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Strongly (-1,1) Rings

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ABSTRACT

In this paper we prove that in a strongly (-1,1) ring of char. $\neq 2,3$ the associator is in the nucleus. Using this, we prove that the nilpotency of the associator in a strongly (-1, 1) ring.

Keywords: Strongly (-1,1) rings, nilpotency.

INTRODUCTION

Algebras of type (γ, δ) were first defined by Albert[1]. When $\gamma = -1$ and $\delta = 1$ we obtain the (-1,1) rings. i.e., a (-1,1) ring satisfies the identities $(x, y, z) + (x, z, y) = 0$ and $(x, y, z) + (y, z, x) + (z, x, y) = 0$. A ring in which $(x, y, z) + (x, z, y) = 0$ and $((x, y), z) = 0$ hold is called a strongly (-1,1) ring. The strongly (-1,1) rings were first introduced by Kleinfeld[2]. Pchelintsev[3] established the nilpotency of the associators in a free (-1,1) ring. In this paper we show that the associator (x, y, z) is in the nucleus of a strongly (-1,1) ring. Using this, we prove the nilpotency of the associator in a strongly (-1,1) ring. We know that a strongly (-1,1) ring is a (-1,1) ring. But the converse need not be true. At the end of this paper, we give an example of a (-1,1) ring which is not strongly (-1,1).

PRELIMINARIES

Throughout this paper R will denote a strongly (-1,1) ring of char. $\neq 2,3$. We shall denote the commutator and the associator by $(x, y) = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$ for all x, y, z in R respectively. The nucleus N of a ring R is defined as $N = \{n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0\}$.The center C of R is defined as $C = \{ c \in N / (c,R) = 0\}$. A ring R is said to be of characteristic $\neq n$ if $nx = 0$ implies $x = 0$, for all $x \in R$ and n is a natural number. A ring R is of characteristic $\neq n$ is simply denoted by char.$\neq n$. A ring is called nilpotent if there is a fixed positive integer t such that every product involving t elements is zero.

A nonassociative ring R is called a strongly (-1,1) ring if it satisfies the following identities:

$$(x, y, z) + (x, z, y) = 0 \tag{1}$$
$$((x, y), z) = 0. \tag{2}$$

i.e., a right alternative ring satisfying the identity (2) is called a strongly (-1,1) ring.
In any ring we have the identity
$(x, y, z) + (y, z, x) + (z, x, y)$
$= ((x, y), z) + ((y, z), x) +((z, x), y).$

Using (2), the above identity becomes

$$(x, y, z) + (y, z, x) + (z, x, y) = 0. \tag{3}$$

We use the Teichmuller identity:

$$(wx, y, z) – (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \tag{4}$$

The following identity also holds in any ring:
$(xy, z) – x(y, z) – (x, z)y – (x, y, z) – (z, x, y) + (x, z, y) = 0.$
Using (1), the above identity becomes

$$(xy, z) – x(y, z) – (x, z)y – 2 (x, y, z) – (z, x, y) = 0. \tag{5}$$

By putting w = n in (4), we obtain

$$(nx, y, z) = n(x, y, z). \tag{6}$$

Now we define U to be the set of all elements u of R which commutes with all elements of R. i.e., $U = \{u\in R / (u, R) = 0\}$. By putting $z = u\in U$, $y = x$ in (5), we get $-2(x, x, u) = 0$. Hence $(x, x, u) = 0$ and $(x, u, x) = 0$ because of (1). Replacing x by x + y in this last two identities gives

$$(x, y, u) = - (y, x, u) \text{ and } (x, u, y) = - (y, u, x), \tag{7}$$
for $u \in U$.

