

# **Reverse Mathematics and Brouwer's Fixed Point Theorem**

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# Introduction

In 1949, B.H. Arnold (Oregon State U.) published a proof of the fundamental theorem of algebra using Brouwer's fixed point theorem.

## **Fundamental theorem of algebra:**

Every nonconstant polynomial with complex coefficients has a zero.

## **Brouwer's fixed point theorem:**

If  $f : I^2 \rightarrow I^2$  is continuous, then for some  $z \in I^2$ ,  $f(z) = z$ . In general, any continuous map of a compact, convex space to itself has a fixed point.

# Reverse Mathematics

**Theorem:** (Simpson [4])  $\mathbf{RCA}_0$  can prove the fundamental theorem of algebra.

**Theorem:** (Shioji and Tanaka [3])  
 $\mathbf{RCA}_0$  proves that these are equivalent:

1. Brouwer's fixed point theorem for  $I^2$ .
2.  $\mathbf{WKL}_0$ .

Conclusions:

1. Brouwer's theorem is a very big hammer to use on the FTA.
2. There should be a restricted (computable) version of Brouwer's thm that could be used in Arnold's proof.

## Terminology for a computable fixed point theorem

A *computably coded continuous function*:  
is encoded by a computable set of 5-tuples,  
each of which defines a  $\delta$  neighborhood and  
an  $\epsilon$  neighborhood that contains  $f(\delta)$ .

A *modulus of uniform continuity for  $f$* :  
is a function  $h$  such that for every  $n$ ,  
$$|x - y| < 2^{-h(n)} \rightarrow |f(x) - f(y)| < 2^{-n}.$$

We write  $f : I^2 \rightarrow I^2$  if a ccc function  $f$  is  
defined at every computable point in  $I^2$ .

Given  $f : I^2 \rightarrow I^2$  we define its *extension*  
 $f^*$  by setting  $f^*(a) = \lim_{x \rightarrow a} f(x)$  at each  
 $a$  where the limit exists, and saying  $f^*$  is  
undefined at other points.

## A computable restriction of Brouwer's fixed point theorem:

Suppose that

- (1)  $f : I^2 \rightarrow I^2$  is a computably coded continuous function,
- (2)  $f$  has a computable modulus of uniform continuity, and
- (3)  $f^*$  has finitely many fixed points.

Then  $f$  has a computable fixed point.

The proof is based on:

**Lemma:** Suppose that

- (1)  $f : I^2 \rightarrow I^2$  is a computably coded continuous function, and
- (2)  $f$  has a computable modulus of uniform continuity.

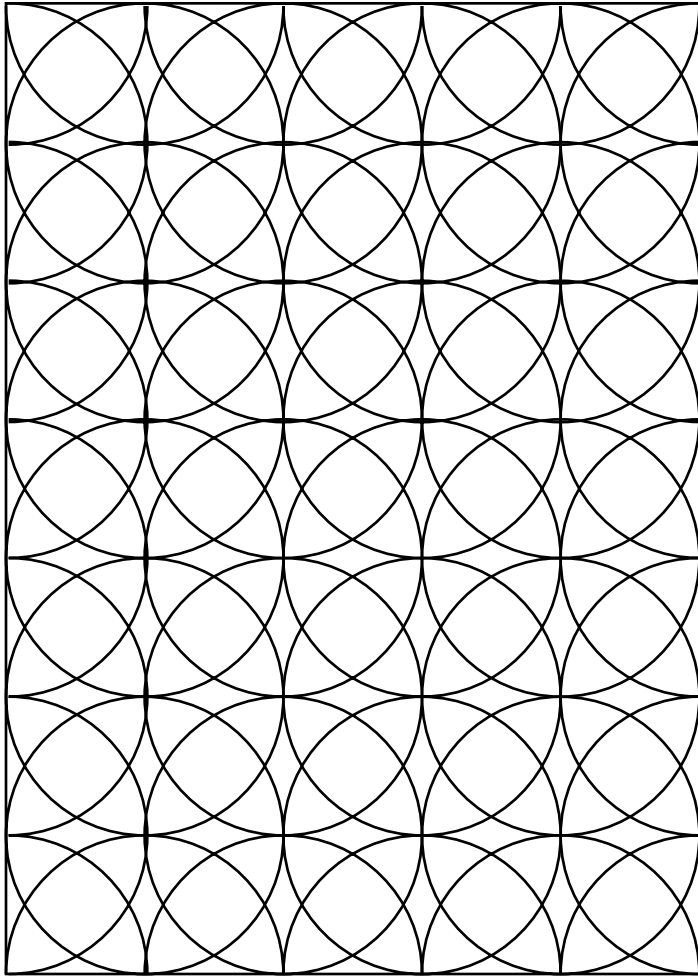
Then every isolated fixed point of  $f^*$  is a computable fixed point of  $f$ .

