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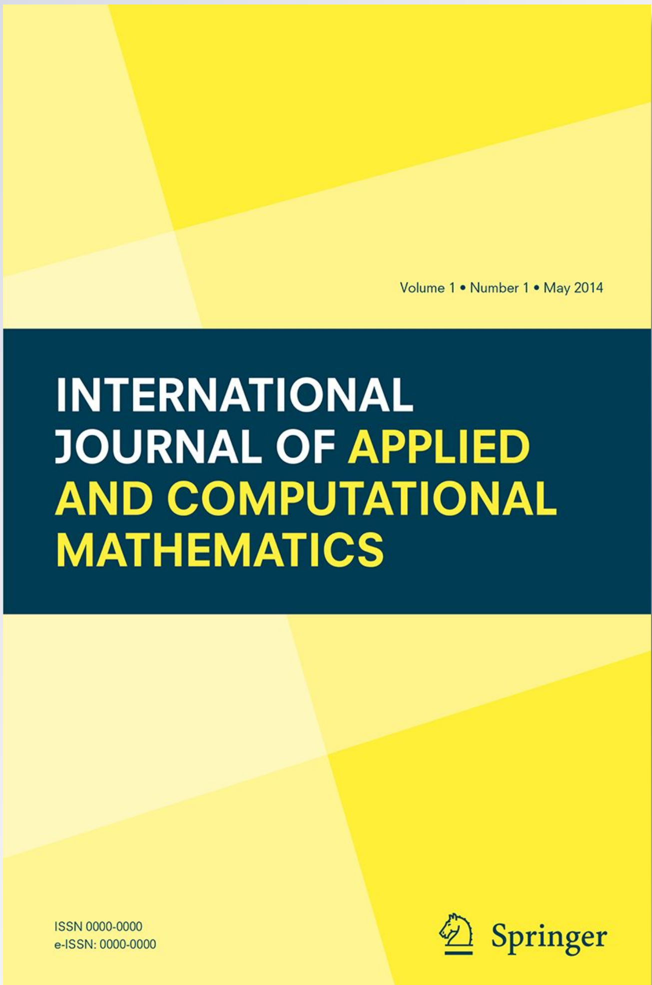

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ORIGINAL PAPER

Approximation of a System of Rational Functional Equations of Three Variables

B. V. Senthil Kumar[1] · Hemen Dutta[2] · S. Sabarinathan[3]

Abstract
This study aims at substantiating the validity of stability results of a system of rational functional equations involving three variables connected with the Ulam stability theory of functional equations. There are some functional equations identified with real-time occurences. This system of functional equations is related with the intensive properties of substances such as density and volume.

Keywords Hyperreal · Reciprocal function · Bi-reciprocal functional equation · Generalized Hyers–Ulam–Rassias stability

Mathematics Subject Classification 39B82 · 39B72

Introduction

The set of real numbers is extended to a set comprising indeterminable and minuscule numbers, which is called as the set of hyperreal numbers. If there exists a positive integer k such that for a hyperreal number v, if $|v| < k$, then v is said to be finite and minuscule if $|v| < \frac{1}{k}$. The symbol $^*\mathbb{R}$ is used to denote the set of hyperreals. The conjecture of real numbers is employed in the standard analysis. The study of standard analysis is calculus. In the same manner, the hypothesis of hyperreals are utilized for non-standard analysis (NSA). The hypothesis of NSA was extended by Robinson in 1960's and it was considered as attentive

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fundamentals for intuitions about minuscules. It can be found that there are plenty of applications of hyperreals in many divisions of science, often with an interesting philosophical dimension. Hyperreals could be used to reevaluate the archive of calculus. A remarkable implementation of NSA is in the didactics of calculus. There are many other exertions of NSA in physics that are related to differential and stochastic equations and also in Markov processes, Lévy Brownian motion, and Sturm-Liouville problems. NSA also finds major role in quantum mechanics which includes quantum field theory and Feynman path integrals. For details of many fascinating and interesting features of hyperreals, one can refer to [13,40].

The conjecture of stability of functional equations is pioneered via the query submitted by Ulam [39] and subsequently his problem was responded by Hyers [18]. Eventually Ulam's query was postulated by Rassias [28] and Gavruta [12] in different adaptations. There are many novel, significant, motivating, interesting problems connected with the stability problem of various forms of functional equations and system of functional equations in different spaces and domains (see [1,2,10,14,15,21–23,25,26]).

The reciprocal, quadratic reciprocal and cubic reciprocal functional equations are studied for the first time in [5,31] and [19], respectively.

Several other type of rational form of functional equations and their stability results are disucssed in [3,4,6–9,16,17,20,24,30,33,34,37,38].

Moreover, the mathematical formulation of several multiplicative inverse functional equations and the classical investigation of various stability results of multiplicative inverse undecic and duodecic functional equations can be found in [36].

In the recent times, the authors of this paper [35] have dealt with a new rational functional equation involving two variables of the form

$$r(u_1+u_2, v_1+v_2) = \frac{\prod_{j=1}^{2}\prod_{k=1}^{2} r(u_j, v_k)}{\sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{1}{r(u_j,v_k)}\prod_{m=1}^{2}\prod_{n=1}^{2} r(u_m, v_n)\right)} \tag{1}$$

to accomplish the solution of (1), examine its various stabilities and illustrate its relevance to designing some systems appearing in physics.

Ravi and first author of this paper [32] dealt the following system of bi-reciprocal functional equations

$$\left.\begin{aligned} r(x+u, y) &= \frac{r(x, y)r(u, y)}{r(x, y)+r(u, y)} \\ r(x, y+v) &= \frac{r(x, y)r(x, v)}{r(x, y)+r(x, v)} \end{aligned}\right\} \tag{2}$$

and validated the existence of various stabilities in Fréchet spaces. It is easy to see that $r(x, y) = \frac{1}{xy}$ is a solution of (2).

In the sequel, we extend the system of bi-reciprocal functional equations (2) of two variables into a system of functional equations involving three variables with the following definition.

Definition 1 A function $T_r : {}^*\mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ is called as tri-reciprocal if T_r fulfills the ensuing system of functional equations

$$\left.\begin{aligned} T_r(u_1+u_2,v,w) &= \frac{T_r(u_1,v,w)T_r(u_2,v,w)}{T_r(u_1,v,w)+T_r(u_2,v,w)},\\ T_r(u,v_1+v_2,w) &= \frac{T_r(u,v_1,w)T_r(u,v_2,w)}{T_r(u,v_1,w)+T_r(u,v_2,w)},\\ T_r(u,v,w_1+w_2) &= \frac{T_r(u,v,w_1)T_r(u,v,w_2)}{T_r(u,v,w_1)+T_r(u,v,w_2)}. \end{aligned}\right\} \tag{3}$$

It can be easily substantiated that the rational function $T_r(u,v,w) = \frac{c}{uvw}$ is a solution of the system (3), where c is a constant.

