

# CSC236 Week 01: Introduction and Basic Induction

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## Contents

<b>1</b>	<b>Why reason about computing</b>	<b>1</b>
<b>2</b>	<b>How to reason about computing</b>	<b>2</b>
<b>3</b>	<b>How to do well in this course</b>	<b>2</b>
<b>4</b>	<b>Assume that you already know</b>	<b>2</b>
<b>5</b>	<b>By December you'll know</b>	<b>2</b>
<b>6</b>	<b>domino fates foretold</b>	<b>2</b>
<b>7</b>	<b>Simple induction outline</b>	<b>3</b>
<b>8</b>	<b>Trominoes</b>	<b>3</b>
<b>9</b>	<b><math>3^n \geq n^3</math>?</b>	<b>4</b>
9.1	Scratch Work . . . . .	4
9.2	Simple Induction . . . . .	4

## 1 Why reason about computing

- You're not just hackers anymore  
Sometimes you need to analyze code before it runs. Sometimes it should never be run!
- Can you test everything?  
Infinitely many inputs: integers, strings, lists.
- Careful, you might get to like it. . . (!\*)

## 2 How to reason about computing

- It's messy...  
interesting problems fight back.  
You need to draft, re-draft, and re-re-draft.  
You need to follow blind alleys until you find a solution.  
You can also find a solution that isn't wrong, but could be better.
- It's art...  
Strive for correctness, clarity, surprise, humor, pathos, and others.

## 3 How to do well in this course

- read the syllabus as a two-way promise
- question, answer, record, synthesize  
try annotating blank slides.
- collaborate with respect  
You need computer science friends who are respectful and constructively critical.

## 4 Assume that you already know

- Chapter 0 material from *Introduction to Theory of Computation*.
- CSC110/111 material, especially proofs and big- $\mathcal{O}$ .

## 5 By December you'll know

- understand and use several flavours of induction.  
some of these flavours will taste new
- Formal languages, regular languages, regular expressions  
Sets of strings
- complexity and correctness of programs — both recursive and iterative

## 6 domino fates foretold

$$[P(0) \wedge (\forall n \in \mathbb{N}, P(n) \implies P(n+1))] \implies \forall n \in \mathbb{N}, P(n)$$

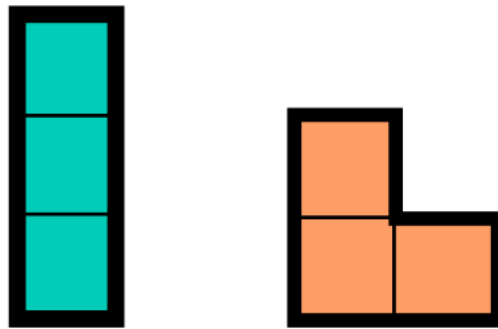
If the initial case works, and each case that works implies its successor works, then all cases work

## 7 Simple induction outline

- inductive step: introduce  $n$  and inductive hypothesis  $H(n)$ 
  - derive conclusion  $C(n)$ : show that  $C(n)$  follows from  $H(n)$ , indicating **where** you use  $H(n)$  and why that is valid.
- Verify base case(s): verify that the claim is true for any cases not covered in the inductive step
- In simple induction  $C(n)$  is just  $H(n + 1)$

## 8 Trominoes

See <https://en.wikipedia.org/wiki/Tromino>



Can an  $n \times n$  square grid, with one subsquare removed, be tiled (covered without overlapping) by “chair” trominoes?

- $1 \times 1$ : Yes.
- $2 \times 2$ : Yes.
- $3 \times 3$ : No. The remaining number of squares is not divisible by 3.
- $4 \times 4$ : Yes.

*Proof:*  $\forall n \in \mathbb{N}$ , define the predicate  $P(n)$  as a  $2^n \times 2^n$  square grid, with one subsquare removed, can be tiled (covered without overlapping) by “chair” trominoes.

- Induction on  $n$

Let  $n$  be an arbitrary, fixed, natural number. (Let  $n \in \mathbb{N}$ ).

Assume  $P(n)$ , that is a  $2^n \times 2^n$  grid, with one square removed can be tiled with "chairs."

I will prove  $P(n + 1)$ , that is a  $2^{n+1} \times 2^{n+1}$  grid, with one square removed can be tiled by chairs.

Let  $G$  be a  $2^{n+1} \times 2^{n+1}$  grid with one square removed. Notice that  $G$  can be decomposed into four  $2^n \times 2^n$  disjoint quadrant grids. We may assume, WLOG (without loss of generality)

that the missing square is in the upper-right quadrant, since otherwise just rotate it there, and rotate back when done. By  $P(n)$  I can tile the upper-right quadrant, minus the missing square. By  $P(n)$  3 more times, I can tile the remaining 3 quadrants, omitting for a moment the 3 tiles nearest the centre of  $G$ , with chairs. The briefly omitted squares form a chair! So I complete the tiling by adding one more chair. Thus  $P(n+1)$ .

- Base Case

A  $2^0 \times 2^0$  grid, with one square removed, is just empty space! This can be tiled with 0 chairs. So  $P(0)$  is true.

