Graph Algorithms

Single-source shortest path in DAG

Shortest paths are always well defined in a DAG, Since there are no negative-weight cycle in a graph

- If the DAG contains a path from u to v, u precedes v in the topological sort
- If u comes before v in the topological order, there is no path from v to u

Single-source shortest path in DAG

DAG-Shortest-Paths(G, s)

Topologically sort the vertices of G

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

for each vertex u, taken in topologically sorted order

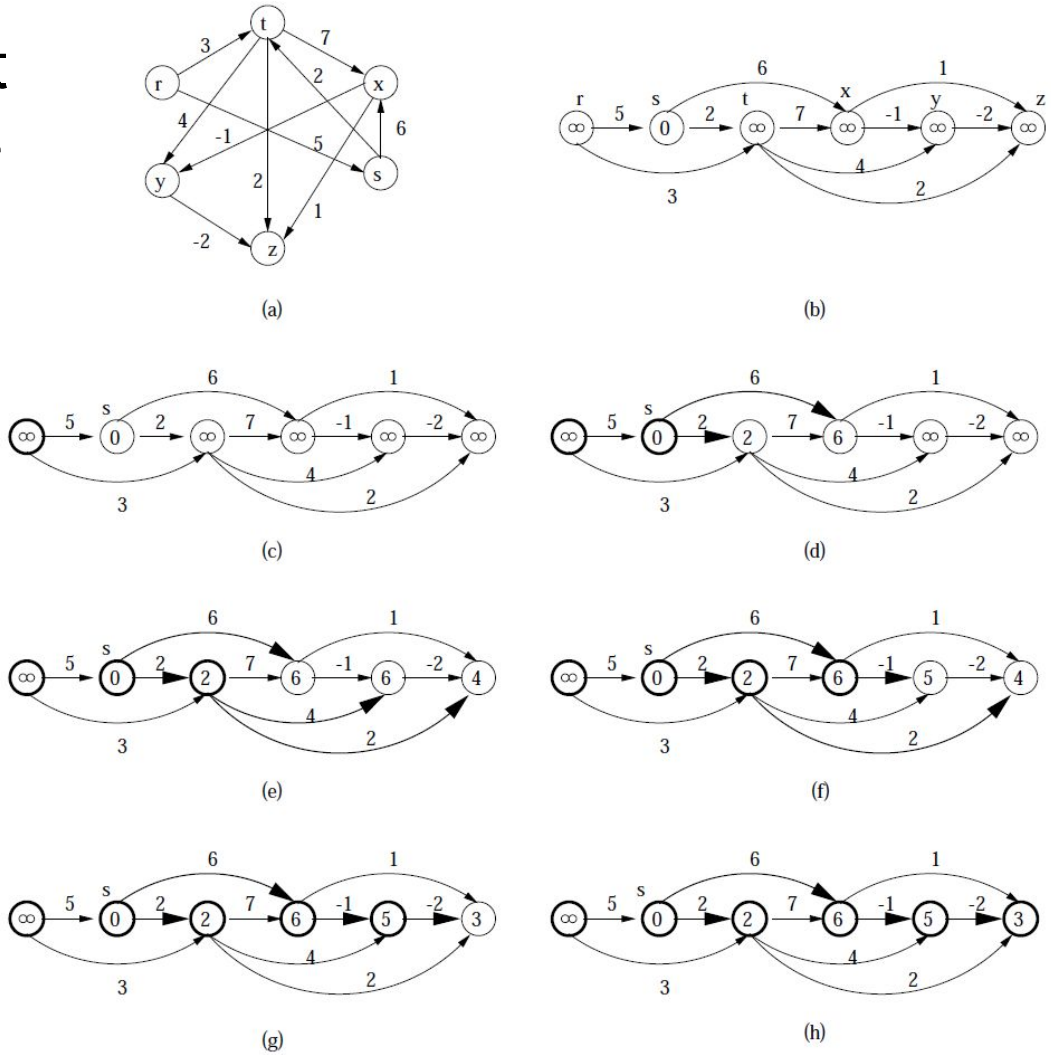
for each $v \in Adj[u]$

if $d[v] > d[u] + w(u, v)$

$d[v] \leftarrow d[u] + w(u, v)$

Single-source shortest path in DAG: Example

Why is it working in graphs with negative edges?



