Notes on "Statistics for Mathematicians" by Victor M. Panaretos

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1 Chapter 1: Regular Probability Models

Definition 1.1: Regular Parametric Probability Models

- X : $\mathbb{R}$-valued random variable
- F_θ : distribution function of X
- θ : a parameter in $\Theta \subseteq \mathbb{R}^p$ (parameter space)

The probability model $\{F_\theta : \theta \in \Theta\}$ will be calld *regular* if one of the two following conditions holds:

1. $\forall \theta \in \Theta$, the distribution F_θ is continuous with density $f(x;\theta)$
2. $\forall \theta \in \Theta$, the distribution F_θ is discrete with probability mass function $f(x;\theta)$ such that $\sum_{x \in \mathbb{Z}} f(x;\theta) = 1$ for all $\theta \in \Theta$.

- The model F_θ cannot switch between continuous and discrete depending on the value of θ.
- $\mathcal{X} := \{x \in \mathbb{R} : f(x;\theta) > 0\}$ is called the *sample space* of X.

1.1 Discrete Regular Models

Definition 1.5 Binomial distribution
A random variable X is said to follow the binomial distribution with parameters $p \in (0,1)$ and $n \in \mathbb{N}$, denoted $X \sim Binom(n,p)$, if

1. $\mathcal{X} = \{0, \ldots, n\}$,
2. $f(x;n,p) = \binom{n}{p}\binom{n}{p} p^x (1-p)^{n-x}$. $f(x;n,p) = \binom{n}{p} p^x (1-p)^{n-x}$.

- The mean: $\mathbb{E}[X] = np$
- The variance: $Var[X] = np(1-p)$
- The moment generating function: $M(t) = (1 - p + pe^t)^n$

Derivations

The mean:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} && (1)\\
&= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} && (2)\\
&= p \sum_{x=1}^{n} x \binom{n}{x} p^{x-1} (1-p)^{n-x} && (3)
\end{aligned}$$

Consider the following polynomial's derivative with respect to P

$$\begin{aligned}
(P+Q)^n &= \sum_{x=0}^{n} \binom{n}{x} P^x Q^{n-x} && (4)\\
\frac{\partial}{\partial P}(P+Q)^n &= n(P+Q)^{n-1} = \sum_{x=0}^{n} x \binom{n}{x} P^{x-1} Q^{n-x} && (5)\\
&= \frac{1}{P}\left(\sum_{x=0}^{n} x \binom{n}{x} P^x Q^{n-x}\right) && (6)\\
nP(P+Q)^{n-1} &= \sum_{x=0}^{n} x \binom{n}{x} P^x Q^{n-x} && (7)
\end{aligned}$$

Putting $P = p$ and $Q = 1 - p$, we have

$$\mathbb{E}[X] = pn(p + 1 - p)^{n-1} = pn \qquad (8)$$

Th vatiance:

$$\begin{aligned}
Var[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 && (9)\\
&= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} - n^2p^2 && (10)
\end{aligned}$$

To calculate the first term in the last line, let us consider the following second derivative of the polynomial $(P+Q)^n$

$$
\begin{aligned}
\frac{\partial^2}{\partial P^2}(P+Q)^n &= n(n-1)(P+Q)^{n-2} \\
&= \sum_{x=2}^{n} x(x-1)\binom{n}{x} P^{x-2}Q^{n-x} \\
&= \sum_{x=2}^{n} x^2 \binom{n}{x} P^{x-2}Q^{n-x} - \sum_{x=2}^{n} x\binom{n}{x} P^{x-2}Q^{n-x} \\
&= \frac{1}{P^2}\left(\sum_{x=2}^{n} x^2\binom{n}{x} P^{x}Q^{n-x} - \sum_{x=2}^{n} x\binom{n}{x} P^{x}Q^{n-x}\right) \\
&= \frac{1}{P^2}\left(\sum_{x=1}^{n} x^2\binom{n}{x} P^{x}Q^{n-x} - \sum_{x=1}^{n} x\binom{n}{x} P^{x}Q^{n-x}\right) \\
&= \frac{1}{P^2}\left(\sum_{x=1}^{n} x^2\binom{n}{x} P^{x}Q^{n-x} - nP(P+Q)^{n-1}\right) \\
\therefore \sum_{x=1}^{n} x^2\binom{n}{x} P^{x}Q^{n-x} &= n(n-1)P^2(P+Q)^{n-2} + nP(P+Q)^{n-1}
\end{aligned}
$$

Putting $P = p$ and $Q = 1-p$, we have

$$
\sum_{x=0}^{n} x^2\binom{n}{x} p^x(1-p)^{n-x} = n(n-1)p^2 + np \tag{11}
$$

Hence

$$
Var[X] = n(n-1)p^2 + np - n^2p^2 = np(1-p) \tag{12}
$$

The moment generating function:

$$
\begin{aligned}
M(t) = \mathbb{E}[e^{tX}] &= \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x(1-p)^{n-x} && (13) \\
&= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x(1-p)^{n-x} && (14) \\
&= (1-p+pe^t)^n && (15)
\end{aligned}
$$