

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE



<http://algs4.cs.princeton.edu>

2.1 ELEMENTARY SORTS

- ▶ *rules of the game*
- ▶ *selection sort*
- ▶ *insertion sort*
- ▶ *shellsort*
- ▶ *shuffling*
- ▶ *convex hull*

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Sorting problem

Ex. Student records in a university.

Chen	3	A	991-878-4944	308 Blair
Rohde	2	A	232-343-5555	343 Forbes
Gazsi	4	B	766-093-9873	101 Brown
Furia	1	A	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman

item →

key →

Sort. Rearrange array of N items into ascending order.

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

Sample sort client 1

Goal. Sort **any** type of data.

Ex 1. Sort random real numbers in ascending order.

seems artificial, but stay tuned for an application

```
public class Experiment
{
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        Double[] a = new Double[N];
        for (int i = 0; i < N; i++)
            a[i] = StdRandom.uniform();
        Insertion.sort(a);
        for (int i = 0; i < N; i++)
            StdOut.println(a[i]);
    }
}
```

```
% java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.363292849257276
0.460954145685913
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```

Sample sort client 2

Goal. Sort **any** type of data.

Ex 2. Sort strings from file in alphabetical order.

```
public class StringSorter
{
    public static void main(String[] args)
    {
        String[] a = In.readStrings(args[0]);
        Insertion.sort(a);
        for (int i = 0; i < a.length; i++)
            StdOut.println(a[i]);
    }
}
```

```
% more words3.txt
bed bug dad yet zoo ... all bad yes
```

```
% java StringSorter words3.txt
all bad bed bug dad ... yes yet zoo
```

Sample sort client 3

Goal. Sort **any** type of data.

Ex 3. Sort the files in a given directory by filename.

```
import java.io.File;
public class FileSorter
{
    public static void main(String[] args)
    {
        File directory = new File(args[0]);
        File[] files = directory.listFiles();
        Insertion.sort(files);
        for (int i = 0; i < files.length; i++)
            StdOut.println(files[i].getName());
    }
}
```

```
% java FileSorter .
Insertion.class
Insertion.java
InsertionX.class
InsertionX.java
Selection.class
Selection.java
Shell.class
Shell.java
ShellX.class
ShellX.java
```

Callbacks

Goal. Sort **any** type of data.

Q. How can `sort()` know how to compare data of type `Double`, `String`, and `java.io.File` without any information about the type of an item's key?

Callback = reference to executable code.

- Client passes array of objects to `sort()` function.
- The `sort()` function calls back object's `compareTo()` method as needed.

Implementing callbacks.

- Java: **interfaces**.
- C: **function pointers**.
- C++: **class-type functors**.
- C#: **delegates**.
- Python, Perl, ML, Javascript: **first-class functions**.

Callbacks: roadmap

client

```
import java.io.File;
public class FileSorter
{
    public static void main(String[] args)
    {
        File directory = new File(args[0]);
        File[] files = directory.listFiles();
        Insertion.sort(files);
        for (int i = 0; i < files.length; i++)
            StdOut.println(files[i].getName());
    }
}
```

object implementation

```
public class File
implements Comparable<File>
{
    ...
    public int compareTo(File b)
    {
        ...
        return -1;
        ...
        return +1;
        ...
        return 0;
    }
}
```

Comparable interface (built in to Java)

```
public interface Comparable<Item>
{
    public int compareTo(Item that);
}
```

key point: no dependence
on File data type

sort implementation

```
public static void sort(Comparable[] a)
{
    int N = a.length;
    for (int i = 0; i < N; i++)
        for (int j = i; j > 0; j--)
            if (a[j].compareTo(a[j-1]) < 0)
                exch(a, j, j-1);
            else break;
}
```

Total order

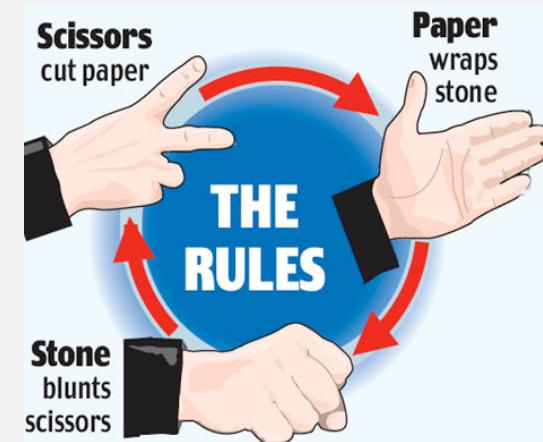
A **total order** is a binary relation \leq that satisfies:

- Antisymmetry: if $v \leq w$ and $w \leq v$, then $v = w$.
- Transitivity: if $v \leq w$ and $w \leq x$, then $v \leq x$.
- Totality: either $v \leq w$ or $w \leq v$ or both.

Ex.

- Standard order for natural and real numbers.
- Chronological order for dates or times.
- Alphabetical order for strings.
- ...

violates totality: `(Double.NaN <= Double.NaN)` is false



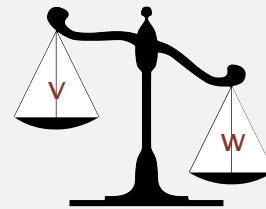
an intransitive relation

Surprising but true. The `<=` operator for `double` is not a total order. (!)

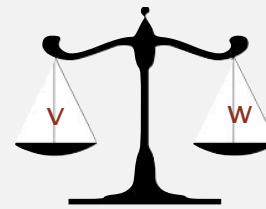
Comparable API

Implement `compareTo()` so that `v.compareTo(w)`

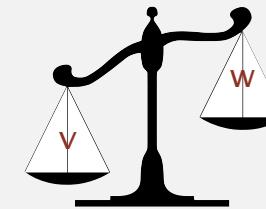
- Is a total order.
- Returns a negative integer, zero, or positive integer if `v` is less than, equal to, or greater than `w`, respectively.
- Throws an exception if incompatible types (or either is `null`).



less than (return -1)



equal to (return 0)



greater than (return +1)

Built-in comparable types. Integer, Double, String, Date, File, ...

User-defined comparable types. Implement the Comparable interface.

Implementing the Comparable interface

Date data type. Simplified version of java.util.Date.

```
public class Date implements Comparable<Date>
{
    private final int month, day, year;

    public Date(int m, int d, int y)
    {
        month = m;
        day   = d;
        year  = y;
    }

    public int compareTo(Date that)
    {
        if (this.year < that.year) return -1;
        if (this.year > that.year) return +1;
        if (this.month < that.month) return -1;
        if (this.month > that.month) return +1;
        if (this.day   < that.day ) return -1;
        if (this.day   > that.day ) return +1;
        return 0;
    }
}
```

only compare dates
to other dates

Two useful sorting abstractions

Helper functions. Refer to data through compares and exchanges.

Less. Is item v less than w ?

```
private static boolean less(Comparable v, Comparable w)
{  return v.compareTo(w) < 0;  }
```

Exchange. Swap item in array a[] at index i with the one at index j.

```
private static void exch(Comparable[] a, int i, int j)
{
    Comparable swap = a[i];
    a[i] = a[j];
    a[j] = swap;
}
```

Testing

Goal. Test if an array is sorted.

```
private static boolean isSorted(Comparable[] a)
{
    for (int i = 1; i < a.length; i++)
        if (less(a[i], a[i-1])) return false;
    return true;
}
```

Q. If the sorting algorithm passes the test, did it correctly sort the array?

A.

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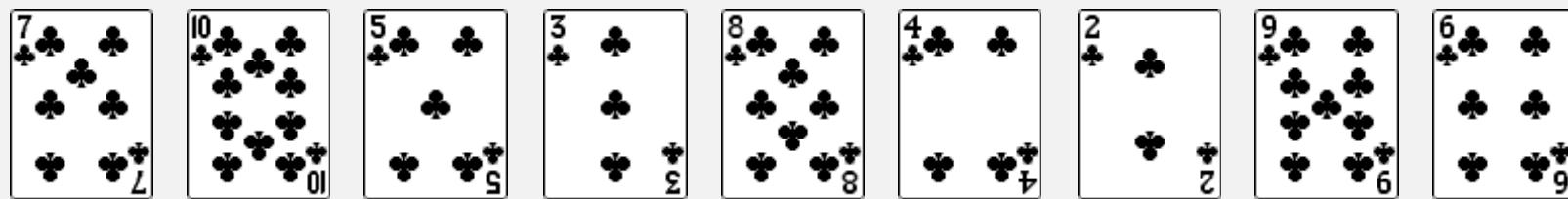
<http://algs4.cs.princeton.edu>

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Selection sort demo

- In iteration i , find index min of smallest remaining entry.
- Swap $a[i]$ and $a[\text{min}]$.



initial



Selection sort

Algorithm. \uparrow scans from left to right.

Invariants.

- Entries to the left of \uparrow (including \uparrow) fixed and in ascending order.
- No entry to right of \uparrow is smaller than any entry to the left of \uparrow .



Selection sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```



- Identify index of minimum entry on right.

```
int min = i;
for (int j = i+1; j < N; j++)
    if (less(a[j], a[min]))
        min = j;
```



- Exchange into position.

```
exch(a, i, min);
```



Selection sort: Java implementation

```
public class Selection
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int min = i;
            for (int j = i+1; j < N; j++)
                if (less(a[j], a[min]))
                    min = j;
            exch(a, i, min);
        }
    }

    private static boolean less(Comparable v, Comparable w)
    { /* as before */ }

    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

Selection sort: mathematical analysis

Proposition. Selection sort uses $(N-1) + (N-2) + \dots + 1 + 0 \sim N^2/2$ compares and N exchanges.

		a[]										
i	min	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Trace of selection sort (array contents just after each exchange)

Running time insensitive to input. Quadratic time, even if input is sorted.
Data movement is minimal. Linear number of exchanges.