In this paper, we prove the existence of the stability results of the system of equations (3) in the setting of hyperreals, associated with the stability theory originated from Ulam, Hyers, Rassias and Gavruta. We also discuss an application of the solution of system of equations (3) with the basic properties of an object.

For the convenience of manipulation, let us symbolize for a given mapping $T_r : {}^*\mathbb{R}\times {}^*\mathbb{R}\times {}^*\mathbb{R} \longrightarrow \mathbb{R}$, the difference operators $\Delta_1, \Delta_2, \Delta_3 : {}^*\mathbb{R}\times {}^*\mathbb{R}\times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} \Delta_1 T_r(u_1,u_2,v,w) &= T_r(u_1+u_2,v,w) - \frac{T_r(u_1,v,w)T_r(u_2,v,w)}{T_r(u_1,v,w)+T_r(u_2,v,w)},\\ \Delta_2 T_r(u,v_1,v_2,w) &= T_r(u,v_1+v_2,w) - \frac{T_r(u,v_1,w)T_r(u,v_2,w)}{T_r(u,v_1,w)+T_r(u,v_2,w)},\\ \Delta_3 T_r(u,v,w_1,w_2) &= T_r(u,v,w_1+w_2) - \frac{T_r(u,v_1,w)T_r(u,v_2,w)}{T_r(u,v_1,w)+T_r(u,v_2,w)} \end{aligned}$$

for all $u_1,u_2,v_1,v_2,w_1,w_2 \in {}^*\mathbb{R}$.

Stability of the System of Equation (3)

In the sequel, we validate the existence of stabilities of the system of equations (3) pertinent to Hyers, Rassias and Gavruta by considering the set of hyperreals as domain. The proceeding result includes the investigation of the stability of system of equations (3) relevant to Gavruta's repercussion.

Theorem 1 *Assume that the functions* $P, Q, R : {}^*\mathbb{R}\times^*\mathbb{R}\times {}^*\mathbb{R}\times^*\mathbb{R} \longrightarrow [0,\infty)$ *satisfy the following conditions:*

$$\left.\begin{aligned} &\sum_{j=0}^{\infty} 8^j P(2^j u, 2^u, 2^j v, 2^j w) < \infty,\\ &\sum_{j=0}^{\infty} 8^j Q(2^{j+1} u, 2^j v, 2^j v, 2^j w) < \infty,\\ &\sum_{j=0}^{\infty} 8^j R(2^{j+1} u, 2^{j+1} v, 2^j w, 2^j w) < \infty \end{aligned}\right\} \tag{4}$$

for all $u, v, w \in {}^*\mathbb{R}$. *Also, let us presume that for a given positive real number* ϵ, *the above functions satisfy the ensuing inequalities:*

$$\left.\begin{aligned}\sum_{j=0}^{\infty} 8^j P(2^j u, 2^u, 2^j v, 2^j w) < \frac{\epsilon}{6},\\ \sum_{j=0}^{\infty} 8^j Q(2^{j+1} u, 2^j v, 2^j v, 2^j w) < \frac{\epsilon}{12}\\ \sum_{j=0}^{\infty} 8^j R(2^{j+1} u, 2^{j+1} v, 2^j w, 2^j w) < \frac{\epsilon}{24}\end{aligned}\right\} \tag{5}$$

for all $u, v, w \in {}^*\mathbb{R}$. Suppose a function $T_r^* : \mathbb{R} \times^* \mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ fulfills

$$|\Delta_1 T_r(u_1, u_2, v, w)| \leq P(u_1, u_2, v, w) \tag{6}$$
$$|\Delta_2 T_r(u, v_1, v_2, w)| \leq Q(u, v_1, v_2, w) \tag{7}$$
$$|\Delta_3 T_r(u, v, w_1, w_2)| \leq R(u, v, w_1, w_2) \tag{8}$$

for all $u_1, u_2, v_1, v_2, w_1, w_2 \in {}^*\mathbb{R}$. Then, there exists a unique tri-reciprocal function T_R : ${}^*\mathbb{R} \times^* \mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies (3) and

$$\begin{aligned}&|T_r(u, v, w) - T_R(u, v, w)|\\ &\leq 2\sum_{j=0}^{\infty} 8^j P(2^j u, 2^j u, 2^j v, 2^j w) + 4\sum_{j=0}^{\infty} 8^j Q(2^{j+1} u, 2^j v, 2^j v, 2^j w)\\ &\quad + 8\sum_{j=0}^{\infty} 8^j R(2^{j+1} u, 2^{j+1} v, 2^j v, 2^j w)\\ &\leq \epsilon\end{aligned} \tag{9}$$

for all $u, v, w \in {}^*\mathbb{R}$.

Proof First, let us substitute $u_1 = u_2 = u$ in (6) and then multiplying by 2 on both sides to obtain

$$|2T_r(2u, v, w) - T_r(u, v, w)| \leq 2P(u, u, v, w) \tag{10}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, letting $v_1 = v_2 = v$ in (7) and then multiplying by 2 on both sides, we obtain

$$|2T_r(u, 2v, w) - T_r(u, v, w)| \leq 2Q(u, v, v, w) \tag{11}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, plugging (w_1, w_2) by (w, w) in (8) and then multiplying by 2 on both sides, we get

$$|2T_r(u, v, 2w) - T_r(u, v, w)| \leq 2R(u, v, w, w) \tag{12}$$

for all $u, v, w \in {}^*\mathbb{R}$. Replacing v by $2v$ in (12) and then multiplying by 2 on both sides, yields

$$|4T_r(u, 2v, 2w) - 2T_r(u, 2v, w)| \leq 4R(u, 2v, w, w) \tag{13}$$

for all $u, v, w \in {}^*\mathbb{R}$. By employing triangle inequality law on (11) and (13), we find

$$|4T_r(u, 2v, 2w) - T_r(u, v, w)| \leq 2Q(u, v, v, w) + 4R(u, 2v, w, w) \tag{14}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, swapping u by $2u$ in (14) and then multiplying by 2, we obtain

$$|8T_r(2u, 2v, 2w) - 2T_r(2u, v, w)| \leq 4Q(2u, v, v, w) + 8R(2u, 2v, w, w) \tag{15}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, again in lieu of triangle inequality law on (10) and (15), we lead to

$$\begin{aligned} &|8T_r(2u, 2v, 2w) - T_r(u, v, w)| \\ &\quad \le 2P(u, u, v, w) + 4Q(2u, v, v, w) + 8R(2u, 2v, w, w) \end{aligned} \tag{16}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, plugging (u, v, w) into $(2u, 2v, 2w)$ in (16) and then multiplying by 8 on both sides, we obtain

