Name: ______________________________

1. Consider a random variable x and constants α and β. Show that the expectation and variance of $\alpha x + \beta$ are

$$\begin{aligned} \mathrm{E}\,(\alpha x + \beta) &= \alpha\,\mathrm{E}\,(x) + \beta, \\ \mathrm{var}\,(\alpha x + \beta) &= \alpha^2\,\mathrm{var}\,(x). \end{aligned}$$

2. Consider two random variables x and y.

   (a) Show that the variance of $\alpha x + y$ is given by

$$\begin{aligned} \mathrm{var}\,(\alpha x + y) &= \alpha^2\,\mathrm{var}\,(x) + \mathrm{var}\,(y) + 2\alpha\,\mathrm{cov}\,(x, y) \\ &= \alpha^2\sigma_x^2 + \sigma_y^2 + 2\alpha\rho\sigma_x\sigma_y, \end{aligned}$$

   where α is a constant, $\sigma_x^2 = \mathrm{var}\,(x)$, $\sigma_y^2 = \mathrm{var}\,(y)$, and the correlation coefficient ρ is defined as $\rho = \mathrm{cov}\,(x, y)/\sigma_x\sigma_y$.

   (b) Using the result of (a), show that the correlation coefficient always lies in the range $\rho \in [-1, 1]$. *Hint*: use the fact that the variance $\mathrm{var}\,(\alpha x + y)$ is always greater than or equal to zero, and consider the cases $\alpha = \pm\sigma_y/\sigma_x$.

3. Consider the exponential PDF,

$$p(x|\xi) = \frac{1}{\xi}e^{-x/\xi}, \quad x \geq 0.$$

   (a) Show that the corresponding cumulative distribution is given by

$$F(x) = P(X \leq x|\xi) = 1 - e^{-x/\xi}, \quad x \geq 0.$$

   (b) Show that the conditional probability to find a value x between x_0 and $x_0 + x'$ given that $x > x_0$ is equal to the (unconditional) probability to find $x \leq x'$. I.e.,

$$P(x \leq x_0 + x'|x > x_0, \xi) = P(x \leq x'|\xi).$$

   (c) Cosmic ray muons produced in the upper atmosphere enter a detector at sea level, and some of them come to rest in the detector and decay. The time difference t between entry into the detector and decay follows an exponential distribution, and the mean value $\langle t \rangle$ is the mean lifetime of the muon (approximately 2.2 μs). Explain why the time that the muon lived prior to entering the detector does not play a role in determining the mean lifetime.

4. (a) Using a random number generator, write a short program to generate 10,000 random values uniformly distributed between zero and one, and display the result as a histogram with 100 bins.

   (b) Modify your program to make an xy scatter plot of 10,000 pairs of consecutively generated values.

5. Consider a "sawtooth" PDF,

$$p(x|x_{\max}) = \begin{cases} 2x/x_{\max}^2, & 0 < x < x_{\max} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Use the transformation method to find the function $x(r)$ to generate 10,000 random numbers according to $p(x|x_{\max})$. Implement the method in a short computer program and make a histogram of the results. (Use, for example, $x_{\max} = 1$.)

   (b) Write a program to generate 10,000 random numbers according to the sawtooth PDF using the acceptance-rejection technique. Plot a histogram of the results.

6. Recall the problem from Homework 1 where someone buys boxes of cereal with a toy ball inside. The balls come in 5 possible colors, and you calculated the probability of a person obtaining their 2 favorite colors given that they bought 6 boxes.

   Perform 10,000 numerical "experiments" in which one of 5 possible colored balls are drawn from 6 boxes of cereal, and use it to estimate the probability that a person will obtain their 2 favorite colors.

7. The Birthday Problem

   (a) You are in a class with 30 students. What is the probability that at least two students share a birthday (month and day)? Assume that a person is equally likely to be born on any given day. Ignore complications like leap years, seasonal effects on birthdays, twins, etc.

   (b) Suppose there is a very strong seasonal effect on birthdays, such that the probability a person is born on a given day $d \in [1, 365]$ is

$$P(d|\tau = 365, I) \propto 1 + \frac{1}{2}\sin\left(\frac{2\pi}{\tau}d\right).$$

   Write a Monte Carlo that calculates the probability that at least two out of 30 people will share a birthday in this case. (Note: there is a seasonal effect on birthdays but it is not this extreme.)