Selection sort: animations

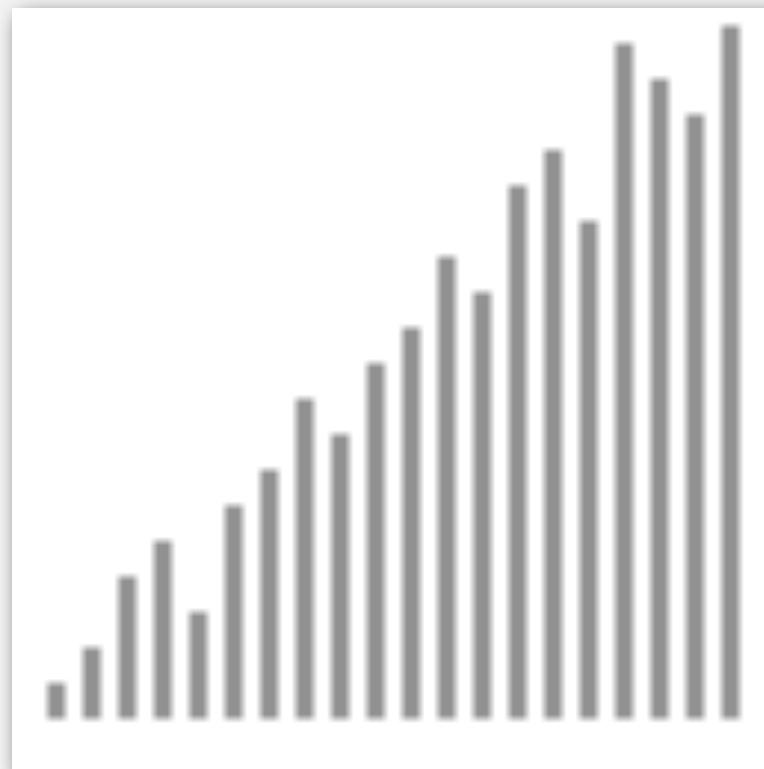
20 random items



<http://www.sorting-algorithms.com/selection-sort>

Selection sort: animations

20 partially-sorted items



<http://www.sorting-algorithms.com/selection-sort>

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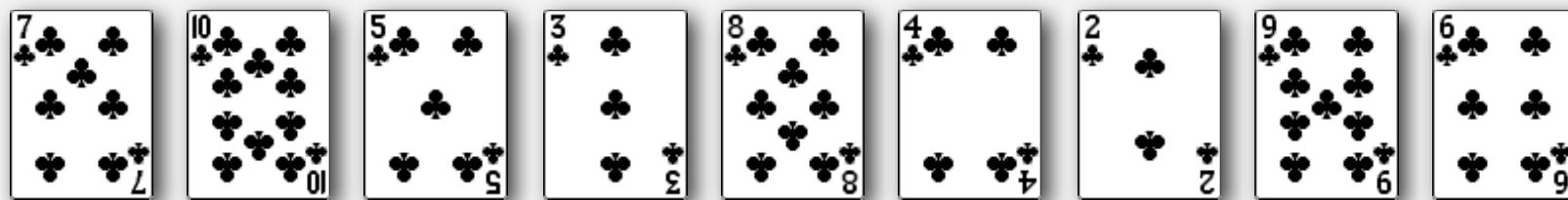
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- ▶ *convex hull*

Insertion sort demo

- In iteration i , swap $a[i]$ with each larger entry to its left.



Insertion sort

Algorithm. \uparrow scans from left to right.

Invariants.

- Entries to the left of \uparrow (including \uparrow) are in ascending order.
- Entries to the right of \uparrow have not yet been seen.



Insertion sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```



- Moving from right to left, exchange $a[i]$ with each larger entry to its left.

```
for (int j = i; j > 0; j--)  
    if (less(a[j], a[j-1]))  
        exch(a, j, j-1);  
    else break;
```



