Notes on BF theories

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1 BF-Theory in D dimensions

Let M be a D dimensional manifold, G be a finite dimanesional Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Let ω be a $\mathfrak{g}$-valued connection on M and $F(\omega) = d\omega + \omega \wedge \omega$ its corresponding curvature. We also introduce another $\mathfrak{g}$-valued (D-2)-form B. Then BF-theory is defined by the following action integral,

$$I_{BF} = \int_M trB \wedge F(\omega) \tag{1.1}$$

where tr is the Killing form of the Lie algebra $\mathfrak{g}$. It is a topological field theory, in the sense that the action does not depend on the metrical structure of the base manifold M. Since the case $D = 3$ is equivalent to the first order formalism of the gravitational theory, the model has been studied intensively so far and some results are known.

Table 1: Ingredients of BF Theory

M	Base manifold	D dimensional
G	compact Lie group	usually $SO(D)$ or $SO(D-1,1)$
P	principal fibre bundle over M with principal group G	
$\mathfrak{g}$	Lie algebra of G	
ω	$\mathfrak{g}$-valued connection	
$F(\omega)$	the curvature of ω	$F = d\omega + \omega \wedge \omega$
B	$\mathfrak{g}$-valued $(D-2)$-form	

2 3D gravity

2.1 Second order formalism

$g \in sym(T^*M \bigotimes T^*M)$ or $g : TM \bigotimes TM \to C(M)$. In local coordinates x^μ, $g = g_{\mu\nu}dx^\mu dx^\nu$. g is assumed to be non-degenerate at all points in M.

$$I_{EH} = \int_M \sqrt{|g|}R(g) \tag{2.1}$$

2.2 Fisrt order formalism

Consider a tirad $e = e^I_\mu \tau_I dx^\mu$. The indices $I = 1, 2, 3$ are thought to be the coordinates of the inner gauge symmetry and $\tau_I = \frac{i}{2}\sigma_I$ where σ_I are the Pauli Matrices. There are nine free variables e^I_μ.

3 4D Gravity

3.1 Bivectors in 4-D and Simplicity Condition

$$I_{PAL} = \int_M tr\left[B \wedge F(\omega)\right] - \lambda_{IJKL} B^{IJ} B^{KL} \tag{3.1}$$

Theorem 3.1. *Assume that e^1, e^2, e^3, e^4 span $\mathbb{R}^4$. A bivector $B = B_{\mu\nu} e^\mu \wedge e^\nu$ with $B_{\mu\nu} \in \mathbb{R}$, can be written as a sum of at most two wedge products of vectors which are linearly independent. In other words, it is either*

$$B = u^1 \wedge v^1 + u^2 \wedge v^2 \tag{3.2}$$

for some linearly independent vectors u^1, u^2, v^1, v^2 or

$$B = u \wedge v \tag{3.3}$$

for some linearly independent vectors u, v. In the latter case, B is said to be simple.

Proof. First, let us collect as many terms with the same index in the left subscript as possible, say starting with subscripts containing 1 and 2.

$$\begin{aligned} B &= B_{\mu\nu} e^\mu \wedge e^\nu \\ &= e^1 \wedge (B_{12}e^2 + B_{13}e^3 + B_{14}e^4) + e^2 \wedge (B_{23}e^3 + B_{24}e^4) + B_{34}e^3 \wedge e^4 \\ &= e^1 \wedge e'^1 + e^2 \wedge e'^2 + B_{34}e^3 \wedge e^4 \end{aligned} \tag{3.4}$$

Since e^1, e^2, e^3, e^4 are linearly independent, so are the vectors e^1, e'^1, e^2, e'^2. Now

$$e'^1 \wedge e'^2 = B_{12}e^2 \wedge e'^2 + C_{12;34}e^3 \wedge e^4. \tag{3.5}$$

Here $C_{12;34} := B_{13}B_{24} - B_{14}B_{23}$.
If $C_{12;34} \neq 0$,

$$e^3 \wedge e^4 = \frac{1}{C_{12;34}}\left(e'^1 \wedge e'^2 - B_{12}e^2 \wedge e'^2\right) \tag{3.6}$$

then

$$B = \left(e^1 - \frac{B_{34}}{C_{12;34}}e'^2\right) \wedge e'^1 + \left(1 - \frac{B_{12}B_{34}}{C_{12;34}}\right) e^2 \wedge e'^2 = u^1 \wedge v^1 + u^2 \wedge v^2 \tag{3.7}$$

where we have defined $u^1 = \left(e^1 - \frac{B_{34}}{C_{12;34}}e'^2\right), u^2 = e'^1, v^1 = \left(1 - \frac{B_{12}B_{34}}{C_{12;34}}\right)e^2, v^2 = e'^2$.
If $C_{12;34} = 0$, we can look for other pairs of indeces to find one ij with $C_{ij;kl} \neq 0$ and then we can apply the same process as described above. Otherwise, we have $C_{ij;kl} = 0$ for all ij. In this case B is a simple bivector. To see this, let us list the coefficients $B_{\mu\nu}$ as the entries of an anti-symmetric matrix

$$\begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{34} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{pmatrix} \tag{3.8}$$

with $B_{\mu\nu} = -B_{\nu\mu}$. Then $C_{ij;kl} = 0$ implies that

$$\begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & k_1B_{13} & k_1B_{14} \\ B_{31} & k_2B_{12} & 0 & k_2B_{14} \\ B_{41} & k_3B_{12} & k_3B_{43} & 0 \end{pmatrix} \tag{3.9}$$

for some constants k_1, k_2, k_3. Then, from the anti-symmetry, we have

$$\begin{cases} k_1 B_{13} &= -k_2 B_{12} \\ k_1 B_{14} &= -k_3 B_{12} \\ k_2 B_{14} &= -k_3 B_{13}. \end{cases} \tag{3.10}$$

Hence, $k_1 B_{13} = k_1 B_{14} = k_2 B_{14} = 0$, which implies that

$$B = e^1 \wedge B_{1\mu} e^\mu = u \wedge v. \tag{3.11}$$

$\square$