EN 2570 - Digital Signal Processing

Assignment 02

Index No : 180529E

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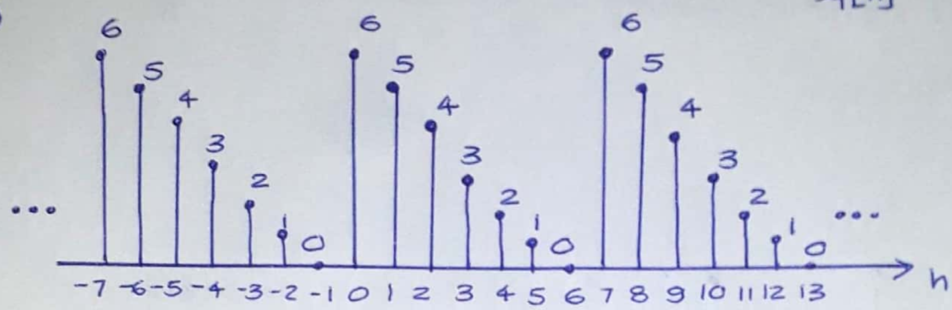
8.21

a)

$\tilde{x}_1[n]$



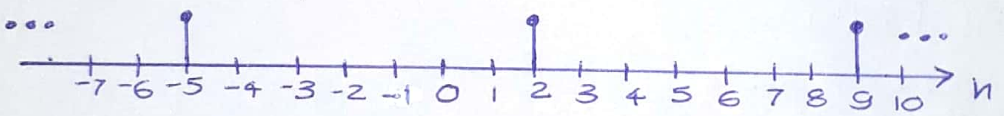
$$x_1[n] = \begin{cases} 6-n & ; 0 \le n \le 6 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\tilde{x}_1[n] = x_1[((n))_7] \text{ (periodic form)}$$

$$x_2[n] = \delta[n-2]$$

$$\tilde{x}_2[n] = x_2[((n))_7] \text{ (periodic form)}$$

$\tilde{x}_2[n]$



$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

Since $\tilde{y}_1[n]$ is the result of periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$,

$$y_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

$$= \sum_{m=0}^{6} \tilde{x}_1[n-m]\tilde{x}_2[m]$$

$$= \sum_{m=0}^{6} x_1[((n-m))_7]\, x_2[((m))_7]$$

$$= \sum_{m=0}^{6} x_1[((n-m))_7]\, \delta[((m))_7 - 2]$$

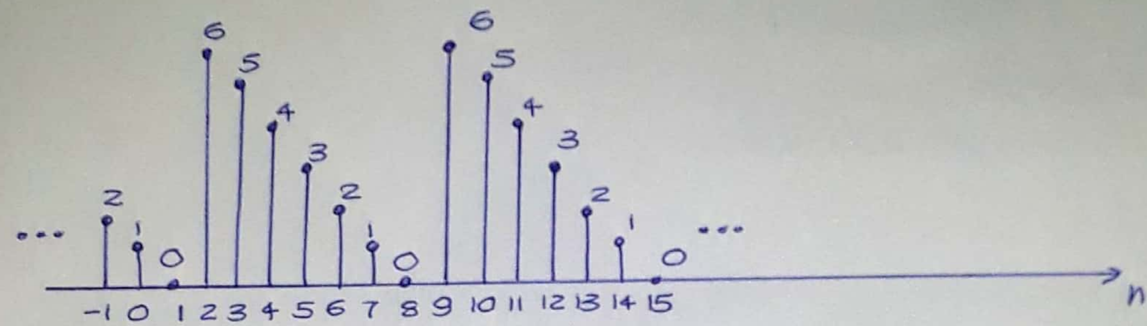
$\tilde{x}_2[n]$ is a periodic impulse shifted by 2.

