Numerical Analysis of the Korteweg de Vires equation

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1 Introduction

A solver for the one dimensional Korteweg-de-Viers equation was written using C and parallized using openmpi. The resultant data was plotted with gnuplot in order to visualize the results. On inspection the plots are consistant with results for the methods and initial conditions used. The algorithms used are embarssingly parallel and so parallisation was trivial, and resulted in a noticable speed up in the code.

2 KdV Equation

The Korteweg-de-Vires is a one dimensional partial differential equation, of the form

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0$$

The equation is primarily used to model waves in shallow water, and other similar phenomena. The equation can be analytically solved via the inverse scattering transform, or one of several numerical methods. The stable solutions of the KdV equation take the form of solitons, waves that move with a fixed shape.

2.1 Initial Conditions

In order to solve the KdV equation one must start with some sort of initial condition defining the state of the system. In this case the initial condition used was

$$u(x) = -12 \cdot csch(x)^2)$$

the values of x used were x from $x = -1.5\pi$ to $x = 1.5\pi$ by steps of $\pi/32$.

3 Numerical Analysis

The techniques used in solving this problem were numerical differentiation using a 5 point finite difference scheme and Runga-Kutta forth order(rk4) time stepping. The KdV equation requires the first and third derivatives of u with respect to x these are calculated using the 5 point finite difference formula

$$\frac{\mathrm{d}u_n}{\mathrm{d}x} = \frac{C_1 u_{n-2} + C_2 u_{n-1} + C_3 u_n + C_4 u_{n+1} + C_5 u_{n+2}}{C_6 \Delta x^{C_7}}$$

in this equation $C_1 - C_7$ constants dependent on the order of the derivative and δx is the x distance between u_n & u_{n+1}. derivatives of u have been found the time step of u can be calculated via the equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u$$

by using rk4 time steping.

4 Code

4.1 Code structure

There are several pieces of source code that were written for this project. Initally code was written in a generic way, the code for the rk4 algorithm was written using function pointers, as was much of the code for the kdv implementation. The reason for this was to let the program be extensible, so for example different in initial conditions could be used. This code was difficult to work with, as it resulted in a large number of variable parameters and function pointers. This code can be found int the kdv.h,kdv.c,main.c and calculus.c and h files. An attempt was made to simplify this code using a structure called udis(for u-discreet) to consolidate much of the variable parts of the code into one manageable block. This also proved to be too difficult to work with. This code can be seen all files with udis in the name. Eventually an attempt was made to write a simple and limited, but working version of the code. This code is located in the kdv-simple.c file, with some code from calculus.c used. This code uses specifics of the kdv equation to simplify much of the code, the rk4 algorithm does not have a time dependent part, and works on full arrays and all function pointers were removed from the code.

4.2 Parallelization

An attempt was made to insure all functions were pure, that is the functions do not modify their arguments. This is the approach used in many functional programming languages such as lisp,haskell,ml,etc. The advantage of this technique is that it makes paralleling the code trivial, for example the function u_discrete which calculates $-\partial_x^3 u + 6u\partial_x u$ can be run in a parallel for loop as the original array is not modified. While in this particular program there was not a huge benefit from this in more complicated programs the benefits of functional programming techniques can be significant as it eliminates much of the trouble with parallel programming, which is coordinating access to mutable data between threads.

4.3 Data Manipulation

Initially the program wrote the data at every time step to a file, this significantly increased the runtime of the program and made working with the data near impossible. So the code now writes the data to file at a predetermined interval which should be set based on the size of the time interval. The data is written to a file named Number.txt where number is the number of time steps so far padded with leading 0's if necessary to reach 6 digits. This data is stored in a directory called kdv-data. The data is visualized using gnuplot, the shell script gnuplot.sh generates a gnuplot script plot.gpi using the current files in kdv-data(and so must be re-run if the data is changed). The file plot.gpi is a script that will generate an animated gif called animate.gif that shows the progression of u through time. The gnuplot.sh script can be modified(via commenting/uncommenting lines) to generate a single .png image containing a superposition of all of the data.

