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METHODOLOGIES AND APPLICATION

Fuzzy approximations of a multiplicative inverse cubic functional equation

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Abstract

The aim of this study is to introduce a new multiplicative inverse cubic functional equation, to accomplish its general solution, to employ Hyers' method for solving its stability problems in Felbin's type fuzzy normed linear spaces, to present an apt example for justifying its stability result is invalid for singular case and to elucidate its interpretation through an application in electromagnetism.

Keywords Multiplicative inverse function · Multiplicative inverse functional equation · Multiplicative inverse quadratic functional equation · Generalized Hyers–Ulam–Rassias stability

1 Introduction

The detailed literature associated with the fundamental query concerning the stability of functional equation and its responses provided by many mathematicians in various adaptations can be referred in Brzdek (1994, 2009), Forti (1995, 2004), Gavruta (1994), Hyers (1941), T. Rassias (1978), J. Rassias (1982) and Ulam (1964). There are various interesting, noteworthy and instigating results concerning different forms of functional equations, and one can refer to Erami (2012), Javadi and Rassias (2012), Park et al. (2012) and Wiwatwanich and Nakmahachalasint (2008).

In recent times, the theory of stability of functional equations is emerging in the area of multiplicative inverse functional equations. The Ulam stability problems for various types of multiplicative inverse functional equations and their applications are available in Bodaghi et al. (2016), Kim et al. (2017), Ravi and Senthil Kumar (2010), Senthil Kumar et al. (2016) and Senthil Kumar and Dutta (2018, 2019a, b).

The prediction of missing values in data processing using local least squares, NRMSE and Pearson correlation is discussed in Al-Janabi and Alkaim (2019). The general properties, summary, advantages and disadvantages of various techniques such as Chi-squared automatic interaction detection, exchange Chi-squared automatic interaction detection, random forest regression and classification, multivariate adaptive regression splines and boosted tree classifiers and regression which are used for prediction of data mining are presented in Al-Janabi and Mahdi (2019).

The primary stabilities of the system of bireciprocal functional equations

$$\left.\begin{aligned} b_r(x_1+x_2,y) &= \frac{b_r(x_1,y)b_r(x_2,y)}{b_r(x_1,y)+b_r(x_2,y)} \\ b_r(x,y_1+y_2) &= \frac{b_r(x,y_1)b_r(x,y_2)}{b_r(x,y_1)+b_r(x,y_2)} \end{aligned}\right\} \tag{1.1}$$

pertinent to Ulam and Hyers are discussed in Fréchet spaces in Ravi and Senthil Kumar (2015). It is easy to see that $b_r(x,y) = \frac{k}{xy}$ is a solution of (1.1), where k is a constant.

The ensuing reciprocal–quadratic functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}} \tag{1.2}$$

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is solved for its general solution, and its various stabilities are studied in Ravi and Suresh (2017).

The authors in Senthil Kumar et al. (2019) have extended the system of bireciprocal functional equations (1.1) of two variables into a system of functional equations involving three variables with the following definition:

Definition 1.1 A mapping $r_T : {}^*\mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \to \mathbb{R}$ is trireciprocal if r_T satisfies the following system of functional equations:

$$\left.\begin{aligned} r_T(a_1+a_2,b,c) &= \frac{r_T(a_1,b,c)r_T(a_2,b,c)}{r_T(a_1,b,c)+r_T(a_2,b,c)}, \\ r_T(a,b_1+b_2,c) &= \frac{r_T(a,b_1,c)r_T(a,b_2,c)}{r_T(a,b_1,c)+r_T(a,b_2,c)}, \\ r_T(a,b,c_1+c_2) &= \frac{r_T(a,b,c_1)r_T(a,b,c_2)}{r_T(a,b,c_1)+r_T(a,b,c_2)} \end{aligned}\right\} \tag{1.3}$$

for all $a_1, a_2, a, b_1, b_2, b, c_1, c_2, c \in {}^*\mathbb{R}$. The function r_T is said to be trireciprocal mapping.

The multiplicative inverse function of the form $r_T(a,b,c) = \dfrac{k}{abc}$, where k is a constant, satisfies the system of Eq. (1.3).

In this paper, we introduce a new multiplicative inverse cubic functional equation of the form

$$r_c(a+b) = \frac{r_c(a)r_c(b)}{r_c(a)+3r_c(a)^{\frac{2}{3}}r_c(b)^{\frac{1}{3}}+3r_c(a)^{\frac{1}{3}}r_c(b)^{\frac{2}{3}}+r_c(b)}. \tag{1.4}$$

We solve Eq. (1.4) and also investigate its classical fundamental stabilities in Felbin's type fuzzy normed spaces.

