Angular Momentum and Racah's formula

hisanobu-nakamura

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1 Introduction

This is a review of Racah's proof of the formula of the Clebsch-Gordan coefficients or Wigner's 3j-symbol and 6j-symbol.

2 Angular Momentum Operators and $\mathfrak{su}(2)$ Representation

$J_x, J_y, J_z \in \mathfrak{su}(2)$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{2.1}$$

The Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ and the ladder operators $J_\pm = J_x \pm iJ_y$

$$\begin{aligned} \left[J^2, J_i\right] &= 0 && (2.2) \\ [J_z, J_\pm] &= \pm J_\pm && (2.3) \\ [J_+, J_-] &= 2J_z && (2.4) \\ J^2 &= J_-J_+ + J_z^2 + J_z && (2.5) \\ &= J_+J_- + J_z^2 - J_z && (2.6) \end{aligned}$$

Let V be a finite dimensional vector space over $\mathbb{C}$ and $\varphi : \mathfrak{su}(2) \to \mathbf{End}(V)$ be its associated representation. The Casimir operator J^2 and J_z commute, so there are simultaneous egenvectors of the operators. It can be shown that, by the finiteness of the dimension V, J_z has a maximal eigenvalue j, which is called the **spin** of the representation. So it is plausible to denote the representation space as V_j. And it can be shown that the simultaneous eigenvalue of J^2 is $j(j+1)$ and invariant of the eigenvalue of J_z. Let us write an eigenvector of J_z with the eigenvalue m as $|jm\rangle \in V_j$. Then, in summary, we have

$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle. \tag{2.7}$$

3 The ladder operators' coefficients

$$\begin{aligned} J_+|jm\rangle &= \sqrt{j(j+1) - m(m+1)}|jm+1\rangle && (3.1) \\ &= \sqrt{(j-m)(j+m+1)}|jm+1\rangle && (3.2) \\ J_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}|jm-1\rangle && (3.3) \\ &= \sqrt{(j+m)(j-m+1)}|jm-1\rangle && (3.4) \end{aligned}$$

It is useful to write $|jm\rangle$ in terms of $(J_-)^k|jj\rangle$. So, let us rewrite the coefficients in simpler notation;

$$J_-|jj-(k-1)\rangle = f(j,k)|jj-k\rangle \tag{3.5}$$

where

$$f(j,k) := \sqrt{k(2j-k+1)}\,, 1 \le k \le 2j$$

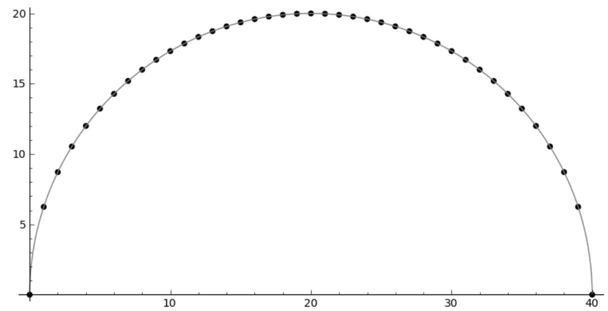

Figure 1: The graph of $f(j = \frac{39}{2}, k)$ where k is the horizontal axis.

We also have

$$J_+|jj-k\rangle = f(j,k)|jj-k+1\rangle \tag{3.6}$$

Note that $D := 2j+1$ is the dimension of the j-th representation space.
Hence

$$(J_-)^k|jj\rangle = F(j,k)|jj-k\rangle \tag{3.7}$$

where $F(j,k) = \prod_{i=1}^{k} f(j,i)$ and evaluated as,

$$\begin{aligned} F(j,k) &= \sqrt{k(2j+1-k)(k-1)(2j+1-(k-1)) \times \cdots \times 2\cdot(2j+1-2)\cdot 1\cdot(2j+1-1)} \\ &= \sqrt{k(D-k)(k-1)(D-(k-1)) \times \cdots \times 2\cdot(D-2)\cdot 1\cdot(D-1)} \\ &= \sqrt{\frac{k!(2j)!}{(2j-k)!}} = k!\sqrt{{}_{2j}C_k} \end{aligned} \tag{3.8}$$

and $F(j,0) = 1$

4 Recursion Relations for Clebsch-Gordan coefficients

The tensor product of two representation space $V_{j_1} \otimes V_{j_2}$ decomposes into the direct sum of irreducible representations V_J where $|j_1 - j_2| \leq J \leq j_1 + j_2$ as

$$V_{j_1} \otimes V_{j_2} = V_{|j_1-j_2|} \oplus \cdots \oplus V_{j_1+j_2} \tag{4.1}$$

