Angular Momentum and Racah's formula

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1 Angular Momentum Operators and $\mathfrak{su}(2)$ Representation

$J_x, J_y, J_z \in \mathfrak{su}(2)$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1.1}$$

The Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ and the ladder operators $J_\pm = J_x \pm iJ_y$

$$\begin{aligned}
\left[J^2, J_i\right] &= 0 && (1.2)\\
[J_z, J_\pm] &= \pm J_\pm && (1.3)\\
[J_+, J_-] &= 2J_z && (1.4)\\
J^2 &= J_-J_+ + J_z^2 + J_z && (1.5)\\
&= J_+J_- + J_z^2 - J_z && (1.6)
\end{aligned}$$

Let V be a finite dimensional vector space over $\mathbb{C}$ and $\varphi : \mathfrak{su}(2) \to \mathbf{End}(V)$ be its associated representation. Simultaneous eigenvector $|jm\rangle$:

$$J^2|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle \tag{1.7}$$

2 The ladder operators' coefficients

$$\begin{aligned}
J_+|jm\rangle &= \sqrt{j(j+1) - m(m+1)}|jm+1\rangle && (2.1)\\
&= \sqrt{(j-m)(j+m+1)}|jm+1\rangle && (2.2)\\
J_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}|jm-1\rangle && (2.3)\\
&= \sqrt{(j+m)(j-m+1)}|jm-1\rangle && (2.4)
\end{aligned}$$

It is useful to write $|jm\rangle$ in terms of $(J_-)^k|jj\rangle$. So, let us rewrite the coefficients in simpler notation;

$$J_-|jj-(k-1)\rangle = f(j,k)|jj-k\rangle \tag{2.5}$$

where

$$f(j,k) := \sqrt{k(2j-k+1)}\,, 1 \le k \le 2j$$

We also have

$$J_+|jj-k\rangle = f(j,k)|jj-k+1\rangle \tag{2.6}$$

Note that $D := 2j+1$ is the dimension of the j-th representation space.
Hence

$$(J_-)^k|jj\rangle = F(j,k)|jj-k\rangle \tag{2.7}$$

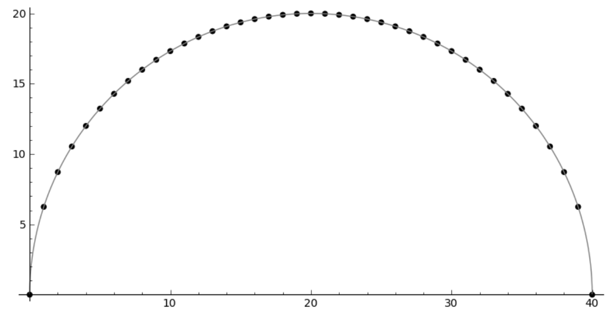

Figure 1: The graph of $f(j = \frac{39}{2}, k)$ where k is the horizontal axis.

where $F(j,k) = \prod_{i=1}^{k} f(j,i)$ and evaluated as,

$$
\begin{aligned}
F(j,k) &= \sqrt{k(2j+1-k)(k-1)(2j+1-(k-1)) \times \cdots \times 2\cdot(2j+1-2)\cdot 1\cdot(2j+1-1)} \\
&= \sqrt{k(D-k)(k-1)(D-(k-1)) \times \cdots \times 2\cdot(D-2)\cdot 1\cdot(D-1)} \\
&= \sqrt{\frac{k!(2j)!}{(2j-k)!}} = k!\sqrt{{}_{2j}C_k} \qquad (2.8)
\end{aligned}
$$

and $F(j,0) = 1$

3 Recursion Relations for Clebsch-Gordan coefficients

$$
|j_1 j_2 JM\rangle = \sum_{m_1,m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 m_1 j_2 m_2|JM\rangle \qquad (3.1)
$$

