Test Questions for Project 2

Test Problems – Prologue

1. These problems are designed to spark the directions of analyses. They are not meant to be a complete set of all possible issues that can arise when encountering a code.
    a) Each problem is designed to explore an issue. For example, if the user enters rubbish, how does the code respond?

2. There is a full set of "rubric" that exists as part of the requirement for the Project. These test questions are supplemental and supportive of that rubric.

Interpolation: Lagrange/Newton's divided differences

Data points entered as:

(0,1),(2,2),(3,4)

(0.1),(2.2),(3,4)

(0,1),(0,2),(0,4)

```
X = [-1.5 3.2; 1.8 3.3; -3.7 1.5; -1.5 1.3; ...
     0.8 1.2; 3.3 1.5; -4.0 -1.0; -2.3 -0.7;
     0 -0.5; 2.0 -1.5; 3.7 -0.8; -3.5 -2.9; ...
     -0.9 -3.9; 2.0 -3.5; 3.5 -2.25];
```

(0,0), (1,0), (0,1), (1,1)

For Lagrange code: User starts with (0,1),(2,2),(3,4) then realizes that data should also include (2,0) AFTER first data was entered. Restart or fixable?

For Newton code: User starts with (0,1),(2,2),(3,4) then realizes that data should also include (2,0) AFTER first data was entered. Restart or fixable?

Apply the following world population figures to estimate the 1980 population, using (a) the straight line through the 1970 and 1990 estimates; (b) the parabola through the 1960, 1970, and 1990 estimates; and (c) the cubic curve through all four data points. Compare with the 1980 estimate of 4452584592.

year	population
1960	3039585530
1970	3707475887
1990	5281653820
2000	6079603571

Interpolate $f(x) = 1/(1 + 12x^2)$ at evenly spaced points in $[-1, 1]$.

Use the method of divided differences to find the degree 4 interpolating polynomial $P_4(x)$ for the data (0.6,1.433329), (0.7,1.632316), (0.8,1.896481), (0.9,2.247908), and (1.0,2.718282). (b) Calculate $P_4(0.82)$ and $P_4(0.98)$. (c) The preceding data come from the function f (x) = e^{x^2}. Use the interpolation error formula to find upper bounds for the error at x = 0.82 and x = 0.98, and compare the bounds with the actual error.

Interpolation: Chebyshev

Data points entered as:

(0,1),(2,2),(3,4)

(0.1),(2.2),(3,4)

(0,1),(0,2),(0,4)

```
X = [-1.5 3.2; 1.8 3.3; -3.7 1.5; -1.5 1.3; ...
     0.8 1.2; 3.3 1.5; -4.0 -1.0; -2.3 -0.7;
     0 -0.5; 2.0 -1.5; 3.7 -0.8; -3.5 -2.9; ...
     -0.9 -3.9; 2.0 -3.5; 3.5 -2.25];
```

(0,0), (1,0), (0,1), (1,1)

Suppose you are designing the **ln** key for a calculator whose display shows six digits to the right of the decimal point. Find the least degree d for which Chebyshev interpolation on the interval $[1, e]$ will approximate within this accuracy.

Let $f(x) = e^{|x|}$. Compare evenly spaced interpolation with Chebyshev interpolation by plotting degree n polynomials of both types on the interval $[-1, 1]$, for $n = 10$ and 20. For evenly spaced interpolation, the left and right interpolation base points should be -1 and 1. By sampling at a 0.01 step size, create the empirical interpolation errors for each type, and plot a comparison. Can the Runge phenomenon be observed in this problem?

Carry out the steps for $f(x) = e^{-x^2}$.

Interpolation: Cubic Splines

Data points entered as:

(0,1),(2,2),(3,4)

(0.1),(2.2),(3,4)

(0,1),(0,2),(0,4)

```
X = [-1.5 3.2; 1.8 3.3; -3.7 1.5; -1.5 1.3; ...
     0.8 1.2; 3.3 1.5; -4.0 -1.0; -2.3 -0.7;
     0 -0.5; 2.0 -1.5; 3.7 -0.8; -3.5 -2.9; ...
     -0.9 -3.9; 2.0 -3.5; 3.5 -2.25];
```

(0,0), (1,0), (0,1), (1,1)

Find the equations and plot the natural cubic spline that interpolates the data points (a) $(0, 3)$, $(1, 5)$, $(2, 4)$, $(3, 1)$ (b) $(-1, 3)$, $(0, 5)$, $(3, 1)$, $(4, 1)$, $(5, 1)$.

