

Angular Momentum and Racah's formula

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<2020-02-26 >

1 Angular Momentum Operators and $\mathfrak{su}(2)$ Representation

$J_x, J_y, J_z \in \mathfrak{su}(2)$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (1.1)$$

The Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ and the ladder operators $J_{\pm} = J_x \pm iJ_y$

$$[J^2, J_i] = 0 \quad (1.2)$$

$$[J_z, J_{\pm}] = \pm J_{\pm} \quad (1.3)$$

$$[J_+, J_-] = 2J_z \quad (1.4)$$

$$J^2 = J_-J_+ + J_z^2 + J_z \quad (1.5)$$

$$= J_+J_- + J_z^2 - J_z \quad (1.6)$$

Let V be a finite dimensional vector space over \mathbb{C} and $\varphi : \mathfrak{su}(2) \rightarrow \mathbf{End}(V)$ be its associated representation. Simultaneous eigenvector $|jm\rangle$:

$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle \quad (1.7)$$

2 The ladder operators' coefficients

$$J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}|jm+1\rangle \quad (2.1)$$

$$= \sqrt{(j-m)(j+m+1)}|jm+1\rangle \quad (2.2)$$

$$J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}|jm-1\rangle \quad (2.3)$$

$$= \sqrt{(j+m)(j-m+1)}|jm-1\rangle \quad (2.4)$$

It is useful to write $|jm\rangle$ in terms of $(J_-)^k|jj\rangle$. So, let us rewrite the coefficients in simpler notation;

$$J_-|jj - (k-1)\rangle = f(j, k)|jj - k\rangle \quad (2.5)$$

where

$$f(j, k) := \sqrt{k(2j - k + 1)}, 1 \leq k \leq 2j$$

We also have

$$J_+|jj - k\rangle = f(j, k)|jj - k + 1\rangle \quad (2.6)$$

Note that $D := 2j + 1$ is the dimension of the j -th representation space.

Hence

$$(J_-)^k|jj\rangle = F(j, k)|jj - k\rangle \quad (2.7)$$

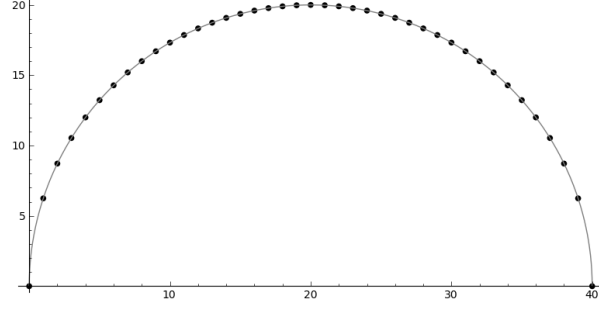


Figure 1: The graph of $f(j = \frac{39}{2}, k)$ where k is the horizontal axis.

where $F(j, k) = \prod_{i=1}^k f(j, i)$ and evaluated as,

$$\begin{aligned}
 F(j, k) &= \sqrt{k(2j+1-k)(k-1)(2j+1-(k-1)) \times \cdots \times 2 \cdot (2j+1-2) \cdot 1 \cdot (2j+1-1)} \\
 &= \sqrt{k(D-k)(k-1)(D-(k-1)) \times \cdots \times 2 \cdot (D-2) \cdot 1 \cdot (D-1)} \\
 &= \sqrt{\frac{k!(2j)!}{(2j-k)!}} = k! \sqrt{2j C_k}
 \end{aligned} \tag{2.8}$$

and $F(j, 0) = 1$

3 Recursion Relations for Clebsch-Gordan coefficients

$$|j_1 j_2 JM\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 m_1 j_2 m_2 | JM \rangle \tag{3.1}$$

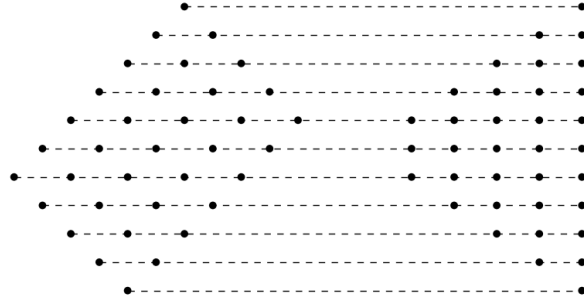


Figure 2: Correspondence between the tensor product $j_1 \otimes j_2$ and J ($j_1 = 3, j_2 = 2$).