Insertion sort: Java implementation

```
public class Insertion
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
            for (int j = i; j > 0; j--)
                if (less(a[j], a[j-1]))
                    exch(a, j, j-1);
                else break;
    }

    private static boolean less(Comparable v, Comparable w)
    { /* as before */ }

    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

Insertion sort: mathematical analysis

Proposition. To sort a randomly-ordered array with distinct keys, insertion sort uses $\sim \frac{1}{4} N^2$ compares and $\sim \frac{1}{4} N^2$ exchanges on average.

Pf. Expect each entry to move halfway back.

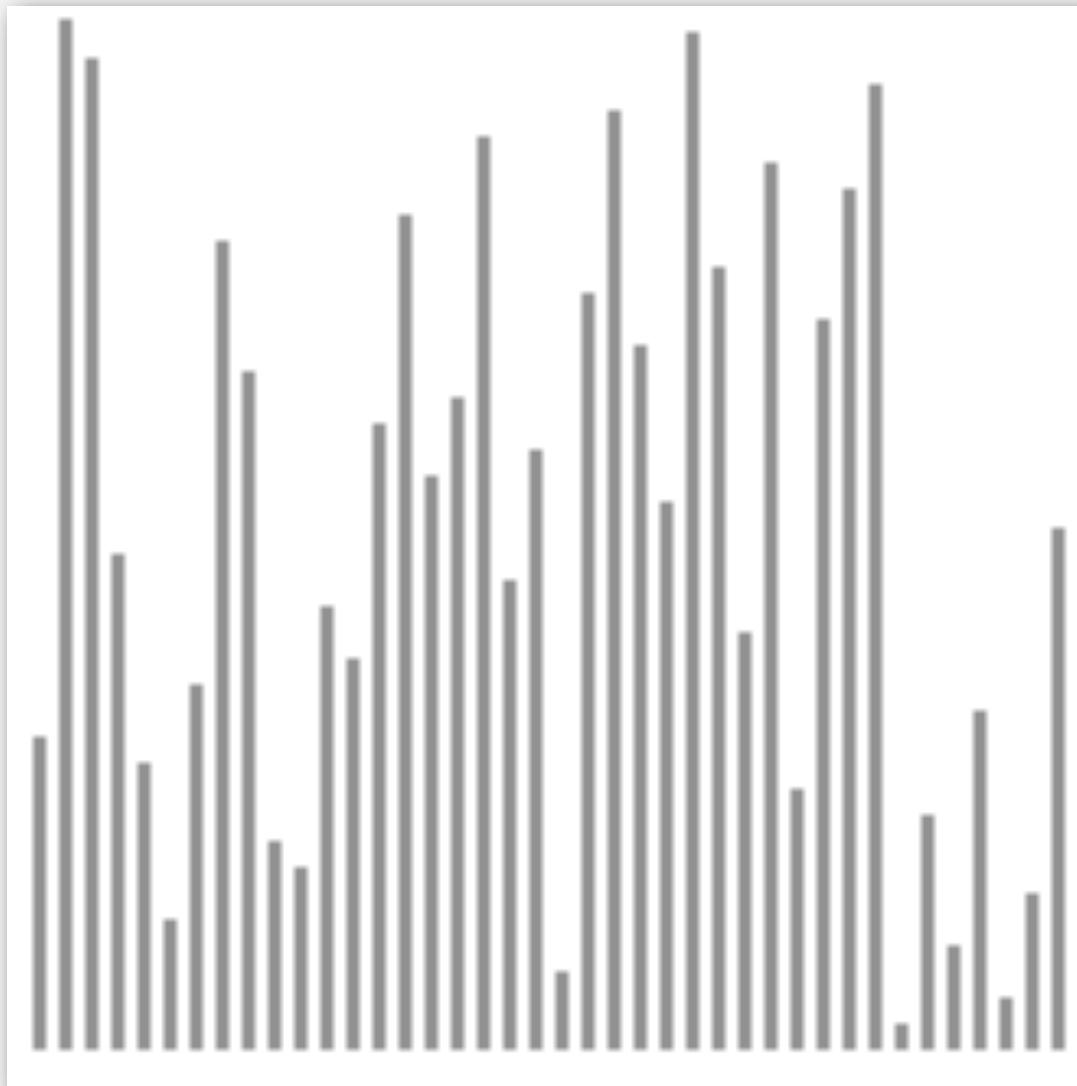
		a[]										
i	j	0	1	2	3	4	5	6	7	8	9	10
1	0	O	S	R	T	E	X	A	M	P	L	E
2	1	O	R	S	T	E	X	A	M	P	L	E
3	3	O	R	S	T	E	X	A	M	P	L	E
4	0	E	O	R	S	T	X	A	M	P	L	E
5	5	E	O	R	S	T	X	A	M	P	L	E
6	0	A	E	O	R	S	T	X	M	P	L	E
7	2	A	E	M	O	R	S	T	X	P	L	E
8	4	A	E	M	O	P	R	S	T	X	L	E
9	2	A	E	L	M	O	P	R	S	T	X	E
10	2	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Trace of insertion sort (array contents just after each insertion)

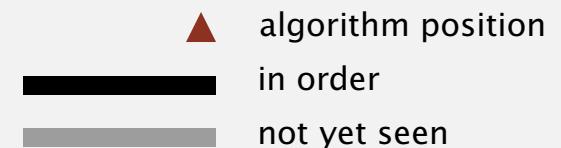
Insertion sort: trace

Insertion sort: animation

40 random items



<http://www.sorting-algorithms.com/insertion-sort>



Insertion sort: best and worst case

Best case. If the array is in ascending order, insertion sort makes $N-1$ compares and 0 exchanges.

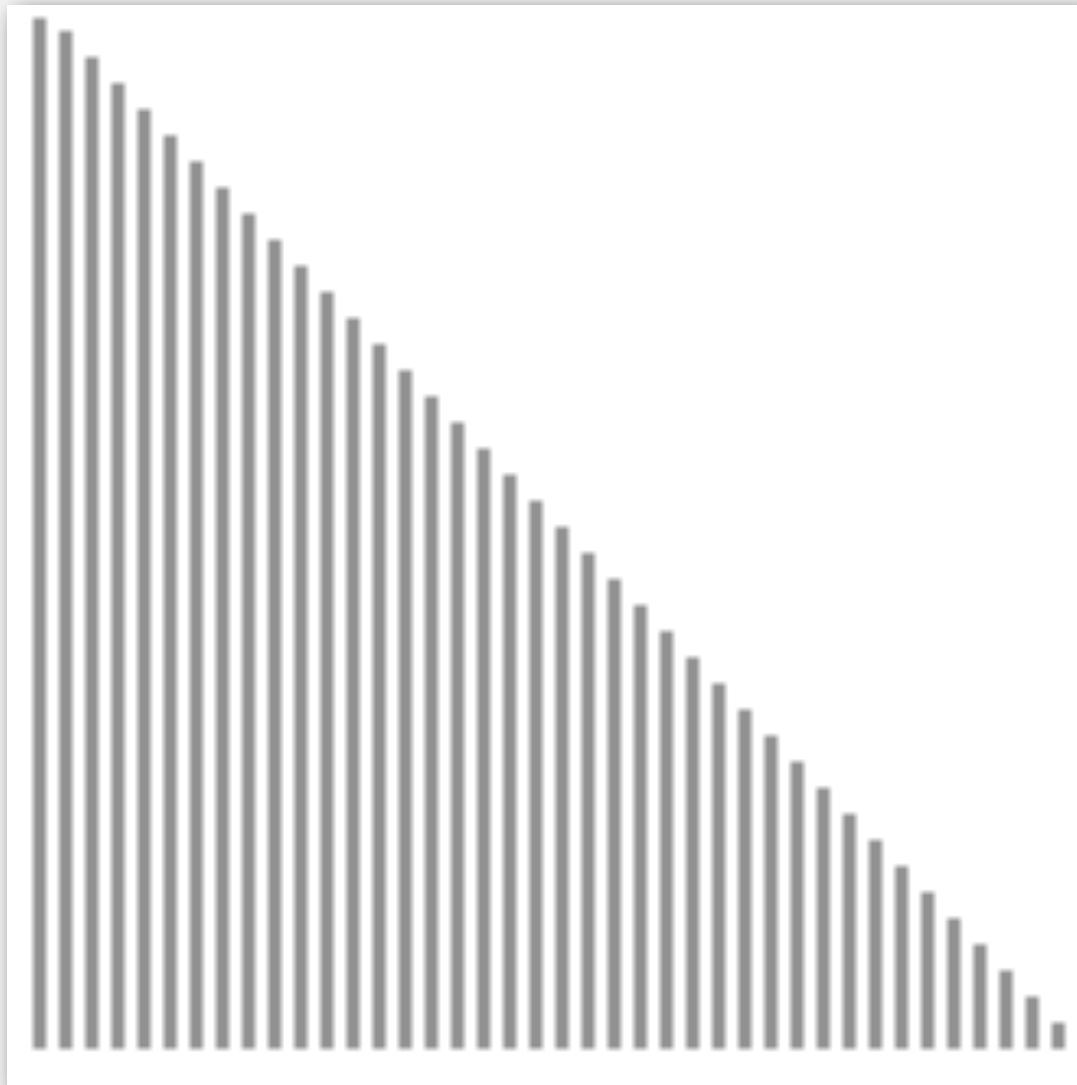
A E E L M O P R S T X

Worst case. If the array is in descending order (and no duplicates), insertion sort makes $\sim \frac{1}{2} N^2$ compares and $\sim \frac{1}{2} N^2$ exchanges.

X T S R P O M L E E A

Insertion sort: animation

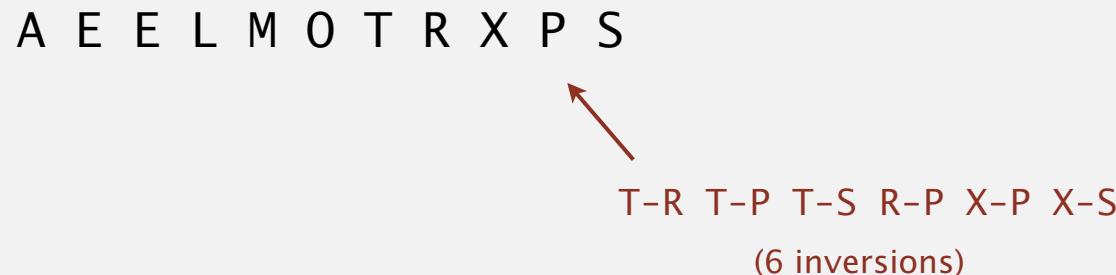
40 reverse-sorted items



<http://www.sorting-algorithms.com/insertion-sort>

Insertion sort: partially-sorted arrays

Def. An **inversion** is a pair of keys that are out of order.



Def. An array is **partially sorted** if the number of inversions is $\leq c N$.

- Ex 1. A subarray of size 10 appended to a sorted subarray of size N .
- Ex 2. An array of size N with only 10 entries out of place.

Proposition. For partially-sorted arrays, insertion sort runs in linear time.

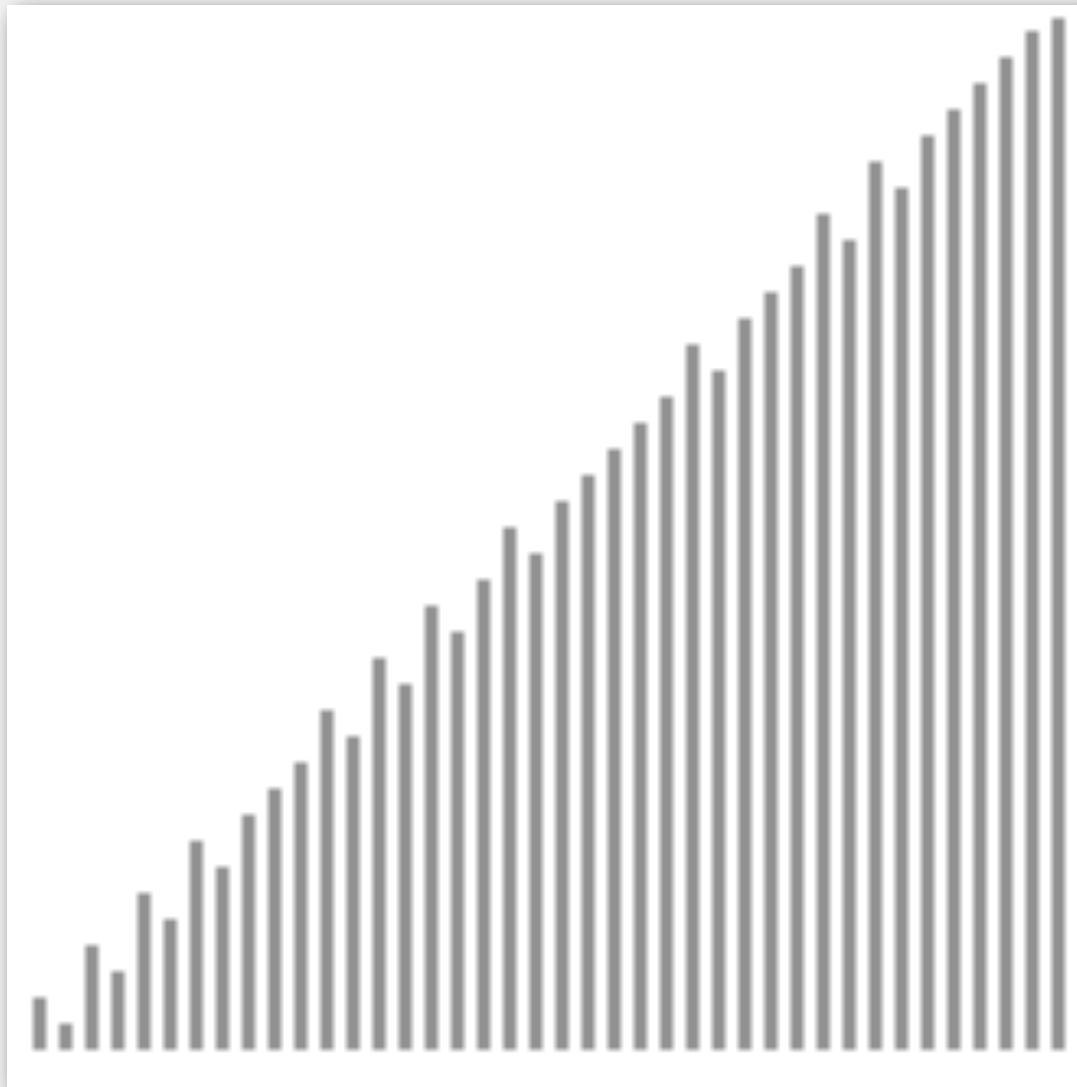
Pf. Number of exchanges equals the number of inversions.

$$\text{number of compares} = \text{exchanges} + (N - 1)$$

↑

Insertion sort: animation

40 partially-sorted items



<http://www.sorting-algorithms.com/insertion-sort>

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Shellsort overview

Idea. Move entries more than one position at a time by *h*-sorting the array.

an *h*-sorted array is *h* interleaved sorted subsequences



Shellsort. [Shell 1959] *h*-sort array for decreasing sequence of values of *h*.

input	S	H	E	L	L	S	O	R	T	E	X	A	M	P	L	E
13-sort	P	H	E	L	L	S	O	R	T	E	X	A	M	S	L	E
4-sort	L	E	E	A	M	H	L	E	P	S	O	L	T	S	X	R
1-sort	A	E	E	E	H	L	L	M	O	P	R	S	S	T	X	

h-sorting

How to *h*-sort an array? Insertion sort, with stride length *h*.

3-sorting an array

M	O	L	E	E	X	A	S	P	R	T
E	O	L	M	E	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
E	E	L	M	O	X	A	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	X	M	S	P	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T
A	E	L	E	O	P	M	S	X	R	T

Why insertion sort?

- Big increments \Rightarrow small subarray.
- Small increments \Rightarrow nearly in order. [stay tuned]

Shellsort example: increments 7, 3, 1

input

S O R T E X A M P L E

7-sort

S O R T E X A M P L E
M O R T E X A S P L E
M O R T E X A S P R L E
M O L T E X A S P R E
M O L E E X A S P R T

3-sort

M O L E E X A S P R T
E O L M E X A S P R T
E E L M O X A S P R T
E E L M O X A S P R T
A E L E O X M S P R T
A E L E O X M S P R T
A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T

1-sort

A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T
A E E L O P M S X R T
A E E L O P M S X R T
A E E L O P M S X R T
A E E L M O P S X R T
A E E L M O P S X R T
A E E L M O P R S X T
A E E L M O P R S T X

result

A E E L M O P R S T X

Shellsort: intuition

Proposition. A g -sorted array remains g -sorted after h -sorting it.

7-sort										3-sort									
S	O	R	T	E	X	A	M	P	L	E	M	O	L	E	E	X	A	S	P
M	O	R	T	E	X	A	S	P	L	E	E	O	L	M	E	X	A	S	P
M	O	R	T	E	X	A	S	P	L	E	E	E	L	M	O	X	A	S	P
M	O	L	T	E	X	A	S	P	R	E	E	E	L	M	O	X	A	S	P
M	O	L	E	E	X	A	S	P	R	T	A	E	L	E	O	X	M	S	P
											A	E	L	E	O	X	M	S	P
											A	E	L	E	O	P	M	S	R
											A	E	L	E	O	P	M	S	X
											A	E	L	E	O	P	M	S	R
											A	E	L	E	O	P	M	S	T
											A	E	L	E	O	P	M	S	X
											A	E	L	E	O	P	M	S	R
											A	E	L	E	O	P	M	S	T

still 7-sorted

Challenge. Prove this fact—it's more subtle than you'd think!

Shellsort: which increment sequence to use?

Powers of two. 1, 2, 4, 8, 16, 32, ...

No.

Powers of two minus one. 1, 3, 7, 15, 31, 63, ...

Maybe.

→ $3x + 1$. 1, 4, 13, 40, 121, 364, ...

OK. Easy to compute.

Sedgewick. 1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, ...

Good. Tough to beat in empirical studies.

merging of $(9 \times 4^i) - (9 \times 2^i) + 1$
and $4^i - (3 \times 2^i) + 1$

Shellsort: Java implementation