Apply $J_+ = j_{1+} + j_{2+}$ take inner product with $\langle j_1 j_2 m_1 m_2|$

$$
\begin{aligned}
&\sqrt{J(J+1)-M(M+1)}\langle j_1 m_1 j_2 m_2|JM+1\rangle \\
&= \sqrt{j_1(j_1+1)-m_1(m_1-1)}\langle j_1 m_1-1 j_2 m_2|JM\rangle + \sqrt{j_2(j_2+1)-m_2(m_2-1)}\langle j_1 m_1 j_2 m_2-1|JM\rangle
\end{aligned} \qquad (3.2)
$$

and $J_- = j_{1-} + j_{2-}$

$$
\begin{aligned}
&\sqrt{J(J+1)-M(M-1)}\langle j_1 m_1 j_2 m_2|JM-1\rangle \\
&= \sqrt{j_1(j_1+1)-m_1(m_1+1)}\langle j_1 m_1+1 j_2 m_2|JM\rangle + \sqrt{j_2(j_2+1)-m_2(m_2+1)}\langle j_1 m_1 j_2 m_2+1|JM\rangle
\end{aligned} \qquad (3.3)
$$

4 Explicit formulae for Clebsch-Gordan coefficinets

4.1 $\langle j_1 j_2 m_1 m_2|JM\rangle$

Define $d = j_1 + j_2 - J$, ($j_1 > j_2$ and $0 \le d \le 2j_2$), and $L = J - M$. Let us determine the coefficients of the top spin state

$$
\begin{aligned}
|JJ\rangle &= a_0|j_1 j_1\rangle|j_2 j_2 - d\rangle + a_1|j_1 j_1 - 1\rangle|j_2 j_2 - d + 1\rangle + \cdots + a_d|j_1 j_1 - d\rangle|j_2 j_2\rangle \\
&= \sum_{i=0}^{d} a_i|j_1 j_1 - i\rangle|j_2 j_2 - d + i\rangle \qquad (4.1)
\end{aligned}
$$

by imposing the top spin condition

$$
J_+|JJ\rangle = 0 \implies a_{i+1} = -\frac{f(j_2, d-i)}{f(j_1, i+1)} a_i \quad (i = 0, \ldots, d-1), \qquad (4.2)
$$

which means

$$\begin{aligned}
a_i &= -\frac{f(j_2, d-(i-1))}{f(j_1, i)} a_{i-1} \quad (i = 1, \dots, d) && (4.3)\\
&= (-1)^i \frac{f(j_2, d-(i-1)) f(j_2, d-(i-2)) \cdots f(j_2, d-1) f(j_2, d)}{f(j_1, i) f(j_1, i-1) \cdots f(j_1, 2) f(j_1, 1)} a_0 && (4.4)\\
&= (-1)^i \frac{F(j_2, d)}{F(j_1, i) F(j_2, d-i)} a_0 && (4.5)
\end{aligned}$$

Here $F(j_2, 0) = 1$.

4.2 Normalisation

Normalisation condition $\langle JJ|JJ\rangle = 1$ yields

$$\begin{aligned}
\frac{1}{a_0^2} &= \sum_{i=0}^{d} \frac{F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d-i)^2} && (4.6)\\
&= 1 + \left[\frac{f(j_2, d)}{f(j_1, 1)}\right]^2 + \cdots + \left[\frac{f(j_2, d-(i-1)) f(j_2, d-(i-2)) \cdots f(j_2, d-1) f(j_2, d)}{f(j_1, i) f(j_1, i-1) \cdots f(j_1, 2) f(j_1, 1)}\right]^2 + \\
&\quad \cdots + \left[\frac{F(j_2, d)}{F(j_1, d)}\right]^2 \\
&= \frac{1}{F(j_1, d)^2} \Big\{ (D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_1 - 1) \cdot 1 + (D_1 - d) \cdot d \cdots (D_1 - 2) \cdot 2 \cdot (D_2 - d) \cdot d + \\
&\quad \cdots + (D_1 - d) \cdot d \cdots (D_1 - i - 1) \cdot (i+1) \cdot (D_2 - (d-i+1)) \cdot (d-i+1) \cdots (D_2 - d) \cdot d + \cdots \Big\} \\
&= \frac{1}{F(j_1, d)^2} \Big\{ \frac{(d!)^2}{d!} (D_1 - d) \cdot (D_1 - 2) \cdot (D_1 - 1) + \frac{(d!)^2}{1!(d-1)!} (D_1 - d) \cdots (D_1 - 2) \cdot (D_2 - d) + \\
&\quad \cdots + \frac{(d!)^2}{i!(d-i)!} (D_1 - d) \cdot (D_1 - i - 1) \cdot (D_2 - (d-i+1)) \cdots (D_2 - d) + \cdots \Big\}
\end{aligned}$$

