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On new approximations and expositions of reciprocal third power mappings

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Abstract: The intention of this work is to deal with new form of reciprocal third power functional equations for their solutions. The Ulam stabilities of these equations are determined in the setting of non-Archimedean fields. A proper instance is demonstrated to show the invalidity of stability result for a very critical case. The interpretation of the equations dealt in this study is associated with a significant hypothesis in electromagnetic theory and the relation of stiffness (or deflection) and length of diving board (or cantilever beam).

Keywords: Reciprocal functional equation, cubic functional equation, Generalized Hyers-Ulam stability, non-Archimedean field.

1 Introduction

The inception and the evolution of Ulam stability results of various equations in different versions are accessible in [1,2,3,4,5,6]. For the past four decades, there are many interesting, new, motivating and important and remarkable results pertinent to various forms of functional, reciprocal or rational type or multiplicative inverse, difference, differential and integral equations, one can refer to [7,8,9,10,11,12,13,14,15,16,17,18,19,20]. Quite recently, the following reciprocal third power functional equation,

$$m_c(\lambda_1+\lambda_2) = \frac{m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3} \tag{1}$$

is dealt in [21] to investigate its fuzzy stabilities and its application using inverse cubic law. It is also prove that the reciprocal third power mapping $m_c(\lambda) = \frac{k}{\lambda^3}$ is a solution of (1). Inspired by the significant results available in the literature, we propose a new reciprocal third power difference functional equation:

$$m_c\left(\frac{1}{p}\sum_{k=1}^{p}\lambda_k\right) - m_c\left(\sum_{k=1}^{p}\lambda_k\right) = \frac{(p^3-1)\prod_{k=1}^{p} m_c(\lambda_k)}{\left[\sum_{k=1}^{p} m_c(\lambda_k)^{\frac{1}{3}}\right]^3} \tag{2}$$

and a reciprocal third power adjoint functional equation:

$$m_c\left(\frac{1}{p}\sum_{k=1}^{p}\lambda_k\right) + m_c\left(\sum_{k=1}^{p}\lambda_k\right) = \frac{(p^3+1)\prod_{k=1}^{p} m_c(\lambda_k)}{\left[\sum_{k=1}^{p} m_c(\lambda_k)^{\frac{1}{3}}\right]^3}. \tag{3}$$

It is not hard to verify that the reciprocal third power mapping $m_c(\lambda) = \frac{1}{\lambda^3}$ is a solution of equations (2) and (3). We illustrate the application of the above equations, for $p = 2$ with (u_1, u_2) replaced with (u, v) in both equations. Hence, we can reduce them in the following forms for two variables:

$$m_c\left(\frac{\lambda_1+\lambda_2}{2}\right) - m_c(\lambda_1+\lambda_2) = \frac{7m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3} \tag{4}$$

and

$$m_c\left(\frac{\lambda_1+\lambda_2}{2}\right) + m_c(\lambda_1+\lambda_2) = \frac{9m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3} \tag{5}$$

respectively. We solve Ulam stability problems for the above equations (4) and (5) in the setting of non-Archimedean fields. We present a suitable counter-example for the invalidity of stability result in

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case of singularity. Then, we elucidate their implications in electromagnetism and in the property of diving board.

In order to prove our main results, we furnish below the fundamental definition of non-Archimedean field.

Definition 1. *Suppose $\mathbb{G}$ is a field provided with a valuation $|\cdot| : \mathbb{G} \longrightarrow [0,\infty)$. Then $\mathbb{G}$ is said to be a non-Archimedean field if the ensuing conditions hold: $|k| = 0$ if and only if $k = 0$, $|k\ell| = |k||\ell|$ and $|k+\ell| \leq sup\{|k|,|\ell|\}$ for all $k,\ell \in \mathbb{G}$.*

In the entire study, we assume that $\mathbb{G}$ and $\mathbb{H}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, we use the symbol $\mathbb{G}^*$ which excludes 0 from $\mathbb{G}$. For the intention of proving the major results in an easy manner, let us define the difference operator $\Delta m_c : \mathbb{G}^* \times \mathbb{G}^* \longrightarrow \mathbb{H}$ as follows:

$$\Delta m_c(\lambda_1,\lambda_2) = m_c\left(\frac{\lambda_1+\lambda_2}{2}\right) - m_c(\lambda_1+\lambda_2) - \frac{7m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3}$$

for all $\lambda_1,\lambda_2 \in \mathbb{G}^*$.

