Cassinian Cookies

hisanobu-nakamura

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1 Generalized Cassinian Curves, Surfaces and Hypersurfaces

1.1 The Curves of Cassini in 2D Plane

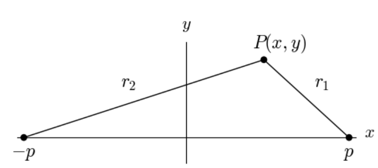

Figure 1:

The *curves of Cassini*, or *Cassinian curevs*, in $\mathbb{R}^2$ are defined as follows: given two poins $p_1, p_2 \in \mathbb{R}^2$, called focus, and a real number $a \in \mathbb{R}$, the Cassinian curve with the focses p_1, p_2 is the set of points $x \in \mathbb{R}^2$ whose product of the Euclidean distances $r_i = \sqrt{|x - p_i|}$, $(i = 1, 2)$ are always constant a^2:

$$r_1 r_2 = a^2. \tag{1.1}$$

They were first introduced by Cassini as orbits of the heavenly bodies, but later discarded as a physical model for astrophysics.

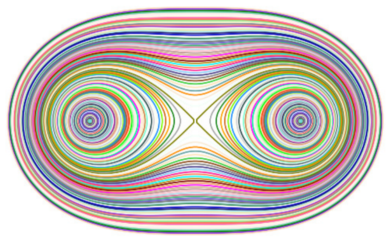

Figure 2: Cassinian curves with varying parameter a

1.2 Generalisations

There are two immediate generalisations of the definition of Cassinian curves: (i) increase the number n of foci or/and (ii) increase the dimensionality D of the ambient space $\mathbb{R}^D$. Let us denote a generic point in the ambient space $\mathbb{R}^D$ by $x = (x^1, x^2, \cdots, x^D)$, n distinct fixed points by $p_i \in \mathbb{R}^D$, $(i = 1, \cdots, n)$ and the values of the distances by $r_i := |x - p_i|$, where $|x|$ is the usual D-dimensional Euclidean distance. Then define

$$F(p_1, \cdots, p_n; x) := \prod_{i=1}^{n} r_i. \tag{1.2}$$

(D-1)-dimensional Cassinian hypersurface associated with p_i's is defined as the set of points $x \in \mathbb{R}^D$, the product of whose distances from x is equal to a constant a^2:

$$C(p_1, \cdots, p_n; a) := \left\{ x \in \mathbb{R}^D \middle| F(p_1, \cdots, p_n; x) = a^2 \right\}. \tag{1.3}$$

We will often use the abbreviation, $C(P; a) = C(p_1, \cdots, p_n; a)$ in the sequel. A further generalisation with arbitrary exponentials can be carried out to obtain *the weighted n-Cassinian curves*, which are defined by:

$$\begin{aligned} F(P; k; x) &:= \prod_{i=1}^{n} r_i^{k_i} \\ C(P; k; a) &:= \left\{ x \in \mathbb{R}^D \middle| F(P; k; x) = a^2 \right\} \end{aligned} \tag{1.4}$$

An example of a weighted Cassinian curve is illustrated in Figure 3. However, in this article, we will concentrate mainly on the case $k_i = 1, \forall i$.

When $n = 2$, we can assume that $p_1 = (p, 0, \ldots, 0)$ and $p_2 = (-p, 0, \ldots, 0)$ without loss of generality. Then

$$r_1 r_2 = \sqrt{(x^1 - p)^2 + (x^2)^2 + \cdots + (x^D)^2} \sqrt{(x^1 + p)^2 + (x^2)^2 + \cdots + (x^D)^2} = \sqrt{(x^1 - p)^2 + R^2} \sqrt{(x^1 + p)^2 + R^2} \tag{1.5}$$

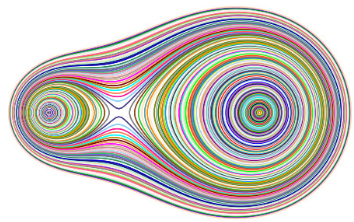

Figure 3: Weighted 2-Cassinian curve: $r_1 r_2^2 = a^2$

with $R^2 = (x^2)^2 + \cdots + (x^D)^2$. Hence, $(D-1)$-dimensional Cassinian hypersurface with two foci is like a lemniscate with its y-axes replaced by a $(D-2)$-sphere S_R^{D-2} of radius R.

