Angular Momentum and Racah's formula

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1 Angular Momentum Operators and $\mathfrak{su}(2)$ Representation

$J_x, J_y, J_z \in \mathfrak{su}(2)$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.1}$$

The Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ and the ladder operators $J_\pm = J_x \pm iJ_y$

$$\begin{align}
\left[J^2, J_i\right] &= 0 \tag{1.2}\\
[J_z, J_\pm] &= \pm J_\pm \tag{1.3}\\
[J_+, J_-] &= 2J_z \tag{1.4}\\
J^2 &= J_-J_+ + J_z^2 + J_z \tag{1.5}\\
&= J_+J_- + J_z^2 - J_z \tag{1.6}
\end{align}$$

Let V be a finite dimensional vector space over $\mathbb{C}$ and $\varphi : \mathfrak{su}(2) \to \mathbf{End}(V)$ be its associated representation. Simultaneous eigenvector $|jm\rangle$:

$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle \tag{1.7}$$

2 The ladder operators' coefficients

$$\begin{align}
J_+|jm\rangle &= \sqrt{j(j+1) - m(m+1)}|jm+1\rangle \tag{2.1}\\
&= \sqrt{(j-m)(j+m+1)}|jm+1\rangle \tag{2.2}\\
J_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}|jm-1\rangle \tag{2.3}\\
&= \sqrt{(j+m)(j-m+1)}|jm-1\rangle \tag{2.4}
\end{align}$$

It is useful to write $|jm\rangle$ in terms of $(J_-)^k|jj\rangle$. So, let us rewrite the coefficients in simpler notation;

$$J_-|jj-(k-1)\rangle = f(j,k)|jj-k\rangle \tag{2.5}$$

where

$$f(j,k) := \sqrt{k(2j-k+1)}\,, 1 \leq k \leq 2j$$

We also have

$$J_+|jj-k\rangle = f(j,k)|jj-k+1\rangle \tag{2.6}$$

Note that $D := 2j+1$ is the dimension of the j-th representation space.
Hence

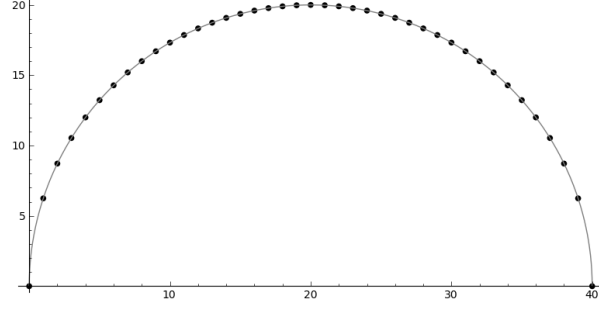$$(J_-)^k|jj\rangle = F(j,k)|jj-k\rangle \tag{2.7}$$

Figure 1: The graph of $f(j = \frac{39}{2}, k)$ where k is the horizontal axis.

where $F(j,k) = \prod_{i=1}^{k} f(j,i)$ and evaluated as,

$$
\begin{aligned}
F(j,k) &= \sqrt{k(2j+1-k)(k-1)(2j+1-(k-1)) \times \cdots \times 2 \cdot (2j+1-2) \cdot 1 \cdot (2j+1-1)} \\
&= \sqrt{k(D-k)(k-1)(D-(k-1)) \times \cdots \times 2 \cdot (D-2) \cdot 1 \cdot (D-1)} \\
&= \sqrt{\frac{k!(2j)!}{(2j-k)!}} = k!\sqrt{_{2j}C_k}
\end{aligned}
\tag{2.8}
$$

and $F(j,0) = 1$

3 Recursion Relations for Clebsch-Gordan coefficients

$$
|j_1 j_2 JM\rangle = \sum_{m_1,m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 m_1 j_2 m_2 | JM\rangle
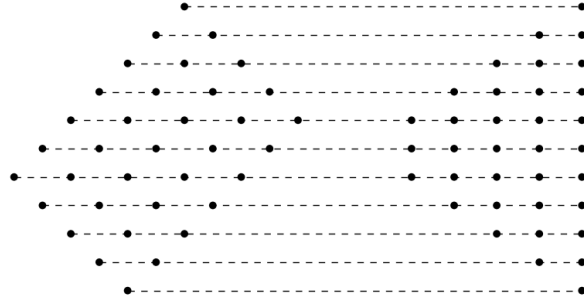\tag{3.1}
$$

Figure 2: Correspondence between the tensor product $j_1 \otimes j_2$ and J ($j_1 = 3, j_2 = 2$).

