Some conservation results for higher order reverse mathematics

Jeffry Hirst, Professor Emeritus
Appalachian State University
Boone, NC

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ESI Reverse Mathematics: New Paradigms

Motivation for conservation results

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Either:
(1) The higher order theory has the same second order consequences as a familiar second order theory. This allows us to work in the expressive higher order theory and draw conclusions about second order arithmetic.

Or:
(2) The higher order theory does not have the same second order consequences as any familiar second order theory.

The base system RCA_0^ω

In his article *Higher order reverse mathematics* [5], Kohlenbach presents axioms for doing reverse mathematics using arithmetic in all finite types. The axioms for RCA_0^ω include:

Feferman's E-PRA^ω which includes axioms for arithmetic on numbers (type 0 objects), functions from numbers to numbers (type $0 \to 0$ objects, also called type 1), functions from type 1 objects to numbers, and so on. . .

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plus, the choice scheme $\mathrm{QF} - \mathrm{AC}^{1,0}$:

$$\forall x^1 \exists y^0 A(x, y) \to \exists Y^{1\to 0} \forall x^1 A(x, Y(x))$$

where $A(x, y)$ is quantifier-free.

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Kohlenbach proves that RCA_0^ω has the same second order theorems as (a function based version of) RCA_0.

The proof uses the type structure ECF (extensional hereditarily continuous functionals) of Troelstra.

Arithmetical comprehension

We can write ACA_0^ω for the higher order subsystem consisting of RCA_0^ω plus the axiom $(\exists^2)$.

The axiom $(\exists^2)$ asserts the existence of a type 2 function $\exists^2$ with the following property:
If $f : \mathbb{N} \to \mathbb{N}$ is any function on the natural numbers, then $\exists^2(f) = 0$ if and only if 0 is in the range of f.

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The proof uses a re-axiomatization of ACA_0^ω that simplifies the verification that the higher order model satisfies the axioms.

A misleading question

$\mathsf{RCA}_0^\omega + (\mathsf{LPO})$ has the same second order theorems as ACA_0.

Does $\mathsf{RCA}_0^\omega + (\mathsf{LLPO})$ have the same second order theorems as WKL_0?

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Does $\mathsf{RCA}_0^\omega + (\mathsf{LLPO})$ have the same second order theorems as WKL_0?

No. Absolutely not.

LPO and LLPO

(LLPO) is an axiom that asserts the existence of a type 2 function LLPO with the following property:

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The article [3] (joint with Carl Mummert) includes a proof of:

Theorem (RCA_0^ω) (LLPO) is equivalent to $(\exists^2)$ (which is (LPO)).

The proof uses Kohlenbach's [5] equivalence of $(\exists^2)$ with the existence of a sequentially discontinuous function. That result is based on Lemma 1 of Grilliot [2].

Weihrauch and higher order

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However, RCA_0^ω proves $(\mathrm{LLPO}) \equiv (\mathrm{LPO})$.

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The reverse math result (roughly) shows us that any realizer for LLPO can be used to compute a realizer for LPO. However, the computation is not uniform, so it doesn't show that LPO can be Weihrauch reduced to LLPO.

Weak Koňig's Lemma

In *Higher order reverse mathematics* [5], Kohlenbach provides a conservation result for WKL_0, based on a fan functional that can compute moduli of uniform continuity.

(MUC): There is a function Ω defined on functions from $2^{\mathbb{N}}$ to 2, such that for any $\varphi : 2^{\mathbb{N}} \to 2$, and any $f_1, f_2 \in 2^{\mathbb{N}}$, if

f_1 and f_2 agree on the first $\Omega(\varphi)$ entries

then $\varphi(f_1) = \varphi(f_2)$.

(MUC) is not restricted to continuous functions, and so is inconsistent with $(\exists^2)$.

Prop. 3.15 [5] (paraphrased): RCA_0^ω+(MUC) is conservative over WKL_0.

Note that Ω is a type 3 object.

More moduli of uniform continuity

Section 8 of *Banach's theorem in HORM* [3] (joint with Carl Mummert), discusses functions that compute moduli of uniform continuity.

($M_{[0,1]}$): There is function M such that if $f : [0, 1] \to \mathbb{R}$ is continuous, then $M(f)$ is a modulus of uniform continuity for f.

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Lemma: RCA_0^ω plus $(\exists^2)$ proves $(M_{[0,1]})$.

Prop: The second order theorems of $\mathsf{RCA}_0^\omega + (M_{[0,1]})$ are exactly the same as those of WKL_0.

Note: $M_{[0,1]}$ is a type 3 object.

Preliminary results

Viewing $2^{\mathbb{N}}$ and 2 as complete separable metric spaces, we can formulate restrictions of (MUC).

$(M_{2^{\mathbb{N}}})$: There is a function M such that if $f : 2^{\mathbb{N}} \to 2$ is continuous, then $M(f)$ is a modulus of uniform continuity for f.

Prop: RCA_0^{ω} plus $(\exists^2)$ proves $(M_{2^{\mathbb{N}}})$. (See [3].)

Conj: The second order theorems of $\mathrm{RCA}_0^{\omega} + (M_{2^{\mathbb{N}}})$ are exactly the same as those of WKL_0.

More preliminary results

Further restrictions of $M_{2^{\mathbb{N}}}$ may be of interest.

($M_{2^{\mathbb{N}},code}$): There is a function M such that if $f : 2^{\mathbb{N}} \to 2$ is a continuous function defined by a traditional RM code (set of quintuples), then $M(f)$ is a modulus of uniform continuity for f.

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Conj: RCA_0^ω plus $(\exists^2)$ proves $(M_{2^{\mathbb{N}},code})$. (Section 8 of [3] plus extraction of functions from codes.)

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Conj: The second order theorems of $\mathrm{RCA}_0^\omega + (M_{2^{\mathbb{N}},code})$ are exactly the same as those of WKL_0.

Conj: RCA_0^ω plus $(M_{2^{\mathbb{N}},code})$ **does not prove** $(M_{2^{\mathbb{N}}})$.

Idea for the proof: $M_{2^{\mathbb{N}},code}$ is a type 2 object. Using an argument based on Grilliot's lemma, prove that if any principle asserting the existing of a type 2 object implies $(M_{2^{\mathbb{N}}})$, then it also implies $(\exists^2)$.

Stronger theories: Π^1_1-CA_0

The theory Π^1_1-CA_0 is related to the Suslin functional.

(SF): There is a function S mapping trees from $\mathbb{N}^{<\mathbb{N}}$ to 2 such that $S(T) = 0$ if and only if T is well-founded.

Theorem (Sakamoto and Yamizaki [6], based on work of Avigad and Feferman) RCA^ω_0 plus (SF) is conservative over Π^1_1-CA_0 for Π^1_3 sentences.

Question: Can Hunter's argument for ACA^ω_0 be adapted? Does a Fefermanesque μ style reaxiomatization streamline the proof?

Stronger theories: ATR_0

In second order arithmetic, there are parallels between WKL_0 and ATR_0.

$\mathrm{WKL}_0 \equiv \Sigma^0_1$-separation $\qquad$ $\mathrm{ATR}_0 \equiv \Sigma^1_1$-separation

Questions:

- What are good candidates for functionals that have the same second order theorems as ATR_0?
- Are the ATR_0 level Weihrauch problems good candidates?
- Is the LPO/LLPO behavior repeated at this level?
- Is there a good candidate of type 2?

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