UNIVERSITY OF OSLO

Project report title

Special Curriculum

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Temporary report structure

1 Introduction

Quick introduction of the promise of quantum computing and the major hurdle of decoherence. This is where I introduce topological quantum computation as a potential solution, highlighting its reliance on non-local, protected qubits.
Goal: Realizing MZMs
Challenge: Mention the top-down approach using nanowires and its experimental difficulties (notes from note section 5.1)
Alternative: Introduce the bottom-up approach QD-SC-QD systems to create a minimal, highly tunable Kitaev chain. Define the concept of PMMs.
Research Question: Objective: "This report will provide the theoretical background for superconductivity in hybrid systems and present an analytical and numerical investigation into the emergence and robustness of PMMs in a double QD system".

2 Theoretical Background: The superconducting State

Notes from note section 1 and note section 2 to build necessary background.

2.1 From Electron Attraction to the Energy Gap:

Short summary of Cooper pair formation, BCS theory, and the concept of the superconducting gap.

2.2 The Bogoliubov-de Gennes Formalism:

Introduce the BdG Hamiltonian as the mathematical tool for describing quasiparticles in systems where superconductivity varies in space (like at an interface). Important for numerical work later.

2.3 Hybrid Systems and the Proximity Effect:

Explain how a semiconductor can inherit superconducting properties from a nearby superconductor. Define Andreev reflection and the formation of ABS as an important phenomena in hybrid systems.

3 Engineering Majorana Modes in a Minimal System

Connect general theory to specific PMM platform, notes note section 4 and note section 5.2

3.1 The Kitaev Chain Model:

Introduce the Kitaev model Hamiltonian. Explain its key features (p-wave pairing) and how it leads to a topological phase with protected, zero energy Majorana edge states.

3.2 The QD-SC-QD System as an Effective Kitaev Chain:

Explain the mapping: The two QDs are the sites, Elastic cotunneling is the hopping term, CAR is the p-wave pairing. See note section 5.2 for details.

4 Analytical and Numerical Investigation of PMMs

Present my work and demonstrate how to apply theory to a concrete problem.

4.1 The Model Hamiltonian:

Present the minimal Hamiltonian for the QD-SC-QD system that i have in my notes note section 5.2.2, for instance Eq 3.54 from chap3.pdf. Define all the terms clearly. This will be the Hamiltonian I use.

$$H = \begin{pmatrix} \epsilon_L & t & 0 & \Delta \\ t & \epsilon_R & -\Delta & 0 \\ 0 & -\Delta & -\epsilon_L & -t \\ \Delta & 0 & -t & -\epsilon_R \end{pmatrix}$$

Where ϵ_L and ϵ_R are the energy levels of the left and right dots, t is the elastic cotunneling amplitude, and Δ is the CAR amplitude. The basis is $(d_L, d_R, d_L^\dagger, d_R^\dagger)$, called the Nambu basis. The operators $d_L^\dagger$ and $d_R^\dagger$ create an electron in the left and right dot, respectively.

4.2 Analytical Work - Finding the Sweet Spot:

Solve the Hamiltonian for the simplest case: two indentical dots with their energy levels at zero ($\epsilon_L = \epsilon_R = 0$). I can write the 4x4 matrix in the Nambu basis $\Psi^\dagger = (d_L, d_R, d_L^\dagger, d_R^\dagger)$. Show analytically that the condition for having two zero energy solutions (the Majoranas) is at $|t| = |\Delta|$. This proves i understand the sweet spot.

Solving the eigenvalue problem $H\Psi = E\Psi$ gives the characteristic polynomial:

$$\det(H - EI) = \begin{pmatrix} \epsilon_L - E & t & 0 & \Delta \\ t & \epsilon_R - E & -\Delta & 0 \\ 0 & -\Delta & -\epsilon_L - E & -t \\ \Delta & 0 & -t & -\epsilon_R - E \end{pmatrix} = 0$$

$$= E^4 - E^2 a + b = 0$$

where

$$a = \epsilon_L^2 + \epsilon_R^2 + 2(t^2 + \Delta^2)$$

and

$$b = (t^2 - \epsilon_L \epsilon_R - \Delta^2)^2$$

The quadratic formula for E^2 gives:

$$E^2 = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

The discriminant simplifies to:

$$a^2 - 4b = [(\epsilon_L + \epsilon_R)^2 + 4t^2][(\epsilon_L - \epsilon_R)^2 + 4\Delta^2]$$

and the eigenvalues are $\pm\sqrt{E^2}$. For zero energy solutions, we need $b = 0$, which requires:

$$t^2 - \epsilon_L \epsilon_R - \Delta^2 = 0$$

At $\epsilon_L = \epsilon_R = 0$, this reduces to the sweet spot condition:

$$|t| = |\Delta|$$

With the conditions we obtained from setting $b = 0$, a becomes $2(t^2 + \Delta^2) = 4t^2$.

