An Algorithm to Solve Linear Plus Linear Fractional Capacitated Transportation Problem with Specified Flow

Dr. Moazzam Arif, Institute of Basic Science, Department of Mathematical Science and Computer Applications, Bundelkhand University, Jhansi, UP, India

ABSTRACT

The present paper presents an algorithm to solve a linear plus linear fractional transportation problem with specified flow. If in addition to the flow constraint, the minimum requirement of each destination is also specified then the situation arises of distributing at minimum cost a certain commodity produced in a country, after keeping reserve stocks, to various states with minimum requirement of each state specified. A related transportation problem is formed in which the flow constraint is replaced by two extra destinations, one for supplementing the total flow up to the specified level ,and the other for identifying the supply points preferred to keep reserves. Optimal basic feasible solution of the related transportation problem so formulated is shown to give an optimal solution of the given problem. A numerical example is included in support of theory.

Keywords
Capacitated transportation problem, linear plus linear fractional transportation problem, specified flow

1 INTRODUCTION

Transportation problem with fractional objective function are widely used as performance measures in many real life situations such as the analysis of financial aspects of transportation enterprises and undertaking, and transportation management situations, where an individual or a group of people is confronted with the hurdle of maintaining good ratios between some crucial and important parameters concerned with the transportation of commodities from certain sources to various destinations. Fractional objective function includes optimisation of ratio of total actual cost to total standard cost, total return to total investment, ratio of risk assets to capital, total tax to total public expenditure on commodity etc. Gupta, Khanna and Puri [1] discussed a paradox in linear fractional transportation problem with mixed constraints and established a sufficient condition for the existence of a paradox. Jain and Saksena [2] studied time minimizing transportation problem with fractional bottleneck objective function which is solved by a lexicographic primal code. Xie, Jia and Jia [3] developed a technique for duration and cost optimisation for transportation problem. In addition to this fractional objective function , if one more linear function is added, then , it makes the problem more realistic. This type of objective function is called as linear plus linear fractional objective function. Khurana and Arora [4] studied linear plus linear fractional transportation problem for restricted and enhanced flow.

Another important class of transportation problems consists of capacitated transportation problem. Many researchers, i.e. Arora and Gupta [5], Misra and Das [6] have contributed in this field. Jain and Arya [7] studied the inverse version of capacitated transportation problem.

Sometimes situations arise due to extra demand in the market that the total flow needs to be enhanced, compelling some factories to increase their production in order to meet the extra demand. The total flow from the factories in the market is now increased by an amount of the extra demand. Khurana and Arora [8] studied enhanced flow in a fixed charge indefinite quadratic transportation problem. Khurana and Arora [4] studied linear plus linear fractional transportation problem with restricted and enhanced flow. This motivated us to study enhanced flow in a linear plus linear fractional capacitated transportation problem.

2 Problem Formulation

$$(\text{P1}) : \min z = \sum_{i\in I} \sum_{j\in J} r_{ij} x_{ij} + \frac{\sum_{i\in I} \sum_{j\in J} s_{ij} x_{ij}}{\sum_{i\in I} \sum_{j\in J} t_{ij} x_{ij}}$$

Subject to

$$a_i \le \sum_{i\in I} X_{ij} \le A_i \ \forall\ i \in I \qquad 1.1$$

$$b_j \le \sum_{j\in J} X_{ij} \le B_j \ \forall j \in J \qquad 1.2$$

$$\text{and integers } \ \forall i \in I,\ j \in J \qquad 1.3$$

I = {1,2,…….,m} is the index set of m origins.

J = {1,2,…….,n} is the index set of n destinations

x_{ij} = number of units transported from i^th^ origin to j^th^ destination.

r_{ij} = transportation cost of one unit of commodity from i^th^ origin to j^th^ destination.

s_{ij} = the sales tax per unit of goods transported from i^th^ origin to j^th^ destination.

t_{ij} = the total public expenditure unit of goods transported from i^th^ origin to j^th^ destination

l_{ij} and u_{ij} are the bounds on number of units to be transported from i^th^ origin to j^th^ destination.

a_i is the availability at the i^th^ origin, $i \in I$

b_j is the bounds on the demand at the jth destination, $j \in J$

It is assumed that $\sum_{i\in I}\sum_{j\in J} t_{ij}x_{ij} > 0$ for every feasible solution satisfying equations above.