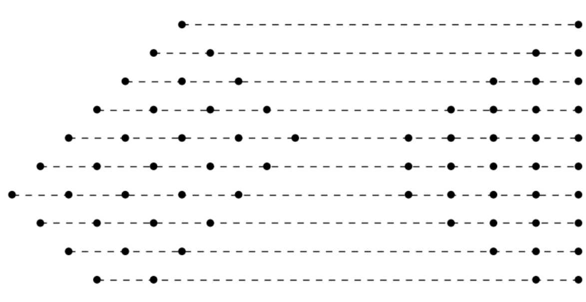

Figure 2: Correspondence between the tensor product $V_{j_1} \otimes V_{j_2}$ and V_J ($j_1 = 3, j_2 = 2$).

One of the elements in $|j_1 j_2 JM\rangle \in V_J$ is expandable by the tensor basis of $|j_1 j_2 m_1 m_2\rangle := |j_1 m_1\rangle \otimes |j_2 m_2\rangle \in V_{j_1} \otimes V_{j_2}$ as

$$|j_1 j_2 JM\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | JM\rangle. \tag{4.2}$$

The coefficients $\langle j_1 j_2 m_1 m_2 | JM\rangle$ are called **Clebsch-Gordan coefficients** and they can be recursively calculated by the following formulae: Apply $J_+ = j_{1+} + j_{2+}$ take inner product with $\langle j_1 j_2 m_1 m_2|$

$$\begin{aligned}
&\sqrt{J(J+1) - M(M+1)}\langle j_1 j_2 m_1 m_2 | JM+1\rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1-1)}\langle j_1 j_2 m_1 - 1 m_2 | JM\rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\langle j_1 j_2 m_1 m_2 - 1 | JM\rangle
\end{aligned} \tag{4.3}$$

and $J_- = j_{1-} + j_{2-}$

$$\begin{aligned}
&\sqrt{J(J+1) - M(M-1)}\langle j_1 j_2 m_1 m_2 | JM-1\rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1+1)}\langle j_1 j_2 m_1 + 1 m_2 | JM\rangle + \sqrt{j_2(j_2+1) - m_2(m_2+1)}\langle j_1 j_2 m_1 m_2 + 1 | JM\rangle
\end{aligned} \tag{4.4}$$

5 Explicit formulae for Clebsch-Gordan coefficinets

5.1 $\langle j_1 j_2 m_1 m_2 | JM\rangle$

Define $d = j_1 + j_2 - J$, ($j_1 \geq j_2$ and $0 \leq d \leq 2j_2$), and $L = J - M$. Let us determine the coefficients of the top spin state

$$\begin{aligned}
|JJ\rangle &= a_0|j_1 j_1\rangle|j_2 j_2 - d\rangle + a_1|j_1 j_1 - 1\rangle|j_2 j_2 - d + 1\rangle + \cdots + a_d|j_1 j_1 - d\rangle|j_2 j_2\rangle \\
&= \sum_{i=0}^{d} a_i |j_1 j_1 - i\rangle|j_2 j_2 - d + i\rangle
\end{aligned} \tag{5.1}$$

by imposing the top spin condition

$$J_+|JJ\rangle = 0 \implies a_{i+1} = -\frac{f(j_2, d-i)}{f(j_1, i+1)} a_i \quad (i = 0, \ldots, d-1), \tag{5.2}$$

which means

$$a_i = -\frac{f(j_2, d-(i-1))}{f(j_1, i)} a_{i-1} \quad (i = 1, \ldots, d) \tag{5.3}$$

$$= (-1)^i \frac{f(j_2, d-(i-1)) f(j_2, d-(i-2)) \cdots f(j_2, d-1) f(j_2, d)}{f(j_1, i) f(j_1, i-1) \cdots f(j_1, 2) f(j_1, 1)} a_0 \tag{5.4}$$

$$= (-1)^i \frac{F(j_2, d)}{F(j_1, i) F(j_2, d-i)} a_0 \tag{5.5}$$

Here $F(j_2, 0) = 1$.