1.3 Polar Coordinates and Compatibility Conditions

We can compose a list of the data $r_i := |x - p_i|$ and think of the list $(r_1, \cdots, r_n)$ as a point in $\mathbb{R}^n_{\geq 0} := \{r = (r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_i \geq 0\}$. We call $(r_1, \cdots, r_n)$ *the multipolar coordinates* of the point x. Every point $x \in \mathbb{R}^D$ has a correspondence in the multipolar coordinate space $\mathbb{R}^n_{\geq 0}$ defining a map $\phi : \mathbb{R}^D \to \mathbb{R}^n_{\geq 0}$ by $x \mapsto (r_1, \ldots, r_n)$. When $D > n$, ϕ is not injective, so a point in $\mathbb{R}^n_{\geq 0}$ corresponds to a $(D-n)$-dimensional hypersurface in $\mathbb{R}^D$. When $D = n$, a point $r \in (r_1, \cdots, r_n)$ specifies two points in $\mathbb{R}^D$ which are symmetrical about the hyperplane containing the foci p_i $(i = 1, \cdots, D)$. When $D < n$, the image $\phi(\mathbb{R}^D) \subset \mathbb{R}^n$ is a D-dimensional hypersurface, which can be specified by some polynomials in r_i and $r_{ij} := |p_i - p_j|$. Let us call those polynomials *compatibility conditions*. Usually, we can construct such expressions more than required, so we need to choose out some suitable set of independent ones. Once such a set of equations are chosen, they define a D dimensional surface S in n-tuple polar coordinate space $r_1 : \cdots : r_n \in \mathbb{R}^n_{pol}$, which represents the points in the original Euclidean space $\mathbb{R}^D$. Then, if we also plot a surface $C(P; a)$ defined by $r_1 \cdots r_n = a^2$ in the multipolar space $\mathbb{R}^n_{pol}$, the intersection $C(P; a) \cap S$ is the set corresponding to a Cassinian surface and for sufficiently small a, there are several disconnected components in general. Observations in 2 and 3 dimensional spaces tell us that there seem to be $n-1$ disconnected components with n foci in $D = 2$ but n components for certain values of a in 3 dimensional space. We have not figured out how it really works yet. So, the followings are some observations in low dimensions for small n.

1.4 The 2D Plane Case

First, let us consider the case $D = 2$ with $n = 2$ foci. Without loss of generality, we can assume the foci are of the form $p_1 = (p, 0), P_2 = (-p, 0)$. Then the bipolar coordinates $r_1 : r_2$ with $|p_1 - p_2| = 2p$ have to satisfy the compatibility conditions, which are, in this case, just triangle inequalities,

$$r_1 + r_2 \geq 2p \tag{1.6}$$
$$r_1 + 2p \geq r_2 \tag{1.7}$$
$$r_2 + 2p \geq r_1. \tag{1.8}$$

These inequalities specify a region in $\mathbb{R}^2_{\geq 0}$ which is the image $\phi(\mathbb{R}^2)$ (Figure 4). Note that the boudary of the shaded region corresponds to the line joining the two foci in the original Euclidean plane. Writing

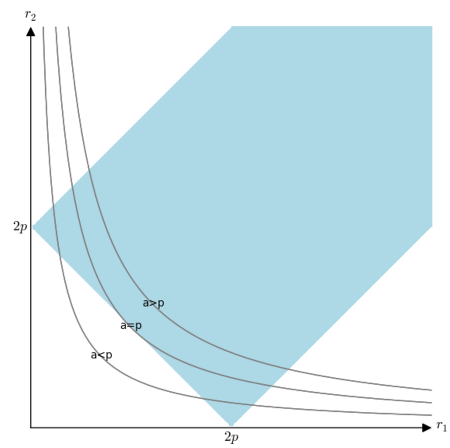

Figure 4: The diagram for the reality region and the cassinian $r_1 r_2 = a^2$ in bipolar coordinates. Only the portions of the curves intersecting with the shaded region are realised in $\mathbb{R}^2$

$r_{12} = |p_1 - p_2|$, this condition can be written symmetrically as

$$(r_1 + r_2 - r_{12})(r_1 - r_2 + r_{12})(-r_1 + r_2 + r_{12}) \geq 0. \tag{1.9}$$

Multiplying both sides by $(r_1 + r_2 + r_{12})$, which is greater than zero, we get

$$(r_1 + r_2 + r_{12})(r_1 + r_2 - r_{12})(r_1 - r_2 + r_{12})(-r_1 + r_2 + r_{12}) \geq 0. \tag{1.10}$$

The l.h.s. of this inequality is actually related to the area A of the triangle whose side lengths are r_1, r_2, r_{12} by the Heron's formula:

$$A = \sqrt{s(s - r_1)(s - r_2)(s - r_{12})}, \quad s = \frac{r_1 + r_2 + r_{12}}{2} \tag{1.11}$$

So (1.10) is equivalent to $16A^2 \geq 0$. Furthermore, the Heron's formula can be obtained by the so-called Cayley-Menger determinant defined as follows:

$$W_2(r_1, r_2; r_{12}) := \begin{vmatrix} 0 & r_1^2 & r_2^2 & 1 \\ r_1^2 & 0 & r_{12}^2 & 1 \\ r_2^2 & r_{12}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 16A^2 \tag{1.12}$$

We will use this formula to obtain compatibility conditions in higher dimensions later. For the case $n = 3$, we have a redundancy in the number of variables r_i, so one degree of freedom needs to be eliminated by some equation. To obtain such an equation, we can naively use the cosine rule for angles $\alpha = \angle BxC, \beta = \angle CxA, \gamma = \angle AxB$,

$$\begin{aligned} f_{23} &:= r_2^2 + r_3^2 - r_{23}^2 = 2r_2 r_3 \cos\alpha && (1.13) \\ f_{31} &:= r_3^2 + r_1^2 - r_{31}^2 = 2r_1 r_3 \cos\beta && (1.14) \\ f_{12} &:= r_1^2 + r_2^2 - r_{12}^2 = 2r_1 r_2 \cos\gamma && (1.15) \\ f_{123} &:= r_1^2 r_2^2 r_3^2, && (1.16) \end{aligned}$$

where $r_{ij} := |p_i - p_j|$ and the fact that $\alpha + \beta + \gamma = 2\pi$ to eliminate cosines, we obtain the compatibility condition for the tripolar coordinates $r_1 : r_2 : r_3$;

$$F_3(r_1, r_2, r_3; r_{12}, r_{23}, r_{31}) := f_{23}^2 r_1^2 + f_{31}^2 r_2^2 + f_{12}^2 r_3^2 - 4f_{123} - f_{23} f_{31} f_{12} = 0. \tag{1.17}$$