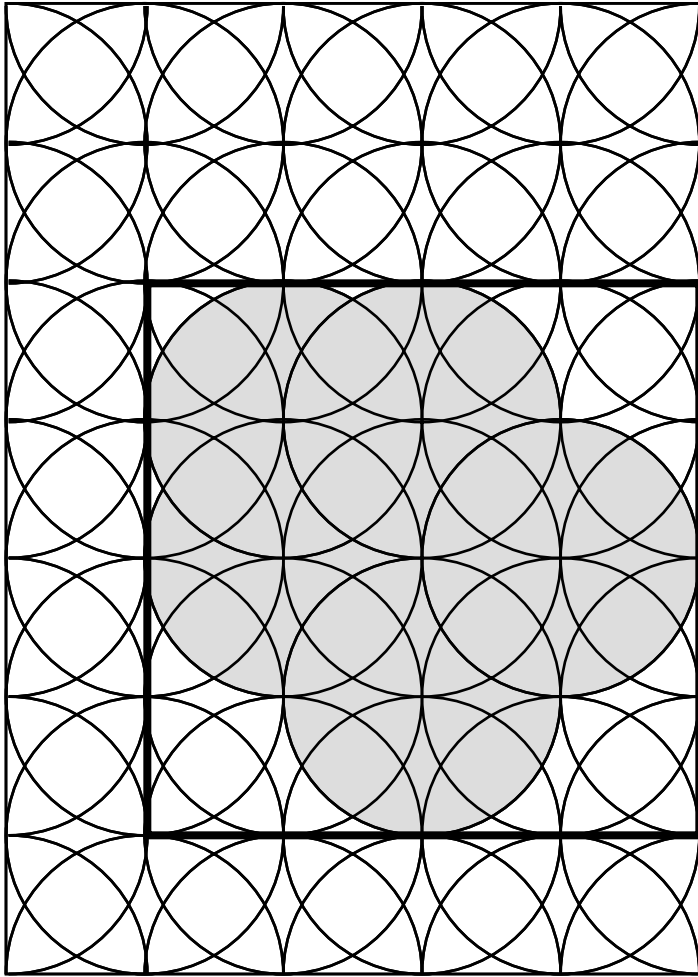
To compute

an isolated zero of  $g(z) = |f(z) - z|$ ,

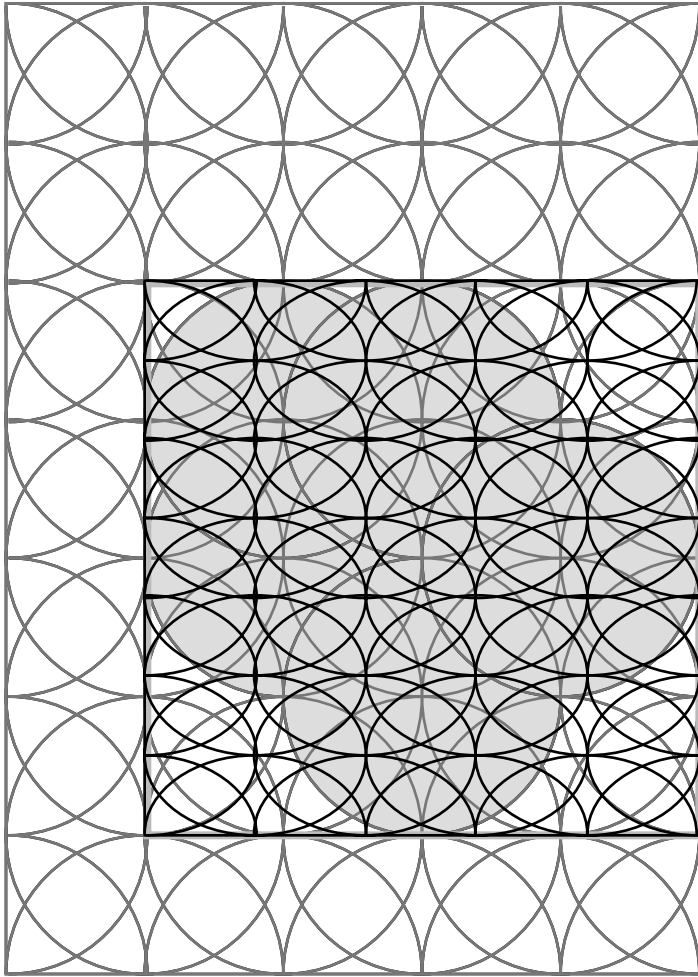
- isolate the zero in a rectangle,
- cover the rectangle with  $\delta$  nhoods,
- mark the nhoods where  $g(\text{center point})$  is very close to 0,
- draw a new smaller rectangle containing the marked nhoods.

.









## Orevkov's Construction

In [2], V.P. Orevkov constructs a ccc function  $f : I^2 \rightarrow I^2$  such that  $f^*$  has no fixed points.

In this construction:

$f^*$  is not total, and

$f$  has no computable modulus  
of uniform continuity.

