

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE



<http://algs4.cs.princeton.edu>

2.3 QUICKSORT

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort.

← last lecture

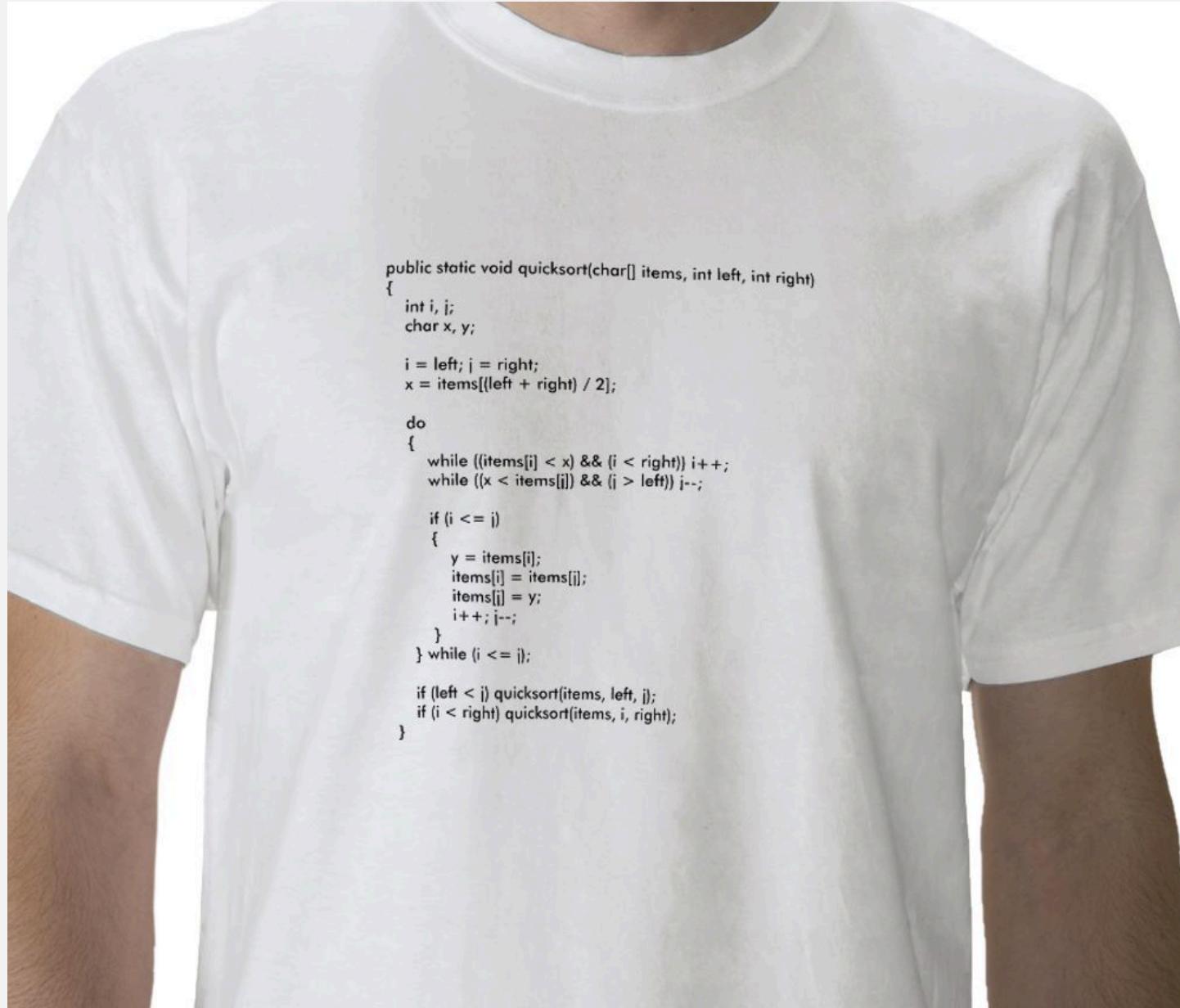
- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

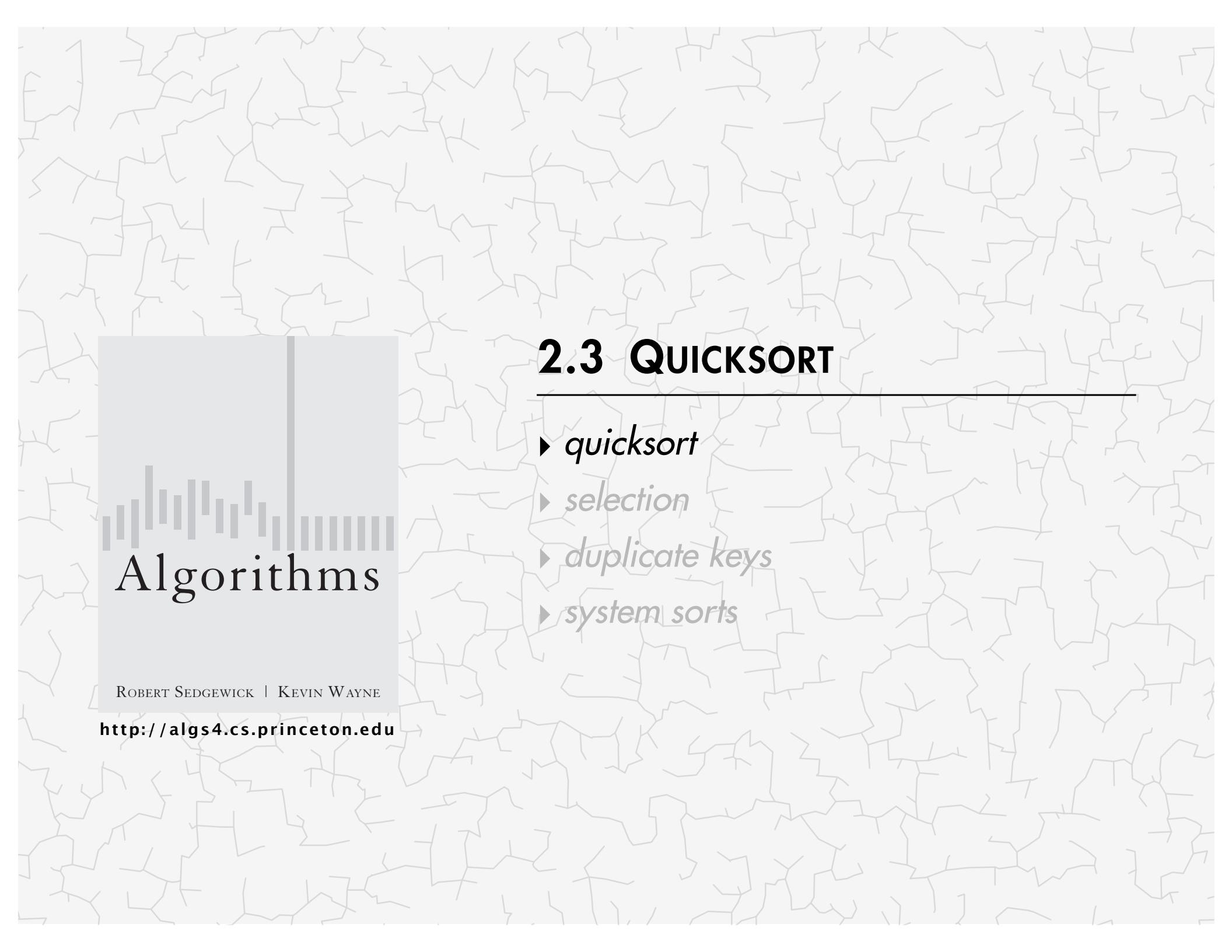
Quicksort.

