Cayley-Menger Determinants

hisanobu-nakamura

<2020-02-26 >

Title: Cayley-Menger Determinants Date: 2020-02-26 Category: math Tags: math, Cayley-Menger

1 Cayley-Menger determinants and the Volume of n-Simplex

Cayley uses the multiplication formula for the determinants of two matrices $A = (a_{ij})$ and $B = (b_{ij})$.

$$\det AB = \det A \det B \tag{1.1}$$

to deduce the volume of simplexes in n dimensions. Given that $\mathbf{p}_i \in \mathbb{R}^n$ $(n = 0, \ldots, n)$

$$A_n = \begin{pmatrix} |\mathbf{p}_0|^2 & -2\mathbf{p}_0 & 1 \\ |\mathbf{p}_2|^2 & -2\mathbf{p}_1 & 1 \\ \vdots & \vdots & \vdots \\ |\mathbf{p}_n|^2 & -2\mathbf{p}_n & 1 \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 1 & \mathbf{p}_0 & |\mathbf{p}_0|^2 \\ 1 & \mathbf{p}_1 & |\mathbf{p}_1|^2 \\ \vdots & \vdots & \vdots \\ 1 & \mathbf{p}_n & |\mathbf{p}_n|^2 \\ 0 & \mathbf{0} & 1 \end{pmatrix} \tag{1.2}$$

$$\begin{aligned} \det A_n B_n &= \det A_n \det B_n \\ &= \det A_n \det B_n^t \\ &= \det A_n B_n^t \\ &= \begin{vmatrix} 0 & r_{01}^2 & r_{02}^2 & \cdots & \cdots & r_{0n}^2 & 1 \\ r_{10}^2 & 0 & r_{12}^2 & \cdots & \cdots & r_{1n}^2 & 1 \\ r_{20}^2 & r_{21}^2 & 0 & \cdots & \cdots & r_{2n}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 0 & r_{n-1,n}^2 & 1 \\ r_{n0}^2 & r_{n1}^2 & r_{n2}^2 & \cdots & r_{n,n-1}^2 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix} \end{aligned} \tag{1.3}$$

$$\begin{aligned} \det A_n &= \begin{vmatrix} -2\mathbf{p}_0 & 1 \\ -2\mathbf{p}_1 & 1 \\ \vdots & \vdots \\ -2\mathbf{p}_n & 1 \end{vmatrix} = (-2)^n \begin{vmatrix} \mathbf{p}_0 & 1 \\ \mathbf{p}_1 - \mathbf{p}_0 & 0 \\ \vdots & \vdots \\ \mathbf{p}_n - \mathbf{p}_0 & 0 \end{vmatrix} \\ &= (-2)^n \begin{vmatrix} \mathbf{p}_{10} \\ \mathbf{p}_{20} \\ \vdots \\ \mathbf{p}_{n0} \end{vmatrix} = (-2)^n n! V_n \end{aligned} \tag{1.4}$$

$$B_n = n! V_n \tag{1.5}$$

where V_n is the volume of the n-simplex spanned by $\mathbf{p}_{i0}$ $(i = 1, \cdots, n)$. So

$$V_n^2 = \frac{1}{(-2)^n (n!)^2} \det A_n B_n^t \tag{1.6}$$

2 Heron's Formula

Note that 2-D version of the (1.6) gives the famous Heron's fromula for the area of a triangle, so this can be considered as the extension of the Heron's formula to higher dimensions. Indeed, putting $a = r_{01}, b = r_{02}, c = r_{12}$,

$$\begin{aligned}
\det A_2B_2 &= \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} && (2.1)\\
&= \begin{vmatrix} a^2 & -2a^2 & -a^2-b^2+c^2 \\ b^2 & -a^2-b^2+c^2 & -2b^2 \\ 1 & 0 & 0 \end{vmatrix} && (2.2)\\
&= -(a^2+b^2-c^2)^2 + 4a^2b^2 && (2.3)\\
&= (a+b+c)(-a+b+c)(a-b+c)(a+b-c) && (2.4)
\end{aligned}$$

yielding the famous Heron's formula for the area of a triangle with side lengths a, b, c

$$V_2 = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.5}$$

where $s = \frac{a+b+c}{2}$.

3 Volume of a Tetrahedron

$$\begin{aligned}
\det A_3B_3 &= \begin{vmatrix} 0 & a^2 & b^2 & d^2 & 1 \\ a^2 & 0 & c^2 & e^2 & 1 \\ b^2 & c^2 & 0 & f^2 & 1 \\ d^2 & e^2 & f^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} && (3.1)\\
&= \begin{vmatrix} -2a^2 & -a^2-b^2+c^2 & -a^2-d^2+e^2 \\ -a^2-b^2+c^2 & -2b^2 & -b^2-d^2+f^2 \\ -a^2-d^2+e^2 & -b^2-d^2+f^2 & -2d^2 \end{vmatrix} && (3.2)\\
&= && (3.3)
\end{aligned}$$

4 Examples

Cayley uses the multiplication formula for the determinants of two matrices $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$.