Find and plot the cubic spline S satisfying $S(0) = 1$, $S(1) = 3$, $S(2) = 3$, $S(3) = 4$, $S(4) = 2$ and with $S''(0) = S''(4) = 0$.

Find and plot the cubic spline S satisfying $S(0) = 1$, $S(1) = 3$, $S(2) = 3$, $S(3) = 4$, $S(4) = 2$ and with $S''(0) = 3$ and $S''(4) = 2$.

Find and plot the cubic spline S satisfying $S(0) = 1$, $S(1) = 3$, $S(2) = 3$, $S(3) = 4$, $S(4) = 2$ and with $S'(0) = 0$ and $S'(4) = 1$.

Find and plot the cubic spline S satisfying $S(0) = 1$, $S(1) = 3$, $S(2) = 3$, $S(3) = 4$, $S(4) = 2$ and with $S'(0) = -2$ and $S'(4) = 1$.

(a) Consider the natural cubic spline through the world population data points in Computer Problem 3.1.1. Evaluate the year 1980 and compare with the correct population. (b) Using a linear spline, estimate the slopes at 1960 and 2000, and use these slopes to find the cubic spline through the data. Plot the spline and estimate the 1980 population.

Recall the carbon dioxide data of Exercise 3.1.17. (a) Find and plot the natural cubic spline through the data, and compute the spline estimate for the CO_2 concentration in 1950. (b) Carry out the same analysis for the parabolically terminated spline.

Interpolation: Bezier

Find a one-piece Bézier spline that has vertical tangents at its endpoints $(-1, 0)$ and $(1, 0)$ and that passes through $(0, 1)$.

Find a one-piece Bézier spline that has a horizontal tangent at endpoint $(0, 1)$ and a vertical tangent at endpoint $(1, 0)$ and that passes through $(1/3, 2/3)$ at $t = 1/3$.

Find the one-piece Bézier space curve $(x(t), y(t), z(t))$ defined by the four points.
(a) $(1, 0, 0), (2, 0, 0), (0, 2, 1), (0, 1, 0)$ (b) $(1, 1, 2), (1, 2, 3), (-1, 0, 0), (1, 1, 1)$
(c) $(2, 1, 1), (3, 1, 1), (0, 1, 3), (3, 1, 3)$

Find the knots and control points for the following Bézier space curves.

(a) $\begin{cases} x(t) = 1 + 6t^2 + 2t^3 \\ y(t) = 1 - t + t^3 \\ z(t) = 1 + t + 6t^2 \end{cases}$ (b) $\begin{cases} x(t) = 3 + 4t - t^2 + 2t^3 \\ y(t) = 2 - t + t^2 + 3t^3 \\ z(t) = 3 + t + t^2 - t^3 \end{cases}$

(c) $\begin{cases} x(t) = 2 + t^2 - t^3 \\ y(t) = 1 - t + 2t^3 \\ z(t) = 2t^3 \end{cases}$

Given $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and (x_4, y_4), show that the equations

$$x(t) = x_1(1 - t)^3 + 3x_2(1 - t)^2 t + 3x_3(1 - t)t^2 + x_4 t^3$$
$$y(t) = y_1(1 - t)^3 + 3y_2(1 - t)^2 t + 3y_3(1 - t)t^2 + y_4 t^3$$

give the Bézier curve with endpoints $(x_1, y_1), (x_4, y_4)$ and control points $(x_2, y_2), (x_3, y_3)$.

Least Squares: Linear

Form the normal equations, and compute the least squares solution and 2-norm error for the following inconsistent systems:

(a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -5 \\ 15 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}$

Consider the world oil production data of Computer Problem 3.2.3. Find the best least squares (a) line, (b) parabola, and (c) cubic curve through the 10 data points and the RMSE of the fits. Use each to estimate the 2010 production level. Which fit best represents the data in terms of RMSE?

Consider the world population data of Computer Problem 3.1.1. Find the best least squares (a) line, (b) parabola through the data points, and the RMSE of the fit. In each case, estimate the 1980 population. Which fit gives the best estimate?

Consider the carbon dioxide concentration data of Exercise 3.1.13. Find the best least squares (a) line, (b) parabola, and (c) cubic curve through the data points and the RMSE of the fit. In each case, estimate the 1950 CO_2 concentration.