$$\begin{aligned} &\left|8^2 T_r(4u, 4v, 4w) - 8T_r(2u, 2v, 2w)\right| \\ &\quad \le 16P(2u, 2u, 2v, 2w) + 32Q(4u, 2v, 2v, 2w) + 64R(4u, 4v, 2w, 2w) \end{aligned} \tag{17}$$

for all $u, v, w \in {}^*\mathbb{R}$. Further by the application of triangle inequality law on (16) and (17), we get

$$\begin{aligned} &\left|8^2 T_r(4u, 4v, 4w) - T_r(u, v, w)\right| \\ &\quad \le 2P(u, u, v, w) + 16P(2u, 2u, 2v, 2w) + 4Q(2u, v, v, w) + 32Q(4u, 2v, 2v, 2w) \\ &\qquad + 8R(2u, 2v, w, w) + 64R(4u, 4v, 2w, 2w) \\ &\quad \le 2\sum_{j=0}^{1} 8^j P(2^j u, 2^j u, 2^j v, 2^j w) + 4\sum_{j=0}^{1} 8^j Q(2^{j+1}u, 2^j v, 2^j v, 2^j w) \\ &\qquad + 8\sum_{j=0}^{1} 8^j R(2^{j+1}u, 2^{j+1}v, 2^j w, 2^j w) \end{aligned}$$

for all $u, v, w \in {}^*\mathbb{R}$. Continuing further in this fashion and utilizing induction arguments on the above inequality, we conclude that

$$\begin{aligned} &\left|8^n T_r(4u, 4v, 4w) - T_r(u, v, w)\right| \\ &\quad \le 2\sum_{j=0}^{n-1} 8^j P(2^j u, 2^j u, 2^j v, 2^j w) + 4\sum_{j=0}^{n-1} 8^j Q(2^{j+1}u, 2^j v, 2^j v, 2^j w) \\ &\qquad + 8\sum_{j=0}^{n-1} 8^j R(2^{j+1}u, 2^{j+1}v, 2^j w, 2^j w) \\ &\quad \le 2\sum_{j=0}^{\infty} 8^j P(2^j u, 2^j u, 2^j v, 2^j w) + 4\sum_{j=0}^{\infty} 8^j Q(2^{j+1}u, 2^j v, 2^j v, 2^j w) \\ &\qquad + 8\sum_{j=0}^{\infty} 8^j R(2^{j+1}u, 2^{j+1}v, 2^j w, 2^j w) \end{aligned} \tag{18}$$

for all $u, v, w \in {}^*\mathbb{R}$. Next is to prove the sequence $\{8^n r(2^n u, 2^n v, 2^n w)\}$ is convergent. For this, let us plug (u, v, w) into $(2^k u, 2^k v, 2^k w)$ in (18) and then multiply by 8^k, then we find for any $n, k > 0$,

$$\begin{aligned} &\left|8^{n+k} T_r(2^{n+k}u, 2^{n+k}v, 2^{n+k}w) - 8^k T_r(2^k u, 2^k v, 2^k w)\right| \\ &\quad = 8^k \left|8^n T_r(2^{n+k}u, 2^{n+k}v, 2^{n+k}w) - T_r(2^k u, 2^k v, 2^k w)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq 2\sum_{j=o}^{\infty} 8^{k+j} T_r P\left(2^{k+j}u, 2^{k+j}u, 2^{k+j}v, 2^{k+j}w\right)\\
&\quad + 4\sum_{j=0}^{\infty} 8^{k+j} Q\left(2^{k+j+1}u, 2^{k+j}u, 2^{k+j}v, 2^{k+j}w\right)\\
&\quad + 8\sum_{j=0}^{\infty} 8^{k+j} R\left(2^{k+j+1}u, 2^{k+j+1}v, 2^{k+j}w, 2^{k+j}w\right)\\
&\quad \longrightarrow 0 \text{ as } k \longrightarrow \infty.
\end{aligned}
$$

From the above result, we conclude that the sequence $\{8^n r(2^n u, 2^n v, 2^n w)\}$ is Cauchy. Since $\mathbb{R}$ is complete, it is obvious that there exists a function $T_R :^* \mathbb{R} \times^* \mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ defined by the existence of the limit $T_R(u, v, w) = \lim_{n\to\infty} 8^n T_r(2^n u, 2^n v, 2^n w)$ for all $u, v, w \in X$. It is clear that (9) is valid for all $u, v, w \in {}^*\mathbb{R}$ by letting $n \longrightarrow \infty$ in (18). Now, let us show that T_R satisfies (3). In fact, switching (u, v, w) to $(2^n u, 2^n v, 2^n w)$ and then multiplying by 8^n in (6), (7) and (8), respectively, we lead to

$$
\begin{aligned}
|\Delta_1 T_R(u_1, u_2, v, w)| &= \lim_{n\to\infty} 8^n \Delta_1 T_r(2^n u_1, 2^n u_2, 2^n v, 2^n w)\\
&\leq \lim_{n\to\infty} 8^n P(2^n u_1, 2^n u_2, 2^n v, 2^n w) = 0,\\
|\Delta_2 T_R(u, v_1, v_2, w)| &= \lim_{n\to\infty} 8^n \Delta_2 T_r(2^n u, 2^n v_1, 2^n v_2, 2^n w)\\
&\leq \lim_{n\to\infty} 8^n P(2^n u, 2^n v_1, 2^n v_2, 2^n w) = 0,\\
|\Delta_3 T_R(u, v, w_1, w_2)| &= \lim_{n\to\infty} 8^n \Delta_3 T_r(2^n u, 2^n v, 2^n w_1, 2^n w_2)\\
&\leq \lim_{n\to\infty} 8^n P(2^n u, 2^n v, 2^n w_1, 2^n w_2) = 0
\end{aligned}
$$

for all $u_1, u_2, v_1, v_2, w_1, w_2 \in {}^*\mathbb{R}$, which shows that T_R is tri-reciprocal. In order to show the uniqueness of T_R, let us presume that $T_R' :^* \mathbb{R} \times^* \mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ is another tri-reciprocal function satisfying (3) and (9), then

$$
\begin{aligned}
&\left|T_R(u, v, w) - T_R'(u, v, w)\right|\\
&\quad = 8^n \left|T_R(2^n u, 2^n v, 2^n w) - T_R'(2^n u, 2^n v, 2^n w)\right|\\
&\quad = 8^n \Big\{ \left|T_R(2^n u, 2^n v, 2^n w) - T_r(2^n u, 2^n v, 2^n w)\right|\\
&\qquad + \left|T_r(2^n u, 2^n v, 2^n w) - T_R'(2^n u, 2^n v, 2^n w)\right| \Big\}\\
&\quad \leq 4\sum_{j=0}^{\infty} 8^{n+j} P(2^{n+j}u, 2^{n+j}u, 2^{n+j}v, 2^{n+j}w)\\
&\qquad + 8\sum_{j=0}^{\infty} 8^{n+j} Q(2^{n+j+1}u, 2^{n+j}v, 2^{n+j}v, 2^{n+j}w)\\
&\qquad + 16\sum_{j=0}^{\infty} 8^{n+j} R(2^{n+j+1}u, 2^{n+j+1}v, 2^{n+j}w, 2^{n+j}w)
\end{aligned}
$$

for all $u, v, w \in {}^*\mathbb{R}$. This implies that $T_R(u, v, w) = T_R'(u, v, w)$ for all $u, v, w \in {}^*\mathbb{R}$. Hence, we conclude that T_R is unique, which completes the proof. □

In the upcoming result, we illustrate the validity of the existence of stability of the system of equations (3) incorporated with addition of exponents of norms.

