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CPFE II -Numerical Methods
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Numerical Methods Assignment

Question 1. Let X_t follow a mean reverting process:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

For the algebraic function: $Y_t = X_t^3$, find the stochastic differewntial equation satisfied by Y_t i.e., dY_t

From Ito's lemma:

$$dY_t = df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t^2 \tag{1}$$

Given: $Y_t = f(X_t) = X_t^3 \implies f'(X_t) = 3X_t^2 \,\mathrm{and}\, \mathrm{f''(X_t) = 6X_t}$

$$dY_t = 3X_t^2 dX_t + 3X_t dX_t^2 \tag{2}$$

For a mean-reverting process, given:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

$$(dX_t)^2 = \kappa^2(\theta - X_t)^2 dt^2 + \sigma^2 dW_t^2 + 2\kappa(\theta - X_t)\sigma\, dt \cdot dW_t$$

From quadratic variation of Brownian motion:

$$dt^2 = 0, \qquad dt\, dW_t = 0, \qquad (dW_t)^2 = dt,$$

$$(dX_t)^2 = 0 + \sigma^2\, dt + 0 = \sigma^2\, dt \tag{3}$$

Using Eq.3 in Eq.2 and $X_t = Y_t^{1/3}$, we obtain the desired stochastic differential equation:

$$\boxed{dY_t = [3Y_t^{2/3}\kappa(\theta - X_t) + 3Y_t^{1/3}\sigma^2] \cdot dt + 3Y_t^{2/3}\sigma \cdot dW_t} \tag{4}$$

Question 2. Let follow a geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

For the exponential function: $Y_t = e^{-X_t}$, find the SDE satisfied by Y_t, i.e., dY_t.
From Ito's lemma:

$$dY_t = df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t^2 \tag{5}$$

Given: $Y_t = f(X_t) = e^{-X_t} \implies f'(X_t) = -e^{-X_t} \,\mathrm{and}\, \mathrm{f''(X_t) = e^{-X_t}}$

$$dY_t = -e^{-X_t}dX_t + \frac{1}{2}e^{-X_t}dX_t^2 \tag{6}$$

For a geometric Brownian motion, given:

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

$$(dX_t)^2 = \mu^2 X_t^2 dt^2 + \sigma^2 X_t^2 dW_t^2 + 2\sigma\mu X_t^2\, dt \cdot dW_t$$

From quadratic variation of Brownian motion:

$$dt^2 = 0, \qquad dt\, dW_t = 0, \qquad (dW_t)^2 = dt,$$

$$(dX_t)^2 = 0 + \sigma^2\, dt + 0 = \sigma^2\, dt \tag{7}$$

Using Eq.7 in Eq.6 and $X_t = -\log Y_t$, we obtain the desired stochastic differential equation:

$$dY_t = -e^{-X_t}[\mu X_t dt + \sigma X_t dW_t] + \frac{1}{2}e^{-X_t}\sigma^2 dt \tag{8}$$

$$dY_t = [-e^{-X_t}\mu X_t + \frac{1}{2}e^{-X_t}\sigma^2]dt - e^{-X_t}\sigma X_t dW_t \tag{9}$$

$$\boxed{dY_t = [\mu Y_t \log Y_t + \frac{1}{2}\sigma^2 Y_t] \cdot dt + \sigma Y_t \log Y_t \cdot dW_t} \tag{10}$$

Question 3. Let follow a geometric Brownian motion:

$$dX_t = adt + bdW_t$$

For the exponential function: $Y_t = \cos\beta X_t$, find the SDE satisfied by Y_t, i.e., dY_t.

