mathbb{N}$.

Proposition 5.7 (RCA_0^ω)**.** $(\mathsf{bB}_\mathbb{N})$ implies (LLPO).

Proof. Assume RCA_0^ω and $(\mathsf{bB}_\mathbb{N})$. Our goal is to prove the existence of the functional $\mathsf{LLPOmin}$. Let $g\colon \mathbb{N} \to \mathbb{N}$ and define bounded injections f_0 and f_1 as follows.

Figure 1 illustrates the construction of f_0 and f_1 for three choices of g. In general, for even inputs like $n = 2m$, let $f_0(n) = n + 2$ and $f_1(n) = n$. For $n = 1$, let $f_0(1) = 0$ and $f_1(1) = 1$. The appearance of 0 in the range of g affects the definitions of f_0 and f_1 on other odd values. Suppose that $n = 2m + 3$. If $(\forall t \leq n - 2)[g(t) \neq 0]$, then let $f_0(n) = n - 2$ and $f_1(n) = n$. If $(\exists t \leq n - 2)[g(t) = 0]$, write $s = (\mu t)[g(t) = 0]$. If s is even, then let

$$f_0(n) = \begin{cases} n - 2, & \text{if } s = n - 3, \\ n, & \text{if } s < n - 3, \end{cases}$$

and let $f_1(n) = n + 2$. If s is odd, then let

$$f_0(n) = \begin{cases} n - 2, & \text{if } s = n - 2, \\ n, & \text{if } s < n - 2, \end{cases}$$

and

$$f_1(n) = \begin{cases} n, & \text{if } s = n - 2, \\ n + 2, & \text{if } s < n - 2. \end{cases}$$

In figure, f_0 is represented by solid arrows and f_1 by dashed arrows. Extending the chains to the left and right, each number has an exiting arrow, so both f_0 and f_1 are total. No number has two entering arrows, so f_0 and f_1 are injective.

Figure 1(a) corresponds to the situation when 0 does not appear in the range of g. Any bijection h satisfying Banach's theorem must either consist of all the (inverses of the) dashed arrows or all the solid arrows. In this situation, $h(1)$ may be 0 or 1.

Figure 1(b) corresponds to the case when $g(2) = 0$ is the first zero in the range of g. In this case, 5 must be in the domain of h, so $h(5) = f_0(5) = 3$. The only bijection satisfying Banach's theorem consists of solid arrows to the left of 5, so $h(1) = 0$.

Figure 1(c) is for the case when $g(3) = 0$ is the first zero in the range of g. Here 5 must be in the range of h, so $h(5) = f_1^{-1}(5) = 5$. The only bijection satisfying Banach's theorem consists of (inverses of the) dashed arrows to the left of 5, and so $h(1) = 1$. If 0 first appears in the range of g at an even value, the the figure for f_0 and f_1 will be a shifted version of the second figure. Odd values yield a shifted version of the third figure.

Because $f_0(n)$ is never less than $n-2$, $b_0(n) = n+2$ is a bounding function for f_0. Similarly, $f_1(n)$ is never less than n, so $b_1(n) = n$ is a bounding function for f_1. Routine verifications show that for any choice of g, f_0 and f_1 will be injections bounded by b_0 and b_1. Suppose that h is any bijection satisfying Banach's theorem for f_0, f_1, b_0, and b_1. If the first 0 in the range

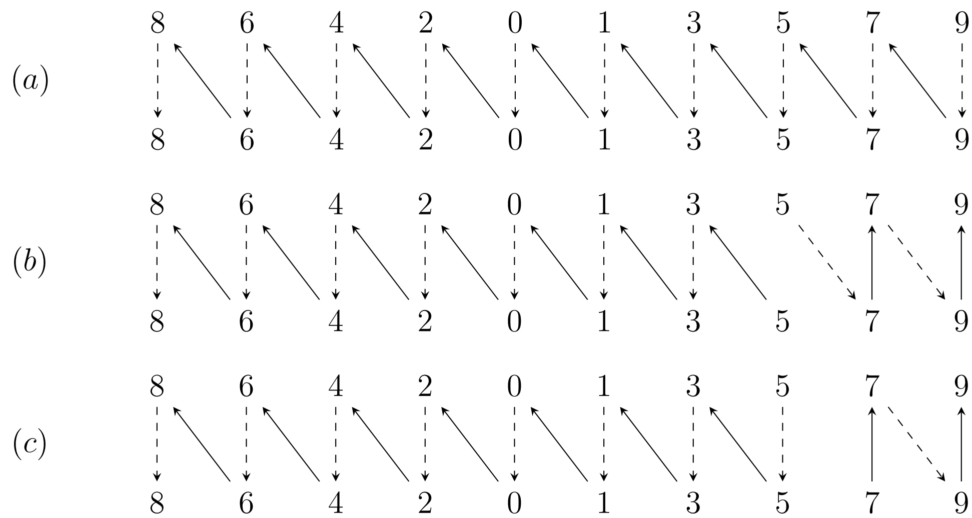

Figure 1: Construction for Proposition 5.6. (a): f_0 (solid) and f_1 (dashed) when 0 is not in the range of g. (b): f_0 (solid) and f_1 (dashed) when $g(2) = 0$. (c): f_0 (solid) and f_1 (dashed) when $g(3) = 0$.

of g occurs at an even value, then $h(1) = 0$. If it occurs at an odd value, then $h(1) = 1$. If 0 is not in the range of g, then $h(1)$ may be either 0 or 1. RCA_0^ω proves the existence of the functional mapping g to the bounded injections $\vec{f_g} = \langle f_1, f_1, b_0, b_1 \rangle$ as defined above. The functional $\mathsf{bB}_\mathbb{N}(\vec{f_g})$ yields the bijection h for $\vec{f_g}$. Consequently, the functional mapping g to $\mathsf{bB}_\mathbb{N}(\vec{f_g})(1)$ (which equals $h(1)$) is $\mathsf{LLPOmin}(g)$. □

Concatenating the preceding arguments yields the desired equivalence theorem and concludes the section.

