Vector Spaces and Linear Transformations

A vector space is a set of vectors that can be added together and scaled by real numbers. The operations of addition and scalar multiplication are defined as follows: given two vectors $\mathbf{u}$ and $\mathbf{v}$, the sum of the two vectors is defined as $\mathbf{u} + \mathbf{v}$, and for a scalar c, the product of the vector $\mathbf{u}$ and the scalar c is defined as $c\mathbf{u}$.

The properties of a vector space are:

1. **Commutativity of Addition**: The order in which we add vectors does not matter.

2. **Associativity of Addition**: When we add three vectors, it doesn't change whether we do it first or second.

3. **Existence of Additive Identity**: There exists a zero vector that, when added to any other vector, leaves the latter unchanged.

4. **Existence of Additive Inverse**: For each vector in the space, there is a vector such that their sum is the zero vector.

Linear transformations are functions between vector spaces that preserve the operations of addition and scalar multiplication.

If $\mathbf{T}$ is a linear transformation from a vector space $\mathcal{V}$ to another vector space $\mathcal{W}$, then for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{V}$ and any real numbers a and b, the following equation holds:

$$\mathbf{T}(a\mathbf{x} + b\mathbf{y}) = a\mathbf{T}(\mathbf{x}) + b\mathbf{T}(\mathbf{y}).$$

This equation is true for all vectors in the domain space.

Eigenvalues and eigenvectors are used to describe how a linear transformation stretches or shrinks a vector. The eigenvalue of an eigenvector represents the amount by which it is scaled under a particular linear transformation.