From Ito's lemma:

$$dY_t = df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t^2 \tag{11}$$

Given: $Y_t = f(X_t) = \cos\beta X_t \implies f'(X_t) = -\beta\sin\beta X_t \,\text{and}\, \mathrm{f''(X_t) = \beta^2\cos\beta X_t}$

$$dY_t = -\beta\sin\beta X_t\, dX_t + \frac{1}{2}\beta^2\cos\beta X_t \cdot dX_t^2 \tag{12}$$

For a geometric Brownian motion, given:

$$dX_t = adt + bdW_t$$
$$(dX_t)^2 = a^2dt^2 + b^2dW_t^2 + 2ab\, dt \cdot dW_t$$

From quadratic variation of Brownian motion:

$$dt^2 = 0, \qquad dt\, dW_t = 0, \qquad (dW_t)^2 = dt,$$

$$(dX_t)^2 = 0 + b^2\, dt + 0 = b^2\, dt \tag{13}$$

Using Eq.13 in Eq.12 and $X_t = -\frac{1}{\beta}\cos^{-1} Y_t$, we obtain the desired stochastic differential equation:

$$dY_t = -\beta\sin\beta X_t \cdot [adt + bdW_t] + \frac{1}{2}\beta^2\cos\beta X_t \cdot b^2 dt \tag{14}$$

$$dY_t = [-\beta a \cdot \sin\beta X_t + \frac{1}{2}\beta^2 b^2 \cdot \cos\beta X_t]dt - \beta b \cdot \sin\beta X_t \cdot dW_t \tag{15}$$

$$\boxed{dY_t = [-\beta a\sqrt{1 - Y_t^2} + \frac{1}{2}b^2\beta^2 Y_t] \cdot dt - \beta b\sqrt{1 - Y_t^2} \cdot dW_t} \tag{16}$$

Question 4. Kolmogorov – Smirnov (KS) test for uniformity: The following 12 observations are obtained from a random number generator that claims to produce samples from a Uniform(0,1) distribution: 0.02, 0.04, 0.07, 0.11, 0.15, 0.19, 0.24, 0.31, 0.39, 0.48, 0.60, 0.85

a. State the null and alternative hypotheses.

Here is the null hypotheses: $\mathcal{H}_\circ : R_i \sim U(0,1)$ and the alternative hypothesis: $\mathcal{H}_1 : R_i \nsim \mathrm{U}(0,1)$

b. Compute the empirical distribution function (EDF).

Random numbers, $X_i = [0.02, 0.04, 0.07, 0.11, 0.15, 0.19, 0.24, 0.31, 0.39, 0.48, 0.60, 0.85]$. The sample size is n = 12. The empirical distribution function (EDF) is:

$$F_n(x) = \frac{\#\{X_i \leq x\}}{n}$$

where, $\mathbb{I}_A$ is the indicator the event A. For the i-th order observation of $x_(i)$,

$$F_n(x_i) = \frac{i}{12}$$

c. Compute the KS statistic

For a Uniform$(0,1)$ distribution, the theoretical cumulative distribution function (CDF) is

$$F(x) = x, \quad 0 \le x \le 1.$$

The Kolmogorov–Smirnov statistic is defined as

$$D_n = \max_{1\le i\le n} \left\{ \left| F_n(x_{(i)}) - x_{(i)} \right|, \ \left| x_{(i)} - F_n(x_{(i-1)}) \right| \right\},$$

where $F_n(x)$ is the empirical distribution function (EDF). For a sample of size $n = 12$, the EDF evaluated at the ordered observations satisfies

$$F_n(x_{(i)}) = \frac{i}{12}.$$

The deviations are computed as follows:

i	$x_{(i)}$	$F_n(x_{(i)}) = \frac{i}{12}$	$\left\|F_n(x_{(i)}) - x_{(i)}\right\|$	$\left\|x_{(i)} - \frac{i-1}{12}\right\|$
1	0.02	0.083	0.063	0.020
2	0.04	0.167	0.127	0.043
3	0.07	0.250	0.180	0.063
4	0.11	0.333	0.223	0.110
5	0.15	0.417	0.267	0.150
6	0.19	0.500	0.310	0.190
7	0.24	0.583	0.343	0.240
8	0.31	0.667	0.357	0.310
9	0.39	0.750	0.360	0.390
10	0.48	0.833	0.353	0.480
11	0.60	0.917	0.317	0.600
12	0.85	1.000	0.150	0.767

