CSC111 Lecture 13: Introduction to Graphs

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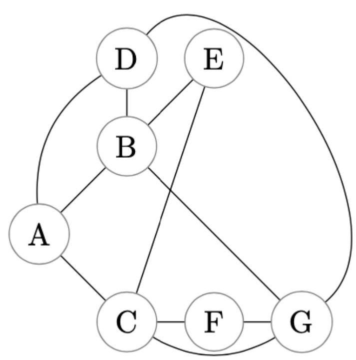
1 Exercise 1: Graph terminology review

One of the tricky things about learning graphs is that there's a lot of terminology to understand. This exercise will give you the opportunity to practice using this terminology on a concrete example. Consider the graph below:



1. How many vertices does this graph have?

   There are 7 vertices: $V = \{A, B, C, D, E, F, G\}$.

2. How many edges does this graph have?

   There are 11 edges: $E = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, D\}, \{B, E\}, \{B, G\}, \{C, E\}, \{C, F\}, \{C, G\}, \{D, G\}, \{F, G\}\}$.

3. List all the vertices that are adjacent to vertex G.

   $\{B, C, D, F\}$.

4. Find a *path* that goes through all vertices of the graph. (Remember that a path cannot have any duplicate vertices.)

   One such path is A, D, G, F, C, E, B.

2 Exercise 2: A property of vertex degrees

Recall that the **degree** of a vertex v, denoted $d(v)$, is its number of neighbours.

Answer the following questions about this definition.

1. Let $G = (V, E)$ be the graph from Exercise 1. Complete the table below showing the *degree* of each vertex. We've done the first row for you.

Vertex	Degree
A	3
B	4
C	4
D	3
E	2
F	2
G	4

2. What is the *sum of the degrees* of all the vertices in the above table?

   $3 + 4 + 4 + 3 + 2 + 2 + 4 = 22$

3. Compare your answer to Question 2 and the *number of edges* of this graph (Question 2 in Exercise 1). What do you notice?

   The sum of the degrees of all the vertices in the graph is equal to double the number of edges in the graph. $\sum\limits_{v \in V} d(v) = 2 \cdot |E|$.

4. Prove the following graph property: for all graphs $G = (V, E)$, $\sum\limits_{v \in V} d(v) = 2 \cdot |E|$. Your proof body can consist of a short explanation written in English.

   Note: $\sum\limits_{v \in V}$ means "sum over all vertices v in V".

   WTS $\forall G = (V, E), \sum\limits_{v \in V} d(v) = 2 \cdot |E|$

   Proof:

   Let $G = (V, E)$ be an arbitrary graph

   For a vertex v, $d(v)$, is the number of edges that "touch" (incident) that vertex.

   We know that each edge touches exactly 2 vertices. This means each edge will be counted by exactly two of the $d(v)$ expressions in the summation. 1 egde will represent 2 total degrees. 4 edges will represent 8 total degrees.

   Thus we know that the sum of the $d(v)$ (over all $v \in V$) is equal to double the number of edges. $\blacksquare$

3 Additional exercises

1. Let $G = (V, E)$ be a graph, and assume that for all $v \in V$, $d(v) \leq 5$. Find and prove a good upper bound (exact, not asymptotic) on the total number of edges, $|E|$, in terms of the number of vertices, $|V|$.

   Formally, you can think of this as proving the following statement (after filling in the blank):

   $$\forall G = (V, E),\ \big(\forall v \in V,\ d(v) \leq 5\big) \Rightarrow |E| \leq \underline{\hspace{3cm}}$$

   See the lecture slide starting titled "Graphs and induction" for the proof