Corollary 1 *Consider a fixed constant $\theta_1 > 0$. Suppose $\alpha < -3$ is a real number. Assume a function $T_r :^* \mathbb{R} \times^* \mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the ensuing inequalities*

$$\left.\begin{aligned} |\Delta_1 T_r(u_1, u_2, v, w)| &\le \theta_1 \left(\|u_1\|^\alpha + \|u_2\|^\alpha + \|v\|^\alpha + \|w\|^\alpha\right), \\ |\Delta_2 T_r(u, v_1, v_2, w)| &\le \theta_1 \left(\|u\|^\alpha + \|v_1\|^\alpha + \|v_2\|^\alpha + \|w\|^\alpha\right), \\ |\Delta_3 T_r(u, v, w_1, w_2)| &\le \theta_1 \left(\|u\|^\alpha + \|v\|^\alpha + \|w_1\|^\alpha + \|w_2\|^\alpha\right), \end{aligned}\right\} \tag{19}$$

for all $u_1, u_2, v_2, w_1, w_2 \in {}^*\mathbb{R}$. Then, there exists a unique tri-reciprocal function $T_R :^* \mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ which satisfies (3) and

$$\begin{aligned} &|T_r(u, v, w) - T_R(u, v, w)| \\ &\le \left(\frac{2\theta_1}{1 - 2^{\alpha+3}}\right) \left\{\left(2 + 6 \cdot 2^\alpha\right)|u|^\alpha + \left(5 + 4 \cdot 2^\alpha\right)|v|^\alpha + 11\,|w|^\alpha\right\} \end{aligned}$$

for all $u, v, w \in {}^*\mathbb{R}$.

Proof Considering

$$\begin{aligned} P(u_1, u_2, v, w) &= \theta_1 \left(\|u_1\|^\alpha + \|u_2\|^\alpha + \|v\|^\alpha + \|w\|^\alpha\right), \\ Q(u, v_1, v_2, w) &= \theta_1 \left(\|u\|^\alpha + \|v_1\|^\alpha + \|v_2\|^\alpha + \|w\|^\alpha\right), \\ \text{and} \qquad R(u, v, w_1, w_2) &= \theta_1 \left(\|u\|^\alpha + \|v\|^\alpha + \|w_1\|^\alpha + \|w_2\|^\alpha\right) \end{aligned}$$

for all $u, u_1, u_2, v, v_1, v_2, w, w_1, w_2 \in {}^*\mathbb{R}$ in Theorem 1, we get

$$\begin{aligned} &|T_r(u, v, w) - T_R(u, v, w)| \\ &\le 2\theta_1 \sum_{j=0}^{\infty} 8^j \left(\left|2^j u\right|^\alpha + \left|2^j u\right|^\alpha + \left|2^j v\right|^\alpha + \left|2^j w\right|^\alpha\right) \\ &\quad + 4\theta_1 \sum_{j=0}^{\infty} 8^j \left(\left|2^{j+1} u\right|^\alpha + \left|2^j v\right|^\alpha + \left|2^j v\right|^\alpha + \left|2^j w\right|^\alpha\right) \\ &\quad + 8\theta_1 \sum_{j=0}^{\infty} 8^j \left(\left|2^{j+1} u\right|^\alpha + \left|2^{j+1} v\right|^\alpha \left|2^j w\right|^\alpha + \left|2^j w\right|^\alpha\right) \\ &\le 2\theta_1 \sum_{j=0}^{\infty} \left(2 \cdot 2^{(\alpha+3)j}\,|u|^\alpha + 2^{(\alpha+3)j}\,|v|^\alpha + 2^{(\alpha+3)j}\,|w|^\alpha\right) \\ &\quad + 4\theta_1 \sum_{j=0}^{\infty} \left(2^\alpha \cdot 2^{(\alpha+3)j}\,|u|^\alpha + 2 \cdot 2^{(\alpha+3)j}\,|v|^\alpha + 2^{(\alpha+3)j}\,|w|^\alpha\right) \\ &\quad + 8\theta_1 \sum_{j=0}^{\infty} \left(2^\alpha \cdot 2^{(\alpha+3)j}\,|u|^\alpha + 2^\alpha \cdot 2^{(\alpha+3)j}\,|v|^\alpha + 2 \cdot 2^{(\alpha+3)j}\,|w|^\alpha\right) \\ &\le \left(\frac{2\theta_1}{1 - 2^{\alpha+3}}\right) \left[\left(2 + 6 \cdot 2^\alpha\right)|u|^\alpha + \left(5 + 4 \cdot 2^\alpha\right)|v|^\alpha + 11\,|w|^\alpha\right] \end{aligned}$$

for all $u, v, w \in {}^*\mathbb{R}$. □

The following corollaries are the stability results of system of equations (3) associated with UGR stability (involving product of powers of norms) [27] and J. Rassias stability [29] (involving mixed product-sum of powers of norms). Since the arguments of arriving at the desired results are similar to Corollary 1, we provide only the results.

