Pointless Topology 勉強ノート

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1 Preliminary

1.1 Topology トポロジー

Let $\mathcal{P}(X)$ denote the power set of X.

定義 1.1 (Topology トポロジー)**.** *A* ***topological space*** *is an ordered pair* (X, τ), $\tau \subseteq \mathcal{P}(X)$ *which satisfies the following properties*

1. $\varnothing \in \tau$ *and* $X \in \tau$.
2. *if* $U, V \in \tau$, *then* $U \cap V \in \tau$.
3. *if* $\forall I, U_i \in \tau$ *forall* $i \in I$, *then* $\bigcup_{i \in I} U_i$.

τ *is called the* ***topology*** *of* X. *The members of the topology* $U \in \tau$ *is said to be* ***open*** *and* $V \subseteq X$ *is said to be* ***closed*** *if* $\exists U$ *open such that* $V = U^c$.

定義 1.2. *If* X *is a topological space and* $x \in X$, *a neighbourhood (abbreviated "nhood") od* x *is a set* U *which contains an open set* V *containing* x. *Thus,* U *is a nhood of* x *iff* $x \in U^\circ$. *The collection* $\mathscr{U}_x$ *of all nhoods of* x *is the nhood system of* x.

定義 1.3 (Separation Axioms 分離公理)**.** *A space* (X, τ) *is called* T_i *,if respectively satisfies the following conditions,*

1. T_0*:* $\forall x, y \in X$ $\exists$ *an open set* $U \in \tau$ *such that* U *contains one of* x, y *and not the other.*
2. T_1*:* $\forall x, y \in X$ $\exists$ *a nhood of each not containing the other.*

例 1.1 (T_0-space)**.** $X = \{a, b\}, \tau = \{\varnothing, \{a\}, X\}$

1.2 Posets, Lattices 半順序集合、束

定義 1.4 (Posets)**.** *A* ***partial order (半順序)*** *on a set* X *is a binary relation* $R \subseteq X \times X$ *satisfying,*

1. $\forall a, aRa$ *(reflexivity, 反射律),*
2. $\forall a, b, c,\ aRb \,\&\, bRc \Rightarrow aRc$ *(transitivity, 推移律),*
3. $\forall a, b,\ aRb \,\&\, bRa \Rightarrow a = b$ *(antisymmetry, 反対称律).*

if moreover

4. $\forall a, b$ either aRb or bRa holds,

*it is said to be a **linear** or **total** order.*
*A **poset** or **partially ordered set**, $(X, \leq)$ is a set with a partial order. If the order of a poset is linear (or total), it is called a **linearly ordered set**, **totally ordered set** or **chain**. A relation that satisfies only (1) and (2) is called **preorder**.*

定義 1.5 (Suprema, infima)**.** *A **supremum** s of a subset $M \subseteq (X, \leq)$ the least upper bound of M, that is*

1. $\forall m \in M, m \leq s$,

2. $\forall m \in M, m \leq x \Rightarrow s \leq x$.

*Similarly, a **infimum** of a subset $M \subseteq (X, \leq)$ the greatest lower bound of M.*
*We also call a supremum a **join** and an infimum **meet** and notate $\sup M, \inf M$ or $\bigvee M, \bigwedge M$ respectively.*
For finite cases, we wirte $a \vee b := \sup\{a, b\}$ or $a_1 \vee \cdots \vee a_n := \sup\{a_1 \ldots a_n\}$ and $a \wedge b := \inf\{a, b\}$ or $a_1 \wedge \cdots \wedge a_n := \inf\{a_1 \ldots a_n\}$.

Since each $x \in X$ is both a lower and an upper bound of the empty set $\varnothing$,

$$\sup \varnothing \text{ is the least element of } X \tag{1.1}$$

and

$$\inf \varnothing \text{ is the greatest element of } X \tag{1.2}$$

We use the symbols 0 or $\perp$ for the former and 1 or $\top$ for the latter.

