How To Prove The Arithmetic Mean And Geometric Mean Inequality

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1 Arithmetic Mean and Geometric Mean Inequality

1.1 Introduction

This is a proof of the famous AM-GM inequality that the author came up with some time ago. I saw somebody used the theorem in a solution to a math olympiad problem, which I don't remember the details of. Although I must have seen it somewhere during my undergrad-math education, I couldn't immediately think of any good proof of it. So, I started proving it by myself. I knew I had to make use of induction, and was vaguely hoping to make the argument as "symmetric" as possible. Here, I mean symmetric by treating all the terms in the arithmetic sum in the same manner. I think I have done my best to make it so. And the surprise was the method of my proof is actually quite similar to the one on Wikipedia. The difference is the progression of the argument is in the opposite direction. They start with the *Lemma1*, which is stated below, whereas I started directly from the arithmetic mean to reach at *Lemma1*. Well, anyway, it turns out that my proof wasn't that original as I expected at the first place... Never mind. Cheer up (for myself) and let's keep up the good mathematical energy and motivation. :)

1.2 The Statement

Here is the arithmetic and geometric formula

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n},$$

where $\forall i \; a_i > 0$.

1.3 Proof of the Statement

OK for $n = 1$. Equality holds. Suppose that the inequality holds for all natural numbers up to n, then

$$\begin{aligned}
a_1 + \cdots + a_{n+1} &= \frac{a_2 + a_3 \cdots + a_{n+1}}{n} + \frac{a_1 + a_3 + \cdots + a_{n+1}}{n} + \cdots \\
&\quad + \frac{a_1 + a_2 + \overset{i}{\breve{\cdots}} + a_{n+1}}{n} + \cdots + \frac{a_1 + \cdots + a_n}{n} \\
&= \sqrt[n]{a_2 \cdots a_{n+1}} + \cdots + \sqrt[n]{a_1 \cdots a_n} \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\frac{\sqrt[n]{a_2 \cdots a_{n+1}}}{\sqrt[n+1]{a_1 \cdots a_{n+1}}} + \cdots + \frac{\sqrt[n]{a_1 \cdots a_n}}{\sqrt[n+1]{a_1 \cdots a_{n+1}}} \right) \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\sqrt[n]{a_2 \cdots a_{n+1}}}{a_1}} + \cdots + \sqrt[n+1]{\frac{\sqrt[n]{a_1 \cdots a_n}}{a_{n+1}}} \right) \\
&= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\alpha_1}{a_1}} + \cdots + \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} \right)
\end{aligned}$$

where $\alpha_i = \sqrt[n]{a_1 \overset{i}{\breve{\cdots}} a_{n+1}}$ (Here, $\overset{i}{\breve{\cdots}}$ means omitting the i-th term in the sum or the product). It is easy to see that

$$\sqrt[n+1]{\frac{\alpha_1}{a_1}} \cdots \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} = 1.$$

So, if we can show the following proposition, we are done.
Lemma1:

If $b_1, \cdots, b_{n+1} > 0$ and $b_1 \cdots b_{n+1} = 1$, then

$$b_1 + \cdots + b_{n+1} \geq n + 1 \tag{1.1}$$

□

Now, by substituting $b_i = \sqrt[n+1]{\frac{\alpha_i}{a_i}}$, it is straightforward to see that

$$\begin{aligned}
a_1 + \cdots + a_{n+1} &= \sqrt[n+1]{a_1 \cdots a_{n+1}} \left(\sqrt[n+1]{\frac{\alpha_1}{a_1}} + \cdots + \sqrt[n+1]{\frac{\alpha_{n+1}}{a_{n+1}}} \right) \\
&\geq \sqrt[n+1]{a_1 \cdots a_{n+1}} (n+1)
\end{aligned}$$

□

1.4 Proof of the Lemma1

Lemma1:

If $b_1, \cdots, b_{n+1} > 0$ and $b_1 \cdots b_{n+1} = 1$, then

$$b_1 + \cdots + b_{n+1} \geq n + 1 \tag{1.2}$$

Proof: First, if $b_1 = \cdots = b_{n+1} = 1$, then the equality holds. We notice that $\exists i$ such that $b_i > 1$ implies that $\exists j$ such that $b_j < 1$. So, let us assume the condition and proceed to prove the statement by induction. For $n = 1$, we can assume that $b_1 = 1 - c_1$ with $0 < c_1 < 1$. Then

$$b_1 + b_2 = b_1 + \frac{1}{b_1} = 1 - c_1 + \frac{1}{1 - c_1} \geq 1 - c_1 + 1 + c_1 = 2.$$

Now, let us suppose that (1.2) holds true for all the natural numbers up to n. Regarding the fact mentioned above, let us suppose that $b_n = 1 + c_n$ and $b_{n+1} = 1 + c_{n+1}$ with $c_n < 0$ and $c_{n+1} > 0$. Then, by the assumption,

$$(b_1 \cdots b_{n-1})(b_n b_{n+1}) = 1 \quad \text{implies} \quad b_1 + \cdots + b_{n-1} + b_n b_{n+1} \geq n.$$

If we can say

$$b_1 + \cdots + b_{n-1} + (b_n + b_{n+1}) \geq \text{or} > b_1 + \cdots b_{n-1} + (b_n b_{n+1} + 1),$$

then the inequality follows. But

$$b_n + b_{n+1} - (b_n b_{n+1} + 1) = -c_n c_{n+1} > 0.$$

Hence, it follows that

$$b_1 + \cdots + b_{n-1} + b_n + b_{n+1} > b_1 + \cdots + b_{n-1} + b_n b_{n+1} + 1 \geq n + 1$$

$\square$