Sometimes because of extra demand in the market, the total flow from the factories in the market is increased. $\sum_{i\in I}\sum_{j\in J} X_{ij} = \sum_{j\in J} r_j < P < \sum_{i\in I} a_i \; x_{ij} \geq 0$ be the specified flow. This flow constraint changes the structure of the transportation problem. The resulting linear plus linear fractional capacitated transportation problem with specified flow is:

$$(P_2): \quad \min z = \sum_{i\in I'}\sum_{j\in J'} r'_{ij} y_{ij} + \frac{\sum_{i\in I'}\sum_{j\in J'} s'_{ij} y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} t'_{ij} y_{ij}}$$

Subject to

$$\left.\begin{array}{l} \sum_{j\in J'} y_{ij} = A'_i \;\forall\, i \in I' \\ \sum_{i\in I'} y_{ij} = B'_j \;\forall j \in J' \\ l_{ij} \leq y_{ij} \leq u_{ij} \quad \text{and integers} \;\; \forall i \in I, j \in J \\ \sum_{i\in I}\sum_{j\in J} X_{ij} = \sum_{j\in J} r_j < P < \sum_{i\in I} a_i \,, \; x_{ij} \geq 0 \;\; i \in I, j \in J \\ l_{ij} \leq y_{ij} \leq u_{ij} \;\; \forall\, i \in I, j \in J \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

$$0 \leq y_{m+l,j} \leq \sum_{i\in I} B_J - b_j ; \forall j \in J$$

$$0 \leq y_{i,n+1} \leq \sum_{j\in J} A_i - a_i \forall i \in I$$

$$y_{m+1,n+1} \geq 0 \;\; A'_i = A_i \;\forall i \in I$$

$$A'_{m+1} = \sum_{j\in J} B_j - P, \; B'_j = B_j \forall j \in J \;\; , B'_{n+1} = \sum_{i\in I} A_i - P$$

$$r'_{ij} = r_{ij}, \forall i \in I, j \in J$$

$$r'_{m+1,j} = r'_{i,n+1} = 0, \forall i \in I, \forall j \in J$$

$$r'_{m+1,n+1} = M$$
$$t'_{ij} = t_{ij}, \forall i \in I, j \in J$$
$$t'_{m+1,j} = t'_{i,n+1} = 0, \forall i \in I, j \in J$$
$$t'_{m+1,n+1} = M$$
$$s'_{ij} = s_{ij}, \forall i \in I, j \in J,$$
$$s'_{m+1,j} = s'_{i,n+1} = 0, \forall i \in I, j \in J;$$
$$s'_{m+1,n+1} = M$$

Where M is a large positive number.

$$I' = \{1, 2, ..., m, m+1\}$$
$$J' = \{1, 2, ..., n, n+1\}$$

3 THEORETICAL DEVELOPMENT

Definition: ***Corner feasible solution:*** A basic feasible solution $\{y_{ij}\}$ $i \in I', j \in J'$ to problem (P_2) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$

Theorem 1:A non corner feasible solution of problem (P_2) cannot provide a basic feasible solution to problem (P_1).

Proof: Let $\{y_{ij}\}_{I' \times J'}$ be a non corner feasible solution to problem (P_2).Then, $y_{m+1,n+1} = \lambda (> 0)$ Thus,

$$\sum_{i \in I'} y_{i,n+1} = \sum_{i \in I} y_{i,n+1} + y_{m+1,n+1}$$

$$= \sum_{i \in I} y_{i,n+1} + \lambda$$

$$= \sum_{i \in I} \sum_{j \in J} A_i - P$$

Therefore, $\sum_{i \in I} y_{i,n+1} = \sum_{i \in I} A_i - (P + \lambda)$

Now, for $i \in I$,

$$\sum_{j \in J'} y_{ij} = A'_i = A_i$$

$$\sum_{i \in I} \sum_{j \in J'} y_{ij} = \sum_{i \in I} A_i$$

The above two relations implies that $\sum_{i\in I}\sum_{j\in J} y_{ij} = P+\lambda$

This implies that total quantity transported from the sources in I to the destinations in J is P + λ >P, a contradiction to assumption that total flow is P and hence $\left\{y_{ij}\right\}_{I'\times J'}$ cannot provide a feasible solution to problem(P_1)

Lemma 1: There is one to one correspondence between a feasible solution of problem (P_1) and a corner feasible solution of problem (P_2).

Proof: Let $\left\{x_{ij}\right\}_{I\times J}$ be a feasible solution of problem (P_1).