Theorem 5.8 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. $(\mathsf{B}_\mathbb{N})$.
3. $(\mathsf{bB}_\mathbb{N})$.

Proof. Proposition 5.6 shows that (1) implies (2). Because $(\mathsf{bB}_\mathbb{N})$ is a restriction of $(\mathsf{B}_\mathbb{N})$, (2) implies (3) is immediate. Proposition 5.7 and Theorem 4.3 show that (3) implies (1). □

6 Banach's Theorem on compact spaces

Our next goal is to analyze the strength of Banach's theorem restricted to uniformly continuous functions on complete separable metric spaces. We formalize complete separable metric spaces in the manner of Simpson [14, II.5].

The space $\hat{A}$ is the collection of rapidly converging sequences of elements of an underlying (countable) set A. The metric is a function $d\colon A \times A \to \mathbb{R}$, extended to $\hat{A}$ by defining $d(\langle a_i \rangle_{i\in\mathbb{N}}, \langle a'_i \rangle_{i\in\mathbb{N}}) = \langle d(a_i, a'_i) \rangle_{i\in\mathbb{N}}$. As in Definition III.2.3 of Simpson [14], a space is *compact* if there is an infinite sequence of finite sequences of points of $\hat{A}$ of the form $\langle \langle x_{ij} : i \leq n_j \rangle : j \in \mathbb{N} \rangle$, such that for all $z \in \hat{A}$ and $j \in \mathbb{N}$, there is an $i \leq n_j$ such that $d(x_{ij}, z) < 2^{-j}$.

Uniform continuity can be witnessed by a modulus of uniform continuity as formalized in Definition IV.2.1 of Simpson [14]. The function $h\colon \mathbb{N} \to \mathbb{N}$ is a modulus of uniform continuity for f if for all k, $|x - y| < 2^{-h(k)}$ implies $|f(x) - f(y)| < 2^{-k}$. If h_f is a modulus of uniform continuity for f and h_g is a modulus of uniform continuity for g, then h defined by $h(n) = \max\{h_f(n), h_g(n)\}$ is a modulus of uniform continuity for f and g. Consequently, a joint modulus can be used to simplify some statements.

Kohlenbach [9] defines two equivalent forms of continuity for functionals of type $1 \to 1$. First, $C^{1\to 1}$ is everywhere sequentially continuous if ([9, Definition 3.3]):

$$(\forall g^1)(\forall \langle g_n \rangle)[\lim_{n\to\infty} g_n = g \to \lim_{n\to\infty} C(g_n) = C(g)].$$

Second, $C^{1\to 1}$ is everywhere ε-δ continuous if ([9, Definition 3.5]):

$$(\forall g^1)(\forall k)(\exists n)(\forall h^1)[d(g,h) < 2^{-n} \to d(C(g), C(h)) < 2^{-k}].$$

This second definition is similar to familiar textbook definitions of continuity for total functions. The use of n and k reduces the type of the quantifiers corresponding to δ and ε. Proposition 3.6 of Kohlenbach [9] proves in RCA_0^ω that C is sequentially continuous if and only if C is ε-δ continuous.

The following portion of Proposition 3.7 of Kohlenbach [9] is very useful in proving reversals.

Proposition 6.1 ([9, Proposition 3.7])**.** The following are equivalent over RCA_0^ω:

1. $(\exists^2)$.

2. There is a functional which is not everywhere sequentially continuous.

3. There is a functional which is not everywhere ε-δ continuous.

For uniformly continuous functionals on compact complete separable metric spaces, it is possible to find ranges using only $(\exists^2)$. Indeed, the next two lemmas show that for Cantor space the existence of ranges is equivalent, a higher order analog of Lemma III.1.3 of Simpson [14]

Lemma 6.2 ($\mathsf{RCA}_0^\omega + (\exists^2)$)**.** Suppose X is a compact complete separable metric space. There is functional R such that if $f\colon X \to X$ is a function with modulus of uniform continuity h, $R(f,h)$ is the characteristic function of the range of f. That is, for all $y \in X$, $R(f,h)(y) \in \{0,1\}$ and $R(F,h)(y) = 1$ if and only if $(\exists x \in X)[f(x) = y]$.

Proof. Working in RCA_0^ω, let X be as hypothesized, with the compactness of X witnessed by $\langle\langle x_{ij} : i \le n_j\rangle : j \in \mathbb{N}\rangle$. Consider a function $f\colon X \to X$ with modulus of uniform continuity h. Informally, a value $y \in X$ is in the range of f if and only if for every m there is an x with $d(F(x), y) < 2^{-m}$. By uniform continuity and compactness, such an x exists if and only if there is an $i \le n_{h(m)}$ such that $d(f(x_{ih(m)}), y) < 2^{-m}$. In RCA_0^ω, for $f\colon X \to X$ and $h\colon \mathbb{N} \to \mathbb{N}$, we may define

$$K(f, h, y, m) = \begin{cases} 1, & \text{if } (\exists i \le n_{h(m)})[d(f(x_{ih(m)}), y) < 2^{-m}] \\ 0, & \text{otherwise.} \end{cases}$$

Viewing K as a function in m with parameters f, h, and y, in $(\exists^2)$ we may define $R(f, h)(y) = \varphi(K(F, h, y, m))$. Informally, by the definition of φ, $R(f, h)(y) = 1$ if and only if for all m there is an x with $d(f(x), y) < 2^{-m}$. Note that the termination of the calculation of $R(f, h)$ does not depend on the continuity of f or the correctness of h.

To complete our proof, we must verify in $\mathsf{RCA}_0^\omega + (\exists^2)$ our informal claim that for each continuous function f with modulus of uniform continuity h, $R(f, h)$ is the characteristic function for the range of f. First, if $R(f, h)(y) = 1$, then there is a sequence $\langle x'_{i_m}\rangle$ such that for every m, $d(f(x'_{i_m}), y) < 2^{-m}$. The principle $(\exists^2)$ implies ACA_0 which implies the Bolzano–Weierstrass theorem (see Theorem III.2.7 of Simpson [14]), so we can thin $\langle x'_{i_m}\rangle$ to a sequence converging to some $x \in X$. By sequential continuity of f, we have $f(x) = y$.