← this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

Quicksort t-shirt





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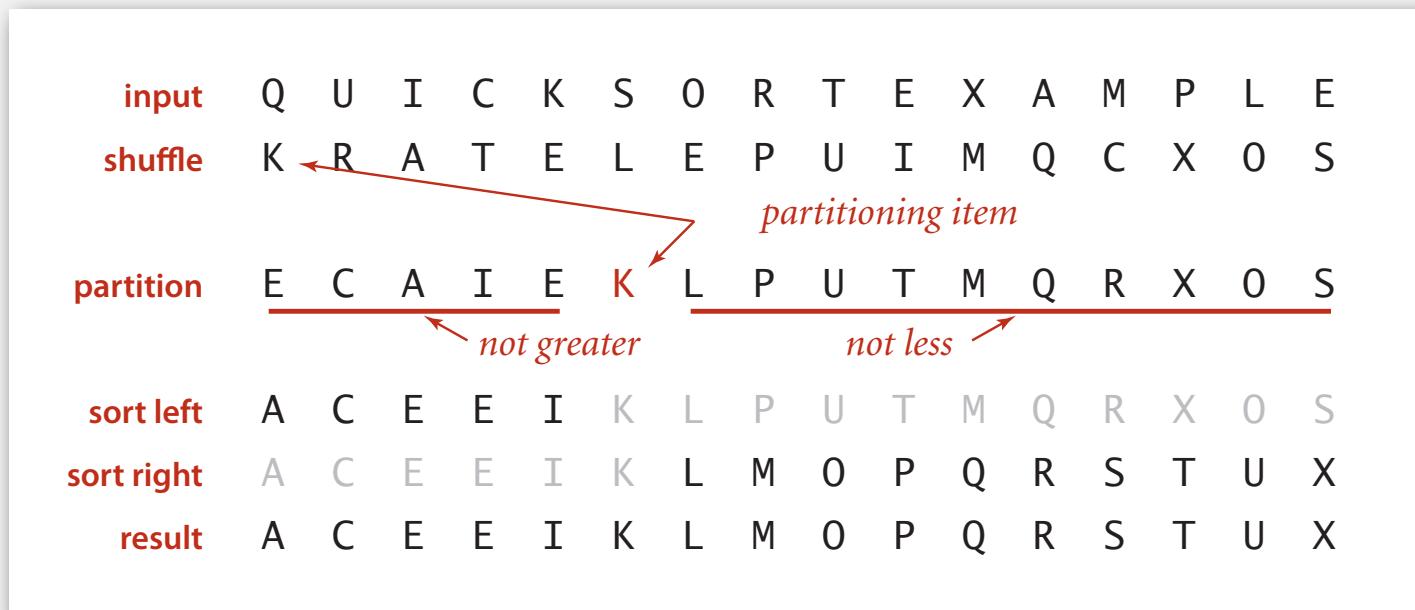
Quicksort

Basic plan.

- **Shuffle** the array.
- **Partition** so that, for some j
 - entry $a[j]$ is in place
 - no larger entry to the left of j
 - no smaller entry to the right of j
- **Sort** each piece recursively.



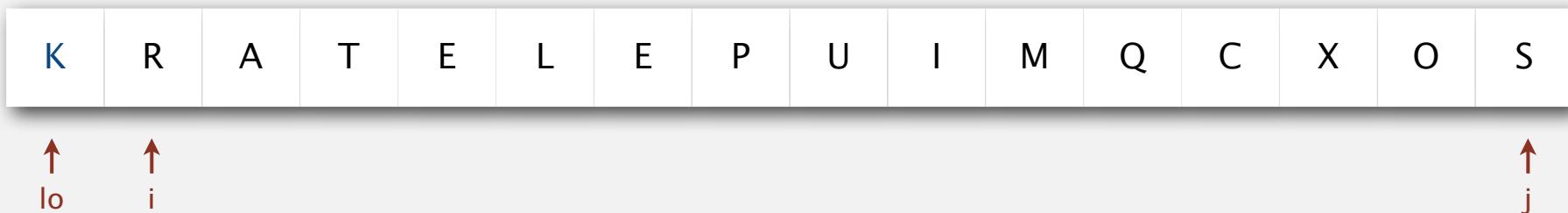
Sir Charles Antony Richard Hoare
1980 Turing Award



Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as $(a[i] < a[lo])$.
- Scan j from right to left so long as $(a[j] > a[lo])$.
- Exchange $a[i]$ with $a[j]$.



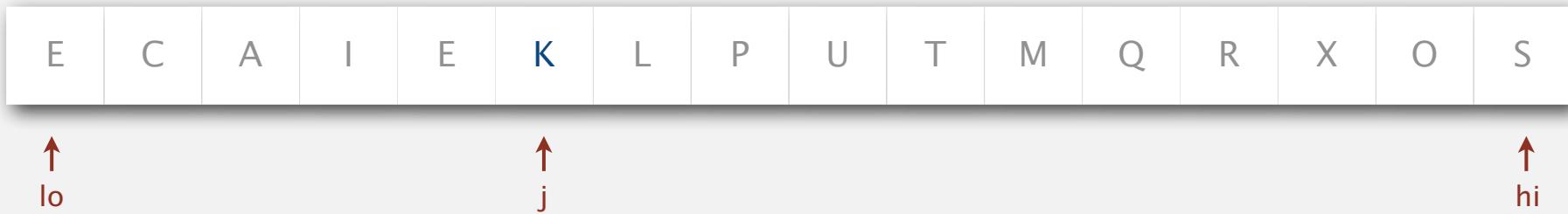
Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as $(a[i] < a[lo])$.
- Scan j from right to left so long as $(a[j] > a[lo])$.
- Exchange $a[i]$ with $a[j]$.

When pointers cross.

- Exchange $a[lo]$ with $a[j]$.



partitioned!

