

# Angular Momentum and Racah's formula

hisnobu-nakamura

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## 1 Introduction

This is a review of Racah's proof of the formula of the Clebsch-Gordan coefficients or Wigner's 3j-symbol and 6j-symbol.

## 2 Angular Momentum Operators and $\mathfrak{su}(2)$ Representation

$J_x, J_y, J_z \in \mathfrak{su}(2)$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (2.1)$$

The Casimir operator  $J^2 = J_x^2 + J_y^2 + J_z^2$  and the ladder operators  $J_{\pm} = J_x \pm iJ_y$

$$[J^2, J_i] = 0 \quad (2.2)$$

$$[J_z, J_{\pm}] = \pm J_{\pm} \quad (2.3)$$

$$[J_+, J_-] = 2J_z \quad (2.4)$$

$$J^2 = J_-J_+ + J_z^2 + J_z \quad (2.5)$$

$$= J_+J_- + J_z^2 - J_z \quad (2.6)$$

Let  $V$  be a finite dimensional vector space over  $\mathbb{C}$  and  $\varphi : \mathfrak{su}(2) \rightarrow \mathbf{End}(V)$  be its associated representation. The Casimir operator  $J^2$  and  $J_z$  commute, so there are simultaneous eigenvectors of the operators. It can be shown that, by the finiteness of the dimension  $V$ ,  $J_z$  has a maximal eigenvalue  $j$ , which is called the **spin** of the representation. So it is plausible to denote the representation space as  $V_j$ . And it can be shown that the simultaneous eigenvalue of  $J^2$  is  $j(j+1)$  and invariant of the eigenvalue of  $J_z$ . Let us write an eigenvector of  $J_z$  with the eigenvalue  $m$  as  $|jm\rangle \in V_j$ . Then, in summary, we have

$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle. \quad (2.7)$$

## 3 The ladder operators' coefficients

$$J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}|jm+1\rangle \quad (3.1)$$

$$= \sqrt{(j-m)(j+m+1)}|jm+1\rangle \quad (3.2)$$

$$J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}|jm-1\rangle \quad (3.3)$$

$$= \sqrt{(j+m)(j-m+1)}|jm-1\rangle \quad (3.4)$$

It is useful to write  $|jm\rangle$  in terms of  $(J_-)^k|jj\rangle$ . So, let us rewrite the coefficients in simpler notation;

$$J_-|jj - (k-1)\rangle = f(j, k)|jj - k\rangle \quad (3.5)$$

where

$$f(j, k) := \sqrt{k(2j - k + 1)}, \quad 1 \leq k \leq 2j$$

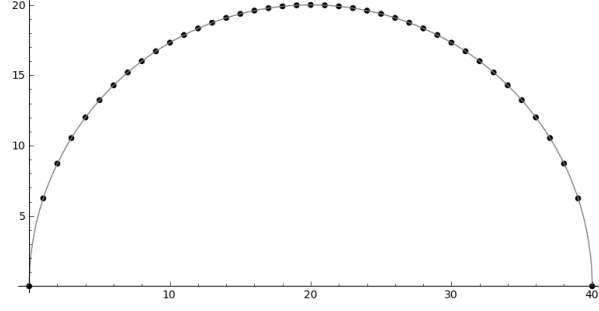


Figure 1: The graph of  $f(j = \frac{39}{2}, k)$  where  $k$  is the horizontal axis.

We also have

$$J_+ |jj - k\rangle = f(j, k) |jj - k + 1\rangle \quad (3.6)$$

Note that  $D := 2j + 1$  is the dimension of the  $j$ -th representation space.  
Hence

$$(J_-)^k |jj\rangle = F(j, k) |jj - k\rangle \quad (3.7)$$

where  $F(j, k) = \prod_{i=1}^k f(j, i)$  and evaluated as,

$$\begin{aligned} F(j, k) &= \sqrt{k(2j+1-k)(k-1)(2j+1-(k-1)) \times \cdots \times 2 \cdot (2j+1-2) \cdot 1 \cdot (2j+1-1)} \\ &= \sqrt{k(D-k)(k-1)(D-(k-1)) \times \cdots \times 2 \cdot (D-2) \cdot 1 \cdot (D-1)} \\ &= \sqrt{\frac{k!(2j)!}{(2j-k)!}} = k! \sqrt{2j C_k} \end{aligned} \quad (3.8)$$

and  $F(j, 0) = 1$

## 4 Recursion Relations for Clebsch-Gordan coefficients

The tensor product of two representation space  $V_{j_1} \otimes V_{j_2}$  decomposes into the direct sum of irreducible representations  $V_J$  where  $|j_1 - j_2| \leq J \leq j_1 + j_2$  as

$$V_{j_1} \otimes V_{j_2} = V_{|j_1-j_2|} \oplus \cdots \oplus V_{j_1+j_2} \quad (4.1)$$