Second, if $R(f, h)(y) = 0$, then for some natural number m, we must have $(\forall i \le n_{h(m)})[d(f(x_{ih(m)}, y) \ge 2^{-m}]$. Suppose by way of contradiction that $f(x) = y$. Choose $i \le n_{h(m)}$ such that $d(x, x_{ih(m)}) < 2^{-h(m)}$. Because f is uniformly continuous, $d(f(x_{ih(m)}), f(x)) < 2^{-m}$. Concatenating inequalities, we have

$$2^{-m} \le d(f(x_{ih(m)}), y) = d(f(x_{ih(m)}), f(x)) < 2^{-m},$$

a contradiction. Thus, $R(f, h)(y) = 0$ implies $(\forall x \in X)[f(x) \ne y]$, completing the proof. □

Lemma 6.3 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. If X is a compact complete separable metric space, then there is functional R such that if $f\colon X \to X$ is a function with modulus of uniform continuity h, $R(f, h)$ is the characteristic function of the range of f.
3. There is functional R such that if $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a function with modulus of uniform continuity h, $R(f, h)$ is the characteristic function of the range of f.

Proof. We will work in RCA_0^ω. By Lemma 6.2, item (1) implies item (2). Item (3) is a special case of item (2), so we need only show that item (3) implies item (1).

In RCA_0^ω, we can prove the existence of the function that maps an arbitrary function $f\colon \mathbb{N} \to \mathbb{N}$ to an element of Cantor space $f'\colon \mathbb{N} \to 2$ so that, for all n, $f'(n) = 1$ if and only if $f(n) > 0$. In terms of the function from the definition of $(\exists^2)$, $\mathsf{R}_{\mathsf{LPO}}(f) = \mathsf{R}_{\mathsf{LPO}}(f')$. RCA_0^ω can also prove the existence of the transformation $S\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for all $f\colon \mathbb{N} \to 2$, $S(f)(n) = 0$ if $(\forall m \leq n)[f(m) \neq 0]$ and $S(f)(n) = 1$ otherwise. Let $\mathcal{C}$ denote the set of functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$. RCA_0^ω proves the existence of the function $T\colon 2^{\mathbb{N}} \to \mathcal{C}$ which maps each $f \in 2^{\mathbb{N}}$ to the constant function in $\mathcal{C}$ that takes the value $S(f)$. For each f, because $T(f)$ is a constant function, the constant 0 function on $\mathbb{N}$, denoted by $z(n) \equiv 0$, is a modulus of uniform continuity for $T(f)$. (Any function could be used as a modulus.) For any $f \in \mathbb{N}^{\mathbb{N}}$, z is in the range of $T(f')$ if and only if 0 is not in the range of f', and this occurs if and only if $\mathsf{R}_{\mathsf{LPO}}(f) = 1$. Using the functional from item (3), we have $\mathsf{R}_{\mathsf{LPO}}(f) = R(T(f'), z)(z)$, so item (3) implies $(\exists^2)$. □

Our proof of Banach's theorem in compact metric spaces requires a functional that can calculate the inverse of a given function. The next two lemmas show that $(\exists^2)$ is sufficient and also necessary for this task.

Lemma 6.4 (RCA_0^ω+$(\exists^2)$)**.** Suppose X is a compact complete separable metric space. There is a function I such that if $f\colon X \to X$ is a function with modulus of uniform continuity h, then $I(f,h)$ is a function that selects elements from the pre-image of f. That is, for all $y \in X$, if there is an p such that$f(p) = y$, then $f(I(f,h)(y)) = y$. In particular, if f is injective, then the restriction of $I(f,h)$ to the range of f is the inverse of f.

Proof. Suppose that the compactness of X is witnessed by the sequence of finite sequences $\langle\langle x_{ij} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$. Thus for all $z \in X$, there is an x_{ij} in $\langle x_{ij} : i \leq n_j\rangle$ such that $d(x_{ij}, z) < 2^{-j}$. Given a function f with a modulus of uniform continuity h, for each y we will calculate a rapidly converging subsequence $p = \langle p_m : m \in \mathbb{N}\rangle$ such that if y is in the range of f then $f(p) = y$. We will argue that this calculation is sufficiently uniform that the desired function I can be found using $\mathsf{RCA}_0^\omega + (\exists^2)$.

Fix $f\colon X \to X$ with modulus of uniform continuity h, so that if $d(t_1, t_2) < 2^{-h(k)}$ then $d(f(t_1), f(t_2)) < 2^{-k}$. Increasing h if necessary, we may assume that $h(k) \geq k + 3$ for all k. Using the witness points for compactness, if y is in the range of f, then for all j we have $(\exists k \leq n_{h(j)})[d(f(x_{k,h(j)}, y) < 2^{-j}]$.

Given f, h, and y as above, we can define the desired $p = \langle p_m : m \in \mathbb{N}\rangle$. If y is not in the range of f, let $p = y$. If y is in the range of f construct $\langle p_m : m \in \mathbb{N}\rangle$ as follows. Let $p_m = x_{ih(m)}$ where $i \leq n_{h(m)}$ is the least integer such that:

1. $d(f(x_{ih(m)}), y) < 2^{-m}$,
2. $(\forall j > m)(\exists k \leq n_{h(j)})[d(f(x_{kh(j)}, y) < 2^{-j} \wedge d(x_{kh(j)}, x_{ih(m)}) < 2^{-m-2}]$, and

3. if $m > 0$, then $d(p_{m-1}, x_{ih(m)}) \le 2^{-m}$.

The third clause ensures that $p = \langle p_m : m \in \mathbb{N}\rangle$ is a rapidly converging Cauchy sequence. By Proposition 3.6 of Kohlenbach [9], $(\exists^2)$ proves that f is sequentially continuous, so the first clause shows that $f(p) = y$. Informally, the second clause guarantees that each p_m is sufficiently close to a pre-image of y that the construction can continue. We verify this next.

