

# Pointless Topology 勉強ノート

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2024年6月13日~

## 1 Preliminary

### 1.1 Topology トポロジー

Let  $\mathcal{P}(X)$  denote the power set of  $X$ .

**定義 1.1** (Topology トポロジー). A **topological space** is an ordered pair  $(X, \tau)$ ,  $\tau \subseteq \mathcal{P}(X)$  which satisfies the following properties

1.  $\emptyset \in \tau$  and  $X \in \tau$ .
2. if  $U, V \in \tau$ , then  $U \cap V \in \tau$ .
3. if  $\forall I, U_i \in \tau$  for all  $i \in I$ , then  $\bigcup_{i \in I} U_i \in \tau$ .

$\tau$  is called the **topology** of  $X$ . The members of the topology  $U \in \tau$  is said to be **open** and  $V \subseteq X$  is said to be **closed** if  $\exists U$  open such that  $V = U^c$ .

**定義 1.2.** If  $X$  is a topological space and  $x \in X$ , a neighbourhood (abbreviated “nhood”) of  $x$  is a set  $U$  which contains an open set  $V$  containing  $x$ . Thus,  $U$  is a nhood of  $x$  iff  $x \in U^\circ$ . The collection  $\mathcal{U}_x$  of all nhoods of  $x$  is the nhood system of  $x$ .

**定義 1.3** (Separation Axioms 分離公理). A space  $(X, \tau)$  is called  $T_i$ , if respectively satisfies the following conditions,

1.  $T_0$ :  $\forall x, y \in X \exists$  an open set  $U \in \tau$  such that  $U$  contains one of  $x, y$  and not the other.
2.  $T_1$ :  $\forall x, y \in X \exists$  a nhood of each not containing the other.

**例 1.1** ( $T_0$ -space).  $X = \{a, b\}, \tau = \{\emptyset, \{a\}, X\}$

### 1.2 Posets, Lattices 半順序集合、束

**定義 1.4** (Posets). A **partial order** (半順序) on a set  $X$  is a binary relation  $R \subseteq X \times X$  satisfying,

1.  $\forall a, aRa$  (reflexivity, 反射律),
2.  $\forall a, b, c, aRb \ \& \ bRc \Rightarrow aRc$  (transitivity, 推移律),
3.  $\forall a, b, aRb \ \& \ bRa \Rightarrow a = b$  (antisymmetry, 反对称律).

if moreover

4.  $\forall a, b$  either  $aRb$  or  $bRa$  holds,

it is said to be a **linear** or **total** order.

A **poset** or **partially ordered set**,  $(X, \leq)$  is a set with a partial order. If the order of a poset is linear (or total), it is called a **linearly ordered set**, **totally ordered set** or **chain**. A relation that satisfies only (1) and (2) is called **preorder**.

**定義 1.5** (Suprema, infima). A **supremum**  $s$  of a subset  $M \subseteq (X, \leq)$  the least upper bound of  $M$ , that is

$$1. \forall m \in M, m \leq s,$$

$$2. \forall m \in M, m \leq x \Rightarrow s \leq x.$$

Similarly, a **infimum** of a subset  $M \subseteq (X, \leq)$  the greatest lower bound of  $M$ .

We also call a supremum a **join** and an infimum **meet** and notate  $\sup M, \inf M$  or  $\bigvee M, \bigwedge M$  respectively.

For finite cases, we write  $a \vee b := \sup\{a, b\}$  or  $a_1 \vee \cdots \vee a_n := \sup\{a_1 \dots a_n\}$  and  $a \wedge b := \inf\{a, b\}$  or  $a_1 \wedge \cdots \wedge a_n := \inf\{a_1 \dots a_n\}$ .

Since each  $x \in X$  is both a lower and an upper bound of the empty set  $\emptyset$ ,

$$\sup \emptyset \text{ is the least element of } X \quad (1.1)$$

and

$$\inf \emptyset \text{ is the greatest element of } X \quad (1.2)$$

We use the symbols  $0$  or  $\perp$  for the former and  $1$  or  $\top$  for the latter.

