# CSC236 Week 02: Complete Induction

Hisbaan Noorani

September 16 – 22, 2021

## Contents



## 1 Complete Induction

- Every natural number greater than 1 has a prime factorization

$$\begin{aligned}
2 &= 2 \\
3 &= 3 \\
4 &= 2 \times 2 \\
5 &= 5 \\
6 &= 2 \times 3 \\
7 &= 7 \\
8 &= 2 \times 2 \times 2 \\
9 &= 3 \times 3 \\
10 &= 2 \times 5
\end{aligned}$$

- How does the factorization of 8 help with the factorization of 9?

  The fact that 8 can be expressed as a product of primes has nothing to do with 9 being a product of primes.

## 2 Notational Convenience

Sometimes you will see the following:

$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \implies P(k)$$

## 3 More dominoes

$$\left(\forall n \in \mathbb{N}, \left[\bigwedge_{k=0}^{k=n-1} P(k)\right] \implies P(n)\right) \implies \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true.

## 4 Complete induction outline

- **Inductive step**: introduce $n$ and state inductive hypothesis $H(n)$
  - **Derive conclusion** $C(n)$: show that $C(n)$ follows from $H(n)$, indicating where you use $H(n)$ and why that is valid.
- **Verify base case(s)**: verify that the claim is true for any cases not covered in the inductive step

This is the same outline as simple induction but we modify the inductive hypothesis, $H(n)$ so that it assumes the main claim for every natural number from the starting point up to $n-1$, and the conclusion, $C(n)$ is now the main claim for $n$.

## 5 Watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(\lfloor\sqrt{n}\rfloor)]^2 + 2f(\lfloor\sqrt{n}\rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 3 \\ f(3) &= 3 \\ f(4) &= 15 \\ f(5 \ldots 15) &= 15 \\ f(16) &= 255 \end{aligned}$$

All of these things are divisible by 3. The square of something that is divisible by 3 is still divisible by 3 and the double of something that is divisible by 3 is still divisible by 3.

### 5.1 For all natural numbers $n > 1, f(n)$ is a multiple of 3?

*Proof:* $\forall n \in \mathbb{N}$, define the predicate $P(n)$ as $f(n)$ is a multiple of 3.

I will prove, using complete induction, that $\forall n > 1, P(n)$.

- Induction on $n$:

  Let $n \in N$. Assume $n > 1$. Also assume that $P(k)$ is true for all natural numbers $k$ less than $n$, and greater than 1.

  Notice that the floor of the square root of $n$ is greater than 1. Also, the square root of $n$ is less than $n$ (since $n > 1 \implies n^2 > n \implies n > \sqrt{n}$).

  Thus by the induction hypothesis, I have $P(\left\lfloor \sqrt(n) \right\rfloor)$, this number is a multiple of 3.

  Let $k \in \mathbb{N}$ s.t. $\lfloor \sqrt{n} \rfloor = 3k$, so $f(n) = (3k)^2 + 2(3k) = 3(3k^2 + 2k)$, a multiple of 3.

  So $P(n)$ follows in both possible cases.

- Base Cases:

  $P(2)$ claims that $f(2) = 3$ is a multiple of 3, which is true.

  $P(3)$ claims that $f(3) = 3$ is a multiple of 3, which is true. We have to prove $P(3)$ in the base case because the floor of the square root of three is not 2, but 1.

And thus $\forall n \in \mathbb{N}$ s.t. $n \geq 2, P(n)$. ■

**Note:** We have to include $P(3)$ as a base case because $\lfloor \sqrt{3} \rfloor$ is not 2, but 1.

# 6 Zero pair-free binary strings

Deonte by $zpfbs\,(n)$ the number of binary strings of length $n$ That contain no pairs of adjacent zeros. What is $zpfbs\,(n)$ for the first few natural numbers $n$?

$$\begin{aligned}
zpfbs\,(0) &= 1 \\
zpfbs\,(1) &= 2 \\
zpfbs\,(2) &= 3 \\
zpfbs\,(3) &= 5 \\
zpfbs\,(4) &= 8 \\
zpfbs\,(5) &= 13 \\
&\cdots \\
zpfbs\,(n) &= zpfbs\,(n-1) + zpfbs\,(n-2)
\end{aligned}$$

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n+2), & n > 1 \end{cases}$$

$\forall n \in \mathbb{N}$, defined predicate $P(n)$ as: $f(n) = zpfbs\,(n)$

Prove by complete induction that for all natural numbers $n$, $P(n)$.

*Proof:* Let $n \in \mathbb{N}$. Assume that $P$ is true for $0, \ldots, n-1$. I will show that $P(n)$ follows.

For the case $n \geq 2$: Partition the zero-pair-free binary strings of length $n$ into those that end in 1 and those that end in 0. Those that end in 1 are simply those of length $n-1$. with a 1 appended, and by $P(n-1)$ (since $n-1 < n$ and $n-1 \geq 0$, $n \geq 1$), there are $f(n-1)$ of these. Those that end in 0 must actually end in 10 (otherwise they are a zero-pair), and by $P(n-2)$ (since $n-2 \leq n$ and $n-2 \geq 0$, $n \geq 2$), there are $f(n-2)$ of these. Altogether there are $f(n-1)+f(n-2)$ zero-pair-free binary strings of length $n$ when $n \geq 2$, which is $P(n)$.

For the base case $n = 0$: There is one binary string (the empty one) of length 0, and it is zero-pair-free, and $f(0) = 1$ and $P(0)$ is true.

For the base case $n = 1$: There are two binary strings of length 1, and neither have pairs of zeros, and $f(1) = 2$ so $P(1)$ is true.

Thus in all possible cases, $P(n)$ follows. $\blacksquare$

## 7 Every natural number greater than 1 has a prime factorization

$\forall n \in \mathbb{N}$, define the predicate $P(n)$ as: $n$ can be expressed as a product of primes.

Prime factorization: represent as product of 1 or more primes.

Prove by complete induction that for all natural numbers $n$, $P(n)$.

*Proof:* Let $n \in \mathbb{N}$ s.t. $n > 1$. Assume $P$ is true for $2, \ldots, n-1$. I will show that $P(n)$ follows.

Case $n$ is composite: By definition, $n$ has a natural number factor $f_1$ such that $1 < f_1 < n$. By $P(f_1)$ (since $1 < f_1 < n$) we know $f_1$ can be expressed as a product of primes. Let $f_2 = \frac{n}{f_1}$, since $f_1 > 1$, we know that $\frac{n}{f_1} < n$, and also since $f_1 < n$, we know that $\frac{n}{f_1} > \frac{f_1}{f_1} = 1$. Since $f_1 > 1$, then $f_1 = \frac{n}{f_2} > 1$, so $n > f_2$. So, by $P(f_2)$, we know that $f_2$ can be expressed as a product of primes. Therefore since $f_1$ and $f_2$ are both products of primes, $n = f_1 \times f_2$ is a product of primes and $P(n)$ follows.

Case $n$ is prime: Then $n$ is its own primes factorization, and $P(n)$ follows.

In all possible cases, $P(n)$ follows. $\blacksquare$