Writing

$$\begin{aligned}
G_i(j_1, j_2, d) &:= \frac{F(j_1, d) F(j_2, d)}{F(j_1, i) F(j_2, d-i)}, && (4.7)\\
&= \sqrt{\frac{(d!)^2}{(d-i)! i!} (D_2 - d)(D_2 - d - 1) \cdots (D_2 - d - i + 1)(D_1 - d) \cdots (D_1 - i + 1)}
\end{aligned}$$

Or, substituting $d = j_1 + j_2 - J$, this can be written as

$$G_i(j_1, j_2, j_1 + j_2 - J) = (-1)^i \sqrt{\frac{((j_1 + j_2 - J)!)^2}{(j_1 + j_2 - J - i)! i!} \frac{(j_2 + J - j_1)!(j_1 + J - j_2)!}{(j_2 + J - j_1 - i)!(2j_1 - i)!}}.$$

In terms of these G_i's, the coefficients a_i become

$$a_i = (-1)^i \frac{G_i(j_1, j_2, d)}{\sqrt{\sum_{i=1}^{d} G_i(j_1, j_2, d)^2}}. \quad (4.8)$$

We want to know the normalising coefficient $N := \frac{1}{\sqrt{\sum_{i=1}^{d} G_i(j_1, j_2, d)^2}}$. In order to simplify the sum

$$\begin{aligned}
\sum_{i=1}^{d} G_i(j_1, j_2, d)^2 &= \sum_{i=1}^{d} \frac{F(j_1, d)^2 F(j_2, d)^2}{F(j_1, i)^2 F(j_2, d-i)^2} \\
&= \frac{(d!)^2}{(2j_1 - d)!(2j_2 - d)!} \sum_{i=1}^{d} \frac{(2j_1 - i)!(2j_2 - d + i)!}{i!(d-i)!}, && (4.9)
\end{aligned}$$

we use a formula due to Racah (mentioned in Messiah[1])

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \tag{4.10}$$

with $a \geq c, b \geq d \geq 0$, where the sum is taken over $-c \leq s \leq d$.
Now substituting $a = 2j_2 - d, b = 2j_1, c = 0, d = d$, we obtain

$$N = \sqrt{\frac{(2j_2 - 2d + 2j_1 + 1)!}{d!(2j_2 - d + 2j_1 + 1)!}} = \sqrt{\frac{(2J+1)!}{(j_1 + j_2 - J)!(j_1 + j_2 + J + 1)!}} \tag{4.11}$$

$$a_i \quad = \quad (-1)^i N G_i(j_1, j_2, d)$$

Now, by multiplying the top-spin state with the ladder operators L times, we obtain the state $|JM\rangle$ with $M = J - L$