```
public class Shell
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;

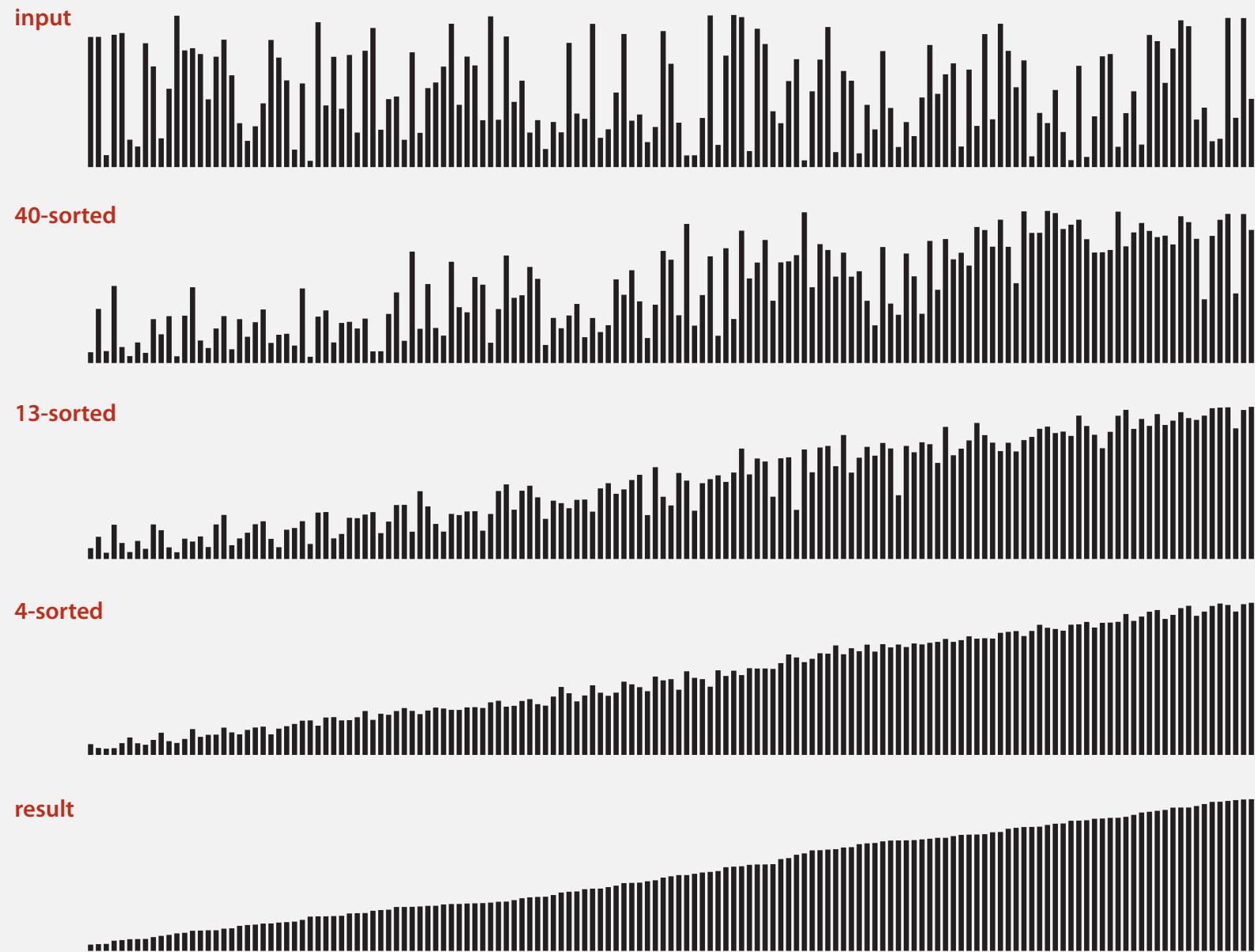
        int h = 1;
        while (h < N/3) h = 3*h + 1; // 1, 4, 13, 40, 121, 364, ...
        ← 3x+1 increment sequence

        while (h >= 1)
        { // h-sort the array.
            for (int i = h; i < N; i++)
            {
                for (int j = i; j >= h && less(a[j], a[j-h]); j -= h)
                    exch(a, j, j-h);
            }
            ← insertion sort

            h = h/3;
        }
        ← move to next increment
    }

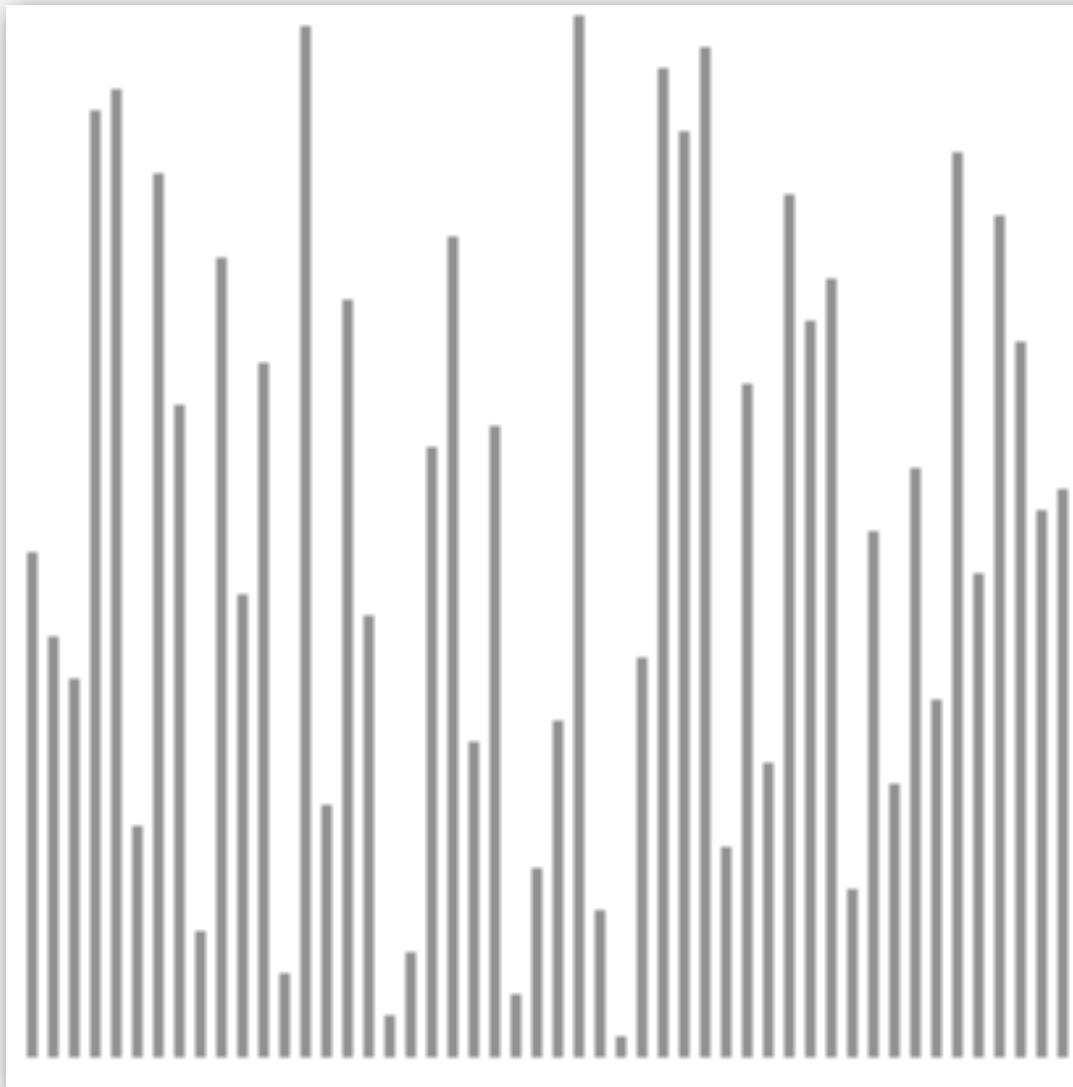
    private static boolean less(Comparable v, Comparable w)
    { /* as before */ }
    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