To initialize the construction, we must find p_0. Suppose $f(t_0) = y$. Because $h(2) \ge 0 + 3$ we can fix an $i \le n_{h(0)}$ with $d(x_{ih(0)}, t_0) < 2^{-3}$. Because h is a modulus of uniform continuity, $d(f(x_{ih(0)}), y) = d(f(x_{ih(0)}), f(t_0)) < 2^{-0}$, so clause (1) is satisfied. For any $j > 0$, there is a $k \le n_{h(j)}$ such that $d(x_{kh(j)}, t_0) < 2^{-h(j)} < 2^{-3}$, and so $d(f(x_{kh(j)}, y) = d(f(x_{kh(j)}), f(t_0)) < 2^{-j}$. For such a j and k,

$$d(x_{kh(j)}, x_{ih(0)}) \le d(x_{kh(j)}, t_0) + d(x_{ih(0)}, t_0) < 2^{-3} + 2^{-3} = 2^{-2},$$

so clause (2) is also satisfied. The third clause is vacuously true. We have shown that for some i, $x_{ih(0)}$ satisfies all three clauses. Let i_0 be the least such i, and set $p_0 = x_{i_0 h(0)}$.

Suppose p_{m-1} has been chosen satisfying all three clauses. By clause (2) for p_{m-1}, we can find a sequence of points $\langle t_{m_j} : j \in \mathbb{N}\rangle$ such that for every j, $d(f(t_{m_j}), y) < 2^{-j}$ and $d(t_{m_j}, p_{m-1}) < 2^{-(m-1)-2} = 2^{-m-1}$. In Theorem III.2.7, Simpson [14] proves the generalization of the Bolzano–Weierstrass theorem for compact metric spaces in ACA_0, so it is also a theorem of $\mathsf{RCA}_0^\omega + (\exists^2)$. Consequently, there is a subsequence of $\langle t_{m_j} : j \in \mathbb{N}\rangle$ converging to a value t_m with $f(t_m) = y$ and $d(t_m, x_{m-1}) \le 2^{-m-1}$. Choose $i \le n_{h(m)}$ so that $d(x_{ih(m)}, t_m) < 2^{-h(m)} < 2^{-m-3}$. Clause (1) holds for $x_{ih(m)}$ because h is a modulus of uniform continuity and $f(t_m) = y$. For any $j > m$, there is a $k \le n_{h(j)}$ such that $d(x_{kh(j)}, t_m) < 2^{-h(j)} \le 2^{-j-3}$ and so $d(f(x_{kh(j)}), y) < 2^{-j}$. For such a j and k,

$$\begin{aligned} d(x_{kh(j)}, x_{ih(m)}) &\le d(x_{kh(j)}, t_0) + d(x_{ih(m)}, t_0) \\ &< 2^{-j-3} + 2^{-m-3} < 2^{-m-2}, \end{aligned}$$

so clause (2) holds for $x_{ih(m)}$. Finally,

$$d(p_{m-1}, x_{ih(m)}) < d(p_{m-1}, t_m) + d(x_{ih(m)}, t_m) < 2^{-m-1} + 2^{-m-3} < 2^{-m}.$$

We have shown that all three clauses hold for some choice of i, so let i_m be the least such i and set $p_m = x_{i_m h(m)}$. This concludes the argument that our construction never halts, yielding the desired pre-image $p = \langle p_m : m \in \mathbb{N}\rangle$.

It remains to show that $\mathsf{RCA}_0^\omega + (\exists^2)$ suffices to prove the existence of the function I from the statement of the lemma. Suppose we are given f with modulus h and a value y from the metric space. By Lemma 6.2, $R(f, h)$ is the characteristic function for the range of f. If y is not in the range of f, output y. Otherwise, begin constructing p, searching for an $x_{ih(m)}$ satisfying clauses (1), (2), and (3) above. By ($\exists^2$), we may use a realizer for LPO to check if clause (2) holds. As argued above, when y is in the range, this process calculates the desired pre-image of y. Summarizing, $\mathsf{RCA}_0^\omega + (\exists^2)$ proves the existence of the function mapping f, h, and y to the desired value. Applying λ-abstraction yields $I(f, h)$. $\square$

The use of ($\exists^2$) in the previous lemma is necessary, as shown by the following reversal.

Lemma 6.5 (RCA_0^ω)**.** The following are equivalent:

1. ($\exists^2$).

2. If X is a compact complete separable metric space, then there is a function I such that if $f\colon X \to X$ is a function with modulus of uniform continuity h, then $I(f, h)$ is a function that selects elements from the pre-image of f.

3. There is a function I such that if $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a function with modulus of uniform continuity h, then $I(h, f)$ is a function that selects elements from the pre-image of f.

Proof. We will work in RCA_0^ω. By Lemma 6.4, item (1) implies item (2). Item (3) is a special case of item (2), so we need only show that item (3) implies item (1). By Proposition 4.2, it suffices to show that (3) implies (LLPO).

Given an input $w\colon \mathbb{N} \to 2$ for $\mathsf{LLPOmin}$, we will show how to construct a function f with modulus of uniform continuity h such that information about the pre-image of f as provided by $I(f, h)$ in item (3) can be used to calculate $\mathsf{LLPOmin}$ for w. In particular, we will control the pre-image of the constant 0 function, denoted $\vec{0} \in 2^{\mathbb{N}}$. If the first t where $w(t) = 0$ is even, we require $f^{-1}(\vec{0}) = \{\vec{0}\}$. If the first t such that $w(t) = 0$ is odd, we require $f^{-1}(\vec{0}) = \{\vec{1}\}$. If 0 is not in the range of w, $f^{-1}(\vec{0})$ will be the set $\{\vec{0}, \vec{1}\}$.

Now we can specify the behavior of f. Let $x\colon \mathbb{N} \to 2$ by and element of $2^{\mathbb{N}}$. Evaluating f at x yields a function $f(x)$, which also maps $\mathbb{N}$ into 2 and is defined as follows.

