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Approximation of multiplicative inverse undecic and duodecic functional equations

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This paper describes the classical investigation of various stability results of multiplicative inverse undecic and duodecic functional equations pertinent to Ulam stabiity theory in non-Archimedean fields via fixed point technique and then presents two refutations that the stability results are invalid for control cases.

KEYWORDS

generalized Ulam-Hyers stability, rational functional equation, reciprocal function

1 | INTRODUCTION

We frequently meet up with the ensuing question that occurs naturally when mathematical and computational techniques are applied in science and engineering: What are the misinterpretations we perpetrate, when a function is reinstated which satisfies an equation near to the correct solutions of that equation? This question is refined as a significant problem by Ulam[1] pertinent to stability of functional equations. As of now, it is a huge vicinity of research and pertaining to functional, difference, differential, and integral equations. For easy understanding, stability of an equation deals with the question: When an approximate solution of an equation is near to the exact solution of that equation? The equation is said to be stable of there is a decisive justification. Subsequently, Ulam's question was first partially answered by Hyers.[2] The result of Hyers was generalized by Aoki,[3] T. M. Rassias,[4] J. M. Rassias[5] in various versions. Gavurta[6] promoted the stability results of T. M. Rassias and J. M. Rassias by replacing the upper bound as a general control function.

The p-adic numbers were discovered by Hensel[7] in 1897 as a number theoretical analogue of power series in complex analysis. He introduced a field with a valuation norm that does not satisfy the Archimedean property.

For the first time, Ravi and the first author of this paper[8] made a breakthrough in the field of stability theory of functional equations by considering rational functional equation. They examined disparate stability results of the following rational functional equation

$$\mathcal{R}(m+n) = \frac{\mathcal{R}(m)\mathcal{R}(n)}{\mathcal{R}(m)+\mathcal{R}(n)} \tag{1.1}$$

in the setting of nonzero real numbers.

There are many significant and remarkable results concerning the stability of different forms of functional equations, one can refer to the studies of Jung, Mirmostafaee, and Park, Shin, Saadati, and Lee.[9-11]

In this investigation, we introduce rational functional equations of the form

$$u(2s+t)+u(2s-t)=\frac{4u(s)u(t)}{\left(4u(t)^{2/11}-u(s)^{2/11}\right)^{11}}\left[\frac{1}{2}\sum_{k=0}^{5}\binom{11}{2k}[u(s)]^{2k/11}[u(t)]^{(11-2k)/11}\right] \tag{1.2}$$

and

$$v(2s+t)+v(2s-t)=\frac{4v(s)v(t)}{\left(4v(t)^{1/6}-v(s)^{1/6}\right)^{12}}\left[\sum_{k=0}^{6}\binom{12}{2k}[v(s)]^{k/6}[v(t)]^{(12-2k)/12}\right]. \tag{1.3}$$

It is easy to verify that the multiplicative inverse undecic function $u(s)=\frac{1}{s^{11}}$ and the multiplicative inverse duodecic function $v(s)=\frac{1}{s^{12}}$ are solutions of the Equations (1.2) and (1.3), respectively. Using fixed point method, we obtain the generalized Ulam-Hyers stabilty of the above Equations (1.2) and (1.3) in non-Archimedean fields and also present suitable counter-examples to portray that the stability results are invalid for control cases.

2 | MATHEMATICAL FORMULATION OF FUNCTIONAL EQUATIONS (1.2) AND (1.3)

In this section, we elucidate the mathematical formulation of Equations (1.2) and (1.3). For an odd positive integer n, the following algebraic identity

$$\frac{1}{(2s+t)^n}+\frac{1}{(2s-t)^n}=\frac{\frac{4}{s^nt^n}}{\left(\frac{4}{t^2}-\frac{1}{s^2}\right)^n}\left[\frac{1}{2}\sum_{k=0}^{(n-1)/2}\binom{n}{2k}\frac{2^{n-2k}}{s^{2k}t^{n-2k}}\right]$$

leads to the following functional equation

$$u(2s+t)+u(2s-t)=\frac{4u(s)u(t)}{\left(4u(t)^{2/n}-u(s)^{2/n}\right)^{n}}\left[\frac{1}{2}\sum_{k=0}^{(n-1)/2}\binom{n}{2k}[u(s)]^{2k/n}[u(t)]^{(n-2k)/n}\right] \tag{2.1}$$

