UNIVERSITY OF OSLO

Project report title

Special Curriculum

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GitHub Link

October 9, 2025

Temporary report structure TODO: Fix citations and references

1 Introduction

Superconductivity, as described by Bardeen-Cooper-Schrieffer (BCS) theory, arises from the formation of Cooper pairs. These are bound pair of electrons with opposite momentum and spin. The pairing leads to an energy gap in the excitation spectrum, which protects the superconducting state from low-energy perturbations. Whem superconductivity is induced in hybrid systems, such as semiconductor proximitized by superconductors, it can give rise to emergent quasiparticle excitations with exotic properties. Majorana zero modes, are one such excitation that can appear at the ends of one-dimensional topological superconductors. These modes are predicted to obey non-Abelian exchange statistics, which makes them suitable for fault-tolerant quantum computation, meaning: information is stored in the joint parity of spatially separated modes, rendering it insesitive to local perturbations.

To identify and manipulate MZMs, it is important to examine their non-local nature and the conditions under which they arise. The quantum information encoded in these modes is associated with fermionic parity rather than local charge, motivating measurements that distinguish between even and odd parity ground states without collapsing the non-local encoding. This conceptual framework directly informs the design and interpretation of experiments in minimal realizations of Majorana systems, where one seeks to reproduce the essential physics in a controlable, highly tunable setup.

In this report, we investigate the Poor Man's Majoranas (PMMs) in a double quantum dot-suprcondcutor-quantum dot (QD-SC-QD) system. PMMs provide a bottom-up, minimal Kitaev chain realization, capturing the key features of MZMs while remaining experimentally accessible and numerically tractable. By combining analytical and numerical approaches, we explore the emergence of zero-energy modes, their spatial localization, and the robustness of their degeneracy under variations in system parameters, providing insight into the feasiblity of realizing topologically protected qubits in such minimal systems.

2 Theoretical Background: The superconducting State

Notes from note section 1 and note section 2 to build necessary background.

2.1 From Electron Attraction to the Energy Gap:

Superconductivity is characterized by the complete loss of electrical resistance below a critical temperature T_c, and the expulsion of magnetic fields (the Meissner effect), which demonstrates that it represents a distinct thermodynamic phase rather than a perfect conductor.
Londong (1937) first introduced a phenomenological description in terms of a macroscopic quantum wavefunction $\psi(r)$, whose rigidity gives rise to the London equations and an exponential decay of magnetic fields inside the superconductor over a characteristic length scale, the London penetration depth $\lambda_L = \sqrt{m/(\mu_0 e^2 |\psi|^2)}$ CITATION[CambridgeBCS.pdf chapter 2].
Microscopically, the origin of superconductivity lies in an effective attraction between electrons mediated by lattice vibrations (phonons). This leads to the formation of Cooper pairs, which are bound states of two electrons with opposite momenta and spins. The exchange of a virtual phonon leads to an attractive interaction for energy transfers smaller than the phonon frequency ω_D,

$$V_{eff}(q,\omega) = \frac{|g_q|^2}{\omega^2 - \omega_q^2}$$

where g_q is the electron-phonon coupling strength zand $\omega = (\epsilon_k - \epsilon_k^{'})$, and the ϵ_α are the single-particle energies. The isotope effect, where T_c varies inversely with the square root of the ionic mass, provides strong evidence for the phonon-mediated pairing mechanism.CITATION[CambridgeBCS.pdf chapter 3].
In his seminal work, Cooper (1956) demonstrated that an arbitrarily weak attractive interaction destabilizes the Fermi sea: two eelectrons of opposite spin and momentum can form a bound state with energy below the Fermi level. The pair is called a Cooper pair, and the operator we use to describe the creation of a Cooper pair is;

$$\Lambda^\dagger = \sum_k \phi(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger$$

and the self-consistency condition

$$1 = g_0 N(0) \int_0^{\omega_D} \frac{d\epsilon}{\sqrt{2\epsilon^2 + E}}$$

yields the binding energy of the Cooper pair:

$$E_B = -2\omega_D e^{-2/(g_0 N(0))}$$

where $N(0)$ is the density of states at the Fermi level and g_0 is the effective interaction strength CITATION[CambridgeBCS.pdf chapter 3.2 and BCS.pdf] Even this simple two-particle result implies that the Fermi surface is unstable toward pair condensation.

