# Pointless Topology 勉強ノート

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# 1 Preliminary

## 1.1 Topology トポロジー

Let $\mathcal{P}(X)$ denote the power set of $X$.

**定義 1.1** (Topology トポロジー)**.** *A **topological space** is an ordered pair* $(X, \tau)$*,* $\tau \subseteq \mathcal{P}(X)$ *which satisfies the following properties*

1. $\varnothing \in \tau$ *and* $X \in \tau$*.*

2. *if* $U, V \in \tau$*, then* $U \cap V \in \tau$*.*

3. *if* $\forall I, U_i \in \tau$ *forall* $i \in I$*, then* $\bigcup_{i \in I} U_i$*.*

$\tau$ *is called the **topology** of* $X$*. The members of the topology* $U \in \tau$ *is said to be **open** and* $V \subseteq X$ *is said to be **closed** if* $\exists U$ *open such that* $V = U^c$*.*

**定義 1.2.** *If* $X$ *is a topological space and* $x \in X$*, a neighbourhood (abbreviated "nhood") od* $x$ *is a set* $U$ *which contains an open set* $V$ *containing* $x$*. Thus,* $U$ *is a nhood of* $x$ *iff* $x \in U^\circ$*. The collection* $\mathscr{U}_x$ *of all nhoods of* $x$ *is the nhood system of* $x$*.*

**定義 1.3** (Separation Axioms 分離公理)**.** *A space* $(X, \tau)$ *is called* $T_i$ *,if respectively satisfies the following conditions,*

1. $T_0$*:* $\forall x, y \in X\ \exists$ *an open set* $U \in \tau$ *such that* $U$ *contains one of* $x, y$ *and not the other.*

2. $T_1$*:* $\forall x, y \in X\ \exists$ *a nhood of each not containing the other.*

**例 1.1** ($T_0$-space)**.** $X = \{a, b\}, \tau = \{\varnothing, \{a\}, X\}$

## 1.2 Posets, Lattices 半順序集合、束

**定義 1.4** (Posets)**.** *A **partial order (半順序)** on a set* $X$ *is a binary relation* $R \subseteq X \times X$ *satisfying,*

1. $\forall a, aRa$ *(reflexivity, 反射律),*

2. $\forall a, b, c,\ aRb\ \&\ bRc \Rightarrow aRc$ *(transitivity, 推移律),*

3. $\forall a, b,\ aRb\ \&\ bRa \Rightarrow a = b$ *(antisymmetry, 反対称律).*

*if moreover*

*4. $\forall a, b$ either $aRb$ or $bRa$ holds,*

*it is said to be a **linear** or **total** order.*
*A **poset** or **partially ordered set**, $(X, \leq)$ is a set with a partial order. If the order of a poset is linear (or total), it is called a **linearly ordered set**, **totally ordered set** or **chain**. A relation that satisfies only (1) and (2) is called **preorder**.*

**定義 1.5** (Suprema, infima)**.** *A **supremum** $s$ of a subset $M \subseteq (X, \leq)$ the least upper bound of $M$, that is*

*1. $\forall m \in M, m \leq s$,*

*2. $\forall m \in M, m \leq x \Rightarrow s \leq x$.*

*Similarly, a **infimum** of a subset $M \subseteq (X, \leq)$ the greatest lower bound of $M$.*
*We also call a supremum a **join** and an infimum **meet** and notate $\sup M, \inf M$ or $\bigvee M, \bigwedge M$ respectively.*
*For finite cases, we wirte $a \vee b := \sup\{a, b\}$ or $a_1 \vee \cdots \vee a_n := \sup\{a_1 \ldots a_n\}$ and $a \wedge b := \inf\{a, b\}$ or $a_1 \wedge \cdots \wedge a_n := \inf\{a_1 \ldots a_n\}$.*

Since each $x \in X$ is both a lower and an upper bound of the empty set $\varnothing$,

$$\sup \varnothing \text{ is the least element of } X \tag{1.1}$$

and

$$\inf \varnothing \text{ is the greatest element of } X \tag{1.2}$$

We use the symbols 0 or $\perp$ for the former and 1 or $\top$ for the latter.