Apply $J_+ = j_{1+} + j_{2+}$ take inner product with $\langle j_1 j_2 m_1 m_2 |$

$$
\begin{aligned}
&\sqrt{J(J+1) - M(M+1)} \langle j_1 m_1 j_2 m_2 | JM+1 \rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1-1)} \langle j_1 m_1 - 1 j_2 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)} \langle j_1 m_1 j_2 m_2 - 1 | JM \rangle
\end{aligned}
\tag{3.2}
$$

and $J_- = j_{1-} + j_{2-}$

$$
\begin{aligned}
&\sqrt{J(J+1) - M(M-1)} \langle j_1 m_1 j_2 m_2 | JM-1 \rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1+1)} \langle j_1 m_1 + 1 j_2 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2+1)} \langle j_1 m_1 j_2 m_2 + 1 | JM \rangle
\end{aligned}
\tag{3.3}
$$

2

4 Explicit formulae for Clebsch-Gordan coefficinets

4.1 $\langle j_1 j_2 m_1 m_2 | J M \rangle$

Define $d = j_1 + j_2 - J$, ($j_1 > j_2$ and $0 \le d \le 2j_2$), and $L = J - M$. Let us determine the coefficients of the top spin state

$$
\begin{aligned}
|JJ\rangle &= a_0|j_1 j_1\rangle|j_2 j_2 - d\rangle + a_1|j_1 j_1 - 1\rangle|j_2 j_2 - d + 1\rangle + \cdots + a_d|j_1 j_1 - d\rangle|j_2 j_2\rangle \\
&= \sum_{i=0}^{d} a_i|j_1 j_1 - i\rangle|j_2 j_2 - d + i\rangle
\end{aligned}
\tag{4.1}
$$

by imposing the top spin condition

$$
J_+|JJ\rangle = 0 \implies a_{i+1} = -\frac{f(j_2, d - i)}{f(j_1, i + 1)}a_i \quad (i = 0, \ldots, d - 1),
\tag{4.2}
$$

which means

$$
\begin{aligned}
a_i &= -\frac{f(j_2, d - (i - 1))}{f(j_1, i)}a_{i-1} \quad (i = 1, \ldots, d) \tag{4.3} \\
&= (-1)^i \frac{f(j_2, d - (i - 1))f(j_2, d - (i - 2)) \cdots f(j_2, d - 1)f(j_2, d)}{f(j_1, i)f(j_1, i - 1) \cdots f(j_1, 2)f(j_1, 1)}a_0 \tag{4.4} \\
&= (-1)^i \frac{F(j_2, d)}{F(j_1, i)F(j_2, d - i)}a_0 \tag{4.5}
\end{aligned}
$$

Here $F(j_2, 0) = 1$.

