# CSC236 Lecture 03: Complete Induction

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## 1 Complete Induction

- Every natural number greater than 1 has a prime factorization

$$\begin{aligned}
2 &= 2 \\
3 &= 3 \\
4 &= 2 \times 2 \\
5 &= 5 \\
6 &= 2 \times 3 \\
7 &= 7 \\
8 &= 2 \times 2 \times 2 \\
9 &= 3 \times 3 \\
10 &= 2 \times 5
\end{aligned}$$

- How does the factorization of 8 help with the factorization of 9?

  The fact that 8 can be expressed as a product of primes has nothing to do with 9 being a product of primes.

## 2 Notational Convenience

Sometimes you will see the following:

$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \implies P(k)$$

## 3 More dominos

$$\left(\forall n \in \mathbb{N}, \left[\bigwedge_{k=0}^{k=n-1} P(k)\right] \implies P(n)\right) \implies \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true.

## 4 Complete induction outline

- **Inductive step**: introduce $n$ and state inductive hypothesis $H(n)$
  - **Derive conclusion** $C(n)$: show that $C(n)$ follows from $H(n)$, indicating hwere you use $H(n)$ and why that is valid.
- **Verify base case(s)**: verify that the claim is true for any cases not covered in the inductie step

This is the same outline as simple induction but we modify the inductive hypothesis, $H(n)$ so that it assuems the main claim for every natural number from the starting point up to $n-1$, and the conclusion, $C(n)$ is now the main claim for $n$.

## 5 Watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ \left[f(\lfloor\sqrt{n}\rfloor)\right]^2 + 2f(\lfloor\sqrt{n}\rfloor) & n > 1 \end{cases}$$

Check a few cases, and makea conjecture:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 3 \\ f(3) &= 3 \\ f(4) &= 15 \\ f(5 \ldots 15) &= 15 \\ f(16) &= 255 \end{aligned}$$

All of these things are divisible by 3. The square of something that is divisible by 3 is still divisible by 3 and the double of something that is divisibly by 3 is still divisibly by 3.

## 6 For all natural numebrs $n > 1, f(n)$ is a multiple of 3?

For natual numbers $n$ define $P(n) : f(n)$ is a multiple of 3.

I will prove, using complete induction, that $\forall n > 1, P(n)$.

*i.* Induction on $n$

Let $n \in N$. Assume $n > 1$. Also assume that $P(k)$ is true for all natural numbers $k$ less than $n$, and greater than 1.

Notice that the floor of the square root of $n$ is greater than 1. Also, the square root of $n$ is less than $n$ (since $n > 1 \implies n^2 > n \implies n > \sqrt{n}$).

Thus by the induction hypothesis, I have $P(\left\lfloor \sqrt(n) \right\rfloor)$, this number is a multiple of 3.

Let $k \in \mathbb{N}$ s.t. $\lfloor \sqrt{n} \rfloor = 3k$, so $f(n) = (3k)^2 + 2\,(3k) = 3\left(3k^2 + 2k\right)$, a multiple of 3.

So $P(n)$ follows in both possible cases.

*ii.* Base Cases

$P(2)$ claims that $f(2) = 3$ is a multiple of 3, which is true.

$P(3)$ claims that $f(3) = 3$ is a multiple of 3, which is true.

## 7 Zero pair free binary strings, zpfbs...

Deonte by $zpfbs\,(n)$ the number of binary strings of length $n$ That contain no paris of adjacent zeros. What is $zpfbs\,(n)$ for the first few natural numbers $n$?

$$\begin{aligned}
zpfbs\,(0) &= 1\\
zpfbs\,(1) &= 2\\
zpfbs\,(2) &= 3\\
zpfbs\,(3) &= 5\\
zpfbs\,(4) &= 8\\
zpfbs\,(5) &= 13\\
&\dots\\
zpfbs\,(n) &= zpfbs\,(n-1) + zpfbs\,(n-2)
\end{aligned}$$