**定義 1.6** (Semilattices, Lattice)**.** *A **meet-semilattice** is a poset $X$ such that $\forall a, b \in X$ there exists an infimum $a \wedge b$.*
*A **join-semilattice** is a poset $X$ such that $\forall a, b \in X$ there exists an supremum $a \vee b$.*
*A **lattice** is a poset $X$ such that $\forall a, b \in X$ both an infimum $a \wedge b$ and a supremum $a \vee b$ exist.*
*A **bounded lattice** is a poset in which all finite subsets have infima and suprema (i.e. a lattice with bottom and top).*
*A poset is a **complete lattice** if every subset has a supremum and an infimum.*

In a bounded semilattice, $\wedge$ or $\vee$ is a binary operation and satisfies the following properties,

$$a \wedge a = a \qquad a \vee a = a \tag{1.3}$$
$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a \tag{1.4}$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad (a \vee b) \vee c = a \vee (b \vee c) \tag{1.5}$$
$$a \wedge 1 = a \qquad a \vee 0 = a. \tag{1.6}$$

In other words, bounded semilattices are commutative monoids (semigroup with unit/identity element) in which every element is idempotent.

**定理 1.1.** *Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on $A$ such that $a \wedge b$ is the join of $a$ and $b$, and $0$ is the least element.*

証明. If such a partial order exists,

$$a \leq b \Leftrightarrow a \vee b = b. \tag{1.7}$$

would be the correspondence. Now, let us verify this connection.

***Reflexivity***

$$a \wedge a = a \Rightarrow a \leq a. \tag{1.8}$$

***Antisymmetry***

If $a \leq b$ and $b \leq a$, then $b = a \wedge b = b \wedge a = a$ by commutativity.

***Transitivity***

If $a \leq b$ and $b \leq c$, then

$$\begin{aligned} a \wedge c &= a \wedge (b \wedge c) && (\because b \leq c) \\ &= (a \wedge b) \wedge c && \text{(associativity)} \\ &= b \wedge c && (\because a \leq b) \\ &= c && (\because b \leq c) \end{aligned}$$

Hence, $a \leq c$.

***Join-uniqueness***

Since $a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$, $a \leq a \wedge b$. Similarly, $b \leq a \wedge b$, so $a \wedge b$ is an upper-bound for $\{a, b\}$. For the leastness, suppose $a \leq c$ and $b \leq c$, then

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ &= a \wedge c \\ &= c. \end{aligned}$$

Hence, $a \wedge b \leq c$. □

A lattice can also be defined purely algebraically in those terms,

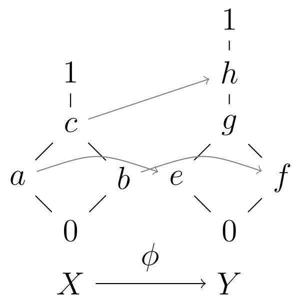
**定義 1.7.** *A* ***lattice*** $(L, \vee, \wedge)$ *is an algebra (a set with two binary operations) that satisfy*

| | | | |
|---|---|---|---|
| $(L1)$ | $a \wedge a = a$ | $a \vee a = a$ | *(idempotency)* |
| $(L2)$ | $a \wedge b = b \wedge a$ | $a \vee b = b \vee a$ | *(commutativity)* |
| $(L3)$ | $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ | $(a \vee b) \vee c = a \vee (b \vee c)$ | *(associativity)* |
| $(L4)$ | $a \vee (a \wedge b) = a$ | $a \wedge (a \vee b) = a$ | *(absorption identities)* |

$(L4)$ is necessary for the two operations $\wedge, \vee$ to be consistent with the corresponding order $\leq$. In fact, $a \wedge b = b$ implies $a \vee b = a \vee (a \wedge b) = a$ by $(L4)$.

For homomorphisms (structure-preserving maps) of (semi)lattices and posets, we need to be a little bit careful since the order-preserving homomorphisms of posets does not always preserve the joins (or meets) as shown in the following example.([Stone] section 1.3 Exercise)

**例 1.2.** *Consider the posets* $X = \{a \leq c, b \leq c\}$ *and* $Y = \{e \leq g, f \leq g, g \leq h\}$*, and a homomorphism* $\phi : X \to Y$ *which maps each element as in the diagram below:*




*Indeed, we have* $a, b \leq c$ *and* $\phi(a), \phi(b) \leq \phi(c)$*, which means the order is preserved, but* $\phi(a \vee b) \neq \phi(a) \vee \phi(b)$*.*

([Stone] Sec.1.5) In most of the lattices we'll consider, the operations $\wedge$ and $\vee$ will satisfy an additional identity, namely the distributive law

$$\text{(i) } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{1.9}$$

for all $a, b, c$.