4.2 Normalisation

Normalisation condition $\langle JJ|JJ\rangle = 1$ yields

$$
\begin{aligned}
\frac{1}{a_0^2} &= \sum_{i=0}^{d} \frac{F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d - i)^2} \tag{4.6} \\
&= 1 + \left[\frac{f(j_2, d)}{f(j_1, 1)}\right]^2 + \cdots + \left[\frac{f(j_2, d - (i - 1))f(j_2, d - (i - 2)) \cdots f(j_2, d - 1)f(j_2, d)}{f(j_1, i)f(j_1, i - 1) \cdots f(j_1, 2)f(j_1, 1)}\right]^2 + \\
&\quad \cdots + \left[\frac{F(j_2, d)}{F(j_1, d)}\right]^2 \\
&= \frac{1}{F(j_1, d)^2}\Big\{(D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_1 - 1) \cdot 1 + (D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_2 - d) \cdot d + \\
&\quad \cdots + (D_1 - d) \cdot d \cdots (D_1 - i - 1) \cdot (i + 1) \cdot (D_2 - (d - i + 1)) \cdot (d - i + 1) \cdots (D_2 - d) \cdot d + \cdots\Big\} \\
&= \frac{1}{F(j_1, d)^2}\Big\{\frac{(d!)^2}{d!}(D_1 - d) \cdot (D_1 - 2) \cdot (D_1 - 1) + \frac{(d!)^2}{1!(d - 1)!}(D_1 - d) \cdots (D_1 - 2) \cdot (D_2 - d) + \\
&\quad \cdots + \frac{(d!)^2}{i!(d - i)!}(D_1 - d) \cdot (D_1 - i - 1) \cdot (D_2 - (d - i + 1)) \cdots (D_2 - d) + \cdots\Big\}
\end{aligned}
$$

Writing

$$
\begin{aligned}
G_i(j_1, j_2, d) &:= \frac{F(j_1, d)F(j_2, d)}{F(j_1, i)F(j_2, d - i)}, \tag{4.7} \\
&= \sqrt{\frac{(d!)^2}{(d - i)!i!}(D_2 - d)(D_2 - d - 1) \cdots (D_2 - d - i + 1)(D_1 - d) \cdots (D_1 - i + 1)}
\end{aligned}
$$

Or, substituting $d = j_1 + j_2 - J$, this can be written as

$$G_i(j_1, j_2, j_1 + j_2 - J) = (-1)^i \sqrt{\frac{((j_1 + j_2 - J)!)^2}{(j_1 + j_2 - J - i)!i!} \frac{(j_2 + J - j_1)!(j_1 + J - j_2)!}{(j_2 + J - j_1 - i)!(2j_1 - i)!}}.$$

In terms of these G_i's, the coefficients a_i become

$$a_i = (-1)^i \frac{G_i(j_1, j_2, d)}{\sqrt{\sum_{i=1}^{d} G_i(j_1, j_2, d)^2}}. \tag{4.8}$$

We want to know the normalising coefficient $N := \frac{1}{\sqrt{\sum_{i=1}^{d} G_i(j_1, j_2, d)^2}}$. In order to simplify the sum

$$\sum_{i=1}^{d} G_i(j_1, j_2, d)^2 = \sum_{i=1}^{d} \frac{F(j_1, d)^2 F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d - i)^2}$$

$$= \frac{(d!)^2}{(2j_1 - d)!(2j_2 - d)!} \sum_{i=1}^{d} \frac{(2j_1 - i)!(2j_2 - d + i)!}{i!(d - i)!}, \tag{4.9}$$

we use a formula due to Racah (mentioned in Messiah[1])

$$\sum_{s} \frac{(a + s)!(b - s)!}{(c + s)!(d - s)!} = \frac{(a + b + 1)!(a - c)!(b - d)!}{(c + d)!(a + b - c - d + 1)!}. \tag{4.10}$$

with $a \geq c, b \geq d \geq 0$, where the sum is taken over $-c \leq s \leq d$.
Now substituting $a = 2j_2 - d, b = 2j_1, c = 0, d = d$, we obtain

$$N = \sqrt{\frac{(2j_2 - 2d + 2j_1 + 1)!}{d!(2j_2 - d + 2j_1 + 1)!}} = \sqrt{\frac{(2J + 1)!}{(j_1 + j_2 - J)!(j_1 + j_2 + J + 1)!}} \tag{4.11}$$

$$a_i = (-1)^i N G_i(j_1, j_2, d)$$

Now, by multiplying the top-spin state with the ladder operators L times, we obtain the state $|JM\rangle$ with $M = J - L$

$$J_-^L |JJ\rangle = (j_{1-} + j_{2-})^L \sum_{h=0}^{d} a_h \times |j_1 j_1 - h\rangle |j_2 j_2 - d + h\rangle$$