Shellsort: visual trace



Shellsort: animation

50 random items

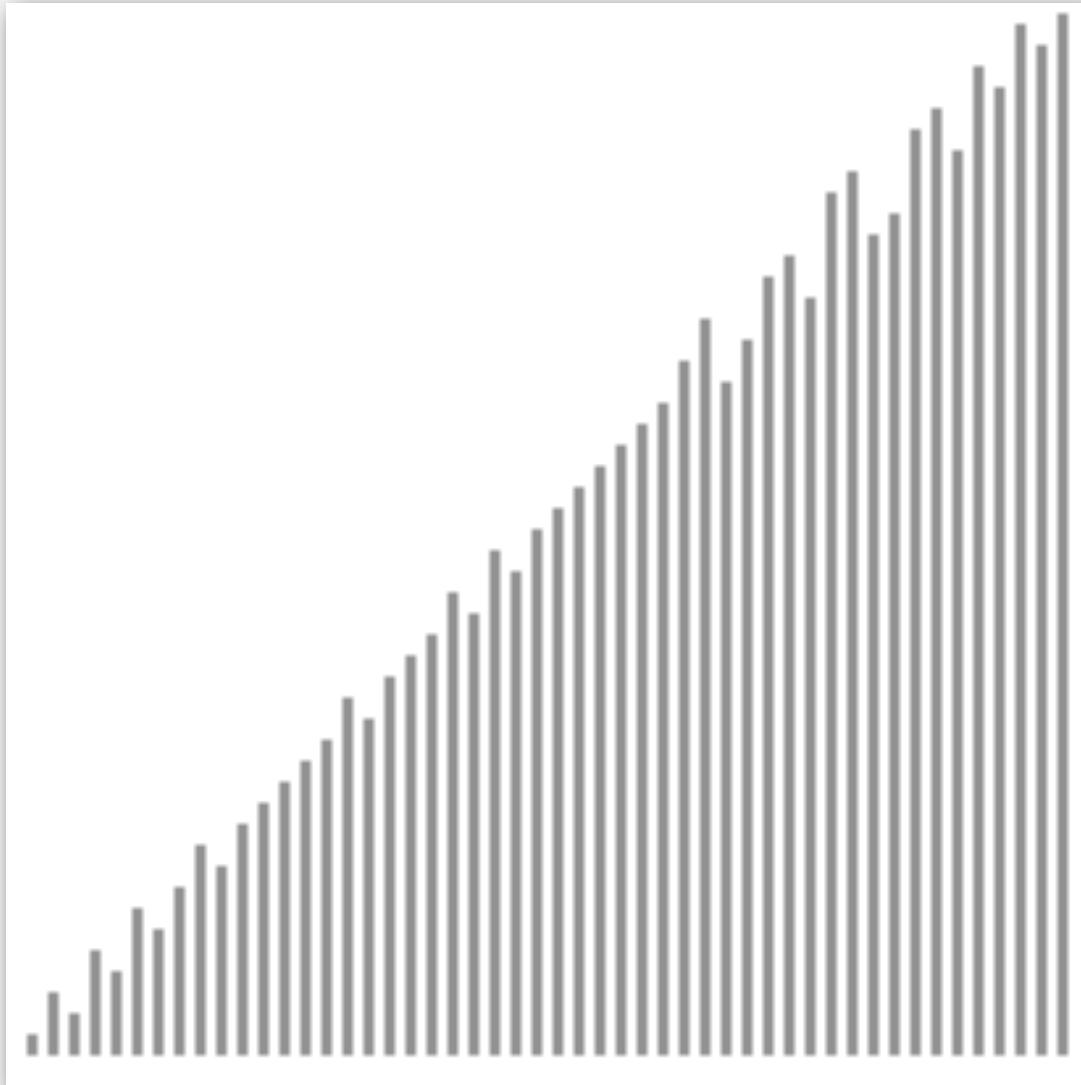


<http://www.sorting-algorithms.com/shell-sort>

- ▲ algorithm position
- ▬ h-sorted
- ▬ current subsequence
- ▬ other elements

Shellsort: animation

50 partially-sorted items



<http://www.sorting-algorithms.com/shell-sort>

- The legend consists of four entries, each with a colored bar followed by text:

 - A red triangle pointing upwards: "algorithm position"
 - A solid black horizontal bar: "h-sorted"
 - A solid grey horizontal bar: "current subsequence"
 - A solid light grey horizontal bar: "other elements"

Shellsort: analysis

Proposition. The worst-case number of compares used by shellsort with the $3x+1$ increments is $O(N^{3/2})$.

Property. Number of compares used by shellsort with the $3x+1$ increments is at most by a small multiple of N times the # of increments used.

N	compares	$N^{1.289}$	$2.5 N \lg N$
5,000	93	58	106
10,000	209	143	230
20,000	467	349	495
40,000	1022	855	1059
80,000	2266	2089	2257

measured in thousands

Remark. Accurate model has not yet been discovered (!)

Why are we interested in shellsort?

Example of simple idea leading to substantial performance gains.

Useful in practice.

- Fast unless array size is huge (used for small subarrays).
- Tiny, fixed footprint for code (used in some embedded systems).
- Hardware sort prototype.

bzip2, /linux/kernel/groups.c

uClibc

Simple algorithm, nontrivial performance, interesting questions.

- Asymptotic growth rate?
- Best sequence of increments? ← open problem: find a better increment sequence
- Average-case performance?

Lesson. Some good algorithms are still waiting discovery.

Algorithms

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2.1 ELEMENTARY SORTS

- ▶ *rules of the game*
- ▶ *selection sort*
- ▶ *insertion sort*
- ▶ ***shellsort***
- ▶ *shuffling*
- ▶ *convex hull*

Algorithms

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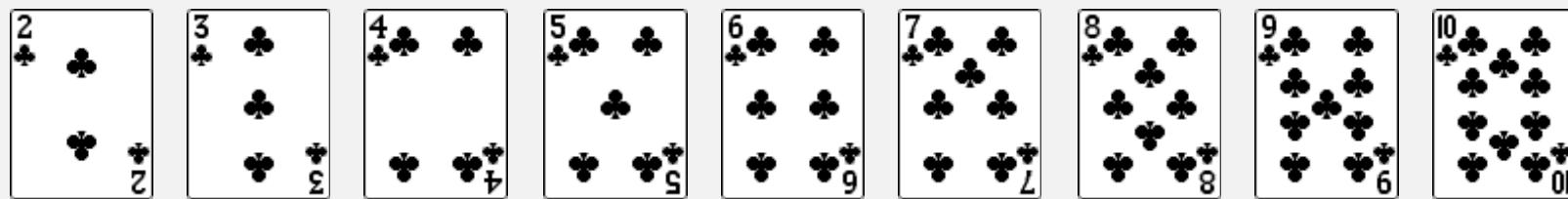
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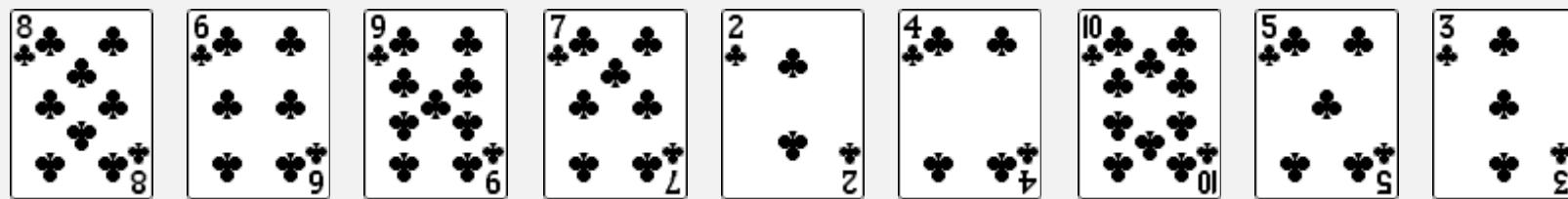
How to shuffle an array

Goal. Rearrange array so that result is a uniformly random permutation.



How to shuffle an array

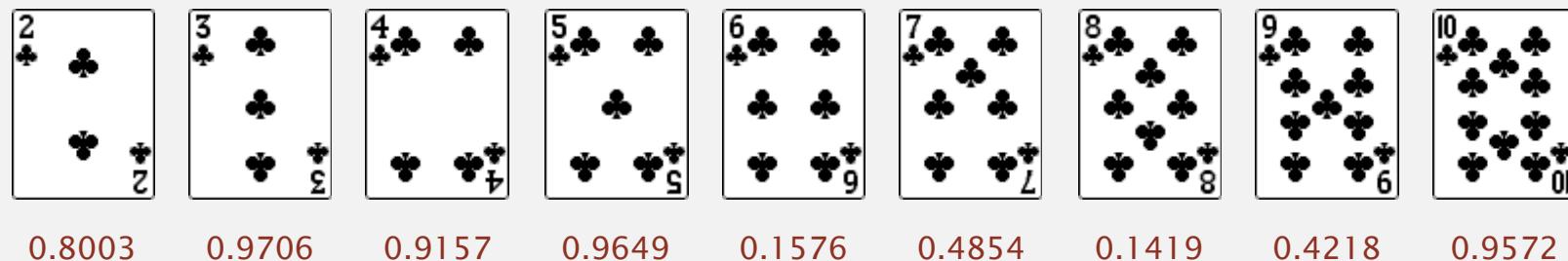
Goal. Rearrange array so that result is a uniformly random permutation.



Shuffle sort

- Generate a random real number for each array entry.
- Sort the array.

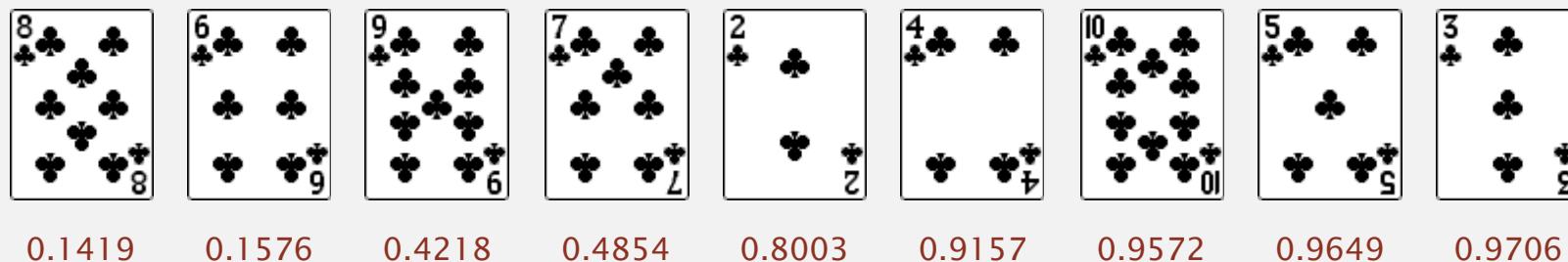
useful for shuffling
columns in a spreadsheet



Shuffle sort

- Generate a random real number for each array entry.
- Sort the array.