1. $f(x)(0) = 0$.

2. For $n > 0$, $f(x)(n)$ is defined by two cases:

   (a) if $n - 1$ is not the least t such that $w(t) = 0$ then

   $$f(x)(n) = \begin{cases} x(n), & \text{if } x(0) = 0, \\ 1 - x(n), & \text{if } x(0) = 1. \end{cases}$$

   (b) if $n - 1$ is the least t such that $w(t) = 0$ then

   $$f(x)(n) = \begin{cases} x(0), & \text{if } n - 1 \text{ is even}, \\ 1 - x(0), & \text{if } n - 1 \text{ is odd}. \end{cases}$$

Routine arguments verify that the pre-image of f satisfies the requirements listed above. Also, if the sequences x and y agree in the first n values, the sequences $f(x)$ and $f(y)$ also agree in the first n values. Thus, the function $h(n) = n$ is a modulus of uniform continuity for f. The construction of f from w is sufficiently uniform that RCA_0^ω proves the existence of a function g mapping each $w \in 2^{\mathbb{N}}$ to its associated function f.

Let I be the function described in item (3) of the statement of the lemma, and let $h(n) = n$ be the identity function on $\mathbb{N}$. The for any $w \in 2^{\mathbb{N}}$, $I(g(w), h)(\vec{0})$ is (an element of $2^{\mathbb{N}}$) equal to $\vec{0}$ if the first 0 in the range of w occurs at an even value, and $\vec{1}$ if the first 0 occurs at an odd value. The sequences coding elements of $2^{\mathbb{N}}$ output by I are rapidly converging sequences of finite approximations to $\vec{0}$ or $\vec{1}$. By the definition of the metric on $2^{\mathbb{N}}$, the first entry in the third finite approximation for any sequence equal to $\vec{0}$ will be 0, and similarly the value 1 can be extracted from any sequence equal to $\vec{1}$. Thus $I(g(w), h)(0)$ uniformly calculates $\mathsf{LLPOmin}$ for w. $\square$

The previous results allow us to formulate and analyze some restrictions of Banach's theorem. For compact complete separable metric spaces, a functional form of Banach's theorem restricted to uniformly continuous functions is equivalent to the functional existence principle $(\exists^2)$. Note that if f and g have moduli of uniform continuity h_f and h_g, then h defined by $h(n) = \max\{h_f(n), h_g(n)\}$ is a modulus of uniform continuity for both f and g. As a notational convenience, we will use common moduli of uniformity for pairs of functions.

Definition 6.6. For a complete separable metric space X, a Banach functional B_X is defined as follows. For injective functions $f\colon X \to X$ and $g\colon X \to X$ with a common modulus of uniformity h, $B_X(f, g, m)$ is a bijective function $H\colon X \to X$ such that for all $x \in X$, $H(x) = f(x)$ or $g(H(x)) = x$. The parenthesized expression (B_X) denotes the principle asserting the existence of a Banach functional for X.

Following our previous pattern, the next result proves a version of Banach's theorem for compact metric spaces using $(\exists^2)$. The reversals and a summary appear in a second result.

Lemma 6.7 ($\mathsf{RCA}_0^\omega + (\exists^2)$)**.** If X is a compact metric space, then (B_X).

Proof. Assume RCA_0^ω and $(\exists^2)$. Suppose X is a complete separable metric space and that $\langle\langle x_{ij} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ witnesses that X is compact. Let f and g be injections of X into X with a common modulus of uniform continuity h. Apply Lemma 6.2 to find the range functionals $R(f, h)$ and $R(g, h)$. Apply Lemma 6.4 to to find pre-image selectors $I(f, h)$ and $I(g, h)$. Because f and g are injections, the restrictions of these functions to the ranges of f and g are inverse functions. Consequently, we will use the shorthand notation f^{-1} and g^{-1}. Note that the pre-image selectors f^{-1} and g^{-1} are defined for all inputs from X, and that if (for example) y is in the range of f, then $f(f^{-1}(y)) = y$.

Our goal is to construct the bijection H in the statement of (B_X). This is achieved by a back-and-forth construction, alternately iterating applications of g^{-1} and f^{-1}, and basing the value of H on the terminating condition of this process.

First we construct a functional that alternately applies g^{-1} and f^{-1}. Using primitive recursion, define $S(x,n)$ by $S(x,0)=x$ and for $n \geq 0$, $S(x,2n+1) = g^{-1}(S(x,2n))$ and $S(x,2n+2) = f^{-1}(S(x,2n+1))$. Calculating a few values yields $S(x,0) = x$, $S(x,1) = g^{-1}(x)$, $S(x,2) = f^{-1}(g^{-1}(x))$, and $S(x,3) = g^{-1}(f^{-1}(g^{-1}(x)))$. As noted above, f^{-1} and g^{-1} are total, so $S(x,n)$ is defined for all x and all n.

A traditional back-and-forth construction using partial inverse functions might halt if, for example, x was not in the range of g, or if $g^{-1}(x)$ was not in the range of f, and so on. Define the function $P\colon X \times \mathbb{N} \to \{0,1\}$ by $P(x,0)=1$ and for $n \geq 0$, $P(x,2n+1) = R(g,h)(S(x,2n))$ and $P(x,2n+2) = R(f,h)(S(x,2n+1))$. Consider the initial stages of the back-and-forth process displayed in the following table.

n=stage	0	1	2	3	$\ldots$
$S(x,n)$	x	$g^{-1}(x)$	$f^{-1}(g^{-1}(x))$	$g^{-1}(f^{-1}(g^{-1}(x)))$	$\ldots$

If x is not in the range of g then $P(x,1)=0$. If $g^{-1}(x)$ is not in the range of f, then $P(x,2)=0$. In general, the least n with $P(x,n)=0$ will be the stage where the traditional back-and-forth process based on partial inverse functions will halt. If the back-and-forth process does not halt, then $P(x,n)=1$ for all n. Writing $P_x(n)$ for $P(x,n)$, we may view $P_x(n)$ as a function from $\mathbb{N}$ into $\{0,1\}$. Apply φ as provided by $(\exists^2)$, and we have $\varphi(P_x(n)) = 0$ if the back-and-forth process halts and $\varphi(P_x(n)) = 1$ if the process never terminates.

