1. Suppose $\vec{x}, \vec{a} \in \mathbb{R}^n \Rightarrow \vec{a}^T\vec{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

Then $\frac{\partial(\vec{a}^T\vec{x})}{\partial x_1} = a_1$; $\frac{\partial(\vec{a}^T\vec{x})}{\partial x_2} = a_2$ ... $\frac{\partial(\vec{a}^T\vec{x})}{\partial x_k} = a_k$

Finally $\frac{\partial(\vec{a}^T\vec{x})}{\partial \vec{x}} = \left[\frac{\partial(\vec{a}^T\vec{x})}{\partial x_1} \quad \frac{\partial(\vec{a}^T\vec{x})}{\partial x_2} \quad \cdots \quad \frac{\partial(\vec{a}^T\vec{x})}{\partial x_n} \right]$

$$= [a_1 \; a_2 \; a_3 \cdots a_n]$$

$$= \vec{a}^T$$

2. Suppose $\vec{a} \in \mathbb{R}^n$; $\mathbb{A} \in \mathbb{R}^{n\times n}$. Let $f = \vec{a}^T\mathbb{A}\vec{a}$: scalar quantity

$$\frac{\partial(\vec{a}^T\mathbb{A}\vec{a})}{\partial \vec{a}} \equiv \frac{\partial f}{\partial \vec{a}} = \left[\frac{\partial f}{\partial a_1} \; \frac{\partial f}{\partial a_2} \cdots \frac{\partial f}{\partial a_k} \cdots \frac{\partial f}{\partial a_n} \right]$$

Expanding $f = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i A_{ij} a_j$

Then $\frac{\partial f}{\partial a_k} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial a_k}\left[a_i A_{ij} a_j\right]$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \frac{\partial}{\partial a_k}\left[a_i a_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\left[\frac{\partial a_i}{\partial a_k} a_j + a_i \frac{\partial a_j}{\partial a_k}\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\left[\delta_{ik} a_j + a_i \delta_{jk}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\delta_{ik} a_j + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\delta_{jk} a_i$$

$$= \sum_{j=1}^{n} A_{kj} a_j + \sum_{i=1}^{n} A_{ik} a_i$$

$$\frac{\partial f}{\partial a_k} = \sum_{i=1}^{n} a_i\left[A_{ki} + A_{ik}\right] \Rightarrow \frac{\partial f}{\partial \vec{a}} = \vec{a}^T\left[\mathbb{A} + \mathbb{A}^T\right]$$

3. Suppose $\vec{y}, \vec{w} \in \mathbb{R}^n$ both dependent of $\vec{z}$.

$$\vec{y}^T \vec{w} = \sum_{i=1}^{n} y_i w_i$$

$$\Rightarrow \frac{\partial(\vec{y}^T \vec{w})}{\partial z_k} = \sum_{i=1}^{n} \left[w_i \frac{\partial y_i}{\partial z_k} + y_i \frac{\partial w_i}{\partial z_k} \right]$$

$$\frac{\partial(\vec{y}^T \vec{w})}{\partial \vec{z}} = \vec{w}^T \frac{\partial \vec{y}}{\partial \vec{z}} + \vec{y}^T \frac{\partial \vec{w}}{\partial \vec{z}}$$

If $\vec{y} = \vec{w}$: $\quad \frac{\partial(\vec{y}^T \vec{y})}{\partial \vec{z}} = 2 \vec{y}^T \frac{\partial \vec{y}}{\partial \vec{z}}$.

Making the substitution: $\vec{y} = \vec{x} - A\vec{s}$ and $\vec{z} = \vec{s}$

$$\frac{\partial (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}} = 2 (\vec{x} - A\vec{s})^T \frac{\partial(\vec{x} - A\vec{s})}{\partial \vec{s}}$$

$$* \quad \frac{\partial(\vec{x} - A\vec{s})}{\partial \vec{s}} = - \frac{\partial(A\vec{s})}{\partial \vec{s}} = -A$$

$$\left\{ \begin{array}{ll} \vec{V} = A\vec{s} & V_i = \sum_k A_{ik} S_k \\ \frac{\partial V_i}{\partial s_j} = A_{ij} & \frac{\partial \vec{V}}{\partial \vec{s}} = A \end{array} \right.$$

Putting all together: $\quad \frac{\partial (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}} = -2 (\vec{x} - A\vec{s})^T A$

Transposing both sides: $\quad \frac{\partial (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}^T} = -2 A^T (\vec{x} - A\vec{s})$

Second derivative

$$\frac{\partial^2 (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}\, \partial \vec{s}^T} = -2 A^T \frac{\partial(\vec{x} - A\vec{s})}{\partial \vec{s}} \overset{\text{Using } (*)}{=} -2 A^T \left[-A \right]$$

$$\frac{\partial^2 (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}\, \partial \vec{s}^T} = 2 A^T A$$