So by relation (1), we have $x_{ij} \le A_i$ (1.4)

Define $\left\{y_{ij}\right\}_{I'\times J'}$ by the following transformation

$$y_{ij} = x_{ij}, i\in I, j\in J \tag{1.5}$$

$$y_{i,n+1} = A_i - \sum_{j\in J} x_{ij}; \forall i \in I \tag{1.6}$$

$$y_{m+1,j} = B_j - \sum_{i\in I} x_{ij}; \forall j \in J \tag{1.7}$$

$$y_{m+1,n+1} = 0 \tag{1.8}$$

It can be shown that $\left\{y_{ij}\right\}$ so defined is a cfs to problem (P_2)

Relation (1) and (1.5) imply that $l_{ij} \le y_{ij} \le u_{ij}; \forall i\in I, j\in J$

Relation (1) and (1.6) imply that $0 \le y_{i,n+1} \le A_i - a_i; \forall \mathrm{i} \in \mathrm{I}$

Relation (1) and (1.7) imply that $0 \le y_{m+1,j} \le B_j - b_j; \forall j \in J$

Relation (1.8) implies that $y_{m+1,n+1} \ge 0$

Also for $i \in I$,relation (1.5) and (1.6) imply that

$$\sum_{j\in J'} y_{ij} = \sum_{j\in J} y_{ij} + y_{i,n+1} = \sum_{j\in J} x_{ij} + \sum_{j\in J} u_{ij} - \sum_{j\in J} x_{ij} = \sum_{j\in J} u_{ij} = a_i$$

For $i = m+1$

$$\sum_{j\in J'} y_{m=1,j} = \sum_{j\in J} y_{ij} + y_{m+1,n+1} = \sum_{j\in J}\left(B_j - \sum_{i\in I} x_{ij}\right)$$

$$=\sum_{i\in I}\sum_{j\in J}u_{ij}-\sum_{i\in I}\sum_{j\in J}x_{ij}$$

$$=\sum_{i\in I}\sum_{j\in J}u_{ij}-P=a'_{m+1}$$

Therefore, $\sum_{j\in J'}y_{ij}=A'_i;\forall i\in I'$

Similarly, it can be shown that $\sum_{i\in I}y_{ij}=B'_j;\forall j\in J'$

Therefore, $\left\{y_{ij}\right\}_{I'\times J'}$ is a cfs to problem (P_2).

Conversely, let $\left\{y_{ij}\right\}_{I'\times J'}$ be a cfs to problem (P_2). Define x_{ij} $,i\in I,\, j\in J$ by the following transformation.

$$x_{ij}=y_{ij},i\in I,\, j\in J \qquad (1.9)$$

It implies that $l_{ij}\le x_{ij}\le u_{ij},i\in I,\, j\in J$

Now for $i\in I$,the source constraints in problem (P_2) imply

$$\sum_{j\in J'}y_{ij}=A'_i=A_i$$

$$\sum_{j\in J}y_{ij}+y_{i,n+1}=A_i$$

$$\Rightarrow a_i\le\sum_{j\in J}y_{ij}\le A_i \qquad (\text{ since } 0\le y_{i,n+1}\le A_i-a_i;\forall i\in I\ )$$

Hence, $\sum_{j\in J}y_{ij}\ge a_i,i\in I$ and subsequently, $\sum_{j\in J}x_{ij}\ge a_i,i\in I$

Similarly, for $j\in J,\sum_{j\in J}y_{ij}\ge B_j;\forall j\in J$ and subsequently, $\sum_{i\in I}x_{ij}\ge B_j;\forall j\in J$

For $i=m+1$

$$\sum_{j\in J'}y_{m+1,j}=A'_{m+1}=\sum_{j\in J}B_j-P$$

$$\Rightarrow\sum_{j\in J}y_{m+1,j}=\sum_{j\in J}B_j-P \text{ because } y_{m+1,n+1}=0 \qquad (1.10)$$

Now for $j\in J$ the destination constraints in problem (P_2) give

$$\sum_{i\in I} y_{ij} + y_{m+1,j} = B_j$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{j\in J} y_{m+1,j} = \sum_{j\in J} B_j$

By relation (1.10), we have

$$\sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij} - \sum_{j\in J} y_{m+1,j} = P$$
$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

Therefore, $\left\{x_{ij}\right\}_{I\times J}$ is a feasible solution to problem (P_2).

Remark 1: If problem (P_2) has cfs, then since $c'_{m+1,n+1} = M$ and $d'_{m+1,n+1} = M$,it follows that non corner feasible solution can not be an optimal solution to problem (P_2).

Lemma 2 : The value of the objective function of problem (P_2) at a feasible solution $\left\{x_{ij}\right\}_{I\times J}$ is equal to the value of the objective function of problem (P_2') at its corresponding cfs $\left\{y_{ij}\right\}_{I'\times J'}$ and conversely.