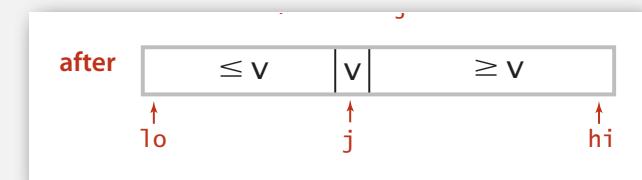
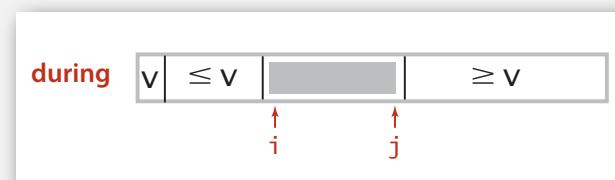
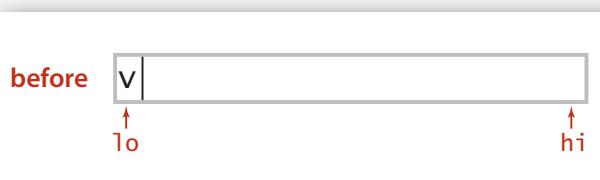
Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))           find item on left to swap
            if (i == hi) break;

        while (less(a[lo], a[--j]))           find item on right to swap
            if (j == lo) break;

        if (i >= j) break;                  check if pointers cross
        exch(a, i, j);                   swap
    }

    exch(a, lo, j);                  swap with partitioning item
    return j;                        return index of item now known to be in place
}
```



Quicksort: Java implementation

```
public class Quick
{
    private static int partition(Comparable[] a, int lo, int hi)
    { /* see previous slide */ }