A company test-markets a new soft drink in 22 cities of approximately equal size. The selling price (in dollars) and the number sold per week in the cities are listed as follows:

city	price	sales/week
1	0.59	3980
2	0.80	2200
3	0.95	1850
4	0.45	6100
5	0.79	2100
6	0.99	1700
7	0.90	2000
8	0.65	4200
9	0.79	2440
10	0.69	3300
11	0.79	2300

city	price	sales/week
12	0.49	6000
13	1.09	1190
14	0.95	1960
15	0.79	2760
16	0.65	4330
17	0.45	6960
18	0.60	4160
19	0.89	1990
20	0.79	2860
21	0.99	1920
22	0.85	2160

(a) First, the company wants to find the "demand curve": how many it will sell at each potential price. Let P denote price and S denote sales per week. Find the line $S = c_1 + c_2 P$ that best fits the data from the table in the sense of least squares. Find the normal equations and the coefficients c_1 and c_2 of the least squares line. Plot the least squares line along with the data, and calculate the root mean square error.

After studying the results of the test marketing, the company will set a single selling price P throughout the country. Given a manufacturing cost of \$0.23 per unit, the total profit (per city, per week) is $S(P - 0.23)$ dollars. Use the results of the preceding least squares approximation to find the selling price for which the company's profit will be maximized.

Let A be the $10 \times n$ matrix formed by the first n columns of the 10×10 Hilbert matrix. Let c be the n-vector $[1, \ldots, 1]$, and set $b = Ac$. Use the normal equations to solve the least squares problem $Ax = b$ for (a) $n = 6$ (b) $n = 8$, and compare with the correct least squares solution $\overline{x} = c$. How many correct decimal places can be computed? Use condition number to explain the results. (This least squares problem is revisited in Computer Problem 4.3.7.)

Let $x_1, \ldots, x_{11}$ be 11 evenly spaced points in $[2, 4]$ and $y_i = 1 + x_i + x_i^2 + \cdots + x_i^d$. Use the normal equations to compute the best degree d polynomial, where (a) $d = 5$ (b) $d = 6$ (c) $d = 8$. Compare with Example 4.5. How many correct decimal places of the coefficients can be computed? Use condition number to explain the results. (This least squares problem is revisited in Computer Problem 4.3.8.)

Least Squares: QR Factorization

Use QR factorization to find the least squares solutions and 2-norm error of the following inconsistent systems:

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 10 \\ 3 \end{bmatrix}$

Use QR factorization to find the least squares solutions and 2-norm error of the following inconsistent systems:

(a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -5 \\ 15 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}$

Let A be the $10 \times n$ matrix formed by the first n columns of the 10×10 Hilbert matrix. Let c be the n-vector $[1, \ldots, 1]$, and set $b = Ac$. Use the QR factorization to solve the least squares problem $Ax = b$ for (a) $n = 6$ (b) $n = 8$, and compare with the correct least squares solution $\overline{x} = c$. How many correct decimal places can be computed? See Computer Problem 4.1.8, where the normal equations are used.

Let $x_1, \ldots, x_{11}$ be 11 evenly spaced points in $[2, 4]$ and $y_i = 1 + x_i + x_i^2 + \cdots + x_i^d$. Use the QR factorization to compute the best degree d polynomial, where (a) $d = 5$ (b) $d = 6$ (c) $d = 8$. Compare with Example 4.5 and Computer Problem 4.1.9. How many correct decimal places of the coefficients can be computed?

Least Square: Nonlinear

3. Find the point (x, y) and distance K that minimizes the sum of squares distance to the circles with radii increased by K, as in Example 4.23 (a) circles with centers $(-1, 0)$, $(1, 0)$, $(0, 1)$, $(0, -2)$ and all radii 1 (b) circles with centers $(-2, 0)$, $(3, 0)$, $(0, 2)$, $(0, -2)$ and all radii 1.

4. Carry out the steps of Computer Problem 3 with the following circles and plot the results
(a) centers $(-2, 0)$, $(2, 0)$, $(0, 2)$, $(0, -2)$, and $(2, 2)$, with radii 1, 1, 1, 1, 2 respectively
(b) centers $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$, $(2, 0)$ and all radii 1.

5. Use the Gauss–Newton Method to fit a power law to the height–weight data of Example 4.10 without linearization. Compute the RMSE.

6. Use the Gauss–Newton Method to fit the blood concentration model (4.21) to the data of Example 4.11 without linearization.

7. Use the Levenberg–Marquardt Method with $\lambda = 1$ to fit a power law to the height–weight data of Example 4.10 without linearization. Compute the RMSE.