And thus  $\forall n \in \mathbb{N}, P(n)$ . ■

## 9 $3^n \geq n^3$ ?

### 9.1 Scratch Work

Check for a few values of  $n$ :

$3^0 = 1 \geq 0 = 0^3$	✓
$3^1 = 3 \geq 1 = 1^3$	✓
$3^2 = 9 \geq 8 = 2^3$	✓
$3^3 = 27 \geq 27 = 3^3$	✓
$3^4 = 81 \geq 64 = 4^3$	✓
$3^{-1} = \frac{1}{3} \geq -1 = -1^3$	✓
$3^{2.5} < 2.5^3$	×

### 9.2 Simple Induction

*Proof:*  $\forall n \in \mathbb{N}$ , define the predicate  $P(n)$  as  $3^n \geq n^3$ .

- Induction on  $n$

Let  $n \in \mathbb{N}$ . Assume  $H(n) : 3^n \geq n^3$ . I will prove  $H(n+1)$  follows, that is  $3^{n+1} \geq (n+1)^3$ .

$$\begin{aligned}
& 3^{n+1} \\
&= 3 \cdot 3^n \\
&\geq 3 \cdot n^3 \\
&= n^3 + n^3 + n^3 \\
&\geq n^3 + 3n^2 + 9n && (\text{since } n \geq 3) \\
&\geq n^3 + 3n^2 + 3n + 6n \\
&= n^3 + 3n^2 + 3n + 1 && (\text{since } 6n \geq 1) \\
&= (n+1)^3
\end{aligned}$$

And thus we have shown that  $\forall n \in \mathbb{N}$  s.t.  $n \geq 3, H(n) \implies H(n+1)$ .

- Base Case

$3^3 \geq 3^3$  so  $P(3)$  holds.

$3^2 \geq 2^3$  so  $P(2)$  holds.

$3^1 \geq 1^3$  so  $P(1)$  holds.

$3^0 \geq 0^3$  so  $P(0)$  holds.

And thus, we have shown  $\forall n \in \mathbb{N}, 3^n \geq n^3$ , as needed. ■

# CSC236 Week 02: Complete Induction

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## Contents

<b>1</b>	<b>Complete Induction</b>	<b>1</b>
<b>2</b>	<b>Notational Convenience</b>	<b>2</b>
<b>3</b>	<b>More dominoes</b>	<b>2</b>
<b>4</b>	<b>Complete induction outline</b>	<b>2</b>
<b>5</b>	<b>Watch the base cases, part 1</b>	<b>2</b>
<b>6</b>	<b>For all natural numbers <math>n &gt; 1</math>, <math>f(n)</math> is a multiple of 3?</b>	<b>3</b>
<b>7</b>	<b>Zero pair-free binary strings</b>	<b>3</b>
<b>8</b>	<b>Every natural number greater than 1 has a prime factorization</b>	<b>4</b>

## 1 Complete Induction

- Every natural number greater than 1 has a prime factorization

$$2 = 2$$

$$3 = 3$$

$$4 = 2 \times 2$$

$$5 = 5$$

$$6 = 2 \times 3$$

$$7 = 7$$

$$8 = 2 \times 2 \times 2$$

$$9 = 3 \times 3$$

$$10 = 2 \times 5$$

- How does the factorization of 8 help with the factorization of 9?

The fact that 8 can be expressed as a product of primes has nothing to do with 9 being a product of primes.

## 2 Notational Convenience

Sometimes you will see the following:

$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \implies P(k)$$

## 3 More dominoes

$$\left( \forall n \in \mathbb{N}, \left[ \bigwedge_{k=0}^{k=n-1} P(k) \right] \implies P(n) \right) \implies \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true.

## 4 Complete induction outline

- **Inductive step:** introduce  $n$  and state inductive hypothesis  $H(n)$ 
  - **Derive conclusion**  $C(n)$ : show that  $C(n)$  follows from  $H(n)$ , indicating where you use  $H(n)$  and why that is valid.
- **Verify base case(s):** verify that the claim is true for any cases not covered in the inductive step

This is the same outline as simple induction but we modify the inductive hypothesis,  $H(n)$  so that it assumes the main claim for every natural number from the starting point up to  $n - 1$ , and the conclusion,  $C(n)$  is now the main claim for  $n$ .

## 5 Watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(\lfloor \sqrt{n} \rfloor)]^2 + 2f(\lfloor \sqrt{n} \rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture:

$$\begin{aligned}f(0) &= 1 \\f(1) &= 1 \\f(2) &= 3 \\f(3) &= 3 \\f(4) &= 15 \\f(5 \dots 15) &= 15 \\f(16) &= 255\end{aligned}$$

All of these things are divisible by 3. The square of something that is divisible by 3 is still divisible by 3 and the double of something that is divisible by 3 is still divisible by 3.

## 6 For all natural numbers $n > 1$ , $f(n)$ is a multiple of 3?

*Proof:*  $\forall n \in \mathbb{N}$ , define the predicate  $P(n)$  as  $f(n)$  is a multiple of 3.

I will prove, using complete induction, that  $\forall n > 1, P(n)$ .

- Induction on  $n$ :

Let  $n \in \mathbb{N}$ . Assume  $n > 1$ . Also assume that  $P(k)$  is true for all natural numbers  $k$  less than  $n$ , and greater than 1.

Notice that the floor of the square root of  $n$  is greater than 1. Also, the square root of  $n$  is less than  $n$  (since  $n > 1 \implies n^2 > n \implies n > \sqrt{n}$ ).

Thus by the induction hypothesis, I have  $P(\lfloor \sqrt{n} \rfloor)$ , this number is a multiple of 3.

