Pointless Topology 勉強ノート

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1 Preliminary

1.1 Topology トポロジー

Let $\mathcal{P}(X)$ denote the power set of X.

定義 1.1 (Topology トポロジー). *A **topological space** is an ordered pair (X, τ), $\tau \subseteq \mathcal{P}(X)$ which satisfies the following properties*

1. $\varnothing \in \tau$ *and* $X \in \tau$.
2. *if* $U, V \in \tau$, *then* $U \cap V \in \tau$.
3. *if* $\forall I, U_i \in \tau$ *forall* $i \in I$, *then* $\bigcup_{i \in I} U_i$.

*τ is called the **topology** of X. The members of the topology $U \in \tau$ is said to be **open** and $V \subseteq X$ is said to be **closed** if $\exists U$ open such that $V = U^c$.*

定義 1.2 (Separation Axioms 分離公理). *A space (X, τ) is called T_i ,if respectively satisfies the following conditions,*

1. T_0*:* $\forall x, y \in X$ $\exists$ *an open set* $U \in \tau$ *such that* U *contains one of* x, y *and not the other.*
2. T_1*:* $\forall x, y \in X$ $\exists$ *a nhood of each not containing the other.*

例 1.1 (T_0-space). $X = \{a, b\}, \tau = \{\varnothing, \{a\}, X\}$

1.2 Posets, Lattices 半順序集合、束

定義 1.3 (Posets). *A **partial order (半順序)** on a set X is a binary relation $R \subseteq X \times X$ satisfying,*

1. $\forall a, aRa$ *(reflexivity, 反射律),*
2. $\forall a, b, c,\ aRb \,\&\, bRc \Rightarrow aRc$ *(transitivity, 推移律),*
3. $\forall a, b,\ aRb \,\&\, bRa \Rightarrow a = b$ *(antisymmetry, 反対称律).*

if moreover

4. $\forall a, b$ *either* aRb *or* bRa *holds,*

*it is said to be a **linear** or **total** order.*
*A **poset** or **partially ordered set**, $(X, \leq)$ is a set with a partial order. If the order of a poset is linear (or total), it is called a **linearly ordered set**, **totally ordered set** or **chain**. A relation that satisfies only (1) and (2) is called **preorder**.*

定義 1.4 (Suprema, infima)**.** *A **supremum** s of a subset $M \subseteq (X, \leq)$ the least upper bound of M, that is*

1. $\forall m \in M, m \leq s,$
2. $\forall m \in M, m \leq x \Rightarrow s \leq x.$

*Similarly, a **infimum** of a subset $M \subseteq (X, \leq)$ the greatest lower bound of M.*
*We also call a supremum a **join** and an infimum **meet** and notate $\sup M, \inf M$ or $\bigvee M, \bigwedge M$ respectively.*
For finite cases, we wirte $a \vee b := \sup\{a, b\}$ or $a_1 \vee \cdots \vee a_n := \sup\{a_1 \dots a_n\}$ and $a \wedge b := \inf\{a, b\}$ or $a_1 \wedge \cdots \wedge a_n := \inf\{a_1 \dots a_n\}$.

Since each $x \in X$ is both a lower and an upper bound of the empty set $\varnothing$,

$$\sup \varnothing \text{ is the least element of } X \tag{1.1}$$

and

$$\inf \varnothing \text{ is the greatest element of } X \tag{1.2}$$

We use the symbols 0 or $\perp$ for the former and 1 or $\top$ for the latter.

定義 1.5 (Semilattices, Lattice)**.** *A **meet-semilattice** is a poset X such that $\forall a, b \in X$ there exists an infimum $a \wedge b$.*
*A **join-semilattice** is a poset X such that $\forall a, b \in X$ there exists an supremum $a \vee b$.*
*A **lattice** is a poset X such that $\forall a, b \in X$ both an infimum $a \wedge b$ and a supremum $a \vee b$ exist.*
*A **bounded lattice** is a poset in which all finite subsets have infima and suprema (i.e. a lattice with bottom and top).*
*A poset is a **complete lattice** if every subset has a supremum and an infimum.*

In a bounded semilattice, $\wedge$ or $\vee$ is a binary operation and satisfies the following properties,

$$a \wedge a = a \qquad a \vee a = a \tag{1.3}$$
$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a \tag{1.4}$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad (a \vee b) \vee c = a \vee (b \vee c) \tag{1.5}$$
$$a \wedge 1 = a \qquad a \vee 0 = a. \tag{1.6}$$

In other words, bounded semilattices are commutative monoids (semigroup with unit/identity element) in which every element is idempotent.

定理 1.1. *Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A such that $a \wedge b$ is the join of a and b, and 0 is the least element.*

証明. Clearly, if such a partial order exists,

$$a \leq b \Leftrightarrow a \vee b = b. \tag{1.7}$$

□

A lattice can also be defined purely algebraically in those terms,

定義 1.6. *A* ***lattice*** $(L, \vee, \wedge)$ *is an algebra (a set with two binary operations) that satisfy*

$$
\begin{array}{llll}
(L1) & a \wedge a = a & a \vee a = a & \textit{(idempotency)} \\
(L2) & a \wedge b = b \wedge a & a \vee b = b \vee a & \textit{(commutativity)} \\
(L3) & (a \wedge b) \wedge c = a \wedge (b \wedge c) & (a \vee b) \vee c = a \vee (b \vee c) & \textit{(associativity)} \\
(L4) & a \vee (a \wedge b) = a & a \wedge (a \vee b) = a & \textit{(absorption identities)}
\end{array}
$$

定義 1.7 (Ideal)**.** *An* ***ideal*** *in a bounded distributive lattice* L *is a subset* $J \subseteq L$ *such that*

$$0 \in J, \tag{1.8}$$

$$a, b \in J \Rightarrow a \vee b \in J, \tag{1.9}$$

$$b \leq a \;\&\; a \in J \Rightarrow b \in J. \tag{1.10}$$

定義 1.8 (Filter)**.** *A* ***filter*** *in a bounded distributive lattice* L *is a subset* $F \subseteq L$ *such that*

$$1 \in F, \tag{1.11}$$

$$a, b \in F \Rightarrow a \wedge b \in F, \tag{1.12}$$

$$b \geq a \;\&\; a \in F \Rightarrow b \in F. \tag{1.13}$$

2 Stone Spaces

3 Spaces and Lattices of Open Sets

We will suppose that all topological spaces that appear here will be T_0.

3.1 Sober spaces

定義 3.1 (meet-irrducibility)**.** *Let* (X, τ) *be a top.space.* $W \in \tau$ *is said to be a* ***meet-irreducible*** *open set if* $U, V \in \tau$ *and* $U \cap V \subseteq W$*, then either* $U \subseteq W$ *or* $V \subseteq W$*.*

定義 3.2 (sober space)**.** X *is said to be* ***sober*** *if all the meet-irreducible open sets are of the form* $X \backslash \overline{\{x\}}$*.*

命題 3.1. *Each Haudorff space is sober.*

証明. Suppose W is meet-irreducible, for contradiction, there exists $x_1, x_2 \notin W$ and $x_i \in U_i, x_j \notin U_i (i \neq j)$. Then $W = (W \cup U_1) \cap (W \cup U_2)$ and $W \cup U_i \nsubseteq W$. □

参考文献

[1] Jorge Picado, Aleš Putlr, Frames and Locales:Topology without points, Birkhäuser.