Apply $J_+ = j_{1+} + j_{2+}$ take inner product with $\langle j_1 j_2 m_1 m_2 |$

$$\begin{aligned}
 &\sqrt{J(J+1) - M(M+1)} \langle j_1 m_1 j_2 m_2 | JM+1 \rangle \\
 &= \sqrt{j_1(j_1+1) - m_1(m_1-1)} \langle j_1 m_1 - 1 j_2 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)} \langle j_1 m_1 j_2 m_2 - 1 | JM \rangle
 \end{aligned} \tag{3.2}$$

and $J_- = j_{1-} + j_{2-}$

$$\begin{aligned}
 &\sqrt{J(J+1) - M(M-1)} \langle j_1 m_1 j_2 m_2 | JM-1 \rangle \\
 &= \sqrt{j_1(j_1+1) - m_1(m_1+1)} \langle j_1 m_1 + 1 j_2 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2+1)} \langle j_1 m_1 j_2 m_2 + 1 | JM \rangle
 \end{aligned} \tag{3.3}$$

4 Explicit formulae for Clebsch-Gordan coefficients

4.1 $\langle j_1 j_2 m_1 m_2 | JM \rangle$

Define $d = j_1 + j_2 - J$, ($j_1 > j_2$ and $0 \leq d \leq 2j_2$), and $L = J - M$. Let us determine the coefficients of the top spin state

$$\begin{aligned} |JJ\rangle &= a_0 |j_1 j_1\rangle |j_2 j_2 - d\rangle + a_1 |j_1 j_1 - 1\rangle |j_2 j_2 - d + 1\rangle + \cdots + a_d |j_1 j_1 - d\rangle |j_2 j_2\rangle \\ &= \sum_{i=0}^d a_i |j_1 j_1 - i\rangle |j_2 j_2 - d + i\rangle \end{aligned} \quad (4.1)$$

by imposing the top spin condition

$$J_+ |JJ\rangle = 0 \implies a_{i+1} = -\frac{f(j_2, d-i)}{f(j_1, i+1)} a_i \quad (i = 0, \dots, d-1), \quad (4.2)$$

which means

$$a_i = -\frac{f(j_2, d-(i-1))}{f(j_1, i)} a_{i-1} \quad (i = 1, \dots, d) \quad (4.3)$$

$$= (-1)^i \frac{f(j_2, d-(i-1))f(j_2, d-(i-2)) \cdots f(j_2, d-1)f(j_2, d)}{f(j_1, i)f(j_1, i-1) \cdots f(j_1, 2)f(j_1, 1)} a_0 \quad (4.4)$$

$$= (-1)^i \frac{F(j_2, d)}{F(j_1, i)F(j_2, d-i)} a_0 \quad (4.5)$$

Here $F(j_2, 0) = 1$.

4.2 Normalisation

Normalisation condition $\langle JJ | JJ \rangle = 1$ yields

$$\begin{aligned} \frac{1}{a_0^2} &= \sum_{i=0}^d \frac{F(j_2, d-i)^2}{F(j_1, i)^2 F(j_2, d-i)^2} \\ &= 1 + \left[\frac{f(j_2, d)}{f(j_1, 1)} \right]^2 + \cdots + \left[\frac{f(j_2, d-(i-1))f(j_2, d-(i-2)) \cdots f(j_2, d-1)f(j_2, d)}{f(j_1, i)f(j_1, i-1) \cdots f(j_1, 2)f(j_1, 1)} \right]^2 + \\ &\quad \cdots + \left[\frac{F(j_2, d)}{F(j_1, d)} \right]^2 \\ &= \frac{1}{F(j_1, d)^2} \left\{ (D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_1 - 1) \cdot 1 + (D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_2 - d) \cdot d + \right. \\ &\quad \left. \cdots + (D_1 - d) \cdot d \cdots (D_1 - i - 1) \cdot (i + 1) \cdot (D_2 - (d - i + 1)) \cdot (d - i + 1) \cdots (D_2 - d) \cdot d + \cdots \right\} \\ &= \frac{1}{F(j_1, d)^2} \left\{ \frac{(d!)^2}{d!} (D_1 - d) \cdot (D_1 - 2) \cdot (D_1 - 1) + \frac{(d!)^2}{1!(d-1)!} (D_1 - d) \cdots (D_1 - 2) \cdot (D_2 - d) + \right. \\ &\quad \left. \cdots + \frac{(d!)^2}{i!(d-i)!} (D_1 - d) \cdot (D_1 - i - 1) \cdot (D_2 - (d - i + 1)) \cdots (D_2 - d) + \cdots \right\} \end{aligned} \quad (4.6)$$