Let  $k \in \mathbb{N}$  s.t.  $\lfloor \sqrt{n} \rfloor = 3k$ , so  $f(n) = (3k)^2 + 2(3k) = 3(3k^2 + 2k)$ , a multiple of 3.

So  $P(n)$  follows in both possible cases.

- Base Cases:

$P(2)$  claims that  $f(2) = 3$  is a multiple of 3, which is true.

$P(3)$  claims that  $f(3) = 3$  is a multiple of 3, which is true.

And thus  $\forall n \in \mathbb{N}$  s.t.  $n \geq 2, P(n)$ . ■

**Note:** We have to include  $P(3)$  as a base case because  $\lfloor \sqrt{3} \rfloor$  is not 2, but 1.

## 7 Zero pair-free binary strings

Deonte by  $zpfbs(n)$  the number of binary strings of length  $n$  That contain no pairs of adjacent zeros. What is  $zpfbs(n)$  for the first few natural numbers  $n$ ?



$$\begin{aligned}
zpbs(0) &= 1 \\
zpbs(1) &= 2 \\
zpbs(2) &= 3 \\
zpbs(3) &= 5 \\
zpbs(4) &= 8 \\
zpbs(5) &= 13 \\
&\dots \\
zpbs(n) &= zpbs(n-1) + zpbs(n-2)
\end{aligned}$$

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n-2), & n > 1 \end{cases}$$

$\forall n \in \mathbb{N}$ , defined predicate  $P(n)$  as:  $f(n) = zpbs(n)$

Prove by complete induction that for all natural numbers  $n$ ,  $P(n)$ .

*Proof:* Let  $n \in \mathbb{N}$ . Assume that  $P$  is true for  $0, \dots, n-1$ . I will show that  $P(n)$  follows.

For the case  $n \geq 2$ : Partition the zero-pair-free binary strings of length  $n$  into those that end in 1 and those that end in 0. Those that end in 1 are simply those of length  $n-1$  with a 1 appended, and by  $P(n-1)$  (since  $n-1 < n$  and  $n-1 \geq 0$ ,  $n \geq 1$ ), there are  $f(n-1)$  of these. Those that end in 0 must actually end in 10 (otherwise they are a zero-pair), and by  $P(n-2)$  (since  $n-2 \leq n$  and  $n-2 \geq 0$ ,  $n \geq 2$ ), there are  $f(n-2)$  of these. Altogether there are  $f(n-1) + f(n-2)$  zero-pair-free binary strings of length  $n$  when  $n \geq 2$ , which is  $P(n)$ .

For the base case  $n = 0$ : There is one binary string (the empty one) of length 0, and it is zero-pair-free, and  $f(0) = 1$  and  $P(0)$  is true.

For the base case  $n = 1$ : There are two binary strings of length 1, and neither have pairs of zeros, and  $f(1) = 2$  so  $P(1)$  is true.

Thus in all possible cases,  $P(n)$  follows. ■

## 8 Every natural number greater than 1 has a prime factorization

$\forall n \in \mathbb{N}$ , define the predicate  $P(n)$  as:  $n$  can be expressed as a product of primes.

Prime factorization: represent as product of 1 or more primes.

Prove by complete induction that for all natural numbers  $n$ ,  $P(n)$ .

*Proof:* Let  $n \in \mathbb{N}$  s.t.  $n > 1$ . Assume  $P$  is true for  $2, \dots, n-1$ . I will show that  $P(n)$  follows.

Case  $n$  is composite: By definition,  $n$  has a natural number factor  $f_1$  such that  $1 < f_1 < n$ . By  $P(f_1)$  (since  $1 < f_1 < n$ ) we know  $f_1$  can be expressed as a product of primes. Let  $f_2 = \frac{n}{f_1}$ , since  $f_1 > 1$ , we know that  $\frac{n}{f_1} < n$ , and also since  $f_1 < n$ , we know that  $\frac{n}{f_1} > \frac{f_1}{f_1} = 1$ . Since  $f_1 > 1$ , then  $f_1 = \frac{n}{f_2} > 1$ , so  $n > f_2$ . So, by  $P(f_2)$ , we know that  $f_2$  can be expressed as a product of primes.

Therefore since  $f_1$  and  $f_2$  are both products of primes,  $n = f_1 \times f_2$  is a product of primes and  $P(n)$  follows.

Case  $n$  is prime: Then  $n$  is its own primes factorization, and  $P(n)$  follows.

In all possible cases,  $P(n)$  follows. ■

# CSC236 Week 03: Recurrences, Structural Induction

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September 23 – 29, 2021

## Contents

<b>1</b>	<b>Recursively defined function</b>	<b>1</b>
<b>2</b>	<b>Prove that <math>\forall n \in \mathbb{N}, f(n) = f'(n)</math></b>	<b>2</b>
<b>3</b>	<b>Define sets inductively</b>	<b>2</b>
<b>4</b>	<b>What can you do with it?</b>	<b>3</b>
<b>5</b>	<b>Other structurally-defined sets</b>	<b>3</b>
<b>6</b>	<b>Structural induction</b>	<b>3</b>
<b>7</b>	<b>Structural induction proofs</b>	<b>4</b>
7.1	Prove $\forall e \in \mathcal{E}, \text{vr}(e) = \text{op}(e) + 1$ . . . . .	4
7.2	Prove $\forall e \in \mathcal{E}, \text{vr}(e) \leq 2^{h(3)}$ . . . . .	4