8. Use the Levenberg–Marquardt Method with $\lambda = 1$ to fit the blood concentration model (4.21) to the data of Example 4.11 without linearization.

Apply Levenberg–Marquardt to fit the model $y = c_1 e^{-c_2(t-c_3)^2}$ to the following data points, with an appropriate initial guess. State the initial guess, the regularization parameter λ used, and the RMSE. Plot the best least squares curve and the data points.
(a) $(t_i, y_i) = \{(-1, 1), (0, 5), (1, 10), (3, 8), (6, 1)\}$
(b) $(t_i, y_i) = \{(1, 1), (2, 3), (4, 7), (5, 12), (6, 13), (8, 5), (9, 2)(11, 1)\}$

Apply Levenberg–Marquardt to fit the model $y = c_1 e^{-c_2 t} \cos(c_3 t + c_4)$ to the following data points, with an appropriate initial guess. State the initial guess, the regularization parameter λ used, and the RMSE. Plot the best least squares curve and the data points. This problem has multiple solutions with the same RMSE, since c_4 is only determined modulo 2π.
(a) $(t_i, y_i) = \{(0, 3), (2, -5), (3, -2), (5, 2), (6, 1), (8, -1), (10, 0)\}$
(b) $(t_i, y_i) = \{(1, 2), (3, 6), (4, 4), (5, 2), (6, -1), (8, -3)\}$

Differentiation: Difference methods/ Extrapolation

1. Make a table of the error of the three-point centered-difference formula for $f'(0)$, where $f(x) = \sin x - \cos x$, with $h = 10^{-1}, \ldots, 10^{-12}$, as in the table in Section 5.1.2. Draw a plot of the results. Does the minimum error correspond to the theoretical expectation?
2. Make a table and plot of the error of the three-point centered-difference formula for $f'(1)$, as in Computer Problem 1, where $f(x) = (1+x)^{-1}$.
3. Make a table and plot of the error of the two-point forward-difference formula for $f'(0)$, as in Computer Problem 1, where $f(x) = \sin x - \cos x$. Compare your answers with the theory developed in Exercise 18.
4. Make a table and plot as in Problem 3, but approximate $f'(1)$, where $f(x) = x^{-1}$. Compare your answers with the theory developed in Exercise 18.
5. Make a plot as in Problem 1 to approximate $f''(0)$ for (a) $f(x) = \cos x$ (b) $f(x) = x^{-1}$, using the three-point centered-difference formula. Where does the minimum error appear to occur, in terms of machine epsilon?

Differentiation: Automatic Differentiation

What is the accuracy for divided differences on f (x) = x^3 using automatic differentiation?

Show how automatic differentiation in the forward mode computes directional derivatives.

Differentiate the following functions in forward mode:

a. f(x) = $e^{g(x)}$ where $g(x) = \frac{3x^3+2x^2-5x-4}{x^2+1}$ for x=3.
b. f(x) = tan(h(x)) + x^4 -2x^3 where h(x) = e^{2x+1}, for x=3.
c. Compute, using the numerical differentiation method, the derivatives of (a) and (b) for h = 10^{-n}, n = 1, 2, … , k , k+1, where k is the value for which $10^{-(k+1)}$ increases the error.
d. Compare the accuracy of the solutions for the two approached of the derivatives using 10^{-k} in the numerical differentiation.
e. Compare the computation effort of the two methods where your measure of "computation effort" is a term you need to define and is a rational and defensible term. For example, time/unit of accuracy would be a defensible term. "It runs faster" is not. Note that time is hard to measure.

Integration: Newton-Coates – Trapezoidal, Simpson

1. Use the composite Trapezoid Rule with $m = 16$ and 32 panels to approximate the definite integral. Compare with the correct integral and report the two errors.

$$\text{(a)} \int_0^4 \frac{x\,dx}{\sqrt{x^2+9}} \quad \text{(b)} \int_0^1 \frac{x^3\,dx}{x^2+1} \quad \text{(c)} \int_0^1 xe^x\,dx \quad \text{(d)} \int_1^3 x^2 \ln x\,dx$$

$$\text{(e)} \int_0^\pi x^2 \sin x\,dx \quad \text{(f)} \int_2^3 \frac{x^3\,dx}{\sqrt{x^4-1}} \quad \text{(g)} \int_0^{2\sqrt{3}} \frac{dx}{\sqrt{x^2+4}}\,dx \quad \text{(h)} \int_0^1 \frac{x\,dx}{\sqrt{x^4+1}}$$

