Elliptic and Theta Functions

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1 About this note

This note contains notes for Kotan seminar on elliptic and theta functions.

2 Examples of Elliptic Integrals

Here, we gather some examples varying from pure mathematics to physics.

2.1 Arc Length of Ellipse

Canonical form ():

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad with\ a \geq b. \tag{1}$$

Parametric form

$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\sin\varphi \\ b\cos\varphi \end{pmatrix}, where\ 0 \leq \varphi < 2\pi \tag{2}$$

The length of a curve $\mathbf{p}(t) = (x(t), y(t))$ is

$$\begin{aligned} s(u) &= \int_0^u \left|\frac{d\mathbf{p}}{dt}\right| dt \qquad (3) \\ &= \int_0^u \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \end{aligned}$$

For the ellipse's case, we have

$$\begin{aligned} s(\varphi) &= \int_0^\varphi \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi \qquad (4) \\ &= a\int_0^u \sqrt{1 - k^2\sin^2\varphi}\, d\varphi \end{aligned}$$

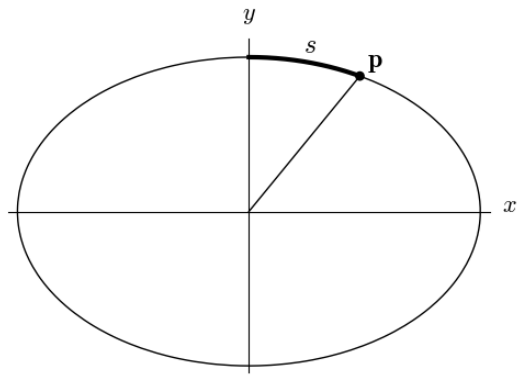

Figure 1: An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

where we defined $k = \sqrt{\frac{a^2 - b^2}{a^2}}$.

$$E(k, \varphi) := \int_0^u \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi \tag{5}$$

$$E(k) := E(k, \pi/2) \tag{6}$$

$E(k, \varphi)$ is called the **second incomplete elliptic integral (2)** and $E(k)$ the **second complete elliptic integral (2)**. The total length s_{total} of the ellipse is given by

$$s_{total} = 4aE(k) \tag{7}$$

Alternatively, putting $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$, the length can also be obtained as

$$\begin{aligned} s &= \int_0^x \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dx \\ &= a\int_0^x \sqrt{1 + \frac{b^2}{a^2}\frac{\frac{x^2}{a^2}}{1 - \frac{x^2}{a^2}}}\, dx \end{aligned} \tag{8}$$

Let us substitute further with $z = \frac{x}{a}$, we have

Table 1: Range correspondence between x and z.

x	$0 \to a$
z	$0 \to 1$

$$\begin{aligned} s &= a\int_0^z \sqrt{1 + \frac{b^2}{a^2}\frac{z^2}{1 - z^2}}\, dx \\ &= a\int_0^z \sqrt{\frac{1 - k^2 z^2}{1 - z^2}}\, dx. \end{aligned} \tag{9}$$

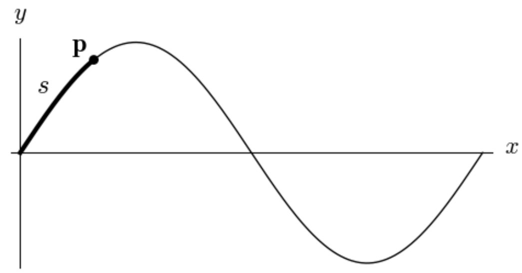

Figure 2: A sine curve $y = a\sin\theta$.

2.2 Arc length of sine curve

The arc length of sine curve $y = a\sin\theta$ is given by,

$$\begin{aligned} s &= \int_0^\theta \sqrt{1+a^2\cos^2\theta}\, d\theta \qquad (10)\\ &= \sqrt{1+a^2}\int_0^\theta \sqrt{1-\frac{a^2}{1+a^2}\sin^2\theta}\, d\theta \\ &= \sqrt{1+a^2}\int_0^\theta \sqrt{1-k^2\sin^2\theta}\, d\theta \\ &= \sqrt{1+a^2}E(k,\theta) \quad \textit{with} \quad k=\sqrt{\frac{a^2}{1+a^2}}. \end{aligned}$$

