Name: ______________________________

1. The earliest determination of Avogadro's number, by Jean Baptiste Perrin, was based on Brownian motion. The experiment was conducted by observing particles of mastic (a substance used in varnish) suspended in water. The particles were spheres of radius $r = 0.52$ μm and had a density of 1.063 g cm^{-3}. By viewing the particles through a microscope, only those in a layer approximately 1 μm thick were in focus; particles outside this layer were not visible. By adjusting the microscope lens, the focal plane could be moved vertically. Photographs were taken at 4 different heights z and the number of particles $n(z)$ counted. The data are shown in the table below.

height z (μm)	number of particles n
0	1880
6	940
12	530
18	305

The gravitational potential energy of a spherical particle of mastic in water is given by

$$E = \frac{4}{3}\pi r^3 \Delta\rho g z,$$

where $\Delta\rho = \rho_{\text{mastic}} - \rho_{\text{water}} = 0.063$ g cm^{-3} is the difference in the density of the mastic and water, and $g = 980$ cm s^{-2} is the acceleration due to gravity. Statistical mechanics predicts that the probability for a particle to be in a state of energy E is proportional to

$$P(E) \propto e^{-E/kT},$$

where k is Boltzmann's constant and T is the absolute temperature. The particles should therefore be distributed in height according to an exponential law, where the number n observed at z can be treated as a Poisson variable with a mean $\nu(z)$. This is found to be

$$\nu(z) = \nu_0 \exp\left(-\frac{4\pi r^3 \Delta\rho g z}{3kT}\right),$$

where ν_0 is the expected number of particles at $z = 0$.

(a) (10 points) Write a computer program to determine the parameters k and ν_0 with the method of maximum likelihood. Use the data given in the table to construct the log-likelihood function based on Poisson probabilities,

$$\ln \mathcal{L}(\nu_0, k) = \sum_{i=1}^{N} (n_i \ln \nu_i - \nu_i - \ln n_i!),$$

where $N = 4$ is the number of measurements. For the temperature use $T = 293$ K.

(b) (5 points) From the value you obtain for k, determine Avogadro's number using the relation

$$N_A = R/k,$$

where R is the gas constant. Perrin used $R = 8.32 \times 10^7$ erg mol^{-1} K^{-1}.

(c) (10 points) How does N_A compare to the current standard value? Comment on possible systematic errors in Perrin's determination of N_A.

2. Consider again Perrin's data on the number of mastic particles as a function of height, where the free parameters in the likelihood are the amplitude of the expected counts ν_0 and Boltzmann's constant k.

   (a) (15 points) Write a Markov Chain Monte Carlo (MCMC) to produce random samples (ν_0, k) from the likelihood function. Assume uniform priors on ν_0 and k. Plot the time evolution of ν_0 and k and indicate the end of the burn-in period. Then make a triangle plot of the joint distribution of ν_0 and k and their marginal distributions using values sampled after the burn-in. Note: you may use the public code emcee, which already implements optimizations such as parallel tempering.

   (b) (10 points) Repeat your MCMC but now use a uniform prior for ν_0 and a Jeffreys prior for k. Plot the results in a triangle plot. How do the joint and marginal likelihoods change?

   (c) (15 points) Sample from the likelihood using a uniform prior for ν_0 and a Jeffreys prior for k, but this time use nested sampling instead of MCMC. Plot the joint and marginal distributions in a triangle plot. Note: you may use the public code nestle to sample the joint posterior PDF.

3. At LEP II, the reaction $e^+e^- \to W^+W^-$ was used to study properties of the W boson. Suppose 10 W bosons are found (i.e. in 5 events), and out of the 10, 2 are found to decay to a muon and a neutrino.

   (a) (5 points) Find the 68.3% central confidence interval for the binomial parameter p for a W to decay to $\mu\nu$ (i.e. $p =$ branching ratio (W $\to \mu\nu$)). Express the answer as $p = \hat{p}^{+d}_{-c}$ where $\hat{p}$ is the ML estimate for p and $[\hat{p} - c, \hat{p} + d]$ is the confidence interval.

   (b) (5 points) Compare the interval from (a) to $\hat{p} \pm \sigma_{\hat{p}}$, where $\sigma_{\hat{p}}$ is the estimate of the standard deviation of $\hat{p}$.

   (c) (5 points) A common mistake is to regard the number of W bosons itself as a random variable and to include its variance in the error for $\hat{p}$ (e.g. using error propagation). Why is this not the correct approach for the error of a branching ratio?