Pointless Topology 勉強ノート

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1 Preliminary

1.1 Topology トポロジー

Let $\mathcal{P}(X)$ denote the power set of X.

定義 1.1 (Topology トポロジー)**.** *A **topological space** is an ordered pair (X, τ), $\tau \subseteq \mathcal{P}(X)$ which satisfies the following properties*

1. $\varnothing \in \tau$ *and* $X \in \tau$.
2. *if* $U, V \in \tau$, *then* $U \cap V \in \tau$.
3. *if* $\forall I, U_i \in \tau$ *forall* $i \in I$, *then* $\bigcup_{i \in I} U_i$.

*τ is called the **topology** of X. The members of the topology $U \in \tau$ is said to be **open** and $V \subseteq X$ is said to be **closed** if $\exists U$ open such that $V = U^c$.*

定義 1.2. *If X is a topological space and $x \in X$, a neighbourhood (abbreviated "nhood") od x is a set U which contains an open set V containing x. Thus, U is a nhood of x iff $x \in U^\circ$. The collection $\mathscr{U}_x$ of all nhoods of x is the nhood system of x.*

定義 1.3 (Separation Axioms 分離公理)**.** *A space (X, τ) is called T_i ,if respectively satisfies the following conditions,*

1. T_0: $\forall x, y \in X$ $\exists$ *an open set $U \in \tau$ such that U contains one of x, y and not the other.*
2. T_1: $\forall x, y \in X$ $\exists$ *a nhood of each not containing the other.*

例 1.1 (T_0-space)**.** $X = \{a, b\}, \tau = \{\varnothing, \{a\}, X\}$

1.2 Posets, Lattices 半順序集合、束

定義 1.4 (Posets)**.** *A **partial order (半順序)** on a set X is a binary relation $R \subseteq X \times X$ satisfying,*

1. $\forall a, aRa$ *(reflexivity, 反射律),*
2. $\forall a, b, c,\ aRb \,\&\, bRc \Rightarrow aRc$ *(transitivity, 推移律),*
3. $\forall a, b,\ aRb \,\&\, bRa \Rightarrow a = b$ *(antisymmetry, 反対称律).*

if moreover

4. $\forall a, b$ either aRb or bRa holds,

it is said to be a ***linear*** *or* ***total*** *order.*
A ***poset*** *or* ***partially ordered set***, *$(X, \leq)$ is a set with a partial order. If the order of a poset is linear (or total), it is called a* ***linearly ordered set***, ***totally ordered set*** *or* ***chain***. *A relation that satisfies only (1) and (2) is called* ***preorder***.

定義 1.5 (Suprema, infima)**.** *A* ***supremum*** *s of a subset $M \subseteq (X, \leq)$ the least upper bound of M, that is*

1. *$\forall m \in M, m \leq s$,*

2. *$\forall m \in M, m \leq x \Rightarrow s \leq x$.*

Similarly, a ***infimum*** *of a subset $M \subseteq (X, \leq)$ the greatest lower bound of M.*
We also call a supremum a ***join*** *and an infimum* ***meet*** *and notate $\sup M, \inf M$ or $\bigvee M, \bigwedge M$ respectively.*
For finite cases, we wirte $a \vee b := \sup\{a, b\}$ or $a_1 \vee \cdots \vee a_n := \sup\{a_1 \ldots a_n\}$ and $a \wedge b := \inf\{a, b\}$ or $a_1 \wedge \cdots \wedge a_n := \inf\{a_1 \ldots a_n\}$.

Since each $x \in X$ is both a lower and an upper bound of the empty set $\varnothing$,

$$\sup \varnothing \text{ is the least element of } X \tag{1.1}$$

and

$$\inf \varnothing \text{ is the greatest element of } X \tag{1.2}$$

We use the symbols 0 or $\perp$ for the former and 1 or $\top$ for the latter.

定義 1.6 (Semilattices, Lattice)**.** *A* ***meet-semilattice*** *is a poset X such that $\forall a, b \in X$ there exists an infimum $a \wedge b$.*
A ***join-semilattice*** *is a poset X such that $\forall a, b \in X$ there exists an supremum $a \vee b$.*
A ***lattice*** *is a poset X such that $\forall a, b \in X$ both an infimum $a \wedge b$ and a supremum $a \vee b$ exist.*
A ***bounded lattice*** *is a poset in which all finite subsets have infima and suprema (i.e. a lattice with bottom and top).*
A poset is a ***complete lattice*** *if every subset has a supremum and an infimum.*

In a bounded semilattice, $\wedge$ or $\vee$ is a binary operation and satisfies the following properties,

$$a \wedge a = a \qquad a \vee a = a \tag{1.3}$$
$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a \tag{1.4}$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad (a \vee b) \vee c = a \vee (b \vee c) \tag{1.5}$$
$$a \wedge 1 = a \qquad a \vee 0 = a. \tag{1.6}$$

In other words, bounded semilattices are commutative monoids (semigroup with unit/identity element) in which every element is idempotent.