By $(\exists^2)$ and Theorem 4.3, we may also use R_{LLPOmin}, a realizer for $\mathsf{LLPOmin}$. Define the functional $T(x)$ by

$$T(x) = \begin{cases} 1, & \text{if } \varphi(P_x(n)) = 1, \\ R_{\mathsf{LLPOmin}}(P_x), & \text{if } \varphi(P_x(n)) = 0. \end{cases}$$

Finally, define the bijection $H(x)$ by

$$H(x) = \begin{cases} f(x), & \text{if } T(x) = 1, \text{ and} \\ g^{-1}(x), & \text{if } T(x) = 0. \end{cases}$$

One can argue that H is the desired bijection by the usual arguments. Briefly, consider the following diagram representing images and pre-images of an element x from X.

$$\begin{tikzcd}
{} \arrow[dr, dashed] & g^{-1}(x) \arrow[dr] & & f(x) \arrow[dr] & & f(g(f(x))) \arrow[dr, dashed] & \\
 & f^{-1}(g^{-1}(x)) \arrow[u] & x \arrow[ur] & & g(f(x)) \arrow[ur] & & {}
\end{tikzcd}$$

Each element of the lower copy of X appears in at least one bipartite subgraph of the sort pictured. Also, for each y in the upper copy of X, we know $y = g^{-1}(g(y))$, so each element in the upper copy of X appears in at least one bipartite subgraph. Because f and g are injective, each element appears in exactly one bipartite subgraph. The choice of the values of $H(x)$ ensure that if the bipartite graph terminates on the left, the left most vertex is either in the lower copy of X and in the domain of H, or in the upper copy of X and in the range of H. Thus H is a bijection of X into itself, satisfying the requirements of (B_X). $\square$

This section concludes with proofs of two reversals for instances of the previous lemma, summarizing the results for Banach's theorem on compact metric spaces in the following theorem.

Theorem 6.8 (RCA_0^ω)**.** The following are equivalent:

1. $(\exists^2)$.
2. If X is a compact metric space, then (B_X).
3. $(\mathsf{B}_{[0,1]})$.
4. $(\mathsf{B}_{2^\mathbb{N}})$.

Proof. The previous lemma proves that item (1) implies item (2). Item (3) and item (4) are special cases of item (2), so we can complete the proof by reversing (3) and (4) to (1). For the first reversal, suppose $B_{[0,1]}(f, g, h)$ is the Banach functional for $[0, 1]$. Consider the injections f and g defined by $f(x) = g(x) = x/2$. Each x in $[0, 1]$ is represented by a rapidly converging sequence of rationals, and dividing each element of the sequence by 2 yields a rapidly converging sequence representing $x/2$. Thus RCA_0^ω proves that f and g are defined and total. The identity function $h(k) = k$ is a modulus of uniform continuity for f and g. Suppose $H = B_{[0,1]}(f, g, h)$ is the bijection satisfying Banach's theorem for f and g. Consider $x = \frac{1}{2}$ and the sequence $x_n = \frac{1}{2} + \frac{1}{2^n}$. For each n, x_n is not in the range of g, so $H(x_n) = f(x_n) = \frac{x_n}{2} = \frac{1}{4} + \frac{1}{2^{n+1}}$. Thus, $\lim_{n\to\infty} H(x_n) = \frac{1}{4}$. The functional H is bijective, so 1 is in the range of H. Fix x with $H(x) = 1$. By the Banach theorem, $H(x) = f(x)$ or $H(x) = g^{-1}(x)$. Because 1 is not in the range of F, $H(x) = g^{-1}(x)$. Thus $1 = g^{-1}(x)$, so $x = \frac{1}{2}$ and $H(\frac{1}{2}) = 1$. Summarizing,

$$H(\lim_{n\to\infty} x_n) = H(\frac{1}{2}) = 1 \neq \frac{1}{4} = \lim_{n\to\infty} H(x_n).$$

Thus H is not sequentially continuous at $x = \frac{1}{2}$, and $(\exists^2)$ follows by Proposition 6.1.

For the final reversal, suppose $B_{2^\mathbb{N}}(f, g, h)$ is the Banach functional for Cantor space. Consider the padding function $P(x)$ that adds a zero after each entry in a binary input string. Formally, $P(x)(n) = x(m)$ if $n = 2m$, and $P(x)(n) = 0$ otherwise. For example,

$$P(\langle 1, 0, 1, 1 \dots \rangle) = \langle 1, 0, 0, 0, 1, 0, 1, 0 \dots \rangle.$$

RCA_0^ω proves that $P(x)$ is defined and total, and that the identity function $h(k) = k$ is a modulus of uniform continuity for $P(x)$. Let $f(x)$ and $g(x)$ both be $P(x)$. Let $H = B_{2^\mathbb{N}}(f, g, h)$ be the bijection satisfying Banach's theorem for f and g. For each n, let σ_n consist of n copies of the string 10, followed by 11, followed by zeros. The double 1 ensures that σ_n is not in the range of $g(x) = P(x)$. Thus, for each n, $H(\sigma_n) = f(\sigma_n) = P(\sigma_n)$, which consists of n copies of the string 1000 followed by 1010, followed by zeros. Thus $\lim_{n\to\infty} H(\sigma_n)$ is the string 1000 repeated infinitely. On the other hand, $\lim_{n\to\infty} \sigma_n$ is $\langle 1, 0, 1, 0 \dots \rangle$. The string $\langle 1, 1, 1 \dots \rangle$ is not in the range of $f(x)$, so $H(g(\langle 1, 1, 1 \dots \rangle)) = \langle 1, 1, 1 \dots \rangle$. Because $g(\langle 1, 1, 1 \dots \rangle) = \langle 1, 0, 1, 0 \dots \rangle$, we have $H(\lim_{n\to\infty} \sigma_n) = H(\langle 1, 0, 1, 0 \dots \rangle) = \langle 1, 1, 1 \dots \rangle$. Thus $H(\lim_{n\to\infty} \sigma_n) \neq \lim_{n\to\infty} H(\sigma_n)$, so H is not sequentially continuous at $x = \langle 1, 0, 1, 0 \dots \rangle$. The principle $(\exists^2)$ follows by Proposition 6.1, completing the reversal and the proof of the theorem. □