5.2 Derivation of Racah's formula

Normalisation condition $\langle JJ|JJ\rangle = 1$ yields

$$\begin{aligned}
\frac{1}{a_0^2} &= \sum_{i=0}^{d} \frac{F(j_2,d)^2}{F(j_1,i)^2 F(j_2,d-i)^2} \qquad (5.6)\\
&= 1 + \left[\frac{f(j_2,d)}{f(j_1,1)}\right]^2 + \cdots + \left[\frac{f(j_2,d-(i-1))f(j_2,d-(i-2))\cdots f(j_2,d-1)f(j_2,d)}{f(j_1,i)f(j_1,i-1)\cdots f(j_1,2)f(j_1,1)}\right]^2 + \\
&\quad \cdots + \left[\frac{F(j_2,d)}{F(j_1,d)}\right]^2 \\
&= \frac{1}{F(j_1,d)^2}\Big\{(D_1-d)\cdot d\cdots(D_1-2)\cdot 2\cdot(D_1-1)\cdot 1 + (D_1-d)\cdot d\cdots(D_1-2)\cdot 2\cdot(D_2-d)\cdot d + \\
&\quad \cdots + (D_1-d)\cdot d\cdots(D_1-i-1)\cdot(i+1)\cdot(D_2-(d-i+1))\cdot(d-i+1)\cdots(D_2-d)\cdot d + \cdots\Big\} \\
&= \frac{1}{F(j_1,d)^2}\Big\{\frac{(d!)^2}{d!}(D_1-d)\cdot(D_1-2)\cdot(D_1-1) + \frac{(d!)^2}{1!(d-1)!}(D_1-d)\cdots(D_1-2)\cdot(D_2-d) + \\
&\quad \cdots + \frac{(d!)^2}{i!(d-i)!}(D_1-d)\cdot(D_1-i-1)\cdot(D_2-(d-i+1))\cdots(D_2-d) + \cdots\Big\}
\end{aligned}$$

Writing

$$\begin{aligned}
G_i(j_1,j_2,d) &:= \frac{F(j_1,d)F(j_2,d)}{F(j_1,i)F(j_2,d-i)}, \qquad (5.7)\\
&= \sqrt{\frac{(d!)^2}{(d-i)!i!}(D_2-d)(D_2-d-1)\cdots(D_2-d-i+1)(D_1-d)\cdots(D_1-i+1)}
\end{aligned}$$

Or, substituting $d = j_1 + j_2 - J$, this can be written as

$$G_i(j_1,j_2,j_1+j_2-J) = (-1)^i\sqrt{\frac{((j_1+j_2-J)!)^2}{(j_1+j_2-J-i)!i!}\frac{(-j_1+j_2+J)!(j_1+J-j_2)!}{(-j_1+j_2+J-i)!(2j_1-i)!}}.$$

In terms of these G_i's, the coefficients a_i become

$$a_i = (-1)^i\frac{G_i(j_1,j_2,d)}{\sqrt{\sum_{i=1}^d G_i(j_1,j_2,d)^2}}. \qquad (5.8)$$

We want to know the normalising coefficient $N := \frac{1}{\sqrt{\sum_{i=1}^d G_i(j_1,j_2,d)^2}}$. In order to simplify the sum

$$\begin{aligned}
\sum_{i=1}^{d} G_i(j_1,j_2,d)^2 &= \sum_{i=1}^{d}\frac{F(j_1,d)^2F(j_2,d)^2}{F(j_1,i)^2F(j_2,d-i)^2} \\
&= \frac{(d!)^2}{(2j_1-d)!(2j_2-d)!}\sum_{i=1}^{d}\frac{(2j_1-i)!(2j_2-d+i)!}{i!(d-i)!}, \qquad (5.9)
\end{aligned}$$

we use a formula due to Racah (mentioned in Messiah[1])