$$\begin{aligned}
J_-^L|JJ\rangle &= (j_{1-} + j_{2-})^L \sum_{h=0}^{d} a_h \times |j_1 j_1 - h\rangle|j_2 j_2 - d + h\rangle \\
F(J,L)|JJ-L\rangle &= \sum_{h=0}^{d} a_h \sum_{l=0}^{L} {}_L C_l \frac{F(j_1, h+l)F(j_2, (L+d)-(l+h))}{F(j_1,h)F(j_2,d-h)} |j_1 j_1 - (h+l)\rangle|j_2 j_2 - (L+d) + (h+l)\rangle \\
|JJ-L\rangle &= \frac{1}{F(J,L)} \sum_{k=0}^{L+d} \left[\sum_{k=h+l, \substack{0\leq h\leq d \\ 0\leq l\leq L}} a_h \times {}_L C_l \frac{F(j_1,k)F(j_2,K-k)}{F(j_1,h)F(j_2,d-h)} \right] |j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle \\
&= \frac{N}{F(J,L)} \sum_{k=0}^{L+d} F(j_1,k)F(j_2,K-k) \left[\sum_{\substack{k=h+l \\ 0\leq h\leq d \\ 0\leq l\leq L}} \frac{(-1)^h {}_L C_l G_h(j_1,j_2,d)}{F(j_1,h)F(j_2,d-h)} \right] |j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle
\end{aligned}$$

where $K = L + d = J - M + j_1 + j_2 - J = j_1 + j_2 - M$. Now, consider the coefficients of $|j_1 j_1 - k\rangle|j_2 j_2 - K + k\rangle$

$$\begin{aligned}
B_k \quad :=& \quad F(j_1,k)F(j_2,K-k) \left[\sum_{\substack{k=h+l \\ 0\leq h\leq d \\ 0\leq l\leq L}} \frac{(-1)^h {}_L C_l G_h(j_1,j_2,d)}{F(j_1,h)F(j_2,d-h)} \right] \\
=& \quad \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!}} \sum_{\substack{k=h+l \\ 0\leq h\leq d \\ 0\leq l\leq L}} (-1)^h {}_L C_l \sqrt{\frac{(2j_1-h)!(2j_2-d+h)!(d!)^2(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!(2j_1-d)!(2j_2-d)!h!(d-h)!}} \\
=& \quad \sqrt{\frac{k!(K-k)!}{(2j_1-k)!(2j_2-K+k)!(2j_1-d)!(2j_2-d)!}} L!d! \sum_{\substack{k=h+l \\ 0\leq h\leq d \\ 0\leq l\leq L}} (-1)^h \frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!}
\end{aligned}$$

The coefficient outside the sum, in terms of j_1, j_2, J, m_1, m_2, M, using the relations $K = L + d = J - M + j_1 + j_2 - J = j_1 + j_2 - M$, $k = j_1 - m_1$, is

$$\sqrt{\frac{(j_1 - m_1)!(j_2 + m_1 - M)!}{(j_1 + m_1)!(j_2 - m_1 + M)!(j_1 - j_2 + J)!(j_2 - j_1 + J)!}} (J - M)!(j_1 + j_2 - J)! \tag{4.12}$$

Multipying by $\frac{N}{F(J,J-M)}$

$$\begin{aligned}
&\sqrt{\frac{(2J+1)(j_1+j_2-J)!}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+j_2+J+1)!}\frac{(j_1-m_1)!(j_2-m_2)!(J+M)!(J-M)!}{(j_1+m_1)!(j_2+m_2)!}} \\
&= \sqrt{(2J+1)}\sqrt{\Delta(j_1j_2J)}\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(J+M)!(J-M)!} \\
&\times\frac{1}{(j_1-j_2+J)!(j_2-j_1+J)!(j_1+m_1)!(j_2+m_2)!}
\end{aligned} \tag{4.13}$$

where we have defined

$$\Delta(abc) := \frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!}. \tag{4.14}$$

