Notes on "Regular Polytopes" by H.S.M. Coxeter

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<2019-06-27 >

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1   About this note

This note contains some interpretations, visualisations, comments, etc, from reading "Regular Polytopes, H.S.M. Coxeter"

2   Jargons

> *congruent*:Two figures are said to be ***congruent*** if the distances between any corresponding pairs of points are equal. Likewise, angles between corresponding pairs of lines are equal. For example, two di-hdra (or trihdral solid angles) are congruent if the three face-angles of one are equal to respective face-angles of the other. Two such tri-hedra are said to be *directly* (or "superposable") congruent if they have the same sense (right- or left-handed), but *enantiomorphous* if they have opposite senses. The same distinction can be applied to figures of any kind, byth following device. (To be continued,,,)

3   The Product Of Three Reflections (§3.1 p35)

I will prove the following statement in a way more understandable to myself than in the book:

> In the product of three reflections, we can always arrange that one
> of the reflecting lines shall be perpendicular to both the others.

Coxeter's proof goes like this:

> The following is perhaps not the simplest proof, but it is one that
> generalizes easily to any number of dimensions. If we regard a con-
> gruent transformation as operating on pencils of parallel rays (in-
> stead of operating on points), we can say that a translation has no
> effect: it leaves every pencil invariant. Since each pencil can be rep-
> resented by that one of its rays which passes through a fixed point O,
> any congruent transformation operating on the rays that emanate
> from O: congruent because of the preservation of angles.
>
> If the given transformation is opposite, so is the induced transfor-
> mation. But the latter, leaving O invariant, can only be a reflection,
> say a reflection in OQ. This leaves O and Q invariant; therefore the
> given transformation leaves the *direction* OQ invariant. Consider
> the product of the given transformation with the reflection in any
> line, p, perpendicular to OQ. This is a direct transformation which
> reverses the direction OQ; i.e., it is a half-turn with the reflection in
> p. But the half-turn is the product of reflectios in two perpendicular
> lines, which may be chosen perpendicular and parallel to p. Thus we
> have altogether three reflections , of which the last two can be com-
> bined to form a translation. The general opposite transformation is
> now reduced to the product of a reflection and a translation which
> commutes, the reflecting line being in the direction of translation.
> This is kind of transformation called a *glide-reflection.*

OK. I think it would be more comprehensive if we add a little bit more explana-
tions to some logical steps hidden between the lines in a modern style. I started
to re-prove the theorem in 2 dimension by using a bra and ket vector notation,
because it is useful for writing inner products like $\langle a|b \rangle$, and the "outer" prod-
ucts like $|a\rangle\langle b|$. A reflection in any dimensions about a plane orthoganal to a
normal vector $|\hat{a}\rangle$ can be expressed using this bra-ket notation as

$$|x'\rangle = |x\rangle - 2|\hat{a}\rangle\langle\hat{a}|x\rangle = (1\!\!1 - 2|\hat{a}\rangle\langle\hat{a}|\,)\,|x\rangle \tag{1}$$

3.1   2D case in terms of complex numbers $\mathbb{C}$

A line $\mathcal{L}(\alpha, d) \subset \mathbb{C}$, is defined by a pair of reals, $0 \leq \alpha < \pi$, and $d \geq 0$,
parametrically as follows:

$$\mathcal{L}(\alpha, d) := \{e^{i\alpha}t + i\,de^{i\alpha} \mid t \in \mathbb{R}\} \tag{2}$$

$e^{i\alpha}$ is the direction of the line and d is the perpendicular distance from the
origin to the line. So the point $i\,de^{i\alpha}$ is the vertical projection of the origin onto
$\mathcal{L}(\alpha, d)$.

3.1.1 Reflections with invariant lines through the origin

First, a reflection $\rho_\alpha : \mathbb{C} \to \mathbb{C}$ with a line passing through the origin, $\mathcal{L}(\alpha, 0)$, as the invariant line is

$$w = \rho_\alpha(z) = e^{i\alpha}\overline{e^{-i\alpha}z} = e^{i2\alpha}\bar{z} \tag{3}$$

Then, the product of **two** such reflections, say ρ_{α_2} and ρ_{α_1} is, (writing $\rho_i = \rho_{\alpha_i}$ for my brevity)

$$w = \rho_2 \circ \rho_1(z) = e^{i2\alpha_2}\overline{e^{i2\alpha_1}\bar{z}} = e^{i2(\alpha_2 - \alpha_1)}z, \tag{4}$$

which is just a rotation. Now, the product of **three** reflections is

$$w = \rho_3 \circ \rho_2 \circ \rho_1(z) = e^{i2\alpha_3}\overline{e^{i2(\alpha_2 - \alpha_1)}z} = e^{i2(\alpha_3 - \alpha_2 + \alpha_1)}\bar{z} = e^{i2\theta}\bar{z}, \tag{5}$$

which is **another reflection** in the line $\mathcal{L}(\theta, 0)$ where $\theta = \alpha_3 - \alpha_2 + \alpha_1$.

