

# CSC236 Week 02: Complete Induction

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## 1 Complete Induction

- Every natural number greater than 1 has a prime factorization

$$2 = 2$$

$$3 = 3$$

$$4 = 2 \times 2$$

$$5 = 5$$

$$6 = 2 \times 3$$

$$7 = 7$$

$$8 = 2 \times 2 \times 2$$

$$9 = 3 \times 3$$

$$10 = 2 \times 5$$

- How does the factorization of 8 help with the factorization of 9?

The fact that 8 can be expressed as a product of primes has nothing to do with 9 being a product of primes.

## 2 Notational Convenience

Sometimes you will see the following:

$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \implies P(k)$$

## 3 More dominoes

$$\left( \forall n \in \mathbb{N}, \left[ \bigwedge_{k=0}^{k=n-1} P(k) \right] \implies P(n) \right) \implies \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true.

## 4 Complete induction outline

- **Inductive step:** introduce  $n$  and state inductive hypothesis  $H(n)$ 
  - **Derive conclusion**  $C(n)$ : show that  $C(n)$  follows from  $H(n)$ , indicating where you use  $H(n)$  and why that is valid.
- **Verify base case(s):** verify that the claim is true for any cases not covered in the inductive step

This is the same outline as simple induction but we modify the inductive hypothesis,  $H(n)$  so that it assumes the main claim for every natural number from the starting point up to  $n - 1$ , and the conclusion,  $C(n)$  is now the main claim for  $n$ .

## 5 Watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(\lfloor \sqrt{n} \rfloor)]^2 + 2f(\lfloor \sqrt{n} \rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture:

$$\begin{aligned}f(0) &= 1 \\f(1) &= 1 \\f(2) &= 3 \\f(3) &= 3 \\f(4) &= 15 \\f(5 \dots 15) &= 15 \\f(16) &= 255\end{aligned}$$

All of these things are divisible by 3. The square of something that is divisible by 3 is still divisible by 3 and the double of something that is divisible by 3 is still divisible by 3.

## 6 For all natural numbers $n > 1$ , $f(n)$ is a multiple of 3?

For natural numbers  $n$  define  $P(n) : f(n)$  is a multiple of 3.

I will prove, using complete induction, that  $\forall n > 1, P(n)$ .

*i.* Induction on  $n$

Let  $n \in \mathbb{N}$ . Assume  $n > 1$ . Also assume that  $P(k)$  is true for all natural numbers  $k$  less than  $n$ , and greater than 1.

Notice that the floor of the square root of  $n$  is greater than 1. Also, the square root of  $n$  is less than  $n$  (since  $n > 1 \implies n^2 > n \implies n > \sqrt{n}$ ).

Thus by the induction hypothesis, I have  $P(\lfloor \sqrt{n} \rfloor)$ , this number is a multiple of 3.

Let  $k \in \mathbb{N}$  s.t.  $\lfloor \sqrt{n} \rfloor = 3k$ , so  $f(n) = (3k)^2 + 2(3k) = 3(3k^2 + 2k)$ , a multiple of 3.

So  $P(n)$  follows in both possible cases.

*ii.* Base Cases

$P(2)$  claims that  $f(2) = 3$  is a multiple of 3, which is true.

$P(3)$  claims that  $f(3) = 3$  is a multiple of 3, which is true.

## 7 Zero Pair-Free Binary Strings

Deonte by  $zpfbs(n)$  the number of binary strings of length  $n$  That contain no paris of adjacent zeros. What is  $zpfbs(n)$  for the first few natural numbers  $n$ ?

$$\begin{aligned}
zpbs(0) &= 1 \\
zpbs(1) &= 2 \\
zpbs(2) &= 3 \\
zpbs(3) &= 5 \\
zpbs(4) &= 8 \\
zpbs(5) &= 13 \\
&\dots \\
zpbs(n) &= zpbs(n-1) + zpbs(n-2)
\end{aligned}$$

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n-2), & n > 1 \end{cases}$$

$\forall n \in \mathbb{N}$ , defined predicate  $P(n)$  as:  $f(n) = zpbs(n)$

Prove by complete induction that for all natural numbers  $n$ ,  $P(n)$ .

Let  $n \in \mathbb{N}$ . Assume that  $P$  is true for  $0, \dots, n-1$ . I will show that  $P(n)$  follows.

For the case  $n \geq 2$ : Partition the zero-pair-free binary strings of length  $n$  into those that end in 1 and those that end in 0. Those that end in 1 are simply those of length  $n-1$  with a 1 appended, and by  $P(n-1)$  (since  $n-1 < n$  and  $n-1 \geq 0$ ,  $n \geq 1$ ), there are  $f(n-1)$  of these. Those that end in 0 must actually end in 10 (otherwise they are a zero-pair), and by  $P(n-2)$  (since  $n-2 \leq n$  and  $n-2 \geq 0$ ,  $n \geq 2$ ), there are  $f(n-2)$  of these. Altogether there are  $f(n-1) + f(n-2)$  zero-pair-free binary strings of length  $n$  when  $n \geq 2$ , which is  $P(n)$ .

For the base case  $n = 0$ : There is one binary string (the empty one) of length 0, and it is zero-pair-free, and  $f(0) = 1$  and  $P(0)$  is true.

For the base case  $n = 1$ : There are two binary strings of length 1, and neither have pairs of zeros, and  $f(1) = 2$  so  $P(1)$  is true.

Thus in all possible cases,  $P(n)$  follows.

## 8 Every natural number greater than 1 has a prime factorization

Each natural number  $n$ , let predicate  $P(n)$  be:  $n$  can be expressed as a product of primes.

Prime factorization: represent as product of 1 or more primes.

Prove by complete induction that for all natural numbers  $n$ ,  $P(n)$ .

Let  $n \in \mathbb{N}$  s.t.  $n > 1$ . Assume  $P$  is true for  $2, \dots, n-1$ . I will show that  $P(n)$  follows.

Case  $n$  is composite: By definition,  $n$  has a natural number factor  $f_1$  such that  $1 < f_1 < n$ . By  $P(f_1)$  (since  $1 < f_1 < n$ ) we know  $f_1$  can be expressed as a product of primes. Let  $f_2 = \frac{n}{f_1}$ , since  $f_1 > 1$ , we know that  $\frac{n}{f_1} < n$ , and also since  $f_1 < n$ , we know that  $\frac{n}{f_1} > \frac{f_1}{f_1} = 1$ . Since  $f_1 > 1$ , then  $f_1 = \frac{n}{f_2} > 1$ , so  $n > f_2$ . So, by  $P(f_2)$ , we know that  $f_2$  can be expressed as a product of primes.

Therefore since  $f_1$  and  $f_2$  are both products of primes,  $n = f_1 \times f_2$  is a product of primes and  $P(n)$  follows.

Case  $n$  is prime: Then  $n$  is its own primes factorization, and  $P(n)$  follows.

In all possible cases,  $P(n)$  follows.