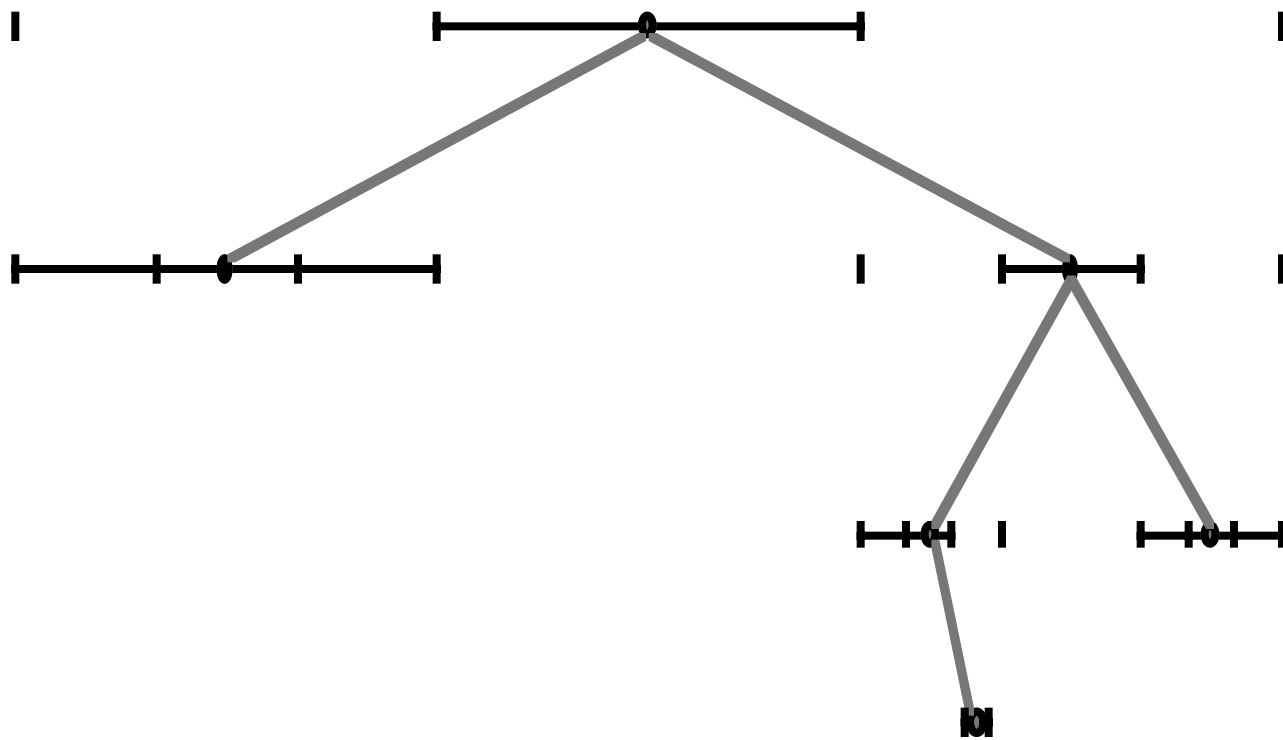
## Building computable counterexamples

Let  $T$  be an infinite computable 0-1 tree with no computable paths.

Using  $T$ , we can build a computable sequence of closed subintervals of  $[0, 1]$  such that

- each pair of intervals intersect in at most one point,
- each computable real in  $[0, 1]$  is contained in one of the intervals, and
