Braiding project for Quantum Computation course

The best resource for this project is:
- [1] C. W. J. Beenakker, "Search for non-Abelian Majorana braiding statistics in superconductors," *SciPost Phys. Lect. Notes*, p. 15, 2020, doi: 10.21468/SciPostPhysLectNotes.15.

Here is another short introductory tutorial:
- https://topocondmat.org/w2_majorana/braiding.html

a) Qubits vs Fermions

The usual model of quantum computation is expressed in terms of qubits and the Pauli operators X, Y and Z. A single fermionic mode is also a two-level system (it is either occupied or not), but is quite different from a qubit. Two key features are fermionic parity conservation and the anti-commutation relations.

Parity conservation means that we can't make superpositions between states of even and odd fermionic numbers. Therefore, a single fermionic mode can't be used as a qubit. It is a two-level system, but we are not allowed to make superpositions between the two states.

The anti-commutation relations imply that fermionic operators are non-local, which means they do not map cleanly to the local Pauli gates.

Let's take system with two fermions, c_1 and c_2. The Hilbert space is 4-dimensional, but because of parity conservation, we can consider the two subspaces of even and odd total parity separately. Let's take the odd parity subspace to be concrete. This is a two-level system, spanned by the states $|{+}{-}\rangle$ (left mode is unoccupied (even parity) and the right mode is occupied (odd parity)) and $|{-}{+}\rangle$. We can write down the standard Pauli operators in this space, i.e. $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- Express the Pauli operators in terms of the fermions c_1 and c_2.
- Define four hermitian Majorana operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ with the mapping $c_k = \frac{\gamma_{2k-2} + i\gamma_{2k-1}}{2}$.
- What are the anti-commutation relations between the Majoranas?
- Express the Pauli operators in terms of Majorana operators.

The references [1], [2] and [3] may be useful.

b) Majorana Exchange gates

Let's take a step towards braiding and define Majorana exchange operators.
- Find unitaries B_{ij} that exchange the Majoranas γ_i and γ_j, i.e. it should act as

$$B_{ij} \gamma_i B_{ij}^\dagger = \gamma_j$$
$$B_{ij} \gamma_j B_{ij}^\dagger = -\gamma_i$$

and trivially on other operators. Express the gates in terms of γ_i and γ_j, as well as in the basis $|{+}{-}\rangle, |{-}{+}\rangle$.

- Check the square B_{ij}^2 of these operators. How are they related to the Pauli operators?

- Express the Hadamard gate in terms of exchange gates.

- Bonus problem: Do the same relations hold in the even parity sector?

c) Jordan-Wigner transform

While we have a map between Majorana operators and Pauli operators acting on a subspace of a specific parity, we need to create yet another map in order to look at the braiding protocol. We saw how four Majoranas can make a qubit. In the braiding protocol, an additional pair of Majoranas will be used to make an exchange. The braiding protocol therefore needs six Majoranas to work. Four to make a qubit, and two extra auxiliary Majoranas to make the braid.

The braiding protocol in [1] is expressed in terms of four Majoranas. One pair is 'half' the qubit and one pair is the auxiliary ones. To interpret the result in terms of a qubit, an additional pair must be added. See Figure 8 in Ref.[4] to see the protocol with all six Majoranas.

To implement the Hamiltonian numerically, we should map the fermions to to Pauli operators and tensor products of those. We already did so, but that was only mapping the operators in a specific parity sector. Now we want to be more general and express it in the full Hilbert space, in a version which is simple to implement numerically.

The general form that the Hamiltonian takes during the braiding protocol is

$$H(t) = \sum_{j=1}^{3} \Delta_j(t)\gamma_0\gamma_j$$

Write down the representation of the Majorana operators in terms of tensor products of local Pauli or spin operators. See Ref.[5] or
https://en.wikipedia.org/wiki/Jordan%E2%80%93Wigner_transformation

d) Implement the Hamiltonian numerically

With the operators expressed as Pauli operators, they can be implemented numerically by a tensor product, usually denoted as 'kron' in most programming languages.

We also need to express the parameters $\Delta_j(t)$ as a function of time. One suggestion is to parameterize it using the following parameters:

- Δ_{max} is a large value that the coupling takes on in those steps of the protocol where it is turned on.
- Δ_{min} is the value it takes when it is switched off (in an ideal experiment this is zero)
- s some parameter that determines how fast one changes from Δ_{min} to Δ_{max}. The particular shape of the step is not that important, but the steepness should be tuneable.
- T is the total time of the protocol.

With $H(t)$ implemented, study how the energy spectrum evolves during the protocol. Does it match expectations?

e) Implement braiding protocol

Now we are ready to solve the Schrödinger equation

$$\partial_t|\psi_t\rangle = iH_t|\psi_t\rangle.$$

The simplest way to do so is to approximate

$$dt\partial_t|\psi_t\rangle \approx |\psi_{t+dt}\rangle - |\psi_t\rangle$$

and express the state at $t + dt$ in terms of the state at t. Then given an initial state, take short steps until reaching the final time T. The numerical error can be monitored by looking at the norm of the state, i.e. $\langle\psi_t|\psi_t\rangle$. How much this deviates from 1 gives a measure of how accurate the solution is.

- Find parameters so that the protocol succeeds and implements the desired exchange gate.
- Calculate the Berry phase with the Berry connection $A = i\langle d\psi|\psi\rangle$.
- Measure parities during the protocol. Do they behave as expected?

f) Errors

Investigate the effect of different kinds of errors.

Bibliography

[1] C. W. J. Beenakker, "Search for non-Abelian Majorana braiding statistics in superconductors," *SciPost Phys. Lect. Notes*, p. 15, 2020, doi: 10.21468/SciPostPhysLectNotes.15.

[2] S. B. Bravyi and A. Y. Kitaev, "Fermionic Quantum Computation," *Annals of Physics*, vol. 298, no. 1, pp. 210–226, May 2002, doi: 10.1006/aphy.2002.6254.

[3] K. Vilkelis, A. L. R. Manesco, J. D. Torres Luna, S. Miles, M. Wimmer, and A. R. Akhmerov, "Fermionic quantum computation with Cooper pair splitters," *SciPost Physics*, vol. 16, no. 5, May 2024, doi: 10.21468/scipostphys.16.5.135.

[4] A. Tsintzis, R. S. Souto, K. Flensberg, J. Danon, and M. Leijnse, "Majorana Qubits and Non-Abelian Physics in Quantum Dot–Based Minimal Kitaev Chains," *PRX Quantum*, vol. 5, no. 1, Feb. 2024, doi: 10.1103/prxquantum.5.010323.

[5] K. Chhajed, "From Ising Model to Kitaev Chain: An Introduction to Topological Phase Transitions," *Resonance*, vol. 26, no. 11, pp. 1539–1558, Nov. 2021, doi: 10.1007/s12045-021-1261-6.