Corollary 2 *Let $\theta_2 > 0$ be a fixed constant and $\alpha < -3$ is a real number. Let a mapping $T_r :^* \mathbb{R} \times^* \mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following inequalities*

$$\left.\begin{aligned} |\Delta_1 T_r(u_1, u_2, v, w)| &\leq \theta_3 \, |u_1|^{\frac{\alpha}{6}} \, |u_2|^{\frac{\alpha}{6}} \, |v|^{\frac{\alpha}{3}} \, |w|^{\frac{\alpha}{3}} \,, \\ |\Delta_2 T_r(u, v_1, v_2, w)| &\leq \theta_3 \, |u|^{\frac{\alpha}{3}} \, |v_1|^{\frac{\alpha}{6}} \, |v_2|^{\frac{\alpha}{6}} \, |w|^{\frac{\alpha}{3}} \,, \\ |\Delta_3 T_r(u, v, w_1, w_2)| &\leq \theta_3 \, |u|^{\frac{\alpha}{3}} \, |v|^{\frac{\alpha}{3}} \, |w_1|^{\frac{\alpha}{6}} \, |w_2|^{\frac{\alpha}{6}} \,, \end{aligned}\right\} \tag{20}$$

for all $u_1, u_2, v_2, w_1, w_2 \in {}^*\mathbb{R}$. Then, a unqiue tri-reciprocal function $T_R :^* \mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ exists and satisfies (3) such that

$$|T_r(u, v, w) - T_R(u, v, w)| \leq \left(\frac{2\theta_2}{1 - 2^{\alpha+3}}\right)\left(1 + 2 \cdot 2^{\frac{\alpha}{3}} + 4 \cdot 2^{\frac{\alpha}{3}}\right) |u|^{\frac{\alpha}{3}} \, |v|^{\frac{\alpha}{3}} \, |w|^{\frac{\alpha}{3}}$$

for all $u, v, w \in {}^*\mathbb{R}$.

Corollary 3 *Assume that $\theta_3 > 0$ is a fixed constant and $\alpha < -3$ is a real number. If a function $T_r :^* \mathbb{R} \times^* \mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the ensuing inequalities*

$$\left.\begin{aligned} &|\Delta_1 T_r(u_1, u_2, v, w)| \\ &\quad \leq \theta_3 \left(|u_1|^{\frac{\alpha}{6}} \, |u_2|^{\frac{\alpha}{6}} \, |v|^{\frac{\alpha}{3}} \, |w|^{\frac{\alpha}{3}} + \left(|u_1|^{\alpha} + |u_2|^{\alpha} + |v|^{\alpha} + |w|^{\alpha}\right)\right), \\ &|\Delta_2 T_r(u, v_1, v_2, w)| \\ &\quad \leq \theta_3 \left(|u|^{\frac{\alpha}{3}} \, |v_1|^{\frac{\alpha}{6}} \, |v_2|^{\frac{\alpha}{6}} \, |w|^{\frac{\alpha}{3}} + \left(|u|^{\alpha} + |v_1|^{\alpha} + |v_2|^{\alpha} + |w|^{\alpha}\right)\right), \\ &|\Delta_3 T_r(u, v, w_1, w_2)| \\ &\quad \leq \theta_3 \left(|u|^{\frac{\alpha}{3}} \, |v|^{\frac{\alpha}{3}} \, |w_1|^{\frac{\alpha}{6}} \, |w_2|^{\frac{\alpha}{6}} + \left(|u|^{\alpha} + |v|^{\alpha} + |w_1|^{\alpha} + |w_2|^{\alpha}\right)\right), \end{aligned}\right\} \tag{21}$$

for all $u_1, u_2, v_2, w_1, w_2 \in {}^*\mathbb{R}$, then a unqiue tri-reciprocal function $T_R :^* \mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \longrightarrow \mathbb{R}$ exists and satisfies (3) such that

$$\begin{aligned} &|T_r(u, v, w) - T_R(u, v, w)| \\ &\quad \leq \left(\frac{2\theta_3}{1 - 2^{\alpha+3}}\right)\left\{\left(1 + 2 \cdot 2^{\frac{\alpha}{6}} + 4 \cdot 2^{\frac{2\alpha}{3}}\right) |u|^{\frac{\alpha}{3}} \, |v|^{\frac{\alpha}{3}} \, |w|^{\frac{\alpha}{3}} \right. \\ &\quad\quad \left. + 2\left(1 + 2^{\alpha} + 2^{\alpha+1}\right) |u|^{\alpha} + \left(5 + 2^{\alpha+3}\right) |v|^{\alpha} + 7 \, |w|^{\alpha} \right\} \end{aligned}$$

for all $u, v, w \in {}^*\mathbb{R}$.

The subsequent theorem is dual result of Theorem 1. Eventhough, the proof of this theorem is analogous to Theorem 1, for the sake of comprehensiveness, we provide the key part of the proof.

Theorem 2 *Assume that the functions* $P, Q, R :^* \mathbb{R} \times {}^*\mathbb{R} \times {}^*\mathbb{R} \times^* \mathbb{R} \longrightarrow [0, \infty)$ *satisfy the following conditions:*

$$\left.\begin{aligned} &\sum_{j=0}^{\infty} \frac{1}{8^i} P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \infty, \\ &\sum_{j=0}^{\infty} \frac{1}{8^j} Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \infty, \\ &\sum_{j=0}^{\infty} \frac{1}{8^j} R\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{w}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \infty \end{aligned}\right\} \tag{22}$$

for all $u, v, w \in {}^*\mathbb{R}$*. Let us assume that for a given positive real number* ϵ*, the above functions satisfy the following inequalities:*

$$\left.\begin{aligned} &\sum_{j=0}^{\infty} \frac{1}{8^i} P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \frac{4\epsilon}{3}, \\ &\sum_{j=0}^{\infty} \frac{1}{8^j} Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \frac{2\epsilon}{3}, \\ &\sum_{j=0}^{\infty} \frac{1}{8^j} R\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{w}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \frac{\epsilon}{3} \end{aligned}\right\} \tag{23}$$

for all $u, v, w \in {}^*\mathbb{R}$*. Suppose a function* $T_r :^* \mathbb{R} \times {}^*\mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies* (6), (7) *and* (8) *for all* $u, u_1, u_2, v, v_1, v_2, w, w_1, w_2 \in {}^*\mathbb{R}$*. Then, there exists a unique tri-reciprocal function* $T_R :^* \mathbb{R} \times {}^*\mathbb{R} \times^* \mathbb{R} \longrightarrow \mathbb{R}$ *which satisfies* (3) *and*

$$\begin{aligned} &|T_r(u, v, w) - T_R(u, v, w)| \\ &\leq \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{8^j} P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) + \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{8^j} Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) \\ &\quad + \sum_{j=0}^{\infty} \frac{1}{8^j} R\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{w}{2^{j+1}}, \frac{w}{2^{j+1}}\right) \\ &\leq \epsilon \end{aligned} \tag{24}$$

for all $u, v, w \in {}^*\mathbb{R}$*.*

Proof First, let us take (u_1, u_2, v, w) as $\left(\frac{u}{2}, \frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right)$ in (6) and then dividing by 4, we get

$$\left|\frac{1}{4} T_r\left(u, \frac{v}{2}, \frac{w}{2}\right) - \frac{1}{8} T_r\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right)\right| \leq \frac{1}{4} P\left(\frac{u}{2}, \frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) \tag{25}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, plugging (u, v_1, v_2, w) into $\left(u, \frac{v}{2}, \frac{v}{2}, \frac{w}{2}\right)$ in (7) and then dividing by 2, we obtain