Writing

$$\begin{aligned} G_i(j_1, j_2, d) &:= \frac{F(j_1, d)F(j_2, d)}{F(j_1, i)F(j_2, d-i)}, \\ &= \sqrt{\frac{(d!)^2}{(d-i)!i!}} (D_2 - d)(D_2 - d - 1) \cdots (D_2 - d - i + 1)(D_1 - d) \cdots (D_1 - i + 1) \end{aligned} \quad (4.7)$$

Or, substituting $d = j_1 + j_2 - J$, this can be written as

$$G_i(j_1, j_2, j_1 + j_2 - J) = (-1)^i \sqrt{\frac{((j_1 + j_2 - J)!)^2}{(j_1 + j_2 - J - i)!i!} \frac{(-j_1 + j_2 + J)!(j_1 + J - j_2)!}{(-j_1 + j_2 + J - i)!(2j_1 - i)!}}.$$

In terms of these G_i 's, the coefficients a_i become

$$a_i = (-1)^i \frac{G_i(j_1, j_2, d)}{\sqrt{\sum_{i=1}^d G_i(j_1, j_2, d)^2}}. \quad (4.8)$$

We want to know the normalising coefficient $N := \frac{1}{\sqrt{\sum_{i=1}^d G_i(j_1, j_2, d)^2}}$. In order to simplify the sum

$$\begin{aligned} \sum_{i=1}^d G_i(j_1, j_2, d)^2 &= \sum_{i=1}^d \frac{F(j_1, d)^2 F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d - i)^2} \\ &= \frac{(d!)^2}{(2j_1 - d)!(2j_2 - d)!} \sum_{i=1}^d \frac{(2j_1 - i)!(2j_2 - d + i)!}{i!(d - i)!}, \end{aligned} \quad (4.9)$$

we use a formula due to Racah (mentioned in Messiah[1])

$$\sum_s \frac{(a + s)!(b - s)!}{(c + s)!(d - s)!} = \frac{(a + b + 1)!(a - c)!(b - d)!}{(c + d)!(a + b - c - d + 1)!}. \quad (4.10)$$

with $a \geq c, b \geq d \geq 0$, where the sum is taken over $-c \leq s \leq d$.

Now substituting $a = 2j_2 - d, b = 2j_1, c = 0, d = d$, we obtain

$$N = \sqrt{\frac{(2j_2 - 2d + 2j_1 + 1)!}{d!(2j_2 - d + 2j_1 + 1)!}} = \sqrt{\frac{(2J + 1)!}{(j_1 + j_2 - J)!(j_1 + j_2 + J + 1)!}} \quad (4.11)$$

$$a_i = (-1)^i N G_i(j_1, j_2, d)$$

Now, by multiplying the top-spin state with the ladder operators L times, we obtain the state $|JM\rangle$ with $M = J - L$

$$\begin{aligned} J_-^L |JJ\rangle &= (j_{1-} + j_{2-})^L \sum_{h=0}^d a_h \times |j_1 j_1 - h\rangle |j_2 j_2 - d + h\rangle \\ F(J, L) |JJ - L\rangle &= \sum_{h=0}^d a_h \sum_{l=0}^L {}_L C_l \frac{F(j_1, h + l) F(j_2, (L + d) - (l + h))}{F(j_1, h) F(j_2, d - h)} |j_1 j_1 - (h + l)\rangle |j_2 j_2 - (L + d) + (h + l)\rangle \\ |JJ - L\rangle &= \frac{1}{F(J, L)} \sum_{k=0}^{L+d} \left[\sum_{k=h+l, \substack{0 \leq h \leq d \\ 0 \leq l \leq L}} a_h \times {}_L C_l \frac{F(j_1, k) F(j_2, K - k)}{F(j_1, h) F(j_2, d - h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle \\ &= \frac{N}{F(J, L)} \sum_{k=0}^{L+d} F(j_1, k) F(j_2, K - k) \left[\sum_{k=h+l, \substack{0 \leq h \leq d \\ 0 \leq l \leq L}} \frac{(-1)^h {}_L C_l G_h(j_1, j_2, d)}{F(j_1, h) F(j_2, d - h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle \end{aligned}$$