One can find that $r_c(a) = k/a^3$ is a solution of Eq. (1.4). Therefore, we call that every solution of Eq. (1.4) is a multiplicative inverse cubic mapping.

The fundamental notions and various developments of Felbin's type fuzzy normed spaces are available in Felbin (1992) and Xiao and Zhu (2002, 2004). For proving our main results, let us present a few significant concepts pertinent to Felbin's fuzzy normed spaces.,

Lemma 1.1 (Xiao and Zhu 2004) *Let $(X, \|\cdot\|, L, R)$ be a fuzzy normed linear space. Suppose that*

$(R-1)$ $R(\alpha,\beta) \le \max(\alpha,\beta)$,
$(R-2)$ *For all $a \in (0,1]$, there exists $b \in (0,\alpha]$ such that $R(b,\lambda) \le a$ for all $\lambda \in (0,a)$,*
$(R-3)$ $lim_{\alpha\to 0^+} R(\alpha,\alpha) = 0$.

Then, $(R-1) \Longrightarrow (R-2) \Longrightarrow (R-3)$, but not conversely.

Lemma 1.2 *Let $(X, \|\cdot\|, L, R)$ be a fuzzy normed linear space satisfying $(R-2)$. Then, we have the following:*

(1) *For each $a \in (0,1]$, $|\cdot\|_a^+$ is a continuous mapping from X into $\mathbb{R}$ at $x \in X$.*
(2) *For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ we have for all $a \in (0,1]$, there exists $b \in (0,a]$; $\|\sum_{i=1}^n x_i\|_a^+ \le \sum_{i=1}^n \|x_i\|_b^+$.*

2 General solution of Eq. (1.4)

In the following results, we solve Eq. (1.4) with $\mathbb{R}^\star$, the space of nonzero real numbers as domain. In order to achieve our main results, we assume the following: $a+b \neq 0$, $r_c(a) + 3r_c(a)^{\frac{2}{3}}r_c(b)^{\frac{1}{3}} + 3r_c(a)^{\frac{1}{3}}r_c(b)^{\frac{2}{3}} + r_c(b) \neq 0$, $r_c(a) \neq 0$, for all $a, b \in \mathbb{R}^\star$.

Theorem 2.1 *Let $r_c : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a mapping such that it satisfies Eq. (1.4). Then, r_c is a multiplicative inverse cubic mapping.*

Proof Plugging b into a in Eq. (1.4), we get

$$r_c(2a) = \frac{1}{8}r_c(a) = \frac{1}{2^3}r_c(a) \tag{2.1}$$

for all $a \in \mathbb{R}^\star$. Now, again reinstating b by $2a$ in (1.4) and employing the above result (2.1), we obtain

$$\begin{aligned} r_c(3a) &= \frac{r_c(a)r_c(2a)}{r_c(a)+3r_c(a)^{\frac{2}{3}}r_c(2a)^{\frac{1}{3}}+3r_c(a)^{\frac{1}{3}}r_c(2a)^{\frac{2}{3}}+r_c(2a)} \\ &= \frac{\frac{1}{8}r_c(a)r_c(a)}{r_c(a)+\frac{3}{2}r_c(a)^{\frac{2}{3}}r_c(a)^{\frac{1}{3}}+\frac{3}{4}r_c(a)^{\frac{1}{3}}r_c(a)^{\frac{2}{3}}} \\ &= \frac{\frac{1}{8}r_c(a)r_c(a)}{r_c(a)+\frac{3}{2}r_c(a)+\frac{3}{4}r_c(a)+\frac{1}{8}r_c(a)} \\ &= \frac{1}{27}r_c(a) = \frac{1}{3^3}r_c(a) \end{aligned}$$

for all $a \in \mathbb{R}^\star$. Let us assume that $r_c(ka) = \frac{1}{k^3}r_c(a)$, for all $a \in \mathbb{R}^\star$ and for any integer $k > 0$. Now, setting $b = ka$ in (1.4), we have

$$\begin{aligned} r_c((k+1)a) &= \frac{r_c(a)r_c(ka)}{r_c(a)+3r_c(a)^{\frac{2}{3}}r_c(ka)^{\frac{1}{3}}+3r_c(a)^{\frac{1}{3}}r_c(ka)^{\frac{2}{3}}+r_c(ka)} \\ &= \frac{\frac{1}{k^3}r_c(a)r_c(a)}{\left(\frac{k^3+3k^2+3k+1}{k^3}\right)r_c(a)} \\ &= \frac{1}{(k+1)^3}r_c(a) \end{aligned}$$

for all $a \in \mathbb{R}^\star$. In lieu of induction hypothesis, we find that $r_c(ma) = \dfrac{1}{m^3}r_c(a)$, for all $a \in \mathbb{R}^\star$ and any integer $m > 0$, which implies that the mapping r_c is multiplicative inverse cubic. □

Theorem 2.2 *Let there exist a symmetric trireciprocal mapping* $r_T : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ *such that a function* $r_c : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *satisfies* (1.4) *if and only if* $r_T(x, y, z) = (r_c(x)r_c(y)r_c(z))^{\frac{1}{3}}$, *for all* $x, y, z \in \mathbb{R}^\star$.