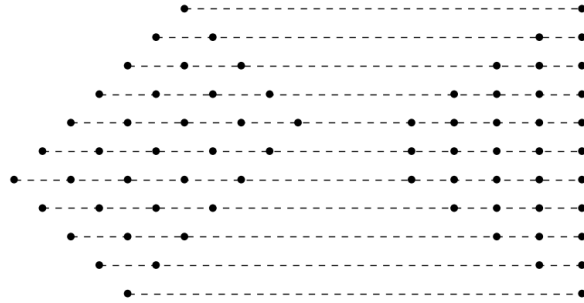


Figure 2: Correspondence between the tensor product  $V_{j_1} \otimes V_{j_2}$  and  $V_J$  ( $j_1 = 3, j_2 = 2$ ).

One of the elements in  $|j_1 j_2 JM\rangle \in V_J$  is expandable by the tensor basis of  $|j_1 j_2 m_1 m_2\rangle := |j_1 m_1\rangle \otimes |j_2 m_2\rangle \in V_{j_1} \otimes V_{j_2}$  as

$$|j_1 j_2 JM\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | JM \rangle. \quad (4.2)$$

The coefficients  $\langle j_1 j_2 m_1 m_2 | JM \rangle$  are called **Clebsch-Gordan coefficients** and they can be recursively calculated by the following formulae: Apply  $J_+ = j_{1+} + j_{2+}$  take inner product with  $\langle j_1 j_2 m_1 m_2 |$

$$\begin{aligned} & \sqrt{J(J+1) - M(M+1)} \langle j_1 j_2 m_1 m_2 | JM+1 \rangle \\ &= \sqrt{j_1(j_1+1) - m_1(m_1-1)} \langle j_1 j_2 m_1 - 1 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)} \langle j_1 j_2 m_1 m_2 - 1 | JM \rangle \end{aligned} \quad (4.3)$$

and  $J_- = j_{1-} + j_{2-}$

$$\begin{aligned} & \sqrt{J(J+1) - M(M-1)} \langle j_1 j_2 m_1 m_2 | JM-1 \rangle \\ &= \sqrt{j_1(j_1+1) - m_1(m_1+1)} \langle j_1 j_2 m_1 + 1 m_2 | JM \rangle + \sqrt{j_2(j_2+1) - m_2(m_2+1)} \langle j_1 j_2 m_1 m_2 + 1 | JM \rangle \end{aligned} \quad (4.4)$$

## 5 Explicit formulae for Clebsch-Gordan coefficients

### 5.1 $\langle j_1 j_2 m_1 m_2 | JM \rangle$

Define  $d = j_1 + j_2 - J$ , ( $j_1 \geq j_2$  and  $0 \leq d \leq 2j_2$ ), and  $L = J - M$ . Let us determine the coefficients of the top spin state

$$\begin{aligned} |JJ\rangle &= a_0 |j_1 j_1\rangle |j_2 j_2 - d\rangle + a_1 |j_1 j_1 - 1\rangle |j_2 j_2 - d + 1\rangle + \cdots + a_d |j_1 j_1 - d\rangle |j_2 j_2\rangle \\ &= \sum_{i=0}^d a_i |j_1 j_1 - i\rangle |j_2 j_2 - d + i\rangle \end{aligned} \quad (5.1)$$

by imposing the top spin condition

$$J_+ |JJ\rangle = 0 \implies a_{i+1} = -\frac{f(j_2, d-i)}{f(j_1, i+1)} a_i \quad (i = 0, \dots, d-1), \quad (5.2)$$

which means

$$a_i = -\frac{f(j_2, d-(i-1))}{f(j_1, i)} a_{i-1} \quad (i = 1, \dots, d) \quad (5.3)$$

$$= (-1)^i \frac{f(j_2, d-(i-1)) f(j_2, d-(i-2)) \cdots f(j_2, d-1) f(j_2, d)}{f(j_1, i) f(j_1, i-1) \cdots f(j_1, 2) f(j_1, 1)} a_0 \quad (5.4)$$

$$= (-1)^i \frac{F(j_2, d)}{F(j_1, i) F(j_2, d-i)} a_0 \quad (5.5)$$

Here  $F(j_2, 0) = 1$ .