$$\left|\frac{1}{2} T_r\left(u, v, \frac{w}{2}\right) - \frac{1}{4} T_r\left(u, \frac{v}{2}, \frac{w}{2}\right)\right| \leq \frac{1}{2} Q\left(u, \frac{v}{2}, \frac{v}{2}, \frac{w}{2}\right) \tag{26}$$

for all $u, v, w \in {}^*\mathbb{R}$. Swapping (u, v, w_1, w_2) into $\left(u, v, \frac{w}{2}, \frac{w}{2}\right)$ in (8), we have

$$\left|T_r(u, v, w) - \frac{1}{2}T_r\left(u, v, \frac{w}{2}\right)\right| \le R\left(u, v, \frac{w}{2}, \frac{w}{2}\right) \tag{27}$$

for all $u, v, w \in {}^*\mathbb{R}$. By employing triangle inequality law on (26) and (27), we find

$$\left|T_r(u, v, w) - \frac{1}{4}T_r\left(u, \frac{v}{2}, \frac{w}{2}\right)\right| \le \frac{1}{2}Q\left(u, \frac{v}{2}, \frac{v}{2}, \frac{w}{2}\right) + R\left(u, v, \frac{w}{2}, \frac{w}{2}\right) \tag{28}$$

for all $u, v, w \in {}^*\mathbb{R}$. Once again, utilizing triangle inequality law on (25) and (28), we lead to

$$\begin{aligned}
&\left|T_r(u, v, w) - \frac{1}{8}T_r\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right)\right| \\
&\quad\le \frac{1}{4}P\left(\frac{u}{2}, \frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) + \frac{1}{2}Q\left(u, \frac{v}{2}, \frac{v}{2}, \frac{w}{2}\right) + R\left(u, v, \frac{w}{2}, \frac{w}{2}\right)
\end{aligned} \tag{29}$$

for all $u, v, w \in {}^*\mathbb{R}$. Now, swapping (u, v, w) into $\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right)$ in (29) and then dividing by 8, we obtain

$$\begin{aligned}
&\left|\frac{1}{8}T_r\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) - \frac{1}{8^2}T_r\left(\frac{u}{2^2}, \frac{v}{2^2}, \frac{w}{2^2}\right)\right| \\
&\quad\le \frac{1}{32}P\left(\frac{u}{2^2}, \frac{u}{2^2}, \frac{v}{2^2}, \frac{w}{2^2}\right) + \frac{1}{16}Q\left(\frac{u}{2}, \frac{v}{2^2}, \frac{v}{2^2}, \frac{w}{2^2}\right) + \frac{1}{8}R\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2^2}, \frac{w}{2^2}\right)
\end{aligned} \tag{30}$$

for all $u, v, w \in {}^*\mathbb{R}$. In lieu of triangle inequality law on (29) and (30), one finds

$$\begin{aligned}
&\left|\frac{1}{8}T_r\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) - \frac{1}{8^2}T_r\left(\frac{u}{2^2}, \frac{v}{2^2}, \frac{w}{2^2}\right)\right| \\
&\quad\le \frac{1}{4}\sum_{j=0}^{1}\frac{1}{8^j}P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) + \frac{1}{2}\sum_{j=0}^{1}\frac{1}{8^j}Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) \\
&\qquad+ \sum_{j=0}^{1}\frac{1}{8^j}R\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{w}{2^{j+1}}, \frac{w}{2^{j+1}}\right)
\end{aligned}$$

for all $u, v, w \in {}^*\mathbb{R}$. Continuing further in this fashion and making use of induction arguments on a positive integer n, we have

$$\begin{aligned}
&\left|\frac{1}{8}T_r\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) - \frac{1}{8^n}T_r\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right)\right| \\
&\quad\le \frac{1}{4}\sum_{j=0}^{n-1}\frac{1}{8^j}P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) + \frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{8^j}Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) \\
&\qquad+ \sum_{j=0}^{n-1}\frac{1}{8^j}R\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{w}{2^{j+1}}, \frac{w}{2^{j+1}}\right) \\
&\quad\le \frac{1}{4}\sum_{j=0}^{\infty}\frac{1}{8^j}P\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right) + \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{8^j}Q\left(\frac{u}{2^j}, \frac{v}{2^{j+1}}, \frac{v}{2^{j+1}}, \frac{w}{2^{j+1}}\right)
\end{aligned}$$

$$+\sum_{j=0}^{\infty}\frac{1}{8^j}R\left(\frac{u}{2^j},\frac{v}{2^j},\frac{w}{2^{j+1}},\frac{w}{2^{j+1}}\right)$$
$$\leq \epsilon$$

for all $u, v, w \in {}^*\mathbb{R}$. The enduring part of the proof is achieved akin to Theorem 1. □

The following result comprises the validity of the stability result of the system of equations (3) concerning the sum of powers of norms, product of powers of norms and mixed product-sum of powers of norms. The following corollaries directly follow from Theorem 2 and hence we omit the proof.

Corollary 4 *Consider a fixed constant $\theta_1 > 0$. Suppose $\alpha > -3$ is a real number. Assume a function $T_r :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ fulfills the inequalities in (19). Then, a unique tri-reciprocal function $T_R :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ exists and satisfies (3) such that*

$$|T_r(u,v,w)-T_R(u,v,w)|$$
$$\leq\left(\frac{2^{\alpha+3}\theta_1}{2^{\alpha+3}-1}\right)\left[\left(\frac{3\cdot 2^{\alpha}+4}{2^{\alpha}+1}\right)|u|^{\alpha}+\left(\frac{2^{\alpha}+5}{4\cdot 2^{\alpha}}\right)|v|^{\alpha}+\left(\frac{14}{4\cdot 2^{\alpha+1}}\right)|w|^{\alpha}\right]$$

for all $u, v, w \in {}^*\mathbb{R}$.

Proof The proof is similar to that of Corollary 1. □

Corollary 5 *Let $\theta_2 > 0$ be a fixed constant and $\alpha < -3$ is a real number. Let a mapping $T_r :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the inequalities (20). Then, a unqiue tri-reciprocal function $T_R :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ exists and satisfies (3) such that*

$$|T_r(u,v,w)-T_R(u,v,w)|\leq\frac{8\theta_2}{2^{\alpha}\left(1-2^{\alpha+3}\right)}\left(1+2^{\frac{\alpha}{3}}+2^{\frac{5\alpha}{3}}\right)|u|^{\frac{\alpha}{3}}|v|^{\frac{\alpha}{3}}|w|^{\frac{\alpha}{3}}$$

for all $u, v, w \in {}^*\mathbb{R}$.