Proof Let $r_T : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a trireciprocal mapping such that

$$r_T(x, y, z) = (r_c(x)r_c(y)r_c(z))^{\frac{1}{3}} \tag{2.2}$$

for all $x, y, z \in \mathbb{R}^\star$. If we set $z = y = x$, in relation (2.2), we get

$$r_T(x, x, x) = r_c(x) \tag{2.3}$$

for all $x \in \mathbb{R}^\star$. Now, applying (2.3), we find

$$r_c(x + y) = r_T(x + y, x + y, x + y). \tag{2.4}$$

Since r_T is a trireciprocal mapping, one can transform (2.4) to Eq. (1.4).

Conversely, if r_c satisfies (1.4). Then, it is clear that r_c is a well-defined function. Hence, from Eq. (1.4), let us define the following:

$$\begin{aligned}
&(r_c(x)r_c(y)r_c(z))^{\frac{1}{3}} \\
&\quad = \frac{1}{6}\Big\{\frac{r_c(x)r_c(y)r_c(z)}{r_c(x+y+z)} - r_c(x) - r_c(y) - r_c(z) \\
&\qquad - 3r_c(x)^{\frac{2}{3}}r_c(y)^{\frac{1}{3}} - 3r_c(x)^{\frac{1}{3}}r_c(y)^{\frac{1}{3}} \\
&\qquad - 3r_c(y)^{\frac{2}{3}}r_c(z)^{\frac{1}{3}} - 3r_c(y)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}} \\
&\qquad - 3r_c(x)^{\frac{2}{3}}r_c(z)^{\frac{1}{3}} - 3r_c(x)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}}\Big\}
\end{aligned} \tag{2.5}$$

for all $x, y, z \in \mathbb{R}^\star$. The expression on the right-hand side of relation 2.5 is well-defined with $r_c(x + y + z) \neq 0$, and consequently, the left-hand side is also well-defined and hence it exists. Now, we consider $r_T(x, y, z) = (r_c(x)r_c(y)r_c(z))^{\frac{1}{3}}$, for all $x, y, z \in \mathbb{R}^\star$. This implies that there exists a symmetric mapping $r_T(x, y, z)$ such that $r_T(x, y, z) = r_T(x, z, y) = r_T(y, z, x) = r_T(y, x, z) = r_T(z, y, x) = r_T(z, x, y)$. Now,

$$\begin{aligned}
&r_T(x_1 + x_2, y, z) = (r_c(x_1 + x_2)r_c(y)r_c(z))^{\frac{1}{3}} \\
&\quad = \left(\frac{r_c(x_1)r_c(x_2)r_c(y)r_c(z)}{r_c(x_1) + 3r_c(x_1)^{\frac{2}{3}}r_c(x_2)^{\frac{1}{3}} + 3r_c(x_1)^{\frac{1}{3}}r_c(x_2)^{\frac{2}{3}} + r_c(x_2)}\right)^{\frac{1}{3}} \\
&\quad = \frac{r_c(x_1)^{\frac{1}{3}}r_c(x_2)^{\frac{1}{3}}r_c(y)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}}}{\left[\left(r_c(x_1)^{\frac{1}{3}} + r_c(x_2)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}} \\
&\quad = \frac{r_c(x_1)^{\frac{1}{3}}r_c(x_2)^{\frac{1}{3}}r_c(y)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}}}{r_c(x_1)^{\frac{1}{3}} + r_c(x_2)^{\frac{1}{3}}} \\
&\quad = \frac{1}{\frac{r_c(x_1)^{\frac{1}{3}}}{r_c(x_1)^{\frac{1}{3}}r_c(x_2)^{\frac{1}{3}}r_c(y)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}}} + \frac{r_c(x_2)^{\frac{1}{3}}}{r_c(x_1)^{\frac{1}{3}}r_c(x_2)^{\frac{1}{3}}r_c(y)^{\frac{1}{3}}r_c(z)^{\frac{1}{3}}}} \\
&\quad = \frac{1}{\frac{1}{r_T(x_1,y,z)} + \frac{1}{r_T(x_2,y,z)}} = \frac{r_T(x_1, y, z)r_T(x_2 y, z)}{r_T(x_1, y, z) + r_T(x_2 y, z)}.
\end{aligned}$$