where $K = L + d = J - M + j_1 + j_2 - J = j_1 + j_2 - M$. Now, consider the coefficients of $|j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle$

$$\begin{aligned}
B_k &:= F(j_1, k) F(j_2, K - k) \left[\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} \frac{(-1)^h {}_L C_l G_h(j_1, j_2, d)}{F(j_1, h) F(j_2, d - h)} \right] \\
&= \sqrt{\frac{k!(K - k)!}{(2j_1 - k)!(2j_2 - K + k)!}} \sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h {}_L C_l \sqrt{\frac{(2j_1 - h)!(2j_2 - d + h)!(d!)^2 (2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!(2j_1 - d)!(2j_2 - d)!h!(d - h)!}} \\
&= \sqrt{\frac{k!(K - k)!}{(2j_1 - k)!(2j_2 - K + k)!(2j_1 - d)!(2j_2 - d)!}} {}^L d! \sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!l!(L - l)!}
\end{aligned}$$

The coefficient outside the sum, in terms of j_1, j_2, J, m_1, m_2, M , using the relations $K = L + d = J - M + j_1 + j_2 - J = j_1 + j_2 - M$, $k = j_1 - m_1$, is

$$\sqrt{\frac{(j_1 - m_1)!(j_2 + m_1 - M)!}{(j_1 + m_1)!(j_2 - m_1 + M)!(j_1 - j_2 + J)!(j_2 - j_1 + J)!}} (J - M)!(j_1 + j_2 - J)! \quad (4.12)$$

Multiplying by $\frac{N}{F(J, J - M)}$

$$\begin{aligned}
&\sqrt{\frac{(2J + 1)(j_1 + j_2 - J)!}{(j_1 - j_2 + J)!(j_2 - j_1 + J)!(j_1 + j_2 + J + 1)!}} \frac{(j_1 - m_1)!(j_2 - m_2)!(J + M)!(J - M)!}{(j_1 + m_1)!(j_2 + m_2)!} \\
&= \sqrt{(2J + 1)} \sqrt{\Delta(j_1, j_2)} \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(J + M)!(J - M)!} \\
&\quad \times \frac{1}{(j_1 - j_2 + J)!(j_2 - j_1 + J)!(j_1 + m_1)!(j_2 + m_2)!} \quad (4.13)
\end{aligned}$$

where we have defined

$$\Delta(abc) := \frac{(a + b - c)!(b + c - a)!(c + a - b)!}{(a + b + c + 1)!}. \quad (4.14)$$

Now, we want to simplify the sum

$$\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!l!(L - l)!} \quad (4.15)$$

furthermore. Putting $k = j_1 - m_1$, $d = j_1 + j_2 - J$ back, we have

$$\begin{aligned}
&\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!l!(L - l)!} \\
&= \sum_l (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1 + l)!(j_2 + J - m_1 - l)!}{l!(j_1 - m_1 - l)!(j_2 - J + m_1 + l)!(J - M - l)!}
\end{aligned}$$

and the sum in the last line is taken over all the values of l with which all the factorial terms containing l makes sense. In order to do so, we are going to use the following formula

$$\frac{a!}{b!c!} = \sum_s \frac{(a - b)!(a - c)!}{s!(a - b - s)!(a - c - s)!(b + c - a + s)!}. \quad (4.16)$$

Now

$$\begin{aligned}
& \sum_l (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!} \\
&= \sum_l (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!} \cdot \frac{(j_2+J-m_1-l)!}{(J-M-l)!(j_1-m_1-l)!} \\
&= \sum_{l,l_1} (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!(j_1-j_2-M-l+l_1)!}
\end{aligned} \tag{4.17}$$

From the last equation (4.17), Racah uses the following formula to proceed with the calculation,