Now, we want to simplify the sum

$$\sum_{\substack{k=h+l\\0\le h\le d\\0\le l\le L}}(-1)^h\frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \tag{4.15}$$

furthermore. Putting $k=j_1-m_1$, $d=j_1+j_2-J$ back, we have

$$\begin{aligned}
&\sum_{\substack{k=h+l\\0\le h\le d\\0\le l\le L}}(-1)^h\frac{(2j_1-h)!(2j_2-d+h)!}{h!(d-h)!l!(L-l)!} \\
&=\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!}
\end{aligned}$$

and the sum in the last line is taken over all the values of l with which all the factrial terms containing l makes sense. In order to do so, we are going to use the following formula

$$\frac{a!}{b!c!}=\sum_s\frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{4.16}$$

Now

$$\begin{aligned}
&\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!(j_2+J-m_1-l)!}{l!(j_1-m_1-l)!(j_2-J+m_1+l)!(J-M-l)!} \\
=\;&\sum_l(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!}\cdot\frac{(j_2+J-m_1-l)!}{(J-M-l)!(j_1-m_1-l)!} \\
=\;&\sum_{l,l_1}(-1)^{j_1-m_1-l}\frac{(j_1+m_1+l)!}{l!(j_2-J+m_1+l)!}\cdot\frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!(j_1-j_2-M-l+l_1)!} \\
=\;&\sum_{l,l_1,l_2}(-1)^{j_1-m_1-l}\frac{(j_1+m_1)!(j_1-j_2+J)!}{l_2!(j_1+m_1-l_2)!(j_1-j_2+J-l_2)!(j_2-J-j_1+l+l_2)!} \\
&\cdot\frac{1}{(j_1-j_2-M+l_1-l)!}\frac{(j_2+m_2)!(j_2+J-j_1)!}{l_1!(j_2+m_2-l_1)!(j_2+J-j_1-l_1)!}
\end{aligned} \tag{4.17}$$

Racah uses the following formula to further simplify the expression,

$$\sum_s(-1)^s\frac{(t-s)!}{s!(x-s)!(z-s)!}=\frac{(t-x)!(t-z)!}{x!z!(t-x-z)!}. \tag{4.18}$$

To use the above formula, we first change the summation variable. Substitute $l' = j_1 - j_2 - M + l_1 - l$, and we have

$$\begin{aligned} j_1 + m_1 + l &= 2j_1 - j_2 - m_2 + l_1 - l' \\ j_2 - J + m_1 + l &= j_1 - J - m_2 + l_1 - l'. \end{aligned}$$

Going back to the main expression,

$$\begin{aligned} &\sum_{l',l_1} (-1)^{-j_2-m_2+l_1-l'} \frac{(2j_1 - j_2 - m_2 + l_1 - l')!}{l'!(j_1 - J - m_2 + l_1 - l')!(j_1 - j_2 - M + l_1 - l')!} \cdot \frac{(j_2 + m_2)!(j_2 + J - j_1)!}{l_1!(j_2 + m_2 - l_1)!(j_2 + J - j_1 - l_1)!} \\ = &\sum_{l_1} (-1)^{j_2+m_2-l_1} \frac{(j_1 + m_1)!(j_1 + J - j_2)!}{(j_1 - J - m_2 + l_1)!(j_1 - j_2 - M + l_1)!(J + M - l_1)!} \cdot \frac{(j_2 + m_2)!(j_2 + J - j_1)!}{l_1!(j_2 + m_2 - l_1)!(j_2 + J - j_1 - l_1)!} \end{aligned}$$

Putting $t = j_2 + m_2 - l_1$ ([2]), we obtain

$$\begin{aligned} &\sum_{\substack{k=h+l \\ 0\le h\le d \\ 0\le l\le L}} (-1)^h \frac{(2j_1 - h)!(2j_2 - d + h)!}{h!(d-h)!l!(L-l)!} \\ = &\sum_t (-1)^t \frac{(j_1 + m_1)!(j_2 + m_2)!(j_1 + J - j_2)!(j_2 + J - j_1)!}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!} \end{aligned}$$