Figure 5 shows a plot of the compatibility surface in the tripolar coordinates space. Note also that the region inside the purple surface corresponds to two points symmetric in the plane which contains the three foci in the original Euclidean 3 dimensional space.
The above compatibility condition can also be stated geometrically: the volume of the tetrahedron formed with the the 6 sides of the lengths $(r_1, r_2, r_3; r_{12}, r_{23}, r_{31})$ is zero i.e. the vertices of the tetrahedron are all in the same 2D plane.

$$W_3(r_1, r_2, r_3; r_{12}, r_{23}, r_{31}) = \begin{vmatrix} 0 & r_1^2 & r_2^2 & r_3^2 & 1 \\ r_1^2 & 0 & r_{12}^2 & r_{13}^2 & 1 \\ r_2^2 & r_{21}^2 & 0 & r_{23}^2 & 1 \\ r_3^2 & r_{31}^2 & r_{32}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{1.18}$$

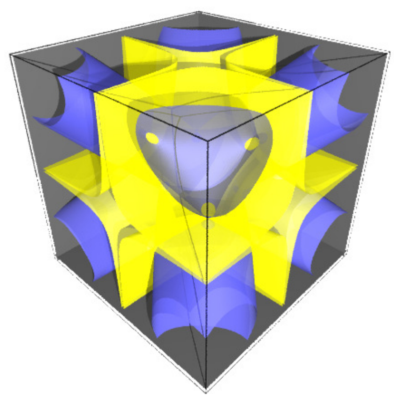

Figure 5: A plot of the compatibility surface (purple) for a triangle in tripolar coordinates and the surface (yellow) of points on cassinian curves . Only one sector is valid, but all of them are shown for the visual purpose.

For $n = 4$ or quadrilateral case, the compatibility condition can be expressed as a pair of simultaneous equations. For example, if we calculate one of the diagonal length l_1, then we have two compatibility conditions for the two triangles

$$W_3(r_1, r_2, r_3; r_{12}, r_{23}, r_{31}) = W_3(r_2, r_3, r_4; r_{23}, r_{34}, r_{42}) = 0 \tag{1.19}$$

And then other two relations which involve the other diagonal must be redundant. In general, compatibility conditions for $n \geq 3$ points $p_i, (i = 1, \cdots, n)$ can be written as

$$F_3(r_1, r_2, r_3) = F_3(r_2, r_3, r_4) = \cdots = F_3(r_{n-3}, r_{n-2}, r_{n-1}) = F_3(r_{n-2}, r_{n-1}, r_n) = 0 \tag{1.20}$$

1.5 The Case in 3D

When $D = 3$, in the cases $n = 2$ and $n = 3$, the singularities appear in the plane where the focuses lie. So the first non-trivial appearance of the singularities occurs when we have $n = 4$ non-coplanar foci, which can be seen as the vertices of a tetrahedron. Th analogy with triangle will lead us to the similar argument for the construction of the compatibility condition for the quadripolar coordinates. That is to divide the tetrahedron $\triangle ABCD$ into four sub-tetrahedra $\triangle xABC$, $\triangle xACD$, $\triangle xABD$ and $\triangle xBCD$ and calculate the solid angles subtended by the vertex x, and then eliminate them by using the condition that they must sum up to 4π. But we are not going to use this rather naive method here. Instead, we resort to the Cayley determinant for the pentachoron.

$$W_4 = \begin{vmatrix} 0 & r_1^2 & r_2^2 & r_3^2 & r_4^2 & 1 \\ r_1^2 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_2^2 & r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & 1 \\ r_3^2 & r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & 1 \\ r_4^2 & r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{1.21}$$

1.5.1 Examples

The graphs of some particular Cassinian curves with varying parameter a are listed below. The images were created by using Sage graphics plot.

Example 1.1. *Figure 6 :* $p_1 = (0,1),\ p_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}),\ p_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$
Figure 7 : $p_1 = (-1,0),\ p_2 = (0,0),\ p_3 = (1,0)$

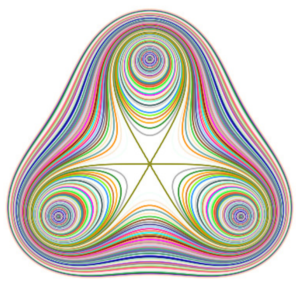

Figure 6:

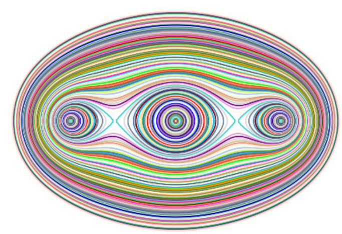

Figure 7:

Example 1.2.