with the rational function $u(s)=\frac{1}{s^n}$ as solution. Similarly, for an even positive integer n, the following algebraic identity

$$\frac{1}{(2s+t)^n}+\frac{1}{(2s-t)^n}=\frac{\frac{4}{s^nt^n}}{\left(\frac{4}{t^2}-\frac{1}{s^2}\right)^n}\left[\sum_{k=0}^{n/2}\binom{n}{2k}\frac{2^{n-2k}}{s^{2k}t^{n-2k}}\right]$$

produces the ensuing functional equation

$$v(2s+t)+v(2s-t)=\frac{4v(s)v(t)}{\left(4v(t)^{2/n}-v(s)^{2/n}\right)^{n}}\left[\sum_{k=0}^{n/2}\binom{n}{2k}[v(s)]^{2k/n}[v(t)]^{(n-2k)/n}\right] \tag{2.2}$$

with the rational function $v(s)=\frac{1}{s^n}$ as solution. From the above Equations (2.1) and (2.2), one can find that the Equations (1.2) and (1.3) are particular cases of $n=11$ and $n=12$, respectively. For $n=3,5,7,9,10$, respectively, the Equation (2.1) generates multiplicative inverse cubic, quintic, septic, and nonic functional equations. In the same way, for $n=2,4,6,8,10$, respectively, the Equation (2.2) produces multiplicative inverse quadratic, quartic, sextic, octic, and decic functional equations.

3 | PRELIMINARIES

We elicit a few primitive ideas of non-Archimedean field in this section. We also furnish the fixed point alternative principle in non-Archimedean adaptation.

Definition 3.1. Suppose $\mathbb{F}$ is a field with a function (valuation) $|\cdot|$ from $\mathbb{F}$ into $[0,\infty)$. Then $\mathbb{F}$ is called a non-Archimedean field if the following conditions hold: (a) $|m|=0$ if and only if $m=0$; (b) $|mn|=|m||n|$; (c) $|m+n|\leq\max\{|m|,|n|\}$ for all $m,n\in\mathbb{F}$.

Obviously, $|1| = |-1| = 1$ and $|m| \leq 1$ for all $m \in \mathbb{N}$. Furthermore, we presume that $|\cdot|$ is nontrivial; that is, there exists an $\alpha_0 \in \mathbb{F}$ such that $|\alpha_0| \neq 0, 1$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions: (a) $||u|| = 0$ if and only if $u = 0$; (b) $||\rho u|| = |\rho|||u||(\rho \in \mathbb{K}, u \in X)$; (c) the strong triangle inequality (ultrametric); namely,

$$||u+v|| \leq \max\{||u||, ||v||\} \qquad (u, v \in X).$$

Then, $(X, ||\cdot||)$ is called a non-Archimedean space. By virtue of the inequality,

$$\|u_n - u_m\| \leq \max\left\{\|u_{j+1} - u_j\| : m \leq j \leq n-1\right\} \qquad (n > m),$$

a sequence $\{u_n\}$ is Cauchy if and only if $\{u_{n+1} - u_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean that every Cauchy sequence is convergent in the space.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{K}$ is the mapping

$$|\tau| = \begin{cases} 0 & \text{if} \quad \beta = 0 \\ \frac{1}{\beta} & \text{if} \quad \beta > 0 \\ -\frac{1}{\beta} & \text{if} \quad \beta < 0 \end{cases}$$

for any $\beta \in \mathbb{K}$.

Example 3.2. For a given prime number p, the p-adic absolute value in $\mathbb{Q}$ is defined as follows: If u is a nonzero rational number, then there exists a unique integer r such that $u = p^r \frac{a}{b}$, where a and b are coprime to p. Set $ord_p(u) = r$, and $|u|_p = p^{-r}$. Then, $ord_p(u)$ is called the p-adic valuation of u and $|u|_p$ is called the p-adic absolute value of u.

