CSC111 Lecture 16: Graph Connectivity and Spanning Trees

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1 Exercise 1: Proving "Lemma 2"

Here's the lemma we just saw in lecture:

Lemma 2: Let $G = (V, E)$ be a graph. If there exists an edge $e \in E$ such that $G - e$ is connected, then that edge e is in a cycle in G, the original graph.

Prove this lemma. *Hint*: the proof body is actually quite short, and can again be written in English! Take the edge e from the assumption and label its endpoints u and v. What can you say about these two vertices in the graph $G - e$?

Proof.

Let $G = (V, E)$ be a graph, and assume there exists an edge e such that $G - e$ is connected. We want to prove e is in a cycle in G.

Let u and v be the endpoints of e. Since we assumed that $G - e$ is connected, there must be a path between u and v in $G - e$. So then let the path equal $u, v_1, v_2, v_3, \ldots, v$

Then the sequence $u, v_1, v_2, v_3, \ldots, v, u$ is a cycle in G that contains e. ■

2 Exercise 2: Proving "Theorem 2 (number of edges in a tree)"

Here's the theorem we just saw in lecture:

Theorem 2 (number of edges in a tree). Let $G = (V, E)$ be a tree with at least one vertex. Then $|E| = |V| - 1$.

Translation into predicate logic, using a variable n:

$$\forall n \in \mathbb{Z}^+,\ \forall G = (V, E), \big(|V| = n \wedge G \text{ is a tree}\big) \implies |E| = n - 1$$

The induction predicate $P(n)$ translates to, "Every tree with n vertices has $n-1$ edges".

In this exercise, you'll prove this theorem using induction on n, the number of vertices in the tree. We've started the proof structure for you.

Proof.

Base case: let $n = 1$.

Let $G = (V, E)$ be a graph, and assume $|V| = 1$ and G is a tree. We want to prove that $|E| = 1 - 1 = 0$

Since there is only one vertex, there cannot be any edges. So $|E| = 0$, as needed ■.

Induction step: let $k \in \mathbb{Z}^+$ and assume that $P(k)$ holds, i.e., every tree with k vertices has $k-1$ edges. We need to prove that $P(k+1)$ is true.

[NOTE: you may assume that every tree with ≥ 2 vertices has at least one vertex of degree 1. Consider removing such a vertex.]

Let $G = (V, E)$. Assume G is a tree, and $|V| = k + 1$. We want to prove that $|E| = k$

Let v be a vertex with degree 1 (from the NOTE above).

Let $G' = (V', E')$ be the graph obtained by removing v from G.

Then G' has no cycles (because G had no cycles), and is still connected (as we only removed v which had degree 1).

This means that G' is a tree. Also it has k vertices, since we just removed 1 vertex from G. By the induction hypothesis, $|E'| = k - 1$.

And hten for G since we removed one edge, we have:

$|E| = |E'| + 1 = (k - 1) + 1 = k$ ■

3 Additional exercises

1. Prove that every tree with ≥ 2 vertices has at least one vertex of degree 1. (*Hint*: consider a path of maximum length in the tree. Pick an endpoint, and prove that it must have degree 1.)

2. Prove that every tree with ≥ 2 vertices has at least *two* vertices of degree 1.

3. Prove or disprove: every tree with ≥ 2 vertices has at least *three* vertices of degree 1.