On a Functional Equation Arising from Subcontrary Mean and Its Pertinences

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Abstract. Modelling equations involving functions is a powerful tool in many physical problems which do not require derivatives of function. The study of solution, stability and application of functional equations is an emerging field in the present scenario of research in abstract and applied mathematics. The purpose of this study is to deal with a new functional equation arising from subcontrary mean (harmonic mean) and its various fundamental stabilities relevant to Ulam's ideology of stability and also its pertinences in different fields such as physics, finance, geometry and in other sciences. We illustrate a numerical example to relate the equation dealt in this study with the fuel economy in automobiles.

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1 Introduction and Preliminaries

The research work on approximating functional, differential and integral inequalities is a hot topic in analysis. The historical background of the stability of mathematical equations is available in the literaure ([5,6,9,11,20]). There are many published papers and textbooks on various types of functional equations, their solutions and stability results; one can refer to [1,7,8,10,12,13,15,19]. The Ulam's approximation of several functional and differential equations are dealt via invariant point technique in ([2–4]). There are many interesting applications of functional equations, especially multiplicative inverse functional equations in various fields [16–18].

In this study, a new multiplicative inverse functional equation arising from subcontrary mean (harmonic mean) of the form

$$m_j\left(\frac{2pq}{p+q}\right) = \frac{1}{2}\left[m_j(p) + m_j(q)\right] \tag{1.1}$$

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is proposed. One can easily verify that the multiplicative inverse function $m_j(p) = \frac{1}{p}$ is a solution of Eq. (1.1). The fundamental stabilities of Eq. (1.1) are solved using fixed point alternative theorem and some applications of Eq. (1.1) are presented. Also, Eq. (1.1) is interpreted with the measurements of fuel economy in automobiles.

2 Stability Results of Subcontrary Mean Functional Equation (1.1)

In this fragmant, let us presume that $q \neq p$, for all $p, q \in \mathbb{R}^\star$. Then we proceed to obtain fundamental stability results of Eq. (1.1) involving a common control mapping as upper bound, a positive fixed constant and sum of powers of norms as upper bounds in the setting of real numbers excluding zero. For the purpose of uncomplicated computation, let us symbolize the operator D as below:

$$Dm_j(p,q) = m_j\left(\frac{2}{\frac{1}{p}+\frac{1}{q}}\right) - \frac{1}{2}\left[m_j(p) + m_j(q)\right].$$

Theorem 2.1. *Consider a function $m_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ with the condition that $m_j(p)$ tends to 0 when p tends to ∞. Also, assume that the function m_j satisfies the following inequality*

$$|Dm_j(p,q)| \leq \upsilon\left(\frac{1}{p}, \frac{1}{q}\right) \tag{2.1}$$

for all $p, q \in \mathbb{R}^\star$, where $\upsilon : \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow [0,\infty)$ is a given function. Suppose there persists $L < 1$ such that the mapping $p \mapsto \Upsilon(p) = \upsilon\left(\frac{1}{p}, 0\right)$ has the property $\Upsilon\left(\frac{p}{2}\right) \leq 2L\Upsilon(p)$ for all $p \in \mathbb{R}^\star$. If the mapping υ has the property

$$\lim_{n\to\infty} 2^{-n}\upsilon\left(2^{-n}p, 2^{-n}q\right) = 0, \tag{2.2}$$

for all $p, q \in \mathbb{R}^\star$, then a unique multiplicative inverse mapping $M_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ exists such that

$$|m_j(p) - M_j(p)| \leq \frac{1}{1-L}\Upsilon(p) \tag{2.3}$$

for all $p \in \mathbb{R}^\star$.

Proof. Let us define a set S as follows: $T = \{\phi : \mathbb{R}^\star \longrightarrow \mathbb{R}, \text{where } \phi \text{ is a mapping}\}$. Assume ρ be the generalized metric on T which is described as:

$$\rho(\psi,\phi) = \rho_\Upsilon(\psi,\phi) = \inf\{\lambda > 0 : |\psi(p) - \phi(p)| \leq \lambda\Upsilon(p),\ \text{for all } p \in \mathbb{R}^\star\}. \tag{2.4}$$