(i) By the definition of $|\cdot|_p$, it is clear that $|u|_p = 0$ if and only if $u = 0$.
(ii) $|u|_p|v|_p = p^{-ord_p(u)}p^{-ord_p(v)}$. By the fundamental theorem of arithmetic, the number of prime factors p in uv is the same as the sum of the factors in u and v individually $(ord_p(u) + ord_p(v))$. Hence, we have $|uv|_p = p^{-ord_p(u)}p^{-ord_p(v)}$.
(iii) $|u+v|_p = p^{-ord_p(u+v)} \leq max\left\{p^{-ord_p(u)}, p^{-ord_p(v)}\right\} = max\left\{|u|_p, |v|_p\right\}$.

Hence, the p-adic absolute value defined above is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n.

Definition 3.3. Let A be a nonempty set and $d : A \times A \to [0, \infty]$ satisfy the ensuing properties: (a) $d(a, b) = 0$ if and only if $a = b$; (b) $d(a, b) = d(b, a)$ (symmetry); and (c) $d(a, c) \leq \max\{d(a, b), d(b, c)\}$ (strong triangle inequality) for all $a, b, c \in A$. Then, (A, d) is called a generalized non-Archimedean metric space. (A, d) is called complete if every Cauchy sequence in A is convergent.

Example 3.4. For each nonempty set A, define

$$d(u, u^\star) = \begin{cases} 0 & \text{if} \quad u = u^\star \\ \infty & \text{if} \quad u \neq u^\star. \end{cases}$$

Then, d is a generalized non-Archimedean metric on A.

Example 3.5. Let A and B be two non-Archimedean spaces over a non-Archimedean field $\mathbb{K}$. If B has a complete non-Archimedean norm over $\mathbb{K}$ and $\phi : A \to [0, \infty)$, for each $s, t : A \to B$, define

$$d(s, t) = \inf\{\delta > 0 : |s(u) - t(u)| \leq \delta\phi(u), \quad \forall u \in A\}.$$

Using[12] Theorem 2.5, Mirmostafaee[13] introduced non-Archimedean version of the alternative fixed point theorem as follows:

Theorem 3.6. *[13](Non-Archimedean Alternative Contraction Principle) If (A, d) is a non-Archimedean generalized complete metric space and $\Gamma : A \to A$ a strictly contractive mapping (that is, $d(\Gamma(x), \Gamma(y)) \leq Ld(y, x)$, for all $x, y \in A$ and a Lipschitz constant $L < 1$), then either*

(i) $d\left(\Gamma^n(x), \Gamma^{n+1}x\right) = \infty$ for all $n \geq 0$, or
(ii) there exists some $n_0 \geq 0$ such that $d\left(\Gamma^n(x), \Gamma^{n+1}(x)\right) < \infty$ for all $n \geq n_0$;

the sequence $\{\Gamma^n(x)\}$ is convergent to a fixed point $x^\star$ of Γ; $x^\star$ is the unique fixed point of Γ in the set $Y = \{y \in X : d(\Gamma^{n_0}(x), y) < \infty\}$ and $d\left(y, x^\star\right) \leq d(y, \Gamma(y))$ for all y in this set.

Throughout this paper, let us assume that $\mathbb{G}$ and $\mathbb{H}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. In the following, we represent $\mathbb{G}^* = \mathbb{G}\backslash\{0\}$, where $\mathbb{G}$ is a non-Archimedean field. For the purpose of simplification, we describe the difference operators $\Delta_1, \Delta_2 : \mathbb{G}^* \times \mathbb{G}^* \to \mathbb{H}$ by

$$\begin{aligned}\Delta_1 u(s,t) &= u(2s+t) + u(2s-t)\\ &\quad - \frac{4u(s)u(t)}{\left(4u(t)^{2/11} - u(s)^{2/11}\right)^{11}} \left[\frac{1}{2}\sum_{k=0}^{5} \binom{11}{2k} [u(s)]^{2k/11}[u(t)]^{(11-2k)/11}\right]\end{aligned}$$

and

$$\begin{aligned}\Delta_2 v(s,t) &= v(2s+t) + v(2s-t)\\ &\quad - \frac{4v(s)v(t)}{\left(4v(t)^{1/6} - v(s)^{1/6}\right)^{12}} \left[\sum_{k=0}^{6} \binom{12}{2k} [v(s)]^{k/6}[v(t)]^{(12-2k)/12}\right]\end{aligned}$$

for all $s, t \in \mathbb{G}^*$.