2 Similarity of equations (4) and (5)

In this section, let us prove that the equations (4) and (5) have reciprocal third power mapping as their solution. In the following result, we show that they are analogous to each other. In the following outcome, we assume that $\lambda_1,\lambda_2,\lambda \in \mathbb{R}^\star$.

Theorem 1. *Let $m_c : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a mapping. Then the following assertions are identical to each other.*

(a) m_c is a solution of (1).

(b) m_c is a solution of (4).

(c) m_c is a solution of (5).

Thus, a reciprocal third power mapping is a solution of equations (4) and (5).

Proof. Let us first assume that m_c is a solution of (1). Now, switching (λ_1,λ_2) to $\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (1) and then multiplying by 8, one attains that

$$m_c\left(\frac{\lambda}{2}\right) = 8m_c(\lambda). \tag{6}$$

Then, reinstating (λ_1,λ_2) by $\left(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\right)$ in (1) and using the result of (6) in the resultant, one acquires that

$$m_c\left(\frac{\lambda_1+\lambda_2}{2}\right) = \frac{8m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3}. \tag{7}$$

It is easy to achieve equation (4) by taking the difference between (7) and (1). Next, let us assume that m_c is a solution of (5). If (λ_1,λ_2) is changed as $\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (5) and on further simplification, one finds that

$$m_c\left(\frac{\lambda}{2}\right) = 8m_c(\lambda). \tag{8}$$

Employing the result of (8) in (4) and then simplifying further, one obtains that

$$m_c(\lambda_1+\lambda_2) = \frac{m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3}. \tag{9}$$

Now, replacing (λ_1,λ_2) by $\left(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\right)$ in (9) and then utilizing (8), one gets that

$$m_c\left(\frac{\lambda_1+\lambda_2}{2}\right) = \frac{8m_c(\lambda_1)m_c(\lambda_2)}{\left[m_c(\lambda_1)^{\frac{1}{3}} + m_c(\lambda_2)^{\frac{1}{3}}\right]^3}. \tag{10}$$

Adding up equations (10) and (9), one arrives at (5). Finally, let us assume that m_c is a solution of (5). Using similar reasoning applied in the aforementioned steps, if (λ_1,λ_2) is considered as $\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (5) and simplified further, one has

$$m_c\left(\frac{\lambda}{2}\right) = 8m_c(\lambda). \tag{11}$$

Using the outcome of (11) in (5), it produces (1). Therefore, m_c is a reciprocal third power mapping.

3 Approximation of reciprocal third power mapping

In the upcoming results, we determine the validity of various fundamental stabilities of equations (4) and (5) associated with Ulam, Hyers, T. Rassias, J. Rassias and Gavruta.

Theorem 2. *Consider a fixed number $\beta \neq \pm 1$. Suppose $m_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ is a mapping satisfies*

$$|\Delta m_c(\lambda_1,\lambda_2)| \leq \xi(\lambda_1,\lambda_2) \tag{12}$$

where $\xi : \mathbb{G}^* \times \mathbb{G}^* \longrightarrow [0,\infty)$ is a function with the condition that

$$\lim_{m\to\infty} \left|\frac{1}{8}\right|^{\beta m} \xi\left(\frac{\lambda_1}{2^{\beta m+\frac{\beta+1}{2}}}, \frac{\lambda_2}{2^{\beta m+\frac{\beta+1}{2}}}\right) = 0 \tag{13}$$