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
```

← shuffle needed for performance guarantee
(stay tuned)

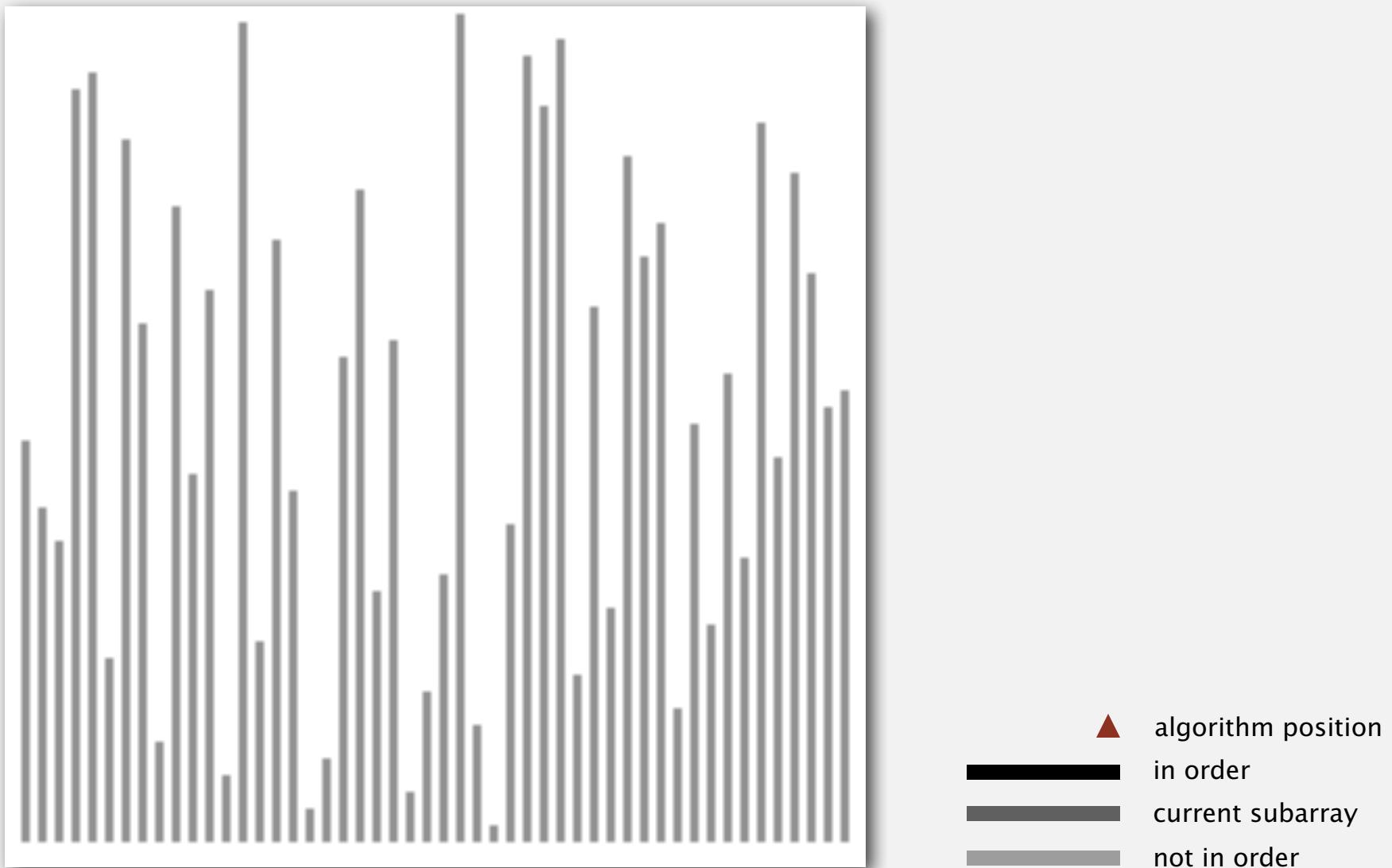
Quicksort trace

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
initial values			Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E	
random shuffle			K	R	A	T	E	L	E	P	U	I	M	Q	C	X	0	S	
0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	0	S	
0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	0	S	
0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	0	S	
0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	0	S	
1	1	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	0	S	
4	4	4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	0	S	
6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	0	S	
7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S	
7	7	8	A	C	E	E	I	K	L	M	M	O	P	T	Q	R	X	U	S
8	8	8	A	C	E	E	I	K	L	M	M	O	P	T	Q	R	X	U	S
10	13	15	A	C	E	E	I	K	L	M	M	O	P	S	Q	R	T	U	X
10	12	12	A	C	E	E	I	K	L	M	M	O	P	R	Q	S	T	U	X
10	11	11	A	C	E	E	I	K	L	M	M	O	P	Q	R	S	T	U	X
10	10	10	A	C	E	E	I	K	L	M	M	O	P	Q	R	S	T	U	X
14	14	15	A	C	E	E	I	K	L	M	M	O	P	Q	R	S	T	U	X
15	15	15	A	C	E	E	I	K	L	M	M	O	P	Q	R	S	T	U	X
result			A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X	

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



<http://www.sorting-algorithms.com/quick-sort>

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The $(j == lo)$ test is redundant (why?), but the $(i == hi)$ test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort ($N \log N$)			quicksort ($N \log N$)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.



lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0	0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O	
2	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O	
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4	4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O	
6	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O	
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8	8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O	
10	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O	
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	
14	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	
A B C D E F G H I J K L M N O																	

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.



			a[]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3}N \ln N$).

Pf. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underset{\text{partitioning}}{(N+1)} + \left(\frac{C_0 + C_{N-1}}{N} \right) + \left(\frac{C_1 + C_{N-2}}{N} \right) + \dots + \left(\frac{C_{N-1} + C_0}{N} \right)$$

- Multiply both sides by N and collect terms: partitioning probability

$$NC_N = N(N+1) + 2(C_0 + C_1 + \dots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N + 1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

- Repeatedly apply above equation:

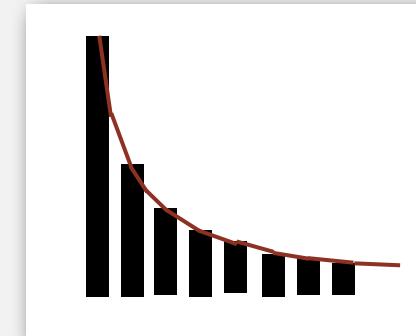
$$\begin{aligned}\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \quad \leftarrow \text{substitute previous equation} \\ &= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \dots + \frac{2}{N+1}\end{aligned}$$

previous equation

← substitute previous equation

- Approximate sum by an integral:

$$\begin{aligned}C_N &= 2(N+1) \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{N+1} \right) \\ &\sim 2(N+1) \int_3^{N+1} \frac{1}{x} dx\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1) \ln N \approx 1.39N \lg N$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \dots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39 N \lg N$.

- 39% more compares than mergesort.
- **But** faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go **quadratic** if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort properties

Proposition. Quicksort is an **in-place** sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by recurring on smaller subarray before larger subarray

Proposition. Quicksort is **not stable**.

Pf.

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.



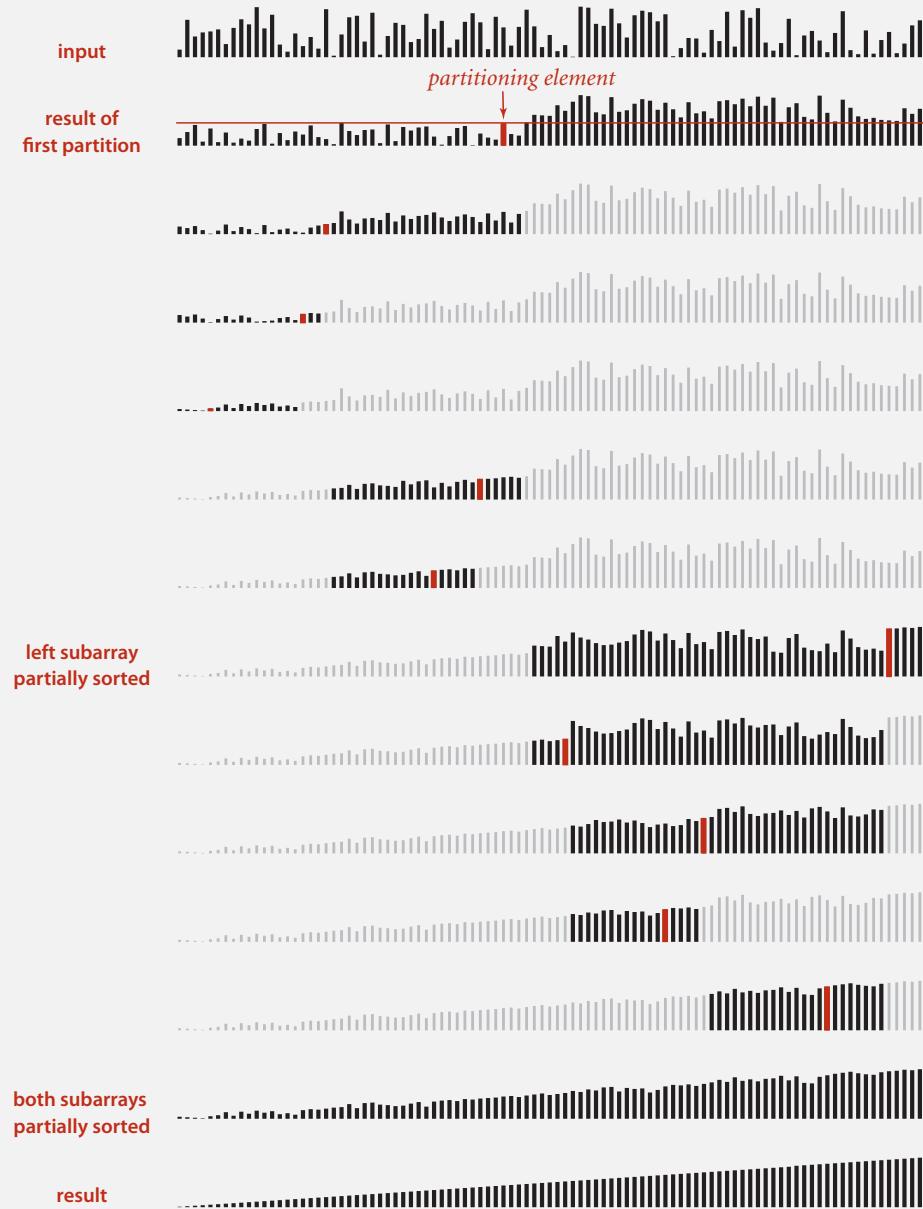
~ $12/7 N \ln N$ compares (slightly fewer)
~ $12/35 N \ln N$ exchanges (slightly more)

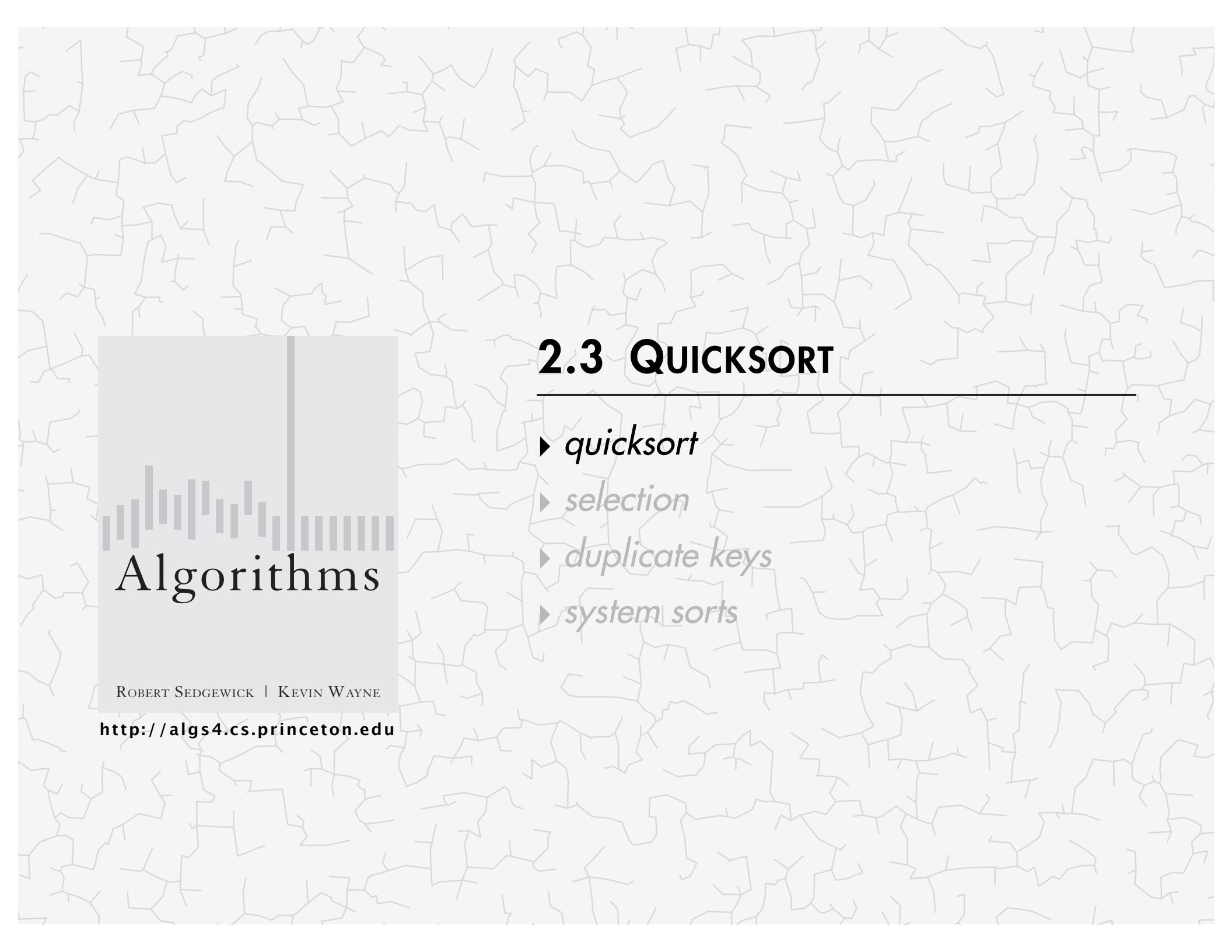
```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;