## 1 Recursively defined function

Recall:

$$f(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ f(n-1) + f(n-2) & n > 1 \end{cases}$$

For comparison, let

$$r_1 = \frac{1 + \sqrt{5}}{2}, r_2 = \frac{1 - \sqrt{5}}{2}, a = \frac{\sqrt{5} + 3}{2\sqrt{5}}, b = \frac{\sqrt{5} - 3}{2\sqrt{5}}$$

... and define

$$f'(n) = ar_1^n + br_2^n$$

Compare the first few values of  $f(n)$  to the first few values of  $f'(n)$

$$\begin{aligned} f(0) &= 1, & f'(0) &= 1 \\ f(1) &= 2, & f'(1) &= 2 \\ f(2) &= 5, & f'(2) &= 5 \\ f(3) &= 8, & f'(3) &= 8 \\ f(4) &= 13, & f'(4) &= 13 \end{aligned}$$

## 2 Prove that $\forall n \in \mathbb{N}, f(n) = f'(n)$

$\forall n \in \mathbb{N}$ , define  $P(n)$  by  $f(n) = f'(n)$ . Prove by complete induction that  $\forall n \in \mathbb{N}, P(n)$ . It helps to verify  $1 + r_1 = r_1^2$  and  $1 + r_2 = r_2^2$ .

*Proof:* Let  $n \in \mathbb{N}$ . Assume the induction hypothesis,  $\forall k \in \mathbb{N}$  s.t.  $k < n$  has  $f(k) = f'(k)$ .

- Case  $n > 1$ :

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) && \text{(by definition)} \\ &= ar_1^{n-1} + br_2^{n-1} + ar_1^{n-2} + ar_2^{n-2} && \text{(by IH, since } 0 \leq n-2, n-1 < n) \\ &= a(r_1^{n-2} + r_1^{n-1}) + b(r_2^{n-2} + r_2^{n-1}) \\ &= a(r_1^{n-2})(1 + r_1) + b(r_2^{n-2})(1 + r_2) \\ &= a(r_1^{n-2})(r_1^2) + b(r_2^{n-2})(r_2^2) \\ &= ar_1^n + br_2^n \\ &= f'(n) \end{aligned}$$

And thus  $\forall n > 1, P(n)$  holds.

- Case  $n = 0$ :  $f(0) = 1 = f'(0)$ , by evaluating  $f'(0)$  so  $P(0)$  holds.
- Case  $n = 1$ :  $f(1) = 2 = f'(1)$ , by evaluating  $f'(1)$  so  $P(1)$  holds.

Thus, in all possible cases,  $P(n)$  holds. ■

## 3 Define sets inductively

One way to define the natural numbers:

$\mathbb{N}$ : The smallest set such that

1.  $0 \in \mathbb{N}$
2.  $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

By **smallest** we mean  $\mathbb{N}$  has no proper subsets that satisfy these two conditions. If we leave out **smallest**, what other sets satisfy the definition?  $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}$ , etc.

## 4 What can you do with it?

The definition in §3 defined the simplest natural number (0) and the rule to produce new natural numbers from old ones (add 1). Proof using Mathematical Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is

1. Show that  $P(0)$  is true for basis, 0.
2. Prove that  $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$ .

## 5 Other structurally-defined sets

Define  $\mathcal{E}$ : The smallest set such that

1.  $x, y, z \in \mathcal{E}$
2.  $e_1, e_2 \in \mathcal{E} \implies (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2), (e_1 \div e_2) \in \mathcal{E}$

Form some expressions in  $\mathcal{E}$ . Count the number of variables (symbols from  $\{x, y, z\}$ ) and the number of operators symbols from  $\{+, -, \times, \div\}$ . Make a conjecture:

$$(x + y), (x \times y) \in \mathcal{E}$$

$$((x + y) \times (x \times y)) \in \mathcal{E}$$

Let  $\text{vr}(e)$  count the number of variables in  $e$  and let  $\text{op}(e)$  count the number of operators in  $e$ .

$$\forall e \in \mathcal{E}, \text{vr}(e) = \text{op}(e) + 1$$

## 6 Structural induction

To prove that a property is true for all  $e \in \mathcal{E}$ , parallel the recursive set definition:

- **Verify the base case(s):** Show that the property is true for the simplest members,  $\{x, y, z\}$ , that is show  $P(x), P(y)$ , and  $P(z)$ .
- **Inductive step:** Let  $e_1$  and  $e_2$  be arbitrary elements of  $\mathcal{E}$ . Assume  $H(\{e_1, e_2\})$ :  $P(e_1)$  and  $P(e_2)$ , that is  $e_1$  and  $e_2$  have the property.
  - **Show that  $C(\{e_1, e_2\})$  follows:**  
All possible combinations of  $e_1$  and  $e_2$  have the property, that is  $P((e_1 + e_2)), P((e_1 - e_2)), P((e_1 \times e_2)), P((e_1 \div e_2))$ .

## 7 Structural induction proofs

### 7.1 Prove $\forall e \in \mathcal{E}, \text{vr}(e) = \text{op}(e) + 1$

*Proof:* For every  $e \in \mathcal{E}$ , define  $P(e)$  as  $\text{vr}(e) = \text{op}(e) + 1$ . Verify the basis: Let  $t \in \{x, y, z\}$ . The  $\text{vr}(t) = 1 = 0 + 1 = \text{op}(t) + 1$ , so  $P(t)$  holds for every element of the basis.