$\therefore$ The resulting sequence $\tilde{y}_1[n]$ can be obtained by shifting $\tilde{x}_1[n]$ by 2.

$$y_1[n] = x_1[((n-2))_7]$$

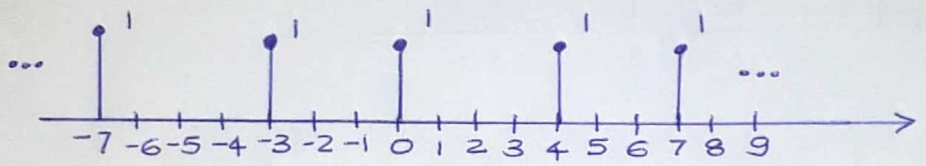
$$\tilde{y}_1[n] = \tilde{x}_1[n-2] \quad //$$

$\bar{y}_1[n]$



b)

$\bar{x}_3[n]$



$$x_3[n] = \begin{cases} \delta[n] + \delta[n-4] & ; 0 \le n \le 6 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\bar{x}_3[n] = x_3[((n))_7] = \delta[((n))_7] + \delta[((n-4))_7]$$

$$\bar{x}_3[n] \xleftrightarrow{DFS} \bar{X}_3[k]$$

$$\bar{y}_2[n] \xleftrightarrow{DFS} \bar{Y}_2[k]$$

$$\bar{Y}_2[k] = \bar{X}_1[k]\,\bar{X}_3[k]$$

From the properties of DFS ;

$$\bar{y}_2[n] = \sum_{m=0}^{6} x_1[((n-m))_7]\, x_3[((m))_7]$$

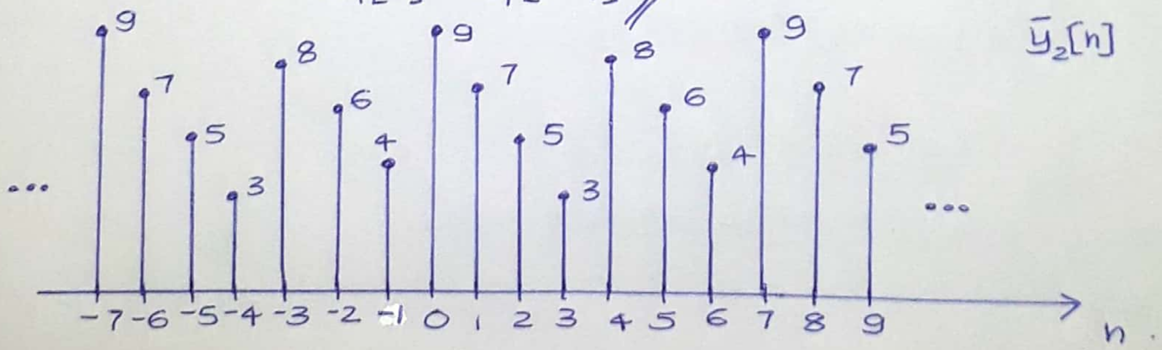
$$= \sum_{m=0}^{6} x_1[((n-m))_7]\,\left[\delta[((m))_7] + \delta[((m-4))_7]\right]$$

$$= \sum_{m=0}^{6} x_1[((n-m))_7]\,\delta[((m))_7] + \sum_{m=0}^{6} x_1[((n-m))_7]\,\delta[((m-4))_7]$$

$$= x_1[((n))_7] + x_1[((n-4))_7]$$

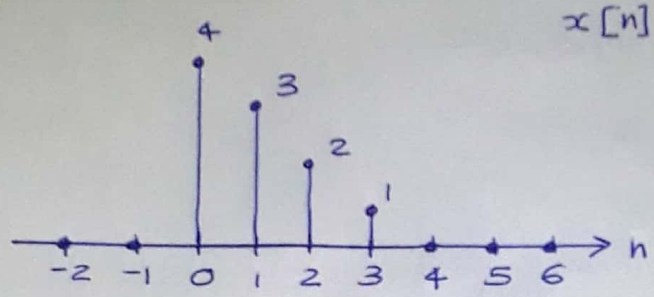
$$= \bar{x}_1[n] + \bar{x}_1[n-4] \;//$$

$\bar{y}_2[n]$



$\boxed{8.23}$

a)



Let $x[n] \xleftrightarrow{DFT} X[k]$

$y[n] \xleftrightarrow{DFT} Y[k]$

From circular shift property of DFT,

Let's consider,

$$x[((n-m))_N] ; \; 0 \le n \le N-1 \xleftrightarrow{DFT} e^{-j(\frac{2\pi}{N})km} X[k] = W_N^{km} X[k]$$