Building on Cooper's insight, Bardeen, Cooper, and Schrieffer (1957) developed a comprehensive microscopic theory of superconductivity CITATION[BCS.pdf]. . The VCS ground state is a coherent superpositon of occupied and unoccupied pair states:

$$|\Psi_{BCS}\rangle = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |0\rangle$$

with normalization $|u_k|^2 + |v_k|^2 = 1$. Minimization of the free energy yields the gap equation:

$$1 = \frac{g_0}{V} \sum_{|\epsilon_k| < \omega_D} \frac{1}{2E_k} \tanh\left(\frac{E_k}{2k_B T}\right)$$

where the quasiparticle excitation spectrum is:

$$E_k = \sqrt{\epsilon_k^2 + \Delta^2}$$

At zero temperature, the gap at the Fermi surface is:

$$\Delta(0) = 2\omega_D e^{-1/(g_0 N(0))}$$

and obeys the universal relation $2\Delta(0)/k_B T_c \approx 3.53$.
The energy gap Δ represents the minimum energy required to break a Cooper pair, explainingt he exponential suppression of heat capacity and resistivity at low temperatures. Physically, it quantifies the rigidity of the condensate. Any single-particle excitation costs a funite energy, enforcing long-range phase coherence.
Superconducting excitations are Bogoliubov quasiparticles, coherent superpositions of electron and hole states, which naturally leads to the Bogoliubov-de Gennes (BdG) formalism. This formalism genereralizes the BCS mean-field theory to spatially inhomogeneous systems and forms the basis for the study of superconducting heterostructures, including those hosting Majorana modes.

2.2 The Bogoliubov-de Gennes Formalism:

The microscopic BCS theory descirbes superconductivity as a coherent condensate of Cooper pairs, but it assumes a uniform order parameter Δ. To describe spatially varying superconducting states, such as those at interfaces or in the presence of impurities, we must generalize the BCS mean-field Hamiltonian to include position dependence. This leads us to the Bogoliubov-de Gennes (BdG) formalism CITATION[CambridgeBCS.pdf chapter 14 and BCS.pdf].
Starting from the BCS mean-field Hamiltonian,

$$H_{MF} = \int d^3r \left[\sum_\sigma \psi_\sigma^\dagger(r) H_0 \psi_\sigma(r) + (\Delta(r)\psi_\uparrow^\dagger(r)\psi_\downarrow^\dagger(r) + h.c.) \right]$$

where $H_0 = \frac{p^2}{2m} - \mu + V(r)$ describes the normal-state single particle dynamics, and $\Delta(r)$ is the superconducting pair potential. We can diagonalize this Hamiltonian by introducing Bogoliubov through canonical transformations:

$$\psi_\uparrow(r) = \sum_n [u_n(r)\gamma_{n\uparrow} - v_n^*(r)\gamma_{n\downarrow}^\dagger], \quad \psi_\downarrow(r) = \sum_n [u_n(r)\gamma_{n\downarrow} + v_n^*(r)\gamma_{n\uparrow}^\dagger]$$

Substituting and deminding that the quasiparticle operators $\gamma_{n\sigma}$ diagonalize the Hamiltonian leads to the BdG equations:

$$\begin{pmatrix} H_0(r) & \Delta(r) \\ \Delta^*(r) & -H_0^*(r) \end{pmatrix} \begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix} = E_n \begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix}$$

Each eigenstate $(u_n(r), v_n(r))$ represents a Bogoliubov quasiparticle, a coherent superposition of electron and hole amplitudes. The corresponding excitation energies $E_n \geq 0$ define the superconducting quasiparticle spectrum.
Because both particle hole components are included, the BdG Hamiltonain possesses an intrinsic particle-hole symmetry.

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG}, \quad \Xi = \tau_x K$$

where τ_x acts in particle-hole space and K is complex conjugation. This symmetry enforces that the spectrum is symmetric about zero energy: if E_n is an eigenvalue, so is $-E_n$.
This doubling of the Hilbert space is not redundant, it reflects the physical reality that the superconducting condensate does not conserve particle number, but rather fermion parity. The eigenstates at $\pm E_n$ correspond to the same physical excitation, and a zero-energy solution ($E_n = 0$) indicates a self-conjugate quasiparticle, $\gamma = \gamma^\dagger$, which is the defining property of a

Majorana modeCITATION[Topocondmat.org].
The BdG framework therefor bridges microscopic BCS theory and modern topological superconductivity. It provides a compact description of inhomogeneous superconductors, quantum dots coupled to superconducting leads, and low dimensional systems where topological phases can emerge.