Let  $e_1, e_2 \in \mathcal{E}$ , and assume  $P(e_1)$  and  $P(e_2)$ . Want to prove that  $P((e_1 \odot e_2))$  where  $\odot \in \{+, -, \times, \div\}$ , that is  $\text{vr}((e_1 \odot e_2)) = \text{op}((e_1 \odot e_2)) + 1$ .

$$\begin{aligned} \text{vr}((e_1 \odot e_2)) &= \text{vr}(e_1) + \text{vr}(e_2) \\ &= \text{op}(e_1) + 1 + \text{op}(e_2) + 1 && \text{(by } P(e_1) \text{ and } P(e_2)) \\ &= [\text{op}(e_1) + \text{op}(e_2)] + 1 \\ &= \text{op}(e_1 \odot e_2) + 1 \end{aligned}$$

So  $P((e_1 \odot e_2))$  follows. ■

### 7.2 Prove $\forall e \in \mathcal{E}, \text{vr}(e) \leq 2^{h(e)}$

Define the heights,  $h(x) = h(y) = h(z) = 0$ , and  $h((e_1 \odot e_2))$  as  $1 + \max(h(e_1), h(e_2))$  if  $e_1, e_2 \in \mathcal{E}$  and  $\odot \in \{+, -, \times, \div\}$ . What's the connection between the unnumber of variables and the height?

$$\begin{aligned} h(x) &= 0, \text{vr}(x) = 1 \\ h((x + y)) &= 1, \text{vr}((x + y)) = 2 \\ h(((x + y) - z)) &= 2, \text{vr}(((x + y) - z)) = 3 \\ h((((x + y) - (z \times x))) &= 3, \text{vr}((((x + y) - (z \times x))) = 8 \end{aligned}$$

*Proof:* For every  $e \in \mathcal{E}$  define  $P(e)$  as  $\text{vr}(e) \leq 2^{h(e)}$ . Let  $a \in \{x, y, z\}$ . Then  $a$  has one variable (itself), and no operators so  $\text{vr}(a) = 1 = 2^0 = 2^{h(a)}$  since  $h$  of any single variable is 0. So  $P(a)$  holds for  $a \in \{x, y, z\}$  (the basis).

Let  $e_1, e_2 \in \mathcal{E}$ . Assume  $P(e_1), P(e_2)$  i.e.  $\text{vr}(e_1) \leq 2^{h(e_1)}$  and  $\text{vr}(e_2) \leq 2^{h(e_2)}$ . We will show that  $P(e_1 \odot e_2)$  is true, where  $\odot \in \{+, -, \times, \div\}$ .

$$\begin{aligned} \text{vr}(e_1 \odot e_2) &= \text{vr}(e_1) + \text{vr}(e_2) \\ &\leq 2^{h(e_1)} + 2^{h(e_2)} && \text{(by } P(e_1) \text{ and } P(e_2)) \\ &\leq 2 \cdot 2^{\max(h(e_1), h(e_2))} \\ &= 2^{1 + \max(h(e_1), h(e_2))} \\ &= 2^{h((e_1 \odot e_2))} && \text{(by definition of } h) \end{aligned}$$

So  $P((e_1 \odot e_2))$  follows. ■

# CSC236 Week 04: Well-Ordering, Induction Pitfalls

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September 30 – October 6, 2021

## Contents

<b>1</b>	<b>Principle of well-ordering</b>	<b>1</b>
<b>2</b>	<b>Well-ordering example</b>	<b>2</b>
<b>3</b>	<b><math>P(n)</math>: Every round-robin tournament with <math>n</math> players with a cycle has a 3-cycle.</b>	<b>2</b>
<b>4</b>	<b>No basis?</b>	<b>3</b>
<b>5</b>	<b>Zero pair free binary strings, again but with well-ordering</b>	<b>4</b>
<b>6</b>	<b>How not to do simple induction</b>	<b>4</b>

## 1 Principle of well-ordering

Every non-empty subset of  $\mathbb{N}$  has a smallest element

- How would you prove this for some  $S \subseteq \mathbb{N}$ ?  
Since  $\mathbb{N}$  is bounded below by 0 and  $0 \in \mathbb{N}$ , we know that any subset  $S$  of  $\mathbb{N}$  is also bounded below by 0. Since it is bounded below, we can say that there is a smallest element.
- Is there something similar for  $\mathbb{Q}$  or  $\mathbb{R}$ ?  
No. For example,  $(0, 1) \subseteq \mathbb{Q}, \{\frac{1}{n} : n \in \mathbb{N}^+\}$ .
- Here is the main part of proving the existence of a unique quotient and remainder

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^+, \exists q, r \in \mathbb{N} \text{ s.t. } m = qn + r \wedge 0 \leq r < n$$

The course notes use Mathematical Induction. Well-ordering seems shorter and clearer.

## 2 Well-ordering example

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^i, R(m, n) = \{r \in \mathbb{N} : \exists q \in \mathbb{N} \text{ s.t. } m = qn + r\}$  has a smallest element.