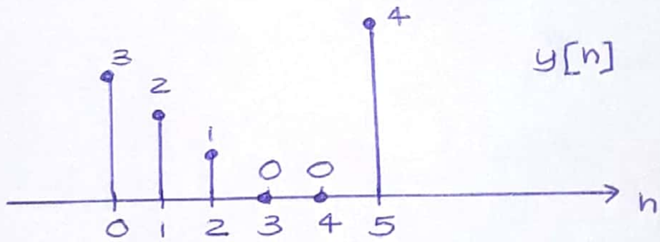
$$Y[k] = W_6^{5k} X[k]$$

$\therefore$ $y[n]$ is a five-sample circular shift of $x[n]$

We can say $N=6$, $m=5$

$$y[n] = \begin{cases} x[((n-5))_6] ; & 0 \le n \le 5 \\ 0 ; & \text{otherwise} \end{cases}$$

$y[0] = 3$
$y[1] = 2$
$y[2] = 1$
$y[3] = 0$
$y[4] = 0$
$y[5] = 4$



b) $w[n] \xleftrightarrow{DFS} W[k] = \text{Im}\{X[k]\}$

From the properties of DFT,

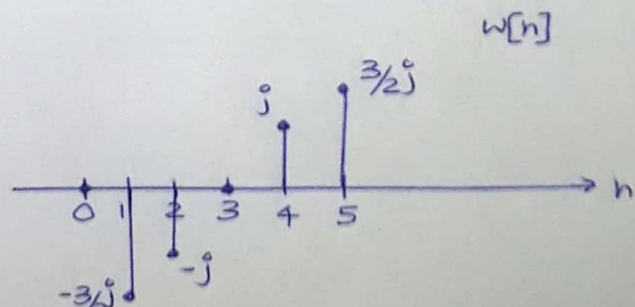
$$\frac{1}{2}\left\{ x[n] - x[((-n))_N] \right\} \xleftrightarrow{DFT} j\,\text{Im}\{X[k]\}$$

$$w[n] = \frac{1}{2j}\left\{ x[n] - x[((-n))_N] \right\}$$

since $((-n))_N = N-n$,

$$w[n] = \frac{1}{2j}\left\{ x[n] - x[N-n] \right\}$$

$w[0] = 0$
$w[1] = -3/2j$
$w[2] = -j$
$w[3] = 0$
$w[4] = j$
$w[5] = 3/2j$



c) According to the analysis equation of DTS;

$$X[K] = \sum_{n=0}^{5} x[n] W_6^{kn} \; ; \; 0 \le K \le 5$$

$$X[K] = 4 + 3W_6^{K} + 2W_6^{2k} + W_6^{3k}$$

$$Q[K] = X[2K+1], \quad K = 0, 1, 2$$

$$Q[K] = 4 + 3W_6^{(2k+1)} + 2W_6^{2(2k+1)} + W_6^{3(2k+1)}$$

$$= 4 + 3W_6^{1} W_6^{2k} + 2W_6^{2} W_6^{4K} + W_6^{3} W_6^{6K}$$

$$\text{Since } W_6^{1} = e^{-j(2\pi/6)^{1}} = e^{-j\pi/3}$$

$$W_6^{2} = e^{-j(2\pi/6)^{2}} = e^{-j2\pi/3}$$

$$W_6^{3} = e^{-j(2\pi/6)^{3}} = e^{-j\pi}$$

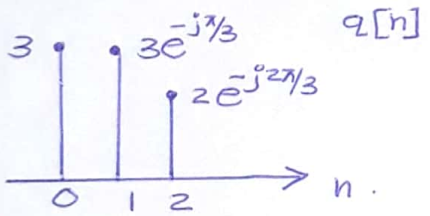
$$Q[K] = 4 + 3e^{-j\pi/3} W_3^{K} + 2e^{-j2\pi/3} W_3^{2K} + e^{-j\pi} W_3^{3K} \quad (\text{Since } W_N^{mK} = W_{N/2}^{mK/2})$$

$$\text{Since } W_3^{3K} = W_3^{0K} = 1 \; \& \; e^{-j\pi} = -1,$$

$$Q[K] = 4 + 3e^{-j\pi/3} W_3^{K} + 2e^{-j2\pi/3} W_3^{2K} + (-1)(1).$$

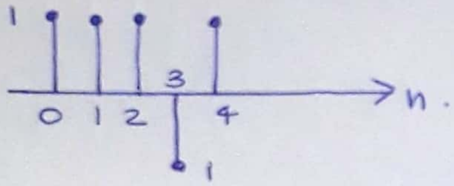
$$Q[K] = 3 + 3e^{-j\pi/3} W_3^{K} + 2e^{-j2\pi/3} W_3^{2K} \quad //$$

