CSC236 Lecture 04: Complete Induction 2

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1 Zero Pair-Free Binary Strings

Deonte by $zpfbs\left(n\right)$ the number of binary strings of length n That contain no paris of adjacent zeros. What is $zpfbs\left(n\right)$ for the first few natural numbers n?

$$\begin{aligned}
zpfbs\left(0\right) &= 1\\
zpfbs\left(1\right) &= 2\\
zpfbs\left(2\right) &= 3\\
zpfbs\left(3\right) &= 5\\
zpfbs\left(4\right) &= 8\\
zpfbs\left(5\right) &= 13\\
&\cdots\\
zpfbs\left(n\right) &= zpfbs\left(n-1\right) + zpfbs\left(n-2\right)
\end{aligned}$$

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n+2), & n > 1 \end{cases}$$

$\forall n \in \mathbb{N}$, defined predicate $P(n)$ as: $f(n) = zpfbs\left(n\right)$

Prove by complete induction that for all natural numbers n, $P(n)$.

Let $n \in \mathbb{N}$. Assume that P is true for $0, \ldots, n-1$. I will show that $P(n)$ follows.

For the case $n \geq 2$: Partition the zero-pair-free binary strings of length n into those that end in 1 and those that end in 0. Those that end in 1 are simply those of length $n-1$. with a 1 appended, and by $P(n-1)$ (since $n-1 < n$ and $n-1 \geq 0$, $n \geq 1$), there are $f(n-1)$ of these. Those that end in 0 must actually end in 10 (otherwise they are a zero-pair), and by $P(n-2)$ (since $n-2 \leq n$ and $n-2 \geq 0$, $n \geq 2$), there are $f(n-2)$ of these. Altogether there are $f(n-1)+f(n-2)$ zero-pair-free binary strings of length n when $n \geq 2$, which is $P(n)$.

For the base case $n = 0$: There is one binary string (the empty one) of length 0, and it is zero-pair-free, and $f(0) = 1$ and $P(0)$ is true.

For the base case $n = 1$: There are two binary strings of length 1, and neither have pairs of zeros, and $f(1) = 2$ so $P(1)$ is true.

Thus in all possible cases, $P(n)$ follows.

2 Every natural number greater than 1 has a prime factorization

Each natural number n, let predicate $P(n)$ be: n can be expressed as a product of primes.

Prime factorization: represent as product of 1 or more primes.

Prove by complete induction that for all natural numbers n, $P(n)$.

Let $n \in \mathbb{N}$ s.t. $n > 1$. Assume P is true for $2, \ldots, n-1$. I will show that $P(n)$ follows.

Case n is composite: By definition, n has a natural number factor f_1 such that $1 < f_1 < n$. By $P(f_1)$ (since $1 < f_1 < n$) we know f_1 can be expressed as a product of primes. Let $f_2 = \frac{n}{f_1}$, since $f_1 > 1$, we know that $\frac{n}{f_1} < n$, and also since $f_1 < n$, we know that $\frac{n}{f_1} > \frac{f_1}{f_1} = 1$. Since $f_1 > 1$, then $f_1 = \frac{n}{f_2} > 1$, so $n > f_2$. So, by $P(f_2)$, we know that f_2 can be expressed as a product of primes. Therefore since f_1 and f_2 are both products of primes, $n = f_1 \times f_2$ is a product of primes and $P(n)$ follows.

Case n is prime: Then n is its own primes factorization, and $P(n)$ follows.

In all possible cases, $P(n)$ follows.