2.3 Hybrid Systems and the Proximity Effect:

Explain how a semiconductor can inherit superconducting properties from a nearby superconductor. Define Andreev reflection and the formation of ABS as an important phenomena in hybrid systems.

3 Engineering Majorana Modes in a Minimal System

Connect general theory to specific PMM platform, notes note section 4 and note section 5.2

3.1 The Kitaev Chain Model:

Introduce the Kitaev model Hamiltonian. Explain its key features (p-wave pairing) and how it leads to a topological phase with protected, zero energy Majorana edge states.

3.2 The QD-SC-QD System as an Effective Kitaev Chain:

Explain the mapping: The two QDs are the sites, Elastic cotunneling is the hopping term, CAR is the p-wave pairing. See note section 5.2 for details.

4 Analytical and Numerical Investigation of PMMs

Present my work and demonstrate how to apply theory to a concrete problem.

4.1 Single particle model:

4.1.1 The Model Hamiltonian:

Present the minimal Hamiltonian for the QD-SC-QD system that i have in my notes note section 5.2.2, for instance Eq 3.54 from chap3.pdf. Define all the terms clearly. This will be the Hamiltonian I use.

$$H = \begin{pmatrix} \epsilon_L & t & 0 & \Delta \\ t & \epsilon_R & -\Delta & 0 \\ 0 & -\Delta & -\epsilon_L & -t \\ \Delta & 0 & -t & -\epsilon_R \end{pmatrix}$$

Where ϵ_L and ϵ_R are the energy levels of the left and right dots, t is the elastic cotunneling amplitude, and Δ is the CAR amplitude. The basis is $(d_L, d_R, d_L^\dagger, d_R^\dagger)$, called the Nambu basis. The operators $d_L^\dagger$ and $d_R^\dagger$ create an electron in the left and right dot, respectively.

4.1.2 Analytical Work - Finding the Sweet Spot:

Solve the Hamiltonian for the simplest case: two indentical dots with their energy levels at zero ($\epsilon_L = \epsilon_R = 0$). I can write the $4x4$ matrix in the Nambu basis $\Psi^\dagger = (d_L, d_R, d_L^\dagger, d_R^\dagger)$. Show analytically that the condition for having two zero energy solutions (the Majoranas) is at $|t| = |\Delta|$.This proves i understand the sweet spot.

Solving the eigenvalue problem $H\Psi = E\Psi$ gives the characteristic polynomial:

$$\det(H - EI) = \begin{pmatrix} \epsilon_L - E & t & 0 & \Delta \\ t & \epsilon_R - E & -\Delta & 0 \\ 0 & -\Delta & -\epsilon_L - E & -t \\ \Delta & 0 & -t & -\epsilon_R - E \end{pmatrix} = 0$$

$$= E^4 - E^2 a + b = 0$$

where

$$a = \epsilon_L^2 + \epsilon_R^2 + 2(t^2 + \Delta^2)$$

and

$$b = (t^2 - \epsilon_L \epsilon_R - \Delta^2)^2$$

The quadratic formula for E^2 gives:

$$E^2 = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

The discriminant simplifies to:

$$a^2 - 4b = [(\epsilon_L + \epsilon_R)^2 + 4t^2][(\epsilon_L - \epsilon_R)^2 + 4\Delta^2]$$

and the eigenvalues are $\pm\sqrt{E^2}$. For zero energy solutions, we need $b = 0$, which requires:

$$t^2 - \epsilon_L \epsilon_R - \Delta^2 = 0$$

At $\epsilon_L = \epsilon_R = 0$, this reduces to the sweet spot condition:

$$|t| = |\Delta|$$

With the conditions we obtained from setting $b = 0$, a becomes $2(t^2+\Delta^2) = 4t^2$.
With $b = 0$, the quadratic formula simplifies to $E^2(E^2 - a) = 0$, giving us two solutions:

$$E^2 = 0 \quad E^2 = a = 4t^2$$

Therefore the four eigenvalues are:

$$E = \{0, 0, 2|t|, -2|t|\}$$

These eigenvalues have corresponding eigenvectors.
For $E = 0$, the eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

and for $E = 2|t|$, the eigenvectors are:

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

The zero-mode eigenvectors v_1 and v_2 correspond to the Majorana operators. The two zero-energy eigenvectors, v_1 and v_2, represent states that are equal superpositions of particle and hole operators on a single dot. In operator form, they correspond to self-conjugate combinations:

$$\gamma_1 \propto d_L + d_L^\dagger, \quad \gamma_2 \propto d_R - d_R^\dagger$$

which satisfy the Majorana condition $\gamma = \gamma^\dagger$. These are the Poor Man's Majorana modes. They are spatially separated zero modes, localized entirely on the left and right dots at the sweet spot $|t| = |\Delta|$ and $\epsilon_L = \epsilon_R = 0$. Their degeneracy at zero energy encodes the nonlocal fermionic parity of the two-dot system, which is the resource we ultimately want for quantum information.
By contrast, the finite-energy solutions

$$E = \pm 2|t|, \quad v_3 \propto (-1, 1, 1, 1)^T, \quad v_4 \propto (1, 1, -1, 1)^T$$

are delocalized Bogoliubov quasiparticles. They form the first excited states above the ground state and define the energy gap protecting the Majorana subspace. Physically, this gap is crucial: as long as it remains finite, the Majorana modes cannot hybridize with bulk excitations, and the zero-energy subspace is robust against small perturbations.

4.1.3 Numerical Simulations - Visualize the Emergence of Majoranas:

Build and diagonalize the Hamiltonian numerically. Band Structure: Plot the two lowest positive energy eigenvalues as a function of a dot's energy ϵ_L (keeping $\epsilon_R = 0$) and $t = \Delta$. Reproduce the plot from Fig 3.5a in chap3.pdf showing two states remaining at zero energy, showcasing the symmetry. Wavefunctions: At the sweet spot ($t = \Delta$ and $\epsilon_L = \epsilon_R = 0$), find the eigenvectors for the two zero-mode energy states. Plot the magnitude of the components. I should be able to see that one zero mode is localized entirely on the left dot and the other is entirely on the right dot, confirming they are spatially separated Majoranas.

4.1.4 Numerical Work - Probing Protection and Robustness:

Detuning t and Δ: Vary t and Δ away from the sweet spot. Plot the lowest energy eigenvalue as I varyt he ratio $\frac{t}{\Delta}$ away from 1. This will show that the zero-energy states immediately split and acquire a finite energy gap, demonstrating their lack of full topological protection. Detuning Dot Energies: Vary ϵ_L and ϵ_R away from zero while keeping $t = \Delta$. The degeneracy should be lifted quadratically. My results here should quantify the robustness of the PMMs to local perturbations. I can conclude that while they are not fully protected like in a true topological system, the degeneracy is protected against certain local perturbations to first order.

4.2 Many-body model:

(Including the Coulomb interaction U)

4.2.1 The Model Hamiltonian:

Present the many-body Hamiltonian for the QD-SC-QD system, including the Coulomb interaction term U. Define all the terms clearly. This will be the Hamiltonian I use.

We have to leave the BdG formalism and work in the many-body basis. This is because the BdG formalism is a mean-field, single-particle approach

that cannot capture electron-electron interactions like the Coulomb repulsion U. The interaction term we use in order to describe the non-local Coulomb interaction betweem the PMMs is: $H_U = U_{LR} n_L n_R$, where $n_\alpha = d_\alpha^\dagger d_\alpha$ is the number operator for dot α.
The full Hamiltonian for the QD-SC-QD system with Coulomb interaction is:

$$H = \sum_{\alpha=L,R} \epsilon_\alpha d_\alpha^\dagger d_\alpha + t(d_L^\dagger d_R + d_R^\dagger d_L) + \Delta(d_R d_L + d_L^\dagger d_R^\dagger) + U_{LR} n_L n_R$$

where ϵ_α are the energy levels of the left and right dots, t is the elastic cotunneling amplitude, Δ is the CAR amplitude, and U_{LR} is the non-local Coulomb interaction between electrons on the two dots. The basis states for the two-dot system are: $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$, $|1,1\rangle$, where $|n_L, n_R\rangle$ indicates the occupation of the left and right dots.

