Disjoint-set data structure

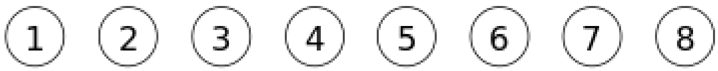
In computer science, a **disjoint-set data structure** (also called a **union–find data structure** or **merge–find set**) is a data structure that tracks a set of elements partitioned into a number of disjoint (non-overlapping) subsets. It provides near-constant-time operations (bounded by the inverse Ackermann function) to add new sets, to merge existing sets, and to determine whether elements are in the same set. In addition to many other uses (see the Applications section), disjoint-sets play a key role in Kruskal's algorithm for finding the minimum spanning tree of a graph.

Disjoint-set/Union-find Forest	
Type	multiway tree
Invented	1964
Invented by	Bernard A. Galler and Michael J. Fischer

Time complexity in big O notation

Algorithm	Average	Worst case
Space	O(n)[1]	O(n)[1]
Search	O($\alpha(n)$)[1]	O($\alpha(n)$)[1]
Merge	O($\alpha(n)$)[1]	O($\alpha(n)$)[1]

Contents



MakeSet creates 8 singletons.

After some operations of *Union*, some sets are grouped together.

History

Disjoint-set forests were first described by Bernard A. Galler and Michael J. Fischer in 1964.[2] In 1973, their time complexity was bounded to $O(\log^*(n))$, the iterated logarithm of n, by Hopcroft and Ullman.[3] (A proof is available here.) In 1975, Robert Tarjan was the first to prove the $O(m\alpha(n))$ (inverse Ackermann function) upper bound on the algorithm's time complexity,[4] and, in 1979, showed that this was the lower bound for a restricted case.[5] In 1989, Fredman and Saks showed that $\Omega(\alpha(n))$ (amortized) words must be accessed by *any* disjoint-set data structure per operation,[6] thereby proving the optimality of the data structure.

In 1991, Galil and Italiano published a survey of data structures for disjoint-sets.[7]

In 1994, Richard J. Anderson and Heather Woll described a parallelized version of Union–Find that never needs to block.[8]

In 2007, Sylvain Conchon and Jean-Christophe Filliâtre developed a persistent version of the disjoint-set forest data structure, allowing previous versions of the structure to be efficiently retained, and formalized its correctness using the proof assistant Coq.[9] However, the implementation is only asymptotic if used ephemerally or if the same version of the structure is repeatedly used with limited backtracking.

Representation

A disjoint-set forest consists of a number of elements each of which stores an id, a parent pointer, and, in efficient algorithms, either a size or a "rank" value.

The parent pointers of elements are arranged to form one or more trees, each representing a set. If an element's parent pointer points to no other element, then the element is the root of a tree and is the representative member of its set. A set may consist of only a single element. However, if the element has a parent, the element is part of whatever set is identified by following the chain of parents upwards until a representative element (one without a parent) is reached at the root of the tree.

Forests can be represented compactly in memory as arrays in which parents are indicated by their array index.

Operations

MakeSet

The *MakeSet* operation makes a new set by creating a new element with a unique id, a rank of 0, and a parent pointer to itself. The parent pointer to itself indicates that the element is the representative member of its own set.

The *MakeSet* operation has $O(1)$ time complexity, so initializing n sets has $O(n)$ time complexity.

Pseudocode:

```
function MakeSet(x) is
    if x is not already present then
        add x to the disjoint-set tree
        x.parent := x
        x.rank   := 0
        x.size   := 1
```

Find

Find(x) follows the chain of parent pointers from *x* up the tree until it reaches a root element, whose parent is itself. This root element is the representative member of the set to which *x* belongs, and may be *x* itself.

Path compression

Path compression flattens the structure of the tree by making every node point to the root whenever *Find* is used on it. This is valid, since each element visited on the way to a root is part of the same set. The resulting flatter tree speeds up future operations not only on these elements, but also on those referencing them.

Tarjan and Van Leeuwen also developed one-pass *Find* algorithms that are more efficient in practice while retaining the same worst-case complexity: path splitting and path halving.[4]

Path halving

Path halving makes every other node on the path point to its grandparent.

Path splitting

Path splitting makes every node on the path point to its grandparent.