Corollary 6 *Assume that $\theta_3 > 0$ is a fixed constant and $\alpha < -3$ is a real number. If a function $T_r :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ satisfies the inequalities (21), then a unqiue tri-reciprocal function $T_R :^* \mathbb{R}\times{}^*\mathbb{R}\times{}^*\mathbb{R} \longrightarrow \mathbb{R}$ exists and satisfies (3) such that*

$$|T_r(u,v,w)-T_R(u,v,w)|$$
$$\leq\left(\frac{\theta_3}{2^{\alpha+3}-1}\right)\left\{\left(2+12\cdot 2^{\frac{2\alpha}{3}}\right)|u|^{\frac{\alpha}{3}}|v|^{\frac{\alpha}{3}}|w|^{\frac{\alpha}{3}}\right.$$
$$\left.+\left(4+4^{\alpha}+2^{\alpha+3}\right)|u|^{\alpha}+\left(10+2^{\alpha+3}\right)|v|^{\alpha}+22|w|^{\alpha}\right\}$$

for all $u, v, w \in {}^*\mathbb{R}$.

Counter-Examples

In this section, employing the notion of the famous counter-example presented in [11], we illustrate some counter-examples that the system of equations (3) is not stable for $\alpha = -3$ in Corollaries 1, 3, 4 and 5.

Consider the function

$$\varphi(u, v, w) = \begin{cases} k\left(|u|^{-3} + |v|^{-3} + |w|^{-3}\right), & \text{for } u, v, w \in (0, \infty) \\ k, & \text{otherwise} \end{cases} \tag{31}$$

where $\varphi : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$. Let $T_r : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ be defined by

$$T_r(u, v, w) = \sum_{n=0}^{\infty} 8^{-n} \varphi(2^{-n}u, 2^{-n}v, 2^{-n}w) \tag{32}$$

for all $u, v, w \in \mathbb{R}^\star$. Then the function T_r serves as a counter-example for the fact that the system of equations (3) is not stable for $\alpha = -3$ in Corollary 1 in the following theorem.

Theorem 3 *If the function $T_r : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ defined in (32) satisfies the following functional inequalities*

$$\left.\begin{aligned} |\Delta_1 T_r(u_1, u_2, v, w)| &\le \frac{12k}{7}\left(|u_1|^{-3} + |u_2|^{-3} + |v|^{-3} + |w|^{-3}\right), \\ |\Delta_2 T_r(u, v_1, v_2, w)| &\le \frac{12k}{7}\left(|u|^{-3} + |v_1|^{-3} + |v_2|^{-3} + |w|^{-3}\right), \\ |\Delta_3 T_r(u, v, w_1, w_2)| &\le \frac{12k}{7}\left(|u|^{-3} + |v|^{-3} + |w_1|^{-3} + |w_2|^{-3}\right), \end{aligned}\right\} \tag{33}$$

for all $u_1, u_2, v_2, w_1, w_2 \in \mathbb{R}^\star$, then there do not exist a tri-reciprocal mapping $T_R : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that

$$|T_r(u, v, w) - T_R(u, v, w)| \le \mu\left(|u|^{-3} + |v|^{-3} + |w|^{-3}\right) \tag{34}$$

for all $u, v, w \in \mathbb{R}^\star$.

Proof Firstly, let us prove that T_r satisfies (34). By computation, we have

$$|T_r(u, v, w)| = \left|\sum_{n=0}^{\infty} 8^{-n}\varphi(2^{-n}x)\right| \le \sum_{n=0}^{\infty} \frac{k}{8^n} = \frac{8k}{7}.$$

Therefore, we see that T_r is bounded by $\frac{8k}{7}$ on $\mathbb{R}$. If $|u_1|^{-3} + |u_2|^{-3} + |v|^{-3} + |w|^{-3} \ge 1$, $|u|^{-3} + |v_1|^{-3} + |v_2|^{-3} + |w|^{-3} \ge 1$ and $|u|^{-3} + |v|^{-3} + |w_1|^{-3} + |w_2|^{-3} \ge 1$ then the left hand side of each inequalities in (33) is less than $\frac{12k}{7}$. Now, suppose that $0 < |u_1|^{-3} + |u_2|^{-3} + |v|^{-3} + |w|^{-3} < 1$. Hence, there exists a positive integer k such that

$$\frac{1}{8^{k+1}} \le |u_1|^{-3} + |u_2|^{-3} + |v|^{-3} + |w|^{-3} < \frac{1}{8^k}. \tag{35}$$

The relation (35) implies $8^k\left(|u_1|^{-3} + |u_2|^{-3} + |v|^{-3} + |w|^{-3}\right) < 1$, or equivalently; $8^k u_1^{-3} < 1, 8^k u_2^{-3} < 1, 8^k v^{-3} < 1, 8^k w^{-3} < 1$. So, $\frac{u_1^3}{8^k} > 1, \frac{u_2^3}{8^k} > 1, \frac{v^3}{8^k} > 1, \frac{w^3}{8^k} > 1$. The last inequalities imply that $\frac{u_1^3}{8^{k-1}} > 8 > 1, \frac{u_2^3}{8^{k-1}} > 8 > 1, \frac{v^3}{8^{k-1}} > 8 > 1, \frac{w^3}{8^{k-1}} > 8 > 1$ and consequently

$$\frac{1}{2^{k-1}}(u_1) > 1, \frac{1}{2^{k-1}}(u_2) > 1, \frac{1}{2^{k-1}}(v) > 1, \frac{1}{2^{k-1}}(w) > 1, \frac{1}{2^{k-1}}(u_1 + u_2) > 1.$$

Therefore, for each value of $n = 0, 1, 2, \ldots, k-1$, we obtain

$$\frac{1}{2^n}(u_1) > 1, \frac{1}{2^n}(u_2) > 1, \frac{1}{2^n}(v) > 1, \frac{1}{2^n}(2) > 1, \frac{1}{2^n}(u_1 + u_2) > 1.$$

and $\Delta_1\varphi(2^{-n}u_1, 2^{-n}u_2, 2^{-n}v, 2^{-n}w) = 0$ for $n = 0, 1, 2, \dots, k-1$. Using (31) and the definition of T_r, we obtain

$$\begin{aligned}|\Delta_1 T_r(u_1, u_2, v, w)| &\leq \sum_{n=k}^{\infty} \frac{k}{8^n} + \frac{1}{2}\sum_{n=k}^{\infty} \frac{k}{8^n} \leq \frac{3k}{2}\frac{1}{8^k}\left(1-\frac{1}{8}\right)^{-1}\\ &\leq \frac{3k}{2}\frac{1}{8^k} \leq \frac{12k}{7}\frac{1}{8^{k+1}} \leq \frac{12k}{7}\left(|u_1|^{-3}+|u_2|^{-3}+|v|^{-3}+|w|^{-3}\right)\end{aligned}$$

for all $u_1, u_2, v, w \in \mathbb{R}^\star$. Therefore, the first inequality in (33) holds good. Similarly, we can show that the second and third inequalities in (33) hold good. Now, we claim that the system of equations (3) is not stable for $\alpha = -3$ in Corollary 1. Assume that there exists a tri-reciprocal mapping $T : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \to \mathbb{R}$ satisfying (34). Therefore, we have

$$|T_r(u, v, w)| \leq (\mu + 1)\left(|u|^{-3} + |v|^{-3} + |w|^{-3}\right). \tag{36}$$