Proof: The value of the objective function of problem (P_2) at a feasible solution $\left\{x_{ij}\right\}_{I\times J}$ is

$$Z = \sum_{i\in I'}\sum_{j\in J'} r'_{ij}\, y_{ij} + \frac{\sum_{i\in I'}\sum_{j\in J'} s'_{ij}\, y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} t'_{ij}\, y_{ij}}$$

$$= \sum_{i\in I}\sum_{j\in J} r_{ij} x_{ij} + \frac{\sum_{i\in I}\sum_{j\in J} s_{ij} x_{ij}}{\sum_{i\in I}\sum_{j\in J} t_{ij} x_{ij}}$$

because

$$r'_{ij} = r_{ij}, \forall i \in I, j \in J$$

$$r'_{m+1,j} = r'_{i,n+1} = 0, \forall i \in I, \forall j \in J$$

$$t'_{ij} = t_{ij}, \forall i \in I, j \in J$$
$$t'_{m+1,j} = t'_{i,n+1} = 0, \forall i \in I, j \in J$$
$$s'_{ij} = s_{ij}, \forall i \in I, j \in J,$$
$$s'_{m+1,j} = s'_{i,n+1} = 0, \forall i \in I, j \in J;$$

$$y_{m+1,n+1} = 0$$

= objective function value of problem (P_1) at $\{x_{ij}\}$.Converse can be proved in a similar way.

Lemma 3: There is a one to one correspondence between the optimal solution among the corner feasible solution to problem (P_2) and the optimal solution to problem(P_1).

Proof : Let $\{\hat{x}_{ij}\}_{I\times J}$ be an optimal solution to problem (P_2) with the value of objective function as Z^0 .Since $\{\hat{x}_{ij}\}_{I\times J}$ is an optimal solution, $\therefore \{x_{ij}\}$ is a feasible solution to problem (P_2) . Then by lemma 1, there exist a corresponding feasible solution $\{\hat{y}_{ij}\}_{I'\times J'}$ is Z^0 [refer to lemma 2]

we will show that $\{\hat{y}_{ij}\}_{I'\times J'}$ is the optimal solution to problem (P_2').

Now, Let if possible, $\{\hat{y}_{ij}\}$ be not an optimal solution to problem (P_2').Therefore there exist a feasible solution $\{y'_{ij}\}$ say to problem (P_2') having the value of objective function $Z' < Z^0$ .Let $\{x'_{ij}\}$ be the corresponding feasible solution to problem (P_2). Then by lemma 2,

$$Z' = \sum_{i\in I'}\sum_{j\in J'} r'_{ij}\, y_{ij} + \frac{\sum_{i\in I'}\sum_{j\in J'} s'_{ij}\, y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} t'_{ij}\, y_{ij}}$$

Which contradicts that $\{\hat{x}_{ij}\}$ is an optimal solution to problem (P_2).

Similarly, starting from an optimal feasible solution to problem (P_2), one can derive an optimal corner feasible solution to problem (P_1) having the same objective function value.

Theorem 2: Optimizing problem (P_2) is equivalent to optimizing problem (P_2) provided problem (P_2) has a feasible solution.

Proof: As problem (P_2) has a feasible solution, by lemma 1, there exists a cfs to problem (P_2').Thus by remark 1, an optimal solution to problem (P_2') will be a cfs. Hence, by lemma 3 , an optimal solution to problem (P_2) can be obtained.

Theorem 3: A feasible solution $X^0 = \left\{ \overset{0}{x}_{ij} \right\}_{I\times J}$ of problem (P1) with objective value $\left[R^0 + \frac{S^0}{T^0} \right]$ will be a local optimum basic feasible solution iff the following conditions hold.

$$\delta_{ij}^{1} = \theta_{ij}(r_{ij} - z^{1}_{ij}) + \frac{\theta_{ij}\left[T^0(s_{ij} - z^{2}_{ij}) - S^0(t_{ij} - z_{ij}^{3})\right]}{T^0\left[T^0 + \theta_{ij}(t_{ij} - z^{3}_{ij})\right]} \geq 0; \forall (i, j) \in N_1$$

$$\delta_{ij}^{2} = -\theta_{ij}(r_{ij} - z^{1}_{ij}) - \frac{\theta_{ij}\left[T^0(s_{ij} - z^{2}_{ij}) - S^0(t_{ij} - z_{ij}^{3})\right]}{T^0\left[T^0 + \theta_{ij}(t_{ij} - z^{3}_{ij})\right]} \geq 0; \forall (i, j) \in N_2$$