We note that the functional R in Lemma 6.2, the functional I in Lemma 6.5, and the functional B in Theorem 6.8 are constructed uniformly in a code for the space X. Hence these functionals could, in principle, be defined with X as a parameter. This is another layer of uniformity in the constructions, although noting the parameter explicitly complicates the notation.

7 Moduli of uniform continuity

This section introduces a function that computes moduli of uniform continuity. As shown below, the strength of the existence of the function lies below $(\exists^2)$, allowing us to streamline the definition of Banach functionals and Theorem 6.8.

Definition 7.1. Suppose X is a compact complete separable metric space and Y is a complete separable metric space. The principle (M) asserts the existence of a function M such that if $f\colon X \to Y$ is continuous, then $M(f)$ is a modulus of uniform continuity for f.

Near the end of his article, Kohlenbach [9] presents a functional form of the fan theorem, denoted by (MUC). He notes that (M) is a consequence of (MUC), MUC is conservative over WKL_0 for second order sentences, and (MUC) is inconsistent with $(\exists^2)$. Because (MUC) proves (M), (M) is also conservative over WKL_0 for second order sentences. The next lemma shows that unlike (MUC), the principle (M) is a consequence of $(\exists^2)$.

Lemma 7.2 (RCA_0^ω)**.** $(\exists^2)$ implies (M).

Proof. Let X be a compact complete separable metric space with compactness witnessed by the sequence of sequences $\langle\langle x_{ij} : i \le n_j\rangle : j \in \mathbb{N}\rangle$. Let Y be a comple separable metric space. We will use d to denote the metric in both spaces. For $f\colon X \to Y$ we can define a prospective value of a modulus of uniform continuity for f at m by setting $(M(f))(m)$ equal to the least n such that:

$$(\forall x_{ij})(\forall x_{i'j'})[d(x_{ij}, x_{i'j'}) < 2^{-n} \to d(f(x_{ij}), f(x_{i'j'})) < 2^{-m-1}] \qquad (1)$$

Informally, $M(f)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ that resembles a modulus of uniform continuity on the compactness witnesses for X. First we will show that $\mathsf{RCA}_0^\omega + (\exists^2)$ suffices to prove the existence of the function M. Then we will verify that if f is continuous, then $M(f)$ is a modulus of uniform continuity for f.

Working in $\mathsf{RCA}_0^\omega + (\exists^2)$, let X and Y be as above, and suppose $f\colon X \to Y$. Recalling the reverse mathematical formalization of inequalities in the reals, the formulas $d(x_{ij}, x_{i'j'}) < 2^{-n}$ and $d(f(x_{ij}), f(x_{i'j'})) > 2^{-m-1}$ are Σ^0_1. Thus RCA_0^ω proves the existence of a function $a(f, m, n, t)$ which is 0 if t codes a witness that there are x_{ij} and $x_{i'j'}$ such that $d(x_{ij}, x_{i'j'}) < 2^{-n}$ and $d(f(x_{ij}), f(x_{i'j'})) > 2^{-m-1}$, and is 1 otherwise. Note that formula (1) holds if $a(f, m, n, t)$ is 1 for all t, and fails if there is a t such that $a(f, m, n, t)$ is 0. As noted in section 4, $(\exists^2)$ implies the existence of the function R_{LPO}. The λ notation $\lambda t.a(f, m, n, t)$ denotes the function that maps each $t \in \mathbb{N}$ to the value $a(f, m, n, t)$. Applying λ abstraction (which is a consequence of RCA_0^ω [9]) and $(\exists^2)$, we can prove the existence of the function $b(f, m, n) = R_{\mathsf{LPO}}(\lambda t.a(f, m, n, t))$. Note that for all f, m, and n, $b(f, m, n) = 1$ if formula (1) holds and $b(f, m, n) = 0$ otherwise. By Proposition 1.5, $(\exists^2)$ proves the existence of Feferman's μ, so by $(\exists^2)$ and an additional application of λ abstraction, we can prove the existence of the function $c(f, m) = \mu(1 - \lambda n.b(f, m, n))$. Note that for each f and m, if there is an n such that formula (1) holds, then $c(f, m)$ is the least such n. If there is no such n, for example if f is discontinuous, then $c(f, m)$ still yields some value, but no useful information. By λ abstraction, $\mathsf{RCA}_0^\omega + (\exists^2)$ proves the existence of $M(f) = \lambda m.c(f, m)$. For every $f\colon X \to Y$, $M(f)$ yields a function from $\mathbb{N}$ to $\mathbb{N}$.