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}. \tag{4.18}$$

To use the above formula, we change the summation variable $l \rightarrow l' = j_1 - j_2 - M + l_1 - l$, and we have

$$\begin{aligned}
j_1 + m_1 + l &= 2j_1 - j_2 - m_2 + l_1 - l' \\
j_2 - J + m_1 + l &= j_1 - J - m_2 + l_1 - l'.
\end{aligned}$$

Putting these terms to (4.17), we get

$$\begin{aligned}
& \sum_{l',l_1} (-1)^{-j_2-m_2+l_1-l'} \frac{(2j_1-j_2-m_2+l_1-l')!}{l'!(j_1-J-m_2+l_1-l')!(j_1-j_2-M+l_1-l')!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!} \\
&= \sum_{l_1} (-1)^{j_2+m_2-l_1} \frac{(j_1+m_1)!(j_1-j_2+J)!}{(j_1-J-m_2+l_1)!(j_1-j_2-M+l_1)!(J+M-l_1)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!}
\end{aligned}$$

In another method that the author found, (4.17) can be further transformed into the following expression by using (4.16)

$$\begin{aligned}
& \sum_{l,l_1,l_2} (-1)^{j_1-m_1-l} \frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!(j_2-J-j_1+l_2+l)!} \\
& \quad \times \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!(j_1-j_2-M+l_1-l)!} \\
&= \sum_{l,l_1,l_2} (-1)^{j_1-m_1} \frac{(-1)^{-l}}{(j_1-j_2-M+l_1-l)!(-j_1+j_2-J+l_2+l)!} \\
& \quad \times \frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!}.
\end{aligned} \tag{4.19}$$

Then, by the use of the formula (see Appendix)

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a, -b) \tag{4.20}$$

(4.17) now also becomes

$$\sum_{l_1} (-1)^{j_2+m_2-l_1} \frac{(j_1+m_1)!(j_1-j_2+J)!}{(j_1-J-m_2+l_1)!(j_1-j_2-M+l_1)!(J+M-l_1)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!}.$$

In either way, putting $t = j_2 + m_2 - l_1$ ([2]), we obtain

$$\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!L!} \\ = \sum_t (-1)^t \frac{(j_1 + m_1)!(j_2 + m_2)!(j_1 - j_2 + J)!(-j_1 + j_2 + J)!}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!},$$

and then the final formula

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = \sqrt{(2J+1)} \sqrt{\Delta(j_1 j_2 J)} \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(J + M)!(J - M)!} \\ \times \sum_t (-1)^t \frac{1}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!} \quad (4.21)$$

Here, notice that the sum takes place in the range

$$\max \{0, -(J - j_2 + m_1), -(J - j_1 - m_2)\} \leq t \leq \min \{j_1 + j_2 - J, j_1 - m_1, j_2 + m_2\} \quad (4.22)$$

By making the substitution $z = a + b - c - t$ in the sum, we have

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = (-1)^{a+b-c} \langle j_2 j_1 m_2 m_1 | JM \rangle \quad (4.23)$$

The Racah symbol

$$(-1)^{a-b-c} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} := \frac{(-1)^{c-\gamma}}{\sqrt{2c+1}} \langle ab\alpha\beta | c - \gamma \rangle \quad (4.24)$$

The Racah formula

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a-b-\gamma} \sqrt{\Delta(abc)} \sqrt{(a + \alpha)!(a - \alpha)!(b + \beta)!(b - \beta)!(c + \gamma)!(c - \gamma)!} \\ \times \sum_t (-1)^t [t!(c - b + t + \alpha)!(c - a + t - \beta)!(a + b - c - t)!(a - t - \alpha)!(b - t + \beta)!]^{-1} \\ (\alpha + \beta + \gamma = 0, \quad |a - b| \leq c \leq a + b)$$

4.3 Some Examples

4.3.1 $J = j_1 + j_2$

$$|JJ\rangle = |j_1 j_1\rangle |j_2 j_2\rangle \quad (4.25)$$

$$(J_-)^k |JJ\rangle = (J_{1-} + J_{2-})^k |j_1 j_1\rangle |j_2 j_2\rangle \quad (4.26)$$

$$|JJ - k\rangle = \sum_{i=0}^k {}_k C_i \frac{F(j_1, i) F(j_2, k - i)}{F(J, k)} |j_1 j_1 - i\rangle |j_2 j_2 - (k - i)\rangle \quad (4.27)$$

$$= \sum_{i=0}^k \sqrt{\frac{2j_1 C_i \cdot 2j_2 C_{k-i}}{2J C_k}} |j_1 j_1 - i\rangle |j_2 j_2 - (k - i)\rangle \quad (4.28)$$