However, we can choose a positive integer m with $m\delta > \mu + 1$. If $u, v, w \in \left(1, 2^{m-1}\right)$ then $2^{-n}u, 2^{-n}v, 2^{-n}w \in (1, \infty)$, for all $n = 0, 1, 2, \dots, m-1$ and thus

$$\begin{aligned}|T_r(u, v, w)| &= \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{8^n} \geq \sum_{n=0}^{m-1} \frac{8^n k\left(u^{-3}+v^{-3}+w^{-3}\right)}{8^n}\\ &= mk\left(u^{-3}+v^{-3}+w^{-3}\right) > (\mu+1)\left(u^{-3}+v^{-3}+w^{-3}\right)\end{aligned}$$

which contradicts (36). Therefore, the system of equations (3) is not stable for $\alpha = -3$ in Corollary 1. □

The upcoming result indicates that the system of equations (3) is not stable for $\alpha = -3$ in Corollary 3. Since the method of proof is similar, we omit the proof.

Theorem 4 *If the function $T_r : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ defined in (32) satisfies the following functional inequalities*

$$\left.\begin{aligned}&|\Delta_1 T_r(u_1, u_2, v, w)|\\ &\quad\leq \theta_3\left(|u_1|^{-\frac{1}{2}}|u_2|^{-\frac{1}{2}}|v|^{-1}|w|^{-1} + \left(|u_1|^{-3}+|u_2|^{-3}+|v|^{-3}+|w|^{-3}\right)\right),\\ &|\Delta_2 T_r(u, v_1, v_2, w)|\\ &\quad\leq \theta_3\left(|u|^{-1}|v_1|^{-\frac{1}{2}}|v_2|^{-\frac{1}{2}}|w|^{-1} + \left(|u|^{-3}+|v_1|^{-3}+|v_2|^{-3}+|w|^{-3}\right)\right),\\ &|\Delta_3 T_r(u, v, w_1, w_2)|\\ &\quad\leq \theta_3\left(|u|^{-1}|v|^{-1}|w_1|^{-\frac{1}{2}}|w_2|^{-\frac{1}{2}} + \left(|u|^{-3}+|v|^{-3}+|w_1|^{-3}+|w_2|^{-3}\right)\right),\end{aligned}\right\} \tag{37}$$

for all $u_1, u_2, v_2, w_1, w_2 \in \mathbb{R}^\star$, then there do not exist a tri-reciprocal mapping $T_R : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that

$$|T_r(u, v, w) - T_R(u, v, w)| \leq \mu\left(|u|^{-1}|v|^{-1}|w|^{-1} + \left(|u|^{-3}+|v|^{-3}+|w|^{-3}\right)\right) \tag{38}$$

for all $u, v, w \in \mathbb{R}^\star$.

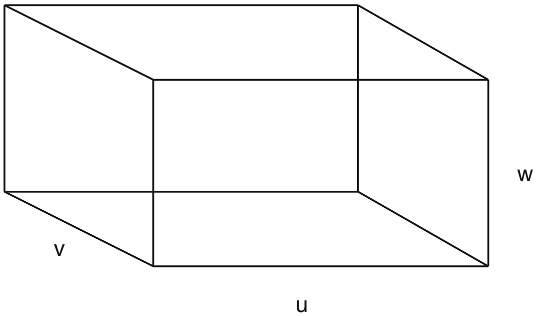
Counter-examples similar to Theorems 3 and 4 can be considered for the non-stability of system of equations in Corollary 4 and 9.

An Exertion of the System of Equation (2)

In this section, we associate the system of functional equations (3) with the relationship between density and volume of any object. We know, density is the mass per unit volume of any object. In this section, we assume that the mass of the object to be a constant. Imagine a cuboid shaped sponge of dimensions of length u, breadth v and height w as in Figure 1. Then the rational function $T_r(u, v, w) = \frac{c}{uvw}$ represents the density of the sponge, which means that if volume increases, then density decreases with constant mass. This can be illustrated in the following occasion. If the sponge is squeezed, then the density of the squeezed sponge is more when compared to the less density of the loosely left sponge.

Now, divide the sponge shown in Figure 1 into two subcuboids along its length as portrayed in Figure 2. Then the densities of the subcuboids are $T_r(u_1, v, w) = \frac{c}{u_1 vw}$ and $T_r(u_2, v, w) = \frac{c}{u_2 vw}$, respectively, where $u_1 + u_2 = u$. Hence, the density of the original cuboid shaped sponge is half of the harmonic mean of the volumes of the two sub cuboids, which satisfies the first member of the system of equations (3) dealt in our study. Similarly, if the original cuboid shaped sponge is divided along its breadth and height, the second and third members of the system of equations (3) satisfy the above fact relating density and volume.

Fig. 1 Cuboid sponge



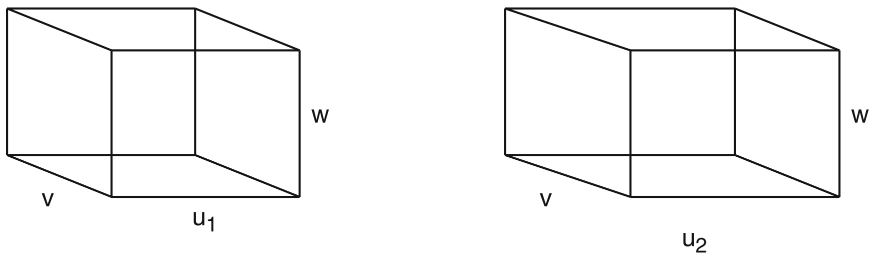


Fig. 2 Division of cuboid sponge into two subcuboids

Conclusion

In this investigation, we have carried out various stabilities of the system of tri-reciprocal functional equations (3). We illustrated counter-examples that the stability results of system of equations (3) do not hold for singular cases. Since the solution of the system (3) is a rational function, both indeterminable and minuscule numbers could be considered in the domain. This is the first attempt that we have considered the set of hyperreals as domain for the investigation of stability results. Also, the notion of non-standard analysis could be identified via functional equations. We have also exhibited the role of the system of functional equations dealt in this study in connection with volume and density of a cuboid shaped sponge.

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