Then $q[n]$ can be obtained as,



$\boxed{8.29}$

$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]$



$C(e^{j\omega}) = X(e^{j\omega}) \cdot X^*(e^{j\omega}) = |X(e^{j\omega})|^2$

autocorrelation function of $x[n]$,

$c[n] = x[n] * x[-n]$

a) $X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega} - e^{-3j\omega} + 4e^{-4j\omega}$

since $x[n]$ is real,

$x[-n] \overset{DTFT}{\longleftrightarrow} X(e^{-j\omega})$

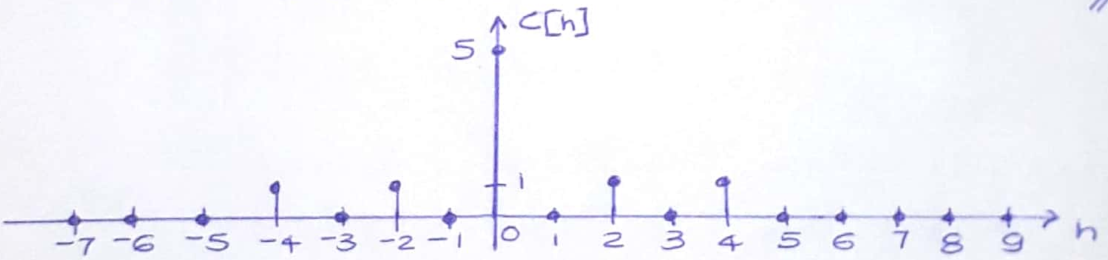
$X(e^{-j\omega}) = 1 + e^{j\omega} + e^{2j\omega} - e^{3j\omega} + e^{4j\omega}$

$C(e^{j\omega}) = (1 + e^{-j\omega} + e^{-2j\omega} - e^{-3j\omega} + e^{-4j\omega})(1 + e^{j\omega} + e^{2j\omega} - e^{3j\omega} + e^{4j\omega})$

$C(e^{j\omega}) = 1 + e^{j\omega} + 2e^{j\omega} - e^{3j\omega} + e^{4j\omega} + e^{-j\omega} + 1 + e^{j\omega} - e^{2j\omega} + e^{3j\omega} + e^{-2j\omega}$
$+ e^{-j\omega} + 1 - e^{j\omega} + e^{2j\omega} - e^{-3j\omega} - e^{-2j\omega} - e^{-j\omega} + 1 - e^{j\omega} + e^{-4j\omega} + e^{-3j\omega}$
$+ e^{-2j\omega} - e^{-j\omega} + 1$

$C(e^{j\omega}) = 5 + e^{2j\omega} + e^{4j\omega} + e^{-2j\omega} + e^{-4j\omega}.$

$c[n] = 5\delta[n] + \delta[n-2] + \delta[n+2] + \delta[n-4] + \delta[n+4]$ //



Since the sequence is even as $c[n] = c[-n]$ for all n.

b)

$x[n] \xrightarrow{\text{5 point DFT}} X_5[k]$

$X_5[k] = \sum_{n=0}^{4} x[n] W_5^{kn}$; $0 \le k \le 4$ & $W_5^{kn} = e^{-j(2\pi/5)kn}$

$X_5[k] = 1 + W_5^{k} + W_5^{2k} - W_5^{3k} + W_5^{4k}$

$X_5^*[k] = 1 + W_5^{-k} + W_5^{-2k} - W_5^{-3k} + W_5^{-4k}$

$C_5[k] = X_5[k] X_5^*[k]$

$C_5[k] = (1 + W_5^{k} + W_5^{2k} - W_5^{3k} + W_5^{4k})(1 + W_5^{-k} + W_5^{-2k} - W_5^{-3k} + W_5^{-4k})$

