Report for Mandatory 1

Dirichlet problem -dispersion coeff ω

We know

$$u(t,x,y) = \sin(k_x x)\sin(k_y y)\cos(\omega t)$$

and

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Which gives us:

$$-\omega^2(\sin(k_x x)\sin(x_y y)\cos(\omega t))$$
$$= c^2(-k_x^2 \sin(k_x x)\sin(x_y y)\cos(\omega t) - k_y^2 \sin(k_x x)\sin(x_y y)\cos(\omega t))$$
$$\omega^2 = c^2(k_x^2 + k_y^2) \Rightarrow \omega = c\sqrt{k_x^2 + k_y^2}$$

Exact solution

We need to find the relation between ω and k_x and k_y, with the exact solutuion:

$$u(x,y,t) = e^{i(k_x x + k_y y + \omega t)}$$

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \nabla^2 u$$
$$\frac{\partial^2 u}{\partial^2 t} = c^2\left(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y}\right)$$

$$\frac{\partial^2 u}{\partial^2 x} = -k_x^2 e^{i(k_x x + k_y y + \omega t)}$$
$$\frac{\partial^2 u}{\partial^2 y} = -k_y^2 e^{i(k_x x + k_y y + \omega t)}$$

And now we can see we get the same relation for wave dispersion:

$$\omega = c\sqrt{k_x^2 + k_y^2}$$

Dispersion coeff

Assume $m_x = m_y$ such that $k_x = k_y = k$. A descrete version of the exact solution will then be:

$$u_{ij}^n = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}$$

Eq(1.3) from the assignment:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right)$$

Pluggin into the lefthand side:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = \frac{e^{ikh(i+j)}}{\Delta t^2} \times \left(e^{-i\tilde{\omega}(n+1)\Delta t} - 2e^{-i\tilde{\omega} n \Delta t} + e^{-i\tilde{\omega}(n-1)\Delta t}\right)$$
$$= \frac{e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}}{\Delta t^2} \times \left(e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t}\right)$$

Lets do the same with the right hand side. We start with one term first:

$$\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} = \frac{e^{-i\tilde{\omega} n \Delta t}}{h^2} \times \left(e^{ikh(i+1+j)} - 2e^{ikh(i+j)} + e^{ikh(i-1+j)}\right)$$
$$= \frac{e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}}{h^2} \times \left(e^{ikh} - 2 + e^{-ikh}\right)$$

We can see that the same result will go for the second term

Now we can put everything together:

$$\frac{e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}}{\Delta t^2} \times \left(e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t}\right)$$
$$= c^2 \frac{e^{-i(\tilde{\omega} n \Delta t - kh(i+j))}}{h^2} \times \left(e^{ikh} - 2 + e^{-ikh} + e^{ikh} - 2 + e^{-ikh}\right)$$

Which becomes

$$\frac{e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t}}{\Delta t^2} = 2c^2\left(\frac{e^{ikh} - 2 + e^{-ikh}}{h^2}\right)$$

Introducing the CFL number $C = \frac{c\Delta t}{h} = 1/\sqrt{2}$

$$e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} = 2C\left(e^{ikh} - 2 + e^{-ikh}\right)$$
$$2\cos(\tilde{\omega}\Delta t) - 2 = 2C^2(2\cos(kh) - 2)$$
$$\cos(\tilde{\omega}\Delta t) = 1 + 2C^2\cos(kh) - 2C^2$$
$$\cos(\tilde{\omega}\Delta t) = \cos(kh)$$

From this we get:

$$\tilde{\omega} = \frac{kh}{\Delta t}$$

From the CFL number, we know that

$$C = \frac{c\Delta t}{h} = \frac{1}{\sqrt{2}}$$

Which in turn gives us: $h = \sqrt{2}c\Delta t$

$$\tilde{\omega} = \frac{kh}{\Delta t} = \frac{k\sqrt{2}c\Delta t}{\Delta t} = \sqrt{2}kc = \omega$$