## 5.2 Derivation of Racah's formula

Normalisation condition  $\langle JJ|JJ\rangle = 1$  yields

$$\begin{aligned}
\frac{1}{a_0^2} &= \sum_{i=0}^d \frac{F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d-i)^2} \\
&= 1 + \left[ \frac{f(j_2, d)}{f(j_1, 1)} \right]^2 + \dots + \left[ \frac{f(j_2, d-(i-1))f(j_2, d-(i-2)) \dots f(j_2, d-1)f(j_2, d)}{f(j_1, i)f(j_1, i-1) \dots f(j_1, 2)f(j_1, 1)} \right]^2 + \\
&\quad \dots + \left[ \frac{F(j_2, d)}{F(j_1, d)} \right]^2 \\
&= \frac{1}{F(j_1, d)^2} \left\{ (D_1 - d) \cdot d \dots (D_1 - 2) \cdot 2 \cdot (D_1 - 1) \cdot 1 + (D_1 - d) \cdot d \dots (D_1 - 2) \cdot 2 \cdot (D_2 - d) \cdot d + \right. \\
&\quad \left. \dots + (D_1 - d) \cdot d \dots (D_1 - i - 1) \cdot (i + 1) \cdot (D_2 - (d - i + 1)) \cdot (d - i + 1) \dots (D_2 - d) \cdot d + \dots \right\} \\
&= \frac{1}{F(j_1, d)^2} \left\{ \frac{(d!)^2}{d!} (D_1 - d) \cdot (D_1 - 2) \cdot (D_1 - 1) + \frac{(d!)^2}{1!(d-1)!} (D_1 - d) \dots (D_1 - 2) \cdot (D_2 - d) + \right. \\
&\quad \left. \dots + \frac{(d!)^2}{i!(d-i)!} (D_1 - d) \cdot (D_1 - i - 1) \cdot (D_2 - (d - i + 1)) \dots (D_2 - d) + \dots \right\}
\end{aligned} \tag{5.6}$$

Writing

$$\begin{aligned}
G_i(j_1, j_2, d) &:= \frac{F(j_1, d)F(j_2, d)}{F(j_1, i)F(j_2, d-i)}, \\
&= \sqrt{\frac{(d!)^2}{(d-i)!i!} (D_2 - d)(D_2 - d - 1) \dots (D_2 - d - i + 1)(D_1 - d) \dots (D_1 - i + 1)}
\end{aligned} \tag{5.7}$$

Or, substituting  $d = j_1 + j_2 - J$ , this can be written as

$$G_i(j_1, j_2, j_1 + j_2 - J) = (-1)^i \sqrt{\frac{((j_1 + j_2 - J)!)^2}{(j_1 + j_2 - J - i)!i!} \frac{(-j_1 + j_2 + J)!(j_1 + J - j_2)!}{(-j_1 + j_2 + J - i)!(2j_1 - i)!}}.$$

In terms of these  $G_i$ 's, the coefficients  $a_i$  become

$$a_i = (-1)^i \frac{G_i(j_1, j_2, d)}{\sqrt{\sum_{i=1}^d G_i(j_1, j_2, d)^2}}. \tag{5.8}$$

We want to know the normalising coefficient  $N := \frac{1}{\sqrt{\sum_{i=1}^d G_i(j_1, j_2, d)^2}}$ . In order to simplify the sum

$$\begin{aligned}
\sum_{i=1}^d G_i(j_1, j_2, d)^2 &= \sum_{i=1}^d \frac{F(j_1, d)^2 F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d-i)^2} \\
&= \frac{(d!)^2}{(2j_1 - d)!(2j_2 - d)!} \sum_{i=1}^d \frac{(2j_1 - i)!(2j_2 - d + i)!}{i!(d-i)!},
\end{aligned} \tag{5.9}$$

we use a formula due to Racah (mentioned in Messiah[1])

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \tag{5.10}$$

with  $a \geq c, b \geq d \geq 0$ , where the sum is taken over  $-c \leq s \leq d$ .