Therefore,

$$D_n = \max = 0.767.$$

d. At the 5% significance level, test whether the sample is consistent with a Uniform(0,1) distribution. Use the critical value: $D_{0.05} = 1.36/\sqrt{n}$

$$D_{0.05} = \frac{1.36}{\sqrt{n}} = \frac{1.36}{\sqrt{12}} \approx 0.392.$$

At the 5% significance level, the sample is **not consistent** with a Uniform$(0,1)$ distribution.

Question 5. A pseudo-random number generator claims to generate observations from a Uniform(0,1) distribution. You generate 120 observations and classify them into 6 equal-width bins as shown below:

No.	Interval	Observed Frequency
1	$[0.0, 0.1667)$	8
2	$[0.1667, 0.3333)$	12
3	$[0.3333, 0.5)$	20
4	$[0.5, 0.6667)$	30
5	$[0.6667, 0.8333)$	32
6	$[0.8333, 1.0]$	18

a. State the null and alternative hypotheses.

- **Null hypothesis H_0:** The pseudo-random number generator produces observations from a Uniform$(0, 1)$ distribution.
- **Alternative hypothesis H_1:** The observations are not drawn from a Uniform$(0, 1)$ distribution.

b. Compute the expected frequencies under the null hypothesis.

The interval $[0, 1]$ is divided into 6 equal-width bins. Under the null hypothesis, each bin has probability

$$p = \frac{1}{6}.$$

With a total of $n = 120$ observations, the expected frequency in each bin is

$$\mathbb{E}_i = np = \frac{120}{6} = 20, \quad i = 1, \ldots, 6.$$

c. Compute the Chi-Square test statistic.

The chi-square test statistic is defined as

$$\chi^2 = \sum_{i=1}^{6} \frac{(O_i - \mathbb{E}_i)^2}{E_i},$$

where O_i and E_i denote the observed and expected frequencies in bin i.

Bin	O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
1	8	20	7.2
2	12	20	3.2
3	20	20	0.0
4	30	20	5.0
5	32	20	7.2
6	18	20	0.2

Thus,

$$\chi^2 = 7.2 + 3.2 + 0 + 5.0 + 7.2 + 0.2 = 22.8.$$

d. At the 5% significance level, determine whether the data is consistent with a Uniform(0,1) distribution. (Use the critical value $\chi^2_{0.05,5}$=11.07).

The number of degrees of freedom is

$$\text{df} = k - 1 = 6 - 1 = 5.$$

At the 5% significance level, the critical value is

$$\chi^2_{0.05,5} = 11.07.$$

Since, $\chi^2 = 22.8 > \chi^2_{0.05,5}$, we reject the null hypothesis. Therefore, at the 5% significance level, the data are **not consistent** with a Uniform$(0, 1)$ distribution.

Question 6. Use Rejection Sampling to estimate the c.d.f of a probability distribution function given by I:

$$I = \int_0^{\pi/2} \sin(x)dx$$

a. Choose a suitable bounding rectangle (envelope).

On the interval $[0, \pi/2]$, $\sin x \in [0, 1]$. The tight rectangle envelope, base: $X \in [0, \pi/2]$ and height $Y \in [0, 1]$, giving an area, $A = \pi/2 \times 1 = \pi/2 = 1.57$.

b. Compute the exact value of the integral and that using Rejection sampling method.

