Inversion of conics

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<2020-06-18 >

Title: Inversion of conics Date: 2020-06-18 Category: math Tags: math

1 Inversion

The images of lines and circles under inversion in circles are again circles and lines. This is the well-known fact of the inversion geometry. But somewhat less known fact about the inversion in circles is that quadratic curves are inversed into certain types of famous quartic curves which have been known from antiguity. In general, the image of a real algebraic curve of degree n, $F(x, y) = 0$ under inversion in unit circle at the origin satisfies an equation $F(\frac{x}{R^2}, \frac{y}{R^2}) = 0$, where $R = \sqrt{x^2 + y^2}$, which is an algebraic expression of degree $2n$. Here we will see how the quadratic curves or conics are mapped under this transformation.

2 Inversions of conics

2.1 Hyperbola

Rectagular hyperbola

$$x^2 - y^2 = 1 \tag{2.1}$$

maps to the Bernouilli's lemniscate

$$(x^2 + y^2)^2 = x^2 - y^2. \tag{2.2}$$

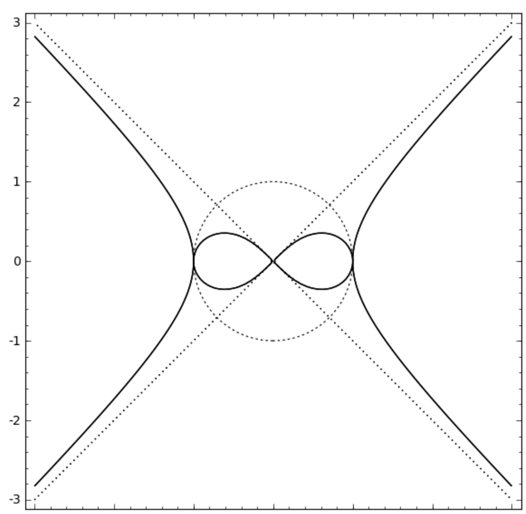

Figure 1: Inversion of the rectangular hyperbola $x^2 - y^2 = 1$.

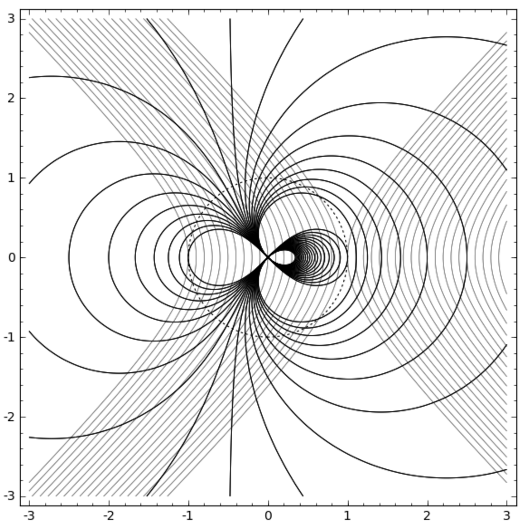

Figure 2: Inversive images of $(x-a)^2 - y^2 = 1$ with varying a

2.2 Parabola

$$y^2 + 1 = x \tag{2.3}$$

maps to a droplet-like curve

$$y^2 + (x^2 + y^2)^2 = x(x^2 + y^2). \tag{2.4}$$

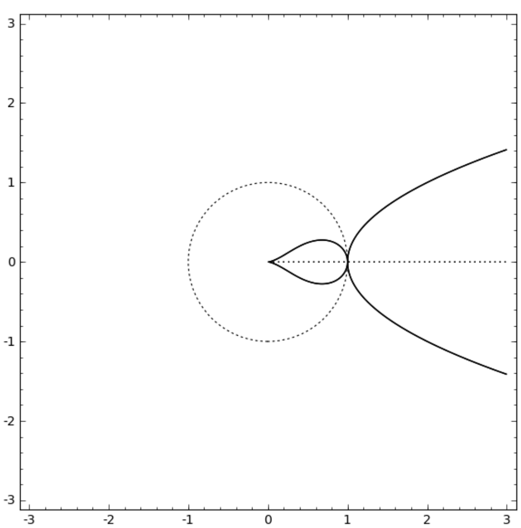

Figure 3: Inversion of parabola $y^2 + 1 = x$

2.3 Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.5}$$

maps to a hippopede

$$(x^2 + y^2)^2 = A^2x^2 + B^2y^2. \tag{2.6}$$

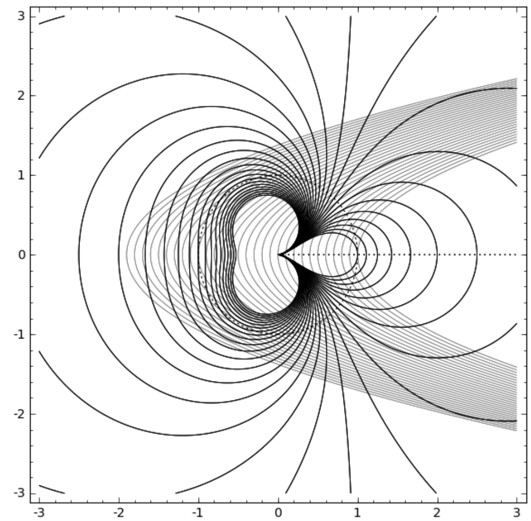

Figure 4: Inversive images of $y^2 = x - a$ with varying a

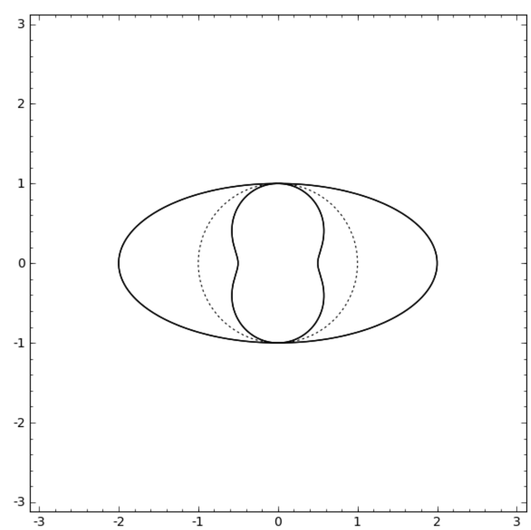

Figure 5: Inversion of ellipse $\frac{x^2}{4} + y^2 = 1$

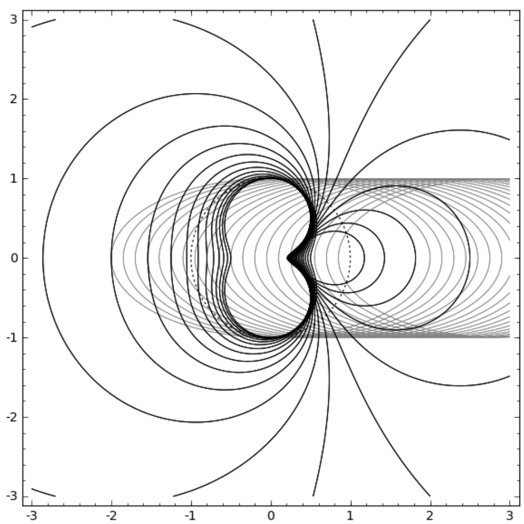

Figure 6: Inversive images of $\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$ with varying c