And then we obtain

$$\begin{aligned} \langle j_1 j_2 m_1 m_2 | JM \rangle &= \sqrt{(2J+1)}\sqrt{\Delta(j_1 j_2 J)}\sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(J + M)!(J - M)!} \\ &\times \sum_t (-1)^t \frac{1}{t!(j_1 + j_2 - J - t)!(j_1 - m_1 - t)!(j_2 + m_2 - t)!(J - j_2 + m_1 + t)!(J - j_1 - m_2 + t)!} \end{aligned} \tag{4.19}$$

Here, notice that the sum tkaes place in the range

$$\max\{0, -(J - j_2 + m_1), -(J - j_1 - m_2)\} \le t \le \min\{j_1 + j_2 - J, j_1 - m_1, j_2 + m_2\} \tag{4.20}$$

By making the substitution $z = a + b - c - t$ in the sum, we have

$$\langle j_1 j_2 m_1 m_2 | JM \rangle = (-1)^{a+b-c} \langle j_2 j_1 m_2 m_1 | JM \rangle \tag{4.21}$$

The Racah symbol

$$(-1)^{a-b-c} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} := \frac{(-1)^{c-\gamma}}{\sqrt{2c+1}} \langle ab\,\alpha\beta | c\ -\gamma \rangle \tag{4.22}$$

The Racah formula

$$\begin{aligned} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \ &(-1)^{a-b-\gamma}\sqrt{\Delta(abc)}\sqrt{(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!} \\ &\times \sum_t (-1)^t [t!(c-b+t+\alpha)!(c-a+t-\beta)!(a+b-c-t)!(a-t-\alpha)!(b-t+\beta)!]^{-1} \\ &(\alpha + \beta + \gamma = 0, \quad |a-b| \le c \le a+b) \end{aligned}$$

4.3 Some Examples

4.3.1 $J = j_1 + j_2$

$$
\begin{aligned}
|JJ\rangle &= |j_1 j_1\rangle |j_2 j_2\rangle && (4.23)\\
(J_-)^k |JJ\rangle &= (J_{1-} + J_{2-})^k |j_1 j_1\rangle |j_2 j_2\rangle && (4.24)\\
|JJ-k\rangle &= \sum_{i=0}^{k} {}_kC_i \frac{F(j_1,i)F(j_2,k-i)}{F(J,k)} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle && (4.25)\\
&= \sum_{i=0}^{k} \sqrt{\frac{{}_{2j_1}C_i \cdot {}_{2j_2}C_{k-i}}{{}_{2J}C_k}} |j_1 j_1 - i\rangle |j_2 j_2 - (k-i)\rangle && (4.26)
\end{aligned}
$$

Rewrite it using $M = J - k = j_1 + j_2 - k$, $m_1 = j_1 - i$, $m_2 = j_2 - (k-i) = -j_1 + M + i = M - m_1$, $k - i = j_2 - m_2$

$$
|j_1 + j_2 M\rangle = \sum_{m_1+m_2=M} \sqrt{\frac{{}_{2j_1}C_{j_1-m_1} \cdot {}_{2j_2}C_{j_2-m_2}}{{}_{2J}C_{j_1+j_2-M}}} |j_1 m_1\rangle |j_2 m_2\rangle \quad (4.27)
$$

$$
\begin{aligned}
\langle j_1 j_2 m_1 m_2 | j_1 + j_2 M\rangle &= \sqrt{\frac{{}_{2j_1}C_{j_1-m_1} \cdot {}_{2j_2}C_{j_2-m_2}}{{}_{2(j_1+j_2)}C_{j_1+j_2-M}}} && (4.28)\\
&= \sqrt{\frac{(2j_1)!(2j_2)!}{(2J)!} \frac{(J+M)!(J-M)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} && (4.29)
\end{aligned}
$$