Similarly, we can prove

$$r_T(x, y_1 + y_2, z) = \frac{r_T(x, y_1, z)r_T(x, y_2, z)}{r_T(x, y_1, z) + r_T(x, y_2, z)}$$

and
$$r_T(x, y, z_1 + z_2) = \frac{r_T(x, y, z_1)r_T(x, y, z_1)}{r_T(x, y, z_1) + r_T(x, y, z_2)}$$

which shows that r_T is symmetric trireciprocal mapping. □

3 Classical stabilities of Eq. (1.4)

In this section, we examine the classical stabilities of Eq. (1.4) in the spirit of Ulam, Hyers, Rassias and Gavruta in the setting of Felbin's type fuzzy normed spaces. Throughout this section, let us assume that $\mathcal{M}$ is a linear space and $(\mathcal{N}, \|\cdot\|, L, R)$ is a fuzzy Banach space satisfying $(R - 2)$. Let us define

$$\begin{aligned}
D_{r_c}(a, b) &= r_c(a + b) \\
&\quad - \frac{r_c(a)r_c(b)}{r_c(a) + 3r_c(a)^{\frac{2}{3}}r_c(b)^{\frac{1}{3}} + 3r_c(a)^{\frac{1}{3}}r_c(b)^{\frac{2}{3}} + r_c(b)}
\end{aligned}$$

for all $a, b \in \mathcal{M}$. In the following results, let $\rho \in \{1, -1\}$ be a fixed number.

Theorem 3.1 *Let* $r_c : \mathcal{M} \longrightarrow \mathcal{N}$ *be a mapping for which there exists a function* $\xi : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{N}^*(\mathbb{R})$ *such that*

$$\sum_{i=0}^{\infty} \frac{1}{8^{i\rho}}\xi\left(\frac{a}{2^{i\rho+\frac{\rho+1}{2}}}, \frac{b}{2^{i\rho+\frac{\rho+1}{2}}}\right) < \infty \tag{3.1}$$

and r_c *satisfies*

$$\|D_{r_c}(a, b)\| \preceq \xi(a, b) \tag{3.2}$$

for all $a, b \in \mathcal{M}$ *and all* $\alpha \in (0, 1]$. *Then, there exists a unique multiplicative inverse cubic mapping* $R_c : \mathcal{M} \longrightarrow \mathcal{N}$ *such that for all* $\alpha \in (0, 1]$,

$$\|r_c(a) - R_c(a)\|_\alpha^+ \preceq \sum_{i=0}^{\infty} \frac{1}{8^{i\rho}} \xi\Big(\frac{a}{2^{i\rho+\frac{\rho+1}{2}}}, \frac{b}{2^{i\rho+\frac{\rho+1}{2}}}\Big) \quad (3.3)$$

for all $a, b \in \mathcal{M}$.

Proof Let $\rho = 1$. Plugging (a, b) into $\left(\frac{a}{2}, \frac{a}{2}\right)$ in (3.2), we get

$$\left\| r_c(a) - \frac{1}{8} r_c\left(\frac{a}{2}\right)\right\|_\alpha^+ \preceq \xi\left(\frac{a}{2}, \frac{a}{2}\right) \quad (3.4)$$

for all $a \in \mathcal{M}$. Now, reinstating a by $\frac{a}{2}$ in (3.4), dividing by 8 and then summing the resulting inequality with in (3.4), one finds

$$\left\| r_c(a) - \frac{1}{8^2} r_c\left(\frac{a}{2^2}\right)\right\|_\alpha^+ \preceq \sum_{i=0}^{1} \frac{1}{8^i} \xi\left(\frac{a}{2^{i+1}}, \frac{a}{2^{i+1}}\right) \quad (3.5)$$

for all $a \in \mathcal{M}$. Proceeding in the similar way and applying induction hypothesis, for an integer $n > 0$, one can arrive

$$\left\| c_a(a) - \frac{1}{8^n} r_c\left(\frac{a}{2^n}\right)\right\|_\alpha^+ \preceq \sum_{i=0}^{n-1} \frac{1}{8^i} \xi\left(\frac{a}{2^{i+1}}, \frac{b}{2^{i+1}}\right) \quad (3.6)$$

for all $a \in \mathcal{M}$. For any positive integer i and $a \in \mathcal{M}$, we have

$$\left\| \frac{1}{8^{i+1}} r_c\left(\frac{a}{2^{i+1}}\right) - \frac{1}{8^i} r_c\left(\frac{a}{2^i}\right)\right\|_\alpha^+ \preceq \frac{1}{8^i} \odot \xi\left(\frac{a}{2^{i+1}}, \frac{a}{2^{i+1}}\right) \quad (3.7)$$

for all $a \in \mathcal{M}$ and all nonnegative integers $n \in \mathbb{N}$. By Lemma 1.2 and inequality (3.7), we conclude that for all $\alpha \in (0, 1]$,