Pseudocode

Pseudocode

Path compression	Path halving	Path splitting

`function Find(x)`
`  if x.parent ≠ x`
`    x.parent := Find(x.parent)`
`  return x.parent`
	`function Find(x)`
	`  while x.parent ≠ x`
	`    x.parent := x.parent.parent`
	`    x := x.parent`
	`  return x`
		`function Find(x)`
		`  while x.parent ≠ x`
		`    x, x.parent := x.parent, x.parent.parent`
		`  return x`

Path compression can be implemented using iteration by first finding the root then updating the parents:

```
function Find(x) is
    root := x
    while root.parent ≠ root
        root := root.parent

    while x.parent ≠ root
        parent := x.parent
        x.parent := root
        x := parent

  return root
```

Path splitting can be represented without multiple assignment (where the right hand side is evaluated first):

```
function Find(x)
    while x.parent ≠ x
        next := x.parent
        x.parent := next.parent
        x := next
    return x
```

or

```
function Find(x)
    while x.parent ≠ x
        prev := x
        x := x.parent
```

```
        prev.parent := x.parent
    return x
```

Union

Union(x,y) uses *Find* to determine the roots of the trees *x* and *y* belong to. If the roots are distinct, the trees are combined by attaching the root of one to the root of the other. If this is done naively, such as by always making *x* a child of *y*, the height of the trees can grow as $O(n)$. To prevent this *union by rank* or *union by size* is used.

by rank

Union by rank always attaches the shorter tree to the root of the taller tree. Thus, the resulting tree is no taller than the originals unless they were of equal height, in which case the resulting tree is taller by one node.

To implement *union by rank*, each element is associated with a rank. Initially a set has one element and a rank of zero. If two sets are unioned and have the same rank, the resulting set's rank is one larger; otherwise, if two sets are unioned and have different ranks, the resulting set's rank is the larger of the two. Ranks are used instead of height or depth because path compression will change the trees' heights over time.

by size

Union by size always attaches the tree with fewer elements to the root of the tree having more elements.

Pseudocode

Pseudocode

Union by rank

```
 function Union(x, y) is
     xRoot := Find(x)
     yRoot := Find(y)

     if xRoot = yRoot then
         // x and y are already in the same
set
         return

     // x and y are not in same set, so we
merge them
     if xRoot.rank < yRoot.rank then
         xRoot, yRoot := yRoot, xRoot // swap
xRoot and yRoot

     // merge yRoot into xRoot
     yRoot.parent := xRoot
     if xRoot.rank = yRoot.rank then
         xRoot.rank := xRoot.rank + 1
```

Union by size

```
 function Union(x, y) is
     xRoot := Find(x)
     yRoot := Find(y)

     if xRoot = yRoot then
         // x and y are already in the same
set
         return

     // x and y are not in same set, so we
merge them
     if xRoot.size < yRoot.size then
         xRoot, yRoot := yRoot, xRoot // swap
xRoot and yRoot

     // merge yRoot into xRoot
     yRoot.parent := xRoot
     xRoot.size := xRoot.size + yRoot.size
```

Time complexity

Without *path compression* (or a variant), *union by rank*, or *union by size*, the height of trees can grow unchecked as $O(n)$, implying that *Find* and *Union* operations will take $O(n)$ time.

Using *path compression* alone gives a worst-case running time of $\Theta(n + f \cdot (1 + \log_{2+f/n} n))$,[10] for a sequence of n *MakeSet* operations (and hence at most $n - 1$ *Union* operations) and f *Find* operations.

Using *union by rank* alone gives a running-time of $O(m \log_2 n)$ (tight bound) for m operations of any sort of which n are *MakeSet* operations.[10]

Using both *path* ***compression***, ***splitting***, *or* ***halving*** and *union by* ***rank*** *or* ***size*** ensures that the amortized time per operation is only $O(m\alpha(n))$[4][5] for m disjoint-set operations on n elements, which is optimal,[6] where $\alpha(n)$ is the inverse Ackermann function. This function has a value $\alpha(n) < 5$ for any value of n that can be written in this physical universe, so the disjoint-set operations take place in essentially constant time.

Proof of O(log*(n)) time complexity of Union-Find

Proof of $O(\log^* n)$ amortized time [11] of Union Find[12][13][14]

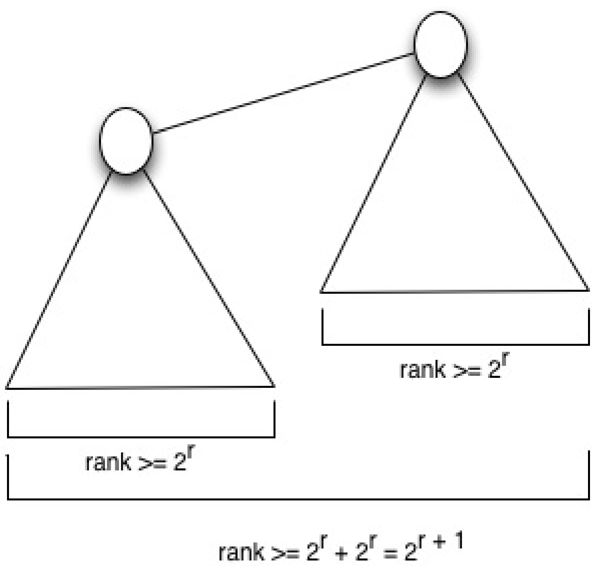
Statement: If m operations, either Union or Find, are applied to n elements, the total run time is $O(m \log^* n)$, where $\log^*$ is the iterated logarithm.