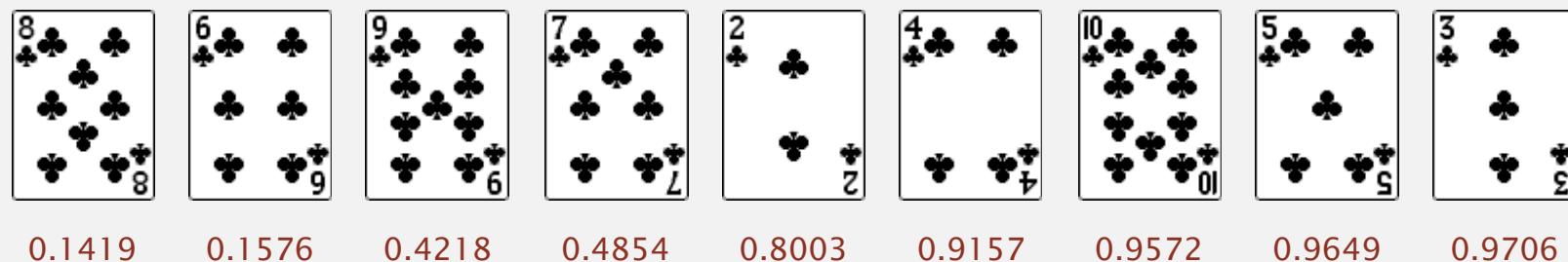
useful for shuffling
columns in a spreadsheet



Shuffle sort

- Generate a random real number for each array entry.
- Sort the array.

useful for shuffling
columns in a spreadsheet



Proposition. Shuffle sort produces a uniformly random permutation
of the input array, provided no duplicate values.

assuming real numbers
uniformly at random

War story (Microsoft)

Microsoft antitrust probe by EU. Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

<http://www.browserchoice.eu>

Select your web browser(s)



Google
chrome

A fast new browser from Google. Try it now!



Safari

Safari for Windows from Apple, the world's most innovative browser.



mozilla
Firefox

Your online security is Firefox's top priority. Firefox is free, and made to help you get the most out of the



Opera™
browser

The fastest browser on Earth. Secure, powerful and easy to use, with excellent privacy protection.



Windows®
Internet Explorer® 8

Designed to help you take control of your privacy and browse with confidence. Free from Microsoft.



appeared last
50% of the time

War story (Microsoft)

Microsoft antitrust probe by EU. Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

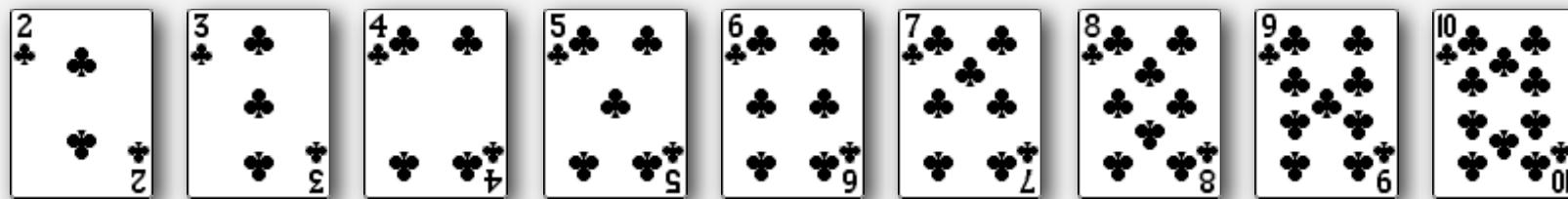
Solution? Implement shuffle sort by making comparator always return a random answer.

```
public int compareTo(Browser that)
{
    double r = Math.random();
    if (r < 0.5) return -1;
    if (r > 0.5) return +1;
    return 0;
}
```

← browser comparator
(should implement a total order)

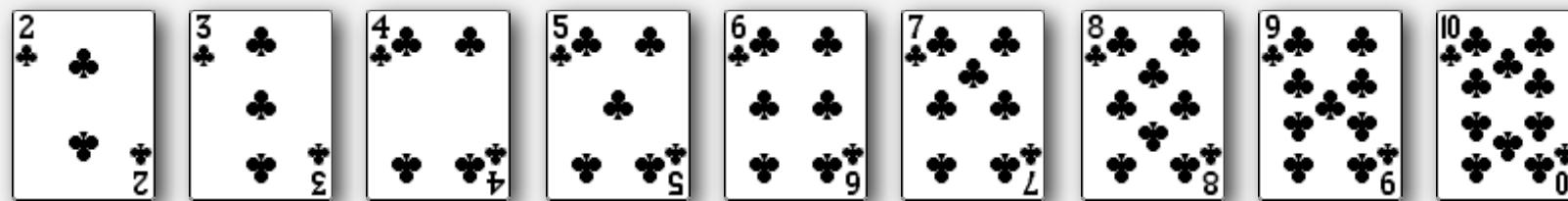
Knuth shuffle demo

- In iteration i , pick integer r between 0 and i uniformly at random.
- Swap $a[i]$ and $a[r]$.



Knuth shuffle

- In iteration i , pick integer r between 0 and i uniformly at random.
- Swap $a[i]$ and $a[r]$.



Proposition. [Fisher-Yates 1938] Knuth shuffling algorithm produces a uniformly random permutation of the input array in linear time.

→ assuming integers
uniformly at random

Knuth shuffle

- In iteration i , pick integer r between 0 and i uniformly at random.
- Swap $a[i]$ and $a[r]$.

common bug: between 0 and $N - 1$
correct variant: between i and $N - 1$

```
public class StdRandom
{
    ...
    public static void shuffle(Object[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int r = StdRandom.uniform(i + 1);           ← between 0 and i
            exch(a, i, r);
        }
    }
}
```

War story (online poker)

Texas hold'em poker. Software must shuffle electronic cards.



How We Learned to Cheat at Online Poker: A Study in Software Security

<http://www.datamation.com/entdev/article.php/616221>

War story (online poker)

Shuffling algorithm in FAQ at www.planetpoker.com

```
for i := 1 to 52 do begin
    r := random(51) + 1;           ← between 1 and 51
    swap := card[r];
    card[r] := card[i];
    card[i] := swap;
end;
```

- Bug 1. Random number r never 52 \Rightarrow 52nd card can't end up in 52nd place.
- Bug 2. Shuffle not uniform (should be between 1 and i).
- Bug 3. random() uses 32-bit seed \Rightarrow 2^{32} possible shuffles.
- Bug 4. Seed = milliseconds since midnight \Rightarrow 86.4 million shuffles.

“The generation of random numbers is too important to be left to chance.”

— Robert R. Coveyou

War story (online poker)

Best practices for shuffling (if your business depends on it).

- Use a hardware random-number generator that has passed both the FIPS 140-2 and the NIST statistical test suites.
- Continuously monitor statistic properties: hardware random-number generators are fragile and fail silently.
- Use an unbiased shuffling algorithm.



Bottom line. Shuffling a deck of cards is hard!

Algorithms

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2.1 ELEMENTARY SORTS

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- ▶ *insertion sort*
- ▶ *shellsort*
- ▶ ***shuffling***
- ▶ *convex hull*

Algorithms

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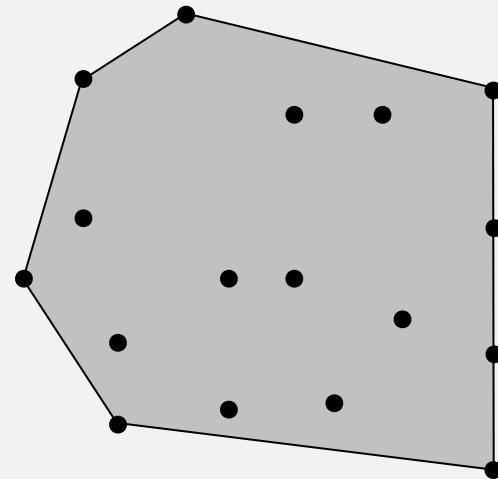
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Convex hull

The **convex hull** of a set of N points is the smallest perimeter fence enclosing the points.

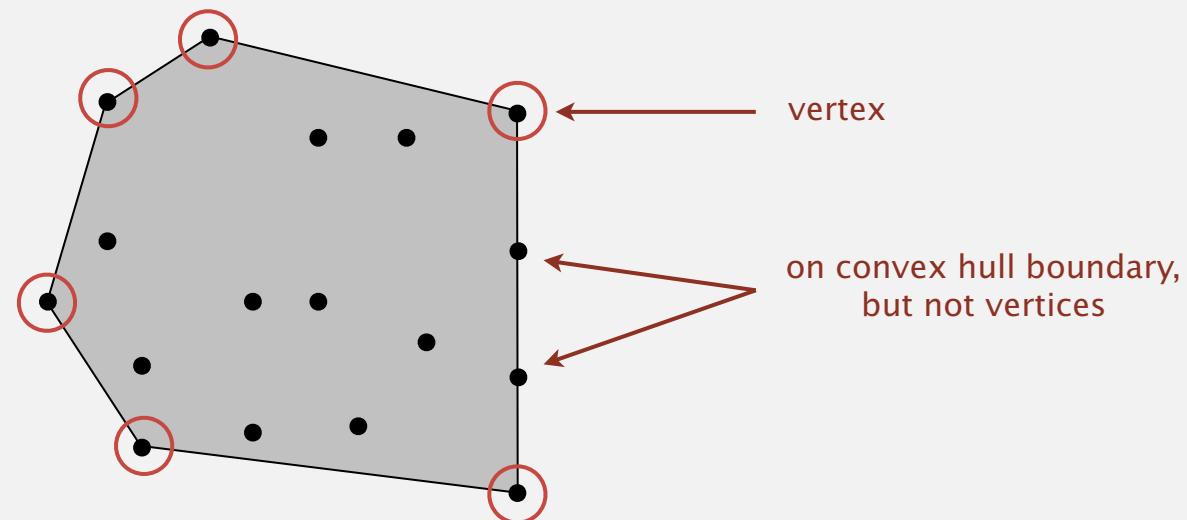


Equivalent definitions.

- Smallest convex set containing all the points.
- Smallest area convex polygon enclosing the points.
- Convex polygon enclosing the points, whose vertices are points in set.

Convex hull

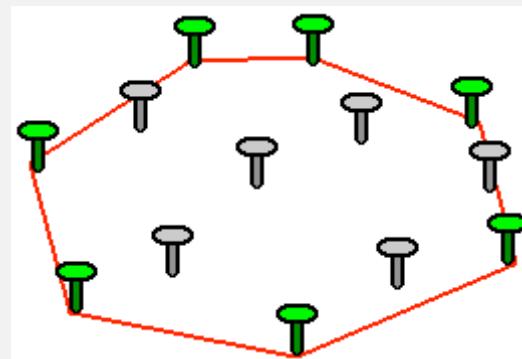
The **convex hull** of a set of N points is the smallest perimeter fence enclosing the points.