Now substituting  $a = 2j_2 - d, b = 2j_1, c = 0, d = d$ , we obtain

$$N = \sqrt{\frac{(2j_2 - 2d + 2j_1 + 1)!}{d!(2j_2 - d + 2j_1 + 1)!}} = \sqrt{\frac{(2J + 1)!}{(j_1 + j_2 - J)!(j_1 + j_2 + J + 1)!}} \tag{5.11}$$

$$a_i = (-1)^i N G_i(j_1, j_2, d)$$

Now, by multiplying the top-spin state with the ladder operators  $L$  times, we obtain the state  $|JM\rangle$  with  $M = J - L$

$$\begin{aligned} J_-^L |JJ\rangle &= (j_{1-} + j_{2-})^L \sum_{h=0}^d a_h \times |j_1 j_1 - h\rangle |j_2 j_2 - d + h\rangle \\ F(J, L) |JJ - L\rangle &= \sum_{h=0}^d a_h \sum_{l=0}^L {}_L C_l \frac{F(j_1, h+l) F(j_2, (L+d)-(l+h))}{F(j_1, h) F(j_2, d-h)} |j_1 j_1 - (h+l)\rangle |j_2 j_2 - (L+d) + (h+l)\rangle \\ |JJ - L\rangle &= \frac{1}{F(J, L)} \sum_{k=0}^{L+d} \left[ \sum_{\substack{k=h+l, \\ 0 \leq h \leq d, \\ 0 \leq l \leq L}} a_h \times {}_L C_l \frac{F(j_1, k) F(j_2, K-k)}{F(j_1, h) F(j_2, d-h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle \\ &= \frac{N}{F(J, L)} \sum_{k=0}^{L+d} F(j_1, k) F(j_2, K-k) \left[ \sum_{\substack{k=h+l, \\ 0 \leq h \leq d, \\ 0 \leq l \leq L}} \frac{(-1)^h {}_L C_l G_h(j_1, j_2, d)}{F(j_1, h) F(j_2, d-h)} \right] |j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle \end{aligned}$$

where  $K = L+d = J-M+j_1+j_2-J = j_1+j_2-M$ . Now, consider the coefficients of  $|j_1 j_1 - k\rangle |j_2 j_2 - K + k\rangle$

$$\begin{aligned} B_k &:= F(j_1, k) F(j_2, K-k) \left[ \sum_{\substack{k=h+l, \\ 0 \leq h \leq d, \\ 0 \leq l \leq L}} \frac{(-1)^h {}_L C_l G_h(j_1, j_2, d)}{F(j_1, h) F(j_2, d-h)} \right] \\ &= \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!}} \sum_{\substack{k=h+l, \\ 0 \leq h \leq d, \\ 0 \leq l \leq L}} (-1)^h {}_L C_l \sqrt{\frac{(2j_1-h)!(2j_2-d+h)!(d!)^2(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!(2j_1-d)!(2j_2-d)!h!(d-h)!}} \\ &= \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!(2j_1-d)!(2j_2-d)!}} L!d! \sum_{\substack{k=h+l, \\ 0 \leq h \leq d, \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \end{aligned}$$

The coefficient outside the sum, in terms of  $j_1, j_2, J, m_1, m_2, M$ , using the relations  $K = L+d = J-M+j_1+j_2-J = j_1+j_2-M$ ,  $k = j_1 - m_1$ , is

$$\sqrt{\frac{(j_1-m_1)!(j_2+m_1-M)!}{(j_1+m_1)!(j_2-m_1+M)!(j_1-j_2+J)!(j_2-j_1+J)!}} (J-M)!(j_1+j_2-J)! \quad (5.12)$$

