GR Note

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Contents

1 Differential geometry

Definition 1.1. *Torsion tensor*

$$\Theta(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.1}$$

Definition 1.2. *Curvature tensor*

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.2}$$

Definition 1.3. *Symmetrised Curvature tensor*

$$R_S(W,X,Y,Z) := g(W, R(X,Y)Z) \tag{1.3}$$

Formula 1.1. *Bianchi's first and second indentities*

$$S(R(X,Y)Z) = S((\nabla_X \Theta)(Y,Z) + \Theta(\Theta(X,Y),Z)) \tag{1.4}$$

$$S((\nabla_X R)(Y,Z) + R(\Theta(X,Y),Z)) = 0 \tag{1.5}$$

where S is the symmetrisation in X, Y, Z.

Proof.

$$\begin{aligned}
S(R(X,Y)Z) &= S(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
&= S(\nabla_X (\nabla_Y Z - \nabla_Y Z) - \nabla_{[X,Y]} Z) \\
&= S(\nabla_X (\Theta(Y,Z)) + \nabla_X [Y,Z] - \nabla_{[Y,Z]} X - [X,[Y,Z]]) \\
&= S(\nabla_X (\Theta)(Y,Z) + \Theta(\nabla_X Y, Z) + \Theta(Y, \nabla_X Z) + \Theta(X,[Y,Z])) \\
&= S(\nabla_X (\Theta)(Y,Z) + \Theta(\nabla_X Y - \nabla_Y X - [X,Y], Z)) \\
&= S(\nabla_X (\Theta)(Y,Z) + \Theta(\Theta(X,Y),Z))
\end{aligned}$$

□

2 Symmetry of spacetimes

Definition 2.1. *A Killing vector X is a vector field, which satisfies the isometry condition,*

$$\mathcal{L}_X g = 0. \tag{2.1}$$

$$\begin{aligned}(\mathcal{L}_X g)(V, W) &= \mathcal{L}_X(g(V, W)) - g(\mathcal{L}_X V, W) - g(V, \mathcal{L}_X W)\\ &= \nabla_X(g(V, W)) - g(\mathcal{L}_X V, W) - g(V, \mathcal{L}_X W)\\ &= g(\nabla_V X, W) + g(V, \nabla_W X)\end{aligned} \tag{2.2}$$

for arbitrary vectors V, W. Hence, a Killing vector field X must satisfy

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \tag{2.3}$$

A propositon for the dimension of freedom of a Killing vector

Proposition 2.1. *The dimension of the freedom of Killing vectors on a manifold with $\dim M = n$ is ,at most, $\frac{n(n+1)}{2}$*

Proposition 2.2. *If X, Y are both Killing vectors, then so is $[X, Y]$.*

Proof.

$$\mathcal{L}_{[X,Y]} g = (\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X) g = 0 \tag{2.4}$$

□

3 Schwarzschild solution

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

Non-vanishing Christoffel symbols are

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right), \quad \Gamma^t_{rt} = \frac{M}{r(r-2M)} = -\Gamma^r_{rr}, \quad \Gamma^r_{\theta\theta} = r - 2M, \quad \Gamma^r_{\phi\phi} = -(r-2M)\sin^2\theta,$$
$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta$$

3.1 Geodesics

The geodesic equations are:

$$\frac{d^2 t}{ds^2} = -2\Gamma^t_{tr}\frac{dt}{ds}\frac{dr}{ds} \tag{3.2}$$

$$\frac{d^2 r}{ds^2} = -\Gamma^r_{tt}\left(\frac{dt}{ds}\right)^2 - \Gamma^r_{rr}\left(\frac{dr}{ds}\right)^2 \Gamma^r_{\theta\theta}\left(\frac{d\theta}{ds}\right)^2 - \Gamma^r_{\phi\phi}\left(\frac{d\phi}{ds}\right)^2 \tag{3.3}$$

$$\frac{d^2 \theta}{ds^2} = -2\Gamma^\theta_{r\theta}\frac{d\theta}{ds}\frac{dr}{ds} - \Gamma^\theta_{\phi\phi}\left(\frac{d\theta}{ds}\right)^2 \tag{3.4}$$

$$\frac{d^2 \phi}{ds^2} = -2\Gamma^\phi_{r\phi}\frac{d\phi}{ds}\frac{dr}{ds} - 2\Gamma^\phi_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} \tag{3.5}$$