定義 1.6 (Semilattices, Lattice)**.** *A **meet-semilattice** is a poset X such that $\forall a, b \in X$ there exists an infimum $a \wedge b$.*
*A **join-semilattice** is a poset X such that $\forall a, b \in X$ there exists an supremum $a \vee b$.*
*A **lattice** is a poset X such that $\forall a, b \in X$ both an infimum $a \wedge b$ and a supremum $a \vee b$ exist.*
*A **bounded lattice** is a poset in which all finite subsets have infima and suprema (i.e. a lattice with bottom and top).*
*A poset is a **complete lattice** if every subset has a supremum and an infimum.*

In a bounded semilattice, $\wedge$ or $\vee$ is a binary operation and satisfies the following properties,

$$a \wedge a = a \qquad a \vee a = a \tag{1.3}$$
$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a \tag{1.4}$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad (a \vee b) \vee c = a \vee (b \vee c) \tag{1.5}$$
$$a \wedge 1 = a \qquad a \vee 0 = a. \tag{1.6}$$

In other words, bounded semilattices are commutative monoids (semigroup with unit/identity element) in which every element is idempotent.

定理 1.1. *Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A such that $a \wedge b$ is the join of a and b, and 0 is the least element.*

証明. If such a partial order exists,

$$a \leq b \Leftrightarrow a \vee b = b. \tag{1.7}$$

would be the correspondence. Now, let us verify this connection.
Reflexivity

$$a \wedge a = a \Rightarrow a \leq a. \tag{1.8}$$

Antisymmetry
If $a \leq b$ and $b \leq a$, then $b = a \wedge b = b \wedge a = a$ by commutativity.
Transitivity
If $a \leq b$ and $b \leq c$, then

$$\begin{aligned} a \wedge c &= a \wedge (b \wedge c) && (\because b \leq c) \\ &= (a \wedge b) \wedge c && \text{(associativity)} \\ &= b \wedge c && (\because a \leq b) \\ &= c && (\because b \leq c) \end{aligned}$$

Hence, $a \leq c$.
Join-uniqueness
Since $a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$, $a \leq a \wedge b$. Similarly, $b \leq a \wedge b$, so $a \wedge b$ is an upper-bound for $\{a, b\}$. For the leastness, suppose $a \leq c$ and $b \leq c$, then

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ &= a \wedge c \\ &= c. \end{aligned}$$

Hence, $a \wedge b \leq c$. □

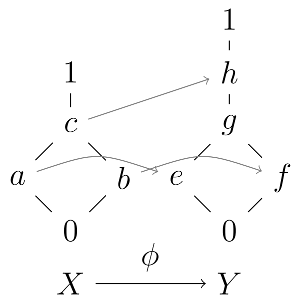
A lattice can also be defined purely algebraically in those terms,

定義 1.7. *A **lattice** $(L, \vee, \wedge)$ is an algebra (a set with two binary operations) that satisfy*

$(L1)$	$a \wedge a = a$	$a \vee a = a$	*(idempotency)*
$(L2)$	$a \wedge b = b \wedge a$	$a \vee b = b \vee a$	*(commutativity)*
$(L3)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \vee b) \vee c = a \vee (b \vee c)$	*(associativity)*
$(L4)$	$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	*(absorption identities)*

$(L4)$ is necessary for the two operations $\wedge, \vee$ to be consistent with the corresponding order $\leq$. In fact, $a \wedge b = b$ implies $a \vee b = a \vee (a \wedge b) = a$ by $(L4)$.
For homomorphisms (structure-preserving maps) of (semi)lattices and posets, we need to be a little bit careful since the order-preserving homomorphisms of posets does not always preserve the joins (or meets) as shown in the following example.([Stone] section 1.3 Exercise)

例 1.2. *Consider the posets $X = \{a \leq c, b \leq c\}$ and $Y = \{e \leq g, f \leq g, g \leq h\}$, and a homomorphism $\phi : X \to Y$ which maps each element as in the diagram below:*



Indeed, we have $a, b \leq c$ and $\phi(a), \phi(b) \leq \phi(c)$, which means the order is preserved, but $\phi(a \vee b) \neq \phi(a) \vee \phi(b)$.