for all $\lambda_1,\lambda_2 \in \mathbb{G}^*$. Then, there exists a unique reciprocal third power mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ which satisfies (4) with the result that

$$|m_c(u) - M_c(u)| \leq \sup\left\{\left|\frac{1}{8}\right|^{p\beta+\frac{\beta-1}{2}} \xi\left(\frac{\lambda}{2^{p\beta+\frac{\beta+1}{2}}}, \frac{\lambda}{2^{p\beta+\frac{\beta+1}{2}}}\right) : p \in \mathbb{N}\cup\{0\}\right\} \tag{14}$$

for all $\lambda \in \mathbb{G}^*$.

Proof. Firstly, we construct a reciprocal third power mapping M_c satisfying (4). For this, let us replace (λ_1, λ_2) by $\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ in (12) to obtain

$$\left| m_c(\lambda) - \frac{1}{2^\beta} m_c\left(\frac{\lambda}{8^\beta}\right) \right| \leq |8|^{\frac{|\beta-1|}{2}} \xi\left(\frac{\lambda}{2^{\frac{\beta+1}{2}}}, \frac{\lambda}{2^{\frac{\beta+1}{2}}}\right) \tag{15}$$

for all $\lambda \in \mathbb{G}^*$. Now, again replace λ with $\frac{\lambda}{2^{\beta m}}$ in (15) and multiply by $\left|\frac{1}{8}\right|^{\beta m}$ in the resultant to acquire

$$\left| \frac{1}{8^{\beta m}} m_c\left(\frac{\lambda}{2^{\beta m}}\right) - \frac{1}{8^{(m+1)\beta}} m_c\left(\frac{\lambda}{2^{(m+1)\beta}}\right) \right| \leq \left|\frac{1}{8}\right|^{\beta m + \frac{\beta-1}{2}} \xi\left(\frac{\lambda}{2^{\beta m + \frac{\beta+1}{2}}}, \frac{\lambda}{2^{\beta m + \frac{\beta+1}{2}}}\right) \tag{16}$$

for all $\lambda \in \mathbb{G}^*$. Application of (13) and (16) produces that the sequence $\left\{\frac{1}{8^{\beta m}} m_c\left(\frac{\lambda}{2^{\beta m}}\right)\right\}$ is Cauchy. In view of the fact that $\mathbb{H}$ is complete, this sequence converges to a mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ given by

$$M_c(\lambda) = \lim_{m\to\infty} \frac{1}{8^{\beta m}} m_c\left(\frac{\lambda}{2^{\beta m}}\right). \tag{17}$$

Also, for each $\lambda \in \mathbb{G}^*$ and for all integers $k > 0$, we have

$$\begin{aligned}
\left| \frac{1}{8^{\beta m}} m_c\left(\frac{\lambda}{2^{\beta m}}\right) - m_c(\lambda) \right| &= \left| \sum_{p=0}^{m-1} \left\{ \frac{1}{8^{(p+1)\beta}} m_c\left(\frac{\lambda}{2^{(p+1)\beta}}\right) - \frac{1}{8^{p\beta}} m_c\left(\frac{\lambda}{2^{p\beta}}\right) \right\} \right| \\
&\leq \sup \left\{ \left| \frac{1}{8^{(p+1)\beta}} m_c\left(\frac{\lambda}{2^{(p+1)\beta}}\right) - \frac{1}{8^{p\beta}} m_c\left(\frac{\lambda}{2^{p\beta}}\right) \right| : 0 \leq p < m \right\} \\
&\leq \sup \left\{ \left|\frac{1}{8}\right|^{p\beta + \frac{\beta-1}{2}} \xi\left(\frac{\lambda}{2^{p\beta + \frac{\beta+1}{2}}}, \frac{\lambda}{2^{p\beta + \frac{\beta+1}{2}}}\right) : 0 \leq p < m \right\}.
\end{aligned} \tag{18}$$