定理 1.1. *Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A such that $a \wedge b$ is the join of a and b, and 0 is the least element.*

証明. If such a partial order exists,

$$a \leq b \Leftrightarrow a \vee b = b. \tag{1.7}$$

would be the correspondence. Now, let us verify this connection.

Reflexivity

$$a \wedge a = a \Rightarrow a \leq a. \tag{1.8}$$

Antisymmetry

If $a \leq b$ and $b \leq a$, then $b = a \wedge b = b \wedge a = a$ by commutativity.

Transitivity

If $a \leq b$ and $b \leq c$, then

$$\begin{aligned} a \wedge c &= a \wedge (b \wedge c) && (\because b \leq c) \\ &= (a \wedge b) \wedge c && \text{(associativity)} \\ &= b \wedge c && (\because a \leq b) \\ &= c && (\because b \leq c) \end{aligned}$$

Hence, $a \leq c$.

Join-uniqueness

Since $a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$, $a \leq a \wedge b$. Similarly, $b \leq a \wedge b$, so $a \wedge b$ is an upper-bound for $\{a, b\}$. For the leastness, suppose $a \leq c$ and $b \leq c$, then

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ &= a \wedge c \\ &= c. \end{aligned}$$

Hence, $a \wedge b \leq c$. □

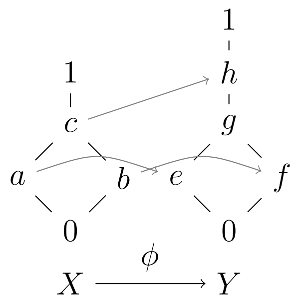
A lattice can also be defined purely algebraically in those terms,

定義 1.7. *A* ***lattice*** $(L, \vee, \wedge)$ *is an algebra (a set with two binary operations) that satisfy*

$(L1)$	$a \wedge a = a$	$a \vee a = a$	*(idempotency)*
$(L2)$	$a \wedge b = b \wedge a$	$a \vee b = b \vee a$	*(commutativity)*
$(L3)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \vee b) \vee c = a \vee (b \vee c)$	*(associativity)*
$(L4)$	$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	*(absorption identities)*

$(L4)$ is necessary for the two operations $\wedge, \vee$ to be consistent with the corresponding order $\leq$. In fact, $a \wedge b = b$ implies $a \vee b = a \vee (a \wedge b) = a$ by $(L4)$.
For homomorphisms (structure-preserving maps) of (semi)lattices and posets, we need to be a little bit careful since the order-preserving homomorphisms of posets does not always preserve the joins (or meets) as shown in the following example.([Stone] section 1.3 Exercise)

例 1.2. *Consider the posets $X = \{a \leq c, b \leq c\}$ and $Y = \{e \leq g, f \leq g, g \leq h\}$, and a homomorphism $\phi : X \to Y$ which maps each element as in the diagram below:*



Indeed, we have $a, b \leq c$ and $\phi(a), \phi(b) \leq \phi(c)$, which means the order is preserved, but $\phi(a \vee b) \neq \phi(a) \vee \phi(b)$.

([Stone] Sec.1.5) In most of the lattices we'll consider, the operations $\wedge$ and $\vee$ will satisfy an additional identity, namely the distributive law

$$\text{(i) } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{1.9}$$

for all a, b, c.

補題 1.2. *If the distributive law (i) holds in a lattice, then so does its dual, i.e. the identity*

$$\textit{(ii) } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.10}$$

証明.

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (i)} \\
&= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{by absorption law} \\
&= a \vee (b \wedge c) && \text{by absorption law}
\end{aligned}$$

□

Note also that in the presence of (i), we can deduce either of the two absorptive law from the other,

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b) \tag{1.11}$$

命題 1.3. *Let a, b, c be three elements of a distributive lattice A. Then there exists at most one $x \in A$ satisfying $x \wedge a = b$ and $x \vee a = c$.*

証明. Suppose both x and y satisfy the conditions. Then,

$$\begin{aligned}
x = x \wedge (x \vee a) = x \wedge c &= x \wedge (y \vee a) \\
&= (x \wedge y) \vee (x \wedge a) \\
&= (x \wedge y) \vee b = x \wedge y
\end{aligned}$$

since $b = x \wedge a = y \wedge a$ is a lower bound for $\{x, y\}$. Similarly, we have $y = x \wedge y$; so $x = y$. □

In any lattice, an element x satisfying $x \wedge a = 0$ and $x \vee a = 1$ is called a ***complement*** od a. The Porposition above tells us that in a distributive lattice, complements are unique when they exist. A ***Boolean algebra*** is a distributive lattice A equipped with an additional unary operation $\neg : A \to A$ such that $\neg a$ is a complement of a. Since $\neg$ is uniquely determined by the other data in the definition, it follows that any lattice homomorphism $f : A \to B$ between Boolean algebras is actually a Boolean algebra homomorphism(i.e. commutes with $\neg$).