([Stone] Sec.1.5) In most of the lattices we'll consider, the operations $\wedge$ and $\vee$ will satisfy an additional identity, namely the distributive law

$$\text{(i) } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{1.9}$$

for all a, b, c.

補題 1.2. *If the distributive law (i) holds in a lattice, then so does its dual, i.e. the identity*

$$\textit{(ii) } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.10}$$

証明.

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (i)} \\
&= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{by absorption law} \\
&= a \vee (b \wedge c) && \text{by absorption law}
\end{aligned}$$

□

Note also that in the presence of (i), we can deduce either of the two absorptive law from the other,

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b) \tag{1.11}$$

命題 1.3. *Let a, b, c be three elements of a distributive lattice A. Then there exists at most one $x \in A$ satisfying $x \wedge a = b$ and $x \vee a = c$.*

証明. Suppose both x and y satisfy the conditions. Then,

$$\begin{aligned}
x = x \wedge (x \vee a) &= x \wedge c = x \wedge (y \vee a) \\
&= (x \wedge y) \vee (x \wedge a) \\
&= (x \wedge y) \vee b = x \wedge y
\end{aligned}$$

since $b = x \wedge a = y \wedge a$ is a lower bound for $\{x, y\}$. Similarly, we have $y = x \wedge y$; so $x = y$. □

In any lattice, an element x satisfying $x \wedge a = 0$ and $x \vee a = 1$ is called a ***complement*** of a. The Porposition above tells us that in a distributive lattice, complements are unique when they exist. A ***Boolean algebra*** is a distributive lattice A equipped with an additional unary operation $\neg : A \to A$ such that $\neg a$ is a complement of a. Since $\neg$ is uniquely determined by the other data in the definition, it follows that any lattice homomorphism $f : A \to B$ between Boolean algebras is actually a Boolean algebra homomorphism(i.e. commutes with $\neg$).

例 **1.3** (Power set)**.** *For any set X, the power set $\mathcal{P}(X)$ of X is a lattice, with $\leq$ interpreted as inclusion, $\wedge$ and $\vee$ as union and intersection of subsets, and 0 and 1 as the empty set and the whole of X. Moreover $\mathcal{P}(X)$ is distributive. Since $\mathcal{P}(X)$ has complements for all its elements, it is a Boolean algebra.*

例 **1.4** (Total Order)**.** *Let A be a totally ordered set with least and greatest elements 0 and 1. Then A is a lattice, with $\wedge$ and $\vee$ interpreted as* min *and* max*. It is distributive;*

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\} \tag{1.12}$$

But if A has more than two elements, it is not a Boolean elgebra; for no element other than 0 and 1 can have a complement.

例 **1.5** (Lattices of subgroups)**.** *Let G be a group. The set subgroups of G, ordered by inclusion, is a lattice in which meet is again interpreted as intersection, but the join of two subgroups is the subgroup generated by their union. This lattice is not in general distributive; for example, if G is the non-cyclic group of order 4, the lattice looks like*

$$\begin{tikzcd}
 & 1 & \\
a \arrow[ur, dash] & b \arrow[u, dash] & c \arrow[ul, dash] \\
 & 0 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

where a, b, c are the three subgroups of order 2, and each of a, b, c has two distict complements.

1.2.1 Boolean Rings and Boolean algebras

命題 **1.4.** *De Morgan's law*

$$\neg(x \wedge y) = \neg x \vee \neg y$$

holds.

証明. We need to say that $\neg x \vee \neg y$ is the complement of $x \wedge y$,

$$\begin{aligned}
&(\neg x \vee \neg y) \wedge (x \wedge y) \\
&= ((\neg x \wedge x \wedge y) \vee (\neg y \wedge x \wedge y)) \\
&= (0 \wedge y) \vee (0 \wedge x) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
&(\neg x \vee \neg y) \vee (x \wedge y) \\
&= (\neg x \vee \neg y \vee x) \wedge (\neg x \vee \neg y \vee y) \\
&= (1 \vee y) \vee (1 \vee x) \\
&= 1
\end{aligned}$$