$$\det AB = \det A \det B \tag{4.1}$$

to deduce relations between distances of points in various situation, such as those of 5 points in three dimensional space, 4 points on a sphere, etc. For example, consider 5 points $\mathbf{p}_i = (x_i, y_i, z_i, w_i) \in \mathbb{R}^4$ in 4 dimensional Euclidean space, and form the following two 6×6 matrices

$$A = \begin{pmatrix} |\mathbf{p}_1|^2 & -2\mathbf{p}_1 & 1 \\ |\mathbf{p}_2|^2 & -2\mathbf{p}_2 & 1 \\ |\mathbf{p}_3|^2 & -2\mathbf{p}_3 & 1 \\ |\mathbf{p}_4|^2 & -2\mathbf{p}_4 & 1 \\ |\mathbf{p}_5|^2 & -2\mathbf{p}_5 & 1 \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \mathbf{p}_1 & |\mathbf{p}_1|^2 \\ 1 & \mathbf{p}_2 & |\mathbf{p}_2|^2 \\ 1 & \mathbf{p}_3 & |\mathbf{p}_3|^2 \\ 1 & \mathbf{p}_4 & |\mathbf{p}_4|^2 \\ 1 & \mathbf{p}_5 & |\mathbf{p}_5|^2 \\ 0 & \mathbf{0} & 1 \end{pmatrix} \tag{4.2}$$

Then, take the determinant of the product of the two matrices

$$\begin{aligned}
\det AB &= \det A \det B \\
&= \det A \det B^t \\
&= \det AB^t \\
&= \begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & r_{15}^2 & 1 \\ r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & r_{25}^2 & 1 \\ r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & r_{45}^2 & 1 \\ r_{51}^2 & r_{52}^2 & r_{53}^2 & r_{54}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}
\end{aligned} \tag{4.3}$$

Then, he set $w_i = 0, (i = 1, \cdots, 5)$ so that the determinant becomes zero and hence obtained a relation among $r_{ij} = |\mathbf{p}_i - \mathbf{p}_j|$. This amounts to restricting the positions of the points $\mathbf{p}_i, (i = 1 \cdots, 5)$ in the 3-dimensional hyperplane defined by $w_i = 0$. However, we can give a more general meaning to the condition $W = 0$. That is, if $W = 0$, then $\mathbf{p}_i$ are in a 3-D hyperplane. We can see it by recognising W as a constant multipple of the 4 dimensional volume of the parallelochoron formed by $\mathbf{p}_i$. Indeed

$$\begin{aligned}
\det A &= \begin{vmatrix} -2\mathbf{p}_1 & 1 \\ -2\mathbf{p}_2 & 1 \\ -2\mathbf{p}_3 & 1 \\ -2\mathbf{p}_4 & 1 \\ -2\mathbf{p}_5 & 1 \end{vmatrix} = \begin{vmatrix} -2(\mathbf{p}_1 - \mathbf{p}_5) & 0 \\ -2(\mathbf{p}_2 - \mathbf{p}_5) & 0 \\ -2(\mathbf{p}_3 - \mathbf{p}_5) & 0 \\ -2(\mathbf{p}_4 - \mathbf{p}_5) & 0 \\ -2\mathbf{p}_5 & 1 \end{vmatrix} \\
&= 16 \begin{vmatrix} \mathbf{p}_{15} \\ \mathbf{p}_{25} \\ \mathbf{p}_{35} \\ \mathbf{p}_{45} \end{vmatrix} = 16V_4
\end{aligned} \tag{4.4}$$

where we defined $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$. Similarly, $\det B = V$. Then we have

$$16V_4^2 = \begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & r_{15}^2 & 1 \\ r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & r_{25}^2 & 1 \\ r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & r_{45}^2 & 1 \\ r_{51}^2 & r_{52}^2 & r_{53}^2 & r_{54}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} \tag{4.5}$$

which gives the volume of the parallelochoron in terms of the lengths of the edges. So, its volume being zero means $\mathbf{p}_{i5}, (i \neq 5)$ are linearly dependent i.e. contained in a 3-D hyperplane.

5 Five points in a plane

For five points in a 2-D plane, we have

$$\begin{vmatrix} 0 & r_{13}^2 & r_{14}^2 & r_{15}^2 & 1 \\ r_{31}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{41}^2 & r_{43}^2 & 0 & r_{45}^2 & 1 \\ r_{51}^2 & r_{53}^2 & r_{54}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_{21}^2 & 0 & r_{23}^2 & r_{24}^2 & 1 \\ r_{31}^2 & r_{32}^2 & 0 & r_{34}^2 & 1 \\ r_{41}^2 & r_{42}^2 & r_{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{5.1}$$

6 References

1. A. Cayley, The Cambridge Mathematical Journal, vol. II, 267-271, 1841 `https://books.google.co.jp/books/about/The_Cambridge_mathematical_journal.html?id=o9xEAAAAcAAJ&redir_esc=y`

2. `http://mathworld.wolfram.com/Cayley-MengerDeterminant.html`