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RINGS WITH ASSOCIATORS IN THE RIGHT NUCLEUS

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In this paper we prove that if R is a simple ring with associator in the right nucleus then R is associative. We extend this result to prime and semi prime rings also.

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1. Introduction

In [2] E. Kleinfeld proved that if R is a semi prime ring such that $(R, R, R) \subseteq N_l \cap N_m \cap N_r$ and $(R, +)$ has no elements of order 2 then R is associative. In [1] Chen-Te Yen has proved that if R is a simple ring of characteristic not 2 with associators in the left nucleus then R is associative. In [3] E. Kleinfeld and Margaret Kleinfeld has shown that assuming associators in the left nucleus, simple rings with identity 1 and characteristic not 2 must be associative. In this paper using the results of paper [1] and [3] we prove that (1) if R is a simple ring with associator in the right nucleus then is associative, (2) if R is a prime ring with associator in the right nucleus then R is associative and (3) if R satisfies the identity $(R, R, (R, R, R)) = 0$ and if for all $a \in R$, $a^2 = 0$ implies $a = 0$, then R must be associative.

2. Preliminaries

Let R be a non associative ring. We shall denote the associator by $(x, y, z) = (xy)\,z - x\,(yz)$ for all $x, y, z \in R$. In any ring R one has the following nuclei:

$$N_l = \{n \in R / (n, R, R) = 0\} \text{ – left nucleus}$$
$$N_m = \{n \in R / (R, n, R) = 0\} \text{ – middle nucleus}$$
$$N_r = \{n \in R / (R, R, n) = 0\} \text{ – right nucleus.}$$

And where the nucleus N is defined by $N = \{ n \in R / (n, R, R) = 0 = (R, n, R) = (R, R, n)\}$.

A ring R is called simple if $R^2 \neq 0$ and the only nonzero ideal of R is itself. A ring is called prime if for any two of its ideals A and B with $AB = 0$ it follows that either $A = (0)$ or $B = (0)$. Since R^2 is a nonzero ideal of R, we have $R^2 = R$. A ring R is called semi prime if the only ideal of R which squares to zero is the zero ideal. Note that each associator is linear in each argument. Thus N_l, N_m and N_r are additive subgroups of $(R, +)$.

We shall make use of the Teichmuller identity.

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w\,(x, y, z) + (w, x, y)\,z \quad \forall w, x, y, z \in R, \tag{1}$$

which is valid in any ring.

As a consequence of (1), we have that N_l, N_m and N_r are associative subrings of R.

Suppose that $n \in R_r$, then with $w = n$ in (1) we obtain

$$(x, y, zn) = (x, y, z)\,n, \quad \forall x, y, z \in R \text{ and } n \in N_r. \tag{2}$$

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Let A be the associator ideal of a ring R. By (1) A can be characterized as all finite sums of associators and right (or left) multiples of associators. Hence, we obtain

$$A = (R, R, R) + (R, R, R)R, \\ = (R, R, R) + R(R, R, R). \tag{3}$$

Lemma 2.1: Let $T = \{t \in R / tR = 0 = Rt\}$. Then T is an ideal of R.

Proof: Let $t_1, t_2 \in T$, then for every $x \in R$ we obtain

$$t_1x = 0,\ t_2x = 0 \quad \text{and} \quad xt_1 = 0 \quad \text{and} \quad xt_2 = 0.$$

Also, $$x(t_1 - t_2) = xt_1 - xt_2 = 0,$$
$$(t_1 - t_2)x = t_1x - t_2x = 0.$$

Now for $r \in R,\ t \in T,\ rT$ and Tr are also in T. Hence $(rt)x = 0$ and $x(tr) = 0$. And hence rt and tr both belong to T. Hence T is an ideal of R.

Theorem 2.2: Let R be a simple ring and satisfies (*) $(R, R, R) \subseteq N_r$. Then R is associative.

Proof: Assume that R is not associative. Then by (3) and (*), we obtain

$$R = R^2 = RA = R(R, R, R)$$

Similarly, we obtain

$$R = AR = (R, R, R)R. \tag{4}$$

Using (1) and (*) we obtain

$$w(x, y, z) + (w, x, y)z \in N_r, \quad \forall w, x, y, z \in R. \tag{5}$$

Then with $y(R, R, R)$ in (5) and applying (*) we obtain

$$w(x, y, z) \in N_r.$$

i.e., $R(R, (R, R, R), R) \subseteq N_r$.

Now using this, (*) and (2) we obtain

$$0 = (R, R, R, (R, (R, R, R), R)) = (R,R,R)(R, (R, R, R), R) \tag{6}$$

and so

$$R(R, R, R)(R, (R, R, R), R) = R(R, R, R)(R, (R, R, R), R) \\ = 0.$$

Using (4) results in

$$R(R, (R, R, R), R) = 0. \tag{7}$$

From (7) and (4) we obtain

$$\begin{aligned} 0 = R(R,(R,R,R),R) &= (R, R, R)R(R, (R, R, R), R) \\ &= (R, A, R)R(R, (R, R, R), R), \text{ since } R \text{ is simple} \\ &= (R, (R, R, R), R)R(R, (R, R, R), R) \\ &= (R, (R, R, R), R)R(R, A, R) \\ &= (R, (R, R, R), R)R(R, R, R) \\ &= (R, (R, R, R), R)R(R, R, R) \\ &= (R, (R, R, R), R)R \end{aligned} \tag{8}$$

From (7) and (8) $(R, (R, R, R), R) \subseteq T$. Since T is an ideal and simplicity of R yields $T = R$ or $T = 0$. But $T = R$ implies $RR = 0$, a contradiction. Thus $T = 0$.