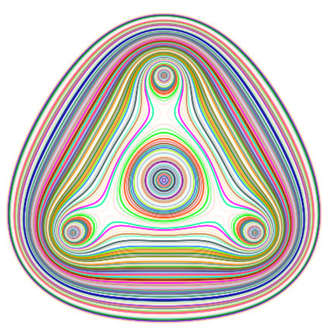

Figure 8:

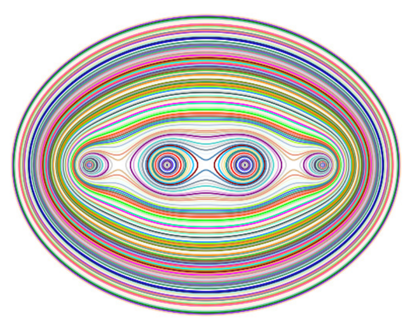

Figure 9:

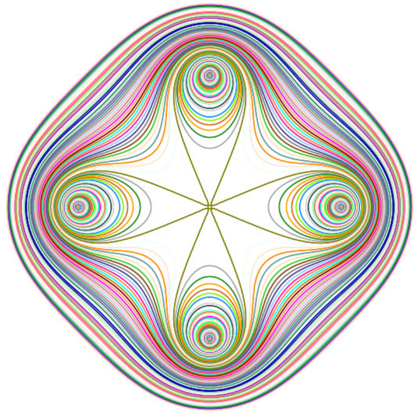

Figure 10:

Example 1.3. $p_1 = (0,1), p_2 = (-\sin\frac{2}{5}\pi, \cos\frac{2}{5}\pi), p_3 = (-\sin\frac{4}{5}\pi, \cos\frac{4}{5}\pi), p_4 = (\sin\frac{4}{5}\pi, \cos\frac{4}{5}\pi), p_5 = (\sin\frac{2}{5}\pi, \cos\frac{2}{5}\pi)$

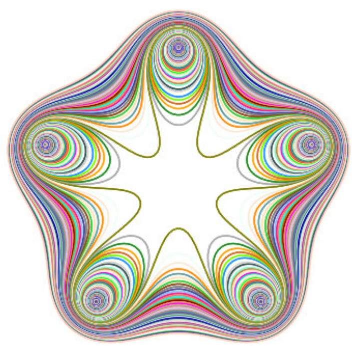

Figure 11:

1.6 Connected Components

Proposition 1.1. *Let $p_i, (i = 1, \cdots, n)$ be distinct points in $\mathbb{R}^D$. Then, for sufficiently small $\delta > 0$, the inverse image $C(p_1, \cdots, p_n; \delta)$ have at least n non-intersecting components homeomorphic to S^{D-1} centred at p_i.*

Proof. Let $r_{\min} := \min_{i,j}\{r_{ij}\}$, $\delta < \frac{r_{\min}}{2}$ and $B(p_i;\delta) := \left\{x \in \mathbb{R}^D \middle| \delta > |x - p_i|\right\}$. Let us denote the minimum pair $r_{ab} = r_{\min}$. Then $B(p_i;\delta) \cap B(p_j;\delta) = \emptyset, (\forall i \neq j)$. We are only interested in their relative positions. So, by multiplying all the coordinates with a suitable constant, we can assume $r_{\min} > 2$ so that $\delta < 1$. Now, consider the inverse image $C(p_1, \cdots, p_n; \delta)$. Because we have assumed $r_{\min} > 2$ and $\delta < 1$, for $x \in B(p_a;\delta)$, we have $r_i = |x - p_i| > 1, (i \neq a)$.

$$r_a < r_a \prod_{i \neq a} r_i = F(p; x). \tag{1.22}$$

Now, consider a (D-1)-sphere centred at p_a with some radius $\rho < \delta$, S_ρ^{D-1}. Take a point $y \in S_\rho^{D-1}$, then connect it with the centre p_a by line joining them $\overrightarrow{\mathbf{yp}_a}$. We want to show that there is a value ρ_0 such that for every point $y \in S_{\rho_0}^{D-1}$, there exists a point $\mathbf{x}_0$ on the line $\overrightarrow{\mathbf{yp}_a}$ which satisfies $F(\mathbf{x}_0) = \rho_0$. To prove this, let us denote a point $\mathbf{x}$ on the line $\overrightarrow{yp_a}$ by $\mathbf{x} = p_a + \rho\hat{r}$, where $\hat{r} := \frac{\overrightarrow{p_a y}}{|\overrightarrow{p_a y}|}$. Then,

$$F(p; \mathbf{x}(\rho)) = \prod_{i=1}^{n} r_i(\rho) = \rho \prod_{i \neq a} (\rho^2 + r_{ia}^2 - 2\rho < \mathbf{r}_{ia}, \hat{r} >)^{\frac{1}{2}} \tag{1.23}$$

is a strictly incresing funtion of ρ for sufficiently small ρ. Indeed

$$\begin{aligned}
\frac{dF}{d\rho} &= \prod_{i \neq a} r_i(\rho) + \rho \sum_{j \neq a} \frac{\rho - < \mathbf{r}_{ja}, \hat{r} >}{r_j(\rho)} \prod_{i \neq a, j} r_i(\rho) \\
&= \prod_{i \neq a} r_i(\rho) \left(1 + \rho \sum_{j \neq a} \frac{< \rho\hat{r} - \mathbf{r}_{ja}, \hat{r} >}{r_j(\rho)^2} \right) \\
&= \prod_{i \neq a} r_i(\rho) \left(1 + \rho \sum_{j \neq a} \frac{< \mathbf{r}_j(\rho), \hat{r} >}{r_j(\rho)^2} \right) \\
&= \prod_{i \neq a} r_i(\rho) \left(1 + \rho < \sum_{j \neq a} \tilde{\mathbf{r}}_j(\rho), \hat{r} > \right)
\end{aligned} \tag{1.24}$$