    int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
    swap(a, lo, m);

    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort with median-of-3 and cutoff to insertion sort: visualization





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Selection

Goal. Given an array of N items, find a k^{th} smallest item.

Ex. Min ($k = 0$), max ($k = N - 1$), median ($k = N/2$).

Applications.

- Order statistics.
- Find the "top k ."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound?  is selection as hard as sorting?
- N upper bound?  is there a linear-time algorithm for each k ?

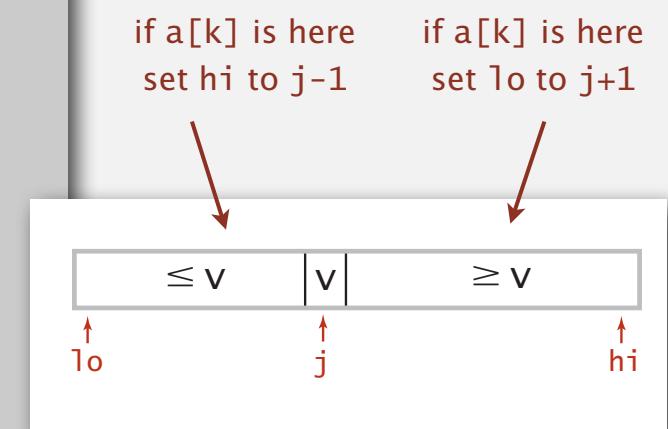
Quick-select

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of j .
- No smaller entry to the right of j .

Repeat in **one** subarray, depending on j ; finished when j equals k .