For a given pair natural numbers  $m, n \neq 0$  does the set  $R(m, n)$  satisfy the conditions for well-ordering?

$$R(m, n) = \{r \in \mathbb{N} : \exists q \in \mathbb{N} \text{ s.t. } m = qn + r\}$$

If so, we still need to be sure that the smallest element,  $r'$  has

1.  $0 \leq r' < n$
2. That  $q'$  and  $r'$  are unique — no other pair of natural numbers would work

... in order to have

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^+, \exists! q', r' \text{ s.t. } m = q'n + r' \wedge 0 \leq r' < n$$

*Proof.* Let  $m \in \mathbb{N}, n \in \mathbb{N}^+$ . The  $R(m, n)$  is a non-empty set of natural numbers. Then there is a smallest element of  $R(m, n)$ , let it be  $r'$ , with corresponding  $q'$  such that  $m = q'n + r'$ .

- Case  $r' < n$ : This automatically satisfies our requirements.
- Case  $r' \geq n$ : Then we can show that there is another  $r''$  such that  $r'' < r'$  and  $r'$  is not the smallest element of the set.

$$\begin{aligned} m &= q'n + r' && \text{(by choice of } r') \\ &= (q' + 1)n + (r' - n) \end{aligned}$$

Let  $r'' = r' - n, q'' = q' - 1$ . Then  $r'' \in R(m, n)$ , since  $m = q''n + r''$

Then  $r'' < r'$  which is a contradiction, and thus the initial assumption that  $r' \geq n$  is false.

So we have a unique pair of natural numbers  $(r', q')$  such that  $m = q'n + r'$  and  $0 \leq r' < n$ . ■

## 3 $P(n)$ : Every round-robin tournament with $n$ players with a cycle has a 3-cycle.

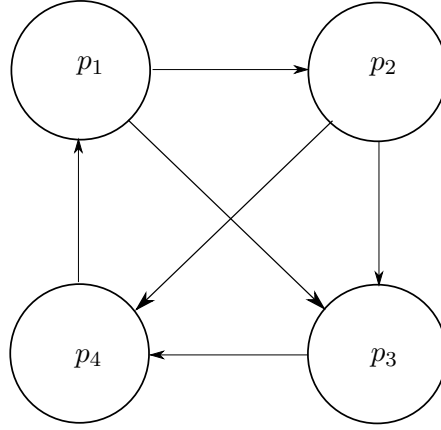
Use: every non-empty subset of  $\mathbb{N}$  has a smallest element

Think of a round-robin as a complete oriented graph  $G = (V, E)$ , where  $V = \{p_1, \dots, p_n\}$  and every pair of distinct vertices has a uniquely directed edge between them.

If there is a cycle  $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow \dots \rightarrow p_n \rightarrow p_1$ , can you find a shorter one? Can you add another directed edge without creating a 3-cycle? No, you can't.

Claim:  $\forall n \in \mathbb{N}^{\geq 3}, P(n)$





*Proof:* Let  $T$  be a tournament with  $n \geq 3$  contestants. Assume  $T$  has at least one cycle. Since there are no 1-cycles or 2-cycles (think about why),  $T$  has a cycle of length at least 3.

Let  $C = \{c \in \mathbb{N}^{\geq 3} : c \text{ is the length of a cycle in } T\}$ . So  $C$  is a non-empty set of natural numbers, so it must have a smallest element  $c'$ . Then, either  $c' = 3$ , in which case we are done, or  $c' > 3$ .

If  $c' > 3$ , there are some sub-cases to consider:

We want to show that this cycle cannot be:  $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow \dots p_{c'} \rightarrow p_1$ . To do this, we can find a smaller 3-cycle that contradicts this is the smallest cycle.

- Case  $p_3 \rightarrow p_1$ : Then there is a shorter cycle,  $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1$ , contradicting  $c'$  being the shortest cycle length.
- Case  $p_1 \rightarrow p_3$ : Then there is a shorter cycle,  $p_1 \rightarrow p_3 \rightarrow \dots \rightarrow p_{c'} \rightarrow p_1$  has length  $c' - 1$ , contradicting  $c'$  being the shortest cycle length.

In both cases, assuming  $c' > 3$  leads to a contradiction, so this assumption is false. ■

## 4 No basis?

What if we try to prove  $\forall n \in \mathbb{N}, \sum_{i=0}^n 2^i = 2^{n+1}$  (!?)

*Pseudo Proof:* Let  $n \in \mathbb{N}$ . Assume  $\sum_{i=0}^n 2^i = 2^{n+1}$  (IH). I will show that  $\sum_{i=0}^{n+1} 2^i = 2^{n+2}$ .

$$\begin{aligned}
 \sum_{i=0}^{n+1} 2^i &= \left[ \sum_{i=0}^n 2^i \right] + 2^{n+1} \\
 &= 2^{n+1} + 2^{n+1} && \text{(by the IH)} \\
 &= 2 \cdot 2^{n+1} \\
 &= 2^{n+2}
 \end{aligned}$$

However, there is no verified base case!! This means that while we can prove the induction step, there is no basis to start the “infinite loop” of induction. For example, Let  $n = 0$ ,  $\sum_{i=0}^n 2^i = \sum_{i=0}^0 2^i = 2^0 = 1 \neq 2^1 = 2^n$ .

## 5 Zero pair free binary strings, again but with well-ordering

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n+2), & n > 1 \end{cases}$$

$P(n)$  can follow simply from  $n$  being a natural number. You do not need to assume to  $P(n-1)$  and  $P(n-2)$  in order to prove this. Since the function is defined recursively, it is tempting to use induction, but you do not always need to.

## 6 How not to do simple induction

Does a graph  $G = (V, E)$  with  $|V| > 0$  have  $|E| = |V| - 1$ ?

