How to prove the arithmetic mean and geometric mean formula

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1 Arithmetic Mean and Geometric Mean Formula

1.1 The Statement

Here is the arithmetic and geometric formula

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n},$$

where $\forall i \ a_i > 0$.

1.2 Proof of the Statement

OK for $n = 1$. Equality holds. Suppose that the inequality holds for all natural numbers up to n, then

$$\begin{aligned}
a_1 + \cdots + a_{n+1} &= \frac{a_2 + a_3 \cdots + a_{n+1}}{n} + \frac{a_1 + a_3 + \cdots + a_{n+1}}{n} + \cdots \\
&\quad + \frac{a_1 + a_2 + \overset{i}{\check{\cdots}} + a_{n+1}}{n} + \cdots + \frac{a_1 + \cdots + a_n}{n} \\
&= \sqrt[n]{a_2 \cdots a_{n+1}} + \cdots + \sqrt[n]{a_1 \cdots a_n} \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\frac{\sqrt[n]{a_2 \cdots a_{n+1}}}{\sqrt[n+1]{a_1 \cdots a_{n+1}}} + \cdots + \frac{\sqrt[n]{a_1 \cdots a_n}}{\sqrt[n+1]{a_1 \cdots a_{n+1}}} \right) \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\sqrt[n]{a_2 \cdots a_{n+1}}}{a_1}} + \cdots + \sqrt[n+1]{\frac{\sqrt[n]{a_1 \cdots a_n}}{a_{n+1}}} \right) \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\alpha_1}{a_1}} + \cdots + \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} \right)
\end{aligned}$$

where $\alpha_i = \sqrt[n]{a_1 \overset{i}{\check{\cdots}} a_{n+1}}$ (Here, $\overset{i}{\check{\cdots}}$ means omitting the i-th term in the sum or the product). It is easy to see that

$$\sqrt[n+1]{\frac{\alpha_1}{a_1}} \cdots \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} = 1.$$

So, if we can show the following proposition, we are done.
Lemma: If $b_1, \cdots, b_{n+1} > 0$, $\exists i$ such that $b_i \neq 1$ and $b_1 \cdots b_{n+1} = 1$, then

$$b_1 + \cdots + b_{n+1} \geq n + 1 \tag{1.1}$$

□

Now, by substituting $b_i = \sqrt[n+1]{\frac{\alpha_i}{a_i}}$, it is straightforward to see that

$$\begin{aligned} a_1 + \cdots + a_{n+1} &= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\alpha_1}{a_1}} + \cdots + \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} \right) \\ &\geq \sqrt[n+1]{a_1 \cdots a_{n+1}}(n+1) \end{aligned}$$

□

1.3 Proof of the Lemma

Lemma: If $b_1, \cdots, b_{n+1} > 0$, $\exists i$ such that $b_i \neq 1$ and $b_1 \cdots b_{n+1} = 1$, then

$$b_1 + \cdots + b_{n+1} \geq n+1 \tag{1.2}$$

Proof: Let us prove by induction. Notice that the assumuption that $\exists i$ such that $b_i > 1$ implies that $\exists j$ such that $b_j < 1$.
So, for $n = 1$, we can assume that $b_1 = 1 - c_1$ with $0 < c_1 < 1$. Then

$$b_1 + b_2 = b_1 + \frac{1}{b_1} = 1 - c_1 + \frac{1}{1 - c_1} \geq 1 - c_1 + 1 + c_1 = 2.$$

Now, let us suppose that (1.2) holds true for all the natural numbers up to n. Regarding the fact mentioned above, let us suppose that $b_n = 1 + c_n$ and $b_{n+1} = 1 + c_{n+1}$ with $c_n < 0$ and $c_{n+1} > 0$. Then, by the assumption,

$$(b_1 \cdots b_{n-1})(b_n b_{n+1}) = 1 \quad \text{implies} \quad b_1 + \cdots + b_{n-1} + b_n b_{n+1} \geq n.$$

If we can say

$$b_1 + \cdots + b_{n-1} + (b_n + b_{n+1}) \geq b_1 + \cdots b_{n-1} + (b_n b_{n+1} + 1),$$

then the inequality follows. But

$$b_n + b_{n+1} - (b_n b_{n+1} + 1) = -c_n c_{n+1} > 0.$$

Hence, it follows that

$$b_1 + \cdots + b_{n-1} + b_n + b_{n+1} \geq b_1 + \cdots + b_{n-1} + b_n b_{n+1} + 1 \geq n+1$$

□