In any strongly (-1,1) ring the following identities hold:

$$(x, (y, z, x)) + (x, (z, y, x)) = 0 \tag{8}$$

$$(x, (y, y, z)) +(z, (y, y, x)) = 0 \tag{9}$$

$$(x, (y, y, z)) - 3(y, (x, z, y)) = 0. \tag{10}$$

MAIN RESULTS

To prove the main theorem first we prove the following lemmas:

Lemma 1 : If R is a strongly (-1,1) ring of char. $\neq$ 2,3, then $(R,(R,R,R)) = 0$.

Proof : From (8), $(x, (y, z, x)) + (x, (z, y, x)) = 0$, this equation becomes

$$(y, (x, x, y)) = 0, \tag{11}$$

since R is of char. $\neq 2$.
Using the right alternative property of R, (11) can be written as

$$(y, (x, y, x)) = 0. \tag{12}$$

By linearizing the identities (12) and (11), we have
$$(y, (x, y, z)) = - (y, (z, y, x)) \tag{13}$$
$$(y, (x, z, y)) = - (y, (z, x, y)). \tag{14}$$

From equations (1), (13), (14) and again (1), we get

$$(y, (y, z, x)) = - (y, (y, x, z)) = (y, (z, x, y)) = -(y, (x, z, y)) = (y, (x, y, z)). \quad (15)$$

Commuting equation (3) with y, we have $(y, (x, y, z) + (y, z, x) + (z, x, y)) = 0$. Using (15), the above equation becomes $3(y, (x, y, z)) = 0$.
Since R is of char. $\neq 3$, we have

$$(y, (x, y, z)) = 0. \quad (16)$$

Using (16), (10) becomes $(x, (y, y, z)) = 0$.

$$\text{Thus} \quad (R, (y, y, z)) = 0. \quad (17)$$

By linearizing equation (17), we obtain

$$(w, (x, y, z)) = - (w, (y, x, z)). \quad (18)$$

Applying equation (1) and (18) repeatedly, we get $(w, (x, y, z)) = -(w, (y, x, z)) = (w, (y, z, x)) = - (w, (z, y, x)) = (w, (z, x, y))$.
Commuting equation (3) with w and applying above equation, we obtain $3(w, (x, y, z)) = 0$. Since R is of char. $\neq 3$, we have $(w, (x, y, z)) = 0$. i.e., $(R, (R, R, R)) = 0$.
This completes the proof of the lemma. □

Lemma 2: If R is a strongly (-1,1) ring of char. $\neq 2$, 3, then the associator is in the nucleus.

Proof : Using equations (1) and (7), we see that for $u \in U$,
$(x, y, u) = - (y, x, u) = (y, u, x) = - (u, y, x) = (u, x, y)$.

$$\text{Thus} \quad (x, y, u) = (y, u, x) = (u, x, y), \quad \text{for } u \in U. \quad (19)$$

The identity (3) gives $(x, y, u) + (y, u, x) + (u, x, y) = 0$. Using (19), the above equation becomes $3(x, y, u) = 0$. Since R is of char. $\neq 3$, we have

$$(x, y, u) = 0, \quad (20)$$

for $u \in U$.
If $u = (r, s, t) \in U$, where $r, s, t \in R$, then (20) yields $(x, y, (r, s, t)) = 0$. From (19), we have $(y, (r, s, t), x) = 0$ and $((r, s, t), x, y) = 0$. Therefore the associator is in the nucleus.

$$\text{i.e., } (R, R, R) \subseteq N. \quad (21)$$

This completes the proof of the lemma. □
Also we have

Lemma 3: In a strongly (-1,1) ring R the following identities are fulfilled:

(i) $(x, y, x) ((x, y, z) + (z, y, x)) = 0$.
(ii) $((x, y, z) + (z, y, x)) (x, y, x) = 0$.
(iii) $2(x, y, x) (z, y, z) + ((x, y, z) + (z, y, x))^2 = 0$.
(iv) $((x, y, z), (z, y, x)) = 0$.

