FAST, CS-4084

Lecture Notes of Quantum Computing

Dr. Faisal Aslam

Faisal Aslam

August 2023

Contents

1 Dirac's Notation and Tensor Product

Bra-ket notation, also known as Dirac notation, is a mathematical notation used in quantum computing and quantum mechanics to represent vectors and matrices. It was introduced by an amazing physicist Paul Dirac and has become a fundamental tool for expressing quantum concepts concisely and efficiently, thus saving time and space in mathematical descriptions.

1.1 Ket Notation

The ket $|0\rangle$ represents a 2-dimensional vector with a value of 1 in its 0th location and 0 in the other location:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.1}$$

Similarly, the ket $|1\rangle$ is a 2-dimensional vector with a value of 1 in its 1st location and 0 in the other location:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2}$$

Together, $|0\rangle$ and $|1\rangle$ form the standard **basis** for a 2D **vector space**. This means that any 2D vector can be expressed as a linear combination of these basis vectors: $\begin{pmatrix} a \\ b \end{pmatrix} = a\,|0\rangle + b\,|1\rangle$, where a and b are scalar coefficients.

1.1.1 Example

Let's write the vector $\begin{pmatrix} i\sqrt{\frac{2}{3}} \\ -i\frac{1}{\sqrt{3}} \end{pmatrix}$ using the standard basis.

Solution

$$\begin{pmatrix} i\sqrt{\frac{2}{3}} \\ -i\frac{1}{\sqrt{3}} \end{pmatrix} = i\sqrt{\frac{2}{3}}\,|0\rangle - i\frac{1}{\sqrt{3}}\,|1\rangle$$

It's important to note that in the standard basis, each vector has only one nonzero entry while the rest of the entries are zeros. The aforementioned concept extends beyond 2D vectors through the use of the tensor product. Let's quickly learn about tensor product.

1.2 Tensor product

Tensor products are a world in their own. In this context, our emphasis is on utilizing them to construct larger matrices from smaller ones.

Example

Given the following two matrices A and B. Find their tensor produce. That is find $A \otimes B$.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Solution

$$A \otimes B = \begin{pmatrix} 0 \times B & -1 \times B \\ 1 \times B & 0 \times B \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

Properties

- Associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
- Distributed: $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
- Scalar floats freely: $(aA) \otimes B = a(A \otimes B) = A \otimes (aB)$

1.3 Extending Ket Notation

By utilizing the tensor product, we can build upon the previous definitions of $|0\rangle$ and $|1\rangle$. Put simply, for any i and j belonging to the set $\{0, 1\}$, the notation $|ij\rangle = |i\rangle \otimes |j\rangle$ represents a vector. This vector has $2^2 = 4$ elements, where the element at the ij-th location has a value of 1, while the rest of the elements are set to zero. For example, $|10\rangle = |1\rangle \otimes |0\rangle$ has a 1 at the 2nd location, with the remaining entries being zeros.

$$\begin{aligned}
|10\rangle &= |1\rangle \otimes |0\rangle \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0\,|0\rangle \\ 1\,|0\rangle \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

Using this extended notation, we can create a standard basis for 4D vectors: $\Big\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \Big\}$. Thus, any 4D vector can be expressed as a linear combination of these basis vectors:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

where a, b, c, and d are scalar coefficients.

1.3.1 Example

Represent $\begin{pmatrix} \frac{i}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ in Bra-Ket notation.

Solution

$$\begin{pmatrix} \frac{i}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{i}{\sqrt{3}}\,|00\rangle + \frac{1}{\sqrt{6}}\,|10\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$$

Similarly, $|0110\rangle$ has $2^4 = 16$ elements, with its 6th element being 1 while the rest of the elements are zeros.

1.4 Bra Notation

Ket notation is used to represent column vectors, whereas Bra notation represents row vectors. Formally, $\langle\psi| = |\psi\rangle^\dagger$, where the $\dagger$ represents the conjugate transpose (also known as the Hermitian transpose) operation. This operation involves taking the transpose of the vector, making it a row vector, and changing the sign of all iotas.

1.4.1 Example

Convert $|\psi\rangle = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ -i\frac{1}{\sqrt{3}} \end{pmatrix}$ to Bra notation.

Solution

$$\begin{aligned}\langle\psi| &= |\psi\rangle^\dagger \\ &= \begin{pmatrix} -i\sqrt{\frac{2}{3}} & i\frac{1}{\sqrt{3}} \end{pmatrix} \\ &= -i\sqrt{\frac{2}{3}}\,\langle 0| + i\frac{1}{\sqrt{3}}\,\langle 1|\end{aligned}$$

We can now represent any row vector using a basis in Bra notation. For instance, the basis of $\left\{\langle 00|, \langle 01|, \langle 10|, \langle 11|\right\}$ can be used to represent the vector $\begin{bmatrix} i\sqrt{\frac{2}{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix} = i\sqrt{\frac{2}{3}}\,\langle 00| - \frac{1}{\sqrt{3}}\,\langle 11|$.