$$F(J, L)|JJ - L\rangle = \sum_{h=0}^{d} a_h \sum_{l=0}^{L} {}_L C_l \frac{F(j_1, h + l)F(j_2, (L + d) - (l + h))}{F(j_1, h)F(j_2, d - h)} |j_1 j_1 - (h + l)\rangle |j_2 j_2 - (L + d) + (h + l)\rangle$$

$$|JJ - L\rangle = \frac{1}{F(J, L)} \sum_{k=0}^{L+d} \left[\sum_{\substack{k=h+l, 0 \leq h \leq d \\ 0 \leq l \leq L}} a_h \times {}_L C_l \frac{F(j_1, k)F(j_2, K - k)}{F(j_1, h)F(j_2, d - h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle$$

$$= \frac{N}{F(J, L)} \sum_{k=0}^{L+d} F(j_1, k)F(j_2, K - k) \left[\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} \frac{(-1)^h {}_L C_l G_h(j_1, j_2, d)}{F(j_1, h)F(j_2, d - h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle$$

4

where $K = L+d = J-M+j_1+j_2-J = j_1+j_2-M$. Now, consider the coefficients of $|j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle$

$$
\begin{aligned}
B_k \;\; &:= \;\; F(j_1,k)F(j_2,K-k)\left[\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} \frac{(-1)^h {}_L C_l G_h(j_1,j_2,d)}{F(j_1,h)F(j_2,d-h)}\right] \\
&= \;\; \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!}} \sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h {}_L C_l \sqrt{\frac{(2j_1-h)!(2j_2-d+h)!(d!)^2(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!(2j_1-d)!(2j_2-d)!h!(d-h)!}} \\
&= \;\; \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!(2j_1-d)!(2j_2-d)!}} L! d! \sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!}
\end{aligned}
$$

The coefficient outside the sum, in terms of j_1, j_2, J, m_1, m_2, M, using the relations $K = L+d = J-M+j_1+j_2-J = j_1+j_2-M$, $k = j_1-m_1$, is

$$
\sqrt{\frac{(j_1-m_1)!(j_2+m_1-M)!}{(j_1+m_1)!(j_2-m_1+M)!(j_1-j_2+J)!(j_2-j_1+J)!}}(J-M)!(j_1+j_2-J)! \tag{4.12}
$$

Multipying by $\frac{N}{F(J,J-M)}$

$$
\begin{aligned}
&\sqrt{\frac{(2J+1)(j_1+j_2-J)!}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+j_2+J+1)!}\frac{(j_1-m_1)!(j_2-m_2)!(J+M)!(J-M)!}{(j_1+m_1)!(j_2+m_2)!}} \\
&= \sqrt{(2J+1)}\sqrt{\Delta(j_1 j_2 J)}\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!} \\
&\quad \times \frac{1}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+m_1)!(j_2+m_2)!}
\end{aligned} \tag{4.13}
$$

where we have defined

$$
\Delta(abc) := \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}. \tag{4.14}
$$

Now, we want to simplify the sum

$$
\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \tag{4.15}
$$

furthermore. Putting $k = j_1-m_1$, $d = j_1+j_2-J$ back, we have

$$
\begin{aligned}
&\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \\
&= \sum_l (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!}
\end{aligned}
$$

and the sum in the last line is taken over all the values of l with which all the factrial terms containing l makes sense. In order to do so, we are going to use the following formula

$$
\frac{a!}{b!c!} = \sum_s \frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{4.16}
$$

5

Now

$$\sum_l (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!}$$

$$= \sum_l (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!} \cdot \frac{(j_2+J-m_1-l)!}{(J-M-l)!(j_1-m_1-l)!}$$

$$= \sum_{l,l_1} (-1)^{j_1-m_1-l} \frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!} \cdot \frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!(j_1-j_2-M-l+l_1)!}$$

$$= \sum_{l,l_1,l_2} (-1)^{j_1-m_1-l} \frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!(j_2-J-j_1+l+l_2)!}$$

$$\cdot \frac{1}{(j_1-j_2-M+l_1-l)!} \frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!} \tag{4.17}$$