**補題 1.2.** *If the distributive law (i) holds in a lattice, then so does its dual, i.e. the identity*

$$\textit{(ii) } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.10}$$

証明.

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (i)} \\
&= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{by absorption law} \\
&= a \vee (b \wedge c) && \text{by absorption law}
\end{aligned}$$

□

Note also that in the presence of (i), we can deduce either of the two absorptive law from the other,

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b) \tag{1.11}$$

**命題 1.3.** *Let* $a, b, c$ *be three elements of a distributive lattice* $A$*. Then there exists at most one* $x \in A$ *satisfying* $x \wedge a = b$ *and* $x \vee a = c$*.*

証明. Suppose both $x$ and $y$ satisfy the conditions. Then,

$$\begin{aligned}
x = x \wedge (x \vee a) = x \wedge c &= x \wedge (y \vee a) \\
&= (x \wedge y) \vee (x \wedge a) \\
&= (x \wedge y) \vee b = x \wedge y
\end{aligned}$$

since $b = x \wedge a = y \wedge a$ is a lower bound for $\{x, y\}$. Similarly, we have $y = x \wedge y$; so $x = y$. □

In any lattice, an element $x$ satisfying $x \wedge a = 0$ and $x \vee a = 1$ is called a ***complement*** of $a$. The Porposition above tells us that in a distributive lattice, complements are unique when they exist. A ***Boolean algebra*** is a distributive lattice $A$ equipped with an additional unary operation $\neg : A \to A$ such that $\neg a$ is a complement of $a$. Since $\neg$ is uniquely determined by the other data in the definition, it follows that any lattice homomorphism $f : A \to B$ between Boolean algebras is actually a Boolean algebra homomorphism(i.e. commutes with $\neg$).

例 **1.3** (Power set)**.** *For any set $X$, the power set $\mathcal{P}(X)$ of $X$ is a lattice, with $\leq$ interpreted as inclusion, $\wedge$ and $\vee$ as union and intersection of subsets, and $0$ and $1$ as the empty set and the whole of $X$. Moreover $\mathcal{P}(X)$ is distributive. Since $\mathcal{P}(X)$ has complements for all its elements, it is a Boolean algebra.*

例 **1.4** (Total Order)**.** *Let $A$ be a totally ordered set with least and greatest elements $0$ and $1$. Then $A$ is a lattice, with $\wedge$ and $\vee$ interpreted as* min *and* max*. It is distributive;*

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\} \tag{1.12}$$

*But if $A$ has more than two elements, it is not a Boolean elgebra; for no element other than $0$ and $1$ can have a complement.*

例 **1.5** (Lattices of subgroups)**.** *Let $G$ be a group. The set subgroups of $G$, ordered by inclusion, is a lattice in which meet is again interpreted as intersection, but the join of two subgroups is the subgroup generated by their union. This lattice is not in general distributive; for example, if $G$ is the non-cyclic group of order 4, the lattice looks like*

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
a \arrow[dr, dash] & b \arrow[d, dash] & c \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

*where $a, b, c$ are the three subgroups of order 2, and each of $a, b, c$ has two distict complements.*

### 1.2.1 Boolean Rings and Boolean algebras

命題 **1.4.** *De Morgan's law*

$$\neg(x \wedge y) = \neg x \vee \neg y$$

*holds.*

証明. We need to say that $\neg x \vee \neg y$ is the complement of $x \wedge y$,

$$\begin{aligned}
&(\neg x \vee \neg y) \wedge (x \wedge y) \\
&= ((\neg x \wedge x \wedge y) \vee (\neg y \wedge x \wedge y)) \\
&= (0 \wedge y) \vee (0 \wedge x) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
&(\neg x \vee \neg y) \vee (x \wedge y) \\
&= (\neg x \vee \neg y \vee x) \wedge (\neg x \vee \neg y \vee y) \\
&= (1 \vee y) \vee (1 \vee x) \\
&= 1
\end{aligned}$$

□

Next, we sketch the equivalence between Boolean algebras and Boolean rings. In any Boolean algebra $A$, we define the ***symmetry difference*** operation $+$ by

$$a + b = (a \wedge \neg b) \vee (b \wedge \neg a). \tag{1.13}$$

**補題 1.5.** *The distributive law* $a \wedge (b+c) = (a \wedge b) + (a \wedge c)$

証明.