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
        int j = partition(a, lo, hi);
        if      (j < k) lo = j + 1;
        else if (j > k) hi = j - 1;
        else            return a[k];
    }
    return a[k];
}
```



Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half:
 $N + N/2 + N/4 + \dots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

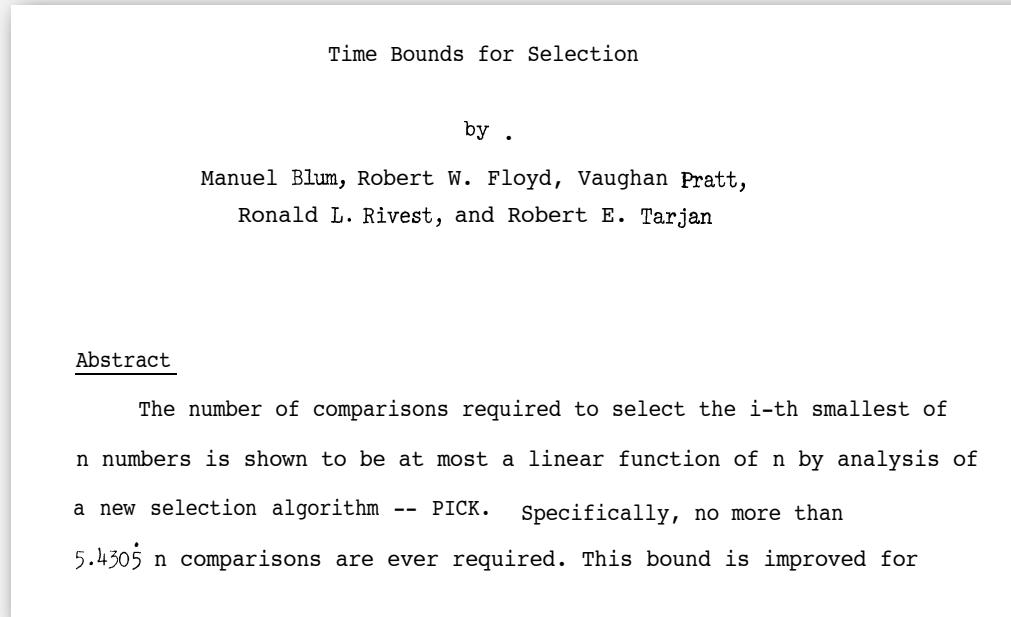
$$C_N = 2N + 2k \ln(N/k) + 2(N-k) \ln(N/(N-k))$$


 $(2 + 2 \ln 2)N$ to find the median

Remark. Quick-select uses $\sim \frac{1}{2}N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] Compare-based selection algorithm whose worst-case running time is linear.



Remark. But, constants are too high \Rightarrow not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

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Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

↑
key

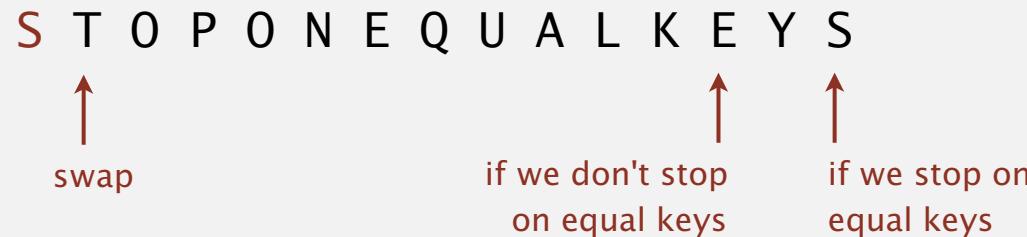
Duplicate keys

Mergesort with duplicate keys. Between $\frac{1}{2}N\lg N$ and $N\lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes **quadratic** unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system
implementation also have this defect



Duplicate keys: the problem

Mistake. Put all items equal to the partitioning item on one side.

Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

B A A B A B B **B** C C C A A A A A A A A A A **A**

Recommended. Stop scans on items equal to the partitioning item.

Consequence. $\sim N \lg N$ compares when all keys equal.

B A A B A **B** C C B C B A A A A A **A** A A A A A A

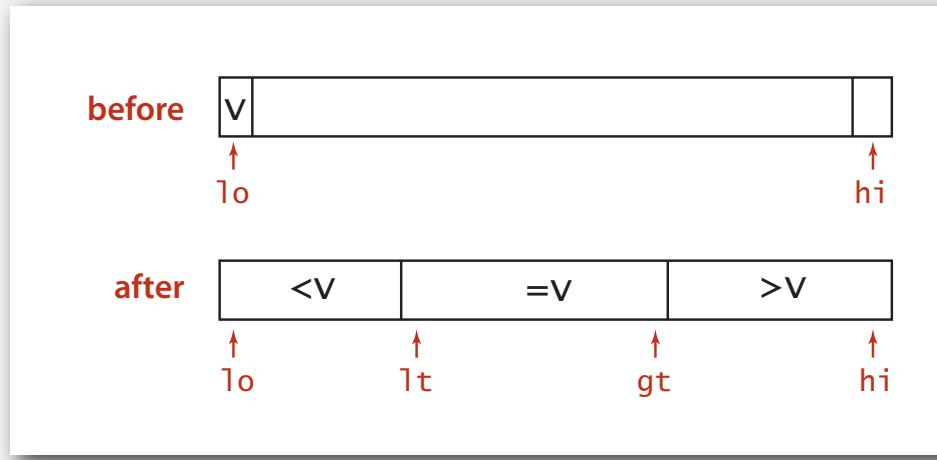
Desirable. Put all items equal to the partitioning item in place.

A A A **B** B B B C C C **A** A A A A A A A A A A A A

3-way partitioning

Goal. Partition array into 3 parts so that:

- Entries between lt and gt equal to partition item v .
- No larger entries to left of lt .
- No smaller entries to right of gt .



Dutch national flag problem. [Edsger Dijkstra]

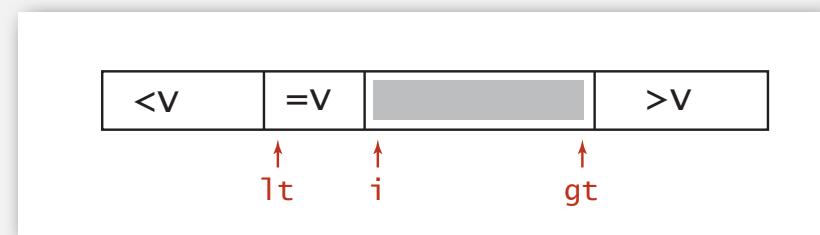
- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system `sort`.

Dijkstra 3-way partitioning demo

- Let v be partitioning item $a[lo]$.
- Scan i from left to right.
 - $(a[i] < v)$: exchange $a[lt]$ with $a[i]$; increment both lt and i
 - $(a[i] > v)$: exchange $a[gt]$ with $a[i]$; decrement gt
 - $(a[i] == v)$: increment i

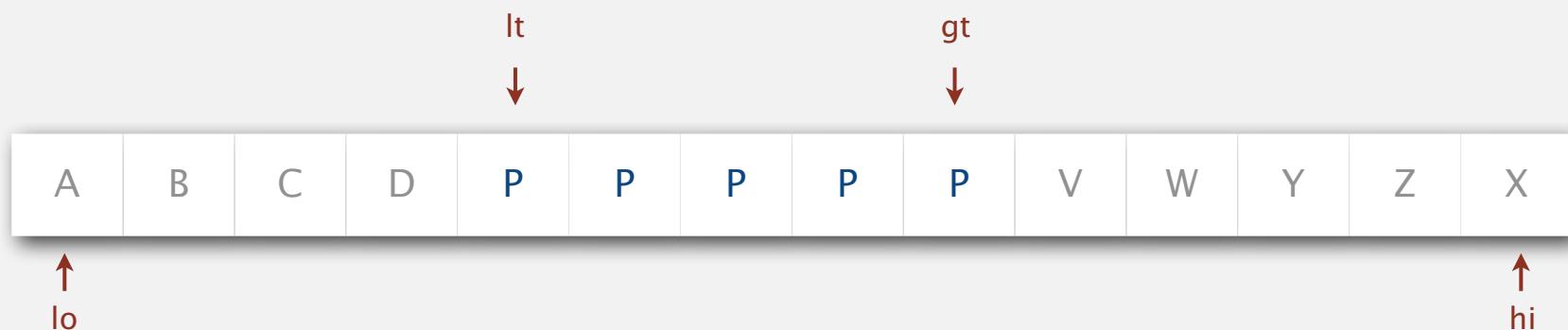


invariant

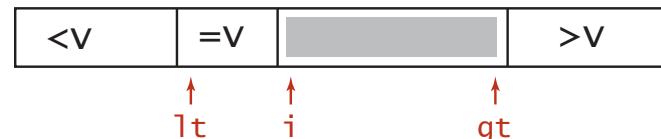


Dijkstra 3-way partitioning demo

- Let v be partitioning item $a[lo]$.
- Scan i from left to right.
 - $(a[i] < v)$: exchange $a[lt]$ with $a[i]$; increment both lt and i
 - $(a[i] > v)$: exchange $a[gt]$ with $a[i]$; decrement gt
 - $(a[i] == v)$: increment i



invariant



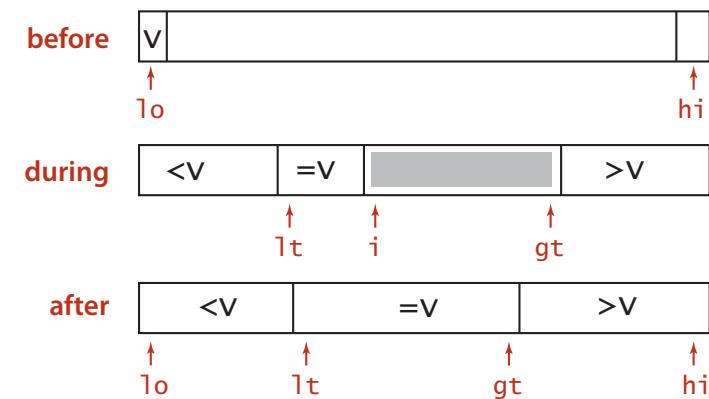
Dijkstra's 3-way partitioning: trace

lt	i	gt	a[]									
			0	1	2	3	4	5	6	7	8	9
0	0	11	R	B	W	W	R	W	B	R	R	W
0	1	11	R	B	W	W	R	W	B	R	R	W
1	2	11	B	R	W	W	R	W	B	R	R	W
1	2	10	B	R	R	W	R	W	B	R	R	W
1	3	10	B	R	R	W	R	W	B	R	R	W
1	3	9	B	R	R	B	R	W	B	R	R	W
2	4	9	B	B	R	R	R	W	B	R	R	W
2	5	9	B	B	R	R	R	W	B	R	R	W
2	5	8	B	B	R	R	R	W	B	R	R	W
2	5	7	B	B	R	R	R	R	B	R	R	W
2	6	7	B	B	R	R	R	R	B	R	R	W
3	7	7	B	B	B	R	R	R	R	R	R	W
3	8	7	B	B	B	R	R	R	R	R	R	W
3	8	7	B	B	B	R	R	R	R	R	R	W

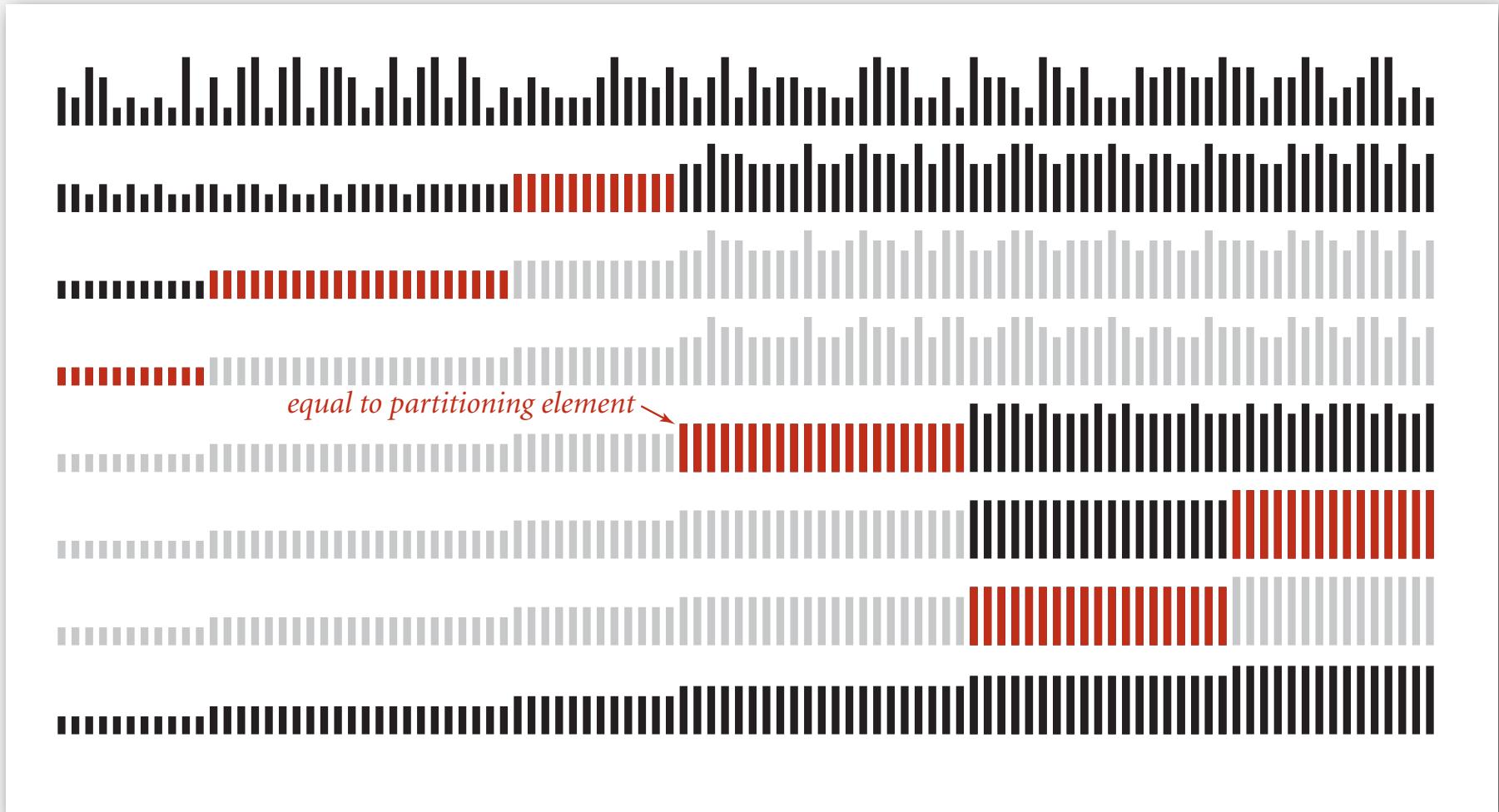
3-way partitioning trace (array contents after each loop iteration)

3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;
    int lt = lo, gt = hi;
    Comparable v = a[lo];
    int i = lo;
    while (i <= gt)
    {
        int cmp = a[i].compareTo(v);
        if      (cmp < 0) exch(a, lt++, i++);
        else if (cmp > 0) exch(a, i, gt--);
        else                i++;
    }
    sort(a, lo, lt - 1);
    sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Duplicate keys: lower bound

Sorting lower bound. If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$$\lg \left(\frac{N!}{x_1! x_2! \cdots x_n!} \right) \sim - \sum_{i=1}^n x_i \lg \frac{x_i}{N}$$

← *N lg N when all distinct;
linear when only a constant number of distinct keys*

compares in the worst case.

Proposition. [Sedgewick-Bentley, 1997]

proportional to lower bound

Quicksort with 3-way partitioning is **entropy-optimal**.

Pf. [beyond scope of course]

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

Algorithms

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2.3 QUICKSORT

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

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Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library. obvious applications
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

- Find the median.
- Identify statistical outliers. problems become easy once items are in sorted order
- Binary search in a database.
- Find duplicates in a mailing list.

- Data compression.
- Computer graphics. non-obvious applications
- Computational biology.
- Load balancing on a parallel computer.

- . . .

Java system sorts

[Arrays.sort\(\)](#).

- Has different method for each primitive type.
- Has a method for data types that implement Comparable.
- Has a method that uses a Comparator.
- Uses tuned quicksort for primitive types; tuned mergesort for objects.

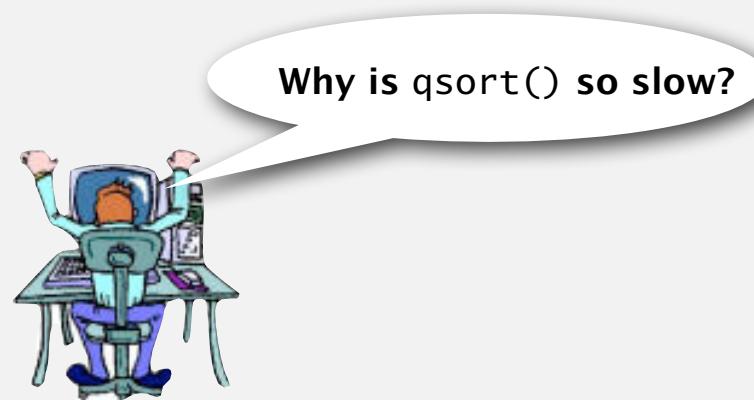
```
import java.util.Arrays;