---

2. Apply the composite Simpson's Rule to the integrals in Computer Problem 1. Use $m = 16$ and 32, and report errors.

3. Use the composite Trapezoid Rule with $m = 16$ and 32 panels to approximate the definite integral.

$$\text{(a)} \int_0^1 e^{x^2}\,dx \quad \text{(b)} \int_0^{\sqrt{\pi}} \sin x^2\,dx \quad \text{(c)} \int_0^\pi e^{\cos x}\,dx \quad \text{(d)} \int_0^1 \ln(x^2+1)\,dx$$

$$\text{(e)} \int_0^1 \frac{x\,dx}{2e^x - e^{-x}} \quad \text{(f)} \int_0^\pi \cos e^x\,dx \quad \text{(g)} \int_0^1 x^x\,dx \quad \text{(h)} \int_0^{\pi/2} \ln(\cos x + \sin x)\,dx$$

4. Apply the composite Simpson's Rule to the integrals of Computer Problem 3, using $m = 16$ and 32.

5. Apply the Composite Midpoint Rule to the improper integrals of Exercise 5, using $m = 10, 100$, and 1000. Compute the error by comparing with the exact value.

6. Apply the Composite Midpoint Rule to the improper integrals of Exercise 6, using $m = 16$ and 32.

7. Apply the Composite Midpoint Rule to the improper integrals

$$\text{(a)} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,dx \quad \text{(b)} \int_0^{\frac{\pi}{2}} \frac{e^x - 1}{\sin x}\,dx \quad \text{(c)} \int_0^1 \frac{\arctan x}{x}\,dx,$$

using $m = 16$ and 32.

8. The arc length of the curve defined by $y = f(x)$ from $x = a$ to $x = b$ is given by the integral $\int_a^b \sqrt{1 + f'(x)^2}\,dx$. Use the composite Simpson's Rule with $m = 32$ panels to approximate the lengths of the curves

$$\text{(a)} \quad y = x^3 \text{ on } [0,1] \quad \text{(b)} \quad y = \tan x \text{ on } [0, \pi/4] \quad \text{(c)} \quad y = \arctan x \text{ on } [0,1].$$

Romberg

1. Use Romberg Integration approximation R_{55} to approximate the definite integral. Compare with the correct integral, and report the error.

$$\text{(a)} \int_0^4 \frac{x\,dx}{\sqrt{x^2+9}} \quad \text{(b)} \int_0^1 \frac{x^3\,dx}{x^2+1} \quad \text{(c)} \int_0^1 xe^x\,dx \quad \text{(d)} \int_1^3 x^2 \ln x\,dx$$

$$\text{(e)} \int_0^\pi x^2 \sin x\,dx \quad \text{(f)} \int_2^3 \frac{x^3\,dx}{\sqrt{x^4-1}} \quad \text{(g)} \int_0^{2\sqrt{3}} \frac{dx}{\sqrt{x^2+4}}\,dx \quad \text{(h)} \int_0^1 \frac{x\,dx}{\sqrt{x^4+1}}\,dx$$

2. Use Romberg Integration to approximate the definite integral. As a stopping criterion, continue until two successive diagonal entries differ by less than 0.5×10^{-8}.

$$\text{(a)} \int_0^1 e^{x^2}\,dx \quad \text{(b)} \int_0^{\sqrt{\pi}} \sin x^2\,dx \quad \text{(c)} \int_0^\pi e^{\cos x}\,dx \quad \text{(d)} \int_0^1 \ln(x^2+1)\,dx$$

$$\text{(e)} \int_0^1 \frac{x\,dx}{2e^x - e^{-x}} \quad \text{(f)} \int_0^\pi \cos e^x\,dx \quad \text{(g)} \int_0^1 x^x\,dx \quad \text{(h)} \int_0^{\pi/2} \ln(\cos x + \sin x)\,dx$$

3. (a) Test the order of the second column of Romberg. If they are fourth-order approximations, how should a log–log plot of the error versus h look? Carry this out for the integral in Example 5.11. (b) Test the order of the third column of Romberg.