Since $\exists M$ such that $\forall \rho \in [0, \delta], | < \sum_{j \neq a} \tilde{\mathbf{r}}_j(\rho), \hat{r} > | < M$, the quantity inside the bracket is positive for sufficiently small ρ so that $\frac{dF}{d\rho} > 0$.

$$\rho_{min} := \min_{\hat{r} \in S_{D-1}} \left\{ \rho : 1 + \rho < \sum_{j \neq a} \tilde{\mathbf{r}}_j(\rho), \hat{r} > > 0 \right\} \tag{1.25}$$

Then, by (1.22), for each point $\mathbf{y}$ on $S_{\rho_{min}}^{D-1}$

$$F(\mathbf{y}) > \rho_{min} \tag{1.26}$$

But, now $F(p; \mathbf{x}(\rho))$ is a strictly increasing function of ρ for any $\hat{r} := \frac{\overrightarrow{p_a y}}{|\overrightarrow{p_a y}|}$, so there exists exactly one ρ_0 such that $F(p; \mathbf{x}(\rho_0(\hat{r}))) = \rho_{min}$ for each $\hat{r} \in S^{D-1}$. This means there is a disconnected component of the inverse image of $C(p_1, \cdots, p_n; \rho_{min}) = 0$ around p_a homeomorphic to S^{D-1}. □

But, the fact is, this is not just good enough to tell you all the components to appear when you vary the value a. For $D = 3$, Cassinian surfaces with the points located on the unit sphere appear to have an extra component which does'nt contain a focus inside. The central component emerges as a point from the critical point at the origin when $a = 1$.

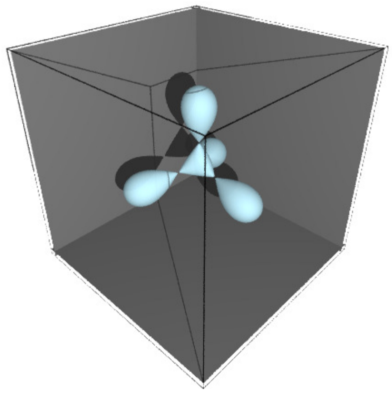

Figure 12: Cassinian surface with foci at vertices of a regular tetrahedron.

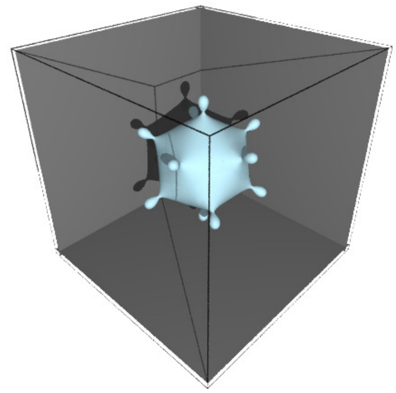

Figure 13: Cassinian surface with foci at vertices of a regular icosahedron.

1.7 Singular points

We want to determine the singular points for the level curve of the function

$$F(x) = \prod_{i=1}^{n} r_i \tag{1.27}$$

that is, points x_0 with $\partial_\mu F(x_0) := \frac{\partial F}{\partial x^\mu}(x_0) = 0$ other than foci where total derivatives are not defined. Thus, we will assume that $x_0 \neq p_i$. Then, $r_i \neq 0$, so we can divide the partial derivatives by $r_1 \cdots r_n$, which yields, as a stationary condition

$$\frac{1}{2r_1 \cdots r_n} \begin{pmatrix} \partial_1 F(x_0) \\ \vdots \\ \partial_D F(x_0) \end{pmatrix} = \sum_{i=1}^{n} \frac{1}{r_i^2} x_i = 0 \tag{1.28}$$

where $x_i = x_0 - p_i$. By writing the image of x_i under a geometric inversion $\mathcal{I}_{S^{D-1}}$ in unit $D-1$ sphere S^{D-1} centred at x_0 as $\tilde{x}_i = \mathcal{I}_{S^{D-1}}(x_i)$, the condition reads as

$$\frac{1}{n} \sum_{i=1}^{n} \tilde{x}_i = 0 \tag{1.29}$$

which translates geometrically as the barycentre of the polygon whose vertices are the inverse images of the foci must coincide with the centre of the inversion. For $n = 2$, it is easy to see that the barycentre is the midpoint of the two foci in any dimension D.
It can be expressed in terms of mechanical language too. If we consider a set of n points with equal mass located at p_i's, the equation (1.28) means that the sum of the centrifugal forces at point x is zero.