And if X^0 is optimal solution of (P2) then

$\delta_{ij}^{1} \geq 0; \forall(\mathrm{i},\mathrm{j}) \in \mathrm{N}_1$ and

$\delta_{ij}^{2} \geq 0; \forall(i,j) \in N_2$ where

$$R^0 = \sum_{i\in I}\sum_{j\in J} r_{ij}x^0{}_{ij},\ S^0 = \sum_{i\in I}\sum_{j\in J} s_{ij}x_{ij},\ T^0 = \sum_{i\in I}\sum_{j\in J} t_{ij}x_{ij}$$

B denotes the set of cells (I,j) which are basic and N_1 and N_2 denotes the set of non basic cells which are at the i lower bounds and upper bounds respectively.

$u_i^1, u_i^2, u_i^3, v_j^1, v_j^2, v_j^3; i \in I, j \in J$

are the dual variables such that

$$u^1{}_i + v^1{}_j = r_{ij};$$
$$u^2{}_i + v^2{}_j = s_{ij};$$
$$u^3{}_i + v^3{}_j = t_{ij}; \forall(\mathrm{i},\mathrm{j}) \in \mathrm{B}$$

$$u^1{}_i + v^1{}_j = z^1{}_{ij};$$
$$u^2{}_i + v^2{}_j = z^2{}_{ij};$$
$$u^3{}_i + v^3{}_j = z^3{}_{ij}; \forall(\mathrm{i},\mathrm{j}) \notin \mathrm{B}.$$

Proof: Let X^0 = $\left\{x_{ij}^{0}\right\}_{I\times J}$ be a basic feasible solution of problem (P1) with equality constraints. Let z^0be the corresponding value of objective function. Then

$$z^0 = \sum_{i\in I}\sum_{j\in J} r_{ij}x^0{}_{ij} + \frac{\sum_{i\in I}\sum_{j\in J} s_{ij}x^0{}_{ij}}{\sum_{i\in I}\sum_{j\in J} t_{ij}x^0{}_{ij}} = [R^0 + \frac{S^0}{T^0}](say)$$

$$= \sum_{i\in I}\sum_{j\in J}\left(r_{ij} - u_i^1 - v^1{}_j\right)x^0{}_{ij} + \sum_{i\in I}\sum_{j\in J}\left(u^1{}_i - v^1{}_j\right)x^0{}_{ij} + \left[\frac{\sum_{i\in I}\sum_{j\in J}\left(s_{ij} - u_i^2 - v^2{}_j\right)x_{ij}{}^0 + \sum_{i\in I}\sum_{j\in J}\left(u_i^2 + v^2{}_j\right)x^0{}_{ij}}{\sum_{i\in I}\sum_{j\in J}\left(t_{ij} - u_i^3 - v^3{}_j\right)x_{ij}{}^0 + \sum_{i\in I}\sum_{j\in J}\left(u_i^3 + v^3{}_j\right)x^0{}_{ij}}\right]$$

$$= \sum_{(\mathrm{i},\mathrm{j})\in \mathrm{N}_1}\left(r_{ij} - u_i^1 - v_j^1\right)l_{ij} + \sum_{(\mathrm{i},\mathrm{j})\in \mathrm{N}_2}\left(r_{ij} - u_i^1 - v_j^1\right)u_{ij} + \sum_{i\in I}\sum_{j\in J}\left(u_i^1 - v^1{}_j\right)x_{ij}{}^0$$

$$+\left[\frac{\sum\sum_{(i,j)\in N_1}\left(s_{ij} - u^2{}_i - v^2{}_j\right)l_{ij} + \sum\sum_{(i,j)\in N_2}\left(s_{ij} - u^2{}_i - v^2{}_j\right)u_{ij} + \sum_{i\in I}\sum_{j\in J}\left(u_1{}^2 + v^2{}_j\right)x_{ij}{}^0}{\sum\sum_{(i,j)\in N_1}\left(t_{ij} - u^3{}_i - v^3{}_j\right)l_{ij} + \sum\sum_{(i,j)\in N_2}\left(t_{ij} - u^3{}_i - v^3{}_j\right)u_{ij} + \sum_{i\in I}\sum_{j\in J}\left(u_1{}^3 + v^3{}_j\right)x_{ij}{}^0}\right]$$

$$= \sum \sum_{(i,j)\in N_1} \left(r_{ij} - z^1{}_{ij}\right) l_{ij} + \sum \sum_{(i,j)\in N_2} \left(r_{ij} - z^1{}_{ij}\right) u_{ij} + \sum_{i\in I} a_i u_i{}^1 + \sum_{j\in J} b_j v^1{}_j$$