□

Next, we sketch the equivalence between Boolean algebras and Boolean rings. In any Boolean algebra A, we define the ***symmetry difference*** operation $+$ by

$$a + b = (a \wedge \neg b) \vee (b \wedge \neg a). \tag{1.13}$$

補題 1.5. *The distributive law* $a \wedge (b+c) = (a \wedge b) + (a \wedge c)$

証明.

$$\begin{aligned}
(a \wedge b) + (a \wedge c) &= ((a \wedge b) \wedge \neg(a \wedge c)) \vee ((a \wedge c) \wedge \neg(a \wedge b)) \\
&= ((a \wedge b) \wedge (\neg a \vee \neg c)) \vee ((a \wedge c) \wedge (\neg a \vee \neg b)) \\
&= ((a \wedge b \wedge \neg a) \vee (a \wedge b \wedge \neg c)) \vee ((a \wedge c \wedge \neg a) \vee (a \wedge c \wedge \neg b)) \\
&= (0 \vee (a \wedge b \wedge \neg c)) \vee (0 \vee (a \wedge c \wedge \neg b)) \\
&= (a \wedge b \wedge \neg c) \vee (a \wedge c \wedge \neg b) \\
&= a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= a \wedge (b+c).
\end{aligned}$$

□

補題 1.6. *The associative law*

$$a + (b+c) = (a+b) + c \tag{1.14}$$

holds.

証明.

$$\begin{aligned}
a + (b+c) &= a + ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= (a \wedge \neg((b \wedge \neg c) \vee (c \wedge \neg b))) \vee (\neg a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b))) \\
&= (a \wedge ((\neg b \vee c) \wedge (\neg c \vee b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee (\neg b \wedge b) \vee (c \wedge \neg c) \vee (c \wedge b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee 0 \vee 0 \vee (c \wedge b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)) \\
&= (a \wedge \neg b \wedge \neg c) \vee (a \wedge c \wedge b) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \\
&= (((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge \neg c) \vee (((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge c) \\
&= ((a+b) \wedge \neg c) \vee (\neg((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge c) \\
&= ((a+b) \wedge \neg c) \vee (\neg(a+b) \wedge c) \\
&= (a+b) + c.
\end{aligned}$$

□

Now for any a, we have

$$a + a = (a \wedge \neg a) \vee (a \wedge \neg a) = 0 \wedge 0 = 0 \tag{1.15}$$

$$a + 0 = (a \wedge 1) \vee (0 \wedge \neg a) = a \vee 0 = a. \tag{1.16}$$

So $(A, +, 0)$ is a commutative group, and $(A, +, \wedge, 0, 1)$ is a commutative ring with 1.

定義 1.8. *A Boolean ring A is a ring with 1 in which every element satisfies $a^2 = a$.*

補題 1.7. *Let A be a Boolean ring, then*

1. *A is commutative.*
2. *Every $a \in A$ satisfies $a + a = 0$.*

証明.

$$\begin{aligned} a+b &= (a+b)^2 \\ &= a^a + ab + ba + b^2 \\ &= a + ab + ba + b. \end{aligned}$$

So $ab+ba=0$. Putting $a=b$, we get $a+a=0$; hence $ab=-ba=ba$. □

So the multiplicative structure $(A,\cdot,1)$ is a semilattice, with partial order defined by $a \leq b$ iff $ab=a$ (c.f. 定理 1.1). Note that 0 is the least element of A for this order.
Now consider $a+b+ab$. We have

$$a(a+b+ab) = a + ab + ab = a \tag{1.17}$$

and

$$b(a+b+ab) = ba + b + ab = b \tag{1.18}$$

so $a+b+ab$ is an upper bound for $\{a,b\}$. But if c is an upper bound for $\{a,b\}$, then

$$(a+b+ab)c = ac + bc + abc = a + b + ab, \tag{1.19}$$

so $a+b+ab$ is the least upper bound. Dnote $a+b+ab$ by $a \vee b$, we thus havea lattice structure $(A,\vee,\cdot,0,1)$. Moreover, by an argument like that of Lemma 1.5, we may verify that $\cdot$ is distributive over $\vee$; and it is also easy to verify that $1+a$ is a complement for a. So A is a Boolean algebra.