4 | VARIOUS STABILITIES OF EQUATIONS (1.2) AND (1.3)

In this section, we obtain various stability results of Equations (1.2) and (1.3) in non-Archimedean fields, using fixed point method.

Definition 4.1. A mapping $u : \mathbb{G}^* \to \mathbb{H}$ is called as multiplicative inverse undecic mapping if u satisfies the Equation (1.2), and hence, Equation (1.2) is called as a multiplicative inverse undecic functional equation. Also, a mapping $v : \mathbb{G}^* \to \mathbb{H}$ is called as multiplicative inverse duodecic mapping if v satisfies the Equation (1.3), and so Equation (1.3) is said to be a multiplicative inverse duodecic functional equation.

Exploration and presumptions on the above definition and Equations (1.2) and (1.3): First, we observe that in the above definition, the equalities $t = 2s$ and $t = -2s$ can not occur since $2s - t$ and $2s + t$ do not belong to $\mathbb{G}^*$. On the other hand, in Equation (1.2) if $4u(t)^{2/11} - u(s)^{2/11} = 0$, then this is equivalent to $u(t) = u(2s)$. Since the multiplicative inverseundecic function $u(s) = \frac{1}{s^{11}}$ is the solution of Equation (1.2), this indicates that $t = 2s$, which is impossible. But, by presuming $u(s) \neq 0$, $u(t) \neq 0$, $v(s) \neq 0$, $v(t) \neq 0$, $4u(t)^{2/11} - u(s)^{2/11} \neq 0$, and $4v(t)^{1/6} - v(s)^{1/6} \neq 0$ for all $s, t \in \mathbb{G}^*$, we can elude the singular cases.

Theorem 4.2. *Let $k \in \{-1, 1\}$. Suppose a mapping $u : \mathbb{G}^* \to \mathbb{H}$ satisfies the inequality*

$$|\Delta_1 u(s,t)| \leq \zeta(s,t) \tag{4.1}$$

for all $s, t \in \mathbb{G}^*$, where $\zeta : \mathbb{G}^* \times \mathbb{G}^* \to \mathbb{H}$ is a given function. If $0 < L < 1$,

$$|3|^{11k}\zeta\left(3^k s, 3^k t\right) \leq L\zeta(s,t) \tag{4.2}$$

for all $s, t \in \mathbb{G}^*$, then there exists a unique multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.2) and

$$|u(s) - \mathcal{U}(s)| \leq L|3|^{11k}\zeta(s,s) \tag{4.3}$$

for all $s \in \mathbb{G}^*$.

Proof. First, let us prove this theorem for the case $k = -1$. Considering t as $\frac{s}{3}$ in Equation (4.1), we obtain

$$\left|u(s) - \frac{1}{3^{11}}u\left(\frac{s}{3}\right)\right| \leq \zeta\left(\frac{s}{3}, \frac{s}{3}\right) \tag{4.4}$$

for all $s \in \mathbb{G}^*$. Let $\mathcal{F} = \{g | g : \mathbb{G}^* \to \mathbb{H}\}$,

$$d(g,h) = \inf\{\beta > 0 : |g(s) - h(s)| \leq \beta\zeta(s,s), \quad \text{for all} \quad s \in \mathbb{G}^*\}. \tag{4.5}$$

Now, let us prove that $(\mathcal{F}, d)$ is complete. Using the idea from,[14] we prove the completeness of $(\mathcal{F}, d)$. Let $\{h_n\}$ be a Cauchy sequence in $(\mathcal{F}, d)$. Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $d(h_m, h_n) \leq \epsilon$ for all $m, n \geq N_\epsilon$. From Equation (4.5), we arrive

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall m, n \geq N_\epsilon, \forall s \in \mathbb{G}^*, |h(s) - h_n(s)| \leq \epsilon \zeta(s, s). \tag{4.6}$$

If s is a fixed number, Equation (4.6) implies that $\{h_n(s)\}$ is a Cauchy sequence in $(\mathbb{H}, |\cdot|)$. Since $(\mathbb{H}, |\cdot|)$ is complete, $\{h_n(s)\}$ converges for all $s \in \mathbb{G}^*$. Therefore, we can define a function $h : \mathbb{G}^* \to \mathbb{H}$ by $h(s) = \lim_{n\to\infty} h_n(s)$ and hence $h \in \mathcal{F}$. Letting $m \to \infty$ in Equation (4.6), we have

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon, \forall s \in \mathbb{G}^* : |h(s) - h_n(s)| \leq \epsilon \zeta(s, s).$$

By considering Equation (4.5), we arrive

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon : d(h, h_n) \leq \epsilon,$$

which implies that the Cauchy sequence $\{h_n\}$ converges to h in $(\mathcal{F}, d)$. Hence, $(\mathcal{F}, d)$ is complete.