Racah uses the following formula to further simplify the expression,

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}. \tag{4.18}$$

To use the above formula, we first change the summation variable. Substitute $l' = j_1 - j_2 - M + l_1 - l$, and we have

$$j_1 + m_1 + l = 2j_1 - j_2 - m_2 + l_1 - l'$$
$$j_2 - J + m_1 + l = j_1 - J - m_2 + l_1 - l'.$$

Going back to the main expression,

$$\sum_{l',l_1} (-1)^{-j_2-m_2+l_1-l'} \frac{(2j_1-j_2-m_2+l_1-l')!}{l'!(j_1-J-m_2+l_1-l')!(j_1-j_2-M+l_1-l')!} \cdot \frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!}$$

$$= \sum_{l_1} (-1)^{j_2+m_2-l_1} \frac{(j_1+m_1)!(j_1+J-j_2)!}{(j_1-J-m_2+l_1)!(j_1-j_2-M+l_1)!(J+M-l_1)!} \cdot \frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!}$$

Putting $t = j_2 + m_2 - l_1$ ([2]), we obtain

$$\sum_{\substack{k=h+l \\ 0 \le h \le d \\ 0 \le l \le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!}$$

$$= \sum_t (-1)^t \frac{(j_1+m_1)!(j_2+m_2)!(j_1+J-j_2)!(j_2+J-j_1)!}{t!(j_1+j_2-J-t)!(j_1-m_1-t)!(j_2+m_2-t)!(J-j_2+m_1+t)!(J-j_1-m_2+t)!}$$

And then we obtain

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = \sqrt{(2J+1)}\sqrt{\Delta(j_1 j_2 J)}\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!}$$

$$\times \sum_t (-1)^t \frac{1}{t!(j_1+j_2-J-t)!(j_1-m_1-t)!(j_2+m_2-t)!(J-j_2+m_1+t)!(J-j_1-m_2+t)!} \tag{4.19}$$

Here, notice that the sum tkaes place in the range

$$\max\{0, -(J-j_2+m_1), -(J-j_1-m_2)\} \le t \le \min\{j_1+j_2-J, j_1-m_1, j_2+m_2\} \tag{4.20}$$

By making the substitution $z = a + b - c - t$ in the sum, we have

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = (-1)^{a+b-c} \langle j_2 j_1 m_2 m_1 | JM \rangle \tag{4.21}$$

The Racah symbol

$$(-1)^{a-b-c} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} := \frac{(-1)^{c-\gamma}}{\sqrt{2c+1}} \langle ab\,\alpha\beta | c -\gamma \rangle \tag{4.22}$$

The Racah formula

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a-b-\gamma} \sqrt{\Delta(abc)} \sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!}$$

$$\times \sum_t (-1)^t [t!(c-b+t+\alpha)!(c-a+t-\beta)!(a+b-c-t)!(a-t-\alpha)!(b-t+\beta)!]^{-1}$$

$$(\alpha + \beta + \gamma = 0, \quad |a - b| \le c \le a + b)$$

4.3 Some Examples

4.3.1 $J = j_1 + j_2$

$$|JJ\rangle = |j_1 j_1\rangle |j_2 j_2\rangle \tag{4.23}$$

$$(J_-)^k |JJ\rangle = (J_{1-} + J_{2-})^k |j_1 j_1\rangle |j_2 j_2\rangle \tag{4.24}$$

$$|JJ-k\rangle = \sum_{i=0}^{k} {}_kC_i \frac{F(j_1, i) F(j_2, k-i)}{F(J, k)} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle \tag{4.25}$$

$$= \sum_{i=0}^{k} \sqrt{\frac{{}_{2j_1}C_i \cdot {}_{2j_2}C_{k-i}}{{}_{2J}C_k}} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle \tag{4.26}$$