- there is a degree preserving isomorphism between the points of  $[0, 1]$  not contained in the union of the intervals and the paths through  $T$ .

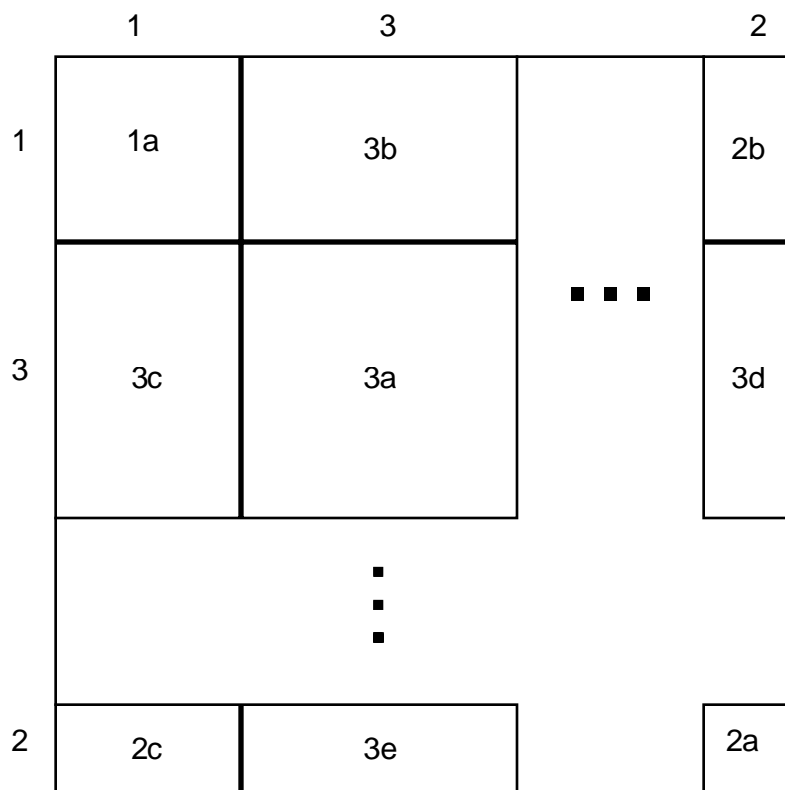
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# The picture for Orevkov's construction

To build a function with no fixed points:

- divide the square using subintervals generated by a tree,
- map each square to the boundary of  $I^2$ ,
- rotate.