$$\begin{aligned}
(a \wedge b) + (a \wedge c) &= ((a \wedge b) \wedge \neg(a \wedge c)) \vee ((a \wedge c) \wedge \neg(a \wedge b)) \\
&= ((a \wedge b) \wedge (\neg a \vee \neg c)) \vee ((a \wedge c) \wedge (\neg a \vee \neg b)) \\
&= ((a \wedge b \wedge \neg a) \vee (a \wedge b \wedge \neg c)) \vee ((a \wedge c \wedge \neg a) \vee (a \wedge c \wedge \neg b)) \\
&= (0 \vee (a \wedge b \wedge \neg c)) \vee (0 \vee (a \wedge c \wedge \neg b)) \\
&= (a \wedge b \wedge \neg c) \vee (a \wedge c \wedge \neg b) \\
&= a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= a \wedge (b+c).
\end{aligned}$$

□

**補題 1.6.** *The associative law*

$$a + (b+c) = (a+b) + c \tag{1.14}$$

*holds.*

証明.

$$\begin{aligned}
a + (b+c) &= a + ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= (a \wedge \neg((b \wedge \neg c) \vee (c \wedge \neg b)) \vee (\neg a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b))) \\
&= (a \wedge ((\neg b \vee c) \wedge (\neg c \vee b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee (\neg b \wedge b) \vee (c \wedge \neg c) \vee (c \wedge b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee 0 \vee 0 \vee (c \wedge b)) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)) \\
&= (a \wedge \neg b \wedge \neg c) \vee (a \wedge c \wedge b) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \\
&= (((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge \neg c) \vee (((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge c) \\
&= ((a+b) \wedge \neg c) \vee (\neg((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge c) \\
&= ((a+b) \wedge \neg c) \vee (\neg(a+b) \wedge c) \\
&= (a+b) + c.
\end{aligned}$$

□

Now for any $a$, we have

$$a + a = (a \wedge \neg a) \vee (a \wedge \neg a) = 0 \wedge 0 = 0 \tag{1.15}$$

$$a + 0 = (a \wedge 1) \vee (0 \wedge \neg a) = a \vee 0 = a. \tag{1.16}$$

So $(A, +, 0)$ is a commutative group, and $(A, +, \wedge, 0, 1)$ is a commutative ring with 1.

**定義 1.8.** *A Boolean ring $A$ is a ring with 1 in which every element satisfies $a^2 = a$.*

**補題 1.7.** *Let $A$ be a Boolean ring, then*

1. *$A$ is commutative.*
2. *Every $a \in A$ satisfies $a + a = 0$.*

証明.

$$\begin{aligned} a+b &= (a+b)^2 \\ &= a^a + ab + ba + b^2 \\ &= a + ab + ba + b. \end{aligned}$$

So $ab+ba=0$. Putting $a=b$, we get $a+a=0$; hence $ab=-ba=ba$. □

So the multiplicative structure $(A,\cdot,1)$ is a semilattice, with partial order defined by $a \leq b$ iff $ab=a$ (c.f. 定理 1.1). Note that 0 is the least element of $A$ for this order.
Now consider $a+b+ab$. We have

$$a(a+b+ab) = a+ab+ab = a \tag{1.17}$$

and

$$b(a+b+ab) = ba+b+ab = b \tag{1.18}$$

so $a+b+ab$ is an upper bound for $\{a,b\}$. But if $c$ is an upper bound for $\{a,b\}$, then

$$(a+b+ab)c = ac+bc+abc = a+b+ab, \tag{1.19}$$

so $a+b+ab$ is the least upper bound. Dnote $a+b+ab$ by $a \vee b$, we thus havea lattice structure $(A,\vee,\cdot,0,1)$. Moreover, by an argument like that of Lemma 1.5, we may verify that $\cdot$ is distributive over $\vee$; and it is also easy to verify that $1+a$ is a complement for $a$. So $A$ is a Boolean algebra.

$$\begin{aligned} ab \vee ac &= (ab+ac+abac) \\ &= ab+ac+abc \\ &= a(b \vee c) \qquad \square \end{aligned}$$

$$a(a+1) = a+a = 0 \qquad\qquad a \vee (1+a) = a+(1+a)+a(1+a) = 1. \quad \square$$

What is the symmetric difference operation in this Boolean algebra?