Adaptive Quadrature

1. Use Adaptive Trapezoid Quadrature to approximate the definite integral within 0.5×10^{-8}. Report the answer with eight correct decimal places and the number of subintervals required.

$$\text{(a)} \int_0^4 \frac{x\,dx}{\sqrt{x^2+9}} \quad \text{(b)} \int_0^1 \frac{x^3\,dx}{x^2+1} \quad \text{(c)} \int_0^1 xe^x\,dx \quad \text{(d)} \int_1^3 x^2 \ln x\,dx$$

$$\text{(e)} \int_0^\pi x^2 \sin x\,dx \quad \text{(f)} \int_2^3 \frac{x^3\,dx}{\sqrt{x^4-1}} \quad \text{(g)} \int_0^{2\sqrt{3}} \frac{dx}{\sqrt{x^2+4}}\,dx \quad \text{(h)} \int_0^1 \frac{x\,dx}{\sqrt{x^4+1}}\,dx$$

6. Use Adaptive Trapezoid Quadrature to approximate the definite integral within 0.5×10^{-8}.

$$\text{(a)} \int_0^1 e^{x^2}\,dx \quad \text{(b)} \int_0^{\sqrt{\pi}} \sin x^2\,dx \quad \text{(c)} \int_0^\pi e^{\cos x}\,dx \quad \text{(d)} \int_0^1 \ln(x^2+1)\,dx$$

$$\text{(e)} \int_0^1 \frac{x\,dx}{2e^x - e^{-x}} \quad \text{(f)} \int_0^\pi \cos e^x\,dx \quad \text{(g)} \int_0^1 x^x\,dx \quad \text{(h)} \int_0^{\pi/2} \ln(\cos x + \sin x)\,dx$$

7. Carry out the steps of Problem 6, using Adaptive Simpson's Quadrature.

8. The probability within σ standard deviations of the mean of the normal distribution is

$$\frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{-x^2/2}\,dx.$$

Use Adaptive Simpson's Quadrature to find, within eight correct decimal places, the probability within (a) 1 (b) 2 (c) 3 standard deviations.

Gaussian Quadrature

Compute the arc length from t = 0 to t = T for a given T $\leq$ 1.

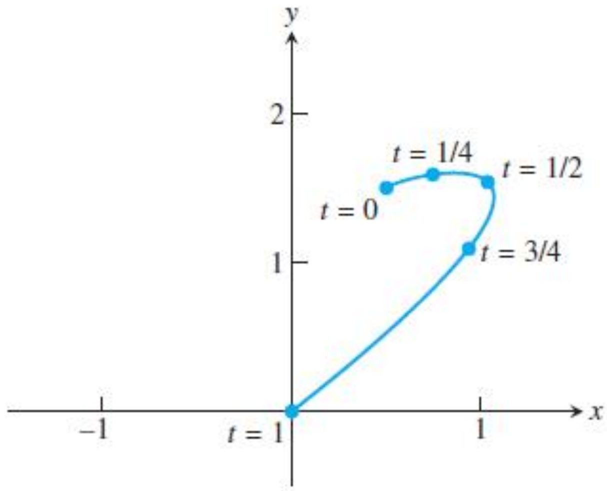

Figure 5.6 Parametrized curve given by Bézier spline. Typically, equal intervals of the parameter t do not divide the path into segments of equal length.

3. Equipartition the path of Figure 5.6 into n subpaths of equal length, for $n = 4$ and $n = 20$. Plot analogues of Figure 5.6, showing the equipartitions. If your computations are too slow, consider speeding up the Adaptive Quadrature with Simpson's Rule, as suggested in Computer Problem 5.4.2.

4. Replace the Bisection Method in Step 2 with Newton's Method, and repeat Steps 2 and 3. What is the derivative needed? What is a good choice for the initial guess? Is computation time decreased by this replacement?

6. Experiment with equipartitioning a path of your choice. Build a design, initial, etc. of your choice out of Bézier curves, partition it into equal arc length segments, and animate as in Step 5.

7. Write a program that traverses the path P according to an arbitrary **progress curve** $C(s), 0 \leq s \leq 1$, with $C(0) = 0$ and $C(1) = 1$. The object is to move along the curve C in such a way that the proportion $C(s)$ of the path's total arc length is traversed between 0 and s. For example, constant speed along the path would be represented by $C(s) = s$. Try progress curves $C(s) = s^{1/3}$, $C(s) = s^2$, $C(s) = \sin s\pi/2$, or $C(s) = 1/2 + (1/2)\sin(2s - 1)\pi/2$, for example.

Consult Wang et al. [2003] and Guenter and Parent [1990] for more details and applications of reparametrization of curves in the plane and space. ✔