$$+ \left[\frac{\sum \sum_{(i,j)\in N_1} \left(s_{ij} - z^2{}_{ij}\right) l_{ij} + \sum \sum_{(i,j)\in N_2} \left(s_{ij} - z^2{}_{ij}\right) u_{ij} + \sum_{i\in I} a_i u_i{}^2 + \sum_{j\in J} b_j v^2{}_j}{\sum \sum_{(i,j)\in N_1} \left(t_{ij} - z^3{}_{ij}\right) l_{ij} + \sum \sum_{(i,j)\in N_2} \left(t_{ij} - z^3{}_{ij}\right) u_{ij} + \sum_{i\in I} a_i u_i{}^3 + \sum_{j\in J} b_j v^3{}_j} \right]$$

Let some non basic variable $x_{i\,j} \in N_1$ undergoes change by an amount θ_{pq} where θ_{pq} is given by min{ $u_{pq} - l_{pq}$; $x^0_{ij} - l_{ij}$ for all basic cells with a ($-\theta$) entry in θ-loop; $u_{ij} - x^0_{ij}$ for all basic cells with a ($+\theta$) entry in θ-loop }

Then new value of the objective function $\hat{z}$ will be given by

$$\hat{z} = \left[R^0 + \theta_{pq}\left(r_{pq} - z^1{}_{pq}\right) \right] + \left[\frac{S^0 + \theta_{pq}\left(s_{pq} - z^2{}_{pq}\right)}{T^0 + \theta_{pq}\left(t_{pq} - z^3{}_{pq}\right)} \right]$$

$$\hat{z} - z^0 = \left[R^0 + \theta_{pq}\left(r_{pq} - z^1{}_{pq}\right) - R^0 \right] + \left[\frac{S^0 + \theta_{pq}\left(s_{pq} - z^2{}_{pq}\right)}{T^0 + \theta_{pq}\left(t_{pq} - z^3{}_{pq}\right)} - \frac{S^0}{T^0} \right]$$

$$= \theta_{pq}(\mathrm{r}_{pq} - \mathrm{z}^1{}_{pq}) + \frac{\theta_{pq}\left[T^0\left(s_{pq} - z_{pq}{}^2\right) - S^0\left(t_{pq} - z_{pq}{}^3\right) \right]}{T^0\left[T^0 + \theta_{pq}\left(t_{pq} - z^3 pq\right) \right]} = \delta^1{}_{pq}\,(say)$$

Similarly when some non basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq} then

$$\hat{z} - z^0 = -\theta_{pq}(\mathrm{r}_{pq} - \mathrm{z}^1{}_{pq}) - \frac{\theta_{pq}\left[T^0\left(s_{pq} - z_{pq}{}^2\right) - S^0\left(t_{pq} - z_{pq}{}^3\right) \right]}{T^0\left[T^0 + \theta_{pq}\left(t_{pq} - z^3 pq\right) \right]} = \delta^2{}_{pq}$$

Hence X^0 will be local optimal solution iff

$$\delta^1{}_{ij} \geq 0; \forall (i,j) \in N_1$$

and $\delta^2{}_{ij} \geq 0; \forall (i,j) \in N_2$. If X0 is a global optimal solution of (P2),then it is an optimal solution and hence the result follows.

4. ALGORITHM

Step 1.starting from the given linear plus linear fractional capacitated transportation problem (P_1) with enhanced flow, form a related transportation problem (P_2) by introducing a dummy source and a dummy destination with

$$a_i^{'} = \sum_{j\in J} u_{ij}; \forall i \in I \ , \ a^{'}_{m+1} = \sum_{i\in I}\ \sum_{j\in J} u_{ij} - P = b^{'}_{n+1} \ , \ b_j^{'} = \sum_{i\in I} u_{ij}; \forall j \in J$$

Step 2 : Find an initial basic feasible solution to (P_2) with respect to variable cost only. Let B be its corresponding basis.

Step 3 : Calculate θ_{ij},

$$u^1{}_i + v^1{}_j = r_{ij};$$
$$u^2{}_i + v^2{}_j = s_{ij};$$
$$u^3{}_i + v^3{}_j = t_{ij}; \forall (\mathrm{i, j}) \in \mathrm{B}$$

$$u^1{}_i + v^1{}_j = z^1{}_{ij};$$
$$u^2{}_i + v^2{}_j = z^2{}_{ij};$$
$$u^3{}_i + v^3{}_j = z^3{}_{ij}; \forall (\mathrm{i, j}) \notin \mathrm{B}.$$

Θ_{ij} = level at which a non basic cell (I,j) enters the basis replacing some basic cell of B.