Abbreviation for the differentiation $\dot{t} := \frac{dt}{ds}$

$$\begin{aligned}
\ddot{t} &= -\frac{2M}{r(r-2M)}\dot{t}\,\dot{r} && (3.6)\\
\ddot{r} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{M}{r(r-2M)}\dot{r}^2 - (r-2M)\,\dot{\theta}^2 - (r-2M)\sin^2\theta\,\dot{\phi}^2 && (3.7)\\
\ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} + \cos\theta\sin\theta\,\dot{\phi}^2 && (3.8)\\
\ddot{\phi} &= -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi} && (3.9)
\end{aligned}$$

(3.8) and (3.9) leads to a pair of angular momentum constants

$$\begin{aligned}
r^2\dot{\phi} + \frac{(r^2\dot{\theta})^2}{a} &= b && (3.10)\\
r^2\sin^2\theta\,\dot{\phi} &= a && (3.11)
\end{aligned}$$

where a, b are constants. In particular, consider the initial condition $\theta_0 = \frac{\pi}{2}$, $\dot{\theta}_0 = 0$. Then $r_0^2\dot{\phi}_0 = a$ and substituting this to the first line, we have $a = b$. Now, equating the L.H.S. of the both equalities, we have

$$r^4\sin^2\theta\cos^2\theta\,\dot{\phi}^2 = -(r^2\dot{\theta})^2 \tag{3.12}$$

which is only possible when $\theta(s) = \frac{\pi}{2}$ for all s, provided $a \neq 0$. In words, if a particle's θ-component of the velocity is zero at $\theta = \frac{\pi}{2}$, then it will always remain so. This means that we can always choose a coordinates set where a given particle's geodosic is in the equatorial plane. Hence, it is sufficient to just think about those geodesics in the plane ($\theta = \frac{\pi}{2}$). Then the second line of the geodesic equations becomes

$$\ddot{r} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{M}{r(r-2M)}\dot{r}^2 - \frac{a^2}{r^4}(r-2M) \tag{3.13}$$

$$\begin{aligned}
\frac{\ddot{t}}{\dot{t}} &= \left\{\frac{1}{r} - \frac{1}{r-2M}\right\}\dot{r}\\
\dot{t} &= \frac{k}{1-\frac{2M}{r}} && (3.14)
\end{aligned}$$

Notice that we can change the constant k by reparametrising $s \mapsto s' = cs$ for $c \neq 0$.

3.2 Timelike geodesics

Geodesic of a particle of mass m

$$\begin{aligned}
-m &= g_{00}(v^0)^2 + g_{11}(v^1)^2 + g_{33}(v^3)^2\\
m &= \frac{k^2}{1-\frac{2M}{r}} - \frac{\dot{r}^2}{1-\frac{2M}{r}} - \frac{a^2}{r^2}\\
\dot{r}^2 &= k^2 - m\left(1-\frac{2M}{r}\right) - \frac{a^2}{r^2}\left(1-\frac{2M}{r}\right) && (3.15)
\end{aligned}$$

By the remark mentioned above, we can set $k = \sqrt{m}$. Then, the equation becomes

$$m\dot{r}^2 = \frac{2mM}{r} - \frac{a^2}{r^2}\left(1-\frac{2M}{r}\right) \tag{3.16}$$

Or we can use the relation $\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{r^2\dot{r}}{a}$ and $u = \frac{1}{r}$, $\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}$ to rewrite (3.15),

$$\left(\frac{du}{d\phi}\right)^2 = \frac{k^2-m}{a^2} - \frac{2M}{a^2}u - u^2\left(1-2Mu\right). \tag{3.17}$$

The right hand side (**??**) can be factorise so that the equation becomes

$$\left(\frac{du}{d\phi}\right)^2 = 2M(u-u_1)(u-u_2)(u-u_3). \tag{3.18}$$

And this can be integrated to give the inverse elliptic function

$$\phi - \phi_0 = \int_{u_0}^{u} \frac{du'}{\sqrt{(u'-u_1)(u'-u_2)(u'-u_3)}}. \tag{3.19}$$

3.3 Null geodesics

Null geodesics are:

$$\begin{aligned} 0 &= g_{00}(v^0)^2 + g_{11}(v^1)^2 + g_{33}(v^3)^2 \\ \dot{r}^2 &= k^2 - \frac{a^2}{r^2} + \frac{2Ma^2}{r^3} \end{aligned} \tag{3.20}$$