$$\begin{aligned} ab \vee ac &= (ab + ac + abac) \\ &= ab + ac + abc \\ &= a(b \vee c) \qquad \square \end{aligned}$$

$$a(a+1) = a + a = 0 \qquad a \vee (1+a) = a + (1+a) + a(1+a) = 1. \quad \square$$

What is the symmetric difference operation in this Boolean algebra?

$$\begin{aligned} (a \wedge \neg b) \vee (b \wedge \neg a) &= (a(1+b)) \vee (b(1+a)) \\ &= (a+ab) \vee (b+ab) \\ &= (a+ab) + (b+ab) + (a+ab)(b+ab) \\ &= a + b + ab + ab + ab + ab \\ &= a + b. \end{aligned}$$

Thus if we start from a Boolean ring and turn it into a Boolean algebra by the definitions

$$\begin{aligned} a \vee b &:= a + b + ab \\ \neg a &:= 1 + a \\ a \overline{+} b &:= a + b \end{aligned}$$

then back into a Boolean ring by defining the addition $+$ as the symmetric difference, we recover the original ring. Similarly if start from a Boolean algebra and go round the other way. Moreover, it is clear from the nature of the constructions that any Boolean algebra homomorphism is also a Boolean ring homomorphism, conversely; so we have proved

定理 1.8. *The category of Boolean algebras is isomorphic to the category of Boolean rings.*

1.3 Heyting Algebras ヘイティング代数

Let a and b be elements of a Boolean algebra, and consider the element $\neg a \vee b$. We have

$$\begin{aligned} c \leq \neg a \vee b \Rightarrow c \wedge a &\leq a \wedge (\neg a \vee b) \\ &= (a \wedge \neg a) \vee (a \wedge b) \\ &= 0 \vee (a \wedge b) = a \wedge b \\ &\leq b; \end{aligned}$$

and conversely

$$\begin{aligned} c \wedge a \leq b \Rightarrow \neg a \vee b &\geq \neg a \vee (a \wedge c) \\ &= (\neg a \vee a) \wedge (\neg a \vee c) \\ &= 1 \wedge (\neg a \vee c) = \neg a \vee c \\ &\geq c. \end{aligned}$$

Thus $\neg a \vee b$ is the unique largest element c satisfying $c \wedge a \leq b$. A lattice A is said to be a ***Heyting algebra*** if, for each pair of elements (a, b), there exists an element $(a \to b)$ such that $c \leq (a \to b)$ iff $c \wedge a \leq b$.

補題 1.9. *Let A be a lattice, $\to$ a binary operation on A. Then $\to$ makes A into a Heyting algebra iff the equations*

$$\begin{aligned} &(i)\ a \to a = 1 \\ &(ii)\ a \wedge (a \to b) = a \wedge b \\ &(iii)\ b \wedge (a \to b) = b \\ &(iv)\ a \to (b \wedge c) = (a \to b) \wedge (a \to c) \end{aligned}$$

hold for all a, b, c in A.

証明. Suppose that A is a Heyting algebra. Then, $(a \to a)$ is the largest element c such that $c \wedge a \leq a$ holds, that is 1. So (i) is true.
For (ii), $a \wedge c \leq b$ implies $a \wedge (a \wedge c) = a \wedge c \leq a \wedge b$. Hence, $a \wedge (a \to b) \leq a \wedge b$. Also, since $(a \wedge b) \wedge a \leq b$, we have $a \wedge b \leq a \to b \ \Rightarrow a \wedge b \leq a \wedge (a \to b)$. So (ii) is true.
For (iii), $a \wedge b \leq b$ implies $b \leq (a \to b)$, so $b \leq b \wedge (a \to b) \leq b$.
For (iv), $a \wedge f \leq (b \wedge c) \leq b, c$ so $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$.
Then $a \wedge (a \to b) \wedge (a \to c) = a \wedge (a \to b) \wedge a \wedge (a \to c) \leq b \wedge c$, so it's true.
For the converse, suppose the equations hold. Then if $c \leq (a \to b)$, we have