例 1.3 (Power set)**.** *For any set X, the power set $\mathcal{P}(X)$ of X is a lattice, with $\leq$ interpreted as inclusion, $\wedge$ and $\vee$ as union and intersection of subsets, and 0 and 1 as the empty set and the whole of X. Moreover $\mathcal{P}(X)$ is distributive. Since $\mathcal{P}(X)$ has complements for all its elements, it is a Boolean algebra.*

例 1.4 (Total Order)**.** *Let A be a totally ordered set with least and greatest elements 0 and 1. Then A is a lattice, with $\wedge$ and $\vee$ interpreted as* min *and* max*. It is distributive;*

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\} \tag{1.12}$$

But if A has more than two elements, it is not a Boolean elgebra; for no element other than 0 and 1 can have a complement.

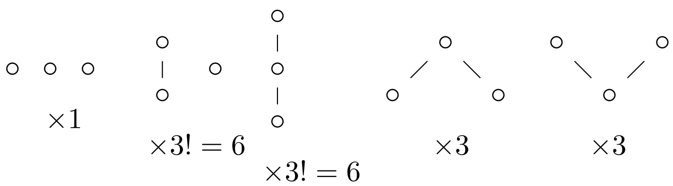
例 1.5 (Lattices of subgroups)**.** *Let G be agroup.*

例 1.6 (Poset)**.** *([Gratzer] Chap.1 Exercise 4) Here are some examples of the possible numbers of partial orders on finite sets:*

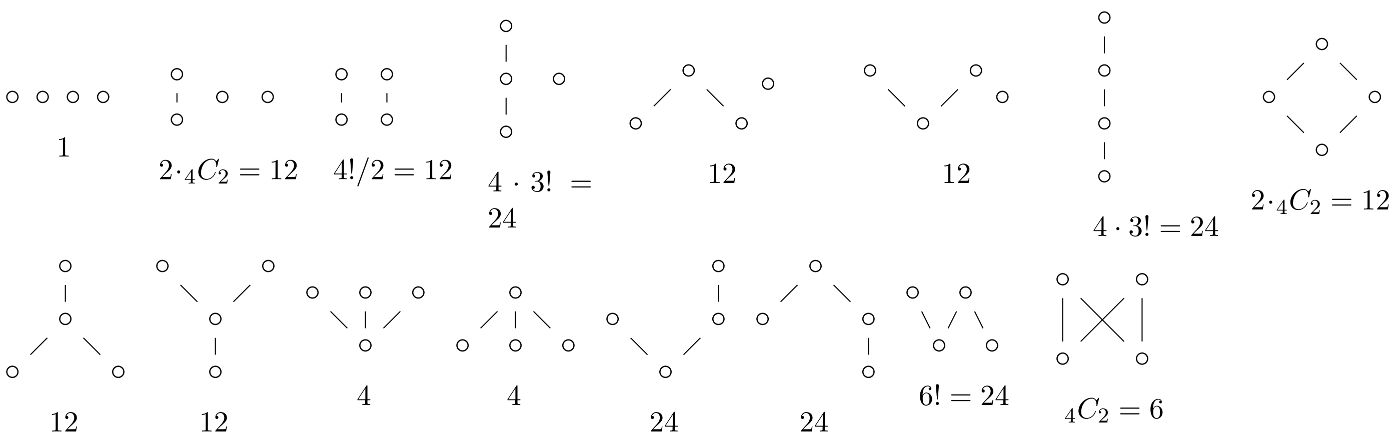
Size 1

Size 2

Size 3



Size 4



1.3 Ideals and Filters イデアルとフィルター

定義 1.8 (Ideal)**.** *An **ideal** in a bounded distributive lattice L is a subset $J \subseteq L$ such that*

$$0 \in J, \tag{1.13}$$

$$a, b \in J \Rightarrow a \vee b \in J, \tag{1.14}$$

$$b \leq a \;\&\; a \in J \Rightarrow b \in J. \tag{1.15}$$

定義 1.9 (Filter)**.** *A **filter** in a bounded distributive lattice L is a subset $F \subseteq L$ such that*

$$1 \in F, \tag{1.16}$$

$$a, b \in F \Rightarrow a \wedge b \in F, \tag{1.17}$$

$$b \geq a \;\&\; a \in F \Rightarrow b \in F. \tag{1.18}$$

2 Stone Spaces

3 Spaces and Lattices of Open Sets

We will suppose that all topological spaces that appear here will be T_0.

3.1 Sober spaces

定義 3.1 (meet-irrducibility)**.** *Let (X, τ) be a top.space. $W \in \tau$ is said to be a **meet-irreducible** open set if $U, V \in \tau$ and $U \cap V \subseteq W$, then either $U \subseteq W$ or $V \subseteq W$.*

定義 3.2 (sober space)**.** *X is said to be **sober** if all the meet-irreducible open sets are of the form $X \backslash \overline{\{x\}}$.*

命題 3.1. *Each Haudorff space is sober.*

証明. Suppose W is meet-irreducible, for contradiction, there exists $x_1, x_2 \notin W$ and $x_i \in U_i, x_j \notin U_i (i \neq j)$. Then $W = (W \cup U_1) \cap (W \cup U_2)$ and $W \cup U_i \nsubseteq W$. □

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