public class StringSort
{
    public static void main(String[] args)
    {
        String[] a = StdIn.readStrings();
        Arrays.sort(a);
        for (int i = 0; i < N; i++)
            StdOut.println(a[i]);
    }
}
```

Q. Why use different algorithms for primitive and reference types?

War story (C qsort function)

AT&T Bell Labs (1991). Allan Wilks and Rick Becker discovered that a `qsort()` call that should have taken seconds was taking minutes.



At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.



Engineering a system sort

Basic algorithm = quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning.
- Partitioning item.
 - small arrays: middle entry
 - medium arrays: median of 3
 - large arrays: Tukey's ninther [next slide]

Engineering a Sort Function

JON L. BENTLEY
M. DOUGLAS McILROY
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.

SUMMARY

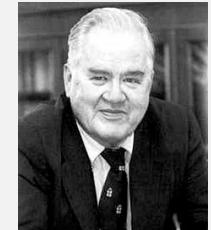
We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

Now widely used. C, C++, Java 6,

Tukey's ninther

Tukey's ninther. Median of the median of 3 samples, each of 3 entries.

- Approximates the median of 9.
- Uses at most 12 compares.



Q. Why use Tukey's ninther?

A. Better partitioning than random shuffle and less costly.

Achilles heel in Bentley-McIlroy implementation (Java system sort)

Q. Based on all this research, Java's system sort is solid, **right?**

A. No: a killer input.

- Overflows function call stack in Java and crashes program.
- Would take quadratic time if it didn't crash first.