Note : $u_i^1, u_i^2, u_i^3, v_j^1, v_j^2, v_j^3; i \in I, j \in J$

are the dual variables which are determined by using above equations and taking one of the u_i 's or v_j 's as zero.

Step 4 : Calculate R^0, S^0, T^0 where

$$R^0 = \sum_{i\in I}\ \sum_{j\in J} r_{ij} x_{ij} \ , \quad S^0 = \sum_{i\in I}\ \sum_{j\in J} s_{ij} x_{ij} \ , \quad T^0 = \sum_{i\in I}\ \sum_{j\in J} t_{ij} x_{ij}$$

Step 5 : Find $\delta^1{}_{ij}; \forall (i, j) \in N_1$

and $\delta^2{}_{ij}; \forall (i, j) \in N_2$ where

$$\delta_{ij}^1 = \theta_{ij}(\mathrm{r}_{ij} - \mathrm{z}^1{}_{ij}) + \frac{\theta_{ij}\left[T^0(\mathrm{s}_{ij} - \mathrm{z}^2{}_{ij}) - \mathrm{S}^0(\mathrm{t}_{ij} - \mathrm{z}_{ij}{}^3)\right]}{T^0\left[T^0 + \theta_{ij}(\mathrm{t}_{ij} - \mathrm{z}^3{}_{ij})\right]} \geq 0; \forall (\mathrm{i, j}) \in \mathrm{N}_1$$

$$\delta_{ij}^2 = -\theta_{ij}(\mathrm{r}_{ij} - \mathrm{z}^1{}_{ij}) - \frac{\theta_{ij}\left[T^0(\mathrm{s}_{ij} - \mathrm{z}^2{}_{ij}) - \mathrm{S}^0(\mathrm{t}_{ij} - \mathrm{z}_{ij}{}^3)\right]}{T^0\left[T^0 + \theta_{ij}(\mathrm{t}_{ij} - \mathrm{z}^3{}_{ij})\right]} \geq 0; \forall (\mathrm{i, j}) \in \mathrm{N}_2$$

Where N_1 and N_2 denotes the set of non-basic cells which are at their lower bounds and upper bounds respectively.
If $\delta^1{}_{ij} \geq 0; \forall (i, j) \in N_1$ and $\delta^2{}_{ij} \geq 0; \forall (i, j) \in N_2$ then the current solution so obtained is optimal solution to (P2) and subsequently to (P1). Then go to next step. Otherwise some $(I,j) \in N_1$ for which $\delta^1{}_{ij} < 0$ or some $(I,j) \in N_2$ for which $\delta^2{}_{ij} < 0$ will enter the basis. Go to step 3.

Step 6 . Find the optimal value of $z = \left[R^0 + \frac{S^0}{T^0} \right]$

5. NUMERICAL ILLUSTRATION

Problem of the Manager of a Cell Phone Manufacturing Company

ABC company produces cell phones. These cell phones are manufactured in the factories (i) located at Haryana, Punjab and Chandigarh. After production, these cell phones are transported to main distribution centres (j) at Kolkata, Chennai and Mumbai. The cartage paid per cell phone is 2, 3 and 4 respectively when the goods are transported from Haryana to Kolkata, Chennai and Mum-bai. Similarly, the cartage paid per cell phone when transported from Punjab to distribution centres at Kolkata, Chennai and Mumbai are 6, 1 and 2 respectively while the figures in case of transportation from Chandigarh is 1, 8 and 4 respectively. In addition to this, the company has to pay sales tax per cell phone. The sales tax paid per cell phone from Haryana to Kolkata, Chennai, Mumbai are 5, 9 and 9 respectively. The tax figures when the goods are transported from Punjab to Kolkata, Chennai and Mum-bai are 4, 6 and 2 respectively. The sales tax paid per unit from Chandigarh to Kolkata, Chennai and Mumbai are 2, 1 and 1 respectively. The total public expenditure per unit when the goods are transported from Haryana to Kolkata, Chennai and Mumbai are 4, 2 and 1 respectively

Solution :

The problem of the manager can be formulated as a 3 × 3 linear plus linear fractional transportation problem (P1) with restricted flow as follows.

Let O_1 and O_2 and O_3 denotes factories at Haryana, Punjab and Chandigarh. D_1, D_2 and $D_3$are the distribution centres at Kolkata, Chennai and Mumbai respectively.