From the above definition of ρ shows that the set T is complete space. Now, define a mapping $\mu : T \longrightarrow T$ by

$$\mu\phi(p) = \frac{1}{2}\phi\left(\frac{p}{2}\right) \qquad (p \in \mathbb{R}^\star) \tag{2.5}$$

for all $\phi \in T$. Next, let us show that μ is a strictly contractive function on the set T. For given $\psi, \phi \in T$, suppose $0 \leq \lambda_{\psi\phi} \leq \infty$ is an arbitrary constant with $\rho(\psi, \phi) \leq \lambda_{\psi\phi}$. Therefore, we have

$$\begin{aligned}
\rho(\psi, \phi) < \lambda_{\psi\phi} &\Longrightarrow |\psi(p) - \phi(p)| \leq \lambda_{\psi\phi}\Upsilon(p), \quad (\forall p \in \mathbb{R}^\star) \\
&\Longrightarrow \left|\frac{1}{2}\psi\left(\frac{p}{2}\right) - \frac{1}{2}\phi\left(\frac{p}{2}\right)\right| \leq \frac{1}{2}\lambda_{\psi\phi}\Upsilon\left(\frac{p}{2}\right), \quad (\forall p \in \mathbb{R}^\star) \\
&\Longrightarrow \left|\frac{1}{2}\psi\left(\frac{p}{2}\right) - \frac{1}{2}\phi\left(\frac{p}{2}\right)\right| \leq L\lambda_{\psi\phi}\Upsilon(p), \quad (\forall p \in \mathbb{R}^\star) \\
&\Longrightarrow \rho(\mu\psi, \mu\phi) \leq L\lambda_{\psi\phi}.
\end{aligned}$$

The above inequality implies that $\rho(\mu\psi, \mu\phi) \leq L\rho(\psi, \phi)$ for all $\psi, \phi \in T$, which inturn indicates that μ is a strictly contractive mapping of T, with the Lipschitz constant L. Now, plugging (p, q) by $(p, 0)$ in (2.1), we get

$$\left|\frac{1}{2}m_j\left(\frac{p}{2}\right) - m_j(p)\right| \leq \upsilon\left(\frac{1}{p}, 0\right) = \Upsilon(p)$$

for all $p \in \mathbb{R}^\star$. Hence (2.4) produces that $\rho(\mu r_j, r_j) \leq 1$. So, by employing the fixed point alternative Theorem, there exists a function $M_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying the following:

(1) M_j is a fixed point of ρ, that is

$$M_j\left(\frac{p}{2}\right) = 2M_j(p) \tag{2.6}$$

for all $p \in \mathbb{R}^\star$. The mapping M_j is the distinctive invariant point of μ in the set $\Gamma = \{\phi \in T : \rho(M_j, m_j) < \infty\}$. This implies that M_j is the unique mapping satisfying (2.6) such that there exists $0 < \lambda < \infty$ satisfying $|M_j(p) - m_j(p)| \leq \lambda\Upsilon(p),\ \forall p \in \mathbb{R}^\star$.

(2) $\rho\left(\mu^n m_j, M_j\right) \to 0$ as $n \to \infty$. Thus, we have

$$\lim_{n\to\infty} 2^{-n} m_j\left(2^{-n}p\right) = M_j(p) \tag{2.7}$$

for all $p \in \mathbb{R}^\star$.

(3) $d(M_j, m_j) \leq \frac{1}{1-L}\rho(M_j, \mu m_j)$, which implies $\rho(M_j, m_j) \leq \frac{1}{1-L}$.

So, the inequality (2.3) holds. On the other hand, from (2.1), (2.2) and (2.7), we have

$$\begin{aligned}
\left|DM_j(p,q)\right| &= \lim_{n\to\infty} 2^{-n}\left|m_j\left(\frac{2}{\frac{1}{2^{-n}p} + \frac{1}{2^{-n}q}}\right) - \frac{1}{2}\left[2^{-n}m_j(2^{-n}p) + 2^{-n}m_j(2^{-n}q)\right]\right| \\
&\leq \lim_{n\to\infty} 2^{-n}\upsilon\left(2^{-n}p, 2^{-n}q\right) = 0
\end{aligned}$$

for all $p, q \in \mathbb{R}^\star$, which shows that M_j is a solution of the Eq. (1.1) and hence $M_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is a multiplicative inverse function. Now, we exhibit that M_j is the distinctive multiplicative inverse mapping satisfying (1.1) and (2.3). Let us suppose that $M_j' : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be one more multiplicative inverse function satisfying (1.1) and (2.3). Since M_j' is an invariant point of μ and $\rho(m_j, M_j') < \infty$, we have $M_j' \in T^\star = \{\psi \in T | \rho(m_j, \psi) < \infty\}$. From the invariant point alternative theorem and since both M_j and M_j' are invariant points of μ, we have $M_j = M_j'$. Therefore, M_j is unique which completes the proof of Theorem 2.1. □

The upcoming theorem is the dual of Theorem 2.1. The proof is obtained by similar arguments as in Theorem 2.1 and so, for completeness, we present only the statement.