```
% more 250000.txt
```

```
0
```

```
218750
```

```
222662
```

```
11
```

```
166672
```

```
247070
```

```
83339
```

```
...
```



250,000 integers
between 0 and 250,000

```
% java IntegerSort 250000 < 250000.txt
```

```
Exception in thread "main"
```

```
java.lang.StackOverflowError
```

```
at java.util.Arrays.sort1(Arrays.java:562)
```

```
at java.util.Arrays.sort1(Arrays.java:606)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
...
```

Java's sorting library crashes, even if
you give it as much stack space as Windows allows

System sort: Which algorithm to use?

Many sorting algorithms to choose from:

Internal sorts.

- Insertion sort, selection sort, bubblesort, shaker sort.
- Quicksort, mergesort, heapsort, samplesort, shellsort.
- Solitaire sort, red-black sort, splaysort, **Yaroslavskiy sort**, psort, ...

External sorts. Poly-phase mergesort, cascade-merge, oscillating sort.

String/radix sorts. Distribution, MSD, LSD, 3-way string quicksort.

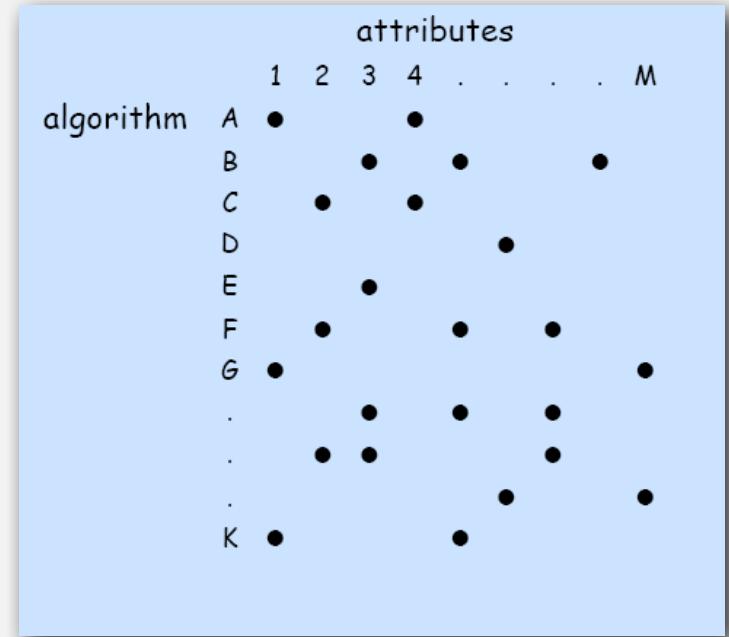
Parallel sorts.

- Bitonic sort, Batcher even-odd sort.
- Smooth sort, cube sort, column sort.
- GPUsort.

System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Is your array randomly ordered?
- Need guaranteed performance?



many more combinations of attributes than algorithms

Elementary sort may be method of choice for some combination.

Cannot cover **all** combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	✓		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	✓	✓	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	✓		?	?	N	tight code, subquadratic
merge		✓	$N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee, stable
quick	✓		$N^2 / 2$	$2N \ln N$	$N \lg N$	$N \log N$ probabilistic guarantee fastest in practice
3-way quick	✓		$N^2 / 2$	$2N \ln N$	N	improves quicksort in presence of duplicate keys
???	✓	✓	$N \lg N$	$N \lg N$	N	holy sorting grail

Algorithms

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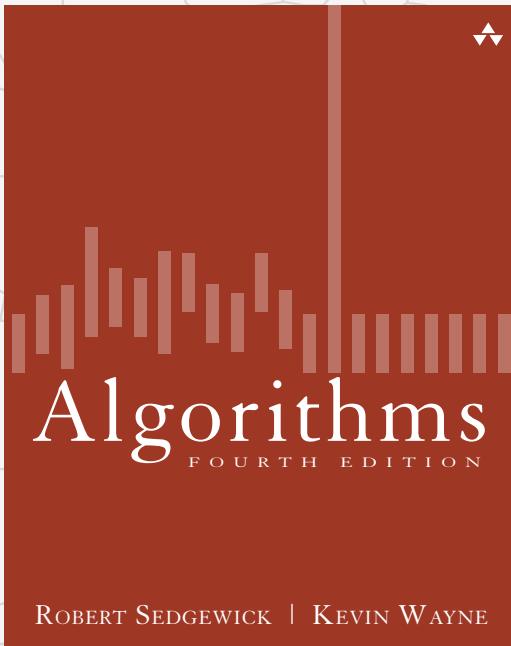
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