Now, employing (17) and allowing $m \to \infty$ in the inequality (18), we observe that the inequality (14) exists. Using (13), (12) and (17), for all $\lambda_1, \lambda_2 \in \mathbb{G}^*$ we have

$$|\Delta m_c(\lambda_1, \lambda_2)| = \lim_{m\to\infty} \left|\frac{1}{8}\right|^{\beta m} \left| \Delta m_c\left(\frac{\lambda_1}{2^{\beta m}}, \frac{\lambda_2}{2^{\beta m}}\right) \right| \leq \lim_{m\to\infty} \left|\frac{1}{8}\right|^{\beta m} \xi\left(\frac{\lambda_1}{2^{\beta m}}, \frac{\lambda_2}{2^{\beta m}}\right) = 0$$

which implies that M_c satisfies equation (4) and therefore it is a reciprocal third power mapping. Next, is to prove the uniqueness assertion of M_c. For this, let us presume that $M_c' : \mathbb{G}^* \longrightarrow \mathbb{H}$ be another reciprocal third power mapping satisfying the approximation (14). Then, we have

$$\begin{aligned}
|M_c(\lambda) - M_c'(\lambda)| &= \lim_{k\to\infty} \left|\frac{1}{2}\right|^{\beta k} \left| M_c\left(\frac{\lambda}{2^{\beta k}}\right) - M_{c'}\left(\frac{\lambda}{2^{\beta k}}\right) \right| \\
&\leq \lim_{k\to\infty} \left|\frac{1}{2}\right|^{\beta k} \sup \left\{ \left| M_c\left(\frac{\lambda}{2^{\beta k}}\right) - m_c\left(\frac{\lambda}{2^{\beta k}}\right) \right|, \left| m_c\left(\frac{\lambda}{2^{\beta k}}\right) - M_{c'}\left(\frac{\lambda}{2^{\beta k}}\right) \right| \right\} \\
&\leq \lim_{k\to\infty} \lim_{m\to\infty} \sup \left\{ \sup \left\{ \left|\frac{1}{8}\right|^{(p+k)\beta + \frac{\beta-1}{2}} \xi\left(\frac{\lambda}{2^{(p+k)\beta + \frac{\beta+1}{2}}}, \frac{\lambda}{2^{(p+k)\beta + \frac{\beta+1}{2}}}\right) : k \leq p \leq m+k \right\} \right\} \\
&= 0
\end{aligned}$$

for all $\lambda \in \mathbb{G}^*$, which shows that M_c is unique. This completes the proof.

The ensuing corollaries are pertinent to other stabilities of equation (4) involving a positive constant, sum of powers of norms, product of different powers of norms and mixed product-sum of powers of norms as upper bounds and their proofs can be accomplished by the application of Theorem 2. Hence, we furnish only the statements. In the following outcomes, let us assume that $m_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ is a mapping.

Corollary 1. *Let $\mu > 0$ be a constant. If the mapping m_c satisfies $|\Delta m_c(\lambda_1, \lambda_2)| \leq \mu$ for all $\lambda_1, \lambda_2 \in \mathbb{G}^*$, then a unique reciprocal third power mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists and satisfies (4) with $|m_c(\lambda) - M_c(\lambda)| \leq \mu$ for all $\lambda \in \mathbb{G}^*$.*

Corollary 2. *Let $\mu \geq 0$ and $q \neq -3$, be fixed constants. If the mapping m_c satisfies $|\Delta m_c(\lambda_1, \lambda_2)| \leq \mu\left(|\lambda_1|^q + |\lambda_2|^q\right)$ for all $\lambda_1, \lambda_2 \in \mathbb{G}^*$, then there exists a unique reciprocal third power mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying (4) and*

$$|m_c(\lambda) - M_c(\lambda)| \leq \begin{cases} \frac{|2|\mu}{|2|^q} |\lambda|^q, & q > -3 \\ |2|\mu|2|^3 |\lambda|^q, & q < -3 \end{cases}$$

for all $\lambda \in \mathbb{G}^*$.