1.5 Matrices in Bra-Ket Notation

Matrices can be conveniently represented using Bra-Ket notation. Let's take a look at an example, specifically $|0\rangle\langle 0|$:

$$\begin{aligned}|0\rangle\langle 0| &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

Observe that $|0\rangle\langle 0|$ yields a 2×2 matrix. In this matrix, the row and column indexed by 0 contain the value 1, while all other entries are zeros. Similarly, for any i and j the matrix $|i\rangle\langle j|$ will have a 1 at the intersection of the i-th row and j-th column, with all other elements being zeros. Let's see another example, to make it more clear. The matrix $|01\rangle\langle 10|$ will be of dimensions $2^2 \times 2^2$ and will have a 1 at the intersection of $(01)_2 = (1)_{10}$-th row and $(10)_2 = (2)_{10}$-th column. That is:

$$
\begin{aligned}
|01\rangle\langle 10| &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

We can create basis to represent matrices, just like vector. For instance, basis to represent 2 matrices is $\Big\{ |0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1| \Big\}$ and basis to represent 4 matrices is $\Big\{ |00\rangle\langle 00|, |00\rangle\langle 01|, |00\rangle\langle 10|, |00\rangle\langle 11|, \ldots \Big\}$.

1.5.1 Example

Represent the matrix $\begin{pmatrix} 0 & 7 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ using Bra-Ket notation.

Solution: $7|00\rangle\langle 01| + |01\rangle\langle 00| + 5|01\rangle\langle 11| + 9|10\rangle\langle 10|$

1.6 Inner Product

The inner product of $|\psi\rangle$ and $|\phi\rangle$, also called the BraKet (it's fun to see it coming together), is mathematically represented as $\langle\psi|\phi\rangle = |\psi\rangle^\dagger \times |\phi\rangle$.

1.6.1 Example

Find the inner product of $|\psi\rangle = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ \frac{-i}{\sqrt{3}} \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

Solution

$$\begin{aligned} \langle\psi|\phi\rangle &= |\psi\rangle^{\dagger} \times |\phi\rangle \\ &= -i\sqrt{\frac{2}{3}} \times \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{3}} \times \frac{i}{\sqrt{2}} \\ &= \frac{-i}{\sqrt{3}} - \frac{1}{\sqrt{6}} \end{aligned}$$

In the above example, please carefully note how the signs of the iotas are changed when computing the Bra of $|\psi\rangle$, whereas the sign remains the same for $|\phi\rangle$.

1.7 Magnitude (Euclidean Norm)

The Euclidean Norm of a vector $|\psi\rangle$ is defined as $\|\,|\psi\rangle\,\| = \sqrt{\langle\psi|\psi\rangle}$.

1.7.1 Example

Find the norm of $|\psi\rangle = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ \frac{-i}{\sqrt{3}} \end{pmatrix}$.

Solution

$$\begin{aligned} \|\,|\psi\rangle\,\| &= \sqrt{\langle\psi|\psi\rangle} \\ &= \sqrt{-i\sqrt{\frac{2}{3}} \times i\sqrt{\frac{2}{3}} + \frac{i}{\sqrt{3}} \times \frac{-i}{\sqrt{3}}} \\ &= \sqrt{\frac{2}{3} + \frac{1}{3}} \\ &= 1 \end{aligned}$$

In the above example, please carefully note how the signs of the iotas are handled.

1.8 Unit Vector

A vector $|\psi\rangle$ is called a unit (or normalized) vector if its norm is 1: $\| \, |\psi\rangle \, \| = 1$.

For instance, the vector $|\psi\rangle = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ \frac{-i}{\sqrt{3}} \end{pmatrix}$ is a unit vector.

1.9 Normalization

An arbitrary vector $|\psi\rangle$ can be convert into a unit vector by dividing it from its norm: $\frac{|\psi\rangle}{\||\psi\rangle\|}$.

1.9.1 Example

Convert vector $|\psi\rangle = \begin{pmatrix} 3i \\ 4 \end{pmatrix}$ to a unit vector.

Solution Let's first calculate its norm: $\| \, |\psi\rangle \, \| = \sqrt{-3i \times 3i + 4 \times 4} = \sqrt{9 + 16} = 5$. Thus, the equivalent unit vector is: $\frac{|\psi\rangle}{\||\psi\rangle\|} = \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix}$

1.10 Orthogonal Vectors

A set of vectors is called orthogonal to each other if the inner product of every pair in the set is zero. For example, $|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ are orthogonal, as $\langle\psi|\phi\rangle = 0$.

1.11 Orthonormal Vectors

A set of vectors is called orthonormal if two conditions are met: i) the inner product of each pair is zero, and ii) each vector is a unit vector. Mathematically,

$$\langle\psi|\phi\rangle = \begin{cases} 0 & |\psi\rangle \neq |\phi\rangle \\ 1 & |\psi\rangle = |\phi\rangle \end{cases}$$

Bibliography