Knowing the condition for the self-intersection points, we want to know the barycentre of a polytope in terms of multipolar coordinates. It is actually easy to obtain;

Proposition 1.2. *Let $p_1 \cdots, p_n$ be distinct points in $\mathbb{R}^D$ and $b := \frac{1}{n}\sum_{i=1}^{n} p_i$ be the barycentre. Then the multipolar coordinates $r_i := |p_i - b|$ of the barycentre is given by*

$$r_i = \frac{1}{n} \sqrt{(n-1) \sum_{i \neq j} r_{ij}^2 - \sum_{\substack{j<k \\ j,k \neq i}} r_{jk}^2} \tag{1.30}$$

Proof.

$$
\begin{aligned}
|p_1 - b|^2 &= \frac{1}{n^2}|(n-1)p_1 - (p_2 + \cdots + p_n)|^2 \\
&= \frac{1}{n^2}|\mathbf{r}_{12} + \cdots + \mathbf{r}_{1n}|^2 \\
&= \frac{1}{n^2}\left\{\sum_{1\neq j} r_{1j}^2 + 2\sum_{\substack{j<k \\ j,k\neq 1}} \mathbf{r}_{1j}\cdot\mathbf{r}_{1k}\right\}
\end{aligned} \tag{1.31}
$$

where $\mathbf{r}_{ij} := p_i - p_j$ and $r_{ij} = |\mathbf{r}_{ij}|$. Then, from the cosine rule, $2\mathbf{r}_{1j}\cdot\mathbf{r}_{1k} = r_{1j}^2 + r_{1k}^2 - r_{jk}^2$,

$$
\begin{aligned}
|p_1 - b|^2 &= \frac{1}{n^2}\left\{\sum_{1\neq j} r_{1j}^2 + \sum_{\substack{j<k \\ j,k\neq 1}} r_{1j}^2 + r_{1k}^2 - r_{jk}^2\right\} \\
&= \frac{1}{n^2}\left\{\sum_{1\neq j} r_{1j}^2 + \sum_{\substack{j<k \\ j,k\neq 1}} (r_{1j}^2 + r_{1k}^2) - \sum_{\substack{j<k \\ j,k\neq 1}} r_{jk}^2\right\} \\
&= \frac{1}{n^2}\left\{\sum_{1\neq j} r_{1j}^2 + (n-2)\sum_{1\neq j} r_{1j}^2 - \sum_{\substack{j<k \\ j,k\neq 1}} r_{jk}^2\right\} \\
&= \frac{1}{n^2}\left\{(n-1)\sum_{1\neq j} r_{1j}^2 - \sum_{\substack{j<k \\ j,k\neq 1}} r_{jk}^2\right\}
\end{aligned}
$$

□

A way to remember (1.30) is that the first sum inside the square root consists of edges connected to p_i and the second sum contains those not connected to p_i. Let us denote the barycentre determined by the data $\{r_{ij}\}_{1\leq i<j\leq n}$, the distances between points p_i's as

$$
Bary(r_{ij}) := r_1 : \cdots : r_n. \tag{1.32}
$$

Then the explicit condition for the stationary points is

$$
Bary\left(\frac{r_{ij}}{r_i r_j}\right) := \frac{1}{r_1} : \cdots : \frac{1}{r_n}. \tag{1.33}
$$

By substituting $u_i = \frac{1}{r_i^2}$, and writing $R_{ij} := r_{ij}^2$

$$
n^2 u_i = (n-1)\sum_{i\neq j} R_{ij}u_i u_j - \sum_{\substack{j<k \\ j,k\neq i}} R_{jk}u_j u_k \tag{1.34}
$$

Hence

$$
\left\{(n-1)\sum_{i\neq j} R_{ij}u_j - n^2\right\} u_i = \sum_{\substack{j<k \\ j,k\neq i}} R_{jk}u_j u_k \tag{1.35}
$$

If the quatity inside the bracket on the l.h.s. is non-zero for all i, we can obtain an equation for u_i of degree $n+1$, by eliminating other indices,

$$
A_{n+1}(R;i)u_i^{n+1} + \cdots + A_1(R;i)u_i + A_0(R;i) = 0. \tag{1.36}
$$

Together with n equations of the above form, we also have combpatibility conditions. And by solving the system of equations or equations and inequalities, we can determine the polar coordinates of the singular points and hence the values a^2 at which they appear.
UNSOLVED:Can we know the number of the positive roots to this equation?
The use of Groebner basis may solve the problem. An observation tells us that it seems that there $n-1$ solutions when all the foci are in the same 2-D plane and n solutions when $n \geq 4$ and all the foci are in the same 3-D hyperplane. What about for $n \geq 5$ when all the foci are in the same 4-D hyperplane?