Rewrite it using $M = J - k = j_1 + j_2 - k$, $m_1 = j_1 - i$, $m_2 = j_2 - (k - i) = -j_1 + M + i = M - m_1$, $k - i = j_2 - m_2$

$$|j_1 + j_2 M\rangle = \sum_{m_1 + m_2 = M} \sqrt{\frac{2j_1 C_{j_1 - m_1} \cdot 2j_2 C_{j_2 - m_2}}{2J C_{j_1 + j_2 - M}}} |j_1 m_1\rangle |j_2 m_2\rangle \quad (4.29)$$

$$\langle j_1 j_2 m_1 m_2 | j_1 + j_2 M \rangle = \sqrt{\frac{2j_1 C_{j_1-m_1} \cdot 2j_2 C_{j_2-m_2}}{2(j_1+j_2) C_{j_1+j_2-M}}} \quad (4.30)$$

$$= \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!} \frac{(J+M)!(J-M)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \quad (4.31)$$

4.3.2 $j_1 = j, j_2 = 1$

We have three possibilities; $J = j + 1, j, j - 1$. Note that any $|J M\rangle$ is expanded by $|j M + 1\rangle|\mathbf{1} - 1\rangle, |j M\rangle|\mathbf{1} 0\rangle, |j M - 1\rangle|\mathbf{1} 1\rangle$.

The case $J = j + 1$.

$$\sqrt{\frac{2j C_{j-M-1}}{2(j+1) C_{j-M+1}}} = \sqrt{\frac{(2j)!}{(j-M-1)!(j+M+1)!} \frac{(j-M+1)!(j+M+1)!}{(2j+2)!}} \quad (4.32)$$

$$= \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}} \quad (4.33)$$

Table 1:	
$ j M + 1\rangle \mathbf{1} - 1\rangle$	$\sqrt{\frac{2j C_{j-M-1}}{2(j+1) C_{j-M+1}}} = \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}}$
$ j M\rangle \mathbf{1} 0\rangle$	$\sqrt{\frac{2 \cdot 2j C_{j-M}}{2(j+1) C_{j-M+1}}} = \sqrt{\frac{2(j-M+1)(j+M+1)}{(2j+2)(2j+1)}}$
$ j M - 1\rangle \mathbf{1} 1\rangle$	$\sqrt{\frac{2j C_{j-M+1}}{2(j+1) C_{j-M+1}}} = \sqrt{\frac{(j+M+1)(j+M)}{(2j+2)(2j+1)}}$

5 $6j$ -symbol

It is defined as a coupling coefficient of three angular momentum

$$\begin{aligned} & |(j_1, (j_2, j_3) j_{23}) J\rangle \\ &= \sum_{j_{12}} [(2j_{12} + 1)(2j_{23} + 1)]^{\frac{1}{2}} (-1)^{j_1+j_2+j_3+J} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} |((j_1, j_2) j_{12}, j_3) J\rangle \end{aligned} \quad (5.1)$$

$6j$ -symbol in terms of $3j$ -symbols

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= \sum_{\substack{\alpha\beta\gamma \\ \delta\epsilon\varphi}} (-1)^{d+e+f+\delta+\epsilon+\varphi} \times \\ &\times \begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix} \begin{pmatrix} e & f & a \\ \epsilon & -\varphi & \alpha \end{pmatrix} \begin{pmatrix} f & d & b \\ \varphi & -\delta & \beta \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \end{aligned} \quad (5.2)$$

Racah's formula ([2])

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= [\Delta(abc)\Delta(aef)\Delta(cde)\Delta(bdf)]^{\frac{1}{2}} \times \\ &\times \sum_x \frac{(x+1)!}{[(p_1-x)(p_2-x)!(p_3-x)!(x-q_1)!(x-q_2)!(x-q_3)!(x-q_4)!]} \end{aligned} \quad (5.3)$$

where $q_1 = a+b+c, q_2 = b+d+f, q_3 = a+e+f, q_4 = d+e+c, p_1 = a+b+d+e, p_2 = b+c+e+f, p_3 = c+a+f+d$. Only values which satisfy the triangle inequalities, $|b-c| \leq a \leq b+c$ are allowed: $(abc), (aef), (dbf), (dec)$. Therefore, there have to be even numbers of half integers at each face. This is equivalent to say that $q_1, q_2, q_3, q_4, p_1, p_2, p_3$ are all integers.

