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Assignment 2

Question 1. Explain the concept of Filtration with an example.

Let Ω be a nonempty set and let $T > 0$ be a fixed number. Assume that for each $t \in [0, T]$ there is a $\sigma - algebra\, \mathcal{F}(t)$ on Ω. If for all $s, t \in [0, T]$ with $s \leq t$, we have:

$$\mathcal{F}(s) \subseteq \mathcal{F}(t),$$

then the collection of $\sigma - algebra$

$$\mathcal{F}(t)_{0 \leq t \leq T}$$

a filtration (on Ω). Alternatively, a filtration represents the information available as time t progresses. More precisely, at any time t, we will know for each set $A \in \mathcal{F}(t)$ whether the actual outcome $\omega \in \Omega$ belongs to A.
Example: Let's say we toss a fair coin twice. The sample space is $\Omega = \{HH, HT, TH, TT\}$. Then:

(i) at time $t = 0$: $\sigma - algebra\, \mathcal{F}_0 = \{\emptyset, \Omega\}$
(ii) at time $t = 1$: $\sigma - algebra\, \mathcal{F}_1 = \sigma(\{\text{first toss} = H\}) = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\}$
(iii) at time $t = 2$: $\sigma - algebra\, \mathcal{F}_2 = \mathcal{P}(\Omega) = \{\emptyset, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{TH, TT\}, \Omega\}$

Then the sequence $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ is filtration in $0 \leq t \leq T$.

Question 2. What are the 3 requirements for a sequence of random variables to become a Martingale? State similarly for sub and super Martingale.

Martingale: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be a filtration on this space, where $T > 0$ is fixed. Consider a stochastic process $\{M(t)\}_{0 \leq t \leq T}$ adapted to the filtration $\{\mathcal{F}_t\}$. The process $M(t)$ is said to be a *martingale* with respect to $\{\mathcal{F}_t\}$ if:

(i) $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$ for all $0 \leq s \leq t \leq T$, implying $M(t)$ has no tendency to rise or fall.
(ii) $M(t)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$,
(iii) $\mathbb{E}[|M(t)|] < \infty$ for all $t \in [0, T]$.

Submartingale: The process is submartingale, if $\mathbb{E}[M(t) \mid \mathcal{F}_s] \geq M(s)$ for all $0 \leq s \leq t \leq T$.

Supermartingale: The process is supermartingale, if $\mathbb{E}[M(t) \mid \mathcal{F}_s] \leq M(s)$ for all $0 \leq s \leq t \leq T$.

Question 3. Prove that a symmetric random walk is a Martingale.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{M_l\}_{l=0,1,2,\ldots}$ be a symmetric random walk, where l takes integer values. The process is defined as

$$M_l = \sum_{j=1}^{l} X_j, \quad l = 1, 2, 3, \ldots$$

with $\{X_j\}$ being a sequence of independent random variables such that

$$X_j = \begin{cases} +1, & \text{with probability } p, \\ -1, & \text{with probability } 1 - p. \end{cases}$$

To verify the martingale property, we express the conditional expectation of M_k using the σ-algebra notation, conditioning on the information available up to time k, i.e. with respect to $\mathcal{F}_k$.

For M_t to be martingale, the following condition has to be true: $\mathbb{E}[M_l|\mathcal{F}_k] = M_k$.

From the L.H.S of the above equation:

$$\begin{aligned}
\mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[M_l - M_k + M_k|\mathcal{F}_k] && \text{add and subtract } M_k. \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k] && \text{using linearity from } \sigma\text{- algebra.} \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k && \mathbb{E}[M_k|\mathcal{F}_k] = M_k, \text{because the } M_k \text{ depends on the first } k \text{ outcomes} \\
&= \mathbb{E}[M_l - M_k] + M_k = M_k && \text{first time is from the independence condition} \\
&= 0 + M_k = M_k \to \text{R.H.S}
\end{aligned}$$

Question 4. Prove that Brownian Motion is a Martingale.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with $\omega \in \Omega$. Then for each $\omega \in \Omega$, suppose $W(t)$ for $t \geq 0$. The process $W(t)$ is a Brownian motion if it satisfies the following condition:

(i) $W(0) = 0$

(ii) All increments, $(W(t_1) - W(t_0)), (W(t_2) - W(t_1)), \ldots (W(t_m) - W(t_{m-1}))$ are independent of each other

(iii) For all the above increments, we have: $\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$ and $\text{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i$

(iv) $W(t)$ and all the increments: $(W(t_1) - W(t_0)), (W(t_2) - W(t_1)), \ldots (W(t_m) - W(t_{m-1}))$ are normally distributed

For Brownian motion to be a martingale, we need to show $\mathbb{E}[W(t)|\mathcal{F}(s)] = W(s)$.
From the L.H.S of the above equation:

$$\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[W(t) - W(s) + W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)] && \text{using the condition for linearity} \\
&= \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + W(s) && \text{as } \mathbb{E}[W(s)|\mathcal{F}(s)] = W(s) \\
&= \mathbb{E}[W(t) - W(s)] + W(s) && \text{the first terms is due to the independence condition} \\
&= 0 + W(s) = W(s) \to \text{R.H.S}
\end{aligned}$$

Question 5. A student is taking a probability class and in each week, he can be either up-to-date or he may have fallen behind. If he is up-to-date in a given week, the probability that he will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If he is behind in the given week, the probability that he will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). Find the Single and 2 step state Transition Matrix for this Markov chain.

Students can have two states - $\{S_1, S_2\} = \{\text{up to date, behind}\}$

If the student is up-to-date this week, i.e., in state S_1:

- probability that he will be up-to-date (in state S_1) in the next week is 0.8
- probability that he will be behind (in state S_2) in the next week is 0.2

If the student is behind in the given week, i.e., in state S_2:

- probability that he will be up-to-date (in state S_1) in the next week is 0.6
- probability that he will be behind (in state S_2) in the next week is 0.4

The single state transition matrix:

$$P = \begin{bmatrix} P(S_1 \to S_1) & P(S_1 \to S_2) \\ P(S_2 \to S_1) & P(S_2 \to S_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}.$$

The 2-step transition matrix:

$$P^2 = P \times P$$

$$P^2 = \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}.$$