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \qquad (5.10)$$

with $a \geq c, b \geq d \geq 0$, where the sum is taken over $-c \leq s \leq d$.
Now substituting $a = 2j_2 - d, b = 2j_1, c = 0, d = d$, we obtain

$$N = \sqrt{\frac{(2j_2-2d+2j_1+1)!}{d!(2j_2-d+2j_1+1)!}} = \sqrt{\frac{(2J+1)!}{(j_1+j_2-J)!(j_1+j_2+J+1)!}} \qquad (5.11)$$

$$a_i = (-1)^i N G_i(j_1, j_2, d)$$

Now, by multiplying the top-spin state with the ladder operators L times, we obtain the state $|JM\rangle$ with $M = J - L$

$$\begin{aligned}
J_-^L|JJ\rangle &= (j_{1-} + j_{2-})^L \sum_{h=0}^{d} a_h \times |j_1 j_1 - h\rangle|j_2 j_2 - d + h\rangle \\
F(J,L)|JJ-L\rangle &= \sum_{h=0}^{d} a_h \sum_{l=0}^{L} {}_L C_l \frac{F(j_1, h+l)F(j_2, (L+d)-(l+h))}{F(j_1,h)F(j_2,d-h)} |j_1 j_1 - (h+l)\rangle|j_2 j_2 - (L+d) + (h+l)\rangle \\
|JJ-L\rangle &= \frac{1}{F(J,L)} \sum_{k=0}^{L+d} \left[\sum_{k=h+l, \substack{0\le h\le d \\ 0\le l\le L}} a_h \times {}_L C_l \frac{F(j_1,k)F(j_2,K-k)}{F(j_1,h)F(j_2,d-h)} \right] |j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle \\
&= \frac{N}{F(J,L)} \sum_{k=0}^{L+d} F(j_1,k)F(j_2,K-k) \left[\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} \frac{(-1)^h {}_L C_l G_h(j_1,j_2,d)}{F(j_1,h)F(j_2,d-h)} \right] |j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle
\end{aligned}$$

where $K = L+d = J-M+j_1+j_2-J = j_1+j_2-M$. Now, consider the coefficients of $|j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle$

$$\begin{aligned}
B_k &:= F(j_1,k)F(j_2,K-k) \left[\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} \frac{(-1)^h {}_L C_l G_h(j_1,j_2,d)}{F(j_1,h)F(j_2,d-h)} \right] \\
&= \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!}} \sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h {}_L C_l \sqrt{\frac{(2j_1-h)!(2j_2-d+h)!(d!)^2(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!(2j_1-d)!(2j_2-d)!h!(d-h)!}} \\
&= \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!(2j_1-d)!(2j_2-d)!}} L!d! \sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!}
\end{aligned}$$

The coefficient outside the sum, in terms of j_1, j_2, J, m_1, m_2, M, using the relations $K = L + d = J - M + j_1 + j_2 - J = j_1 + j_2 - M$, $k = j_1 - m_1$, is

$$\sqrt{\frac{(j_1-m_1)!(j_2+m_1-M)!}{(j_1+m_1)!(j_2-m_1+M)!(j_1-j_2+J)!(j_2-j_1+J)!}} (J-M)!(j_1+j_2-J)! \tag{5.12}$$

Multiplying by $\frac{N}{F(J,J-M)}$

$$\begin{aligned}
&\sqrt{\frac{(2J+1)(j_1+j_2-J)!}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+j_2+J+1)!} \frac{(j_1-m_1)!(j_2-m_2)!(J+M)!(J-M)!}{(j_1+m_1)!(j_2+m_2)!}} \\
&= \sqrt{(2J+1)}\sqrt{\Delta(j_1 j_2 J)}\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!} \\
&\times \frac{1}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+m_1)!(j_2+m_2)!}
\end{aligned} \tag{5.13}$$

where we have defined

$$\Delta(abc) := \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}. \tag{5.14}$$