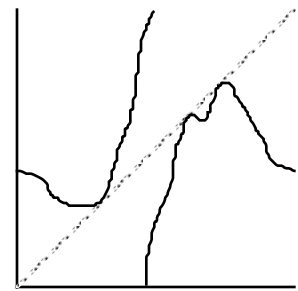


**Question:** If  $f : I^2 \rightarrow I^2$  is ccc and  $f^*$  is total must  $f$  have a computable fixed point?

**Partial answer:**

**Theorem:** If  $f : I^2 \rightarrow \partial I^2$  is ccc and  $f^*$  is total, then  $f$  has a computable fixed point.

Sketch: If  $f : D \rightarrow \partial D$  and  $f^*$  is total, then  $f^*$  has a fixed point on  $\partial D$ . Graph  $f^* : \partial D \rightarrow \partial D$  as a function of radians.



This sort of thing can't happen:

So  $f^*$  must cross  $y = x$  someplace. Apply the computable version of the IVT to find the fixed point.

**Theorem:** Given any infinite computable tree  $T$  with no computable paths, there is a computably coded continuous function  $f : I^2 \rightarrow \partial I^2$  such that  $f^*$  is total, the only computable fixed point of  $f^*$  is  $(0, 0)$ , and there is a degree preserving isomorphism between the noncomputable fixed points of  $f^*$  and the infinite paths through  $T$ .

Sketch: Fix  $T$ . Construct a computably coded continuous function  $g : [0, 1] \rightarrow [0, 1]$  such that the maximum of  $g$  is 1, and there is a degree preserving isomorphism between  $\{x \mid g^*(x) = 1\}$  and the infinite paths through  $T$ .

Define  $f(x, y) = (x \cdot g(x), 0)$ . The only fixed points of  $f$  occur where  $y = 0$  and either  $x = 0$  or  $g(x) = 1$ .

## The original task...

We wanted a restricted version of Brouwer's Theorem that could be used in **RCA**<sub>0</sub> to formalize Arnold's proof of the fundamental theorem of algebra.

Our computable version uses the condition “ $f^*$  has finitely many fixed points.”

We could use this to produce a computable analog of Arnold's proof, but  $f^*$  can't be formalized in **RCA**<sub>0</sub>.

We need another version of the fixed point theorem.



**Theorem ( $\mathbf{RCA}_0$ )** Suppose that:

- (1)  $f : I^2 \rightarrow I^2$  is a total continuous function,
- (2)  $f$  has a modulus of uniform continuity, and
- (3) there is an integer  $m$  and sequences  $\langle n_k \rangle_{k \in \mathbb{N}}$  and  $\langle \langle B_{k,i} \rangle_{i < m_k} \rangle_{k \in \mathbb{N}}$  such that for each  $k$ ,  $m_k < m$ , each  $B_{k,i}$  is an open ball of radius at most  $2^{-k}$  contained in exactly one ball in the list  $\langle B_{k-1,i} \rangle_{i < m_{k-1}}$ , and for every rational point  $z$  exterior to  $\cup_{i < m_k} B_{k,i}$  we have  $|f(z) - z| > 2^{-n_k}$ .

Then  $f$  has a fixed point in  $I^2$ .

What does (3) mean?

Except at  $m$  spots,  $|f(z) - z|$  is nicely bounded away from 0.

## Arnold's proof

To show that  $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$  has a zero, define

$$R = 2 + |a_1| + \cdots + |a_n|, \text{ and}$$

$$g(z) = \begin{cases} z - f(z)/(Re^{i(n-1)\theta r} & \text{for } |z| \leq 1 \\ z - f(z)/(Rz^{n-1}) & \text{for } |z| > 1. \end{cases}$$

$g(z)$  is continuous and maps the closed disk of radius  $R$  into itself.

By Brouwer's fixed point theorem,  $g(z)$  has a fixed point, which is also a zero of  $f(z)$ .

## Bibliography

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4. S. G. SIMPSON, *Subsystems of Second Order Arithmetic*, Springer-Verlag, Berlin, Heidelberg, 1999.