$$\begin{aligned} (a \wedge \neg b) \vee (b \wedge \neg a) &= (a(1+b)) \vee (b(1+a)) \\ &= (a+ab) \vee (b+ab) \\ &= (a+ab)+(b+ab)+(a+ab)(b+ab) \\ &= a+b+ab+ab+ab+ab \\ &= a+b. \end{aligned}$$

Thus if we start from a Boolean ring and turn it into a Boolean algebra by the definitions

$$\begin{aligned} a \vee b &:= a+b+ab \\ \neg a &:= 1+a \\ a \overline{+} b &:= a+b \end{aligned}$$

then back into a Boolean ring by defining the addition + as the symmetric difference, we recover the original ring. Similarly if start from a Boolean algebra and go round the other way. Moreover, it is clear from the nature of the constructions that any Boolean algebra homomorphism is also a Boolean ring homomorphism, conversely; so we have proved

**定理 1.8.** *The category of Boolean algebras is isomorphic to the category of Boolean rings.*

## 1.3 Heyting Algebras ヘイティング代数

Let $a$ and $b$ be elements of a Boolean algebra, and consider the element $\neg a \vee b$. We have

$$\begin{aligned} c \leq \neg a \vee b \Rightarrow c \wedge a &\leq a \wedge (\neg a \vee b) \\ &= (a \wedge \neg a) \vee (a \wedge b) \\ &= 0 \vee (a \wedge b) = a \wedge b \\ &\leq b; \end{aligned}$$

and conversely

$$\begin{aligned} c \wedge a \leq b \Rightarrow \neg a \vee b &\geq \neg a \vee (a \wedge c) \\ &= (\neg a \vee a) \wedge (\neg a \vee c) \\ &= 1 \wedge (\neg a \vee c) = \neg a \vee c \\ &\geq c. \end{aligned}$$

Thus $\neg a \vee b$ is the unique largest element $c$ satisfying $c \wedge z \leq b$. A lattice $A$ is said to be a ***Heyting algebra*** if, for each pair of elements $(a, b)$, there exists an element $(a \to b)$ such that $c \leq (a \to b)$ iff $c \wedge a \leq b$.

**補題 1.9.** *Let $A$ be a lattice, $\to$ a binary operation on $A$. Then $\to$ makes $A$ into a Heyting algebra iff the equations*

$$\begin{aligned} &\text{(i) } a \to a = 1 \\ &\text{(ii) } a \wedge (a \to b) = a \wedge b \\ &\text{(iii) } b \wedge (a \to b) = b \\ &\text{(iv) } a \to (b \wedge c) = (a \to b) \wedge (a \to c) \end{aligned}$$

*hold for all $a, b, c$ in $A$.*

**証明**. Suppose that $A$ is a Heyting algebra. Then, $(a \to a)$ is the largest element $c$ such that $c \wedge a \leq a$ holds, that is 1. So (i) is true.
For (ii), $a \wedge c \leq b$ implies $a \wedge (a \wedge c) = a \wedge c \leq a \wedge b$. So (ii) is true.
For (iii), $a \wedge b \leq b$ implies $b \leq (a \to b)$, so $b \leq b \wedge (a \to b) \leq b$.
For (iv), $a \wedge f \leq (b \wedge c) \leq b$ and $c$ so $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$.
Then $a \wedge (a \to b) \wedge (a \to c) = a \wedge (a \to b) \wedge a \wedge (a \to c) \leq b \wedge c$, so it's true.
For the converse, suppose the equations hold. Then if $c \leq (a \to b)$, we have

$$a \wedge c \leq a \wedge (a \to b) \leq a \wedge b \leq b$$

conversely, if $c \wedge a \leq b$ then

$$\begin{aligned} c &= c \wedge (a \to c) && \text{by (iii)} \\ &\leq (a \to a) \wedge (a \to c) && \text{by (i)} \\ &= a \to (a \wedge c) && \text{by (iv)} \\ &\leq a \to b \end{aligned}$$

since $a \to (-)$ is order-preserving;
$\because$ if $b \leq c$ then $b = b \wedge c$

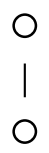
$$\begin{aligned} a \to b &= a \to (b \wedge c) \\ &= (a \to b) \wedge (a \to c) \end{aligned}$$