$$\left\| \frac{1}{8^n} r_c\left(\frac{a}{2^n}\right) - \frac{1}{8^m} r_c\left(\frac{a}{2^m}\right)\right\|_\alpha^+ \preceq \sum_{i=m}^{n-1} \frac{1}{8^i} \xi\left(\frac{a}{2^{i+1}}, \frac{a}{2^{i+1}}\right) \quad (3.8)$$

for all $a \in \mathcal{M}$. Taking the limit as $n \to \infty$ in (3.7) and considering (1.4), it follows that the sequence $\left\{\frac{1}{8^n} r_c\left(\frac{a}{2^n}\right)\right\}$ is a Cauchy sequence for each $a \in \mathcal{M}$. Since $\mathcal{N}$ is complete, we can define $R_c : \mathcal{M} \longrightarrow \mathcal{N}$ by

$$R_c(a) = \lim_{n\to\infty} \frac{1}{8^n} r_c\left(\frac{a}{2^n}\right). \quad (3.9)$$

To show that R_c satisfies in (1.4), replacing (a, b) by $(2^{-n}a, 2^{-n}b)$ in (3.2) and dividing by 8^n, we get

$$\left\|8^{-n} D_{r_c}\left(2^{-n}a, 2^{-n}b\right)\right\|_\alpha^+ \preceq 8^{-n}\xi\left(2^{-n}a, 2^{-n}b\right) \quad (3.10)$$

for all $a, b \in \mathcal{M}$ and for all positive integer n. Now, using (3.9) in (3.10), we see that R_c satisfies (1.4), for all $a, b \in \mathcal{M}$. Taking limit $n \to \infty$ in (3.6), we arrive (3.3). Now, it remains to show that R_c is uniquely defined. Let $R_c' : \mathcal{M} \longrightarrow \mathcal{N}$ be another multiplicative inverse cubic mapping which satisfies (1.4) and inequality (3.3). Clearly, $R_c'(2^{-n}a) = 8^n R_c'(a)$, $R_c(2^{-n}a) = 8^n R_c(a)$ and using (3.3), we have

$$\begin{aligned}
&\left\| R_c'(a) - R_c(a)\right\|_\alpha^+ \\
&= 8^{-n} \odot \left\| R_c'(2^{-n}a) - R_c(2^{-n}a)\right\|_\alpha^+ \\
&\le 8^{-n} \odot \left\| R_c'(2^{-n}a) - r_c(2^{-n}a)\right\|_\alpha^+ \\
&\quad + \left\| r_c(2^{-n}a) - R_c(2^{-n}a)\right\|_\alpha^+ \\
&\le 2 \sum_{i=0}^{\infty} \frac{1}{8^{n+i}} \xi\left(\frac{a}{2^{n+i+1}}, \frac{a}{2^{n+i+1}}\right) \\
&\le 2 \sum_{i=n}^{\infty} \frac{1}{8^i} \xi\left(\frac{a}{2^{i+1}}, \frac{a}{2^{i+1}}\right)
\end{aligned} \quad (3.11)$$

for all $a \in \mathcal{M}$. Allowing $n \to \infty$ in (3.11), we find R_c is a unique.

Similar proof is obtained for the case $\rho = -1$. □

Theorem 3.2 *Let $R(a, b) \le max(a, b)$ and $L(a, b) \ge min(a, b)$. Let $r_c : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping for which there exists a function $\xi : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{N}^*(\mathbb{R})$ satisfying (3.1) and (3.2) for all $a, b \in \mathcal{M}$ and all $\alpha \in (0, 1]$. Then, there exists a unique multiplicative inverse cubic mapping $R_c : \mathcal{M} \longrightarrow \mathcal{N}$ such that*

$$\|r_c(a) - R_c(a)\|_\alpha^+ \preceq \bar{\xi}(a, a)$$

for all $a \in \mathcal{M}$, where $\bar{\xi}(a, 0)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that

$$a_\alpha = \sum_{i=0}^{\infty} \frac{1}{8^{i\rho}} \xi\left(\frac{a}{2^{i\rho+\frac{\rho+1}{2}}}, \frac{a}{2^{i\rho+\frac{\rho+1}{2}}}\right)_\alpha^-,$$

$$b_\alpha = \sum_{i=0}^{\infty} \frac{1}{8^{i\rho}} \xi\left(\frac{a}{2^{i\rho+\frac{\rho+1}{2}}}, \frac{a}{2^{i\rho+\frac{\rho+1}{2}}}\right)_\alpha^+$$

for all $a \in \mathcal{M}$.