Rewrite it using $M = J - k = j_1 + j_2 - k$, $m_1 = j_1 - i$, $m_2 = j_2 - (k-i) = -j_1 + M + i = M - m_1$, $k - i = j_2 - m_2$

$$|j_1 + j_2 M\rangle = \sum_{m_1 + m_2 = M} \sqrt{\frac{{}_{2j_1}C_{j_1 - m_1} \cdot {}_{2j_2}C_{j_2 - m_2}}{{}_{2J}C_{j_1 + j_2 - M}}} |j_1 m_1\rangle |j_2 m_2\rangle \tag{4.27}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 + j_2 M\rangle = \sqrt{\frac{{}_{2j_1}C_{j_1 - m_1} \cdot {}_{2j_2}C_{j_2 - m_2}}{{}_{2(j_1+j_2)}C_{j_1 + j_2 - M}}} \tag{4.28}$$

$$= \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!} \frac{(J+M)!(J-M)!}{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!}} \tag{4.29}$$

4.3.2 $j_1 = j$, $j_2 = 1$

We have three possibilities; $J = j + 1$, j, $j - 1$. Note that any $|J\,M\rangle$ is expanded by $|j\,M+1\rangle|\mathbf{1}\,-1\rangle$, $|j\,M\rangle|\mathbf{1}\,0\rangle$, $|j\,M-1\rangle|\mathbf{1}\,1\rangle$.
The case $J = j + 1$.

$$\sqrt{\frac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{(2j)!}{(j-M-1)!(j+M+1)!} \frac{(j-M+1)!(j+M+1)!}{(2j+2)!}} \tag{4.30}$$

$$= \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}} \tag{4.31}$$

7

<div align="center">Table 1:</div>

$\lvert j\ M+1\rangle\lvert \mathbf{1}\ -1\rangle$	$\sqrt{\dfrac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\dfrac{(j-M+1)(j-M)}{(2j+2)(2j+1)}}$
$\lvert j\ M\rangle\lvert \mathbf{1}\ 0\rangle$	$\sqrt{\dfrac{2\cdot{}_{2j}C_{j-M}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\dfrac{2(j-M+1)(j+M+1)}{(2j+2)(2j+1)}}$
$\lvert j\ M-1\rangle\lvert \mathbf{1}\ 1\rangle$	$\sqrt{\dfrac{{}_{2j}C_{j-M+1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\dfrac{(j+M+1)(j+M)}{(2j+2)(2j+1)}}$

5 $6j$-symbol

It is defined as a coupling coefficient of three angular momentum

$$\lvert(j_1,(j_2,j_3)j_{23})J\rangle$$
$$= \sum_{j_{12}}[(2j_{12}+1)(2j_{23}+1)]^{\frac{1}{2}}(-1)^{j_1+j_2+j_3+J}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}\lvert((j_1,j_2)j_{12},j_3)J\rangle \tag{5.1}$$

$6j$-symbol in terms of $3j$-symbols

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum_{\substack{\alpha\beta\gamma \\ \delta\epsilon\varphi}}(-1)^{d+e+f+\delta+\epsilon+\varphi}\times$$
$$\times\begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix}\begin{pmatrix} e & f & a \\ \epsilon & -\varphi & \alpha \end{pmatrix}\begin{pmatrix} f & d & b \\ \varphi & -\delta & \beta \end{pmatrix}\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \tag{5.2}$$

Racah's formula ([2])

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = [\Delta(abc)\Delta(aef)\Delta(cde)\Delta(bdf)]^{\frac{1}{2}}\times$$
$$\times\sum_x\frac{(x+1)!}{[(p_1-x)(p_2-x)!(p_3-x)!(x-q_1)!(x-q_2)!(x-q_3)!(x-q_4)!]} \tag{5.3}$$

where $q_1 = a+b+c, q_2 = b+d+f, q_3 = a+e+f, q_4 = d+e+c, p_1 = a+b+d+e, p_2 = b+c+e+f, p_3 = c+a+f+d$. Only values which satisfy the triangle inequalities, $\lvert b-c\rvert \leq a \leq b+c$ are allowed:$(abc),(aef),(dbf),(dec)$. Therefore, there have to be even numbers of half integers at each face. This is equivalente to say that $q_1,q_2,q_3,q_4,p_1,p_2,p_3$ are all integers.