Corollary 3. *Let $a, b : q = a + b \neq -3$ and $\mu \geq 0$ be real numbers. Suppose the mapping m_c satisfies $|\Delta m_c(\lambda_1, \lambda_2)| \leq \mu |\lambda_1|^a |\lambda_2|^b$ for all $\lambda_1, \lambda_2 \in \mathbb{G}^*$. Then, a unique reciprocal third power mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying (4) exists with the result that*

$$|m_c(\lambda) - M_c(\lambda)| \leq \begin{cases} \frac{\mu}{|2|^q} |\lambda|^q, & q > -3 \\ \mu|2|^3 |\lambda|^q, & q < -3 \end{cases}$$

for all $\lambda \in \mathbb{G}^*$.

Corollary 4. *Let $\mu \geq 0$ and $q \neq -3$ be real numbers. Suppose the mapping m_c satisfies*

$$|\Delta m_c(\lambda_1, \lambda_2)| \leq \mu\left(|\lambda_1|^{\frac{q}{2}} |\lambda_2|^{\frac{q}{2}} + \left(|\lambda_1|^q + |\lambda_2|^q\right)\right)$$

for all $\lambda_1, \lambda_2 \in \mathbb{G}^*$. Then, there exists a unique reciprocal third power mapping $M_c : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying (4) with the result that

$$|m_c(\lambda) - M_c(\lambda)| \leq \begin{cases} \frac{|3|\mu}{|2|^q} |\lambda|^q, & q > -3 \\ |3|\mu|2|^3 |\lambda|^q, & q < -3 \end{cases}$$

for all $\lambda \in \mathbb{G}^*$.

Motivated from the excellent counter-example presented in [22], we prove the failure of stability result of equation (4) for a very singular case $q = -3$ in Corollary 2 in the setting of non-zero real numbers. Let us consider the following function:

$$\nu(\lambda) = \begin{cases} \frac{\beta}{\lambda^3}, & \text{for } \lambda \in (1, \infty) \\ \beta, & \text{otherwise} \end{cases} \tag{19}$$

where $\nu : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $m_c : \mathbb{R}^* \longrightarrow \mathbb{R}$ be a mapping defined by

$$m_c(\lambda) = \sum_{k=0}^{\infty} 2^{-3k} \nu(2^{-k}\lambda) \tag{20}$$

for all $\lambda \in \mathbb{R}^\star$. In the ensuing theorem, the mapping m_c becomes an example to prove that equation (4) is not stable for $q = -3$ in Corollary 2.

Theorem 3. *If the mapping $m_c : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in (19) satisfies the inequality*

$$|\Delta m_c(\lambda_1, \lambda_2)| \leq \frac{184\beta}{7}\left(|\lambda_1|^{-3} + |\lambda_2|^{-3}\right) \tag{21}$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}^\star$, then a reciprocal third power mapping $M_c : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $C > 0$ do not exist such that

$$|m_c(\lambda) - M_c(\lambda)| \leq C\,|\lambda|^{-3} \tag{22}$$

for all $\lambda \in \mathbb{R}^*$.

Proof. Firstly, let us show that m_c satisfies (21). From the definition of m_c, we have $|m_c(\lambda)| = \left|\sum_{k=0}^{\infty} 2^{-3k}\nu(2^{-k}\lambda)\right| \leq \sum_{k=0}^{\infty} \frac{\beta}{2^{3k}} = \frac{8\beta}{7}$, which implies that the real number $\frac{8\beta}{7}$ is an upper bound for the mapping m_c. When $|\lambda_1|^{-3} + |\lambda_2|^{-3} \geq 1$, then $|\Delta m_c(\lambda_1, \lambda_2)| < \frac{184\beta}{7}$. Now, when we suppose that $0 < |\lambda_1|^{-3} + |\lambda_2|^{-3} < 1$, then there exists a positive integer j such that

$$\frac{1}{2^{3(j+1)}} \leq |\lambda_1|^{-3} + |\lambda_2|^{-3} < \frac{1}{2^{3j}}. \tag{23}$$