**定義 1.6** (Semilattices, Lattice). A **meet-semilattice** is a poset  $X$  such that  $\forall a, b \in X$  there exists an infimum  $a \wedge b$ .

A **join-semilattice** is a poset  $X$  such that  $\forall a, b \in X$  there exists a supremum  $a \vee b$ .

A **lattice** is a poset  $X$  such that  $\forall a, b \in X$  both an infimum  $a \wedge b$  and a supremum  $a \vee b$  exist.

A **bounded lattice** is a poset in which all finite subsets have infima and suprema (i.e. a lattice with bottom and top).

A poset is a **complete lattice** if every subset has a supremum and an infimum.

In a bounded semilattice,  $\wedge$  or  $\vee$  is a binary operation and satisfies the following properties,

$$a \wedge a = a \quad a \vee a = a \quad (1.3)$$

$$a \wedge b = b \wedge a \quad a \vee b = b \vee a \quad (1.4)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad (a \vee b) \vee c = a \vee (b \vee c) \quad (1.5)$$

$$a \wedge 1 = a \quad a \vee 0 = a. \quad (1.6)$$

In other words, bounded semilattices are commutative monoids (semigroup with unit/identity element) in which every element is idempotent.

**定理 1.1.** Let  $(A, \vee, 0)$  be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on  $A$  such that  $a \wedge b$  is the join of  $a$  and  $b$ , and  $0$  is the least element.

証明. If such a partial order exists,

$$a \leq b \Leftrightarrow a \vee b = b. \quad (1.7)$$

would be the correspondence. Now, let us verify this connection.

**Reflexivity**

$$a \wedge a = a \Rightarrow a \leq a. \quad (1.8)$$

**Antisymmetry**

If  $a \leq b$  and  $b \leq a$ , then  $b = a \wedge b = b \wedge a = a$  by commutativity.

**Transitivity**

If  $a \leq b$  and  $b \leq c$ , then

$$\begin{aligned} a \wedge c &= a \wedge (b \wedge c) && (\because b \leq c) \\ &= (a \wedge b) \wedge c && (\text{associativity}) \\ &= b \wedge c && (\because a \leq b) \\ &= c && (\because b \leq c) \end{aligned}$$

Hence,  $a \leq c$ .

**Join-uniqueness**

Since  $a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$ ,  $a \leq a \wedge b$ . Similarly,  $b \leq a \wedge b$ , so  $a \wedge b$  is an upper-bound for  $\{a, b\}$ . For the leastness, suppose  $a \leq c$  and  $b \leq c$ , then

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ &= a \wedge c \\ &= c. \end{aligned}$$

Hence,  $a \wedge b \leq c$ . □

A lattice can also be defined purely algebraically in those terms,

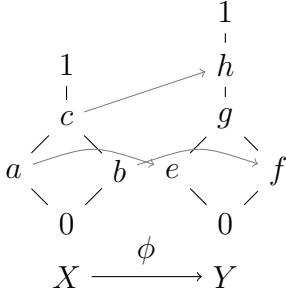
**定義 1.7.** A **lattice**  $(L, \vee, \wedge)$  is an algebra (a set with two binary operations) that satisfy

$$\begin{array}{lll} (L1) & a \wedge a = a & a \vee a = a \quad (\text{idempotency}) \\ (L2) & a \wedge b = b \wedge a & a \vee b = b \vee a \quad (\text{commutativity}) \\ (L3) & (a \wedge b) \wedge c = a \wedge (b \wedge c) & (a \vee b) \vee c = a \vee (b \vee c) \quad (\text{associativity}) \\ (L4) & a \vee (a \wedge b) = a & a \wedge (a \vee b) = a \quad (\text{absorption identities}) \end{array}$$

(L4) is necessary for the two operations  $\wedge, \vee$  to be consistent with the corresponding order  $\leq$ . In fact,  $a \wedge b = b$  implies  $a \vee b = a \vee (a \wedge b) = a$  by (L4).