Convex hull output. Sequence of vertices in counterclockwise order.

Convex hull: mechanical algorithm

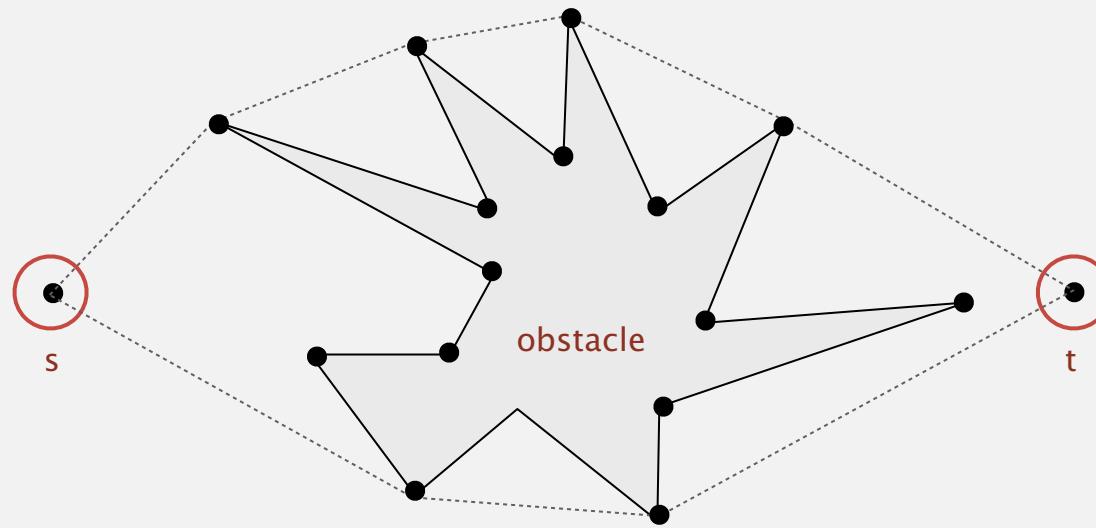
Mechanical algorithm. Hammer nails perpendicular to plane; stretch elastic rubber band around points.



http://www.idlcoyote.com/math_tips/convexhull.html

Convex hull application: motion planning

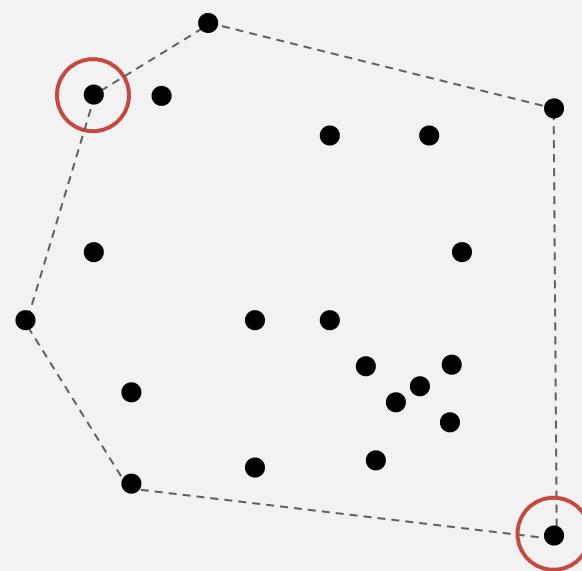
Robot motion planning. Find shortest path in the plane from s to t that avoids a polygonal obstacle.



Fact. Shortest path is either straight line from s to t or it is one of two polygonal chains of convex hull.

Convex hull application: farthest pair

Farthest pair problem. Given N points in the plane, find a pair of points with the largest Euclidean distance between them.

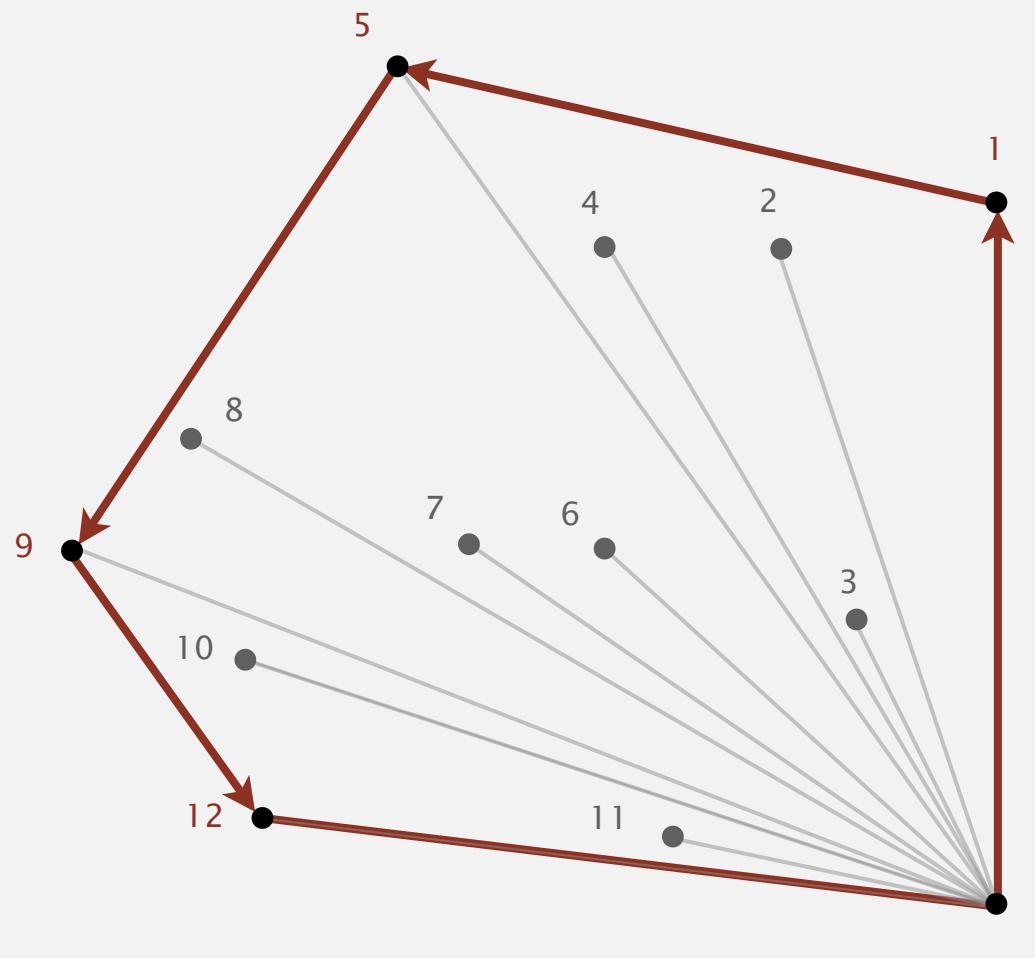


Fact. Farthest pair of points are extreme points on convex hull.

Convex hull: geometric properties

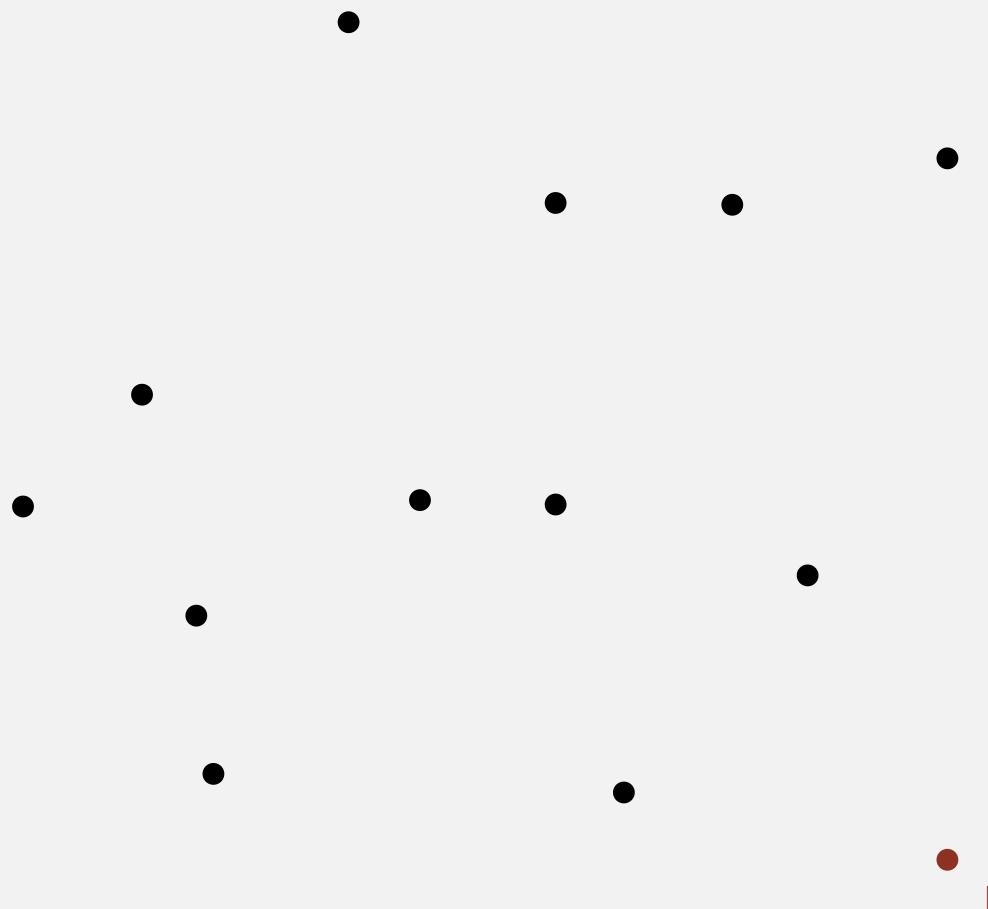
Fact. Can traverse the convex hull by making only counterclockwise turns.

Fact. The vertices of convex hull appear in increasing order of polar angle with respect to point p with lowest y -coordinate.



