Elliptic and Theta Functions

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1 About this note

This note contains notes for Kotan seminar on elliptic and theta functions.

2 Examples of Elliptic Integrals

Here, we gather some examples varying from pure mathematics to physics

2.1 Arc Length of Ellipse

Canonical form ():

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad with\ a \geq b. \tag{1}$$

Parametric form

$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\sin\varphi \\ b\cos\varphi \end{pmatrix}, where\ 0 \leq \varphi < 2\pi \tag{2}$$

The length of a curve $\mathbf{p}(t) = (x(t), y(t))$ is

$$\begin{aligned} s(u) &= \int_0^u \left|\frac{d\mathbf{p}}{dt}\right| dt \qquad (3) \\ &= \int_0^u \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \end{aligned}$$

For the ellipse's case, we have

$$
\begin{aligned}
s(\varphi) &= \int_0^{\varphi} \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}\, d\varphi && (4) \\
&= a \int_0^{u} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi
\end{aligned}
$$

where we defined $k = \sqrt{\frac{a^2 - b^2}{a^2}}$.

$$
\begin{aligned}
E(k, \varphi) &:= \int_0^{u} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi && (5) \\
E(k) &:= E(k, \pi/2) && (6)
\end{aligned}
$$

$E(k, \varphi)$ is called the **second incomplete elliptic integral (2)** and $E(k)$ the **second complete elliptic integral (2)**. The total length s_{total} of the ellipse is given by

$$s_{total} = 4aE(k) \qquad (7)$$

Alternatively, putting \$y $= \pm$ b $\sqrt{\{1-\frac{\{2}{x}\}\{a^2\}\}}$, the length can also be obtained as

$$
\begin{aligned}
s &= \int_0^{x} \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dx && (8) \\
&= a \int_0^{u} \sqrt{1 + \frac{b^2}{a^2} \frac{\frac{x^2}{a^2}}{1 - \frac{x^2}{a^2}}}\, dx
\end{aligned}
$$

Let us substitute further with $z = \frac{x}{a}$,

x	$0 \to a$
z	$0 \to 1$