Theorem 2.2. *Suppose that the mapping* $m_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *satisfies the condition* $m_j(\infty) = 0$ *and the inequality* (2.1), *where* $\upsilon : \underbrace{\mathbb{R}^\star \times \mathbb{R}^\star}_{m \ times} \longrightarrow [0, \infty)$ *is a specified function. Suppose there is an* $L < 1$ *exists such that the function* $p \mapsto \Upsilon(p) = 2\upsilon\left(\frac{2}{p}, 0\right)$ *has the property* $\Upsilon(2p) \leq \frac{1}{2} L \Upsilon(p)$, *for all* $p \in \mathbb{R}^\star$. *If the mapping* υ *has the property* $\lim_{n\to\infty} 2^n \upsilon(2^n p, 2^n q) = 0$, *for all* $p, q \in \mathbb{R}^\star$, *then a unique multiplicative inverse function* $M_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *exists such that* $|m_j(p) - M_j(p)| \leq \frac{1}{1-L}\Upsilon(p)$ *for all* $p \in \mathbb{R}^\star$.

The following corollary is the investigation of various stabilities of Eq. (1.1) pertinent to UHS and UHR stability. The proof directly follow from the above theorems.

Corollary 2.3. *Let* $m_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *be a function. Let there exists a constant* η (*not depending on* p, q)≥ 0 *and real numbers* $\ell \neq -1$ *and* $\delta \geq 0$ *such that the functional inequality*

$$|Dm_j(p,q)| \leq \begin{cases} \frac{\eta}{2} \\ \delta\left(\left|\frac{1}{p}\right|^\ell + \left|\frac{1}{q}\right|^\ell\right) \end{cases}$$

holds for all $p, q \in \mathbb{R}$. *Then a distinctive multiplicative inverse function* $M_j : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *exists which satisfies* (1.1) *and*

$$|m_j(\alpha) - M_j(\alpha)| \leq \begin{cases} \eta \\ \frac{2^{\ell+1}\delta}{|2^{\ell+1}-1|}\left|\frac{1}{p}\right|^\ell, \quad \ell \neq -1 \end{cases}$$

for all $p \in \mathbb{R}^\star$.

Proof. The proof is obtained by considering $\upsilon\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{\eta}{2}, \delta\left(\left|\frac{1}{p}\right|^\ell + \left|\frac{1}{q}\right|^\ell\right)$ for all $p, q \in \mathbb{R}^\star$, and then selecting $L = \frac{1}{2}$ in Theorems 2.1 and 2.2. □

3 Applications of Equation (1.1)

We summarize here various applications of Eq. (1.1) in many other fields where harmonic mean is involved.

- In chemistry, the density of an alloy is the harmonic mean of densitiies of its constituents. Hence Eq. (1.1) can be used to estimate the density of the alloy.
- In an electrical circuit, the effective resistance of two resistors connected in parallel is the harmonic mean of the resistance of the parallel resistor. Hence to calculate the effective resistance of an electric circuit, we can apply Eq. (1.1).
- In finance, the price-earning ratio is the harmonic mean of data points since it gives equal weight to each data point. Thus we can estimate price-earning ratio using Eq. (1.1).
- In geometry, consider an incircle in a triangle. Then the radius of the incircle is equal to the one-third of the harmonic mean of altitudes of the triangle. In this situation, Eq. (1.1) can be utilized.
- In computer science, to evaluate algorithms and systems especially in machine learning and information retrieval, the harmonic mean of the precision and the recall is employed as an accumulated performance score. Hence to evaluate algorithms and systems, we can use Eq. (1.1).

4 Interpretation of Equation (1.1)

We wind up this study with an interpretation of Eq. (1.1) with the standard measurements of fuel economy in automobiles.

There are two standard measurements used for fuel economy in automobiles. They are miles per gallon and litres per 100 km. It is clear that the dimensions of these quantitites are reciprocal to each other. Hence, the calculation of average value of the fuel economy of a car in one measurement implies the harmonic mean of the other. In other words, the average value of fuel economy expressed in litres per 100 km to miles per gallon will produce the harmonic mean of the fuel economy expressed in miles per gallon.

4.1 Numerical Example

Suppose there are two cars with fuel economy 10 L/100 km and 20 L/100 km, respectively. Let $m_j(p)$ and $m_j(q)$ denote fuel economy of the cars. Since $m_j(p) = \frac{1}{p}$, the value of the right hand side of Eq. (1.1) is

$$\frac{1}{2}[m_j(p) + m_j(q)] = \frac{1}{2}\left[\frac{1}{10} + \frac{1}{20}\right] = 0.075 \quad \text{L/100 km.}$$

Now,

$$\begin{aligned} p &= \frac{1}{10} \quad \text{L/100 km} = 2824.81 \quad \text{miles/gal,} \\ q &= \frac{1}{20} \quad \text{L/100 km} = 5649.81 \quad \text{miles/gal.} \end{aligned}$$

Then, $\frac{2pq}{p+q} = 3766.41 \quad \text{miles/gal.}$

But, $0.075\,\text{L/100 km} = 3766.41 \quad \text{miles/gal.}$

Thus, the arithmetic mean of $\frac{1}{p}$ and $\frac{1}{q}$ is obtained by the mapping of harmonic mean of p and q by the solution of Eq. (1.1).

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