For homomorphisms (structure-preserving maps) of (semi)lattices and posets, we need to be a little bit careful since the order-preserving homomorphisms of posets does not always preserve the joins (or meets) as shown in the following example. ([Stone] section 1.3 Exercise)

**例 1.2.** Consider the posets  $X = \{a \leq c, b \leq c\}$  and  $Y = \{e \leq g, f \leq g, g \leq h\}$ , and a homomorphism  $\phi : X \rightarrow Y$  which maps each element as in the diagram below:



Indeed, we have  $a, b \leq c$  and  $\phi(a), \phi(b) \leq \phi(c)$ , which means the order is preserved, but  $\phi(a \vee b) \neq \phi(a) \vee \phi(b)$ .

([Stone] Sec.1.5) In most of the lattices we'll consider, the operations  $\wedge$  and  $\vee$  will satisfy an additional identity, namely the distributive law

$$(i) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (1.9)$$

for all  $a, b, c$ .

**補題 1.2.** *If the distributive law (i) holds in a lattice, then so does its dual, i.e. the identity*

$$(ii) \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad (1.10)$$

**証明.**

$$\begin{aligned} (a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (i)} \\ &= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{by absorption law} \\ &= a \vee (b \wedge c) && \text{by absorption law} \end{aligned}$$

□

Note also that in the presence of (i), we can deduce either of the two absorptive law from the other,

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b) \quad (1.11)$$

**命題 1.3.** *Let  $a, b, c$  be three elements of a distributive lattice  $A$ . Then there exists at most one  $x \in A$  satisfying  $x \wedge a = b$  and  $x \vee a = c$ .*

**証明.** Suppose both  $x$  and  $y$  satisfy the conditions. Then,

$$\begin{aligned} x &= x \wedge (x \vee a) = x \wedge c = x \wedge (y \vee a) \\ &= (x \wedge y) \vee (x \wedge a) \\ &= (x \wedge y) \vee b = x \wedge y \end{aligned}$$

since  $b = x \wedge a = y \wedge a$  is a lower bound for  $\{x, y\}$ . Similarly, we have  $y = x \wedge y$ ; so  $x = y$ . □

In any lattice, an element  $x$  satisfying  $x \wedge a = 0$  and  $x \vee a = 1$  is called a **complement** of  $a$ . The Proposition above tells us that in a distributive lattice, complements are unique when they exist. A **Boolean algebra** is a distributive lattice  $A$  equipped with an additional unary operation  $\neg : A \rightarrow A$  such that  $\neg a$  is a complement of  $a$ . Since  $\neg$  is uniquely determined by the other data in the definition, it follows that any lattice homomorphism  $f : A \rightarrow B$  between Boolean algebras is actually a Boolean algebra homomorphism (i.e. commutes with  $\neg$ ).

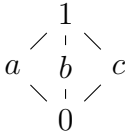
**例 1.3** (Power set). *For any set  $X$ , the power set  $\mathcal{P}(X)$  of  $X$  is a lattice, with  $\leq$  interpreted as inclusion,  $\wedge$  and  $\vee$  as union and intersection of subsets, and 0 and 1 as the empty set and the whole of  $X$ . Moreover  $\mathcal{P}(X)$  is distributive. Since  $\mathcal{P}(X)$  has complements for all its elements, it is a Boolean algebra.*

**例 1.4** (Total Order). *Let  $A$  be a totally ordered set with least and greatest elements 0 and 1. Then  $A$  is a lattice, with  $\wedge$  and  $\vee$  interpreted as min and max. It is distributive;*

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\} \quad (1.12)$$

*But if  $A$  has more than two elements, it is not a Boolean algebra; for no element other than 0 and 1 can have a complement.*