The above relation (23) yields $2^{3j}\left(|\lambda_1|^{-3} + |\lambda_2|^{-3}\right) < 1$, and further produces $2^{3j}\lambda_1^{-3} < 1, 2^{3j}\lambda_2^{-3} < 1$. Hence, we have $\frac{\lambda_1}{2^j} > 1$, $\frac{\lambda_2}{2^j} > 1$. From the last two inequalities, we find that $\frac{\lambda_1}{2^{j-1}} > 2 > 1$, $\frac{\lambda_2}{2^{j-1}} > 2 > 1$ and as a result, we have, $\frac{1}{2^{j-1}}(\lambda_1) > 1$, $\frac{1}{2^{j-1}}(\lambda_2) > 1$, $\frac{1}{2^{j-1}}(\lambda_1 + \lambda_2) > 1$, $\frac{1}{2^{j-1}}\left(\frac{\lambda_1+\lambda_2}{2}\right) > 1$. Thus, for every $k = 0, 1, 2, \ldots, j-1$, we obtain,
$\frac{1}{2^k}(\lambda_1) > 1$, $\frac{1}{2^k}(\lambda_2) > 1$, $\frac{1}{2^k}(\lambda_1+\lambda_2) > 1$, $\frac{1}{2^k}\left(\frac{\lambda_1+\lambda_2}{2}\right) > 1$
and $\Delta\nu(2^{-k}\lambda_1, 2^{-k}\lambda_2) = 0$ for $k = 0, 1, 2, \ldots, j-1$. Using (19) and the definition of m_c, we obtain,
$|\Delta m_c(\lambda_1, \lambda_2)| \leq \sum_{k=j}^{\infty} \frac{\beta}{2^{3k}} + \sum_{k=j}^{\infty} \frac{\beta}{2^{3k}} + \sum_{k=j}^{\infty} \frac{7\beta}{8 \cdot 2^{3k}} \leq \frac{184\beta}{8} \sum_{k=j}^{\infty} \frac{1}{2^{3k}} \leq \frac{184\beta}{7}\left(|\lambda_1|^{-3} + |\lambda_2|^{-3}\right)$
for all $\lambda_1, \lambda_2 \in \mathbb{R}^*$. Hence, the inequality (21) holds. Now, we assert that the equation (4) is not stable for $q = -3$ in Corollary 2. For this, let us consider a reciprocal third power mapping $m_c : \mathbb{R}^* \longrightarrow \mathbb{R}$ exists and satisfies (22). Then, we have

$$|m_c(\lambda)| \leq (C+1)|\lambda|^{-3}. \tag{24}$$

But it is possible to choose a positive integer m with $m\beta > C + 1$. If $\lambda \in \left(1, 2^{m-1}\right)$ then $2^{-k}\lambda \in (1, \infty)$ for all $k = 0, 1, 2, \ldots, m-1$ and thus,

$$|m_c(\lambda)| = \sum_{k=0}^{\infty} \frac{\nu(2^{-k}\lambda)}{2^{3k}} \geq \sum_{k=0}^{m-1} \frac{\frac{2^{3k}\beta}{\lambda^3}}{2^{3k}} = \frac{m\beta}{\lambda^3} > (C+1)\lambda^{-3}$$

which contradicts (24) and hence this concludes that the equation (4) is not stable for $q = -3$ in Corollary 2.

Remark. The stability results concerning equation (5) and an illustration of a counter-example for singular case to show non-stability of equation (5) can be achieved similar to the results of equation (4).