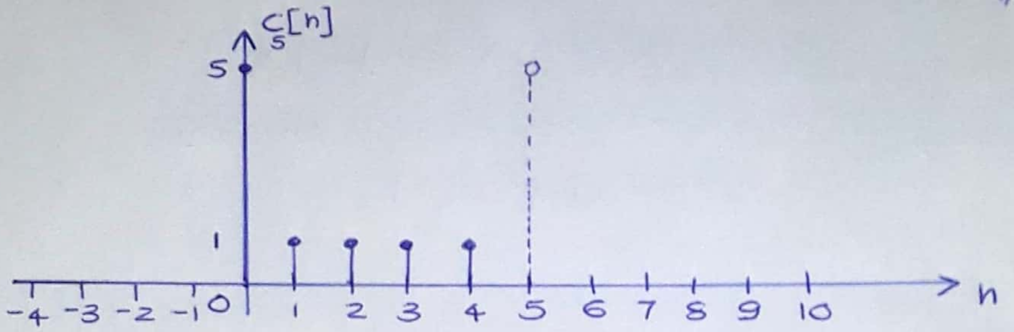
$C_5[k] = 1 + W_5^{-k} + W_5^{-2k} - W_5^{-3k} + W_5^{-4k} + W_5^{k} + 1 + W_5^{-k} - W_5^{-2k} + W_5^{-3k}$
$+ W_5^{2k} + W_5^{k} + 1 - W_5^{-k} + W_5^{-2k} - W_5^{3k} - W_5^{2k} - W_5^{k} + 1 - W_5^{-k}$
$+ W_5^{4k} + W_5^{3k} + W_5^{2k} - W_5^{k} + 1$

$C_5[k] = 5 + W_5^{2k} + W_5^{4k} + W_5^{-2k} + W_5^{-4k}$

$$C_5[k] = 5 + W_5^{2k} + W_5^{4k} + W_5^{3k} + W_5^{k} \quad (\text{Since } W_N^{-kn} = W_N^{k(N-n)})$$

$$C_5[k] = 5 + W_5^{k} + W_5^{2k} + W_5^{3k} + W_5^{4k} \ ; \ 0 \le k \le 4.$$

$$C_5[n] = 5\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] \ //$$



$$C_5[n] = \begin{cases} \sum_{k=-\alpha}^{\alpha} c[n-5\gamma] & ; \ 0 \le n \le 4 \\ 0 & ; \ \text{otherwise} \end{cases}$$

c) $$X_N[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \ ; \ 0 \le k \le N-1.$$

2N point DFT of $x[n]$

$$X_{2N}(k) = \sum_{n=0}^{2N-1} x[n] e^{-j(2\pi/2N)(2kn)}$$

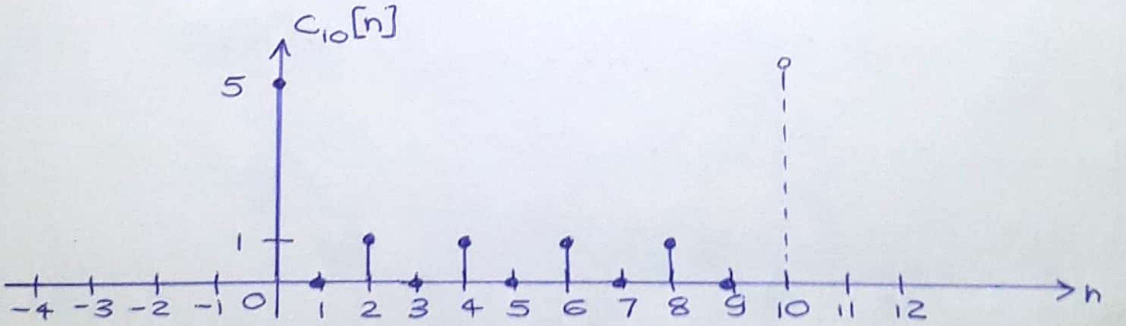
$$= \sum_{n=0}^{2N-1} x[n] W_{2N}^{2kn} \ ; \ 0 \le k \le 2N-1.$$

$$X_{10}[k) = 1 + W_{10}^{2k} + W_{10}^{4k} - W_{10}^{6k} + W_{10}^{8k} \ ; \ 0 \le k \le 9$$

$$X_{10}^{*}[k] = 1 + W_{10}^{-2k} + W_{10}^{-4k} - W_{10}^{-6k} + W_{10}^{-8k} \ ; \ 0 \le k \le 9.$$

$$C_{10}[k] = 5 + W_{10}^{2k} + W_{10}^{4k} + W_{10}^{6k} + W_{10}^{8k} \ ; \ 0 \le k \le 9.$$