Now, we want to simplify the sum

$$\sum_{\substack{k=h+l\\0\le h\le d\\0\le l\le L}}(-1)^h\frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \tag{5.15}$$

furthermore. Putting $k=j_1-m_1$, $d=j_1+j_2-J$ back, we have

$$\begin{aligned}&\sum_{\substack{k=h+l\\0\le h\le d\\0\le l\le L}}(-1)^h\frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!}\\&=\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!}\end{aligned}$$

and the sum in the last line is taken over all the values of l with which all the factrial terms containing l makes sense. In order to do so, we are going to use the following formula

$$\frac{a!}{b!c!}=\sum_s\frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{5.16}$$

Now

$$\begin{aligned}&\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!}\\=&\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!}\cdot\frac{(j_2+J-m_1-l)!}{(J-M-l)!(j_1-m_1-l)!}\\=&\sum_{l,l_1}(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!}\cdot\frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!(j_1-j_2-M-l+l_1)!}\end{aligned} \tag{5.17}$$

From the last euqation (5.17), Racah uses the following formula to proceed with the calculation,

$$\sum_s(-1)^s\frac{(t-s)!}{s!(x-s)!(z-s)!}=\frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}. \tag{5.18}$$

To use the above formula, we change the summation variable $l\to l'=j_1-j_2-M+l_1-l$, and we have

$$\begin{aligned}j_1+m_1+l&=2j_1-j_2-m_2+l_1-l'\\j_2-J+m_1+l&=j_1-J-m_2+l_1-l'.\end{aligned}$$

Putting these terms to (5.17), we get

$$\begin{aligned}&\sum_{l',l_1}(-1)^{-j_2-m_2+l_1-l'}\frac{(2j_1-j_2-m_2+l_1-l')!}{l'!(j_1-J-m_2+l_1-l')!(j_1-j_2-M+l_1-l')!}\cdot\frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!}\\=&\sum_{l_1}(-1)^{j_2+m_2-l_1}\frac{(j_1+m_1)!(j_1-j_2+J)!}{(j_1-J-m_2+l_1)!(j_1-j_2-M+l_1)!(J+M-l_1)!}\cdot\frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!}\end{aligned}$$

In another method that the author found, (5.17) can be further transformed into the following expression by using (5.16)

$$
\begin{aligned}
&\sum_{l,l_1,l_2} (-1)^{j_1-m_1-l} \frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!(j_2-J-j_1+l_2+l)!} \\
&\qquad \times \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(j_2-j_1+J-l_1)!(j_1-j_2-M+l_1-l)!} \\
&= \sum_{l,l_1,l_2} (-1)^{j_1-m_1} \frac{(-1)^{-l}}{(j_1-j_2-M+l_1-l)!(-j_1+j_2-J+l_2+l)!} \\
&\qquad \times \frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!}.
\end{aligned}
\tag{5.19}
$$

Then, by the use of the formula (see Appendix)

$$
\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a,-b) \tag{5.20}
$$

(5.17) now also becomes

$$
\sum_{l_1} (-1)^{j_2+m_2-l_1} \frac{(j_1+m_1)!(j_1-j_2+J)!}{(j_1-J-m_2+l_1)!(j_1-j_2-M+l_1)!(J+M-l_1)!} \cdot \frac{(j_2+m_2)!(-j_1+j_2+J)!}{l_1!(j_2+m_2-l_1)!(-j_1+j_2+J-l_1)!}.
$$