Define $\rho : \mathcal{F} \to \mathcal{F}$ by $\rho(g)(s) = 3^{-11} g\left(3^{-11}s\right)$ for all $s \in \mathbb{G}^*$ and $g \in \mathcal{F}$. Then, ρ is strictly contractive on $\mathcal{F}$, in fact if $|g(s) - h(s)| \leq \beta \zeta(s, s), \quad (s \in \mathbb{G}^*)$; then by Equation (4.2), we obtain

$$\begin{aligned} |\rho(g)(s) - \rho(h)(s)| &= |3|^{-11} \left| g\left(3^{-1}s\right) - h\left(3^{-1}s\right) \right| \\ &\leq \beta |3|^{-11} \zeta\left(3^{-1}s, 3^{-1}s\right) \leq \beta L \zeta(s, s) \quad (s \in \mathbb{G}^*). \end{aligned}$$

From the above, we conclude that $(\rho(g), \rho(h)) \leq Ld(g, h) \quad (g, h \in \mathcal{F})$. Hence, d is strictly contractive mapping with Lipschitiz constant L. Using Equation (4.4), we have

$$|\rho(u)(s) - u(s)| = \left| 3^{-11} u\left(3^{-1}s\right) - u(s) \right| \leq \zeta\left(\frac{s}{3}, \frac{s}{3}\right) \leq |3|^{11} L \zeta(s, s) \quad (s \in \mathbb{G}^*).$$

This indicates that $d(\rho(u), u) \leq L|3|^{11}$. Because of Theorem 3.6 (ii), ρ has a unique fixed point $\mathcal{U} : \mathbb{G}^* \to \mathbb{H}$ in the set $Y = \{g \in \mathcal{F} : d(u, g) < \infty\}$ and for each $s \in \mathbb{G}^*$,

$$\mathcal{U}(s) = \lim_{n\to\infty} \rho^n u(s) = \lim_{n\to\infty} 3^{-11n} u\left(3^{-n}s\right) \quad (s \in \mathbb{G}^*).$$

Therefore, for all $s, t \in \mathbb{G}^*$,

$$\begin{aligned} |\Delta_1 u(s, t)| &= \lim_{m\to\infty} |3|^{-11m} \left| \Delta_1 u\left(3^{-m}s, 3^{-m}t\right) \right| \\ &\leq \lim_{m\to\infty} |3|^{-11m} \zeta\left(3^{-m}s, 3^{-m}t\right) \\ &\leq \lim_{m\to\infty} L^m \zeta(s, t) = 0, \end{aligned}$$

which shows that $\mathcal{U}$ is multiplicative inverse undecic mapping. By Theorem 3.6 (ii), we have $d(u, \mathcal{U}) \leq d(\rho(u), u)$, that is, $|u(s) - (U)(s)| \leq |3|^{-11} L \zeta(s, s) \quad (s \in \mathbb{G}^*)$. Let $\mathcal{U}' : \mathbb{G}^* \to \mathbb{H}$ be a multiplicative inverse undecic mapping that satisfies (4.3); then $\mathcal{U}'$ is a fixed point of ρ in $\mathcal{F}$. However, by Theorem 3.6, ρ has only one fixed point in Y. Similarly, the theorem can be proved for the case $k = 1$. □

The succeeding corollaries are immediate repercussions of Theorem 4.2. In the following corollaries, we assume that $|2| < 1$ for a non-Archimdean field $\mathbb{G}$.

