CSC236 Week 04: Well-Ordering, Induction Pitfalls

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Contents

1 Principle of well-ordering

Every non-empty subset of $\mathbb{N}$ has a smallest element

- How would you prove this for some $S \subseteq \mathbb{N}$?

  Since $\mathbb{N}$ is bounded below by 0 and $0 \in \mathbb{N}$, we know that any subset S of $\mathbb{N}$ is also bounded below by 0. Since it is bounded below, we can say that there is a smallest element.

- Is there something similar for $\mathbb{Q}$ or $\mathbb{R}$?

  No. For example, $(0, 1) \subseteq \mathbb{Q}, \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$.

- Here is the main part of proving the existence of a unique quotient and remainder

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^+, \exists q, r \in \mathbb{N} \text{ s.t. } m = qn + r \land 0 \leq r < n$$

  The course notes use Mathematical Induction. Well-ordering seems shorter and clearer.

2 Well-ordering example

$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^i, let R(m, n) = \{r \in \mathbb{N} : \exists q \in \mathbb{N} \text{ s.t. } m = qn + r\}$ has a smallest element

For a given pair natural numbers $m, n \neq 0$ does the set $R(m, n)$ staisfy the conditions for well-ordering?

$$R(m, n) = \{r \in \mathbb{N} : \exists q \in \mathbb{N} \text{ s.t. } m = qn + r\}$$

If so, we still need to be sure that the smallest element, r' has

1. $0 \leq r' < n$
2. That q' and r' are unique — no other pari of natural numbers would work

... in order to have

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}^{+}, \exists! q', r' \text{ s.t. } m = q'n + r' \wedge 0 \leq r < n$$

Proof. Let $m \in \mathbb{N}, n \in \mathbb{N}^{+}$. The $R(m, n)$ is a non-empty set of natural numbers. Then there is a smallest element of $R(m, n)$, let it be r', with corresponding q' such that $m = q'n + r'$.

- Case $r' < n$: This automatically satisfies our requirements.
- Case $r' \geq n$: Then we can show that there is another r'' such that $r'' < r'$ and r' is not the smallest element of the set.

$$\begin{aligned} m &= q'n + r' && \text{(by choice of } r'\text{)} \\ &= \left(q' + 1\right) n + \left(r' - n\right) \end{aligned}$$

Let $r'' = r' - n, q'' = q' - 1$. Then $r'' \in R(m, n)$, since $m = q''n + r''$

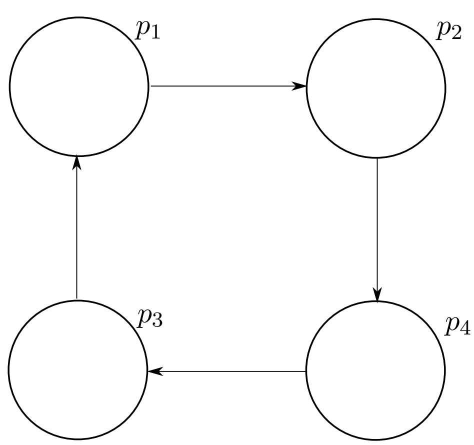
Then $r'' < r'$ which is a contradiction, and thus the inital asusmption that $r' \geq n$ is false.

So we have a pair of natural number (r', q') such that $m = q'n + r'$ and $0 \leq r' < n$. ■

3 $P(n)$: Every round-robin tornament with n players with a cycle has a 3-cycle.

Use: every non-empty subset of $\mathbb{N}$ has a smallest leement

Think of a round-robin as a complete oriented graph $G = (V, E)$, where $V = \{p_1, \ldots, p_n\}$ and every pair of distinct vertices has a unieuqly directed edge between them. Claim: $\forall n \in \mathbb{N}^{\geq 3}, P(n)$



If there is a cycle $p_1 \to p_2 \to p_3 \to \cdots \to p_n \to p_1$, can you find a shorter one? Can you add another directed edge without creating a 3-cycle? No, you can't.

Let T be a tournament wiht $n \geq 3$ contestants. Assume T has at least one cycle. Since there are no 1-cycles or 2-cycles (why?), T has a cycle of length at least 3. Let $C = \left\{c\ in \mathbb{N}^{\geq 3} : c \text{ is the length of a cycle in } T\right\}$. So C is a non-empty set of natural numbers, so it must have a smallest element c'.