Outer measures of Vitali's nonmeasurable sets

Jeffry L. Hirst
Jonathan E. Loss

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Abstract

We use transfinite recursion and diagonalization to construct a Vitali nonmeasurable set of outer measure r for each $r \in (0, 1]$.

In 1905, Vitali proved the existence of a subset of $[0, 1]$ with no Lebesgue measure [2]. His construction can be summarized in two sentences. For $x, y \in [0, 1]$, define an equivalence relation by saying $x \sim y$ precisely when $x - y$ is rational. Any set consisting of exactly one real from each equivalence class is nonmeasurable. We call a set constructed in this fashion a *Vitali nonmeasurable set*.

The Lebesgue outer measure of a set A, denoted by $\mu^*(A)$, is defined by

$$\mu^*(A) = \inf_{A \subset \cup_{n \in \mathbb{N}}(a_n, b_n)} \sum_{n \in \mathbb{N}} (b_n - a_n)$$

where the infimum is taken over all possible countable sequences of open intervals $\langle (a_n, b_n) \mid n \in \mathbb{N} \rangle$ such that $A \subset \cup_{n \in \mathbb{N}}(a_n, b_n)$. If X is any Vitali nonmeasurable subset of $[0, 1]$ and $\overline{X}$ is its relative complement, then it is easy to show that $\mu^*(\overline{X}) = 1$. (See the discussion following the theorem and proof.) The outer measure of X is more varied. At the extremes, since every set of zero outer measure has zero Lebesgue measure and X is nonmeasurable, we have $\mu^*(X) > 0$. Also, $\mu^*([0, 1]) = 1$, so $\mu^*(X) \leq 1$. In fact, $\mu^*(X)$ can be any value in $(0, 1]$, as shown in the following theorem.

Theorem 1. *For every $r \in [0, 1)$ there is a Vitali nonmeasurable set with $\mu^*(X) = r$.*

Proof. We will prove the theorem for $r = 1$ and generalize in the last paragraph. If $\mu^*(X) < 1$, then there is a family $I = \langle (a_n, b_n) \mid n \in \mathbb{N}\rangle$ of open intervals with *rational* endpoints such that $X \subset \cup_{n\in\mathbb{N}}(a_n, b_n)$ and $\sum_{n\in\mathbb{N}}(b_n - a_n) < 1$. Let $\mathfrak{c}$ denote the cardinality of the reals and let $\langle I_\alpha \mid \alpha < \mathfrak{c}\rangle$ be an enumeration of all countable families of open intervals with rational endpoints whose lengths sum to a value less than 1. We will use transfinite recursion to construct a Vitali nonmeasurable set X so that for each $\alpha < \mathfrak{c}$, I_α is not a cover of X. Our construction can be characterized as a diagonalization argument.

Let x_0 be any element not in the union of the intervals in I_0, so that $x_0 \in \overline{\cup I_0}$. Using the equivalence relation from Vitali's construction, let $[x_0] = \{y \in [0, 1] \mid y \sim x_0\}$. If x_α is defined for all $\alpha < \beta < \mathfrak{c}$, let x_β be any element of $\overline{\cup I_\beta} - \cup_{\alpha<\beta}[x_\alpha]$. Such an x_β always exists by the following cardinality argument. The set $\overline{\cup I_\beta}$ is a Lebesgue measurable subset of $[0, 1]$ with positive measure, and therefore has cardinality $\mathfrak{c}$. (See the discussion following the proof for more on this.) For each α, $[x_\alpha]$ is countable, so $| \cup_{\alpha<\beta} [x_\alpha]| = \sup\{\aleph_0, |\beta|\} < \mathfrak{c}$. Thus $\overline{\cup I_\beta} - \cup_{\alpha<\beta}[x_\alpha]$ is nonempty and x_β can be selected.

From the construction, if $\alpha \neq \beta$ then $x_\alpha \not\sim x_\beta$. Let X be any Vitali nonmeasurable set containing $\{x_\alpha \mid \alpha < \mathfrak{c}\}$. Since no countable sequence of intervals in $\langle I_\alpha \mid \alpha < \mathfrak{c}\rangle$ covers X, we have $\mu^*(X) \not< 1$, and so $\mu^*(X) = 1$.

To generalize the result, fix $r \in (0, 1]$. Carry out the preceding construction in $[0, r]$. Since every Vitali equivalence class of $[0, 1]$ contains a Vitali equivalence class of $[0, r]$, the resulting set is a Vitali nonmeasurable subset of $[0, 1]$ with outer measure r. $\square$

The preceding proof uses the fact that every subset of $[0, 1]$ of positive Lebesgue measure has cadinality $\mathfrak{c}$. This is a trivial consequence of the countable additivity of measure and the continuum hypothesis, but can also be proved directly in ZFC. By Theorem B of §16 in Halmos [1], if S is a set of positive Lebesgue measure, then the difference set $D = \{x - y \mid x, y \in S\}$ contains an open interval. Thus, $|D| = \mathfrak{c}$. Since S is infinite, $|S| = |D|$, so $|S| = \mathfrak{c}$.

This same theorem from Halmos can be applied to show that if X is a Vitali nonmeasurable set then $\mu^*(\overline{X}) = 1$. Suppose by way of contradiction that $\mu^*(\overline{X}) < 1$ and let $\langle (a_n, b_n) \mid n \in \mathbb{N}\rangle$ be an open cover of $\overline{X}$ with $\sum_{n\in\mathbb{N}}(b_n - a_n) < 1$. The complement of the union of the cover is a subset of X with positive Lebesgue measure. By the theorem of Halmos, the difference set for this subset of X must contain an interval, and therefore X must contain two elements that differ by a nonzero rational. This contradicts the fact that X contains exactly one element from each Vitali equivalence class.

Bibliography

[1] Paul R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, N. Y., 1950. MR **0033869** (11,504**d**)

[2] Giuseppe Vitali, *Sul problema della misura dei gruppi di punti di una retta* (1905); reprinted in Giuseppe Vitali, *Opere sull'analisi reale e complessa*, Edizioni Cremonese, Florence, 1984. Carteggio. [Correspondence]; Edited and with an introduction by Luigi Pepe, Carlo Pucci, Arturo Vaz Ferreira and Edoardo Vesentini; With a biographical essay by Pepe. MR **777329 (86e:**01077).