Corollary 4.3. *Let δ(independent of s, t)≥ 0 be a constant exists for a mapping $u : \mathbb{G}^* \to \mathbb{H}$ such that the functional inequality $|\Delta_1 u(s, t)| \leq \delta$ for all $s, t \in \mathbb{G}^*$. Then, there exists a unique multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.2) and $|u(s) - \mathcal{U}(s)| \leq \delta$ for all $s \in \mathbb{G}^*$.*

Proof. Assuming $\zeta(s, t) = \delta$ and selecting $L = |3|^{-11}$ in Theorem 4.2, we get the desired result. □

Corollary 4.4. *Let $\beta \neq -11$ and $k_1 \geq 0$ be real numbers exists for a mapping $u : \mathbb{G}^* \to \mathbb{H}$ such that $|\Delta_1 u(s,t)| \leq k_1 \left(|s|^\beta + |t|^\beta\right)$ for all $s, t \in \mathbb{G}^*$. Then, there exists a unique multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.2) and*

$$|u(s) - \mathcal{U}(s)| \leq \begin{cases} \frac{|2|k_1}{|3|^\beta}|s|^\beta, & \beta > -11 \\ |2|k_1|3|^{11}|s|^\beta, & \beta < -11 \end{cases}$$

for all $s \in \mathbb{G}^*$.

Proof. Consider $\zeta(s,t) = k_1\left(|s|^\beta + |t|^\beta\right)$ in Theorem 4.2 and the assume $L = |3|^{-\beta-11}$, $\beta > -11$ and $L = |3|^{\beta+11}$, $\beta < -11$, respectively, for each case of k, the proof follows directly. □

Corollary 4.5. *Let $k_2 \geq 0$ and $\beta \neq -11$ be real numbers, and $u : \mathbb{G}^* \to \mathbb{H}$ be a mapping satisfying the functional inequality $|\Delta_1 u(s,t)| \leq k_2|s|^{\beta/2}|t|^{\beta/2}$ for all $s, t \in \mathbb{G}^*$. Then, there exists a unique multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.2) and*

$$|u(s) - \mathcal{U}(s)| \leq \begin{cases} \frac{k_2}{|3|^\beta}|s|^\beta, & \beta > -11 \\ |3|^{11}k_2|s|^\beta, & \beta < -11 \end{cases}$$

for all $s \in \mathbb{G}^*$.

Proof. It is easy to prove this corollary, by taking $\zeta(s,t) = k_2|s|^{\beta/2}|t|^{\beta/2}$ and then choosing $L = |3|^{-\beta-11}$, $\beta > -11$ and $L = |3|^{\beta+11}$, $\beta < -11$, respectively for each case k in Theorem 4.2. □

In the sequel, by applying fixed point technique, we investigate various stabilities of Equation (1.3) in non-Archimedean fields. Since the proof of the following results are similar to the results obtained for the Equation (1.2), for the sake of completeness, we state only theorems and skip their proofs.

Theorem 4.6. *Let $k \in \{-1, 1\}$. Let $v : \mathbb{G} \to \mathbb{H}$ be a mapping satisfying the inequality $|\Delta_2 v(s,t)| \leq \eta(s,t)$ for all $s, t \in \mathbb{G}^*$, where $\eta : \mathbb{G}^* \times \mathbb{G}^* \to \mathbb{H}$ is a given function. If $0 < L < 1$, $|3|^{12k}\eta\left(3^k s, 3^k t\right) \leq L\eta(s,t)$ for all $s, t \in \mathbb{G}^*$, then there exists a unique multiplicative inverse duodecic mapping $\mathcal{V} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.3) and $|v(s) - \mathcal{V}(s)| \leq L|3|^{12k}\eta(s,s)$ for all $s \in \mathbb{G}^*$.*

Corollary 4.7. *Let ρ (independent of x, y)≥ 0 be a constant. Suppose a mapping $v : \mathbb{G}^* \to \mathbb{H}$ satisfies the inequality $|\Delta_2 v(s,t)| \leq \rho$ for all $x, y \in \mathbb{G}^*$. Then, there exists a unique multiplicative inverse duodecic mapping $\mathcal{V} : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.3) and $|v(x) - \mathcal{V}(x)| \leq \rho$ for all $x \in \mathbb{G}^*$.*

Corollary 4.8. *Let $\beta \neq -12$ and $\theta_1 \geq 0$ be real numbers. If $v : \mathbb{G}^* \to \mathbb{H}$ is a mapping satisfies the inequality $|\Delta_2 v(s,t)| \leq \theta_1\left(|s|^\beta + |t|^\beta\right)$ for all $s, t \in \mathbb{G}^*$, then there exists a unique multiplicative inverse duodecic mapping $v : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.3) and*

$$|v(s) - \mathcal{V}(s)| \leq \begin{cases} \frac{|2|\theta_1}{|3|^\beta}|s|^\beta, & \beta > -12 \\ |2|\theta_1|3|^{12}|s|^\beta, & \beta < -12 \end{cases}$$

for all $s \in \mathbb{G}^*$.