Multiplying by  $\frac{N}{F(J, J-M)}$

$$\begin{aligned} &\sqrt{\frac{(2J+1)(j_1+j_2-J)!}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+j_2+J+1)!}} \frac{(j_1-m_1)!(j_2-m_2)!(J+M)!(J-M)!}{(j_1+m_1)!(j_2+m_2)!} \\ &= \sqrt{(2J+1)} \sqrt{\Delta(j_1 j_2 J)} \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!} \\ &\quad \times \frac{1}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+m_1)!(j_2+m_2)!} \end{aligned} \quad (5.13)$$

where we have defined

$$\Delta(abc) := \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}. \quad (5.14)$$

Now, we want to simplify the sum

$$\sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!l!(L - l)!} \quad (5.15)$$

furthermore. Putting  $k = j_1 - m_1$ ,  $d = j_1 + j_2 - J$  back, we have

$$\begin{aligned} & \sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d - h)!l!(L - l)!} \\ &= \sum_l (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1 + l)!(j_2 + J - m_1 - l)!}{l!(j_1 - m_1 - l)!(j_2 - J + m_1 + l)!(J - M - l)!} \end{aligned}$$

and the sum in the last line is taken over all the values of  $l$  with which all the factorial terms containing  $l$  makes sense. In order to do so, we are going to use the following formula

$$\frac{a!}{b!c!} = \sum_s \frac{(a - b)!(a - c)!}{s!(a - b - s)!(a - c - s)!(b + c - a + s)!}. \quad (5.16)$$

Now

$$\begin{aligned} & \sum_l (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1 + l)!(j_2 + J - m_1 - l)!}{l!(j_1 - m_1 - l)!(j_2 - J + m_1 + l)!(J - M - l)!} \\ &= \sum_l (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1 + l)!}{l!(j_2 - J + m_1 + l)!} \cdot \frac{(j_2 + J - m_1 - l)!}{(J - M - l)!(j_1 - m_1 - l)!} \\ &= \sum_{l, l_1} (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1 + l)!}{l!(j_2 - J + m_1 + l)!} \cdot \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(-j_1 + j_2 + J - l_1)!(j_1 - j_2 - M - l + l_1)!} \end{aligned} \quad (5.17)$$

From the last equation (5.17), Racah uses the following formula to proceed with the calculation,

$$\sum_s (-1)^s \frac{(t - s)!}{s!(x - s)!(z - s)!} = \frac{(t - x)!(t - z)!}{x!z!(t - x - z)!}. \quad (5.18)$$

To use the above formula, we change the summation variable  $l \rightarrow l' = j_1 - j_2 - M + l_1 - l$ , and we have

$$\begin{aligned} j_1 + m_1 + l &= 2j_1 - j_2 - m_2 + l_1 - l' \\ j_2 - J + m_1 + l &= j_1 - J - m_2 + l_1 - l'. \end{aligned}$$

Putting these terms to (5.17), we get

$$\begin{aligned} & \sum_{l', l_1} (-1)^{-j_2 - m_2 + l_1 - l'} \frac{(2j_1 - j_2 - m_2 + l_1 - l')!}{l'!(j_1 - J - m_2 + l_1 - l')!(j_1 - j_2 - M + l_1 - l')!} \cdot \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(j_2 - j_1 + J - l_1)!} \\ &= \sum_{l_1} (-1)^{j_2 + m_2 - l_1} \frac{(j_1 + m_1)!(j_1 - j_2 + J)!}{(j_1 - J - m_2 + l_1)!(j_1 - j_2 - M + l_1)!(J + M - l_1)!} \cdot \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(j_2 - j_1 + J - l_1)!} \end{aligned}$$

In another method that the author found, (5.17) can be further transformed into the following expression by using (5.16)

$$\begin{aligned}
& \sum_{l, l_1, l_2} (-1)^{j_1 - m_1 - l} \frac{(j_1 + m_1)!(j_1 - j_2 + J)!}{l_2!(j_1 + m_1 - l_2)!(j_1 - j_2 + J - l_2)!(j_2 - J - j_1 + l_2 + l)!} \\
& \times \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(j_2 - j_1 + J - l_1)!(j_1 - j_2 - M + l_1 - l)!} \\
& = \sum_{l, l_1, l_2} (-1)^{j_1 - m_1} \frac{(-1)^{-l}}{(j_1 - j_2 - M + l_1 - l)!(-j_1 + j_2 - J + l_2 + l)!} \\
& \times \frac{(j_1 + m_1)!(j_1 - j_2 + J)!}{l_2!(j_1 + m_1 - l_2)!(j_1 - j_2 + J - l_2)!} \cdot \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(j_1 - j_2 + J - l_1)!}.
\end{aligned} \tag{5.19}$$