Exact value of the integral: $I = \int_0^{\pi/2} \sin(x)dx = \cos x|_0^{\pi/2} = 1$

Value of the integral using rejection sampling method is computed as follows -
a) Sample $X \sim \text{Uniform}(0, \pi/2)$
b) Sample $Y \sim \text{Uniform}(0, 1)$
c) Accept the point if $Y \leq \sin X$
Let N, and N_{acc} be the total number of samples and the total number of accepted samples, respectively.

$$\frac{N_{acc}}{N} \simeq \frac{\int_0^{\pi/2} \sin x dx}{A} = \frac{\hat{I}}{\pi/2} \implies \hat{I} = \frac{\pi}{2} \times \frac{N_{acc}}{N}$$

Rejection sampling estimator is given by: $\hat{I} = \frac{\pi}{2} \times \frac{N_{acc}}{N}$. The acceptance probability is given by $\mathbb{P}(\text{accept}) = \frac{1}{\pi/2} \simeq 64\%$. Therefore, 64% of samples are accepted while the 36% are rejected.

c. Comment on efficiency.
A tight envelope leads to moderate efficiency, but rejection sampling is still less efficient than direct Monte Carlo integration, since some samples are discarded.

Question 7. Estimate the integral $I = \int_0^{\pi/2} x^2 e^x dx$, using Python implementation of Control variates method.
a. Choose the control variate: $Y = U^2$, where U is a uniform random number

$$I = \int_0^{\pi/2} x^2 e^x dx = \int_0^1 v^2 e^{v\pi/2} \frac{\pi^3}{8} dv$$
$$\text{assuming : x} = \frac{\pi}{2}\text{v}$$
$$I = \int_0^1 g(v)dv;$$
$$\text{where, g(v)} = \text{v}^2\text{e}^{\text{v}\pi/2}\frac{\pi^3}{8}$$

Now we introduce the control varaiate $Y \equiv h(U) = U^2$, with a known expected value:

$$\mathbb{E}(h(U)) = \int_0^1 U^2 dU = 1/3$$

b. Derive the optimal control coefficient (hint : 'c' or 'β')
The control variate estimator is defined as: $Z = g(v) + c\,[h(v) - \mu]$, where $\mathbb{E}(h(v)) = \mu$

$$\mathbb{V}ar(Z) = \mathbb{V}(g(v) + ch(v)) = \mathbb{V}ar(g(v)) + c^2\mathbb{V}ar(h(v)) + 2c\,\text{Cov(g(v), h(v))}$$

The optimum value of c is given by:

$$\frac{\partial}{\partial c}\mathbb{V}ar(Z) = 0 \implies c = -\frac{\mathrm{Cov(g(v), h(v))}}{\mathbb{V}ar(h(v))} = \beta$$

c. Find the mean, variance using simple MC method and mean, variance using control variate method and compare them.
Mean and variance using the *Monte Carlo* method:

$$I = \int_0^1 g(v)dv;$$

$$\text{where, } \mathrm{g(v) = v^2 e^{v\pi/2}}\frac{\pi^3}{8}$$

The integral is the expected value of $g(U) = \frac{\pi^3}{8}U^2 e^{U\pi/2}$, with $U \sim \mathcal{N}(0, 1)$. Using the sample of size n denote the points in the samples as

Mean and variance using the *control varaiate* $Y \equiv h(U) = U^2$, we estimate:

$$\mathbb{E}(h(U)) = \int_0^1 U^2 dU = 1/3$$

$$\mathbb{E}(h(U)^2) = \int_0^1 U^4 dU = 1/5$$

$$\mathbb{V}ar(h(U)) = \frac{1}{5} - (\frac{1}{3})^2 = 0.0889$$

d. Find the true value using of the integral by using integration by parts multiple times and compare with mean from above two methods.
True method: Using integration by parts

$$I = \int_0^{\pi/2} x^2 e^x dx = x^2 e^x|_0^{\pi/2} - \int_0^{\pi/2} 2xe^x dx = \frac{\pi^2}{4}e^{\pi/2} - 2\int_0^{\pi/2} xe^x dx$$

$$= \frac{\pi^2}{4}e^{\pi/2} - 2[\frac{\pi}{2}e^{\pi/2} - \int_0^{\pi/2} e^x dx] = \frac{\pi^2}{4}e^{\pi/2} - 2\frac{\pi}{2}e^{\pi/2} + 2e^{\pi/2} - 2 = 4.3778$$