Corollary 4.9. *Let $v : \mathbb{G}^* \to \mathbb{H}$ be a mapping and $\theta_2 \geq 0$ and $\beta \neq -12$ be real numbers. If the mapping v satisfies the functional inequality $|\Delta_2 v(s,t)| \leq \theta_2|s|^{\beta/2}|t|^{\beta/2}$ for all $s, t \in \mathbb{G}^*$, then there exists a unique multiplicative inverse duodecic mapping $v : \mathbb{G}^* \to \mathbb{H}$ satisfying the functional Equation (1.3) and*

$$|v(s) - \mathcal{V}(s)| \leq \begin{cases} \frac{\theta_2}{|3|^\beta}|s|^\beta, & \beta > -12 \\ |3|^{12}\theta_2|s|^\beta, & \beta < -12 \end{cases}$$

for all $s \in \mathbb{G}^*$.

5 | REFUTATIONS

We bring this paper to an end with two refutations. The famous counter-example provided by Gajda[15] induced to prove that the stability results of Equations (1.2) and (1.3) do not hold for control cases. In this section, we show that the Equations (1.2) and (1.3) are not valid for $\beta = -11$ in Corollary 4.4 and $\beta = -12$ in Corollary 4.8, respectively.

Example 5.1. Let us cogitate the function

$$\chi(s) = \begin{cases} \frac{c}{s^{11}}, & \text{for } s \in (1, \infty) \\ c, & \text{elsewhere} \end{cases} \tag{5.1}$$

where $\chi : \mathbb{R}^* \to \mathbb{R}$. Let $u : \mathbb{R}^* \to \mathbb{R}$ be a function defined as follows:

$$u(s) = \sum_{m=0}^{\infty} 177147^{-m} \chi(3^{-n}s), \tag{5.2}$$

for all $s \in \mathbb{R}$. Suppose the mapping $u : \mathbb{R}^* \to \mathbb{R}$ described in Equation (5.2) satisfies the functional inequality

$$|\Delta_1 u(s,t)| \leq \frac{265721\ c}{88573} \left(|s|^{-11} + |t|^{-11}\right) \tag{5.3}$$

for all $s, t \in \mathbb{R}^*$. We prove that there does not exist a multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{R}^* \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|u(s) - \mathcal{U}(s)| \leq \delta |s|^{-11} \tag{5.4}$$

for all $s \in \mathbb{R}^*$. Firstly, let us prove that u satisfies Equation (5.3). Using Equation (5.1), we have

$$|u(s)| = \left| \sum_{m=0}^{\infty} 177147^{-m} \chi(3^{-n}s) \right| \leq \sum_{m=0}^{\infty} \frac{c}{177147^m} = \frac{177147}{177146} c.$$

We observe that u is bounded by $\frac{177147\quad c}{177146}$ on $\mathbb{R}$. If $|s|^{-11} + |t|^{-11} \geq 1$, then the left hand side of Equation (5.3) is less than $\frac{265721\ c}{88573}$. Now, suppose that $0 < |s|^{-11} + |t|^{-11} < 1$. Hence, there exists a positive integer m such that

$$\frac{1}{177147^{m+1}} \leq |s|^{-11} + |t|^{-11} < \frac{1}{177147^m}. \tag{5.5}$$

