Reflections of Pinor group representations

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1 Clifford Algebra and Pinor group

Let $\mathrm{Cl}(N)$ be a Clifford algebra of dimension N, and $e_i \in \mathrm{Cl}(N),\ i = 1, ..., N$ be a orthogonal basis. Let us consider a unit vector $n = \sum_{i=1}^N n_i e_i$ with $||n|| = 1$ and its action on a vector $x = \sum_{i=1}^N x_i e_i$ determined by the Pinor representation $\alpha : Pin(N) \to O(N)$:

$$\alpha(n)(x) := nx^t n.$$

Let's calculate the explicit expression of this action!

$$\begin{aligned}
\alpha(n)(x) &= \sum_{i,j,k=1}^{N} n_i x_j n_k e_i e_j e_k \\
&= \sum_i n_i x_i n_i e_i e_i e_i + \sum_{i \neq j} n_i x_i n_j e_i e_i e_j + \sum_{i \neq j} n_i x_j n_i e_i e_j e_i + \sum_{i \neq j} n_j x_i n_i e_j e_i e_i + \sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k \\
&= -\sum_i n_i x_i n_i e_i - \sum_{i \neq j} n_i x_i n_j e_j + \sum_{i \neq j} n_i x_j n_i e_j - \sum_{i \neq j} n_j x_i n_i e_j + \sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k \\
&= -\sum_{i,j=1}^{N} n_i x_i n_j e_j + \sum_{i,j=1}^{N} n_i x_j n_i e_j - \sum_{i,j}^{N} n_j x_i n_i e_j + \sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k \\
&= \sum_{i=1}^{N} x_i e_i - 2\sum_{i,j}^{N} n_i x_i n_j e_j + \sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k \qquad (1)
\end{aligned}$$

The last term is zero, because

$$\sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k = -\sum_{i \neq j \neq k} n_i x_j n_k e_k e_j e_i = -\sum_{i \neq j \neq k} n_i x_j n_k e_i e_j e_k \qquad (2)$$

Hence,

$$\alpha(n)(x) = \sum_{i=1}^{N} x_i e_i - 2\sum_{i,j=1}^{N} n_i x_i n_j e_j = x - 2(n, x)n \qquad (3)$$

, which is a reflection in n.