Lemma 1: As the find function follows the path along to the root, the rank of node it encounters is increasing.

> Proof: claim that as Find and Union operations are applied to the data set, this fact remains true over time. Initially when each node is the root of its own tree, it's trivially true. The only case when the rank of a node might be changed is when the Union by Rank operation is applied. In this case, a tree with smaller rank will be attached to a tree with greater rank, rather than vice versa. And during the find operation, all nodes visited along the path will be attached to the root, which has larger rank than its children, so this operation won't change this fact either.

Lemma 2: A node u which is root of a subtree with rank r has at least 2^r nodes.

> Proof: Initially when each node is the root of its own tree, it's trivially true. Assume that a node u with rank r has at least 2^r nodes. Then when two trees with rank r Union by Rank and form a tree with rank $r + 1$, the new node has at least $2^r + 2^r = 2^{r+1}$ nodes.

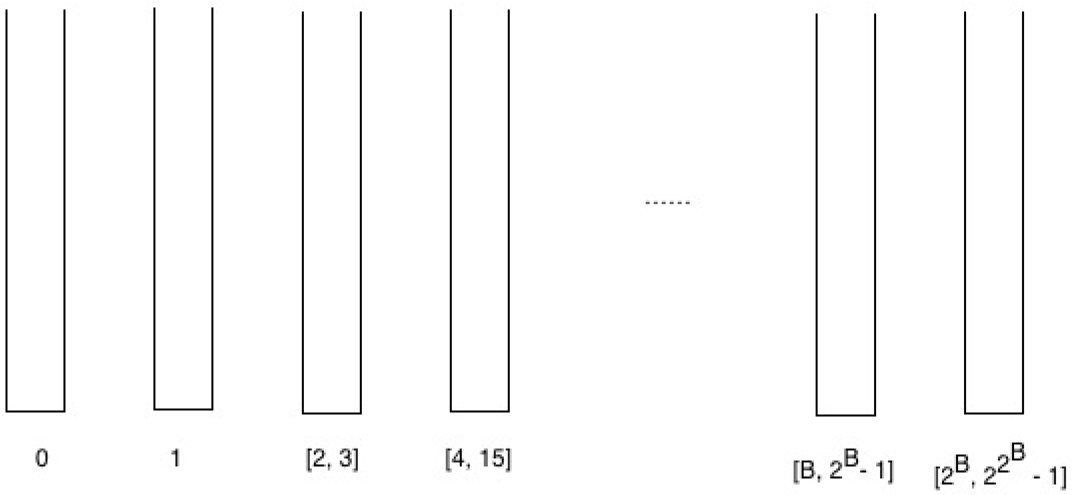


Lemma 3: The maximum number of nodes of rank r is at most $n/2^r$.

> Proof: From lemma 2, we know that a node u which is root of a subtree with rank r has at least 2^r nodes. We will get the maximum number of nodes of rank r when each node with rank r is the root of a tree that has exactly 2^r nodes. In this case, the number of nodes of rank r is $n/2^r$

For convenience, we define "bucket" here: a bucket is a set that contains vertices with particular ranks.

We create some buckets and put vertices into the buckets according to their ranks inductively. That is, vertices with rank 0 go into the zeroth bucket, vertices with rank 1 go into the first bucket, vertices with ranks 2 and 3 go into the second bucket. If the Bth bucket contains vertices with ranks from interval $[r, 2^r - 1]$ = [r, R - 1] then the (B+1)st bucket will contain vertices with ranks from interval $[R, 2^R - 1]$.



Proof of $O(\log^* n)$ Union Find

We can make two observations about the buckets.

1. The total number of buckets is at most $\log^* n$

   > Proof: When we go from one bucket to the next, we add one more two to the power, that is, the next bucket to $[B, 2^B - 1]$ will be $[2^B, 2^{2^B} - 1]$

2. The maximum number of elements in bucket $[B, 2^B - 1]$ is at most $2n/2^B$

   > Proof: The maximum number of elements in bucket $[B, 2^B - 1]$ is at most $n/2^B + n/2^{B+1} + n/2^{B+2} + \ldots + n/2^{2^B - 1} \leq 2n/2^B$

Let F represent the list of "find" operations performed, and let

$$T_1 = \sum_F (\text{link to the root})$$

$$T_2 = \sum_F (\text{number of links traversed where the buckets are different})$$

$$T_3 = \sum_F \text{(number of links traversed where the buckets are the same)}.$$

Then the total cost of m finds is $T = T_1 + T_2 + T_3$

Since each find operation makes exactly one traversal that leads to a root, we have $T_1 = O(m)$.