The ensuing corollary is a direct outcome of Theorem 3.1 associated with Hyers–Ulam (involving a positive constant), Hyers–Ulam–T. Rassias stability (involving sum of powers of norms), Ulam–Hyers–J. Rassias stability (incorporated with product of different powers of norms) and Ulam–Hyers–Gavruta-J. Rassias stability (containing mixed product–sum of powers of norms) of Eq. (1.4).

Corollary 3.1 *Let ϵ be a nonnegative fuzzy real number and $R(a,b) \leq max(a,b)$ and $L(a,b) \geq min(a,b)$. Let $p, \alpha, \beta, \lambda$ be real numbers such that $p \neq -3$ and $\lambda = \alpha + \beta \neq -3$. Suppose that the mapping $r_c : \mathcal{M} \longrightarrow \mathcal{N}$ satisfies the inequality*

$$\|D_{r_c}(a,b)\|_{\alpha}^{+} \preceq \begin{cases} \epsilon \\ \epsilon \otimes \left(\|a\|_{\mathcal{M}}^{p} \oplus \|b\|_{\mathcal{M}}^{p}\right) \\ \epsilon \otimes \left(\|a\|_{\mathcal{M}}^{\alpha} \otimes \|b\|_{\mathcal{M}}^{\beta}\right) \\ \epsilon \otimes \left(\|a\|_{\mathcal{M}}^{p} \oplus \|b\|_{\mathcal{M}}^{p} \oplus \left[\|a\|_{\mathcal{M}}^{\frac{p}{2}} \otimes \|b\|_{\mathcal{M}}^{\frac{p}{2}}\right]\right) \end{cases}$$

for all $a, b \in \mathcal{M}$. Then, there exists a unique multiplicative inverse cubic mapping $R_c : \mathcal{M} \longrightarrow \mathcal{N}$ satisfying

$$\|r_c(a) - R_c(a)\|_{\alpha}^{+} \preceq \begin{cases} \frac{8}{7}\epsilon_{\beta}^{+} & \\ \frac{16\epsilon_{\beta}^{+}\left(\|a\|_{\beta}^{+}\right)^{p}}{|2^{3+p}-1|} & \text{for } p \neq -3 \\ \frac{8\epsilon_{\beta}^{+}}{|2^{\lambda+3}-1|}\left(\|a\|_{\alpha}^{+}\right)^{\lambda} & \text{for } \lambda \neq -3 \\ \frac{24\epsilon_{\beta}^{+}}{|2^{p+3}-1|}\left(\|a\|_{\alpha}^{+}\right)^{p} & \text{for } p \neq -3 \end{cases} \tag{3.12}$$

for all $a \in \mathcal{M}$.

Now, motivated by the excellent counter-example provided by Gajda (1991), we will provide here a counter-example to illustrate that the stability of Eq. (1.4) fails for the critical case $p = -3$ (second result in 3.12) of Corollary 3.1 in the setting of nonzero real numbers. The following theorem proves the nonstability of Eq. (1.4).

Example 3.3 Let $\xi : \mathbb{R}^{\star} \longrightarrow \mathbb{R}$ be a function defined by

$$\xi(x) = \begin{cases} \frac{\epsilon}{a^3}, & \text{if } a \in (1, \infty) \\ \epsilon & \end{cases}$$

where $\epsilon > 0$ is a constant and a function $r_c : \mathcal{M} \longrightarrow \mathcal{N}$ by

$$r_c(a) = \sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}a) \quad \text{for all } a \in \mathbb{R}^{\star}.$$

Then, r_c satisfies the functional inequality

$$\left|D_{r_c}(a,b)\right| \leq \frac{72}{7}\epsilon\left(\left|\frac{1}{a}\right|^{3} + \left|\frac{1}{b}\right|^{3}\right) \tag{3.13}$$

for all $a, b \in \mathcal{M}$. Then, there does not exist a multiplicative inverse cubic mapping $R_c : \mathbb{R}^{\star} \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that

$$|r_c(a) - R_c(a)| \leq \beta\left|\frac{1}{a}\right|^{3} \quad \text{for all } a \in \mathcal{M}. \tag{3.14}$$

Proof Firstly, we observe that $|r_c(a)| \leq \sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}a) = \sum_{k=0}^{\infty} \frac{\epsilon}{8^k} = \frac{8\epsilon}{7}$. This implies that r_c is bounded. Next, let us show that r_c satisfies (3.13). Suppose if $\left|\frac{1}{a}\right|^{3} + \left|\frac{1}{b}\right|^{3} \geq 1$, then the left-hand side of (3.13) is less than $\frac{72\epsilon}{7}$. Now, assume that $0 < \left|\frac{1}{a}\right|^{3} + \left|\frac{1}{b}\right|^{3} < 1$. Then, there exists a positive integer r such that

$$\frac{1}{8^{r+1}} \leq \left|\frac{1}{a}\right|^{3} + \left|\frac{1}{b}\right|^{3} < \frac{1}{8^r}, \tag{3.15}$$

so that $8^r\left(\frac{1}{a}\right)^{3} < 1,\ 8^r\left(\frac{1}{b}\right)^{3} < 1$ and consequently

$$\frac{1}{8^{r-1}}(a) > 1, \quad \frac{1}{8^{r-1}}(b) > 1, \quad \frac{1}{8^{r-1}}(a+b) > 1$$