4.3.2 $j_1 = j,\ j_2 = \mathbf{1}$

We have three possibilities; $J = j+1,\ j,\ j-1$. Note that any $|J\ M\rangle$ is expanded by $|j\ M+1\rangle|\mathbf{1}\ -1\rangle$, $|j\ M\rangle|\mathbf{1}\ 0\rangle$, $|j\ M-1\rangle|\mathbf{1}\ 1\rangle$.
The case $J = j+1$.

$$
\begin{aligned}
\sqrt{\frac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} &= \sqrt{\frac{(2j)!}{(j-M-1)!(j+M+1)!} \frac{(j-M+1)!(j+M+1)!}{(2j+2)!}} && (4.30)\\
&= \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}} && (4.31)
\end{aligned}
$$

Table 1:

$\|j\ M+1\rangle\|\mathbf{1}\ -1\rangle$	$\sqrt{\frac{{}_{2j}C_{j-M-1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{(j-M+1)(j-M)}{(2j+2)(2j+1)}}$
$\|j\ M\rangle\|\mathbf{1}\ 0\rangle$	$\sqrt{\frac{2 \cdot {}_{2j}C_{j-M}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{2(j-M+1)(j+M+1)}{(2j+2)(2j+1)}}$
$\|j\ M-1\rangle\|\mathbf{1}\ 1\rangle$	$\sqrt{\frac{{}_{2j}C_{j-M+1}}{{}_{2(j+1)}C_{j-M+1}}} = \sqrt{\frac{(j+M+1)(j+M)}{(2j+2)(2j+1)}}$

5 $6j$-symbol

It is defined as a coupling coefficient of three angular momentum

$$\begin{aligned} & |(j_1,(j_2,j_3)j_{23})J\rangle \\ = & \sum_{j_{12}}[(2j_{12}+1)(2j_{23}+1)]^{\frac{1}{2}}(-1)^{j_1+j_2+j_3+J}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23}\end{array}\right\}|((j_1,j_2)j_{12},j_3)J\rangle \end{aligned} \tag{5.1}$$

$6j$-symbol in terms of $3j$-symbols

$$\begin{aligned} \left\{\begin{array}{ccc} a & b & c \\ d & e & f\end{array}\right\} = & \sum_{\substack{\alpha\beta\gamma \\ \delta\epsilon\varphi}}(-1)^{d+e+f+\delta+\epsilon+\varphi}\times \\ & \times\left(\begin{array}{ccc} d & e & c \\ \delta & -\epsilon & \gamma\end{array}\right)\left(\begin{array}{ccc} e & f & a \\ \epsilon & -\varphi & \alpha\end{array}\right)\left(\begin{array}{ccc} f & d & b \\ \varphi & -\delta & \beta\end{array}\right)\left(\begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma\end{array}\right) \end{aligned} \tag{5.2}$$

Racah's formula ([2])

$$\begin{aligned} \left\{\begin{array}{ccc} a & b & c \\ d & e & f\end{array}\right\} = & [\Delta(abc)\Delta(aef)\Delta(cde)\Delta(bdf)]^{\frac{1}{2}}\times \\ \times & \sum_x \frac{(x+1)!}{[(p_1-x)(p_2-x)!(p_3-x)!(x-q_1)!(x-q_2)!(x-q_3)!(x-q_4)!]} \end{aligned} \tag{5.3}$$

where $q_1=a+b+c, q_2=b+d+f, q_3=a+e+f, q_4=d+e+c, p_1=a+b+d+e, p_2=b+c+e+f, p_3=c+a+f+d$. Only values which satisfy the triangle inequalities, $|b-c|\le a\le b+c$ are allowed:$(abc),(aef),(dbf),(dec)$. Therefore, there have to be even numbers of half integers at each face. This is equivalente to say that $q_1,q_2,q_3,q_4,p_1,p_2,p_3$ are all integers.