In $D=2$ case, we can use complex numbers to obtain the same result. For that end, let us consider the polynomial function $P(z)$

$$w = P(z) = (z-p_1)(z-p_2)\cdots(z-p_n) \tag{1.37}$$

Then, Cassinian curves are defined to be the set

$$C(P,a) := \{z \in \mathbb{C} |\, |P(z)| = a\}. \tag{1.38}$$

It can also be seen as the inverse image of a circle of radius a centred at the origin. And the singular points are simply the zeros of the derivative $\frac{dP}{dz}$: that is

$$\frac{dP}{dz} = \sum_{i=1}^{n} (z-p_1) \overset{\overset{i}{\smile}}{\cdots} (z-p_n) = 0 \tag{1.39}$$

where $\overset{\overset{i}{\smile}}{\cdots}$ means i-th product is omitted. It is obvious $\frac{dP}{dz}(p_i) \neq 0$ for all p_i. So, we can assume $z \neq p_i$ and have

$$\sum_{i=1}^{n} \frac{1}{z-p_1} = 0 \tag{1.40}$$

which is the same as the geometric inversion except that the orientation is reversed in this case.

1.7.1 Some explicit calculations

Let us consider the case $n=3$, where we can work again in the 2 dimensional plane which contains the foci. Then, for a triangle $\triangle ABC$ with sides' lengths (a,b,c) and a point on the plane, let us call the triplet $x:y:z$ of the distances AP, BP and CP respectively, the tripolar coordinates. Then the tripolar coordinates of the barycentre of $\triangle ABC$ is given by

$$Bary(a,b,c) := \frac{1}{3}\sqrt{2(b^2+c^2)-a^2} : \frac{1}{3}\sqrt{2(c^2+a^2)-b^2} : \frac{1}{3}\sqrt{2(a^2+b^2)-c^2} \tag{1.41}$$

Notice that $Bary(ka,kb,kc) = k\ Bary(a,b,c)$. Now, from inversion geometry, the lengths (a',b',c') of the sides of $\triangle \tilde{A}\tilde{B}\tilde{C}$ are given by

$$(a',b',c') = \left(\frac{a}{yz}, \frac{b}{zx}, \frac{c}{xy}\right). \tag{1.42}$$

Therefore, the stationary condition now reads as

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = Bary(a',b',c') = Bary\left(\frac{a}{yz}, \frac{b}{zx}, \frac{c}{xy}\right). \tag{1.43}$$

multiplying both sides by xyz, we get

$$yz : zx : xy = Bary(ax, by, cz). \tag{1.44}$$

The solutions for these equations should give us the stationary points.

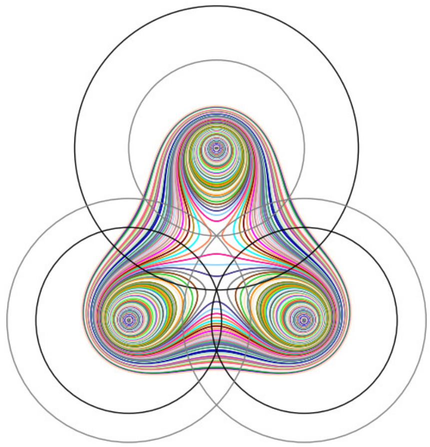

Figure 14: The position of the singular points are calculated in tripolar coordinates and then plotted as the points of intersections of three circles centred at the foci. There are two such singular points where three circles of the same colour meet.

2 Surfaces of Arbitrary Genus Constructed from Generalised Cassinian Curves

Refinement of a statement made in a problem in the book (Morris [1] p28. problem 12)

Theorem 2.1. *If a curve defined by $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$, is closed and has $n-1$ crossings, then we can construct a genus n surface in $\mathbb{R}^3$ by setting*

$$F(x, y)^2 - (r^2 - z^2) = 0 \tag{2.1}$$

for some r.

Proof. First, we want to show for some $r > 0$, if $F^{-1}(r)$ is connected regular (a Jordan curve) then $F^{-1}(-r)$ consist of n compnents (the case $F^{-1}(-r)$ is connected is really the same if set $F' = -F$). Then, from the factorisation

$$(F(x, y) + \sqrt{r^2 - z^2})(F(x, y) - \sqrt{r^2 - z^2}) = 0 \tag{2.2}$$

We can see that the level cruves at $z = \pm r$ have $n - 1$ crossings and for $z \in (-r, r)$ split into the outer curve $F^{-1}(\sqrt{r^2 - z^2})$ and the inner n curves $F^{-1}(-\sqrt{r^2 - z^2})$. □

CONSIDERATION: Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ has $n-1$ crossings (the number of critical points may be less than the number of crossings), say $\{p_i\}$, and every inverse image $F^{-1}(a)$ for $a \in F(\mathbb{R}^2)$ is closed and denote $z_i = F(p_i)$ and set $M := \max\{z_i\}$ and \$ m :=min {zi\}\$. For $a > M$, $F^{-1}(a)$ is connected(?),and does $F^{-1}(a < m)$ have n components? ANSWER: In general, $F^{-1}(a > M)$ is not connected, but if $F^{-1}(a)$ is connected for $a > M$, then .

Question: If $F^{-1}(a > M)$ is connected , by suitably adjusting the constant, we can assume $M = 0$. If we pick up a $r < m - M$, $r \in F(\mathbb{R}^2)$, then a surface defined by

$$F(x, y)^2 + (r^2 - z^2) = 0 \tag{2.3}$$

has genus n ?

