Ex2 8

Dienstag, 20. Oktober 2020 18:32

8. Let

$$\begin{aligned} I_n : C([a,b]) &\to C([a,b]) \\ f &\mapsto \sum_{i=0}^{n} f(x_i) l_i(\cdot) \end{aligned}$$

be the Lagrange interpolation operator. Prove that its operator norm defined by

$$\|I_n\|_\infty := \sup_{f \in C([a,b]), f \neq 0} \frac{\|I_n f\|_\infty}{\|f\|_\infty}$$

evaluates to

$$\|I_n\|_\infty = \sup_{x \in [a,b]} \sum_{i=0}^{n} |l_i(x)|.$$

Hint: part of the work was already done in class

let $f(x)$ only be functions where

$$\|f(x)\|_\infty = 1$$

This restriction is allowed as we can scale any $f(x)$ with $\alpha \in \mathbb{R}$ such that

$$\|\alpha f(x)\|_\infty = \alpha \|f(x)\|_\infty = 1$$

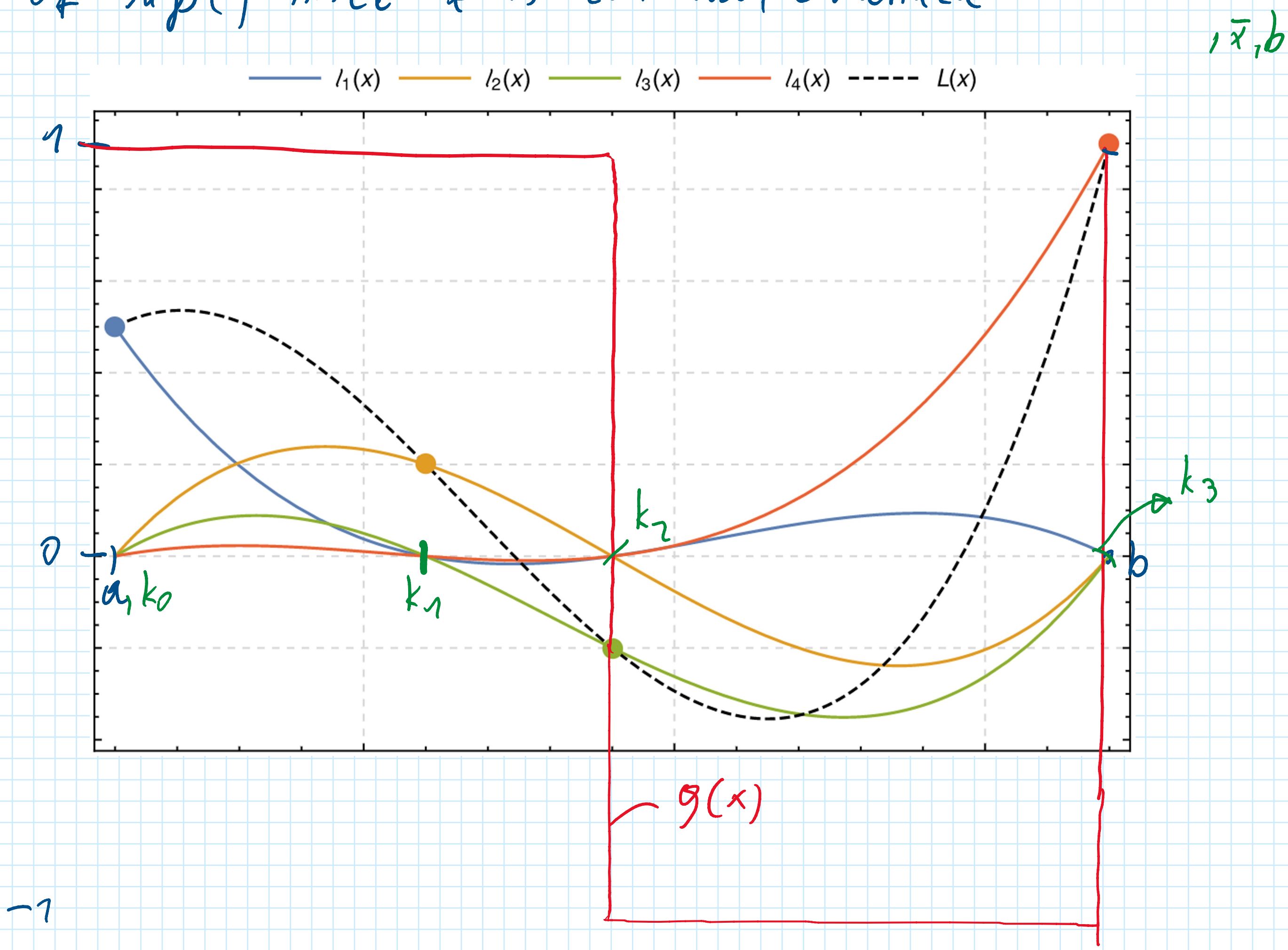
Also I_n is a linear operator

$$\Rightarrow \sup \frac{\|I_n \alpha f\|_\infty}{\|\alpha f\|_\infty} = \sup \frac{\alpha}{\alpha} \frac{\|I_n f\|_\infty}{\|f\|_\infty}$$

Next we define $g(x)$ such that

$$g(x) = \begin{cases} 1 : & k_i < x < k_{i+1} \text{ if } f(k_i) \geq 0 \\ -1 : & \text{otherwise} \end{cases}$$

furthermore we use max() instead of sup() since f is contious, bounded



$$\Rightarrow |f(x)| \leq |g(x)| \quad \forall x \in [a,b]$$

Since $\|f\|_\infty = \|g\|_\infty \overset{!}{=} 1$

$$\Rightarrow \exists \bar{x} \in [a,b] : f(\bar{x}) = g(\bar{x}) = 1$$

$$\Rightarrow \sup \frac{\|I_n f\|}{\|f\|_\infty} = \max \underbrace{\frac{1}{\|f\|_\infty}}_{\overset{!}{=}1} \sum^{i} |f(k_i) L_i| =$$

$$= \max \sum^{i} \Big| \underbrace{\frac{g(k_i)}{\|g\|}}_{=1} L_i \Big| = \underline{\underline{\max \sum^{i} |L_i| \quad \square}}$$

$$\underbrace{\|g\|}_{=\pm 1}$$