Therefore, for each $k = 0, 1, \ldots, r-1$, we have

$$\frac{1}{8^r}(a) > 1, \quad \frac{1}{8^r}(b) > 1, \quad \frac{1}{8^r}(a+b) > 1.$$

and $D_{r_c}(8^{-k}a, 8^{-k}b) = 0$ for $k = 0, 1, 2, \ldots, r-1$. From the definition of r_c and (3.15), we obtain

$$\begin{aligned}
&\left|D_{r_c}(a,b)\right| \\
&= \Bigg|\sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}(a+b)) \\
&\quad - \frac{\sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}a)\sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}b)}{\sum_{k=0}^{\infty} 8^{-k}\left(\xi(2^{-k}a)+3\xi(2^{-k}a)^{\frac{2}{3}}\xi(2^{-k}b)^{\frac{1}{3}}+3\xi(2^{-k}a)^{\frac{1}{3}}\xi(2^{-k}b)^{\frac{2}{3}}+\xi(2^{-k}b)\right)}\Bigg| \\
&= \Bigg|\sum_{k=r}^{\infty} 8^{-k}\xi(2^{-k}(a+b)) \\
&\quad - \frac{\sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}a)\sum_{k=0}^{\infty} 8^{-k}\xi(2^{-k}b)}{\sum\limits_{k=0}^{\infty} 8^{-k}\left(\xi(2^{-k}a)+3\xi(2^{-k}a)^{\frac{2}{3}}\xi(2^{-k}b)^{\frac{1}{3}}+3\xi(2^{-k}a)^{\frac{1}{3}}\xi(2^{-k}b)^{\frac{2}{3}}+\xi(2^{-k}b)\right)}\Bigg| \\
&\le \sum_{k=r}^{\infty}\frac{\epsilon}{8^r}+\frac{1}{8}\sum_{k=r}^{\infty}\frac{\epsilon}{8^r} \le \frac{9\epsilon}{7}\frac{1}{8^k} \le \frac{72\epsilon}{7}\frac{1}{8^{k+1}} \le \frac{72\epsilon}{7}\left(\left|\frac{1}{a}\right|^3+\left|\frac{1}{b}\right|^3\right)
\end{aligned}$$

for all $a, b \in \mathbb{R}^{\star}$. Therefore, r_c satisfies (3.13) for all $a, b \in \mathbb{R}^{\star}$. We assert that equation (1.4) is not stable for $p = -3$ in Corollary 3.1 in the second result of (3.12). Suppose there exist a multiplicative inverse cubic mapping $R_c : \mathbb{R}^{\star} \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ satisfying (3.14). Since r_c is bounded, in view of Corollary 3.1, $R_c(a)$ must have the form $R_c(a) = \frac{k}{a^3}$ for any $a \in \mathcal{M}$. Thus, we obtain that

$$|r_c(a)| \le (\beta + |k|)\left|\frac{1}{a}\right|^3. \tag{3.16}$$

But we can choose a positive integer m with $m\epsilon > \beta + |r|$. If $a \in \left(0, 2^{m-1}\right)$, then $2^{-k}a \in (1,\infty)$ for all $k = 0, 1, \ldots, m-1$. For this a, we get

$$\begin{aligned}
r_c(a) &= \sum_{k=0}^{\infty}\frac{\xi(2^{-k}a)}{8^k} \ge \sum_{k=0}^{m-1}\frac{(2^{-k}a)^3}{8^k} = m\epsilon\frac{1}{a^3} \\
&> (\beta + |k|)\frac{1}{a^3}
\end{aligned}$$

which contradicts (3.16). Therefore, Eq. (1.4) is not stable when $p = -3$ in the sense of Ulam, Hyers and T. Rassias in Corollary 3.1 in the second result of (3.12). □

4 Pertinence of Eq. (1.4) in electromagnetism

In this section, we associate Eq. (1.4) with the well-known inverse cube law occurring in electromagnetism. The magnetic field strength due to a dipole from a point obeys inverse cube law. Suppose F_1 is the field strength between the center of a dipole and another point P_1 at a distance a units, which is shown in Fig. 1.

Then, $F_1 = \frac{k}{a^3}$, where k is a constant. Let F_2 be the field strength between the center of a dipole and another point P_2 at a distance b units, which is provided in Fig. 2.

