CSC236 Week 01: Introduction and Basic Induction

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Contents

1 Why reason about computing

- You're not just hackers anymore

  Sometimes you need to analyze code before it runs. Sometimes it should never be run!

- Can you test everything?

  Infinitely many inputs: integers, strings, lists.

- Careful, you might get to like it. . . (?!*)

2 How to reason about computing

- It's messy. . .

  interesting problems fight back.

  You need to draft, re-draft, and re-re-draft.

You need to follow blind alleys until you find a solution.

You can also find a solution that isn't wrong, but could be better.

- It's art...

  Strive for correctness, clarity, surprise, humour, pathos, and others.

3 How to do well in this course

- Read the syllabus as a two-way promise
- Question, answer, record, synthesize. Try annotating blank slides.
- Collaborate with respect. You need computer science friends who are respectful and constructively critical.

4 Assume that you already know

- Chapter 0 material from *Introduction to Theory of Computation.*
- CSC110/111 material, especially proofs and big-$\mathcal{O}$.

5 By December you'll know

- Understand and use several flavours of induction. Some of these flavours will taste new.
- Formal languages, regular languages, regular expressions.
- Complexity and correctness of programs — both recursive and iterative.

6 Domino fates foretold

$$[P(0) \wedge (\forall n \in \mathbb{N}, P(n) \implies P(n+1))] \implies \forall n \in \mathbb{N}, P(n)$$

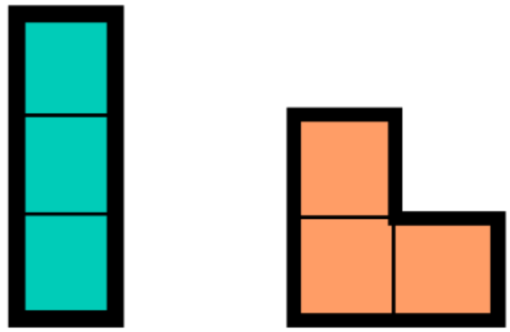
If the initial case works, and each case that works implies its successor works, then all cases work

7 Simple induction outline

- Inductive step: introduce n and inductive hypothesis $H(n)$
  - Derive conclusion $C(n)$: show that $C(n)$ follows from $H(n)$, indicating **where** you use $H(n)$ and why that is valid.
- Verify base case(s): verify that the claim is true for any cases not covered in the inductive step
- In simple induction $C(n)$ is just $H(n+1)$

8 Trominoes

See `https://en.wikipedia.org/wiki/Tromino`



Can an $n \times n$ square grid, with one subsquare removed, be tiled (covered without overlapping) by "chair" trominoes?

- 1×1: Yes.
- 2×2: Yes.
- 3×3: No. The remaining number of squares is not divisible by 3.
- 4×4: Yes.

Proof: $\forall n \in \mathbb{N}$, define the predicate $P(n)$ as a $2^n \times 2^n$ square grid, with one subsquare removed, can be tiled (covered without overlapping) by "chair" trominoes.

- Induction on n

  Let n be an arbitrary, fixed, natural number. (Let $n \in \mathbb{N}$).

  Assume $P(n)$, that is a $2^n \times 2^n$ grid, with one square removed can be tiled with "chairs."

  I will prove $P(n+1)$, that is a $2^{n+1} \times 2^{n+1}$ grid, with one square removed can be tiled by chairs.

  Let G be a $2^{n+1} \times 2^{n+1}$ grid with one square removed. Notice that G can be decomposed into four $2^n \times 2^n$ disjoint quadrant grids. We may assume, without loss of generality, that the missing square is in the upper-right quadrant, since otherwise just rotate it there, and rotate back when done. By $P(n)$ I can tile the upper-right quadrant, minus the missing square. By $P(n)$ 3 more times, I can tile the remaining 3 quadrants, omitting for a moment the 3 tiles nearest the centre of G, with chairs. The briefly omitted squares form a chair! So I complete the tiling by adding one more chair. Thus $P(n+1)$.

- Base Case

  A $2^0 \times 2^0$ grid, with one square removed, is just empty space! This can be tiled with 0 chairs. So $P(0)$ is true.

And thus $\forall n \in \mathbb{N}, P(n)$. ■

9 $3^n \geq n^3$?

9.1 Scratch Work

Check for a few values of n:

$$\begin{aligned}
3^0 &= 1 \geq 0 = 0^3 && \checkmark \\
3^1 &= 3 \geq 1 = 1^3 && \checkmark \\
3^2 &= 9 \geq 8 = 2^3 && \checkmark \\
3^3 &= 27 \geq 27 = 3^3 && \checkmark \\
3^4 &= 81 \geq 64 = 4^3 && \checkmark \\
3^{-1} &= \frac{1}{3} \geq -1 = -1^3 && \checkmark \\
3^{2.5} &< 2.5^3 && \times
\end{aligned}$$

9.2 Simple Induction

Proof: $\forall n \in \mathbb{N}$, define the predicate $P(n)$ as $3^n \geq n^3$.

- Induction on n

  Let $n \in \mathbb{N}$. Assume $H(n)\ :\ 3^n \geq n^3$. I will prove $H(n+1)$ follows, that is $3^{n+1} \geq (n+1)^3$.

$$\begin{aligned}
3^{n+1} &= 3 \cdot 3^n \\
&\geq 3 \cdot n^3 \\
&= n^3 + n^3 + n^3 \\
&\geq n^3 + 3n^2 + 9n && \text{(since } n \geq 3\text{)} \\
&\geq n^3 + 3n^2 + 3n + 6n \\
&= n^3 + 3n^2 + 3n + 1 && \text{(since } 6n \geq 1\text{)} \\
&= (n+1)^3
\end{aligned}$$

  And thus we have shown that $\forall n \in \mathbb{N}$ s.t. $n \geq 3, H(n) \implies H(n+1)$.

- Base Case

  $3^3 \geq 3^3$ so $P(3)$ holds.
  $3^2 \geq 2^3$ so $P(2)$ holds.
  $3^1 \geq 1^3$ so $P(1)$ holds.
  $3^0 \geq 0^3$ so $P(0)$ holds.

And thus, we have shown $\forall n \in \mathbb{N}, 3^n \geq n^3$, as needed. ■