In either way, putting $t = j_2 + m_2 - l_1$ ([2]), we obtain

$$
\begin{aligned}
&\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \\
&= \sum_t (-1)^t \frac{(j_1+m_1)!(j_2+m_2)!(j_1-j_2+J)!(-j_1+j_2+J)!}{t!(j_1+j_2-J-t)!(j_1-m_1-t)!(j_2+m_2-t)!(J-j_2+m_1+t)!(J-j_1-m_2+t)!},
\end{aligned}
$$

and then the final formula

$$
\begin{aligned}
\langle j_1 j_2 m_1 m_2 | JM \rangle &= \sqrt{(2J+1)}\sqrt{\Delta(j_1 j_2 J)}\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!} \\
&\times \sum_t (-1)^t \frac{1}{t!(j_1+j_2-J-t)!(j_1-m_1-t)!(j_2+m_2-t)!(J-j_2+m_1+t)!(J-j_1-m_2+t)!}
\end{aligned}
\tag{5.21}
$$

Here, notice that the sum takes place in the range

$$
\max\{0, -(J-j_2+m_1), -(J-j_1-m_2)\} \le t \le \min\{j_1+j_2-J, j_1-m_1, j_2+m_2\} \tag{5.22}
$$

By making the substitution $t' = j_1 + j_2 - J - t$ in the sum, we have

$$
\langle j_1 j_2 m_1 m_2 | JM \rangle = (-1)^{j_1+j_2-J} \langle j_2 j_1 m_2 m_1 | JM \rangle \tag{5.23}
$$

The Wigner 3j-symbol

$$
\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} := \frac{(-1)^{a-b-\gamma}}{\sqrt{2c+1}} \langle ab\, \alpha\beta | c\ -\gamma \rangle \tag{5.24}
$$

The Racah formula

$$
\begin{aligned}
\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} &= (-1)^{a-b-\gamma}\sqrt{\Delta(abc)}\sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!} \\
&\times \sum_t (-1)^t [t!(c-b+t+\alpha)!(c-a+t-\beta)!(a+b-c-t)!(a-t-\alpha)!(b-t+\beta)!]^{-1} \\
&(\alpha+\beta+\gamma = 0, \quad |a-b| \le c \le a+b)
\end{aligned}
$$

5.3 Some Examples

5.3.1 $J = j_1 + j_2$

$$
\begin{aligned}
|JJ\rangle &= |j_1 j_1\rangle |j_2 j_2\rangle && (5.25)\\
(J_-)^k |JJ\rangle &= (J_{1-} + J_{2-})^k |j_1 j_1\rangle |j_2 j_2\rangle && (5.26)\\
|JJ-k\rangle &= \sum_{i=0}^{k} {}_k C_i \frac{F(j_1,i)F(j_2,k-i)}{F(J,k)} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle && (5.27)\\
&= \sum_{i=0}^{k} \sqrt{\frac{{}_{2j_1}C_i \cdot {}_{2j_2}C_{k-i}}{{}_{2J}C_k}} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle && (5.28)
\end{aligned}
$$

Rewrite it using $M = J - k = j_1 + j_2 - k$, $m_1 = j_1 - i$, $m_2 = j_2 - (k-i) = -j_1 + M + i = M - m_1$, $k - i = j_2 - m_2$

$$
|j_1 + j_2 M\rangle = \sum_{m_1+m_2=M} \sqrt{\frac{{}_{2j_1}C_{j_1-m_1} \cdot {}_{2j_2}C_{j_2-m_2}}{{}_{2J}C_{j_1+j_2-M}}} |j_1 m_1\rangle |j_2 m_2\rangle \qquad (5.29)
$$

$$
\begin{aligned}
\langle j_1 j_2 m_1 m_2 | j_1 + j_2 M\rangle &= \sqrt{\frac{{}_{2j_1}C_{j_1-m_1} \cdot {}_{2j_2}C_{j_2-m_2}}{{}_{2(j_1+j_2)}C_{j_1+j_2-M}}} && (5.30)\\
&= \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!} \frac{(J+M)!(J-M)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} && (5.31)
\end{aligned}
$$

5.3.2 $j_1 = j,\ j_2 = \mathbf{1}$

We have three possibilities; $J = j+1,\ j,\ j-1$. Note that any $|J\ M\rangle$ is expanded by $|j\ M+1\rangle|\mathbf{1}\ -1\rangle$, $|j\ M\rangle|\mathbf{1}\ 0\rangle$, $|j\ M-1\rangle|\mathbf{1}\ 1\rangle$.
The case $J = j+1$.