The arc length of a slightly more general form of $y = a\sin b\theta$

$$\begin{aligned} s &= \int_0^\theta \sqrt{1+a^2b^2\cos^2 b\theta}\, d\theta \qquad (11)\\ &= \frac{\sqrt{1+(ab)^2}}{b}\int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\, d\varphi \quad (\varphi = b\theta) \\ &= \frac{\sqrt{1+(ab)^2}}{b}E(k,\varphi) \quad \textit{with} \quad k=\sqrt{\frac{(ab)^2}{1+(ab)^2}} \end{aligned}$$

2.3 Arc length of quadratic curve

Although the integral is not of elliptic type, it is interesting to calculate the arc length of a quadratic curve of the form $y = ax^2$.

$$\begin{aligned} s &= \int_0^x \sqrt{1+4a^2x^2}\, dx \qquad (12)\\ &= \frac{1}{2a}\int_0^z \sqrt{1+z^2}\, dz \quad (z = 2ax) \\ &= \frac{1}{2a}\left[\int_0^z \frac{dz}{\sqrt{1+z^2}} + \int_0^z \frac{z^2dz}{\sqrt{1+z^2}}\right] \quad \text{(integration by parts on the 2nd term)} \\ &= \frac{1}{2a}\left[\ln\left(\sqrt{1+z^2}+z\right) + z\sqrt{1+z^2} - \int_0^z \sqrt{1+z^2}\right] \end{aligned}$$

So

$$
\begin{aligned}
s &= \frac{1}{4a}\left[\ln\left(\sqrt{1+z^2}+z\right)+z\sqrt{1+z^2}\right] \\
&= \frac{1}{4a}\left[\sinh^{-1} z+z\sqrt{1+z^2}\right] \\
&= \frac{1}{4a}\left[\sinh^{-1} 2ax+2ax\sqrt{1+4a^2x^2}\right]
\end{aligned} \tag{13}
$$

The result is just a combination of "elementary functions".

2.4 Arc length of cubic curves

Without loss of generality, we can consider the arc length of a cubic curve with the form $y = ax^3 + bx$.

$$
\begin{aligned}
s &= \int_0^x \sqrt{1+(3ax^2+b)^2}\, dx \\
&= \int_0^x \sqrt{P(x)}\, dx
\end{aligned} \tag{14}
$$

where $P(x) = (3ax+b)^2$ is a quadratic polynomial in x. This IS actually an elliptic integral. But we have to reduce this to some form so that we can express the integral in terms of three kinds of the elliptic integrals we will define soon (second of which has already defined as $E(k, \varphi)$). We will hopefully learn how to reduce the general elliptic integral of the form

$$\int R(x, y(x))\, dx \tag{15}$$

where $y^2 = P(x)$ is a polynomial in x of degree three or four and $R(x, y)$ is a rational function in x and y.

2.5 Arc length of lemniscate

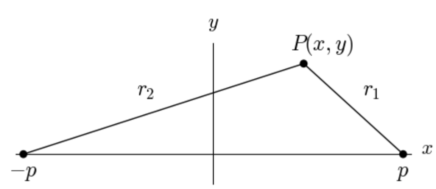

Figure 3: A sine curve $r_1 r_2 = a^2$.

A curve of Cassini is defined as a collection of points $P(x, y)$ whose product of the distances from two given points F_1, F_2 , called foci, is a constant a^2. For simplicity, let us suppose $F_1 = (p, 0), F_2 = (-p, 0)$ for some $p > 0$ and let $r_1 = \overline{PF_1}, r_2 = \overline{PF_2}$, then

$$r_1 r_2 = a^2. \tag{16}$$

Squaring both sides, we have

$$
\begin{aligned}
((x-p)^2+y^2)((x+p)^2+y^2) &= a^4 \\
(x^2+y^2)^2 - 2p^2(x^2-y^2) &= a^4 - p^4
\end{aligned} \tag{17}
$$

In polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, we have

$$\begin{aligned} r_{1,2} &= \sqrt{(x \mp p)^2 + y} = \sqrt{r^2 \pm 2pr\cos\theta + p^2} \\ r_1 r_2 &= \sqrt{(r^2 + p^2)^2 - 4p^2 r^2 \cos^2\theta} \\ r^4 + 2p^2 r^2 (1 - 2\cos^2\theta) &= a^4 - p^4 \\ r^4 - 2p^2 r^2 \cos 2\theta &= a^4 - p^4 \end{aligned} \tag{18}$$

When $a = p$, we have the **lemniscate**

$$r^2 = 2p^2 \cos 2\theta \tag{19}$$

and in coordinates

$$(x^2 + y^2)^2 = 2p^2(x^2 - y^2) \tag{20}$$