Hence, $a \to b \leq a \to c$. $\square$

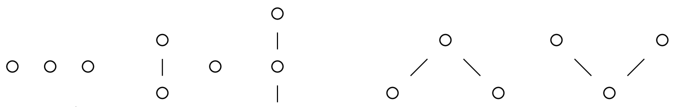
例 **1.6** (Poset)**.** *([Gratzer] Chap.1 Exercise 4) Here are some examples of the possible numbers of partial orders on finite sets:*
***Size 1***



***Size 2***

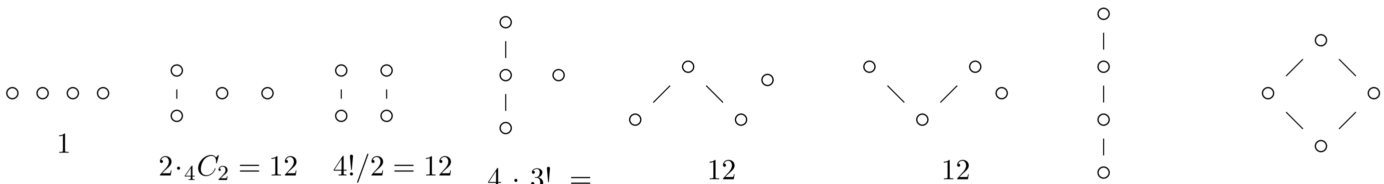


***Size 3***

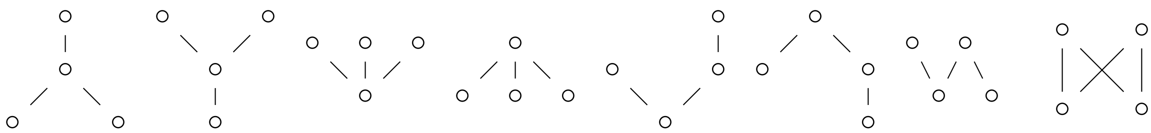


$\times 1$ $\quad \times 3! = 6$ $\quad \times 3! = 6$ $\quad \times 3$ $\quad \times 3$

***Size 4***



$1$ $\quad 2 \cdot {}_4C_2 = 12$ $\quad 4!/2 = 12$ $\quad 4 \cdot 3! = 24$ $\quad 12$ $\quad 12$ $\quad 4 \cdot 3! = 24$ $\quad 2 \cdot {}_4C_2 = 12$



$12$ $\quad 12$ $\quad 4$ $\quad 4$ $\quad 24$ $\quad 24$ $\quad 6! = 24$ $\quad {}_4C_2 = 6$

## 1.4 Ideals and Filters イデアルとフィルター

定義 **1.9** (Ideal)**.** *An **ideal** in a bounded distributive lattice $L$ is a subset $J \subseteq L$ such that*

$$0 \in J, \tag{1.20}$$

$$a, b \in J \Rightarrow a \vee b \in J, \tag{1.21}$$

$$b \leq a \;\&\; a \in J \Rightarrow b \in J. \tag{1.22}$$

定義 **1.10** (Filter)**.** *A **filter** in a bounded distributive lattice $L$ is a subset $F \subseteq L$ such that*

$$1 \in F, \tag{1.23}$$

$$a, b \in F \Rightarrow a \wedge b \in F, \tag{1.24}$$

$$b \geq a \;\&\; a \in F \Rightarrow b \in F. \tag{1.25}$$

# 2 Stone Spaces

# 3 Spaces and Lattices of Open Sets

We will suppose that all topological spaces that appear here will be $T_0$.

## 3.1 Sober spaces

**定義 3.1** (meet-irrducibility)**.** *Let* $(X, \tau)$ *be a top.space.* $W \in \tau$ *is said to be a* ***meet-irreducible*** *open set if* $U, V \in \tau$ *and* $U \cap V \subseteq W$*, then either* $U \subseteq W$ *or* $V \subseteq W$*.*

**定義 3.2** (sober space)**.** $X$ *is said to be* ***sober*** *if all the meet-irreducible open sets are of the form* $X \backslash \overline{\{x\}}$*.*

**命題 3.1.** *Each Haudorff space is sober.*

証明. Suppose $W$ is meet-irreducible, for contradiction, there exists $x_1, x_2 \notin W$ and $x_i \in U_i, x_j \notin U_i (i \neq j)$. Then $W = (W \cup U_1) \cap (W \cup U_2)$ and $W \cup U_i \nsubseteq W$. □

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