Proof : Since $(R,R,R) \subseteq N$, we have
$(x, x, (x, y, z)) = 0$.
Using (9), we have $((x, y, z), (x, x, y)) = - (y, (x, x, (x, y, z)) = 0$.
Using (1), the above equation becomes

$$((x, y, z), (x, y, x)) = 0. \quad (22)$$

Now using (4), (1), (4) and (6), we obtain

$$\begin{aligned}((x, y, z)x, y, x) &= - (x(y, z, x), y, x)\\ &= (x(y, x, z), y, x)\\ &= - ((x, y, x)z, y, x)\\ &= - (x, y, x) (z, y, x).\end{aligned}$$

Thus $((x, y, z)x, y, x) = - (x, y, x) (z, y, x)$ or

$$(x, y, z)\,(x, y, x) = -\,(x, y, x)\,(z, y, x).$$

Using (22), this equation becomes $(x, y, x)(x, y, z) = -(x, y, x)(z, y, x)$. Therefore $(x, y, x)((x, y, z) + (z, y, x)) = 0$. Thus (i) proved. Similarly we can prove the identity (ii). We obtain the identity (iii) by linearizing (i) with $x = x + z$. Further, from (22), we have

$$((x, y, x), (x, y, z)) = 0. \qquad (23)$$

Interchanging x and z in (iii) and subtracting the identity so obtained from (iii), we get

$$((x, y, x), (z, y, z)) = 0. \qquad (24)$$

On linearizing the identity (23) with $x = x + z$, we obtain $((z, y, x), (x, y, z)) + ((x, y, x), (z, y, z)) = 0$. Hence (iv) follows by using (24).
This completes the proof of the lemma. □

Lemma 4 : In a strongly (-1,1) ring R it follows from the equations
$ac = bc = ca = cb = 0$ that $(ab)c = 0$.

Proof : From (1) we have
$(ab)c = a(bc) + (a, b, c) = (a, b, c) = -(a, c, b) = -(ac)b + a(cb) = 0$. □

Using this we prove the nilpotency of the associator in the following theorem:

Theorem 1: In every strongly (-1,1) ring
$(x, y, z)^4 = 0$.

Proof : By substituting $p = (x, y, x)$, $q = (z, y, z)$, $r = (x, y, z)$, $s = (z, y, x)$ in Lemma 3, we observe that (i) $p(r + s) = 0$, (ii) $(r + s)p = 0$, (iii) $2pq + (r + s)^2 = 0$, (iv) $(r, s) = 0$. Interchanging x and z in (i) and (ii), we get $q(r + s) = 0$ and $(r + s)q = 0$.
By Using (i), (ii) and Lemma 4, we have $(pq)(r + s) = 0$. Multiplying both sides of the identity (iii) on the right by $(r + s)$, we get

$$(r + s)^3 = 0. \qquad (25)$$

Now consider an element of the form $(r - 2s)^3$.
By using (4), (3) and (25), we have

$$\begin{aligned}(r - 2s)^3 &= ((x, y, z) - 2(z, y, x))^3\\ &= (-(x, z, y) - 2\,(z, y, x))^3\\ &= ((z, y, x) + (y, x, z) - 2\,(z, y, x))^3\\ &= ((y, x, z) - (z, y, x))^3\\ &= -\,((y, z, x) + (z, y, x))^3\\ &= ((y, x, z) + (z, x, y))^3\\ &= 0\end{aligned}$$

i.e., $(r - 2s)^3 = 0. \qquad (26)$

Now we observe that if x and y are commuting elements of a right alternative ring, then
$(x + y)^3 = x^3 + y^3 + 2(yx^2 + xy^2) + x^2y + y^2x$ and then it follows from (26) that
$0 = (r - 2s)^3 + (s - 2r)^3$
$= r^3 - 8s^3 + 2(4rs^2 - 2sr^2) - 2r^2s + 4s^2r + s^3 - 8r^3 + 2(4sr^2 - 2rs^2) - 2s^2r + 4r^2s$
$= -7\,(r^3 + s^3) + 2\,(2rs^2 - 2sr^2) + 2s^2r + 2r^2s$
or
$$7\,(r^3+s^3) = 2\,(2(rs^2 + sr^2) + r^2 s + s^2r). \qquad (27)$$

