Notes on "Topoi" by Goldblatt

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1 Chapter 3: Arrows instead of epsilons

1.1 3.1. Monic arrows

An arrow $f : a \to b$ in a category $\mathcal{C}$ is *monic* in $\mathcal{C}$ if for any pair $g, h : c \rightrightarrows a$ of $\mathcal{C}$ -arrows, equality $f \circ g = f \circ h$ implies that $g = h$. The symbolism $f : a \rightarrowtail b$ is used to indicate that f is monic.

1.1.1 3.1. Exercise Solutions

In any category
(1) $g \circ f$ is monic if both f and g are monic.
(2) If $g \circ f$ is monic, then so is f.
Solutions
(1)
If $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$, then

$$\begin{aligned} (g \circ f) \circ h_1 &= (g \circ f) \circ h_2 \\ g \circ (f \circ h_1) &= g \circ (f \circ h_2) \\ f \circ h_1 &= f \circ h_2 \quad (\because \text{g is monic}) \\ h_1 &= h_2 \quad (\because \text{f is monic}) \end{aligned}$$

Therefore, $(g \circ f)$ is monic.
(2)
If $f \circ h_1 = f \circ h_2$, then

$$\begin{aligned} g \circ (f \circ h_1) &= g \circ (f \circ h_2) \\ (g \circ f) \circ h_1 &= (g \circ f) \circ h_2 \\ h_1 &= h_2 \quad (\because f \circ g \text{ is monic}) \end{aligned}$$

Hence, f is monic.

1.2 3.2. Epic arrows

An arrow $f : a \to b$ is *epic* (right-cancellable) in a category $\mathcal{C}$ if for any pair $g, h : b \rightrightarrows c$, the equality $g \circ f = h \circ f$ implies $g = h$ i.e. whenever a diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ b \arrow[r, "h"] & c \end{tikzcd}$$

commutes, then $g = h$. The notation $f : a \twoheadrightarrow b$ is used for epic arrows.

1.2.1 3.2. Exercises(Duals of exercises in 3.1)

In any category
(1) $g \circ f$ is epic if both f and g are epic.
(2) If $g \circ f$ is epic, then so is g.
Solutions
(1)
Suppose that f and g are both epic, then if $h_1 \circ (g \circ f) = h_2 \circ (g \circ f)$,

$$\begin{aligned} h_1 \circ (g \circ f) &= h_2 \circ (g \circ f) \\ (h_1 \circ g) \circ f &= (h_2 \circ g) \circ f \\ h_1 \circ f &= h_2 \circ f \\ h_1 &= h_2. \end{aligned}$$

Hence, $g \circ f$ is epic.
(2)
Suppose $g \circ f$ is epic. Then, if $h_1 \circ g = h_2 \circ g$,

$$\begin{aligned} (h_1 \circ g) \circ f &= (h_2 \circ g) \circ f \\ h_1 \circ (g \circ f) &= h_2 \circ (g \circ f) \\ h_1 &= h_2. \end{aligned}$$

Hence, g is epic.

1.3 3.8 Products

DEFINITION A product in a category $\mathcal{C}$ of two objects a and b is a $\mathcal{C}$ -object $a \times b$ together with a pair $(pr_a : a \times b \to a,\ pr_b : a \times b \to b)$ of $\mathcal{C}$ -arrows such that for any pair of $\mathcal{C}$ -arrows of the form $(f : c \to a,\ g : c \to b)$ there is exactly one arrow $\langle a, g \rangle : c \to a \times b$ making

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd}$$

commute, i.e. such that $pr_a \circ \langle f, g \rangle = f$, $pr_b \circ \langle f, g \rangle = g$. $\langle f, g \rangle$ is the product arrow of f and g with respect to the projections pr_a, pr_b.