2.1 Examples of Cassinian Cookies

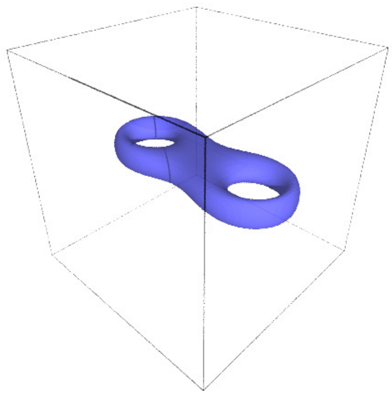

Figure 15:

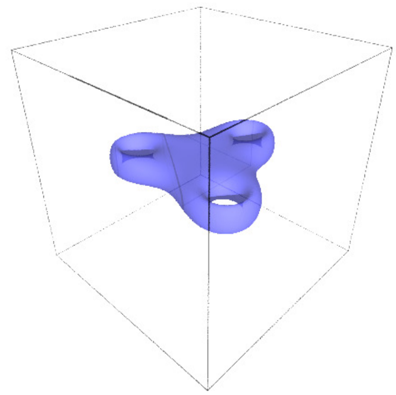

Figure 16:

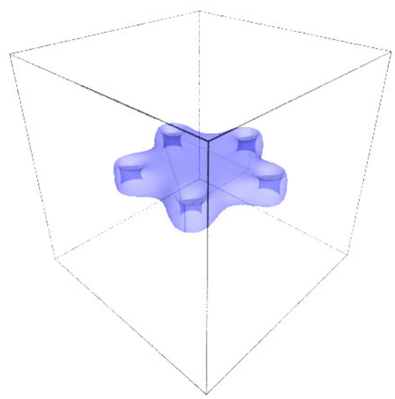

Figure 17:

3 Cassinian Cookies in 4D

$$F(x^1, x^2, x^3)^2 - (r^2 - (x^4)^2) = 0 \tag{3.1}$$

When $n = 2$, we can assume that $p_1 = (p, 0, 0)$ and $p_2 = (-p, 0, 0)$ without loss of generality. $r_1 = \sqrt{(x^1 - p)^2 + (x^2)^2 + (x^3)^2} = \sqrt{(x^1 - p)^2 + R^2}$, $r_2 = \sqrt{(x^1 + p)^2 + (x^2)^2 + (x^3)^2} = \sqrt{(x^1 + p)^2 + R^2}$ where $R^2 = (x^2)^2 + (x^3)^2$

$$F(x^1, x^2, x^3) = r_1 r_2 - a^2 \tag{3.2}$$

4 The Limit $n \to \infty$ with Points on Unit Circle

Consider points of regular n-gon $\{p_i = (\cos\theta_i, \sin\theta_i) | \theta_i = \frac{2i\pi}{n},\ i = 0, \cdots, n-1\}$. And set

$$F_n(x) := \prod_{i=0}^{n-1} |x - p_i| \tag{4.1}$$

What will happen when we take the limit $n \to \infty$?

A Compatibility Conditions For A Triangle

Use the cosine rule for each side:

$$r_{12}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\alpha \tag{A.1}$$
$$r_{31}^2 = r_3^2 + r_1^2 - 2r_3 r_1 \cos\beta \tag{A.2}$$
$$r_{23}^2 = r_2^2 + r_3^2 - 2r_2 r_3 \cos\gamma \tag{A.3}$$

and also the fact that the total angle is $\alpha + \beta + \gamma = 2\pi$ so

$$\cos(\alpha + \beta + \gamma) = 1. \tag{A.4}$$

By addition theorem, this implies

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma. \tag{A.5}$$

Equating the first three expressions to cosines and then substituing them into (A.5) yields the compatibility condition for triangles.

B Solid Angles and Compatibility Conditions For A Tetrahedron

The spherical cosine rule: consider a triangle on a sphere i.e. a region on a sphere bounded by three distinct greater circles. Let us denote the length of the three sides by α, β and γ, and the corresponding opposite dihedral angles by A, B and C.

$$\cos C = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta} \tag{B.1}$$

Let us denote the solid angles above respectively by Θ_{ABC}, Θ_{ACD}, Θ_{ABD} and Θ_{BCD}. And let us denote by, for example, θ_{BAC} the dihedral angle subtended by the greater circles on the unit sphere centred at x, which pass through the projected points of A, B and C. Then it follows that

$$\Theta_{ABC} = \theta_{BAC} + \theta_{ACB} + \theta_{ABC} - \pi \tag{B.2}$$
$$\Theta_{ACD} = \theta_{ACD} + \theta_{CDA} + \theta_{CAD} - \pi \tag{B.3}$$
$$\Theta_{ABD} = \theta_{ABD} + \theta_{BDA} + \theta_{BAD} - \pi \tag{B.4}$$
$$\Theta_{BCD} = \theta_{BCD} + \theta_{CBD} + \theta_{BDC} - \pi \tag{B.5}$$

And then the compatibility condition for the tetrahedron is

$$\Theta_{ABC} + \Theta_{ABD} + \Theta_{ACD} + \Theta_{BCD} = 4\pi \tag{B.6}$$

But we also have $\theta_{BAC} + \theta_{CAD} + \theta_{DAB} = 2\pi$ for each vertices.

References

[1] Morris W. Hirsch, Differential Topology, Springer Verlag, 1976

[2] A. Cayley, The Cambridge Mathematical Journal, vol. II, 267-271, 1841