Example 5.1. *$\{\ j_1=j_2=\frac{1}{2}, j_3=1\ \}$ The resultant angular momenta are $J=0,1,2$.*

$$\begin{aligned} \left(\mathbf{\tfrac{1}{2}}\otimes\mathbf{\tfrac{1}{2}}\right)\otimes\mathbf{1} &= \left(\left[\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{0}\right]\oplus\left[\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{1}\right]\right)\otimes\mathbf{1} \\ &= \left[\left(\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{0},\mathbf{1}\right)\mathbf{1}\right]\oplus\left[\left(\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{1},\mathbf{1}\right)\mathbf{0}\right]\oplus\left[\left(\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{1},\mathbf{1}\right)\mathbf{1}\right]\oplus\left[\left(\left(\mathbf{\tfrac{1}{2}},\mathbf{\tfrac{1}{2}}\right)\mathbf{1},\mathbf{1}\right)\mathbf{2}\right] \end{aligned}$$

$$\begin{aligned} \mathbf{\tfrac{1}{2}}\otimes\left(\mathbf{\tfrac{1}{2}}\otimes\mathbf{1}\right) &= \mathbf{\tfrac{1}{2}}\otimes\left(\left[\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{1}{2}}\right]\oplus\left[\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{3}{2}}\right]\right) \\ &= \left[\left(\mathbf{\tfrac{1}{2}},\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{1}{2}}\right)\mathbf{0}\right]\oplus\left[\left(\mathbf{\tfrac{1}{2}},\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{1}{2}}\right)\mathbf{1}\right]\oplus\left[\left(\mathbf{\tfrac{1}{2}},\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{3}{2}}\right)\mathbf{1}\right]\oplus\left[\left(\mathbf{\tfrac{1}{2}},\left(\mathbf{\tfrac{1}{2}},\mathbf{1}\right)\mathbf{\tfrac{3}{2}}\right)\mathbf{2}\right] \end{aligned}$$

A Binomial Coefficients

The addition theorem for binomial coefficients

$$\sum_s \binom{x}{s}\binom{y}{z-s}=\binom{x+y}{z} \tag{A.1}$$

Putting $x=a-b$, $y=b$, $z=a-c$, we have

$$\frac{a!}{b!c!}=\sum_s\frac{(a-b)!(a-c)!}{s!(a-b-s)!(a-c-s)!(b+c-a+s)!}. \tag{A.2}$$

If $y < 0$,

$$\binom{y}{z-s} = (-1)^{z-s}\binom{z-s-y-1}{z-s}. \tag{A.3}$$

Then (A.1) can be transformed into

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = (-1)^z\binom{x+y}{z},\ (x+y \geq 0) \tag{A.4}$$

or

$$\sum_s (-1)^s \binom{x}{s}\binom{z-s-y-1}{z-s} = \binom{z-x-y-1}{z},\ (x+y < 0) \tag{A.5}$$

Putting $y = z - t - 1$, (A.4) and (A.5) become

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = (-1)^z\frac{(t-z)!(x+z-t-1)!}{x!z!(x-t-1)!},\ (x > t \geq z \geq 0). \tag{A.6}$$

$$\sum_s (-1)^s \frac{(t-s)!}{s!(x-s)!(z-s)!} = \frac{(t-x)!(t-z)!}{x!z!(t-x-z)!},\ (t \geq x, z \geq 0). \tag{A.7}$$

The following formula is referenced in [1] due to Racah, but I have not succeeded in finding out a proof or any mention to this formula so far in [2]. It still needs to be verified somehow.

$$\sum_s \frac{(a+s)!(b-s)!}{(c+s)!(d-s)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}. \tag{A.8}$$

References

[1] Albert Messiah, Quantum Mechanics, Dover (1995)

[2] Giulio Racah, Theory of Complex Spectra.I&II, Phys. Rev. 61&62 (1942)

[3] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients" in Spectroscopic and group theoretical methods in physics, (ed. Bloch et al), North Holland Publ. Co., Amsterdam, 1968