Single-source shortest path in DAG: Runtime: Θ(V+E)

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Θ(V+E)

Single-source shortest path in DAG: Correctness

Theorem. When the algorithm terminates, $d[v] = \delta(s, v)$ for all vertices $v \in V$

Proof.

- If v is not reachable from s, then $d[v] = \delta(s, v) = \infty$
- If v is reachable from s, there is a shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$ where $v_0 = s$ and $v_k = v$.
- The algorithm process the vertices in topologically sorted order
- Therefore, the edges on p are relaxed in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$
- We can prove by induction on the number of relaxation steps that $d[v] = \delta(s, v)$

Single-source shortest path in DAG: Correctness

- **Theorem.** After the k-th edge of path p is relaxed, we have $d[v_k] = \delta(s, v_k)$
- **Proof by induction:** induction on the number of relaxation steps.
- **Induction hypothesis:** After the i-th edge of path p is relaxed, $d[v_i] = \delta(s, v_i)$
- **Base Case: i=0**
  - before any edge of p have been relaxed, we have $d[v_0] = d[s] = 0 = \delta(s, s)$
- **Induction step.** Assuming $d[v_{i-1}] = \delta(s, v_{i-1})$ after the (i-1)-th edge was relaxed → we want to show that $d[v_i] = \delta(s, v_i)$ after the i-th edge is relaxed
  - $\mathbf{d[v_i] \le \delta(s, v_i)}$
    - After relaxing edge (v_{i-1},v_i), we have $d[v_i] \le d[v_{i-1}] + w(v_{i-1},v_i)$
      - before relaxing the edge, there are two cases
        - $d[v_i] > d[v_{i-1}] + w(v_{i-1},v_i)$ if this is the case the algorithm does the following
          - $d[v_i] = d[v_{i-1}] + w(v_{i-1},v_i)$
        - $d[v_i] \le d[v_{i-1}] + w(v_{i-1},v_i)$ if this is the case, no change happen and the property hols
      - $d[v_i] <= d[v_{i-1}] + w(v_{i-1},v_i) = \mathbf{\delta(s, v_{i-1}) + w(v_{i-1},v_i) = \delta(s, v_i)}$ (subpaths of shortest path are also shortest path)
  - $\mathbf{d[v_i] \ge \delta(s, v_i)}$
- Therefore $\mathbf{d[v_i] = \delta(s, v_i)}$

Single-source shortest path in DAG: Correctness

Theorem. d[v] $\geq \delta(s, v)$ for all v

Proof by induction: induction on the number of relaxation steps.

Induction hypothesis: After j relaxation steps, d[v] $\geq \delta(s, v)$ for all v.

Base Case: after initialization, d[v] = $\infty \rightarrow$ d[v] $\geq \delta(s, v)$

d[s] = 0 $\geq$ $\delta(s, s) = 0$

Induction step. Assuming that induction hypothesis is true for j, we want to prove it is true for j+1:

Consider relaxation of edge (u,v). There are two cases:

- d[v] does not change $\rightarrow$ d[v] $\geq \delta(s, v)$ (induction assumption)
- d[v] will change: d[v] = d[u] + w(u,v) $\geq \delta(s, u)$ + w(u,v) $\geq \delta(s, v)$

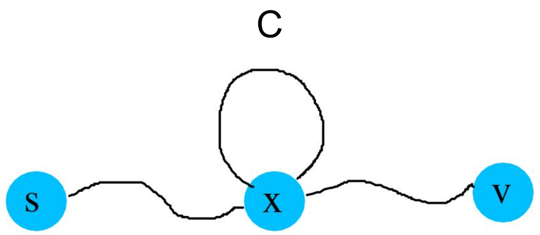
induction

Triangle inequality

Bellman-Ford Algorithm

Bellman-ford Algorithm

- If G has no negative cycles, then there exists a shortest path from s to any node u that uses at most n-1 edges.
- **Proof.** Suppose there exists a shortest path from s to u consisting of n or more edges
  - A path of length at least n must visit at least n+1 nodes
  - There exists a node x that is repeated (pigeonhole principle) → There is a cycle C
  - Can remove C without increasing cost of path



Bellman-ford Algorithm

Intuition. Although Dijkstra's algorithm may not compute all distances in one pass, it will compute the distance to some vertices correctly, e.g. first vertex on a shortest path.

