EM 314 - Assignment 1

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This assignment covers the topics Taylor series, floating point arithmetic and Big $\mathcal{O}$ notation .

Theory

1. For small values of x, the approximation $\sin x \approx x$ is often used. Estimate the error in using this formula with the aid of Taylor's Theorem. For what range of values of x will this approximation give correct results rounded to six decimal places?

2. Consider the floating point number set $\mathbb{F} \subset \mathbb{R}$ such that $\mathbb{F}(\beta, t, L, U)$. Here β is the base, t is the number of digits in the mantissa and $[L, U]$ is the range of variation of the exponent. Show that the set $\mathbb{F}$ contains precisely $2(\beta - 1)\beta^{t-1}(U - L + 1)$ elements.
   Hint: Recall that a floating point number has the form $\pm(.\alpha_1 \ldots \alpha_t) \times \beta^E$.

3. Consider the following approximation f'_h for the derivative f' of a function $f(x)$.
   $$f'_h(x) = \frac{1}{h}[f(x+h) - f(x)]$$
   Let $E_h(x) = |f'(x) - f'_h(x)|$ be the associated error. Show that $E_h(x) = \mathcal{O}(h)$.

Computer Experiments

4. This is an experiment to estimate the cost of finding the determinant of a square matrix A. Use a single GNU Octave/MATLAB script `q4.m` to perform the following tasks.
   (a) Generate a sequence of random square matrices A, of size n, $n = 500, 1000, 1500, \ldots, 5000$.
   Hint: `rand(n)`.
   (b) For each n, compute $\det A$ and measure the CPU time t needed.
   Hint: `cputime()` or `tic()` / `toc()`.
   (c) Plot the points $(\log t, \log n)$ on a graph. *Hint*: `loglog()`.
   (d) Find the best-fit line for the points in part (c). Plot the graph of this line on the same figure. *Hint*: `polyfit()`.
   (e) Let $t = Cn^\alpha$, where C is a constant. Estimate α from your results in part (d).
   Hint: As $t = Cn^\alpha$, α denotes the gradient of the graph of $\ln t$ vs $\ln n$
   (f) It is known that the classical methods to compute $\det A$ yields a computational cost $\mathcal{O}(n^3)$. Compare your experimental results with this theoretical result. If there is some discrepancy between the two, what could be the reasons?

5. Here, we experiment on the approximation f'_h in Question 3, with

$$f(x) = \ln x \text{ and } x = 3.$$

Use a single GNU Octave / MATLAB script `q5.m` to perform the tasks in this question.

(a) Let $N = 10$. Generate a sequence $\{h_k\}$, $k = 1, 2, \ldots N$, such that $h_k = \frac{1}{2^k}$. For each k, compute $f'_{h_k}(x)$ and E_{h_k}.
Note: Tabulate your results in the following form:

k	h_k	$f'_{h_k}(x)$	E_{h_k}

(b) Observe that $E_{h_k} \to 0$. Assuming $E_h \propto h^\gamma$, use a log-log plot (as in Question 4) to find γ. Do you obtain $E_h = \mathcal{O}(h)$ as expected?

(c) Let $N = 40$. Repeat part (a).

(d) Observe that $E_{h_k} \nrightarrow 0$ if $N = 40$. Explain why.

(e) Plot $\log(E_{h_k})$ against $\log(h_k)$. Using the graph, estimate the h_k (say h_{min}) that minimizes E_{h_k}.

Remark. For GNU Octave / MATLAB, machine epsilon $\epsilon_m \approx 10^{-16}$. (this value is stored in a built-in variable `eps`.). It can be shown that $h_{min} = \mathcal{O}\left(\sqrt{\epsilon_m}\right)$, which is $\sim 10^{-8}$.

Overcoming the limitations in numerical differentiation due to finite precision is a challenging and an important current research topic. An alternative is "automatic differentiation" (see, for example, `http://www.neidinger.net/SIAMRev74362.pdf`).
Yet another method using the complex plane is found here:
`https://sinews.siam.org/Details-Page/differentiation-without-a-difference`

Submission Guidelines:

- Produce a document that includes solutions, code, output and figures appropriately. Use a suitable cover page that includes the course title and the course code, assignment number, your name and the registration number.
- Submit your document in **PDF** format, online via **FEeLS**.
- **Due date**: November 5 (Monday), 2018, 11:55 pm. *Please note that marks will be deducted for late submissions.*