With $b = 0$, the quadratic formula simplifies to $E^2(E^2 - a) = 0$, giving us two solutions:

$$E^2 = 0 \quad E^2 = a = 4t^2$$

Therefore the four eigenvalues are:

$$E = \{0, 0, 2|t|, -2|t|\}$$

These eigenvalues have corresponding eigenvectors.
For $E = 0$, the eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

and for $E = 2|t|$, the eigenvectors are:

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

The zero-mode eigenvectors v_1 and v_2 correspond to the Majorana operators. The two zero-energy eigenvectors, v_1 and v_2, represent states that are equal superpositions of particle and hole operators on a single dot. In operator form, they correspond to self-conjugate combinations:

$$\gamma_1 \propto d_L + d_L^\dagger, \quad \gamma_2 \propto d_R - d_R^\dagger$$

which satisfy the Majorana condition $\gamma = \gamma^\dagger$. These are the Poor Man's Majorana modes. They are spatially separated zero modes, localized entirely on the left and right dots at the sweet spot $|t| = |\Delta|$ and $\epsilon_L = \epsilon_R = 0$. Their degeneracy at zero energy encodes the nonlocal fermionic parity of the two-dot system, which is the resource we ultimately want for quantum

information.

By contrast, the finite-energy solutions

$$E = \pm 2|t|, \quad v_3 \propto (-1, 1, 1, 1)^T, \quad v_4 \propto (1, 1, -1, 1)^T$$

are delocalized Bogoliubov quasiparticles. They form the first excited states above the ground state and define the energy gap protecting the Majorana subspace. Physically, this gap is crucial: as long as it remains finite, the Majorana modes cannot hybridize with bulk excitations, and the zero-energy subspace is robust against small perturbations.

4.3 Numerical Simulations - Visualize the Emergence of Majoranas:

Build and diagonalize the Hamiltonian numerically. Band Structure: Plot the two lowest positive energy eigenvalues as a function of a dot's energy ϵ_L (keeping $\epsilon_R = 0$) and $t = \Delta$. Reproduce the plot from Fig 3.5a in chap3.pdf showing two states remaining at zero energy, showcasing the symmetry. Wavefunctions: At the sweet spot ($t = \Delta$ and $\epsilon_L = \epsilon_R = 0$), find the eigenvectors for the two zero-mode energy states. Plot the magnitude of the components. I should be able to see that one zero mode is localized entirely on the left dot and the other is entirely on the right dot, confirming they are spatially separated Majoranas.

4.4 Numerical Work - Probing Protection and Robustness:

Detuning t and Δ: Vary t and Δ away from the sweet spot. Plot the lowest energy eigenvalue as I varyt he ratio $\frac{t}{\Delta}$ away from 1. This will show that the zero-energy states immediately split and acquire a finite energy gap, demonstrating their lack of full topological protection. Detuning Dot Energies: Vary ϵ_L and ϵ_R away from zero while keeping $t = \Delta$. The degeneracy should be lifted quadratically. My results here should quantify the robustness of the PMMs to local perturbations. I can conclude that while they are not fully protected like in a true topological system, the degeneracy is protected against certain local perturbations to first order.

5 Discussion and Conclusion

5.1 Synthesize my Findings:

Summarize what we learned from the calculations. " The analysis confirmed that zero-energy Majorana modes emerge under fine-tuned conditions... Numerical simulations revealed that this degeneracy is fragile and splits when deviating from the sweet spot, quantifying the limited protection of PMMs."

5.2 Connect to Broader Context:

Briefly discuss how the properties you explored (the energy splitting, wavefunction overlap) would affect potential braiding operatios (note section 5.2.3) and what this means for using PMMs in quantum computing.

5.3 Future Directions:

Suggest next steps for a full thesis. This include modeling disorder, investigating longer chains (three or more dots), or numerically simulate braiding protocol.