3.1.2 General reflections

A more general reflection $\rho_{\alpha,d} : \mathbb{C} \to \mathbb{C}$ with the invariant line not passing through the origin $\mathcal{L}(\alpha, d)$ can be constructed as follows:

1. Apply a rotation $e^{-i\alpha}$

2. Translate $-id$

3. Appply reflection in real axis, i.e. take conjugate $z \mapsto \bar{z}$

4. Translate back id

5. Rotate back $e^{i\alpha}$.

Applying all the transformations, we have, for a general reflection in $\mathcal{L}(\alpha, d)$, as

$$w = \rho_{\alpha,d}(z) = e^{i\alpha}\{\overline{e^{-i\alpha}z - i\,d} + i\,d\} = e^{i2\alpha}\bar{z} + 2i\,de^{i\alpha}. \tag{6}$$

The product of **two** such reflections is,

$$
\begin{aligned}
w = \rho_{\alpha_2,d_2} \circ \rho_{\alpha_1,d_1}(z) &= e^{i2(\alpha_2 - \alpha_1)}z - 2i\,(d_1 e^{i2\alpha_2 - \alpha_1} - d_2 e^{i\alpha_2}) \\
&= e^{i2\theta}z - 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1}),
\end{aligned} \tag{7}
$$

where $\theta = \alpha_2 - \alpha_1$. Now let's look for a fixed point c of this transformation. Then it must satisfy

$$
\begin{aligned}
c &= \rho_{\alpha_2,d_2} \circ \rho_{\alpha_1,d_1}(c) \\
(e^{i2\theta} - 1)c &= 2i\,e^{i\theta}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1}) \\
(e^{i\theta} - 1)(e^{i\theta} + 1)c &= 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1})
\end{aligned}
$$

For c to have a definite value, $e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$ have to hold. And those cases are when $\theta = 0$ or π but the latter is impossible because $0 \leq \theta < \pi$. This

implies the two reflection lines are parallel.

Hence, when $\alpha = 0$ or π, the fixed point of the transformation (the centre of the rotation) is

$$c = 2i\,e^{i\theta}\frac{(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1})}{(e^{i\theta} - 1)(e^{i\theta} + 1)}$$

In the case $\alpha = 0$, we have translations: noticing that $\alpha_2 = \alpha_1 = \alpha$

$$w = z - 2i\,(d_1 - d_2). \tag{8}$$

We can see the direction of the resultant translation is perpendicular to the reflection lines.

Next, we proceed to the product of **three** general reflections. Let the three reflections be ρ_{α_1,d_1}, ρ_{α_2,d_2} and ρ_{α_3,d_3}. The composition of them is (again, writing $\rho_i = \rho_{\alpha_i,d_i}$ for my brevity)

$$
\begin{aligned}
w &= \rho_3 \circ \rho_2 \circ \rho_1(z) \\
&= e^{i2(\alpha_3 - \alpha_2 + \alpha_1)}\bar{z} + 2i\,(d_1 e^{i(2\alpha_3 - 2\alpha_2 + \alpha_1)} - d_2 e^{i(2\alpha_3 - \alpha_2)} + d_3 e^{i\alpha_3}) \\
&= e^{i2\theta}\bar{z} + 2i\,e^{i\theta}(d_1 e^{i\theta_{23}} - d_2 e^{i\theta_{13}} + d_3 e^{i\theta_{12}}),
\end{aligned}
\tag{9}
$$

where $\theta = \alpha_3 - \alpha_2 + \alpha_1$ and $\theta_{ij} = \alpha_j - \alpha_i$.

According to Coxeter's proof, the product of three reflections is a glide-reflection. So, let us calculate the length by which the gliding occurs. The gliding occurs in the same direction as the reflection, which is $e^{i2\theta}$ appeared in the above equation. If a point $z \in \mathbb{C}$ is on the reflection line, it will be fixed by the reflection and only displaced by the translation. Therefore, its image will be of the form $z + le^{i\theta}$ for some $l \in \mathbb{R}$. Equating this with the image $\rho_3 \circ \rho_2 \circ \rho_1(z)$ yields

$$
\begin{aligned}
z + le^{i\theta} &= e^{i2\theta}\bar{z} + 2i\,e^{i\theta}(d_1 e^{i\theta_{23}} - d_2 e^{i\theta_{13}} + d_3 e^{i\theta_{12}}) \\
e^{-i\theta}z - e^{i\theta}\bar{z} &= -l + 2i\,(d_1 e^{i\theta_{23}} - d_2 e^{i\theta_{13}} + d_3 e^{i\theta_{12}}).
\end{aligned}
\tag{10}
$$

The L.H.S. of the last equation is a pure imaginary number, so the real part of the R.H.S. must be 0. This fixes the value of l as

$$l = -2\{d_1 \sin\theta_{23} - d_2 \sin\theta_{13} + d_3 \sin\theta_{12}\}. \tag{11}$$

The meaning of l is actually the magnitude of the translation (or the gliding part of the glide-reflection). Also, remembering the expression of a reflection is $w = \rho_{\alpha,d}(z) = e^{i2\alpha}\bar{z} + 2i\,de^{i\alpha}$ with d being real, we see that the real part of $(d_1 e^{i\theta_{23}} - d_2 e^{i\theta_{13}} + d_3 e^{i\theta_{12}})$ is the vertical distance from the origin to the reflection line, namely

$$d = d_1 \cos\theta_{23} - d_2 \cos\theta_{13} + d_3 \cos\theta_{12}. \tag{12}$$