$$C_{10}[n] = 5\delta[n] + \delta[n-2] + \delta[n-4] + \delta[n-6] + \delta[n-8] \ //$$



$$C_{10}[n] = \begin{cases} \sum_{\gamma=-\alpha}^{\alpha} c[n-10\gamma] & ; \ 0 \le n \le 9 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

d) $c_{10}[n] \xrightarrow{DFT} C_{10}[k]$

From the shifting property of DFT,

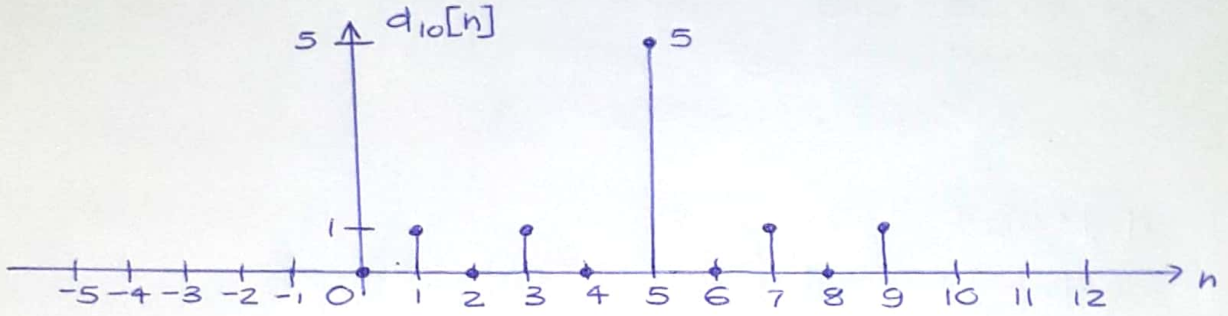
$$c_{10}[((n-m))_{10}] \xrightarrow{DFT} e^{-j(2\pi/10)km} C_{10}[k] = W_{10}^{km} C_{10}[k]$$

$; 0 \le n \le 9$.

$$D_{10}[k] = W_{10}^{5k} C_{10}[k] = W_{10}^{5k} X_{10}[k] X_{10}^{*}[k]$$

by comparison,

$$d_{10}[n] = \begin{cases} c_{10}[((n-5))_{10}] & ; 0 \le n \le 9 \\ 0 & ; \text{otherwise} \end{cases}$$



$\boxed{8.31}$

a) $x[n] = 2\delta[n] + \delta[n-1] - \delta[n-2]$.

$x[n] \xrightarrow{DTFT} X(e^{j\omega})$

$X(e^{j\omega}) = 2 + e^{-j\omega} - e^{-2j\omega}$ //

$y[n] \xrightarrow{DTFT} Y(e^{j\omega})$

$x[-n] \xrightarrow{DTFT} Y(e^{j\omega})$ (Since $y[n] = x[-n]$)

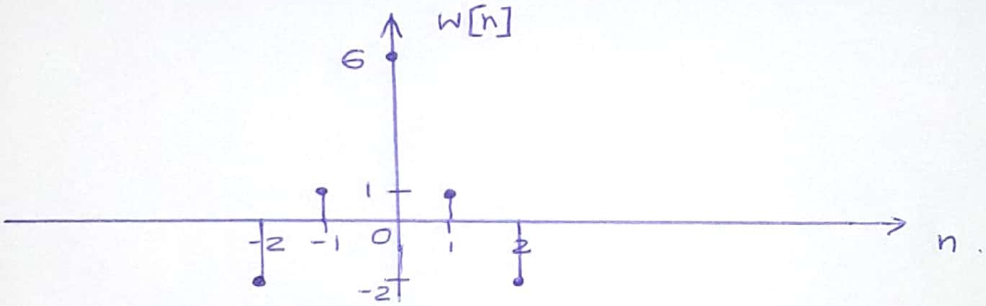
$Y(e^{j\omega}) = 2 + e^{j\omega} - e^{2j\omega}$ /

b) If $W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$,

$$\begin{aligned}
W(e^{j\omega}) &= (2+e^{-j\omega}-e^{-2j\omega})(2+e^{j\omega}-e^{2j\omega}) \\
&= 4 + 2e^{j\omega} - 2e^{2j\omega} + 2e^{-j\omega} + 1 - e^{j\omega} - 2e^{-2j\omega} - e^{-j\omega} + 1. \\
&= 6 + e^{j\omega} + e^{-j\omega} - 2e^{+2j\omega} - 2e^{-2j\omega} \\
&= -2e^{-2j\omega} + e^{-j\omega} + 6 + e^{j\omega} - 2e^{2j\omega} \quad //
\end{aligned}$$

c) $w[n] = x[n] * y[n]$.