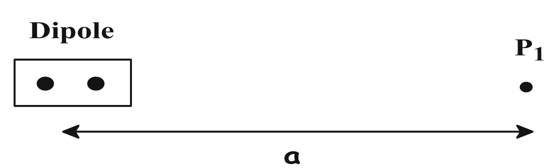

Fig. 1 Field strength of a dipole and a point at a distance a units

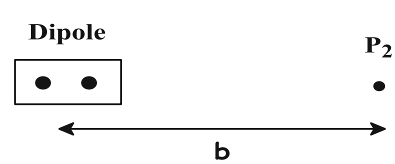

Fig. 2 Field strength of a dipole and a point at a distance b units

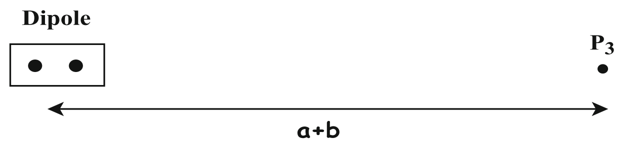

Fig. 3 Field strength of a dipole and a point at a distance a+b units

Then, $F_2 = \dfrac{k}{b^3}$, where k is a constant. Now, assume F_3 is the field strength between the center of a dipole and another point P_3 at a distance $a + b$ units, which is given in Fig. 3.

Then, $F_3 = \dfrac{k}{(a+b)^3}$, where k is a constant. Then, F_1, F_2 and F_3 can be related as follows:

$$F_3 = \frac{F_1 F_2}{F_1 + 3F_1^{\frac{2}{3}} F_2^{\frac{1}{3}} + 3F_1^{\frac{1}{3}} F_2^{\frac{2}{3}} + F_2}. \tag{4.1}$$

From Eq. (4.1), we observe that the relation connecting F_1, F_2 and F_3 is pertinent to Eq. (1.4) with $F_1 = r_c(a)$, $F_2 = r_c(b)$ and $F_3 = r_c(a + b)$. Thus, if the field strengths between the dipole and any two different points are known, then we can measure the field strength between the dipole and a third point at a distance of sum of the distances of the dipole with first point and the dipole with second point.

5 Hypothesis, limitations and justification of the method exercised in this study

5.1 Hypothesis

We implemented the direct method initiated by Hyers to solve Ulam stability problems of equations. In this method, the solution of the functional inequality (3.2) is directly derived from the given function r_c as R_c, which implies that there exists an approximate solution to the functional equation (1.4) as per Ulam's question. Hence, this direct method is very much useful to approach Ulam stability problems.

5.2 Limitations

Since a solution of functional equation (1.4) is a multiplicative inverse cubic mapping, we have to avoid zero and hence we assume some additional conditions so that the function r_c satisfying (1.4) does not have infinity as image of any argument. Also, the stability result fails for singular cases discussed in Example 3.3.

5.3 Justification

This is our first attempt to exercise this direct method to analyze stability results of (1.4) in Felbin's type fuzzy normed spaces. The advantage of this method is that it is easy to construct an approximate solution near to the exact solution of Eq. (1.4). This method can be applied to solve stability problems of some other multiplicative inverse functional equations. The results obtained and the method adopted in this study would be useful for other researchers to carry out further investigations. Since there are lot of applications of rational functions in various fields including physics, economics, business, medicine, digital image processing, chemistry, etc., the study of this type of equations has a lot of scope for other researchers. For some functional equations, this direct method (Hyers' method) fails to produce approximate solution from the assumed function. In such cases, we cannot apply this method. This is one of the disadvantages of the method.

6 Conclusion

In this study, we have dealt with a new multiplicative inverse cubic functional equation (1.4) to obtain its solution. The algorithm of the powerful tool (direct method) devised in Hyers (1941) and employed to achieve our main results is as follows:

1. We have constructed a Cauchy sequence $\left\{\frac{1}{8^n} r_c\left(\frac{a}{2^n}\right)\right\}$, which converges to the mapping R_c by the completeness of $\mathcal{N}$.
2. We have proved that the mapping R_c satisfies (1.4).
3. We have shown that the mapping R_c is unique.

The mapping R_c is a multiplicative inverse cubic mapping, and r_c is an approximate solution to (1.4). Thus, we have accomplished the results concerning Ulam, Hyers, T. Rassias, J. Rassias and Gavruta stabilities of Eq. (1.4). We have also demonstrated that Eq. (1.4) is not stable for a singular case through a counterexample. The significance of Eq. (1.4) is elucidated by means of real-time occurrence in electromagnetism.

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Compliance with ethical standards

Conflict of interest Author 1 declares that he has no conflict of interest. Author 2 declares that he has no conflict of interest. Author 3 declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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