- Base case: Easy
- Inductive step: What happens if you try to extend an arbitrary graph with  $|V| = n$  to one with  $|V| = n + 1$ ? The claim breaks.
- Variations: What happens if you restrict the claim to connected graphs? If we make the graph cyclic, then the statement breaks again. What about acyclic connected graphs? In this final case, it becomes true.
- Take-home: Decompose rather than construct... except in structural induction. You never want to take an object of size  $n$  and extend it to size  $n + 1$ . Instead, you want to take an object of size  $n + 1$ , and find something related to  $n$  (your inductive hypothesis) inside of it.

# CSC236 Week 05: Languages: Definitions

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October 7 – October 13, 2021

## Contents

1	Some definitions	1
2	More notation — string operations	2
3	Language operations	2
4	Another way to define languages	2
5	Regular expression to languages:	3
6	Regex examples	3

## 1 Some definitions

- **alphabet:** Finite, non-empty set of symbols, e.g.  $\{a, b\}$  or  $\{0, 1, -1\}$ . Conventionally denotes  $\Sigma$ .
- **string:** Finite (including empty) sequence of symbols over an alphabet: abba is a string over  $\{a, b\}$ . Convention:  $\varepsilon$  is the empty string, never an allowed symbol,  $\Sigma^*$  is set of all strings over  $\Sigma$ .
- **language:** Subset of  $\Sigma^*$  for some alphabet  $\Sigma$ . Possibly empty, possibly empty, possibly infinite subset. E.e.  $\{\}, \{aa, aaa, aaaa, \dots\}$ .

**N.B.:**  $\{\} \neq \{\varepsilon\}$ .  $|\{\}| = 0 \neq 1 = |\{\varepsilon\}|$

Many problems can be reduced to languages: logical formulas, identifiers fro compilation, natural language processing. Key question is recognition:

Given language  $L$  ans string  $s$ , is  $s \in L$ ?

## 2 More notation — string operations

- **string length:** denotes  $|s|$ , is the number of symbols in  $s$ , e.g.  $|bba| = 3$ .
- $s = t$ : if and only if  $|s| = |t|$ , and  $s_i = t_i$ , for  $0 \leq i < |s|$ .
- $s^R$ : reversal of  $s$  is obtained by reversing symbols of  $s$ , e.g.  $1011^R = 1101$ .
- $st$  **or**  $s \circ t$ : concatenation of  $s$  and  $t$  — all characters of  $s$  followed by all those of  $t$ , e.g.  $bba \circ bb = bbabb$ .
- $s^k$ : denotes  $s$  concatenated with itself  $k$  times, e.g.  $ab^3 = ababab$ ,  $101^0 = \varepsilon$ .
- $\Sigma^n$ : all strings of length  $n$  over  $\Sigma$ ,  $\Sigma^*$  denotes all strings over  $\Sigma$ .

## 3 Language operations

- $\bar{L}$ : Complement of  $L$ , i.e.  $\Sigma^* - L$ . If  $L$  is a language of strings over  $\{0, 1\}$  that start with 0, then  $\bar{L}$  is the language of strings that begin with 1 plus the empty string.
- $L \cup L'$ : Union.
- $L \cap L'$ : Intersection.
- $L - L'$ : Difference.
- $\text{Rev}(L)$ :  $\text{cbrs}^R : s \in L$
- **concatenation:**  $LL'$  or  $L \circ L' = \{rt : r \in L, r \in L'\}$ . Special cases  $L\{\varepsilon\} = L = \{\varepsilon\}L$ , and  $L\{\} = \{\}L$ .
- **exponentiation:**  $L^k$  is concatenation of  $L$ ,  $k$  times. Special case,  $L^0 = \{\varepsilon\}$ , including  $L = \{\}$ .
- **Kleene star:**  $L^* = L^0 \cup L^1 \cup L^2 \cup \dots$

## 4 Another way to define languages

In addition to the set description  $L = \{\dots\}$ .

Definition: The regular expressions (regexps or REs) over alphabet  $\Sigma$  is the smallest set such that

- $\emptyset, \varepsilon$ , and  $x$ , for every  $x \in \Sigma$  are REs over  $\Sigma$ .
- If  $T$  and  $S$  are REs over  $\Sigma$ , then so are:
  - $(T + S)$  (union) — lowest precedence operator
  - $(TS)$  (concatenation) — middle precedence operator
  - $T^*$  (star) — highest precedence

## 5 Regular expression to languages:

The  $L(R)$ , the language denoted (or described) by  $R$  is defined by structural induction.

- Basis; If  $R$  is a regular expression by the basis of the definition of regular expressions, then define  $L(R)$ :
  - $L(\emptyset) = \emptyset$  (the empty language – no strings!)
  - $L(\varepsilon) = \{\varepsilon\}$  (the language consisting of just the empty string)
  - $L(x) = \{x\}$  (the language consisting of the one-symbol string)
- Induction step: If  $R$  is a regular expression by the induction step of the definition, then define  $L(R)$ :
  - $L((T + S)) = L(S) \cup L(T)$
  - $L((TS)) = L(S)L(T)$
  - $L(T^*) = L(T)^*$

We are assuming about  $(S + T)$  and  $(ST)$  above?