**例 1.5** (Lattices of subgroups). *Let  $G$  be a group. The set subgroups of  $G$ , ordered by inclusion, is a lattice in which meet is again interpreted as intersection, but the join of two subgroups is the subgroup generated by their union. This lattice is not in general distributive; for example, if  $G$  is the non-cyclic group of order 4, the lattice looks like*



*where  $a, b, c$  are the three subgroups of order 2, and each of  $a, b, c$  has two distinct complements.*

### 1.2.1 Boolean Rings and Boolean algebras

**命題 1.4.** *De Morgan's law*

$$\neg(x \wedge y) = \neg x \vee \neg y$$

*holds.*

**証明.** We need to say that  $\neg x \vee \neg y$  is the complement of  $x \wedge y$ ,

$$\begin{aligned} & (\neg x \vee \neg y) \wedge (x \wedge y) \\ &= ((\neg x \wedge x \wedge y) \vee (\neg y \wedge x \wedge y)) \\ &= (0 \wedge y) \vee (0 \wedge x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} & (\neg x \vee \neg y) \vee (x \wedge y) \\ &= (\neg x \vee \neg y \vee x) \wedge (\neg x \vee \neg y \vee y) \\ &= (1 \vee y) \vee (1 \vee x) \\ &= 1 \end{aligned}$$

□

Next, we sketch the equivalence between Boolean algebras and Boolean rings. In any Boolean algebra  $A$ , we define the **symmetry difference** operation  $+$  by

$$a + b = (a \wedge \neg b) \vee (b \wedge \neg a). \quad (1.13)$$

補題 1.5. *The distributive law  $a \wedge (b + c) = (a \wedge b) + (a \wedge c)$*

証明.

$$\begin{aligned}
(a \wedge b) + (a \wedge c) &= ((a \wedge b) \wedge \neg(a \wedge c)) \vee ((a \wedge c) \wedge \neg(a \wedge b)) \\
&= ((a \wedge b) \wedge (\neg a \vee \neg c)) \vee ((a \wedge c) \wedge (\neg a \vee \neg b)) \\
&= ((a \wedge b \wedge \neg a) \vee (a \wedge b \wedge \neg c)) \vee ((a \wedge c \wedge \neg a) \vee (a \wedge c \wedge \neg b)) \\
&= (0 \vee (a \wedge b \wedge \neg c)) \vee (0 \vee (a \wedge c \wedge \neg b)) \\
&= (a \wedge b \wedge \neg c) \vee (a \wedge c \wedge \neg b) \\
&= a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= a \wedge (b + c).
\end{aligned}$$

□

補題 1.6. *The associative law*

$$a + (b + c) = (a + b) + c \quad (1.14)$$

*holds.*

証明.

$$\begin{aligned}
a + (b + c) &= a + ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= (a \wedge \neg((b \wedge \neg c) \vee (c \wedge \neg b))) \vee (\neg a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b))) \\
&= (a \wedge ((\neg b \vee c) \wedge (\neg c \vee b))) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee (\neg b \wedge b) \vee (c \wedge \neg c) \vee (c \wedge b))) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge c \wedge \neg b)) \\
&= (a \wedge ((\neg b \wedge \neg c) \vee 0 \vee 0 \vee (c \wedge b))) \vee ((\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)) \\
&= (a \wedge \neg b \wedge \neg c) \vee (a \wedge c \wedge b) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \\
&= (((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge \neg c) \vee (((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge c) \\
&= ((a + b) \wedge \neg c) \vee (\neg(a + b) \wedge c) \\
&= ((a + b) \wedge \neg c) \vee (\neg(a + b) \wedge c) \\
&= (a + b) + c.
\end{aligned}$$

□

Now for any  $a$ , we have

$$a + a = (a \wedge \neg a) \vee (a \wedge \neg a) = 0 \wedge 0 = 0 \quad (1.15)$$

$$a + 0 = (a \wedge 1) \vee (0 \wedge \neg a) = a \vee 0 = a. \quad (1.16)$$

So  $(A, +, 0)$  is a commutative group, and  $(A, +, \wedge, 0, 1)$  is a commutative ring with 1.

