RINGS WITH COMMUTATORS AND THE SQUARE OF EVERY ELEMENT IS IN THE NUCLEUS

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Abstract : In this paper, we prove that $(x,y,z) + (y,z,x) + (z,x,y)$ is in the nucleus N for all elements x,y,z of the ring R. Using this, we prove that a prime ring of char.$\neq 2$ is either associative or a Lie ring.

Keywords : Nonassociative ring, Prime ring, Nucleus, Center, Characteristic and Associator.

Introduction : Kleinfeld [2] considered rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and char. $\neq 2$ it was shown that such rings are either associative or have the property that $x^2 = 0$, for every element x of the ring. In this paper we prove that $(x,y,z) + (y,z,x) + (z,x,y)$ is in the nucleus for all elements x,y,z of the ring R. Using this, it is proved that if R is a prime ring of char. $\neq 2$, then it is either associative or a Lie ring. At the end of this section we give an example in which char. $\neq 2$ is necessary in theorem 1.

Priliminaries : Let R be a nonassociative ring. We shall denote the commutator and the associator by $(x,y) = xy-yx$ and $(x,y,z) = (xy)z-x(yz)$ for all x,y,z in R respectively. The nucleus N of a ring R is defined as $N = \{n \in R \ / \ (n,R,R) = (R,n,R) = (R,R,n) = 0\}$ [1]. The center C of R is defined as $C = \{ c \in N \ / \ (c, R) = 0\}$. A ring R is said to be of characteristic □ n if $nx = 0$ implies $x = 0$, for all $x \in R$ and n is a natural number. A ring R is of characteristic □ n is simply denoted by char.□□□n. A ring R is said to be prime if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$. A Lie ring is a ring in which the multiplication is anticommutative, i.e., $x^2 = 0$ and the Jacobi identity $(xy)z + (yz)x + (zx)y = 0$, for all x,y,z in R is satisfied.

Main Results : Throughout this paper we consider a ring R with commutators and square of every element is in the nucleus.

i.e., $(R,R) \subset N$ ... (1)

and $x^2 \in N$, for all $x \in R$. ... (2)

If $S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y)$, we have the identity

$$(xy,z) + (yz,x) + (zx,y) = S(x,y,z). \quad ...(3)$$

Using (1) in the above identity, we have

$$S(x,y,z) \subset N.$$

i.e., $(x,y,z) + (y,z,x) + (z,x,y) \in N$, ... (4)

for all x, y, z in R.

Now we prove the following lemma.

Lemma 1 : Let R be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of char. $\neq 2$. Then either R is associative or $N^2 = 0$.

Proof : For all $r,s \in R$,

$rs + sr = (r + s)^2 - r^2 - s^2$ must be in N.

Select $n,n' \in N$, and $x, y, z, \in R$.

Then using the above equation

$(n(n'x + xn'), y, z) = 0$. So that

$$(nn'x,y,z) = - (n\, x\, n',y,z).$$

Similarly $(nxn',y,z) = - (xnn',y,z)$ and

$$(xnn',y,z) = - (nn'x,y,z).$$

By combining the above three equalities it follows that $2\,(nn'x,y,z) = 0$. Since R is of char. $\neq 2$, we get

$(n\,n'x,y,z) = 0$. ... (5)

Now
$$\begin{aligned}(nx,y,z) &= ((nx)y)z - (nx)\,(yz)\\ &= (n(xy))z - n(x(yz))\\ &= n((xy)z) - n(x(yz))\\ &= n(x,y,z).\end{aligned}$$

By replacing n by n n' in the above equation, we get

$(n\,n'x,y,z) = n\,n'(x,y,z)$. ... (6)

The combination of (5) and (6) yields $n\,n'(x,y,z) = 0$. So that

$N^2(R,R,R) = 0$. ... (7)

Let A be the ideal generated by N^2, and I the ideal generated by all associators (R,R,R). We have $n\,n'x = n\,(n'x + x\,n') - (nx + xn)n' + xnn'$. So that $N^2R \subset RN^2 + N^2$. Consequently

$$A = R\,N^2 + N^2.$$

In an arbitrary ring

$I = (R,R,R) + (R,R,R)R$. It follows from (7), that $AI = 0$. Since R is prime either

$A = 0$, or $I = 0$. If $A = 0$, then $N^2 = 0$. On the other hand if $I = 0$, then R is associative. This completes the proof of the lemma.

Theorem1. Let R be a prime ring of char. $\neq 2$ satisfying $r^2 \in N$ for all $r \in R$. Then either R is associative or $r^2 = 0$, for all $r \in R$.

Proof : By considering the case $N \neq R$, from the lemma 1 we have $N^2 = 0$.

Let $K = N + NR$.

Since $rn = (rn + nr) - nr \in K$ and

$snr = (sn + ns)r - nsr \in K$, for all $n \in N$ and $r,s \in R$, K must be an ideal of R.

Moreover if

$n' \in N$, $nrn' = n\,(rn' + n'r) - nn'r = 0$, since $N^2 = 0$. Therefore $K^2 = 0$. But R is prime and so $K = 0$. But then $N = 0$, whence $r^2 = 0$, for all $r \in R$. This completes the proof of the theorem.

Theorem 2 : Let R be a prime ring of char. $\neq 2$ satisfying (i) $x^2 \in N$ for all $x \in R$, and (ii) $(x,y,z) + (y,z,x) + (z,x,y) \in N$ for all $x,y,z \in R$. Then R is either associative or a Lie ring.

Proof : Assume that R satisfies (i) and (ii).

By considering the case $N \neq R$, from Theorem 1 we have $x^2 = 0$, for all $x \in R$. Consequently R is anti-commutative. i.e., $xy = -yx$.

For any $n \in N$, $n(xy) = (nx)y = (-xn)y = -x(ny) = x(yn) = (xy)n$,

And also $n(xy) = (-xy)n$.

Using the above two equations, we get $2n(xy) = 0$. Since R is of char. $\neq 2$, we have $n(xy) = 0$. Thus $NR^2 = 0$. The set T of all $t \in R$ such that $tR = 0$, forms an ideal of R which must be zero since R is prime. Since $NR \subset T$, we obtain $NR = 0$ and subsequently $N = 0$.

In any anti-commutative ring $(x,y,z) + (y,z,x) + (z,x,,y) = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)$ which equals twice the Jacobian of x, y, z. Then because of (ii), we have $2((xy)z + (yz)x + (zx)y) \in N$. Since R is of char. $\neq 2$, we have $((xy)z + (yz)x + (zx)y) \in N$. Since $N = 0$, we conclde that the well known Jacobi identity holds and thus R is a Lie ring.

Now we give an example in which char. $\neq 2$ is necessary in theorem 1.

Example 1 : Let 1, x, y be basis elements of the algebra R over an arbitrary field F, where $xy = 1$, $yx = x^2 = y^2 = 0$.

For any $\alpha, \beta, \gamma \in F$,

$$(\alpha + \beta x + \gamma y)^2 = \alpha^2 + 2\alpha\beta x + 2\alpha\gamma y + \beta\gamma$$
$$= 2\alpha(\alpha + \beta x + \gamma y) + \beta\gamma - \alpha^2.$$

Thus R is quadratic over F. Clearly R is simple, power associative, and that all commutators of R are contained in F. R is not associative since $(x,y,y) = y$. Also $(x+y)^2 = 1 \neq 0$. If F happens to be a field of char. = 2 then $r^2 \in F$ for every $r \in R$. Therefore Theorem 1 fails to hold for rings of char. = 2.

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