CSC236 Week 03: Recurrences, Structural Induction

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Contents

1 Recursively defined function

Recall:

$$f(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ f(n-1) + f(n-2) & n > 1 \end{cases}$$

For comparison, let

$$r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2},\ a = \frac{\sqrt{5}+3}{2\sqrt{5}},\ b = \frac{\sqrt{5}-3}{2\sqrt{5}}$$

. . . and define

$$f'(n) = ar_1^n + br_2^n$$

Compare the first few values of $f(n)$ to the first few values of $f'(n)$

$$\begin{aligned} f(0) &= 1, & f'(0) &= 1 \\ f(1) &= 2, & f'(1) &= 2 \\ f(2) &= 5, & f'(2) &= 5 \\ f(3) &= 8, & f'(3) &= 8 \\ f(4) &= 13, & f'(4) &= 13 \end{aligned}$$

2 Prove that $\forall n \in \mathbb{N}, f(n) = f'(n)$

$\forall n \in \mathbb{N}$, define $P(n)$ by $f(n) = f'(n)$. Prove by complete induction that $\forall n \in \mathbb{N}, P(n)$. It helps to verify $1 + r_1 = r_1^2$ and $1 + r_2 = r_2^2$.

Proof: Let $n \in \mathbb{N}$. Assume the induction hypothesis, $\forall k \in \mathbb{N}$ s.t. $k < n$ has $f(k) = f'(k)$.

- Case $n > 1$:

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) && \text{(by definition)} \\ &= ar_1^{n-1} + br_2^{n-1} + ar_1^{n-2} + ar_2^{n-2} && \text{(by IH, since } 0 \leq n-2, n-1 < n\text{)} \\ &= a\left(r_1^{n-2} + r_1^{n-1}\right) + b\left(r_2^{n-2} + r_2^{n-1}\right) \\ &= a\left(r_1^{n-2}\right)(1 + r_1) + b\left(r_2^{n-2}\right)(1 + r_2) \\ &= a\left(r_1^{n-2}\right)\left(r_1^2\right) + b\left(r_2^{n-2}\right)\left(r_2^2\right) \\ &= ar_1^n + br_2^n \\ &= f'(n) \end{aligned}$$

  And thus $\forall n > 1, P(n)$ holds.

- Case $n = 0$: $f(0) = 1 = f'(0)$, by evaluating $f'(0)$ so $P(0)$ holds.
- Case $n = 1$: $f(1) = 2 = f'(1)$, by evaluating $f'(1)$ so $P(1)$ holds.

Thus, in all possible cases, $P(n)$ holds. ■

3 Define sets inductively

One way to define the natural numbers:

$\mathbb{N}$: The smallest set such that

1. $0 \in \mathbb{N}$
2. $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

By **smallest** we mean $\mathbb{N}$ has no proper subsets that satisfy these two conditions. If we leave out **smallest**, what other sets satisfy the definition? $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z},$ etc.

4 What can you do with it?

The definition in §3 defined the simplest natural number (0) and the rule to produce new natural numbers from old ones (add 1). Proof using Mathematical Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is

1. Show that $P(0)$ is true for basis, 0.
2. Prove that $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$.

5 Other structurally-defined sets

Define $\mathcal{E}$: The smallest set such that

1. $x, y, z \in \mathcal{E}$
2. $e_1, e_2 \in \mathcal{E} \implies (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2), (e_1 \div e_2) \in \mathcal{E}$

Form some expressions in $\mathcal{E}$. Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators symbols from $\{+, -, \times, \div\}$. Make a conjecture:

$$(x + y), (x \times y) \in \mathcal{E}$$

$$((x + y) \times (x \times y)) \in \mathcal{E}$$

Let $\mathrm{vr}\,(e)$ count the number of variables in e and let $\mathrm{op}\,(e)$ count the number of operators in e.

$$\forall e \in \mathcal{E}, \mathrm{vr}\,(e) = \mathrm{op}\,(e) + 1$$

6 Structural induction

To prove that a property is true for all $e \in \mathcal{E}$, parallel the recursive set definition:

- **Verify the base case(s)**: Show that the property is true for the simplest members, $\{x, y, z\}$, that is show $P(x), P(y),$ and $P(z)$.
- **Inductive step**: Let e_1 and e_2 be arbitrary elements of $\mathcal{E}$. Assume $H(\{e_1, e_2\})$: $P(e_1)$ and $P(e_2)$, that is e_1 and e_2 have the property.
  - **Show that $C(\{e_1, e_2\})$ follows:**
    All possible combinations of e_1 and e_2 have the property, that is $P((e_1 + e_2)), P((e$

7 Structural induction proofs

7.1 Prove $\forall e \in \mathcal{E}, \mathrm{vr}(e) = \mathrm{op}(e) + 1$

Proof: For every $e \in \mathcal{E}$, define $P(e)$ as $\mathrm{vr}(e) = \mathrm{op}(e) + 1$. Verify the basis: Let $t \in \{x, y, z\}$. The $\mathrm{vr}(t) = 1 = 0 + 1 = \mathrm{op}(t) + 1$, so $P(t)$ holds for every element of the basis.

Let $e_1, e_2 \in \mathcal{E}$, and assume $P(e_1)$ and $P(e_2)$. Want to prove that $P((e_1 \odot e_2))$ where $\odot \in \{+, -, \times, \div\}$, that is $\mathrm{vr}((e_1 \odot e_2)) = \mathrm{op}((e_1 \odot e_2)) + 1$.

$$\begin{aligned}
\mathrm{vr}((e_1 \odot e_2)) &= \mathrm{vr}(e_1) + \mathrm{vr}(e_2) \\
&= \mathrm{op}(e_1) + 1 + \mathrm{op}(e_2) + 1 && \text{(by } P(e_1) \text{ and } P(e_2)) \\
&= [\mathrm{op}(e_1) + \mathrm{op}(e_2)] + 1 \\
&= \mathrm{op}(e_1 \odot e_2) + 1
\end{aligned}$$

So $P((e_1 \odot e_2))$ follows. ■

7.2 Prove $\forall e \in \mathcal{E}, \mathrm{vr}(e) \leq 2^{h(3)}$

Define the heights, $h(x) = h(y) = h(z) = 0$, and $h((e_1 \odot e_2))$ as $1 + \max(h(e_1), h(e_2))$ if $e_1, e_2 \in \mathcal{E}$ and $\odot \in \{+, -, \times, \div\}$. What's the connection between the unmber of variables and the height?

$$\begin{aligned}
h(x) &= 0, \mathrm{vr}(x) = 1 \\
h((x + y)) &= 1, \mathrm{vr}((x + y)) = 2 \\
h(((x + y) - z)) &= 2, \mathrm{vr}(((x + y) - z)) \\
h(((x + y) - (z \times x))) &= 2, \mathrm{vr}(((x + y) - (z \times x))) = 4 \\
h((((x + y) - (z \times x)) \div ((x + y) - (z \times x)))) &= 3, \mathrm{vr}((((x + y) - (z \times x)) \div ((x + y) - (z \times x)))) = 8
\end{aligned}$$

Proof: For every $e \in \mathcal{E}$ define $P(e)$ as $\mathrm{vr}(e) \leq 2^{h(e)}$. Let $a \in \{x, y, z\}$. Then a has one variable (itself), and no operators so $\mathrm{vr}(a) = 1 = 2^0 = 2^{h(a)}$ since h of any single vairable is 0. So $P(a)$ holds for $a \in \{x, y, z\}$ (the basis).

Let $e_1, e_2 \in \mathcal{E}$. Assume $P(e_1), P(e_2)$ i.e. $\mathrm{vr}(e_1) \leq 2^{h(e_1)}$ and $\mathrm{vr}(e_2) \leq 2^{h(e_2)}$. We will show that $P(e_1 \odot e_2)$ is true, where $\odot \in \{+, -, \times, \div\}$.

$$\begin{aligned}
\mathrm{vr}(e_1 \odot e_2) &= \mathrm{vr}(e_1) + \mathrm{vr}(e_2) \\
&\leq 2^{h(e_1)} + 2^{h(e_2)} && \text{(by } P(e_1) \text{ and } P(e_2)) \\
&\leq 2 \cdot 2^{\max(h(e_1), h(e_2))} \\
&= 2^{1 + \max(h(e_1), h(e_2))} \\
&= 2^{h((e_1 \odot e_2))} && \text{(by definition of } h)
\end{aligned}$$

So $P((e_1 \odot e_2))$ follows. ■