4 Splitting of spacetime manifold \$ M $\simeq$ R$\times$ σ\$

Lorentzian manifold \$(M,g)\$
A connection $\nabla : TM \to T^*M \times TM$ compatible with g.
The definition of compatibility:

$$\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{4.1}$$
$$\tag{4.2}$$

The relation with Lie derivative (torsion free condition)

$$\nabla_X Y - \nabla_Y X = [X,Y] = \mathcal{L}_X Y \tag{4.3}$$

Hyperbolicity condition $M \simeq \mathbb{R} \times \sigma$ where σ is a 3-D manifold. Let n be a unit normal ($g(n,n) = -1$) to $Im(\sigma)$ in M

Definition 4.1. *Internal metric $q_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$*

$$q_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu \tag{4.4}$$
$$K_{\mu\nu} := q_\mu^\rho q_\nu^\sigma \nabla_\rho n_\sigma \tag{4.5}$$

$$\begin{aligned} 2K_{\mu\nu} &= q_\mu^\rho q_\nu^\sigma 2\nabla_{(\rho} n_{\sigma)} \\ &= q_\mu^\rho q_\nu^\sigma (\mathcal{L}_n g)_{\rho\sigma} = q_\mu^\rho q_\nu^\sigma (\mathcal{L}_n q + s\mathcal{L}_n n \otimes n)_{\rho\sigma} \\ &= q_\mu^\rho q_\nu^\sigma (\mathcal{L}_n q)_{\rho\sigma} = (\mathcal{L}_n q)_{\mu\nu} \end{aligned} \tag{4.6}$$

We can regard q_μ^ρ as a vector field $q_\mu = q_\mu^\rho \partial_\rho$. Then,

$$\begin{aligned} d(g(q_\mu, q_\nu))(n) &= \nabla_n(g(q_\mu, q_\nu)) = g(\nabla_n q_\mu, q_\nu) + g(q_\mu, \nabla_n q_\nu) \\ &= g(\nabla_{q_\mu} n, q_\nu) + g(q_\mu, \nabla_{q_\nu} n) + g(\mathcal{L}_n q_\mu, q_\nu) + g(q_\mu, \mathcal{L}_n q_\nu) \end{aligned}$$

Here, we used (4.3) with \$$\nabla_{\mathrm{n}} \mathrm{q}_\mu = \nabla_{\mathrm{q}_\mu} \mathrm{n} + \mathrm{L}_{\mathrm{n}} \mathrm{q}_\mu$\$
Comparing with

$$d(g(q_\mu, q_\nu))(n) = \mathcal{L}_n(g(q_\mu, q_\nu)) = (\mathcal{L}_n g)(q_\mu, q_\nu) + g(\mathcal{L}_n q_\mu, q_\nu) + g(q_\mu, \mathcal{L}_n q_\nu)$$

we see

$$(\mathcal{L}_n g)(q_\mu, q_\nu) = g(\nabla_{q_\mu} n, q_\nu) + g(q_\mu, \nabla_{q_\nu} n) \tag{4.7}$$

or in indices

$$(\mathcal{L}_n g)_{\rho\sigma} q_\mu^\rho q_\nu^\sigma = q_\mu^\rho q_\nu^\sigma (\nabla_\rho n_\sigma + \nabla_\sigma n_\rho) \tag{4.8}$$

In words, this is the change of $q_{\mu\nu}$ along n^μ projected to the tangent subspace of σ in M.

5 2+1 dimensional gravity

5.1 Angle between two spacelike planes(faces)

Let S_1, S_2 be two spacelike faces in 3-D Minkowski spacetime and n_1 and n_2 be their unit normals respectively. Then we can write $n_i = (\cosh\theta_i, \sinh\theta_i\cos\phi_i, \sinh\theta_i\sin\phi_i)$. The inner product is

$$n_1 \cdot n_2 \quad = \quad \cosh\theta_1\cosh\theta_2 - \cos\phi\sinh\theta_1\sinh\theta_2$$

This number can be interpreted as the timelike component of the image of the unit vector under a boost relative to the observer comoving with n_1. That is, the vector

$$\begin{aligned} n_3 \quad &:= \quad \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 & 0 \\ -\sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{pmatrix} n_2 \\ \begin{pmatrix} \cosh\theta_3 \\ \sinh\theta_3\cos\phi_3 \\ \sinh\theta_3\sin\phi_3 \end{pmatrix} \quad &= \quad \begin{pmatrix} \cosh\theta_1\cosh\theta_2 - \cos\phi\sinh\theta_1\sinh\theta_2 \\ -\sinh\theta_1\cosh\theta_2 + \cos\phi\cosh\theta_1\sinh\theta_2 \\ \sin\phi\sinh\theta_2 \end{pmatrix} \end{aligned} \tag{5.1}$$