Then, by the use of the formula (see Appendix)

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a, -b) \tag{5.20}$$

(5.17) now also becomes

$$\sum_{l_1} (-1)^{j_2 + m_2 - l_1} \frac{(j_1 + m_1)!(j_1 - j_2 + J)!}{(j_1 - J - m_2 + l_1)!(j_1 - j_2 - M + l_1)!(J + M - l_1)!} \cdot \frac{(j_2 + m_2)!(-j_1 + j_2 + J)!}{l_1!(j_2 + m_2 - l_1)!(j_1 - j_2 + J - l_1)!}.$$

In either way, putting  $t = j_2 + m_2 - l_1$  ([2]), we obtain

$$\begin{aligned}
& \sum_{\substack{k=h+l \\ 0 \leq h \leq d \\ 0 \leq l \leq L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d-h)!l!(L-l)!} \\
& = \sum_t (-1)^t \frac{(j_1 + m_1)!(j_2 + m_2)!(j_1 - j_2 + J)!(-j_1 + j_2 + J)!}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!},
\end{aligned}$$

and then the final formula

$$\begin{aligned}
\langle j_1 j_2 m_1 m_2 | JM \rangle &= \sqrt{(2J+1)} \sqrt{\Delta(j_1 j_2 J)} \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(J + M)!(J - M)!} \\
&\times \sum_t (-1)^t \frac{1}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!}
\end{aligned} \tag{5.21}$$

Here, notice that the sum takes place in the range

$$\max \{0, -(J - j_2 + m_1), -(J - j_1 - m_2)\} \leq t \leq \min \{j_1 + j_2 - J, j_1 - m_1, j_2 + m_2\} \tag{5.22}$$

By making the substitution  $t' = j_1 + j_2 - J - t$  in the sum, we have

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = (-1)^{j_1 + j_2 - J} \langle j_2 j_1 m_2 m_1 | JM \rangle \tag{5.23}$$

The Wigner 3j-symbol

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} := \frac{(-1)^{a-b-\gamma}}{\sqrt{2c+1}} \langle ab \alpha \beta | c - \gamma \rangle \tag{5.24}$$

The Racah formula

$$\begin{aligned}
\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} &= (-1)^{a-b-\gamma} \sqrt{\Delta(abc)} \sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!} \\
&\times \sum_t (-1)^t [t!(c-b+t+\alpha)!(c-a+t-\beta)!(a+b-c-t)!(a-t-\alpha)!(b-t+\beta)!]^{-1} \\
&(\alpha + \beta + \gamma = 0, \quad |a-b| \leq c \leq a+b)
\end{aligned}$$

### 5.3 Some Examples

#### 5.3.1 $J = j_1 + j_2$

$$|JJ\rangle = |j_1 j_1\rangle |j_2 j_2\rangle \quad (5.25)$$

$$(J_-)^k |JJ\rangle = (J_{1-} + J_{2-})^k |j_1 j_1\rangle |j_2 j_2\rangle \quad (5.26)$$

$$|JJ - k\rangle = \sum_{i=0}^k {}_k C_i \frac{F(j_1, i) F(j_2, k-i)}{F(J, k)} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle \quad (5.27)$$

$$= \sum_{i=0}^k \sqrt{\frac{{}_{2j_1} C_i \cdot {}_{2j_2} C_{k-i}}{{}_J C_k}} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle \quad (5.28)$$

Rewrite it using  $M = J - k = j_1 + j_2 - k$ ,  $m_1 = j_1 - i$ ,  $m_2 = j_2 - (k-i) = -j_1 + M + i = M - m_1$ ,  $k-i = j_2 - m_2$

$$|j_1 + j_2 M\rangle = \sum_{m_1+m_2=M} \sqrt{\frac{{}_{2j_1} C_{j_1-m_1} \cdot {}_{2j_2} C_{j_2-m_2}}{{}_J C_{j_1+j_2-M}}} |j_1 m_1\rangle |j_2 m_2\rangle \quad (5.29)$$

$$\langle j_1 j_2 m_1 m_2 | j_1 + j_2 M \rangle = \sqrt{\frac{{}_{2j_1} C_{j_1-m_1} \cdot {}_{2j_2} C_{j_2-m_2}}{{}_{2(j_1+j_2)} C_{j_1+j_2-M}}} \quad (5.30)$$

$$= \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!} \frac{(J+M)!(J-M)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \quad (5.31)$$