Let the cartage be denoted by r_{ij}'s (i = 1, 2, 3 and j = 1, 2 and 3). The sales tax paid per cell phone when trans-
ported from factories (i) to distribution centres (j) is de-noted by s_{ij}. The total public expenditure per cell phone for I = 1, 2 and 3 and j = 1, 2 and 3 is denoted by t_{ij}. Then

$r_{11}2, r_{12}\ \ 3, r_{13}\ \ 4, r_{21}\ \ 6, r_{22}\ \ 1,$

$r_{23}2, r_{31}\ \ 1, r_{32}\ \ 8, r_{33}\ \ 4$

$s_{11}5, s_{12}\ \ 9, s_{13}\ \ 9, s_{21}\ \ 4, s_{22}\ \ 6,$

$s_{23}2, s_{31}2, s_{32}1, s_{33}\ \ 1$

$t_{11}4, t_{12}\ \ 2, t_{13}\ \ 1, t_{21}\ \ 3, t_{22}\ \ 7,$

$t_{23}4, t_{31}\ \ 2, t_{32}\ \ 9, t_{33}\ \ 4$

Let x_{ij} be the number of cell phones transported from the i^{th} factory to the j^{th} distribution centre.

Since factory at Haryana can produce a minimum of 3 and a maximum of 30 cell phones in a month, it can be formulated mathematically as $3 \quad \overset{3}{\underset{j\ 1}{\quad}} x_{1j}30.$

Similarly, $10 \quad \overset{3}{\underset{j\ 1}{\quad}} x_{2j}40,\ 10 \quad \overset{3}{\underset{j\ 1}{\quad}} x_{3j}50.$

Since the minimum and maximum monthly requirement of cell phones at Kolkata are 5 and 30 respectively,

3

The above data can be represented in the form of **Table 1** as follows.

Table 1. Problem (P1).

	D1	D2	D3	*Ai*
O1	2 5 4	3 9 2	4 9 1	30
O_2	6 4 3	1 6 7	2 2 4	40
O3	1 2 2	8 1 9	4 1 4	50
B_j	30	20	30	

Note: values in the upper left corners are r_{ij}'s and values in upper right corners are s_{ij}'s and values in lower right corners are t_{ij}'s for I = 1,2,3 and j = 1,2,3.

Also we have

$0\ x_{14}\ 27$, $0\ x_{24}\ 30$, $0\ x_{34}\ 40$, $0\ x_{41}\ 25$, $0\ x_{42}\ 15$, $0\ x_{43}\ 25$

Now we find an initial basic feasible solution of problem (P2) which is given in **Table 2** below.

Table 2. A basic feasible solution of problem (P2).

	D_1	D_2	D_3	D_4	u_i^1	u_i^2	u_i^3
O_1	2 5 14	3 9 22	4 9 0 1	0 0 **27** 0	0	0	0
O_2	6 4 3	1 6 $\overline{1}$ 5 7	2 2 **5** 4	0 0 **20** 0	0	0	0
O_3	1 2 **17**2	8 1 9	4 1 4	0 0 **33** 0	0	0	0
O_4	0 0 **12** 0	0 0 **3** 0	0 0 $\overline{2}$ 5 0	M M M	−1	−2	−2
v_j^1	1	1	2	0			
v_j^2	2	2	2	0			
v_j^3	2	2	4	0			

Note: Entries of the form a and b represent non basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

Table 3. Calculation of $_{ij}^{1}$ and$_{ij}^{2}$

NB	O_1D_1	O_1D_2	O_1D_3	O_2D_1	O2D2	O3D2	O3D3	O4D3
ij	7	3	4	7	3	3	4	3
r_{ij} z_{ij}^{1}	1	2	2	5	0	7	2	-1
s_{ij} z_{ij}^{2}	3	7	7	2	4	−1	−1	0
t_{ij} z_{ij}^{3}	2	0	−3	1	5	7	0	−2
$_{ij}^{1}$	7.04	6.125	8.25	35.04		20.879	7.976	
$_{ij}^{2}$					0.0138			2.967

Therefore, the company should send 1 cell phone from Haryana to Kolkata, 2 units from Haryana to Chennai. The number of cell phones to be shipped from factory at Punjab to Chennai and Mumbai centres are 15 and 5respectively. Factory at Chandigarh should send 17 units to Kolkata only. The total cartage paid is 50, total sales tax paid is 157 and total public expenditure is 167.

6. CONCLUSION

This paper deals with a linear plus linear fractional transportation problem where in the total transportation flow is restricted to a known specified level. A related transportation problem is formulated and it is shown that it exited an optimal solution. An algorithm is presented and tested by a real life example of a manufacturing company.

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