Theory.

Q1. From the error estimate of bisection methode,

$$e_k = |x_k - x_*| \leqslant \frac{b-a}{2^{k+1}}$$

but in bisection method wee need error to be less than tolerance.

$$e_k < \tau$$

$$\therefore \frac{b-a}{2^{k+1}} < \tau$$

$$\frac{b-a}{\tau} < 2^{k+1}$$

$$\log_2 \left(\frac{b-a}{\tau}\right) < \log_2 2^{k+1}$$

$$\log_2 \left(\frac{b-a}{\tau}\right) < k+1$$

$$\therefore k > \log_2 \left(\frac{b-a}{\tau}\right) - 1$$

Q2. (a) $g(x) = e^{-x}$

$$g'(x) = -e^{-x} \quad < 0 \quad \forall x.$$

$\forall (x)$ $g'(x)$ is negative.

therefore $g(x)$ is is monotonically decreasing function.

$$|g'(x)| = e^{-x}$$

when $x = \ln 1.1$.

$$\Rightarrow |g'(x)|_{max}$$

$$e^{-\ln 1.1} = \frac{1}{1.1} < 1$$

$\therefore |g'(x)| < 1$ on G.

$\therefore g(x)$ is a contraction on G

Q2. (b). $g(x)$ is a continuous monotonically decreasing function on G.

and, $g(\ln 1.1) = e^{-\ln(1.1)} = 1/1.1 = 0.9091$

$g(\ln 3) = e^{-\ln(3)} = 1/3 = 0.3333$

$G = [\ln 1.1, \ln 3] = [0.0953, 1.0986]$

$[0.3333, 0.9091] \subseteq [\ln 1.1, \ln 3]$

$\therefore \; g : G \longrightarrow G.$

(c). g is a contraction an $g : G \longrightarrow G$.

therefore from banach fixed point theorem,

then G contains a unique solution x_*

moreover starting with any $x_0 \in G$, the sequence generated by the iterative procedure

$x_{n+1} = g(x_n)$ converges to x_*

Q3. (a) $g(x) = \tan^{-1}(2x)$

$$g'(x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$$

let $x_k \in [-1/2, 1/2]$

$$e_{k+1} = |x_{k+1} - 0| = |g(x_k) - g(0)|$$
$$= g'(\xi)\, |x_k - 0|$$
$$= g'(\xi) \cdot e_k \quad \text{for some } \xi \in (-1/2, 1/2)$$

$$|g'(x)|_{max} = \frac{2}{1+4 \cdot 0} = 2$$

but $g'(x) > 1 \quad \forall x \in (-1/2, 1/2)$ and $e_{k+1} > e_k$

$\therefore$ Therefore the fixed point iteration will not converge to $x_* = 0$.

Q3. (b) (i) $x_0 = 2.$, Fixed point iteration.

1^{st} iteration $x_1 = \tan^{-1}(2 \cdot 2)$

$= 1.3258.$

$e_1 = |e_1 - e_0|$

$= |1.3258 - 2|$

$= 0.6742.$

2^{n} iteration $x_2 = \tan^{-1}(2 \times 1.3258)$

$= 1.2102.$

$e_2 = |e_2 - e_1|$

$= |1.2102 - 1.3258|$

$= 0.1156$

(b) (ii) $x_0 = 2$, Newton's method.

$g(x) = \tan^{-1} 2x - x$

$g'(x) = \frac{2}{1+4x^2} - 1$

$= \frac{1-4x^2}{1+4x^2}$

Newton's method sequence.

$$x_{n+1} = x_n - \frac{(\tan^{-1} 2x_n - x_n)(1+4x_n^2)}{1-4x_n^2}$$

1^{st} iteration.

$$x_1 = 2 - \frac{(\tan^{-1} 4 - 2)(1+4\times4)}{(1-4\times4)}$$

$= 1.2359$

$e_1 = |2 - 1.2359| = 0.7641$

2^{n} iteration $x_2 = 1.2359 - \frac{[\tan^{-1}(2\times1.2359) - 1.2359][1+4\times1.2359^2]}{[1-4\times1.2359^2]}$

$= 1.1670$

$e_2 = |1.1670 - 1.2359| = 0.0689.$