Notes on "Topoi" by Goldblatt

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1 Chapter 3: Arrows instead of epsilons

1.1 3.1. Monic arrows

An arrow $f: a \to b$ in a category \mathcal{C} is *monic* in \mathcal{C} if for any pair $g, h: c \rightrightarrows a$ of \mathcal{C} -arrows, equality $f \circ g = f \circ h$ implies that g = h. The symbolism $f: a \rightarrowtail b$ is used to indicate that f is monic.

1.1.1 3.1. Exercise Solutions

In any category

- (1) $g \circ f$ is monic if both f and g are monic.
- (2) If $g \circ f$ is monic, then so is f.

Solutions

(1)

If $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$, then

$$\begin{array}{rcl} (g\circ f)\circ h_1 &=& (g\circ f)\circ h_2 \\ g\circ (f\circ h_1) &=& g\circ (f\circ h_2) \\ f\circ h_1 &=& f\circ h_2 \quad (\because \text{g is monic}) \\ h_1 &=& h_2 \quad (\because \text{f is monic}) \end{array}$$

Therefore, $(g \circ f)$ is monic.

(2)

If $f \circ h_1 = f \circ h_2$, then

$$g \circ (f \circ h_1) = g \circ (f \circ h_2)$$

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2$$

$$h_1 = h_2 \quad (\because f \circ g \text{ is monic})$$

Hence, f is monic.

1.2 3.2. Epic arrows

An arrow $f:a\to b$ is epic (right-cancellable) in a category $\mathcal C$ if for any pair $g,h:b\rightrightarrows c$, the equality $g\circ f=h\circ f$ implies g=h i.e. whenever a diagram

$$\begin{array}{ccc}
a & \xrightarrow{f} & b \\
f \downarrow & & \downarrow g \\
b & \xrightarrow{h} & c
\end{array}$$

commutes, then g = h. The notation $f : a \rightarrow b$ is used for epic arrows.

1.2.1 3.2. Exercises (Duals of exercises in 3.1)

In any category

(1) $g \circ f$ is epic if both f and g are epic.

(2) If $g \circ f$ is epic, then so is g.

Solutions

(1)

Suppose that f and g are both epic, then if $h_1 \circ (g \circ f) = h_2 \circ (g \circ f)$,

$$h_1 \circ (g \circ f) = h_2 \circ (g \circ f)$$

$$(h_1 \circ g) \circ f = (h_2 \circ g) \circ f$$

$$h_1 \circ f = h_2 \circ f$$

$$h_1 = h_2.$$

Hence, $g \circ f$ is epic.

(2)

Suppose $g \circ f$ is epic. Then, if $h_1 \circ g = h_2 \circ g$,

$$(h_1 \circ g) \circ f = (h_2 \circ g) \circ f$$

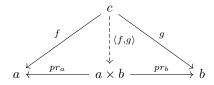
$$h_1 \circ (g \circ f) = h_2 \circ (g \circ f)$$

$$h_1 = h_2.$$

Hence, g is epic.

1.3 3.8 Products

DEFINITION A product in a category \mathcal{C} of two objects a and b is a \mathcal{C} -object $a \times b$ together with a pair $(pr_a : a \times b \to a, pr_b : a \times b \to b)$ of \mathcal{C} -arrows such that for any pair of \mathcal{C} -arrows of the form $(f : c \to a, g : c \to b)$ there is exactly one arrow $\langle a, g \rangle : c \to a \times b$ making



commute, i.e. such that $pr_a \circ \langle f, g \rangle = f$, $pr_b \circ \langle f, g \rangle = g$. $\langle f, g \rangle$ is the product arrow of f and g with respect to the projections pr_a , pr_b .

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