8.7 Problems

1. A particle is confined to move along the surface of a hemispherical cavity. Using the conservation of angular momentum, show that the particle behaves as if it is in the effective potential

$$U_e(\theta) = \frac{L^2}{2m\ell^2 \sin^2\theta} - mg\ell\cos\theta,$$

where m is the mass of the particle, L is its angular momentum, ℓ is the radius of the cavity, and θ refers to the spherical coordinate. Determine the equilibrium position of the mass, as well as the frequency of small oscillations about this equilibrium. Indicate using a diagram how the mass evolves in time given some initial conditions.

2. The orbit of a mass is given by $r = a(1+\cos\theta)$.

   (a) Determine the central force which leads to this orbit.

   (b) Determine the total cross section of capture if the object is under the influence of this force and starts out with a speed v at infinity. You do not need to find the differential cross section to do this part.

3. Consider a mass orbiting in a circular trajectory under the influence of an attractive central force $F(r) = \frac{1}{r^n}$. What constraint must be placed on n for the orbits to be stable?

4. Consider the earth-sun system. To account for interstellar dust, we assume that the universe is isotropically filled with fine dust of density ρ. We will also assume that the mass of the earth is m and the mass of the sun is M.

   (a) Show that the net central force acting on the earth is

$$F_{net} = -\frac{GMm}{r^2} - mkr,$$

   where k is a constant for you to find. Assume that the earth does not lose energy or pick up dust as it orbits the sun.

   (b) Show that for small radial perturbations from a stable circular orbit of radius R_0, the motion of the earth can be modeled as a precessing ellipse. Assuming that the force due to the dust is much smaller than the force due to the sun, show that the frequency at which the ellipse precesses is given by

$$\Omega \approx 2\pi\rho\sqrt{\frac{R_0^3 G}{M}}.$$

5. When one adds a correction δU to the potential $U = -\frac{\alpha}{r}$, the bound orbits of a planet-star system are no longer closed. Instead, as you saw in the previous example, the orbits exhibit precession.

   (a) Show that we can write the angle as a function of radius in the following way:

$$\Delta\theta = -2\frac{\partial}{\partial L}\int_{r_{min}}^{r_{max}} \sqrt{2m(E-U) - \frac{L^2}{r^2}}\, dr.$$

   This allows us to expand the integrand as a Taylor series to first order in δU. Note that writing $\Delta\theta$ in terms of a derivative helps to avoid divergences in the integral.

   (b) Show that after a full period the ellipse rotates by a small angle, which is approximately given by

$$\delta\theta \approx \frac{\partial}{\partial L}\left(\frac{2m}{L}\int_0^{\pi} r^2 \delta U \, d\theta\right)$$

   (c) Calculate this angle for $\delta U = \frac{\beta}{r^2}$ and $\delta U = \frac{\gamma}{r^3}$. (You can also obtain an exact solution for $\delta U = \frac{\beta}{r^2}$).

6. An object elliptically orbits a planet under the influence of gravity. At the perigee (the closest point to the planet), the object fires its thrusters with an impulse I in the radial direction, transitioning it to another elliptical orbit. Determine the new eccentricity, semimajor axis, and orientation of the orbit in terms of the previous values, e and a. You may also express your answers in terms of the initial energy of the object E, as well as the initial angular momentum about the planet L.

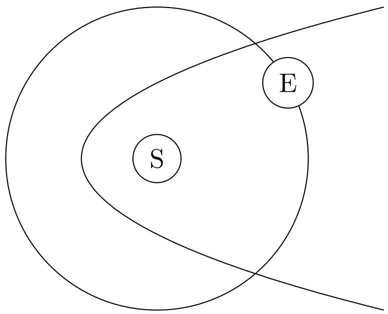

Figure 8.17: A comet spends some time inside Earth's orbit as it passes the sun.

7. Figure 8.17 depicts a comet moving in a parabolic trajectory about the sun, reaching a minimum distance before flying off to infinity. The earth orbits around the sun, with an orbital period T. What is the maximum time the comet can spend within the circular trajectory defining Earth's orbit? Assume that the interaction between the comet and the Earth is negligible.

8. Consider a V-shaped structure with its apex at the origin and an apex angle of χ, as in Figure 8.18. A particle at a distance R has an initial velocity vector (v_x, v_y) such that $v_x < 0$ and $v_y \neq 0$. A central force with potential energy $-\frac{\beta}{r^3}$ acts on the particle.

   (a) Determine the minimum distance between the particle and the origin.

   (b) Under what condition will the particle reach the origin, and under what condition will it eventually leave the structure?

9. Using the conservation of angular momentum, rewrite the equation of motion for the relative position in a two-body orbit as

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{L^2}F(r) + r.$$

10. N particles are placed in an N-gon whose radius is R. The particles all have equal mass m.