$$
\begin{aligned}
\sqrt{\frac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} &= \sqrt{\frac{(2j)!}{(j-M-1)!(j+M+1)!} \frac{(j-M+1)!(j+M+1)!}{(2j+2)!}} && (5.32)\\
&= \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}} && (5.33)
\end{aligned}
$$

Table 1:

$\lvert j\ M+1\rangle\lvert\mathbf{1}\ -1\rangle$	$\sqrt{\frac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}}$
$\lvert j\ M\rangle\lvert\mathbf{1}\ 0\rangle$	$\sqrt{\frac{2 \cdot {}_{2j}C_{j-M}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{2(j-M+1)(j+M+1)}{(2j+2)(2j+1)}}$
$\lvert j\ M-1\rangle\lvert\mathbf{1}\ 1\rangle$	$\sqrt{\frac{{}_{2j}C_{j-M+1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{(j+M+1)(j+M)}{(2j+2)(2j+1)}}$

6 $6j$-symbol

$6j$-symbol is defined as a coupling coefficient between representation vectors produced as a result of adding three angular momentum j_1, j_2, j_3. There is an ambiguity in taking the tensor products. There are three ways to pick the first two representation spaces V_{j_k} and V_{j_l} to form $V_{j_k} \otimes V_{j_l}$ and decompose it into the direct sum of irreducible representations $\bigoplus_{j_{kl}} V_{j_{kl}}$. Then, we take tensor products again with each of the representation space $V_{j_{kl}}$ and V_{j_m}. The final direct sum $\left(\bigoplus_{j_{kl}} V_{j_{kl}}\right) \otimes V_{j_m} = \bigoplus_J n_J V_J$, where n_J is the multiplicity of V_J, is independent of the way we chose the first two j_k and j_l.

$$\begin{aligned} & |(j_1,(j_2,j_3)j_{23})JM\rangle \\ = & \sum_{j_{12}} [(2j_{12}+1)(2j_{23}+1)]^{\frac{1}{2}} (-1)^{j_1+j_2+j_3+J} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{array} \right\} |((j_1,j_2)j_{12},j_3)JM\rangle \end{aligned} \tag{6.1}$$

Example 6.1. $j_1 = j_2 = \frac{1}{2}, j_3 = 1$. *The resultant angular momenta are* $J = 0, 1, 2$.

$$\begin{aligned} \left(\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}\right) \otimes \mathbf{1} &= \left(\left[\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{0}\right] \oplus \left[\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{1}\right]\right) \otimes \mathbf{1} \\ &= \left[\left(\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{0}, \mathbf{1}\right)\mathbf{1}\right] \oplus \left[\left(\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{1}, \mathbf{1}\right)\mathbf{0}\right] \oplus \left[\left(\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{1}, \mathbf{1}\right)\mathbf{1}\right] \oplus \left[\left(\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\mathbf{1}, \mathbf{1}\right)\mathbf{2}\right] \end{aligned}$$

$$\begin{aligned} \mathbf{\frac{1}{2}} \otimes \left(\mathbf{\frac{1}{2}} \otimes \mathbf{1}\right) &= \mathbf{\frac{1}{2}} \otimes \left(\left[\left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{1}{2}}\right] \oplus \left[\left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{3}{2}}\right]\right) \\ &= \left[\left(\mathbf{\frac{1}{2}}, \left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{1}{2}}\right)\mathbf{0}\right] \oplus \left[\left(\mathbf{\frac{1}{2}}, \left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{1}{2}}\right)\mathbf{1}\right] \oplus \left[\left(\mathbf{\frac{1}{2}}, \left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{3}{2}}\right)\mathbf{1}\right] \oplus \left[\left(\mathbf{\frac{1}{2}}, \left(\mathbf{\frac{1}{2}}, \mathbf{1}\right)\mathbf{\frac{3}{2}}\right)\mathbf{2}\right] \end{aligned}$$