#### 5.3.2 $j_1 = j$ , $j_2 = 1$

We have three possibilities;  $J = j+1$ ,  $j$ ,  $j-1$ . Note that any  $|J M\rangle$  is expanded by  $|j M+1\rangle |\mathbf{1} -1\rangle$ ,  $|j M\rangle |\mathbf{1} 0\rangle$ ,  $|j M-1\rangle |\mathbf{1} 1\rangle$ .

The case  $J = j+1$ .

$$\sqrt{\frac{{}_{2j} C_{j-M-1}}{{}_{2(j+1)} C_{j-M+1}}} = \sqrt{\frac{(2j)!}{(j-M-1)!(j+M+1)!} \frac{(j-M+1)!(j+M+1)!}{(2j+2)!}} \quad (5.32)$$

$$= \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}} \quad (5.33)$$

Table 1:

$ j M+1\rangle  \mathbf{1} -1\rangle$	$\sqrt{\frac{{}_{2j} C_{j-M-1}}{{}_{2(j+1)} C_{j-M+1}}} = \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}}$
$ j M\rangle  \mathbf{1} 0\rangle$	$\sqrt{\frac{2 \cdot {}_{2j} C_{j-M}}{{}_{2(j+1)} C_{j-M+1}}} = \sqrt{\frac{2(j-M+1)(j+M+1)}{(2j+2)(2j+1)}}$
$ j M-1\rangle  \mathbf{1} 1\rangle$	$\sqrt{\frac{{}_{2j} C_{j-M+1}}{{}_{2(j+1)} C_{j-M+1}}} = \sqrt{\frac{(j+M+1)(j+M)}{(2j+2)(2j+1)}}$



## 6 $6j$ -symbol

It is defined as a coupling coefficient of three angular momentum

$$\begin{aligned} & |(j_1, (j_2, j_3)j_{23})JM\rangle \\ &= \sum_{j_{12}} [(2j_{12} + 1)(2j_{23} + 1)]^{\frac{1}{2}} (-1)^{j_1 + j_2 + j_3 + J} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} |((j_1, j_2)j_{12}, j_3)JM\rangle \end{aligned} \quad (6.1)$$

$6j$ -symbol in terms of  $3j$ -symbols

$$\begin{aligned} \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} &= \sum_{\substack{\alpha\beta\gamma \\ \delta\epsilon\varphi}} (-1)^{d+e+f+\delta+\epsilon+\varphi} \times \\ &\times \begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix} \begin{pmatrix} e & f & a \\ \epsilon & -\varphi & \alpha \end{pmatrix} \begin{pmatrix} f & d & b \\ \varphi & -\delta & \beta \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \end{aligned} \quad (6.2)$$

Racah's formula ([2])

$$\begin{aligned} \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} &= [\Delta(abc)\Delta(aef)\Delta(cde)\Delta(bdf)]^{\frac{1}{2}} \times \\ &\times \sum_x \frac{(x+1)!}{[(p_1-x)(p_2-x)!(p_3-x)!(x-q_1)!(x-q_2)!(x-q_3)!(x-q_4)!]} \end{aligned} \quad (6.3)$$

where  $q_1 = a+b+c, q_2 = b+d+f, q_3 = a+e+f, q_4 = d+e+c, p_1 = a+b+d+e, p_2 = b+c+e+f, p_3 = c+a+f+d$ . Only values which satisfy the triangle inequalities,  $|b-c| \leq a \leq b+c$  are allowed:  $(abc), (aef), (dbf), (dec)$ . Therefore, there have to be even numbers of half integers at each face. This is equivalent to say that  $q_1, q_2, q_3, q_4, p_1, p_2, p_3$  are all integers.