$w[n] = -2\delta[n+2] + \delta[n+1] + 6\delta[n] + \delta[n-1] - 2\delta[n-2]$



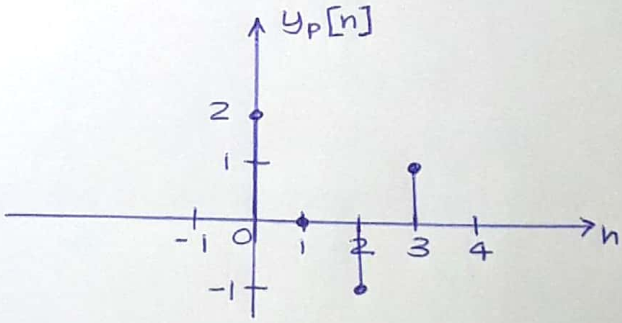
d) $y_p[n] = x[((-n))_4]$; $0 \le n \le 3$

$y_p[0] = x[((0))_4] = 2$

$y_p[1] = x[((-1))_4] = 0$

$y_p[2] = x[((-2))_4] = x[2] = -1$

$y_p[3] = x[((-3))_4] = x[1] = 1$.



e) Let $w_p[n] = x[n]$ ④ $y_p[n]$

$w_p[n] \xleftrightarrow{DFT} X[k]\,Y_p[k]$

$X[k] = \sum_{n=0}^{3} x[n] W_4^{kn}$

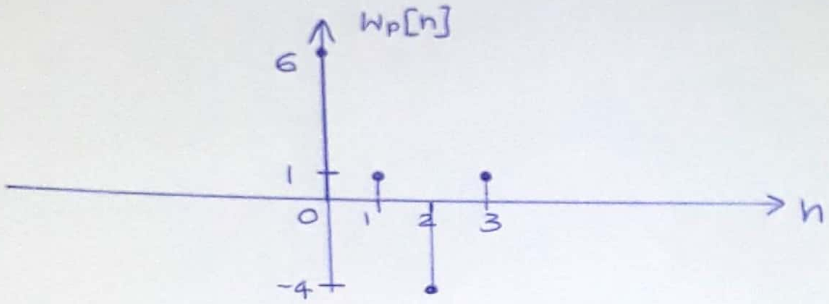
$X[k] = \sum_{n=0}^{3} (2\delta[n] + \delta[n-1] - \delta[n-2]) W_4^{kn}$; $0 \le k \le 3$

$X[k] = 2 + W_4^{k} - W_4^{2k}$.

$$\begin{aligned}
Y_p[k] &= \sum_{n=0}^{3} y_p[n] W_4^{kn} \\
&= \sum_{n=0}^{3} (2\delta[n] - \delta[n-2] + \delta[n-3]) W_4^{kn} \quad ; 0 \le k \le 3 \\
&= 2 - W_4^{2k} + W_4^{3k}
\end{aligned}$$

$$W_p[k] = X[k]\,Y_p[k]$$
$$= (2+W_4^{k}-W_4^{2k})(2-W_4^{2k}+W_4^{3k})$$
$$= 4-2W_4^{2k}+2W_4^{3k}+2W_4^{k}-W_4^{3k}+W_4^{4k}-2W_4^{2k}+W_4^{4k}-W_4^{5k}.$$
$$= 4-2W_4^{2k}+2W_4^{3k}+2W_4^{k}-W_4^{3k}+1-2W_4^{2k}+1-W_4^{k}$$
$$= 6+W_4^{k}-4W_4^{2k}+W_4^{3k}.$$

$$w_p[n] = 6\delta[n]+\delta[n-1]-4\delta[n-2]+\delta[n-3]$$



$w_p[n]$ is periodic. One periodic is given above.

f) $x[n]$ has 3 contiguous non-zero samples.
$y_p[n]$ has 3 contiguous non-zero samples.
Therefore, minimum DFT length $= 3+3-1=5$
So, in order to avoid aliasing N must be greater than or equal 5. $(N \geq 5)$