定義 1.8. *A Boolean ring  $A$  is a ring with 1 in which every element satisfies  $a^2 = a$ .*

補題 1.7. *Let  $A$  be a Boolean ring, then*

1.  *$A$  is commutative.*

2. *Every  $a \in A$  satisfies  $a + a = 0$ .*

証明.

$$\begin{aligned}
a + b &= (a + b)^2 \\
&= a^a + ab + ba + b^2 \\
&= a + ab + ba + b.
\end{aligned}$$

So  $ab + ba = 0$ . Putting  $a = b$ , we get  $a + a = 0$ ; hence  $ab = -ba = ba$ .  $\square$

So the multiplicative structure  $(A, \cdot, 1)$  is a semilattice, with partial order defined by  $a \leq b$  iff  $ab = a$  (c.f. 定理 1.1). Note that 0 is the least element of  $A$  for this order.

Now consider  $a + b + ab$ . We have

$$a(a + b + ab) = a + ab + ab = a \quad (1.17)$$

and

$$b(a + b + ab) = ba + b + ab = b \quad (1.18)$$

so  $a + b + ab$  is an upper bound for  $\{a, b\}$ . But if  $c$  is an upper bound for  $\{a, b\}$ , then

$$(a + b + ab)c = ac + bc + abc = a + b + ab, \quad (1.19)$$

so  $a + b + ab$  is the least upper bound. Dnote  $a + b + ab$  by  $a \vee b$ , we thus have a lattice structure  $(A, \vee, \cdot, 0, 1)$ . Moreover, by an argument like that of Lemma 1.5, we may verify that  $\cdot$  is distributive over  $\vee$ ; and it is also easy to verify that  $1 + a$  is a complement for  $a$ . So  $A$  is a Boolean algebra.

$$\begin{aligned}
ab \vee ac &= (ab + ac + abac) \\
&= ab + ac + abc \\
&= a(b \vee c)
\end{aligned} \quad \square$$

$$a(a + 1) = a + a = 0 \quad a \vee (1 + a) = a + (1 + a) + a(1 + a) = 1. \quad \square$$

What is the symmetric difference operation in this Boolean algebra?

$$\begin{aligned}
(a \wedge \neg b) \vee (b \wedge \neg a) &= (a(1 + b)) \vee (b(1 + a)) \\
&= (a + ab) \vee (b + ab) \\
&= (a + ab) + (b + ab) + (a + ab)(b + ab) \\
&= a + b + ab + ab + ab + ab \\
&= a + b.
\end{aligned}$$

Thus if we start from a Boolean ring and turn it into a Boolean algebra by the definitions

$$\begin{aligned}
a \vee b &:= a + b + ab \\
\neg a &:= 1 + a \\
a \overline{+} b &:= a + b
\end{aligned}$$

then back into a Boolean ring by defining the addition  $+$  as the symmetric difference, we recover the original ring. Similarly if start from a Boolean algebra and go round the other way. Moreover, it is clear from the nature of the constructions that any Boolean algebra homomorphism is also a Boolean ring homomorphism, conversely; so we have proved

定理 1.8. *The category of Boolean algebras is isomorphic to the category of Boolean rings.*

例 1.6 (Poset). ([Gratzer] Chap.1 Exercise 4) *Here are some examples of the possible numbers of partial orders on finite sets:*