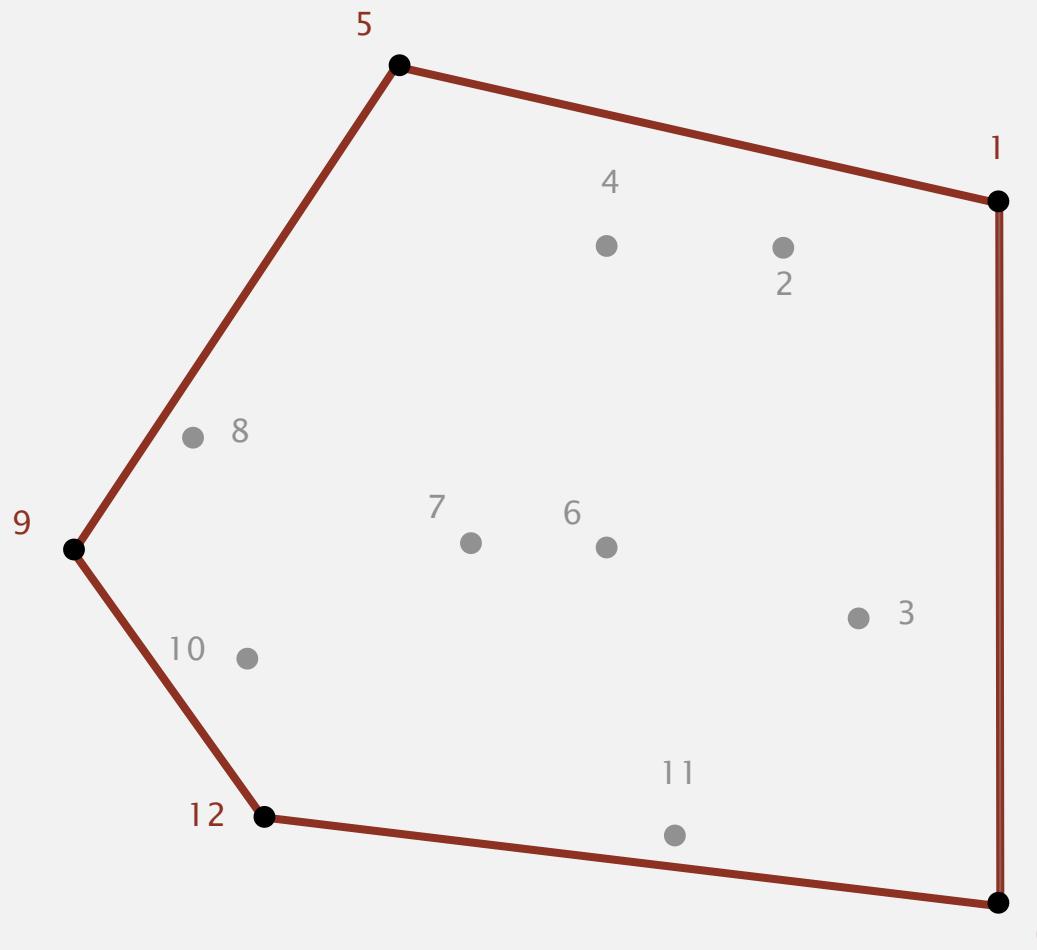
Graham scan demo

- Choose point p with smallest y -coordinate.
- Sort points by polar angle with p .
- Consider points in order; discard unless it creates a ccw turn.



Graham scan demo

- Choose point p with smallest y -coordinate.
- Sort points by polar angle with p .
- Consider points in order; discard unless it creates a ccw turn.



Graham scan: implementation challenges

Q. How to find point p with smallest y -coordinate?

A. Define a total order, comparing by y -coordinate. [next lecture]

Q. How to sort points by polar angle with respect to p ?

A. Define a total order **for each** point p . [next lecture]

Q. How to determine whether $p_1 \rightarrow p_2 \rightarrow p_3$ is a counterclockwise turn?

A. Computational geometry. [next two slides]

Q. How to sort efficiently?

A. Mergesort sorts in $N \log N$ time. [next lecture]

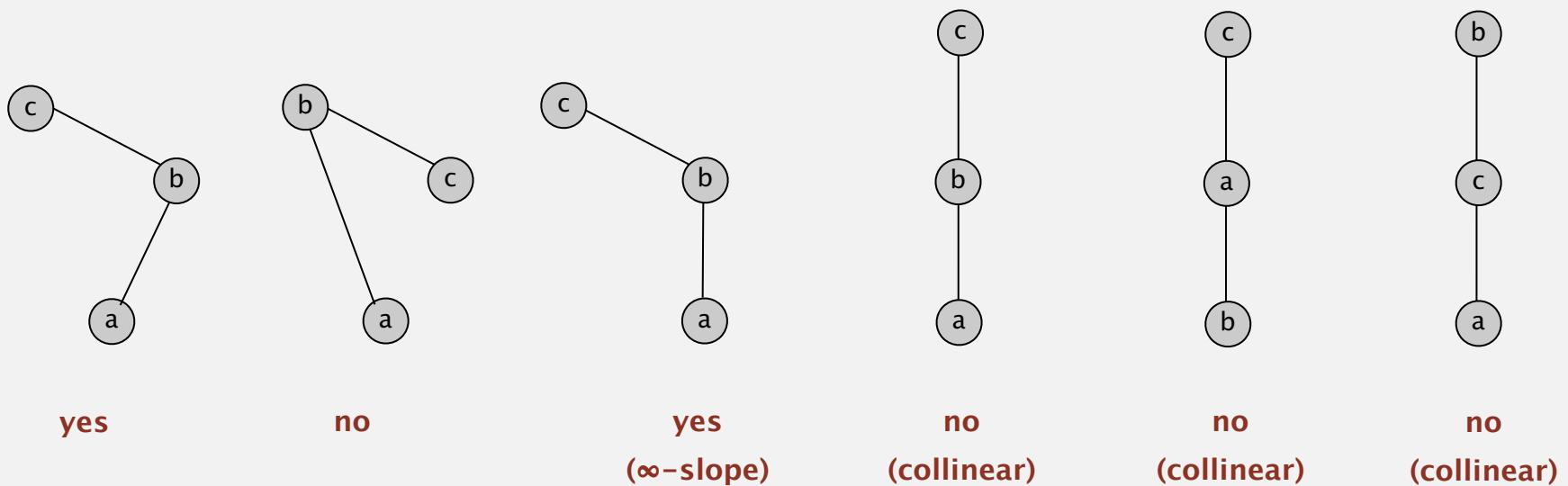
Q. How to handle degeneracies (three or more points on a line)?

A. Requires some care, but not hard. [see booksite]

Implementing ccw

CCW. Given three points a , b , and c , is $a \rightarrow b \rightarrow c$ a counterclockwise turn?

is c to the left of the ray $a \rightarrow b$



Lesson. Geometric primitives are tricky to implement.

- Dealing with degenerate cases.
- Coping with floating-point precision.

Implementing ccw

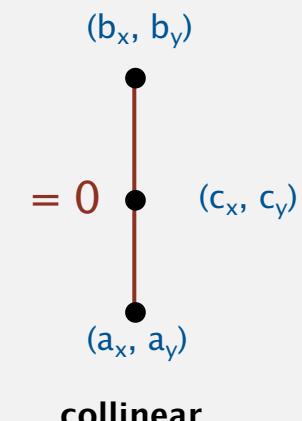
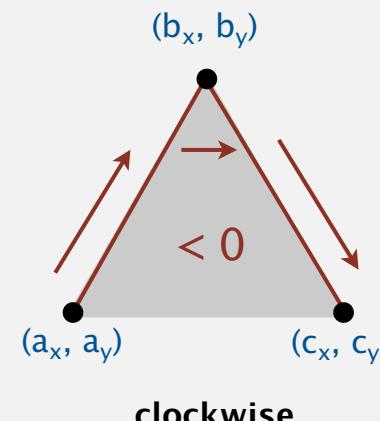
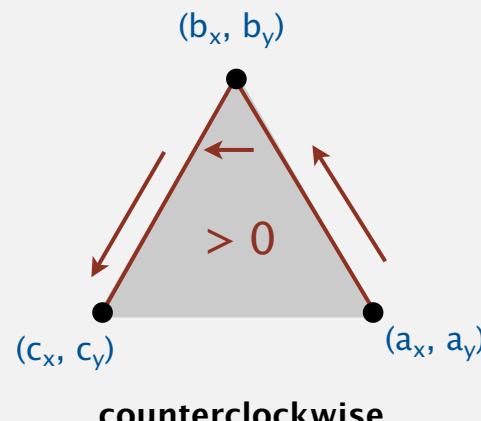
CCW. Given three points a , b , and c , is $a \rightarrow b \rightarrow c$ a counterclockwise turn?

- Determinant (or cross product) gives 2x signed area of planar triangle.

$$2 \times \text{Area}(a, b, c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = (b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)$$

$(b - a) \times (c - a)$

- If signed area > 0 , then $a \rightarrow b \rightarrow c$ is counterclockwise.
- If signed area < 0 , then $a \rightarrow b \rightarrow c$ is clockwise.
- If signed area $= 0$, then $a \rightarrow b \rightarrow c$ are collinear.



Immutable point data type

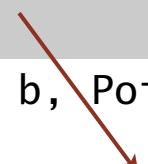
```
public class Point2D
{
    private final double x;
    private final double y;

    public Point2D(double x, double y)
    {
        this.x = x;
        this.y = y;
    }

    ...

    public static int ccw(Point2D a, Point2D b, Point2D c)
    {
        double area2 = (b.x-a.x)*(c.y-a.y) - (b.y-a.y)*(c.x-a.x);
        if      (area2 < 0) return -1; // clockwise
        else if (area2 > 0) return +1; // counter-clockwise
        else                  return 0; // collinear
    }
}
```

danger of
floating-point
roundoff error



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Graham scan: implementation

Simplifying assumptions. No three points on a line; at least 3 points.

```
Stack<Point2D> hull = new Stack<Point>();  
  
Arrays.sort(p, Point2D.Y_ORDER);      ← p[0] is now point with lowest y-coordinate  
Arrays.sort(p, p[0].BY_POLAR_ORDER);   ← sort by polar angle with respect to p[0]  
  
hull.push(p[0]);    ← definitely on hull  
hull.push(p[1]);  
for (int i = 2; i < N; i++) {  
    Point2D top = hull.pop();  
    while (Point2D.ccw(hull.peek(), top, p[i]) <= 0)  
        top = hull.pop();  
    hull.push(top);  
    hull.push(p[i]);    ← add p[i] to putative hull  
}
```

discard points that would
create clockwise turn



Running time. $N \log N$ for sorting and linear for rest.

Pf. $N \log N$ for sorting; each point pushed and popped at most once.