regular-polytopes

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1 About this note

This note contains some interpretations, visualisations, comments, etc, from reading "Regular Polytopes, H.S.M. Coxeter"

2 Jargons

> *confruent*:Two figures are said to be ***congruent*** if the distances between any corresponding pairs of points are equal. Likewise, angles between corresponding pairs of lines are equal. For example, two dihdra (or trihdral solid angles) are congruent if the three face-angles of one are equal to respective face-angles of the other. Two such trihdra are said to be *directly* (or "superposable") congruent if they have the same sense (right- or left-handed), but *enantiomorphous* if they have opposite senses. The same distinction can be applied to figures of any kind, byth following device. (To be continued,,,)

3 The Product Of Three Reflections (§3.1 p35)

I will prove the following statement in a way more understandable to myself than in the book:

> In the product of thre reflections, we can always arrange that one of the reflecting lines shall be perpendicular to both the others.

Coxeter's proof goes like this:

> The following is perhaps not the simplest proof, but it is one that generalizes easily to any number of dimensions. If we regard a congruent transformation as operating on pencils of parallel rays (instead of operating on points), we can say that a translation has no effect: it leaves every pencil invariant. Since each pencil can be represented by that one of its rays which passes through a fixed point O, any congruent transformation operating on the rays that emanate from O: congruent because of the preservation of angles.
>
> If the given transformation is opposite, so is the induced transformation. But the latter, leaving O invariant, can only be a reflection, say a reflection in OQ. This leaves O and Q invariant; therefore the given transformation leaves the *direction* OQ invariant. Consider the product of the given transformation with the reflection in any line, p, perpendicular to OQ. This is a direct transformation which reverses the direction OQ; i.e., it is a half-turn with the reflection in p. But the half-turn is the product of reflectios in two perpendicular lines, which may be chosen perpendicular and parallel to p. Thus we have altogether three reflections , of which the last two can be combined to form a translation. The general opposite transformation is now reduced to the product of a reflection and a translation which commutes, the reflecting line being in the direction of translation. This is kind of transformation called a *glide-reflection.*

OK. I think it would be more comprehensive if we add a little bit more explanations to some logical steps in a modern style.

> If the given transformation is opposite, so is the induced transformation. But the latter, leaving O invariant, can only be a reflection, say a reflection in OQ.

This part needs an explanation. This means the action of any three reflections on pencils of rays is **a reflection** in some line. Actually, you can see this fact by viewing the product of the three reflections as a product of a rotation and a reflection (or vice-versa) because a product of two reflections is a rotation. Let us call them rotation $\rho(\theta)$ (θ is the rotation angle) and $\lambda(\overrightarrow{OQ})$

3.1 2D case in terms of complex numbers $\mathbb{C}$

A line $\mathcal{L}(\alpha, d) \subset \mathbb{C}$, is defined by a pair of reals, $0 \leq \alpha \leq 2\pi$, and $d \geq 0$, parametrically as follows:

$$\mathcal{L}(\alpha, d) := \{e^{i\alpha}t + i\,de^{i\alpha} \mid t \in \mathbb{R}\} \tag{1}$$

α is the direction of the line and d is the perpendicular distance from the origin to the line. So the point ide^{α} is the vertical projection of the origin onto $\mathcal{L}(\alpha, d)$.

First, a reflection $\rho_\alpha : \mathbb{C} \to \mathbb{C}$ with a line passing through the origin, $\mathcal{L}(\alpha, 0)$, as the invariant line is

$$w = \rho_\alpha(z) = e^{i\alpha}\overline{e^{-i\alpha}z} = e^{i2\alpha}\bar{z} \tag{2}$$

Then, the product of **two** such reflections is

$$w = \rho_{\alpha_2} \circ \rho_{\alpha_1}(z) = e^{i2\alpha_2}\overline{e^{i2\alpha_1}\bar{z}} = e^{i2(\alpha_2-\alpha_1)}z, \tag{3}$$

which is just a rotation. Now, the product of **three** reflections is

$$w = \rho_{\alpha_3} \circ \rho_{\alpha_2} \circ \rho_{\alpha_1}(z) = e^{i2\alpha_3}\overline{e^{i2(\alpha_2-\alpha_1)}z} = e^{i2(\alpha_3-\alpha_2+\alpha_1)}\bar{z}, \tag{4}$$

which is **another reflection** in the line $\mathcal{L}(\alpha, 0)$ where $\alpha = \alpha_3 - \alpha_2 + \alpha_1$.
A more general reflection $\rho_{\alpha,d} : \mathbb{C} \to \mathbb{C}$ with the invariant line not passing through the origin $\mathcal{L}(\alpha, d)$ can be constructed as follows:

1. Apply a rotation $e^{-i\alpha}$
2. Translate $-id$
3. Appply reflection in real axis, i.e. take conjugate $z \mapsto \bar{z}$
4. Translate back id
5. Rotate back $e^{i\alpha}$.

Applying all the transformations, we have, for a general reflection in $\mathcal{L}(\alpha, d)$, as

$$w = \rho_{\alpha,d}(z) = e^{i\alpha}\{\overline{e^{-i\alpha}z - i\,d} + i\,d\} = e^{i2\alpha}\bar{z} + 2i\,de^{i\alpha}. \tag{5}$$

The two product of **two** such reflections is,

$$\begin{aligned} w = \rho_{\alpha_2,d_2} \circ \rho_{\alpha_1,d_1}(z) &= e^{i2(\alpha_2-\alpha_1)}z - 2i\,(d_1 e^{i2\alpha_2-\alpha_1} + d_2 e^{i\alpha_2}) \\ &= e^{i2\alpha}z - 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1}), \end{aligned} \tag{6}$$

where $\alpha = \alpha_2 - \alpha_1$. Now let's look for a fixed point c of this transformation. Then it must satisfy

$$\begin{aligned} c &= \rho_{\alpha_2,d_2} \circ \rho_{\alpha_1,d_1}(c) \\ (e^{i2\alpha} - 1)c &= 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1}) \\ (e^{i\alpha} - 1)(e^{i\alpha} + 1)c &= 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1}) \end{aligned}$$

For c to have a definite value, $e^{i\alpha} \neq 1$ and $e^{i\alpha} \neq -1$ have to hold. And those cases are when $\alpha = 0$ or π, which means the two reflection lines are parallel.
Hence, when $\alpha = 0$ or π, the fixed point of the transformation (the centre of the rotation) is

$$c = 2i\,e^{i\alpha}\frac{(d_1 e^{i\alpha_2} - d_2 e^{i\alpha_1})}{(e^{i\alpha} - 1)(e^{i\alpha} + 1)}$$

In the case $\alpha = 0, \pi$, we have translations: noticing that $\alpha_2 = \alpha_1$ or $\alpha_2 = \alpha_1 + \pi$,

$$w = e^{i2\alpha}z \mp 2i\,e^{i\alpha}(d_1 e^{i\alpha_2} \mp d_2 e^{i\alpha_1}). \tag{7}$$