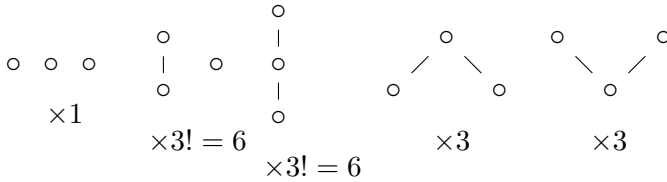
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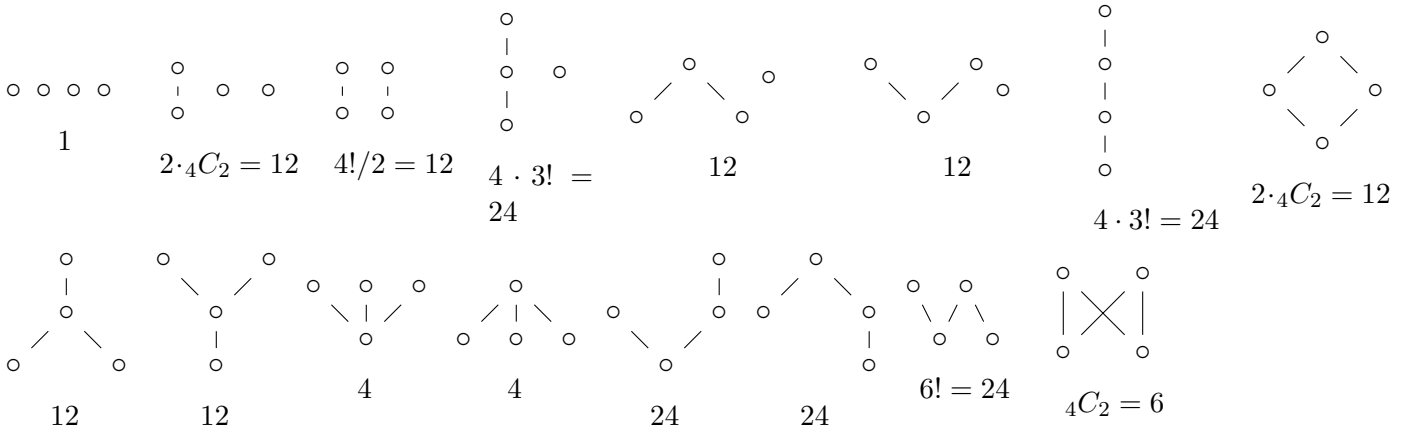
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### 1.3 Ideals and Filters イデアルとフィルター

定義 1.9 (Ideal). An **ideal** in a bounded distributive lattice  $L$  is a subset  $J \subseteq L$  such that

$$0 \in J, \quad (1.20)$$

$$a, b \in J \Rightarrow a \vee b \in J, \quad (1.21)$$

$$b \leq a \text{ \& } a \in J \Rightarrow b \in J. \quad (1.22)$$

定義 1.10 (Filter). A **filter** in a bounded distributive lattice  $L$  is a subset  $F \subseteq L$  such that

$$1 \in F, \quad (1.23)$$

$$a, b \in F \Rightarrow a \wedge b \in F, \quad (1.24)$$

$$b \geq a \text{ \& } a \in F \Rightarrow b \in F. \quad (1.25)$$



## 2 Stone Spaces

## 3 Spaces and Lattices of Open Sets

We will suppose that all topological spaces that appear here will be  $T_0$ .

### 3.1 Sober spaces

**定義 3.1** (meet-irreducibility). *Let  $(X, \tau)$  be a top.space.  $W \in \tau$  is said to be a **meet-irreducible** open set if  $U, V \in \tau$  and  $U \cap V \subseteq W$ , then either  $U \subseteq W$  or  $V \subseteq W$ .*

**定義 3.2** (sober space).  *$X$  is said to be **sober** if all the meet-irreducible open sets are of the form  $X \setminus \overline{\{x\}}$ .*

**命題 3.1.** *Each Hausdorff space is sober.*

**証明.** Suppose  $W$  is meet-irreducible, for contradiction, there exists  $x_1, x_2 \notin W$  and  $x_i \in U_i, x_j \notin U_i (i \neq j)$ . Then  $W = (W \cup U_1) \cap (W \cup U_2)$  and  $W \cup U_i \not\subseteq W$ .  $\square$

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