## 6 Regexp examples

- $L(0 + 1) = L(0) \cup L(1) = \{0, 1\}$
- $L(br0 + 1^*)$  All binary strings over  $\{0, 1\}$
- $L((01)^*) = \{\varepsilon, 01, 0101, 010101, \dots\}$
- $L(0^*1^*)$  0 or more 0s followed by 0 or more 1s
- $L(0^* + 1^*)$  0 or more 0s or 0 or more 1s
- $L((0 + 1)(0 + 1)^*)$  Non-empty binary strings over  $\{0, 1\}$

# CSC236 Week 06: Automata and Languages

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October 14 – October 20, 2021

## Contents

<b>1</b>	<b><math>L = \{x \in \{0, 1\}^* \mid x \text{ begins and ends with a different bit}\}</math></b>	<b>1</b>
<b>2</b>	<b>RE identities</b>	<b>1</b>
<b>3</b>	<b>Turnstile finite-state machine</b>	<b>2</b>

## 1 $L = \{x \in \{0, 1\}^* \mid x \text{ begins and ends with a different bit}\}$

The language  $L'((0(0+1)^*1) \cup (1(1+0)^*0))$  should be the same language as the one listed above.

If we want to prove this, we can show that  $\forall x \in L, x \in L' \wedge \forall x \in L', x \in L$ . This is the same as showing that  $L \subseteq L' \wedge L' \subseteq L$ , which is equivalent to  $L = L'$ , what we are trying to show.

*Proof:* First we show that  $L' \subseteq L$ : Let  $x \in L'$ . Then either  $x = 1y0$  where  $y \in \{0, 1\}^*$  or  $x = 0w1$ , where  $w \in \{1, 0\}^*$ . Without loss of generality, assume  $x = 1y0$ , otherwise just replace 1 with 0, 0 with 1, and  $y$  with  $w$ .

Then  $1 \in L(1)$ ,  $0 \in L(0)$ , and  $y \in L(0+1)^*$ , since it is the concatenation of 0 or more strings from  $L(0+1)$ . So  $x \in L(1)L(0+1)^*L(0)$ , so it begins with 1 and ends with 0, which are different, so  $x \in L$ .

Next we show that  $L \subseteq L'$ : left as an exercise to the reader

So since we have shown that  $L \subseteq L' \wedge L' \subseteq L$ ,  $L = L'$ . ■

## 2 RE identities

Some of these follow from set properties, others require some proof

- Commutativity of union:  $R + S \equiv S + R$
- Associativity of union:  $(R + S) + T \equiv R + (S + T)$
- Associativity of concatenation:  $(RS)T \equiv R(ST)$

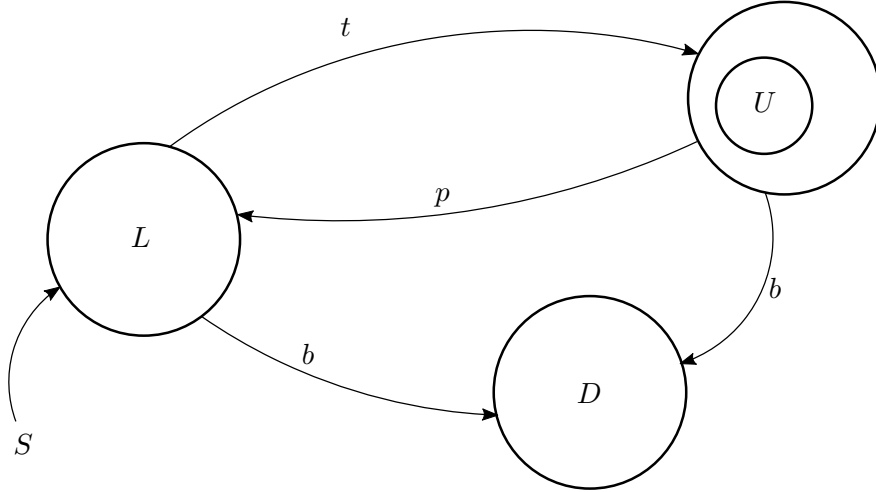
- Left distributivity:  $R(S + T) \equiv RS + RT$
- Right distributivity:  $(S + T)R \equiv SR + ST$
- Identity for union:  $R + \emptyset \equiv R$
- Identity for concatenation:  $R\varepsilon \equiv R \equiv \varepsilon R$
- Annihilator for concatenation:  $\emptyset R \equiv \emptyset \equiv R\emptyset$
- Idempotence of Kleene star:  $(R^*)^* \equiv R^*$

### 3 Turnstile finite-state machine

Let our alphabet be  $\{t, p, b\}$ , where  $p$  denotes push,  $t$  denotes tap or token, and  $b$  denotes bicycle.

We will study three different states:  $Q = \{U, L, D\}$ .  $U$  denotes unlocked,  $L$  denotes locked,  $D$  stands for dead (or jammed or deactivated).

$\Sigma^*$  is all strings over  $\{t, p, b\} = \{t, p, b\}^*$



Is  $tptppt$  accepted? We start at  $L$ , go to  $U$  with the first  $t$ , go back to  $L$  with  $p$  and so on. When we have a  $p$  but we are at  $L$ , then nothing happens as pushing on a locked turnstile will do nothing. Similarly, when we have a  $t$  when we are at a  $U$ , then nothing will happen as using a tap/token on an unlocked turnstile will also do nothing. Following these rules, we arrive at the following:

$$L \xrightarrow{t} U \xrightarrow{p} L \xrightarrow{t} U \xrightarrow{p} L \xrightarrow{p} L \xrightarrow{t} U$$

And thus the combination is accepted.