4 Elucidation of equations (4) and (5) in real time

In this section, we interprete the insinuation of equations (4) and (5) in real time such as electromagnetic theory and construction of spring or diving board in a swimming pool.

- We associate the well-known magnetostatic reciprocal third power law with equations (4) and (5). This law states that the magnetic field strength M_{λ_1} due to a dipole from a point at a distance λ_1 is proportional to the reciprocal third power of λ_1. Then, $M_{\lambda_1} = \frac{K}{\lambda_1^3}$, where K is a constant of proportionality. Similarly, suppose λ_2 is the distance between them, then, $M_{\lambda_2} = \frac{K}{\lambda_2^3}$. Now, suppose a dipole is at a distance $(\lambda_1 + \lambda_2)$ from a point, then, $M_{\lambda_1+\lambda_2} = \frac{K}{(\lambda_1+\lambda_2)^3}$. Suppose the distance $\lambda_1 + \lambda_2$ is halved, then the magnetic field strength is $M_{\frac{\lambda_1+\lambda_2}{2}} = \frac{8K}{(\lambda_1+\lambda_2)^3}$. Now,

$$
\begin{aligned}
M_{\frac{\lambda_1+\lambda_2}{2}} - M_{\lambda_1+\lambda_2} &= \frac{7K}{(\lambda_1+\lambda_2)^3} \\
&= \frac{7K}{\lambda_1^3 + 3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 + \lambda_2^3} \\
&= \frac{\frac{7K}{\lambda_1^3\lambda_2^3}}{\left(\frac{1}{\lambda_1^3} + \frac{3}{\lambda_1^{\frac{2}{3}}\lambda_2^{\frac{1}{3}}} + \frac{3}{\lambda_1^{\frac{1}{3}}\lambda_2^{\frac{2}{3}}} + \frac{1}{\lambda_2^3}\right)} \\
&= \frac{7M_{\lambda_1}M_{\lambda_2}}{M_{\lambda_1} + 3M_{\lambda_1}^{\frac{2}{3}}M_{\lambda_2}^{\frac{1}{3}} + 3M_{\lambda_1}^{\frac{1}{3}}M_{\lambda_2}^{\frac{2}{3}} + M_{\lambda_2}} = \frac{7M_{\lambda_1}M_{\lambda_2}}{\left[M_{\lambda_1}^{\frac{1}{3}} + M_{\lambda_2}^{\frac{1}{3}}\right]^3}.
\end{aligned}
\tag{25}
$$

The aforementioned equation (25) connecting the field strength with different circumstatnces can be interpreted through equation (4). Analogous to this association, equation (5) can be expounded with the sum of field strengths $M_{\frac{\lambda_1+\lambda_2}{2}}$ and $M_{\lambda_1+\lambda_2}$.

- The stiffness or deflection of a diving board or cantilever beam is proportional to the cube of its thickness and inversely proportional to the cube of its length. Thus, the solution of equations (4) and (5) can be related with the stiffness or deflection as it is a function of reciprocal third power of length. Suppose if S is the stiffness of a diving board with length ℓ, then we have $S = \frac{K}{\ell^3}$, where K is constant of proportionality. Equation (4) interprets that it is a relation between difference of the stiffness of diving boards of lengths $\frac{\lambda_1+\lambda_2}{2}$ and $\lambda_1 + \lambda_2$ and the stiffness of diving boards of lengths λ_1 and λ_2. Similarly, we can infer the meaning of equation (5) with the sum of stiffness of diving boards in different situation.

5 Conclusion

In this study, we have introduced new reciprocal third power difference and adjoint functional equations (4) and (5). We have also proved the stability results pertinent to the results of Ulam, Hyers, T. Rassias, J. Rassias and Gavruta. We have illustrated that the equation (4) is unstable for a singular case by means of a counter-example. The relevance of equations (4) and (5) is interpreted through magnetic field strength in electromagnetic theory and relation between stiffness and length of a diving board in a swimming pool.

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