How many iterations of dijkstra algorithm is required?

If there is no negative-weight cycle

- shortest path is a simple path
- Shortest path is of length at most n-1

→At most n-1 iterations of Dijkstra is needed

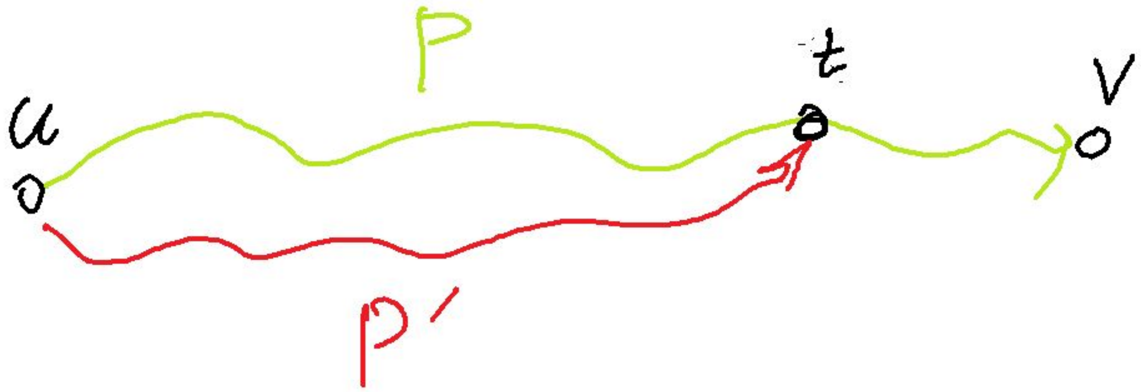
→Each iteration starts at the next node in the shortest path

Bellman-ford Algorithm: dynamic programming approach

- The problem has the optimal substructure property:
  - All subpaths of a shortest path are shortest paths.
- Can we solve the problem using dynamic programming?
- Can we solve the problem recursively?
- What is the subproblem?

Bellman-ford Algorithm: dynamic programming approach

- P = shortest path from u to v with at most i edges
- P = P’ + (t, v)
    - P’: (shortest path from u to t with at most i-1 edge)



Bellman-ford Algorithm: dynamic programming approach

D(i,v) = weight of a shortest path from **s** to **v** that uses at most **i** edges

- Goal D(n-1, v) for each v
  - If there is no negative cycle, then there exists a shortest path that is simple

$$D(i,v) = min \begin{cases} D(i-1,v) & \text{shortest path uses at most i-1 edges} \\ min_{(u,v)\in E}\{D(i-1,u) + w(u,v)\} & \text{shortest path uses exactly i edges} \end{cases}$$

$$D(0,s) = 0$$

$$D(0,v) = \infty \text{ where } v \neq s$$

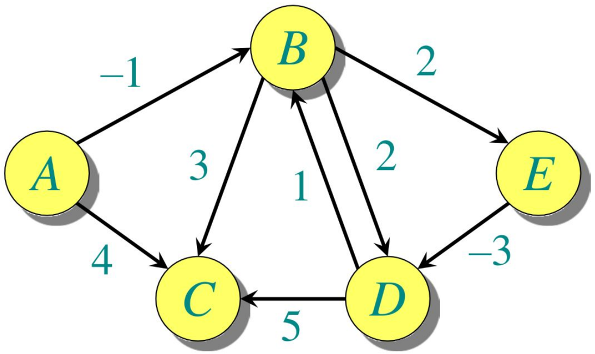
Bellman-ford Algorithm: dynamic programming approach

```
For each node v ∈ V
        M [0, v] = ∞
M [0, s] = 0
for i = 1 to n – 1
        for each node v ∈ V
                M [i, v] = M [i – 1, v]
                for each edge (u, v) ∈ E :
                        M [i, v] ← min { M [i, v], M [i – 1, u] + w(u, v) }.
```

- Runtime: $O(nm)$
- Space Complexity: $O(n^2)$
  - Could be improved to $O(n)$
  - To compute M[i, v] only M[i-1,v] values are needed