$$a \wedge c \leq a \wedge (a \to b) \leq a \wedge b \leq b$$

conversely, if $c \wedge a \leq b$ then

$$\begin{aligned} c &= c \wedge (a \to c) && \text{by (iii)} \\ &\leq (a \to a) \wedge (a \to c) && \text{by (i)} \\ &= a \to (a \wedge c) && \text{by (iv)} \\ &\leq a \to b \end{aligned}$$

since $a \to (-)$ is order-preserving;
$\because$ if $b \leq c$ then $b = b \wedge c$

$$\begin{aligned} a \to b &= a \to (b \wedge c) \\ &= (a \to b) \wedge (a \to c) \end{aligned}$$

Hence, $a \to b \leq a \to c$. □

([Stone] Sec. 1.11) In a Boolean algebra A, we can recover the unary operation $\neg$ from the binary operation $\to$, since $\neg a = (a \to 0)$.
$\because$ In a Boolean algebra, $\neg a$ always exists. From the definition of $\to$, we have

$$c \leq (a \to 0) \text{ iff } c \wedge a = 0$$

so $\neg a \leq (a \to 0)$ since $(\neg a) \wedge a = 0$. But we also have,

$$\begin{aligned} 1 \geq (\neg a) \vee a &\geq (a \to 0) \vee a \\ \neg a &\geq (a \to 0). \end{aligned}$$

Therefore, $\neg a = (a \to 0)$. □
In a general Heyting algebra, we take this as the definition of $\neg$, and call $\neg a$ the ***negation*** (or the ***peudocomplement***) of a. It is clear that $a \wedge \neg a = 0$ (in fact $\neg a$ is the largest element of A with this property), but in general we do not have $a \vee \neg a = 1$.

補題 1.10. *1. A Heyting algebra is distributive.*

2. A Heyting algebra A is a Boolean algebra iff $\neg\neg a = a$ for all $a \in A$.

証明. (i) Since $a \wedge (-)$ is order-preserving, we have $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. But

$$\begin{aligned} a \to ((a \wedge b) \vee (a \wedge c)) &\geq (a \to (a \wedge b)) \vee (a \to (a \wedge c)) \\ &\geq b \vee c, \end{aligned}$$

hence $(a \wedge b) \vee (a \wedge c) \geq a \wedge (b \vee c)$.
(ii) If A is a Boolean algebra, then the identity $\neg\neg a = a$ is clear from the uniqueness of complements. Conversely, suppose $\neg\neg a = a$ holds in a Heyting algebra A; since we know A is distributive, we need only verify the identity $a \vee \neg a = 1$. But the given condition implies that $\neg$ is a bijection $A \to A$, and it is clearly order-preserving, so the De Morgan's law hold. Thus on negating the equation $a \wedge \neg a = 0$, we obtain $\neg a \vee \neg\neg a = \neg a \vee a = 1$. □

1.3.1 Exercises

1.3.2 The Strictness of Lattice Inclusions

例 1.6 (A Heyting Algebra which is not Boolean)**.** *Let A be a total order with least and greatest elements, considered as a lattice. Then A is a Heyting algebra with implication defined by*

$$a \to b = \begin{cases} 1 \text{ when } a \leq b \\ b \text{ otherwise.} \end{cases} \tag{1.20}$$

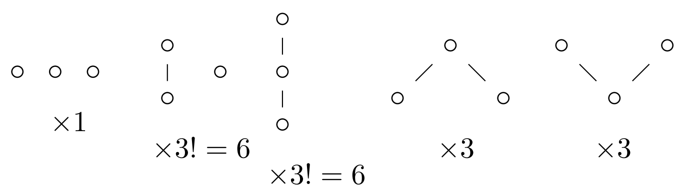
However, A is not Boolean, since $\neg\neg a = 1$ for all $a \neq 0$.