Hence, the inequality Equality (5.5) generates $177147^m \left(|s|^{-11} + |t|^{-11}\right) < 1$, or equivalently; $177147^m s^{-11} < 1, 177147^m t^{-11} < 1$. So, $\frac{s^{11}}{177147^m} > 1$, $\frac{t^{11}}{177147^m} > 1$. Hence, the last inequalities imply $\frac{s^{11}}{177147^{m-1}} > 177147 > 1$, $\frac{t^{11}}{177147^{m-1}} > 177147 > 1$, and as a result, we find $\frac{1}{3^{m-1}}(s) > 1$, $\frac{1}{3^{m-1}}(t) > 1$, $\frac{1}{3^{m-1}}(2s+t) > 1$, $\frac{1}{3^{m-1}}(2s-t) > 1$. Hence, for every value of $m = 0, 1, 2, \ldots, n-1$, we obtain $\frac{1}{3^n}(s) > 1$, $\frac{1}{3^n}(t) > 1$, $\frac{1}{3^n}(2s+t) > 1$, $\frac{1}{3^n}(2s-t) > 1$ and $\Delta_1 u(3^{-n}s, 3^{-n}t) = 0$ for $m = 0, 1, 2, \ldots, n-1$. Applying Equation (5.1) and the definition of u, we obtain

$$\begin{aligned} |\Delta_1 u(s,t)| &\leq \sum_{m=n}^{\infty} \frac{c}{177147^m} + \sum_{m=n}^{\infty} \frac{c}{177147^m} + \frac{177148}{177147} \sum_{m=n}^{\infty} \frac{c}{177147^m} \\ &\leq \frac{531442\ c}{177147} \frac{1}{177147^m} \left(1 - \frac{1}{177147}\right)^{-1} \leq \frac{531442\ c}{177146} \frac{1}{177146^m} \\ &\leq \frac{531442\ c}{177146} \frac{1}{177146^{m+1}} \\ &\leq \frac{265721\ c}{88573} \left(|s|^{-11} + |t|^{-11}\right) \end{aligned}$$

for all $s, t \in \mathbb{R}^*$. This means that the inequality (5.3) holds. We claim that the multiplicative inverse undecic functional Equation (1.2) is not stable for $\beta = -11$ in Corollary 4.4. Assume that there exists a multiplicative inverse undecic mapping $\mathcal{U} : \mathbb{R}^* \to \mathbb{R}$ satisfying Equation (5.4). So we have

$$|u(s)| \leq (\delta + 1)|s|^{-11}. \tag{5.6}$$

Moreover, it is possible to choose a positive integer m with the condition $mc > \delta + 1$. If $x \in \left(1, 3^{m-1}\right)$, then $3^{-n}x \in (1, \infty)$ for all $m = 0, 1, 2, \dots, n-1$ and thus

$$|u(s)| = \sum_{m=0}^{\infty} \frac{\chi(3^{-m}s)}{177147^m} \geq \sum_{m=0}^{n-1} \frac{\frac{177147^m c}{s^{11}}}{177147^m} = \frac{mc}{s^{11}} > (\delta+1)s^{-11}$$

which contradicts Equation (5.6). Therefore, the multiplicative inverse undecic functional Equation (1.2) is not stable for $\beta = -11$ in Corollary 4.4.

Similar to Equation (5.1), the following example acts as a refutation that the Equation (1.3) is unstable for $\beta = -12$ in Corollary 4.9.

Example 5.2. Define the function $\xi : \mathbb{R}^* \to \mathbb{R}$ via

$$\xi(s) = \begin{cases} \frac{\lambda}{s^{12}} & \text{for } s \in (1, \infty) \\ c, & \text{otherwise} \end{cases}. \tag{5.7}$$

Let $\nu : \mathbb{R}^* \to \mathbb{R}$ be defined by

$$\nu(s) = \sum_{m=0}^{\infty} 531441^{-m} \xi(3^{-m}s)$$

for all $s \in \mathbb{R}$. Suppose the function ν satisfies the functional inequality

$$|\Delta_2 \nu(s,t)| \leq \frac{398581\ \lambda}{132860} \left(|s|^{-12} + |t|^{-12}\right)$$

for all $s, t \in \mathbb{R}^*$. Then, there do not exist a multiplicative inverse duodecic mapping $\mathcal{V} : \mathbb{R}^* \to \mathbb{R}$ and a constant $\eta > 0$ such that $|\nu(s) - \mathcal{V}(s)| \leq \eta |s|^{-12}$ for all $s \in \mathbb{R}^*$.

6 | CONCLUSION

In this study, we have proved that the stability results of multiplicative inverse undecic and duodecic functional equations are valid related to the Ulam stability hypothesis in non-Archimedean fields through fixed point method and portrayed two suitable counter-examples that the results fail for singular cases.

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