Bellman-ford Algorithm: DP: example

	A	B	C	D	E
D(0, v)	0	∞	∞	∞	∞
D(1,v)	0	-1	4	∞	∞
D(2,v)	0	-1	2	1	1
D(3,v)	0	- 1	2	-2	1
D(4,v)	0	-1	2	- 2	1



Bellman-ford algorithm: simple version

```
For each node v ∈ V
        d[v] = ∞
d[s] = 0
for i = 1 to n – 1
        for each edge (u, v) ∈ E :
                If d[v] > d[u] + w(u,v)
                        d[v] ← d[u] + w(u, v)
                        parent[v] = u
```

- Re-use same d[v]
- Runtime: O(nm)
- Space Complexity: O(n)
- The set of {v, parent[v]} form a shortest path tree
- Allows to recover path s to v backward from v
- How to detect negative weight cycle reachable from s
  - Run 1 more iteration and see if any d value changes

Bellman-ford algorithm: Proof of correctness

Theorem. At the end, $D(n-1, v)$ is the cost of the shortest path from s to v with at most n-1 edges for all $v \in V$

Proof. Proof by induction.

Induction hypothesis. $D(k,v)$ is the cost of the shortest path from s to v with at most k edges for all $v \in V$

Base Case: i=0

Induction step: Assuming $D(i-1, v)$ is the cost of the shortest path from s to v with at most i-1 edges for all $v \in V$, prove that $D(i, v)$ is the cost of the shortest path from s to v with at most i edges for all $v \in V$

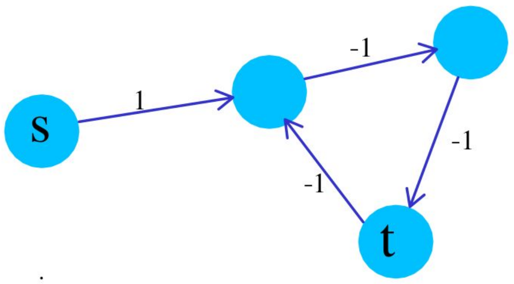
Detecting negative cycles

- Given a directed graph G=(V, E) with edge-weights w_e (can be negative), determine if G contains a negative cycle.
- We reduce this to a slightly different problem and will use Bellman-Ford algorithm to solve it
- **Problem**. Given G and source s , find if there is negative cycle on a path from s to v for any node v

Negative Cycles

Claim 1. If there is a negative cycle on a $s \rightarrow t$ path, then $D(k, v) \rightarrow -\infty$ as $k \rightarrow \infty$ for some $v \in V$

Example: $D(t, 3) = -1$, $D(t, 6) = -4$, $D(t, 9) = -7$



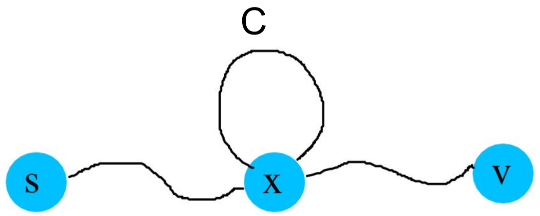
Negative Cycles

- **Claim 2.** If the graph does not have any negative cycle, then D(n, v) = D(n-1, v) for all v ∈ V
  - **Proof**. Any cycle is non-negative, so we can assume that any shortest path from s to v has no cycle and thus it is of length at most n-1

- **Claim 3.** If D(n, v) = D(n-1, v) for all v ∈ V, then the graph has no negative cycles
  - **Proof.** We can show that D(k, v) is finite when k goes to infinity for all v ∈ V
  - By claim 1, there are no negative cycles in graph
  -
- A graph has no negative cycles iff D(n, v) = D(n-1, v) for all v ∈ V
  - →There is an O(mn) algorithm for checking

Algorithm for detecting Negative Cycles

- **Lemma**. If D(n, v) < D(n-1, v) for some v, then any shortest path from s to v contains a negative cycle.
- **Proof.** by contradiction.
  - Suppose G does not contains a negative cycle
  - Since D(n, v) < D(n-1, v), the shortest path from s to v has exactly n edges.
    - Otherwise, D(n, v) = D(n-1, v) (according to the algorithm)
  - By pigeonhole principle, a path of length n must have a repeated vertex, and thus a cycle c. We claim that C must be a negative cycle
  - If C has non-negative weight, removing it would give us a shortest path with less than n edges →contradiction: the path contained exactly n edges.
- there is a negative cycle. How do we find it?