Question 8. A cash-or-nothing digital call option pays a fixed cash amount Q at maturity if the underlying asset price is above the strike price. Otherwise, it pays nothing. You are required to price a digital call option using finite difference methods i.e. Explicit, Implicit and Crank Nicholson under the Black–Scholes framework.
Terms:
- Current stock price: S0=100
- Strike price: K=100
- Cash payout: Q=10
- Risk-free interest rate: r=5%
- Volatility: σ=20%
- Time to maturity: T = 1 year

Use below conditions:
a) Terminal Condition (Payoff)

$$V(s,t) = \begin{cases} Q & S \geq K \\ 0 & S < K \end{cases}$$

b) Boundary conditions:
At S = 0; V(0, t) = 0. At large stock price, $S = S_{max} : V(S_{max}, t) = Qe^{-r(T-t)}$

Given in the problem:

Under the Black–Scholes model, the price $V(S,t)$ of a derivative written on an underlying asset S satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad 0 < S < S_{\max},\ 0 \le t < T.$$

Key Parameters.

$$\begin{aligned} S_0 &= 100, \\ K &= 100, \\ Q &= 10, \\ r &= 0.05, \\ \sigma &= 0.20, \\ T &= 1. \end{aligned}$$

(a) Terminal Condition (Payoff). At maturity $t = T$,

$$V(S,T) = \begin{cases} Q, & S \ge K, \\ 0, & S < K. \end{cases}$$

(b) Boundary Conditions.

$$V(0,t) = 0,$$

and at a sufficiently large stock price $S = S_{\max}$,

$$V(S_{\max}, t) = Qe^{-r(T-t)}.$$

Numerical solving scheme: Spatial and temporal discretisation.

$$S_i = i\Delta S, \quad i = 0, 1, \ldots, M,$$

$$t_n = n\Delta t, \quad n = 0, 1, \ldots, N,$$

with

$$\Delta S = \frac{S_{\max}}{M}, \qquad \Delta t = \frac{T}{N}.$$

Let,

$$V_i^n \approx V(S_i, t_n).$$

Finite Difference Coefficients. For notational convenience, define

$$\begin{aligned} a_i &= \frac{1}{2}\Delta t \left(\sigma^2 i^2 - ri\right), \\ b_i &= 1 - \Delta t \left(\sigma^2 i^2 + r\right), \\ c_i &= \frac{1}{2}\Delta t \left(\sigma^2 i^2 + ri\right). \end{aligned}$$

Explicit Finite Difference Scheme The explicit scheme advances the solution as

$$V_i^{n+1} = a_i V_{i-1}^n + b_i V_i^n + c_i V_{i+1}^n.$$

Implicit Finite Difference Scheme

$$-a_i V_{i-1}^{n+1} + \left(1 + \Delta t(\sigma^2 i^2 + r)\right) V_i^{n+1} - c_i V_{i+1}^{n+1} = V_i^n.$$

This results in a tridiagonal linear system that must be solved at each time step.

Crank–Nicolson Scheme

The Crank–Nicolson scheme is obtained by averaging the explicit and implicit schemes:

$$\begin{aligned}
-\frac{1}{2}a_i V_{i-1}^{n+1} + \left(1 + \frac{1}{2}\Delta t(\sigma^2 i^2 + r)\right) V_i^{n+1} - \frac{1}{2}c_i V_{i+1}^{n+1} \\
= \frac{1}{2}a_i V_{i-1}^{n} + \left(1 - \frac{1}{2}\Delta t(\sigma^2 i^2 + r)\right) V_i^{n} + \frac{1}{2}c_i V_{i+1}^{n}.
\end{aligned}$$

Numerical Procedure.

(i) Initialise V_i^N using the terminal payoff.
(ii) Impose boundary conditions at each time level.
(iii) March backward in time from $t = T$ to $t = 0$.
(iv) Interpolate to obtain the option price $V(S_0, 0)$.