例 **1.7** (A Distributive Lattice which is not Heyting)**.** *Let X be an infinite set, and let A be the subset of the power set PX consisting of all finite subsets of X with X itself. It is easy to see that A is a sublattice of PX, and therefore distributive. But A is not Heyting algebra, for if a is a non-empty finite subset of X, the set of members of A having empty intersection tiwh a has no largest member.*

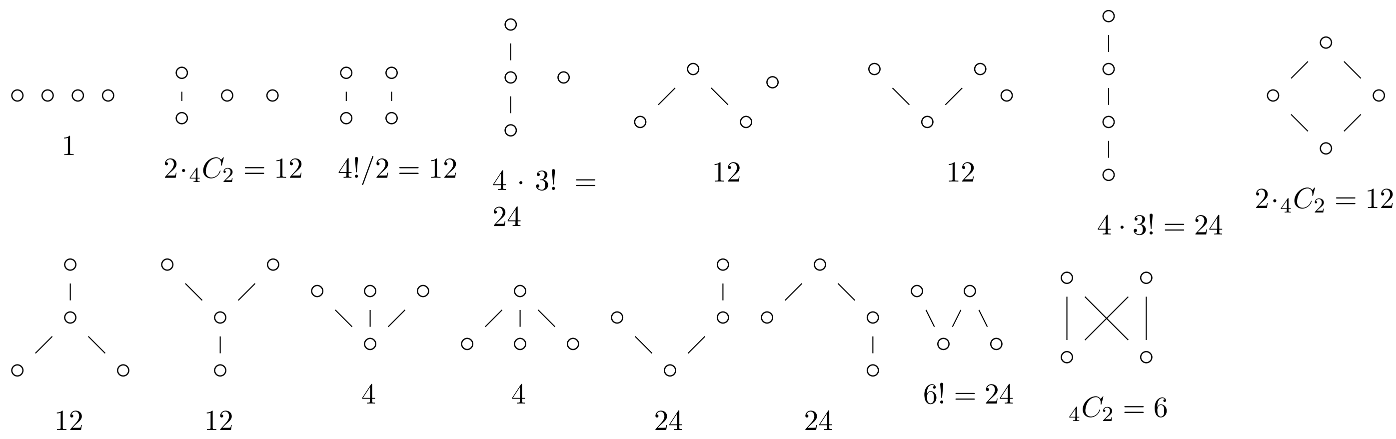
例 **1.8** (Poset)**.** *([Gratzer] Chap.1 Exercise 4) Here are some examples of the possible numbers of partial orders on finite sets:*
Size 1

Size 2

Size 3



Size 4



1.4 Ideals and Filters イデアルとフィルター

定義 **1.9** (Ideal)**.** *An **ideal** in a bounded distributive lattice L is a subset $J \subseteq L$ such that*

$$0 \in J, \tag{1.21}$$

$$a, b \in J \Rightarrow a \vee b \in J, \tag{1.22}$$

$$b \leq a \,\&\, a \in J \Rightarrow b \in J. \tag{1.23}$$

定義 **1.10** (Filter)**.** *A **filter** in a bounded distributive lattice L is a subset $F \subseteq L$ such that*

$$1 \in F, \tag{1.24}$$

$$a, b \in F \Rightarrow a \wedge b \in F, \tag{1.25}$$

$$b \geq a \;\&\; a \in F \Rightarrow b \in F. \tag{1.26}$$

2 Stone Spaces

3 Spaces and Lattices of Open Sets

We will suppose that all topological spaces that appear here will be T_0.

3.1 Sober spaces

定義 **3.1** (meet-irrducibility)**.** *Let (X, τ) be a top.space. $W \in \tau$ is said to be a **meet-irreducible** open set if $U, V \in \tau$ and $U \cap V \subseteq W$, then either $U \subseteq W$ or $V \subseteq W$.*

定義 **3.2** (sober space)**.** *X is said to be **sober** if all the meet-irreducible open sets are of the form $X \backslash \overline{\{x\}}$.*

命題 **3.1.** *Each Haudorff space is sober.*

証明. Suppose W is meet-irreducible, for contradiction, there exists $x_1, x_2 \notin W$ and $x_i \in U_i, x_j \notin U_i (i \neq j)$. Then $W = (W \cup U_1) \cap (W \cup U_2)$ and $W \cup U_i \nsubseteq W$. □

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