Algorithm for detecting Negative Cycles

- So, to detect a negative cycle reachable from s:
  - We run one more iteration, and check if any d value changes.
  - By tracing out the parents using the stored information, we can find P and thus the cycle C. This gives an O(mn) time algorithm to find a negative cycle, using theta(n^2) space

All pairs shortest path problem

- **Input**.
  - A directed graph G = (V, E) with a weight on each edge
  - The edge weight could be negative, but there is not negative-weight cycle
- **Output**: The shortest path distance from u to v for all pairs of u,v $\in V$.

- **Brute-force solution.**
  - Apply Bellman-Ford on each node u $\in V$
  - **Runtime** = (n. mn) = O(n^2m)
  - **Floyd-Warshal** algorithm: O(n^3)

All pairs shortest path problem: First solution

- Subproblem is a path to the predecessor node. To find the optimal solution, we try all possible predecessor nodes x

$$D_i(u,v) = min \begin{cases} D_{i-1}(u,v) & \text{shortest path uses at most i-1 edges} \\ min_{x \in V}\{D_{i-1}(u,x) + w(x,v)\} & \text{shortest path uses exactly i edges} \end{cases}$$

$$D_i(u,v) = min_{x \in V}\{D_{i-1}(u,x) + w(x,v)\}$$

$$D_0(u,u) = 0$$
$$D_0(u,v) = \infty \text{ where } u \neq v$$
$$D_0(u,v) = w(u,v) \text{ where } (u,v) \in E$$

Runtime: $O(n^4)$

All pairs shortest path problem

- $V = \{1, 2, \ldots, n\}$
- **Subproblems are paths in which all interior nodes are in {1..k-1}**
  - We restrict paths to u
  - To find the optimal solution, try all ways to use node k as an interior node
- $D_k[i, j]$ = weight of shortest ij path using only intermediate vertices in $\{1 \ldots k\}$
  - Goal. finding $D_n[i, j]$
- Let P be a min-weight i,j -path in which all interior nodes are in $\{1, \ldots, k\}$
- There are two cases
  - Case 1: k is not used in P
    - Interior nodes are all in {1, …, k-1}
  - Case 2: k is used in P
    - Interior nodes on paths i to k and k to j are all in {1, …, k-1}

All pairs shortest path problem

- $D_k[i, k]$ = weight of shortest ij path using only intermediate vertices in {1…k}
  - Goal. finding $D_n[i, j]$
- **Base cases:**
  - $D_0[i, j]$: shortest path length from i to j without using intermediate vertices
  - $D_0[i, j] = 0$ if u=v
  - $D_0[i, j] = w(u,v)$ if $(u,v) \in E$
  - $D_0[i, j] = \infty$ otherwise

$$D_k[i, j] = \min \begin{cases} D_{k-1}[i, k] + D_{k-1}[k, j] & \text{use vertex k} \\ D_{k-1}[u, v] & \text{don't use vertex k} \end{cases}$$

- Correctness: this considers all possibilities for k_i. Then induction on i.

All pairs shortest path problem

```
Initialize D_0[i, j] as above
for k from 0 to n-1 do
    for i from 1 to n do
        for j from 1 to n do
            D_k[i, j] := min{D_{k−1}[i, j], D_{k−1}[i, k] + D_{k−1}[k, j]}
```

- Runtime: $O(n^3)$
- Space: $O(n^3)$
    - Need to store two n-by-n arrays, and the original graph.
    - As with Bellman-Ford, we don't really need to store all n of the D_k

All pairs shortest path problem

```
Initialize D_0[i, j] as above
for k from 0 to n-1 do
    for i from 1 to n do
        for j from 1 to n do
            D[i, j] := min{D[i, j], D[i, k] + D[k, j]}
```

- Runtime: $O(n^3)$
- Space: $O(n^2)$

All pairs shortest path problem

- What if we want the actual path?
  - Along with D[u, v], compute Next[u, v] = the first vertex after u on a shortest u to v path.
  - If we update D[u, v] = D[u, i] + D[i, v] then also update Next[u, v] := Next[u, i].
  - Exercise. Check how this works.