6.1 $6j$-symbol in terms of $3j$-symbols

$$\begin{aligned} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} &= \sum_{\substack{\alpha\beta\gamma \\ \delta\epsilon\varphi}} (-1)^{d+e+f+\delta+\epsilon+\varphi} \times \\ & \times \left(\begin{array}{ccc} d & e & c \\ \delta & -\epsilon & \gamma \end{array} \right) \left(\begin{array}{ccc} e & f & a \\ \epsilon & -\varphi & \alpha \end{array} \right) \left(\begin{array}{ccc} f & d & b \\ \varphi & -\delta & \beta \end{array} \right) \left(\begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \end{array} \right) \end{aligned} \tag{6.2}$$

Racah's formula ([2])

$$\begin{aligned} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} &= [\Delta(abc)\Delta(aef)\Delta(cde)\Delta(bdf)]^{\frac{1}{2}} \times \\ &\times \sum_x \frac{(x+1)!}{[(p_1-x)(p_2-x)!(p_3-x)!(x-q_1)!(x-q_2)!(x-q_3)!(x-q_4)!]} \end{aligned} \tag{6.3}$$

where $q_1 = a+b+c, q_2 = b+d+f, q_3 = a+e+f, q_4 = d+e+c, p_1 = a+b+d+e, p_2 = b+c+e+f, p_3 = c+a+f+d$. Only values which satisfy the triangle inequalities, $|b-c| \leq a \leq b+c$ are allowed:$(abc), (aef), (dbf), (dec)$. Therefore, there have to be even numbers of half integers at each face. This is equivalente to say that $q_1, q_2, q_3, q_4, p_1, p_2, p_3$ are all integers.

A Addition of Binomial Coefficients

The addition formula for binomial coefficients is given as

$$\sum_s \binom{x}{s}\binom{y}{z-s} = \binom{x+y}{z}. \tag{A.1}$$

This formula can be derived by comparing the coefficients of the polynomials on the both sides of $(X+Y)^{x+y} = (X+Y)^x(X+Y)^y$. Putting $x = a-b$, $y = b$, $z = a-c$ in (A.1), we have

$$\frac{a!}{b!c!} = \sum_s \frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{A.2}$$

If $y < 0$,

$$\binom{y}{z-s} = (-1)^{z-s}\binom{z-s-y-1}{z-s}. \tag{A.3}$$

This can be obtained by using $(X+Y)^y = X^y(1+\frac{Y}{X})^y$ and Taylor-expanding $(1+\frac{Y}{X})^y$ around $\frac{Y}{X} = 0$. Then (A.1) can be transformed into

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = (-1)^z\binom{x+y}{z},\ (x+y \geq 0) \tag{A.4}$$

or

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = \binom{z-x-y-1}{z},\ (x+y < 0) \tag{A.5}$$

Putting $y = z-t-1$, (A.4) and (A.5) become

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = (-1)^z\frac{(t-z)!(x+z-t-1)!}{x!z!(x-t-1)!},\ (x > t \geq z \geq 0). \tag{A.6}$$

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!},\ (t \geq x, z \geq 0). \tag{A.7}$$

The following formula is referenced in [1],[2]. It can be obtained by letting $y < 0$ and $x < 0$ in (A.1):

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \tag{A.8}$$

This can be verified by applying (A.3) on both sides of (A.1). Then we will have

$$\sum_s \binom{a+s}{c+s}\binom{b-s}{d-s} = \binom{a+b+1}{c+d} \tag{A.9}$$

after a change of summation variable, and this is just (A.8).
The following formula

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a\delta(a,-b) \tag{A.10}$$

is a slight generalisation of a simple binomial coefficients formula

$$(X-Y)^K = \sum_{l=0}^{K}(-1)^l\binom{K}{l}X^{K-l}Y^l \tag{A.11}$$

(A.10) is obtained when we set $X = Y = 1$.

References

[1] Albert Messiah, Quantum Mechanics, Dover (1995)

[2] Giulio Racah, Theory of Complex Spectra.I&II, Phys. Rev. 61&62 (1942)

[3] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients" in *Spectroscopic and group theoretical methods in physics*, (ed. Bloch et al), North Holland Publ. Co., Amsterdam, 1968