Also, from the bound above on the number of buckets, we have $T_2 = O(m\log^* n)$.

For T_3, suppose we are traversing an edge from u to v, where u and v have rank in the bucket $[B, 2^B - 1]$ and v is not the root (at the time of this traversing, otherwise the traversal would be accounted for in T_1). Fix u and consider the sequence $v_1,v_2,...,v_k$ that take the role of v in different find operations. Because of path compression and not accounting for the edge to a root, this sequence contains only different nodes and because of Lemma 1 we know that the ranks of the nodes in this sequence are strictly increasing. By both of the nodes being in the bucket we can conclude that the length k of the sequence (the number of times node u is attached to a different root in the same bucket) is at most the number of ranks in the buckets B, i.e. at most $2^B - 1 - B < 2^B$.

Therefore, $T_3 \leq \sum_{[B,2^B-1]} \sum_u 2^B$.

From Observations 1 and 2, we can conclude that $T_3 \leq \sum_B 2^B \frac{2n}{2^B} \leq 2n \log^* n$.

Therefore, $T = T_1 + T_2 + T_3 = O(m \log^* n)$.

Applications

Disjoint-set data structures model the partitioning of a set, for example to keep track of the connected components of an undirected graph. This model can then be used to determine whether two vertices belong to the same component, or whether adding an edge between them would result in a cycle. The Union–Find algorithm is used in high-performance implementations of unification.[15]

This data structure is used by the Boost Graph Library to implement its Incremental Connected Components (http://www.boost.org/libs/graph/doc/incremental_components.html) functionality. It is also a key component in implementing Kruskal's algorithm to find the minimum spanning tree of a graph.

Note that the implementation as disjoint-set forests doesn't allow the deletion of edges, even without path compression or the rank heuristic.

Sharir and Agarwal report connections between the worst-case behavior of disjoint-sets and the length of Davenport–Schinzel sequences, a combinatorial structure from computational geometry.[16]

See also

- Partition refinement, a different data structure for maintaining disjoint sets, with updates that split sets apart rather than merging them together
- Dynamic connectivity

References

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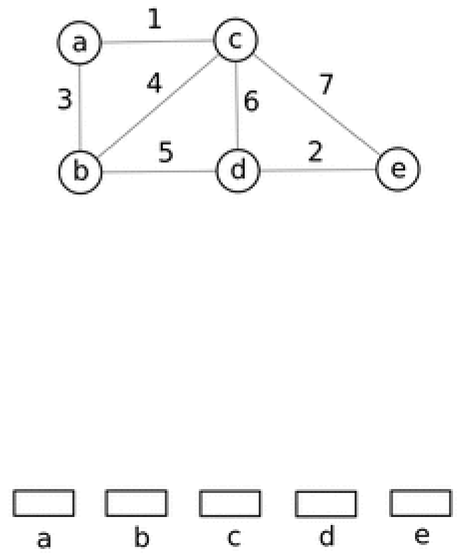

A demo for Union-Find when using Kruskal's algorithm to find minimum spanning tree.

12. Tarjan, Robert Endre (1975). "Efficiency of a Good But Not Linear Set Union Algorithm" (http://portal.acm.org/citation.cfm?id=321884). *Journal of the ACM*. **22** (2): 215–225. doi:10.1145/321879.321884 (https://doi.org/10.1145%2F321879.321884).
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External links

- C++ implementation (http://www.boost.org/libs/disjoint_sets/disjoint_sets.html), part of the Boost C++ libraries
- A Java implementation with an application to color image segmentation, Statistical Region Merging (SRM), IEEE Trans. Pattern Anal. Mach. Intell. 26(11): 1452–1458 (2004) (http://www.lix.polytechnique.fr/~nielsen/Srmjava.java)
- Java applet: A Graphical Union–Find Implementation (http://www.cs.unm.edu/~rlpm/499/uf.html), by Rory L. P. McGuire
- A Matlab Implementation (https://github.com/USNavalResearchLaboratory/TrackerComponentLibrary/blob/master/Container%20Classes/DisjointSet.m) which is part of the Tracker Component Library
- Python implementation (http://code.activestate.com/recipes/215912-union-find-data-structure/)
- Visual explanation and C# code (http://www.mathblog.dk/disjoint-set-data-structure/)