Example 5.1. $\{ j_1 = j_2 = \frac{1}{2}, j_3 = 1 \}$ *The resultant angular momenta are $J = 0,1,2$.*

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\mathbf{1} = \left(\left[\left(\frac{1}{2},\frac{1}{2}\right)\mathbf{0}\right]\oplus\left[\left(\frac{1}{2},\frac{1}{2}\right)\mathbf{1}\right]\right)\otimes\mathbf{1}$$
$$= \left[\left(\left(\frac{1}{2},\frac{1}{2}\right)0,1\right)\mathbf{1}\right]\oplus\left[\left(\left(\frac{1}{2},\frac{1}{2}\right)1,1\right)\mathbf{0}\right]\oplus\left[\left(\left(\frac{1}{2},\frac{1}{2}\right)1,1\right)\mathbf{1}\right]\oplus\left[\left(\left(\frac{1}{2},\frac{1}{2}\right)1,1\right)\mathbf{2}\right]$$

$$\frac{1}{2}\otimes\left(\frac{1}{2}\otimes\mathbf{1}\right) = \frac{1}{2}\otimes\left(\left[\left(\frac{1}{2},1\right)\frac{1}{2}\right]\oplus\left[\left(\frac{1}{2},1\right)\frac{3}{2}\right]\right)$$
$$= \left[\frac{1}{2},\left(\left(\frac{1}{2},1\right)\frac{1}{2}\right)\mathbf{0}\right]\oplus\left[\frac{1}{2},\left(\left(\frac{1}{2},1\right)\frac{1}{2}\right)\mathbf{1}\right]\oplus\left[\frac{1}{2},\left(\left(\frac{1}{2},1\right)\frac{3}{2}\right)\mathbf{1}\right]\oplus\left[\frac{1}{2},\left(\left(\frac{1}{2},1\right)\frac{3}{2}\right)\mathbf{2}\right]$$

A   Addition of Binomial Coefficients

The addition formula for binomial coefficients is given as

$$\sum_s \binom{x}{s}\binom{y}{z-s} = \binom{x+y}{z}. \tag{A.1}$$

This formula can be derived by comparing the coefficients of the polynomials on the both sides of $(X+Y)^{x+y} = (X+Y)^x(X+Y)^y$. Putting $x = a-b$, $y = b$, $z = a-c$ in (A.1), we have

$$\frac{a!}{b!c!} = \sum_s \frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{A.2}$$

If $y < 0$,

$$\binom{y}{z-s} = (-1)^{z-s}\binom{z-s-y-1}{z-s}. \tag{A.3}$$

Then (A.1) can be transformed into

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = (-1)^z \binom{x+y}{z}, \ (x+y \geq 0) \tag{A.4}$$

or

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = \binom{z-x-y-1}{z}, \ (x+y < 0) \tag{A.5}$$

Putting $y = z - t - 1$, (A.4) and (A.5) become

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = (-1)^z \frac{(t-z)!(x+z-t-1)!}{x!z!(x-t-1)!}, \ (x > t \geq z \geq 0). \tag{A.6}$$

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}, \ (t \geq x, z \geq 0). \tag{A.7}$$

The following formula is referenced in [1] due to Racah, but I have not succeeded in finding out a proof or any mention to this formula so far in [2]. It still needs to be verified somehow.

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \tag{A.8}$$

The following formula

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a,-b) \tag{A.9}$$

is a slight generalisation of a simple binomial coefficients formula

$$(X-Y)^K = \sum_{l=0}^{K} (-1)^l \binom{K}{l} X^{K-l} Y^l \tag{A.10}$$

(A.9) is obtained when we set $X = Y = 1$.

References

[1] Albert Messiah, *Quantum Mechanics*, Dover (1995)

[2] Giulio Racah, *Theory of Complex Spectra.I&II*, Phys. Rev. 61&62 (1942)

[3] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients" in *Spectroscopic and group theoretical methods in physics*, (ed. Bloch et al), North Holland Publ. Co., Amsterdam, 1968