Hence $(R, (R, R, R), R) = 0$. Hence the associator (R, R, R) are now in the middle and the right nucleus. So we may use [2] to obtain a contradiction. Now we show that the associator (R, R, R) is also in the left nucleus.

For, replace n by $((a, b, c), d, e)$ in (2) where $a, b, c, d, e \in R$ and $((a, b, c), d, e) \in N_r$ by the hypothesis. Thus,

$$(x, y, z\,((a, b, c), d, e)) = (x, y, z)\,((a, b, c), d, e). \tag{9}$$

Now applying (1) we obtain,

$$\begin{aligned}(x, y, z\,((a, b, c), d, e)) + (x, y, (z, (a, b, c), d)\,e) &= (x, y, (z\,(a, b, c), d, e)) - (x, y, (z, (a, b, c)\,d, e)) \\ &\quad + (x, y, (z, (a, b, c), de)).\end{aligned} \tag{10}$$

$$(x, y, z\,((a, b, c), d, e)) + (x, y, (z, (a, b, c), d)\,e) = 0,$$

$(x, y, z\,((a, b, c), d, e)) = -(x, y, (z, (a, b, c), d)\,e)$, using the fact that associators are in the right and the middle nucleus we obtain $(x, y, (z, (a, b, c), d)\,e) = 0$ implying, $(x, y, z\,((a, b, c), d, e)) = 0$. Hence we obtain,

$$0 = (R, R, R\,((R, R, R), R, R)) = (R, R, R)\,((R, R, R), R, R) \quad \text{by (2)}$$

$$A\,((R, R, R), R, R) = 0. \tag{11}$$

Since $A = R$ we obtain

$$R\,((R, R, R), R, R) = 0. \tag{12}$$

Form (12) and (4) we obtain

$$\begin{aligned}0 &= (R, R, R)\,R\,((R, R, R), R, R) \\ &= (A, R, R)\,R\,((R, R, R), R, R) \\ &= ((R, R, R), R, R))\,R\,((R, R, R), R, R) \\ &= ((R, R, R), R, R)\,R\,(A, R, R) \\ &= ((R, R, R), R, R)\,R\,(R, R, R) \text{ since } R \text{ is simple.} \\ &= ((R, R, R), R, R)\,R.\end{aligned} \tag{13}$$

Thus from (12) and (13) $(R, R, (R, R, R)) \subseteq T$. Since T is an ideal simplicity of R yields $T = R$ or $T = 0$. But $T = R$ implies $RR = 0$ which is again a contradiction. Thus $T = 0$ and so $((R, R, R), R, R) = 0$ and hence we obtain the associators now in the left nucleus as well. Hence $(R, R, R) \subseteq N_l \cap N_m \cap N_r$. Thus [2] applies. This completes the proof of the Theorem.

Theorem 2.3: Let R be a prime ring and satisfies (*) $(R,R,R) \subseteq N_r$. Then R is associative.

Proof: Let R be not associative. Then from (6)

$$\begin{aligned}0 &= (R, R, R, (R, (R, R, R), R)) \\ &= (R, R, R)\,(R, (R, R, R), R), \\ &= A\,(R, (R, R, R), R).\end{aligned}$$

From Lemma 2.1 $(R, (R, R, R), R) \subseteq T$ and T is an ideal of R. Hence we obtain $AT = 0$. But since R is prime we either have $A = 0$ or $T = 0$. But A being an associator ideal is not equal to zero. Hence we have $T = 0$ implying $(R, (R, R, R), R) = 0$. i.e., (R, R, R) are in the middle and the right nucleus. Now from (11), $0 = A\,(R, R, (R, R, R))$ But $((R, R, R), R, R) \subseteq T$ and $AT = 0$. But since R is prime we either have $A = 0$ or $T = 0$. But A being an associator ideal is not equal to zero. Hence we have $T = 0$ implying $((R, R, R), R, R) = 0$. i.e. (R, R, R) are in the left nucleus as well which results in $(R, R, R) \subseteq N$. At this point we make use of the result in [2] to obtain a contradiction.

Theorem 2.4: If R satisfies the identity $(R, R, (R, R, R)) = 0$ and if for all aR, $a^2 = 0$ implies $a = 0$ then R must be associative.

Proof: As in earlier computation, we deduce that $(R, R, R)(R, (R, R, R), R) = 0$. Thus $(v, (w, x, y), z)^2 = 0$. Hence $(v, (w, x, y), z) = 0$, proving that (R, R, R) is in the middle nucleus of R. This implies $(R, R, R)((R, R, R), R, R) = 0$. Then $((x, y, z), v, w)^2 = 0$. So that (R, R, R) is in N, the nucleus of R. Now we use [2] to see that R is associative.

References

[1] Chen-Te.Yen, (2002), Simple Rings of Characteristic Not 2 with Associators in the Left Nucleus are Associative, *Tamkang Journal of Mathematics*, Vol. 33, Pp. 93-95.

[2] E. Kleinfeld, (1986), A Class of Rings which are Very Nearly Associative, *Amer. Math. Monthly*, Vol. 93, No. 1, Pp. 720-722.

[3] E. Kleinfeld, and Margaret Kleinfeld, (1999), Simple Rings with (R, R, R) in the Left Nucleus, *Tamkang Journal of Mathematics*, Vol. 30, No. 1, Pp. 35-37.

[4] R. D. Schafer, (1966), An Introduction to Non-Associative Algebra, *Pure and Appl. Math.*, Academic Press, NewYork.