Example 5.1. $\{ j_1 = j_2 = \frac{1}{2}, j_3 = 1 \}$ The resultant angular momenta are $J = 0, 1, 2$.

$$\begin{aligned} \left(\frac{1}{2} \otimes \frac{1}{2} \right) \otimes 1 &= \left(\left[\left(\frac{1}{2}, \frac{1}{2} \right) 0 \right] \oplus \left[\left(\frac{1}{2}, \frac{1}{2} \right) 1 \right] \right) \otimes 1 \\ &= \left[\left(\left(\frac{1}{2}, \frac{1}{2} \right) 0, 1 \right) 1 \right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2} \right) 1, 1 \right) 0 \right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2} \right) 1, 1 \right) 1 \right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2} \right) 1, 1 \right) 2 \right] \\ \frac{1}{2} \otimes \left(\frac{1}{2} \otimes 1 \right) &= \frac{1}{2} \otimes \left(\left[\left(\frac{1}{2}, 1 \right) \frac{1}{2} \right] \oplus \left[\left(\frac{1}{2}, 1 \right) \frac{3}{2} \right] \right) \\ &= \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1 \right) \frac{1}{2} \right) 0 \right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1 \right) \frac{1}{2} \right) 1 \right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1 \right) \frac{3}{2} \right) 1 \right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1 \right) \frac{3}{2} \right) 2 \right] \end{aligned}$$

A Addition of Binomial Coefficients

The addition formula for binomial coefficients is given as

$$\sum_s \binom{x}{s} \binom{y}{z-s} = \binom{x+y}{z}. \quad (\text{A.1})$$

This formula can be derived by comparing the coefficients of the polynomials on the both sides of $(X+Y)^{x+y} = (X+Y)^x (X+Y)^y$. Putting $x = a-b, y = b, z = a-c$ in (A.1), we have

$$\frac{a!}{b!c!} = \sum_s \frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \quad (\text{A.2})$$

If $y < 0$,

$$\binom{y}{z-s} = (-1)^{z-s} \binom{z-s-y-1}{z-s}. \quad (\text{A.3})$$

Then (A.1) can be transformed into

$$\sum_s (-1)^s \binom{x}{s} \binom{z-s-y-1}{z-s} = (-1)^z \binom{x+y}{z}, \quad (x+y \geq 0) \quad (\text{A.4})$$

or

$$\sum_s (-1)^s \binom{x}{s} \binom{z-s-y-1}{z-s} = \binom{z-x-y-1}{z}, \quad (x+y < 0) \quad (\text{A.5})$$

Putting $y = z-t-1$, (A.4) and (A.5) become

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = (-1)^z \frac{(t-z)!(x+z-t-1)!}{x!z!(x-t-1)!}, \quad (x > t \geq z \geq 0). \quad (\text{A.6})$$

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}, \quad (t \geq x, z \geq 0). \quad (\text{A.7})$$

The following formula is referenced in [1] due to Racah, but I have not succeeded in finding out a proof or any mention to this formula so far in [2]. It still needs to be verified somehow.

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \quad (\text{A.8})$$

The following formula

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a, -b) \quad (\text{A.9})$$

is a slight generalisation of a simple binomial coefficients formula

$$(X - Y)^K = \sum_{l=0}^K (-1)^l \binom{K}{l} X^{K-l} Y^l \quad (\text{A.10})$$

(A.9) is obtained when we set $X = Y = 1$.

References

- [1] *Albert Messiah, Quantum Mechanics, Dover (1995)*
- [2] *Giulio Racah, Theory of Complex Spectra. I&II, Phys. Rev. 61&62 (1942)*
- [3] *G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients" in Spectroscopic and group theoretical methods in physics, (ed. Bloch et al), North Holland Publ. Co., Amsterdam, 1968*