**Example 6.1.**  $\{j_1 = j_2 = \frac{1}{2}, j_3 = 1\}$  The resultant angular momenta are  $J = 0, 1, 2$ .

$$\begin{aligned} \left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes 1 &= \left(\left[\left(\frac{1}{2}, \frac{1}{2}\right) 0\right] \oplus \left[\left(\frac{1}{2}, \frac{1}{2}\right) 1\right]\right) \otimes 1 \\ &= \left[\left(\left(\frac{1}{2}, \frac{1}{2}\right) 0, 1\right) 1\right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2}\right) 1, 1\right) 0\right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2}\right) 1, 1\right) 1\right] \oplus \left[\left(\left(\frac{1}{2}, \frac{1}{2}\right) 1, 1\right) 2\right] \\ \frac{1}{2} \otimes \left(\frac{1}{2} \otimes 1\right) &= \frac{1}{2} \otimes \left(\left[\left(\frac{1}{2}, 1\right) \frac{1}{2}\right] \oplus \left[\left(\frac{1}{2}, 1\right) \frac{3}{2}\right]\right) \\ &= \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1\right) \frac{1}{2}\right) 0\right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1\right) \frac{1}{2}\right) 1\right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1\right) \frac{3}{2}\right) 1\right] \oplus \left[\left(\frac{1}{2}, \left(\frac{1}{2}, 1\right) \frac{3}{2}\right) 2\right] \end{aligned}$$

## A Addition of Binomial Coefficients

The addition formula for binomial coefficients is given as

$$\sum_s \binom{x}{s} \binom{y}{z-s} = \binom{x+y}{z}. \quad (A.1)$$

This formula can be derived by comparing the coefficients of the polynomials on the both sides of  $(X+Y)^{x+y} = (X+Y)^x (X+Y)^y$ . Putting  $x = a-b, y = b, z = a-c$  in (A.1), we have

$$\frac{a!}{b!c!} = \sum_s \frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \quad (A.2)$$

If  $y < 0$ ,

$$\binom{y}{z-s} = (-1)^{z-s} \binom{z-s-y-1}{z-s}. \quad (\text{A.3})$$

This can be obtained by using  $(X+Y)^y = X^y(1+\frac{Y}{X})^y$  and Taylor-expanding  $(1+\frac{Y}{X})^y$  around  $\frac{Y}{X} = 0$ . Then (A.1) can be transformed into

$$\sum_s (-1)^s \binom{x}{s} \binom{z-s-y-1}{z-s} = (-1)^z \binom{x+y}{z}, \quad (x+y \geq 0) \quad (\text{A.4})$$

or

$$\sum_s (-1)^s \binom{x}{s} \binom{z-s-y-1}{z-s} = \binom{z-x-y-1}{z}, \quad (x+y < 0) \quad (\text{A.5})$$

Putting  $y = z - t - 1$ , (A.4) and (A.5) become

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = (-1)^z \frac{(t-z)!(x+z-t-1)!}{x!z!(x-t-1)!}, \quad (x > t \geq z \geq 0). \quad (\text{A.6})$$

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}, \quad (t \geq x, z \geq 0). \quad (\text{A.7})$$

The following formula is referenced in [1],[2]. It can be obtained by letting  $y < 0$  and  $x < 0$  in (A.1):

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \quad (\text{A.8})$$

This can be verified by applying (A.3) on both sides of (A.1). Then we will have

$$\sum_s \binom{a+s}{c+s} \binom{b-s}{d-s} = \binom{a+b+1}{c+d} \quad (\text{A.9})$$

after a change of summation variable, and this is just (A.8).

The following formula

$$\sum_s \frac{(-1)^s}{(a+s)!(b-s)!} = (-1)^a \delta(a, -b) \quad (\text{A.10})$$

is a slight generalisation of a simple binomial coefficients formula

$$(X-Y)^K = \sum_{l=0}^K (-1)^l \binom{K}{l} X^{K-l} Y^l \quad (\text{A.11})$$

(A.10) is obtained when we set  $X = Y = 1$ .

## References

- [1] Albert Messiah, *Quantum Mechanics*, Dover (1995)
- [2] Giulio Racah, *Theory of Complex Spectra.I&II*, *Phys. Rev.* 61&62 (1942)
- [3] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients" in *Spectroscopic and group theoretical methods in physics*, (ed. Bloch et al), North Holland Publ. Co., Amsterdam, 1968