From (25) and by using (iv), we have

$$r^3 + s^3 + 2(rs^2 + sr^2) + r^2 s + s^2r = 0. \qquad (28)$$

Comparing the identities (27) and (28), we obtain
$$r^3 + s^3 = 0 \qquad (29)$$

and $2(rs^2 + sr^2) + r^2s + s^2r = 0. \qquad (30)$

By using (29) and (30), we calculate $(r\text{-}2s)^3$,
$(r\text{-}2s)^3 = r^3 – 8s^3 + 2(4rs^2–2sr^2) – (2r^2s+4s^2r)$
$= 9r^3 – 12sr^2 – 6r^2s + 4(2(rs^2+sr^2) + r^2s + s^2r)$
$= 9r^3 – 12sr^2 – 6r^2s.$

On the other hand, by using 26 we have $(r – 2s)^3 = 0$. Consequently,

$$3r^3 = 4sr^2 + 2r^2s. \qquad (31)$$

Further, by using (6) it follows from (iv) that $((r^2, s), r) = - ((s, r), r^2) – ((r, r^2), s) = 0$.
Thus $r \cdot (r^2s – sr^2) = (r^2s – sr^2)\cdot r$.
From (4), (5), (29) and (iv), we transform right hand side of last identity as follows:

$(r^2s – sr^2)r = (r^2s)r – sr^3 = r(rs\cdot r) – sr^3$
$= r(sr\cdot r) – sr^3 = r(sr^2) – sr^3 = r(sr^2) + s^4$

Therefore we have $r(r^2s – sr^2) = r(sr^2) + s^4$, hence

$$r(r^2s) – 2rs^2 = s^4. \qquad (32)$$

Similarly,
$$2r(r^2os) = 2r^3s + 2(rs)\, r^2 = - 2s^4 + 2sr^3 = - 4s^4. \qquad (33)$$

(where aob = ab + ba is the Jordan product of the elements a and b.)
Adding (31) and (32), we get $3r(r^2s) = - 3s^4$.

$$\text{i.e.,} \quad r(r^2s) = - s^4. \qquad (34)$$

Comparing (32) and (33), we get

$$r(sr^2) = - s^4. \qquad (35)$$

On multiplying both sides of (31) on left by r and using (34) and (35) we have $3r^4 = 4r(sr^2) + 2r(r^2s) = - 6s^4$. Hence, it follows that

$$r^4 + 2s^4 = 0. \qquad (36)$$

Interchanging x and z in (36), we get

$$s^4 + 2r^4 = 0. \qquad (37)$$

Finally, from (36) and (37) we easily prove that $s^4 = 0$.
Hence the theorem is proved. □

Now we give an example of a (-1,1) ring, which is not a strongly (-1,1) ring.

Example : Let R be the ring defined by the following multiplication table.

	e	a	b	c	d	h
e	e	0	0	0	0	0
a	h	c	a	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
h	h	0	0	0	0	0

It is easily seen that R is a (-1,1) ring.

Now $((a, b), b) = (ab – ba,b)$
$= (ab, b) – (ba, b)$
$= (ab)b – b(ab) – (ba)b + b(ba)$
$= (a)b – b(a) – (0)b + b(0)$
$= a – 0 – 0 + 0$
$= a \neq 0.$

Since $((a, b), b) \neq 0$, R is not strongly (-1,1).

REFERENCES

1. Albert A. A., "Almost alternative algebras", *Portugal. Math.* 8, 23-26 (1949).

2. Kleinfeld E., “On a class of right alternative rings”, *Math. Zectschr*. 87, 12-16 (1965).
3. Pchelintsev S. V., “Nilpotency of the associators in a free (-1,1) ring, *Algebra i logika, Vol*.13, No. 2, 217-223 (1974).