We can now proceed to construct the Hamiltonian matrix in this many-body basis, where it becomes:

$$H = \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & \epsilon_R & t & 0 \\ 0 & t & \epsilon_L & 0 \\ \Delta & 0 & 0 & \epsilon_L + \epsilon_R + U_{LR} \end{pmatrix}$$

4.2.2 Analytical Work - Exploring the many body Hamiltonian:

The many-body Hamiltonian above is written in the basis order:

$$|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$$

Due to order and practicality, we can try to rewrite the Hamiltonian in a block-diagonal manner. To do so, we can first investigat the matrix elements we have. The even sector is spanned by the states $|0,0\rangle$ and $|1,1\rangle$, which correspond to the matrix elements $H_{11}, H_{14}, H_{41}, H_{44}$. The odd sector is spanned by the states $|0,1\rangle$ and $|1,0\rangle$, which correspond to the matrix elements $H_{22}, H_{23}, H_{32}, H_{33}$.
If we reorder the basis as [even, odd] = $[|0,0\rangle, |1,1\rangle, |0,1\rangle, |1,0\rangle]$, we obtain:

$$H = \begin{pmatrix} 0 & \Delta & 0 & 0 \\ \Delta & \epsilon_L + \epsilon_R + U_{LR} & 0 & 0 \\ 0 & 0 & \epsilon_R & t \\ 0 & 0 & t & \epsilon_L \end{pmatrix}$$

This is now block-diagonal, with the even sector in the top-left $2x2$ block and the odd sector in the bottom-right $2x2$ block.

In other words, we have:

$$H = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}$$

with:

$$H_{even} = \begin{pmatrix} 0 & \Delta \\ \Delta & \epsilon_L + \epsilon_R + U_{LR} \end{pmatrix}, \quad H_{odd} = \begin{pmatrix} \epsilon_R & t \\ t & \epsilon_L \end{pmatrix}$$

Our next step is to find an expression for the eigenvalues of each sector. For simplicity we will call $S = \epsilon_L + \epsilon_R + U_{LR}$. Starting with the even sector, we solve the characteristic polynomial:

$$\det(H_{even} - EI) = \begin{pmatrix} -E & \Delta \\ \Delta & S - E \end{pmatrix} = 0$$

This gives us the quadratic equation:

$$E^2 - SE + (\Delta^2) = 0$$

Using the quadratic formula, we find the eigenvalues for the even sector:

$$E_{even} = \frac{S \pm \sqrt{S^2 + 4\Delta^2}}{2} = \frac{\epsilon_L + \epsilon_R + U_{LR} \pm \sqrt{(\epsilon_L + \epsilon_R + U_{LR})^2 + 4\Delta^2}}{2}$$

Next, we solve the characteristic polynomial for the odd sector:

$$\det(H_{odd} - EI) = \begin{pmatrix} \epsilon_R - E & t \\ t & \epsilon_L - E \end{pmatrix} = 0$$

This gives us the quadratic equation:

$$(\epsilon_R - E)(\epsilon_L - E) - t^2 = 0$$

$$E^2 - (\epsilon_R - \epsilon_L)E + \epsilon_R\epsilon_L - t^2 = 0$$

Using the quadratic formula, we find the eigenvalues for the odd sector:

$$E_{odd} = \frac{(\epsilon_L + \epsilon_R) \pm \sqrt{(\epsilon_L - \epsilon_R)^2 + 4t^2}}{2}$$

4.2.3 Analytical Work - Finding the Conditions for a Many-body Sweet Spot:

To find the conditions for having two degenerate ground states (one from each sector), we set the lowest eigenvalue from each sector equal to each other:

$$E_{\text{even,min}} = E_{\text{odd,min}}$$

4.2.4 How Majorana like?

This section is per now just notes: Fill in the blanks along the way.
$|e\rangle, |o\rangle$ are the ground states of the even and odd sectors, respectively. Find them:

Three (four) things to look for to see if we found topologically protected Majoranas:
1) Degeneracy of ground states: We need $E_{even,min} = E_{odd,min} \quad \delta E = 0$
2-3) With the number operators n_L and n_R, we can calculate the expectation values $\langle e| n_{L(R)} |e\rangle$ and $\langle o| n_{L(R)} |o\rangle$. The difference in occupation between the two ground states on each dot should be minimal, ideally zero. This indicates that the Majorana modes are non-local and do not localize charge on either dot.
4) Find the Majorana polarization $MP = 1$

5 Discussion and Conclusion

5.1 Synthesize my Findings:

Summarize what we learned from the calculations. " The analysis confirmed that zero-energy Majorana modes emerge under fine-tuned conditions... Numerical simulations revealed that this degeneracy is fragile and splits when deviating from the sweet spot, quantifying the limited protection of PMMs."

5.2 Connect to Broader Context:

Briefly discuss how the properties you explored (the energy splitting, wavefunction overlap) would affect potential braiding operations (note section 5.2.3) and what this means for using PMMs in quantum computing.

5.3 Future Directions:

Suggest next steps for a full thesis. This include modeling disorder, investigating longer chains (three or more dots), or numerically simulate braiding protocol.