It remains to show that if f is continuous then $M(f)$ is a modulus of uniform continuity for f. Fix a continuous $f\colon X \to Y$ and $m \in \mathbb{N}$. Let $n = M(f)(m)$. Suppose that $u, v \in X$ satisfy $d(u, v) < 2^{-n}$. Choose $\delta < 2^{-n} - d(u, v)$. Because f is continuous and $\langle\langle x_{ij} : i \le n_j\rangle : j \in \mathbb{N}\rangle$ is dense in X, we can find an x_{ij} such that $d(x_{ij}, u) < \delta/2$ and $d(f(x_{ij}), f(u)) < 2^{-m-2}$. Similarly, find $x_{i'j'}$ such that $d(x_{i'j'}, v) < \delta/2$ and $d(f(x_{i'j'}), f(v)) < 2^{-m-2}$. By the triangle inequality,

$$d(x_{ij}, x_{i'j'}) \le d(x_{ij}, u) + d(u, v) + d(v, x_{i'j'}) < \delta/2 + d(u, v) + \delta/2 < 2^{-n}.$$

Because $d(x_{ij}, x_{i'j'}) < 2^{-n}$, and because $(M(f))(m) = n$, formula (1) holds, so

$d(f(x_{ij}), f(x_{i'j'})) < 2^{-m-1}$. By the triangle inequality,

$$\begin{aligned} d(f(u), f(v)) &< d(f(u), f(x_{ij})) + d(f(x_{ij}, f(x_{i'j'})) + d(f(x_{i'j'}), f(v)) \\ &< 2^{-m-2} + 2^{-m-1} + 2^{-m-2} = 2^{-m}. \end{aligned}$$

Summarizing, when f is continuous and $M(f)(m) = n$, if $d(u, v) < 2^{-n}$ then $d(f(u), f(v) < 2^{-m}$. Thus $M(f)$ is a modulus of uniform continuity for f. □

The principle (M) allows us to reformulate Theorem 6.8, stripping all reference to moduli of uniform continuity.

Theorem 7.3 (RCA_0^ω)**.** The principle $(\exists^2)$ is equivalent to the statement that for every compact complete separable metric space X, there is a function B'_X that maps each pair of injections from X to X to a bijection satisfying Banach's theorem.

Proof. Assuming $(\exists^2)$, by Lemma 7.2 we may use the function M to calculate moduli of uniform continuity for f and g. The pointwise maximum function $\max(M(f), M(g))$ is a joint modulus of uniform continuity for f and g. If $B_X(f, g, m)$ is the function provided by Theorem 6.8 part (2), then the function defined by $B'_X(f, g) = B_x(f, g, \max(M(f), M(g)))$ is the desired Banach function. The converse is immediate from Theorem 6.8. □

Because (M) is a consequence of (MUC), the principle (M) does not imply $(\exists^2)$. That is, the converse of Lemma 7.2 is not true. The next two results show that like (MUC), the second order theorems of (M) are exactly those of WKL_0. As part of that proof, the next lemma allows us to change representations of functions, with the eventual goal of applying a traditional reverse mathematics result to show that (M) implies WKL_0.

Lemma 7.4 (RCA_0^ω)**.** Suppose X and Y are complete separable metric spaces. Suppose that Φ is a code for a totally defined continuous function as described in Definition II.6.1 of Simpson [14]. Then there is a function $f\colon X \to Y$ such that for all n, a, r, b, and s, if $(n, a, r, b, s) \in \Phi$ then $d(f(a), b) \leq s$.

Proof. Working in RCA_0^ω, suppose X, Y, and Φ are as above. Fix $x \in X$. Because x is in the domain of the function defined by Φ, for each m we can find $(n, a, r, b, s) \in \Phi$ (occurring first in some fixed enumeration of quintuples) such that $d(x, a) < r$ and $s < 2^{-m-1}$. Set $f(x)(m) = b$. The sequence $\langle f(x)(m) : m \in \mathbb{N}\rangle$ is a rapidly converging sequence of elements of Y converging to the desired value of $f(x)$. RCA_0^ω proves the existence of f.

We now verify the last sentence of the lemma. Suppose $(n, a, r, b, s) \in \Phi$. Let $\varepsilon > 0$ and choose m so that $2^{-m-1} < \min\{\varepsilon/2, s\}$. Let $(n', a', r', b', s') \in \Phi$ be the quintuple witnessing the the value for $f(a)(m)$. Then $d(a, a') < r'$ and $s' < 2^{-m-1} < \varepsilon/2$. Let $r_0 = \min\{r, r' - d(a, a')\}$. Then the ball $B(a, r_0)$ is a subset of $B(a, r)$, and is also a subset of $B(a', r')$. Applying property (2) of Simpson's Definition II.6.1, we have $(a, r_0)\Phi(b, s)$ and $(a, r_0)\Phi(b', s')$. By property (1) of Simpson's definition, $d(b,b') \le s + s' < s + \varepsilon/2$. By the choice of m, $d(b', f(a)) \le 2^{-m} < \varepsilon/2$. By the triangle inequality $d(f(a), b) < s + \varepsilon$. Because ε was an arbitrary positive value, $d(f(a), b) \le s$. $\square$

The preceding lemma allows us to completely characterize the second order theory of (M).

Proposition 7.5. The second order theorems of $\mathsf{RCA}_0^\omega + (\mathsf{M})$ are exactly the same as those of WKL_0.

Proof. As noted before, (M) is a consequence of Kohlenbach's (MUC), and so any second order theorem provable using (M) is provable in WKL_0. It remains to show that (M) implies WKL_0. By Theorem IV.2.3 of Simpson [14], it suffices to show that if f is a continuous function (coded by Φ) on $[0,1]$, then f is uniformly continuous. Suppose Φ codes a continuous function on $[0,1]$. By Lemma 7.4, RCA_0^ω proves that there is a function $f\colon [0,1] \to \mathbb{R}$ matching the values of the coded function. Applying (M), the function $M(f)$ is a modulus of uniform continuity for f, and so also for the function coded by Φ. Thus Φ codes a uniformly continuous function on $[0,1]$. $\square$

We conclude by pointing out the potential and limitations of this section. The principle (M) can be viewed as a higher order analogue of WKL_0 in much the same fashion that $(\exists^2)$ is a higher order analogue of ACA_0. A number of Skolemized forms of statements equivalent to WKL_0 may be equivalent to (M) over RCA_0^ω. (But not all, as witnessed by Kohlenbach's UWKL. See Proposition 4.6.) However, (M) may not be the only reasonable candidate for a WKL_0 analogue. For example, reformulating (M) as a function on second order continuous function codes yields an alternative principle $(\mathsf{M_c})$. It seems likely that Proposition 7.5 holds for $(\mathsf{M_c})$, but it is possible that neither (M) nor $(\mathsf{M_c})$ proves the other over RCA_0^ω.

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