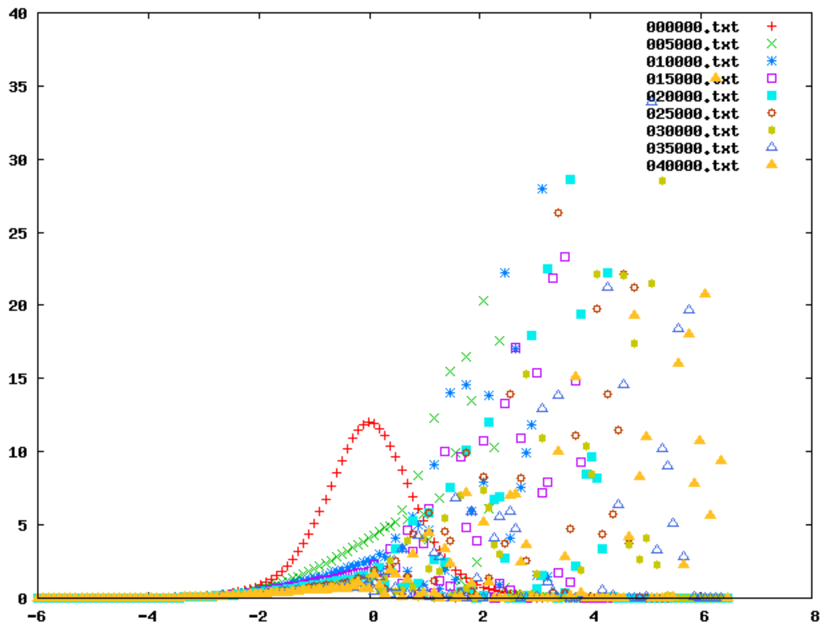
5 Data

Unfortunately despite the best efforts of the author, the data generated does not match the expected values. What is expected is a soliton solution, consisting of a single pulse moving from left to right. As can be see in the example plot below this is not at all like the data that was generated.

5.1 Error Analysis

It is likely that the cause of this error is a small programming error. The author believes the basic algorithms used to have been programmed correctly and reasonable values used for the initial conditions and the space and time steps. Because the data is generated via iteration a small error (for example the addition/omission

Figure 1: KdV data



of a minus sign or accidental shifting of array indices) could cause the results to be completely different than the expected values.

6 Conclusion

While the actual result of this project was not achieved much of the work done was solid and the knowledge gained through this project will be useful in the future. A simple analysis of the development process and its errors would be that the code was developed using bottom up programming which ultimately did not fit the programming language being used. A much more in-depth analysis is included as an appendix. A key lesson learned from this project is to start with writing specific code (in this case code specific to the kdv equation) and then after that code works to generalize to different problems (in this case other 1-D partial differential equations). Ultimately this project could have benefited from a clearer structure from the outset which likely would have enabled more accurate results.

7 Appendix:Analysis of Development

Ultimately the development process for this project was flawed considering the programming language used. The way in which the author writes code is conducive to the use of a bottom up style of development. This style of development is best done using a functional style, with incremental testing using a read eval print loop(repl). The repl used in development was cling a c++ interpreter developed by cern for use in the root programing environment. Cling is likely the most sophisticated c/c++ currently available, it is based off llvm and its c compilier clang and can run almost all c code in an interactive environment (the interpreter is for c++ but as c++ is a near superset of c, c code runs fine). The issue with this development technique for c code comes not from a lack of an interpreter as one might think but from the language itself. As functions in c are a sort of second class object(they can't be created at runtime but can be passed as arguements via function pointers) traditional functional programming techniques are difficult in c. The biggest difficulty in using bottom up programming in c is that incremental testing is very difficult for several reasons, for example the fact that c code will not compile if references functions or datatypes that have not been declared yet, this forces functions and datatypes to be determined first, and losing the fluidity and flexibility of bottonm up programming. A final note on the use of c is that of macros, coming from lisp c macros are very dangerous, in lisp macros are a fundamental, flexible and ultimately safe part of the language that allow code to be generated dynamically. C macros are ultimately just text substitution, the c pre-processor does not understand c and so macros are not subject to the rules of c which can easily lead to broken code. Without macros bottom up programming is difficult, and somewhat more inefficient, yet using c macros comes with a set of issues that make their use prohibitive.

There are several solutions to these issues to be considered in considered in development of future high performance programs. There are two obvious solutions, either do not use C or do not use functional programming. The latter of these is not a likely solution as it violates the way in which the author thinks of programs. The former however is viable, while traditionally functional programming languages have been seen as slow advancement in compiler technology and programming techniques have rendered this untrue. Good performance can be obtained with a number of functional programming languages, but only one can achieve performance equal or better that c, that language is ats. ATS uses theorem proving to prove the safety of things like array bounds, pointer arithmetic, buffer overflows and division by zero at compile time, this allows the compiled program to achieve speeds equal to equivalent c code. Another benefit of ATS is that it pushes error checking to compile time forcing incremental testing. Using ATS is the best solution as it allows functional programming with no speed penalty, however if for some reason this is not a viable option better use of cling and safe use of the c pre-processor could be used to mitigate some of the challenges faced during programming in this project