Square Root Difference Theorem

hisanobu-nakamura

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1 The Statement

Prove the following statetment: $\forall n, k(>0) \in \mathbb{N}, \exists N \in \mathbb{N}$ such that

$$(\sqrt{n+1} - \sqrt{n})^k = \sqrt{N+1} - \sqrt{N} \tag{1}$$

$$(\sqrt{n+1} - \sqrt{n})^k = \sqrt{N+1} - \sqrt{N} \tag{2}$$

Proof

First, let us rephrase the statement in a more tactible way. Take logarithm of the L.H.S. equation

$$\begin{aligned} \ln(\sqrt{n+1} - \sqrt{n})^k &= k \ln\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right) \\ &= -k \ln(\sqrt{n+1} + \sqrt{n}) \end{aligned}$$

Then, with $x = \sqrt{n}$, the logarithm at the last line is equal to $\sinh^{-1} x$. So, let us denote $\phi = -\sinh^{-1} x$. Now, the statement reads that, for ϕ such that $\sinh^2 \phi$ is a natural number, prove that $\sinh^2 k\phi$ is also a natural number. In fact, $\sinh k\phi$ can be expressed as a polynomial in terms of $\sinh \phi = \sqrt{n}$ and $\cosh \phi = \sqrt{n+1}$ with integer coefficients. But in order for $\sinh^2 k\phi$ to be natural, $\sinh k\phi$ must factorize into a natural factor and an irrational factor such as $\sinh \phi$, $\cosh \phi$ or $\sinh \phi \cosh \phi$ The trigonometric-hyperbolic conversion rule:

$$\cos i\phi = \cosh \phi, \quad \sin i\phi = i \sinh \phi \tag{3}$$

This leads to

$$(x+y)^k = \sum_{l+l'=k} \binom{k}{l} x^l y^{l'} \tag{4}$$

$$\begin{aligned} (x+y)^{2m} &= x^{2m} + \binom{2m}{1} \underline{x^{2m-1} y} + \binom{2m}{2} x^{2m-2} y^2 + \\ &\cdots + \binom{2m}{2m-2} x^2 y^{2m-2} + \binom{2m}{2m-1} \underline{x y^{2m-1}} + y^{2m} \end{aligned} \tag{5}$$

$$\begin{aligned} (x+y)^{2m+1} &= x^{2m+1} + \binom{2m+1}{1} \underline{x^{2m} y} + \binom{2m+1}{2} x^{2m-1} y^2 + \\ &\cdots + \binom{2m+1}{2m-1} \underline{x^2 y^{2m-1}} + \binom{2m+1}{2m} x y^{2m} + \underline{y^{2m+1}} \end{aligned} \tag{6}$$