Graph Algorithms

Minimum Spanning Tree

Input: a connected, undirected graph G = (V, E) with weights w: E → R on the edges

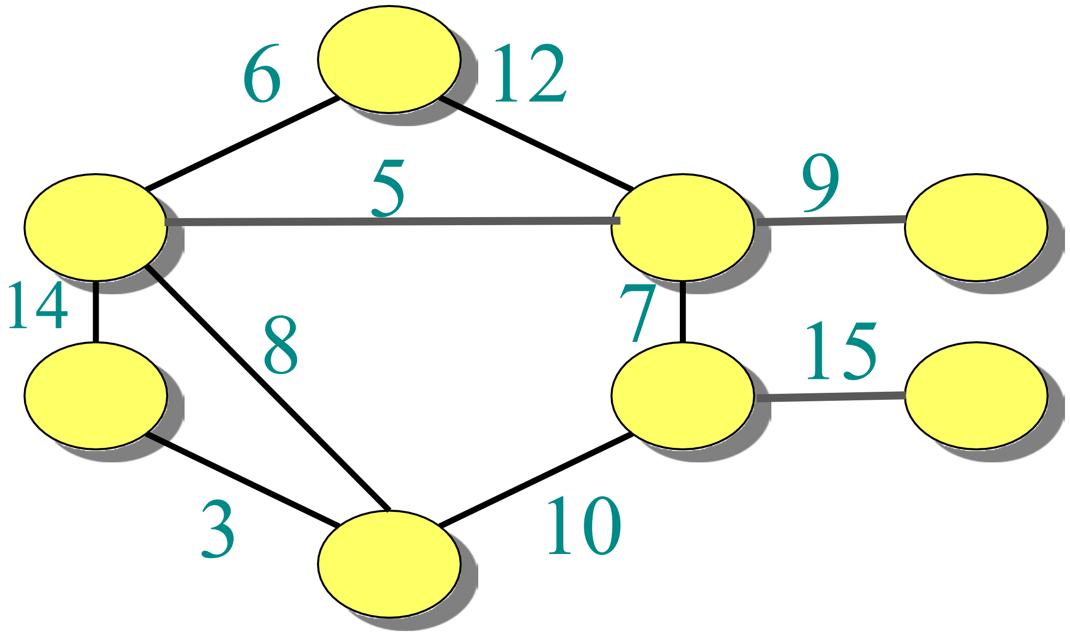
Output: a minimum spanning tree T

- A **spanning tree** of G is a graph (V, T ⊆ E) such that (V,T) is a tree
  - A tree: a connected graph with no cycle
- The weight of a tree:

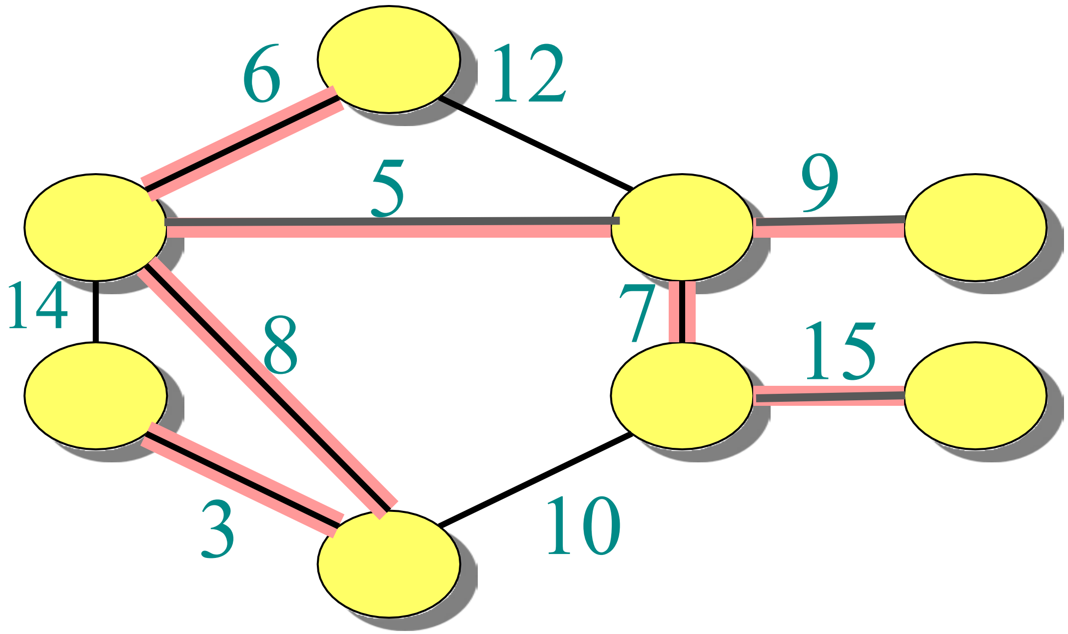
$$w(T) = \sum_{(u,v)\in T} w(u,v)$$

- A **minimum spanning tree**: a tree of minimum weight:
  - subset of edges (of size n − 1) that connects all the vertices and has minimum weight

Example of MST



Example of MST



The edges on spanning tree

The weight of the above tree is 6+5+8+3+7+9+15

Minimum Spanning Trees

There are many greedy algorithms for finding MSTs:

- Borůvka's algorithm (1926)
- Kruskal's algorithm (1956)
- Prim's algorithm (1930, rediscovered 1957)

We will explore Kruskal's algorithm and Prim's algorithm in this course.

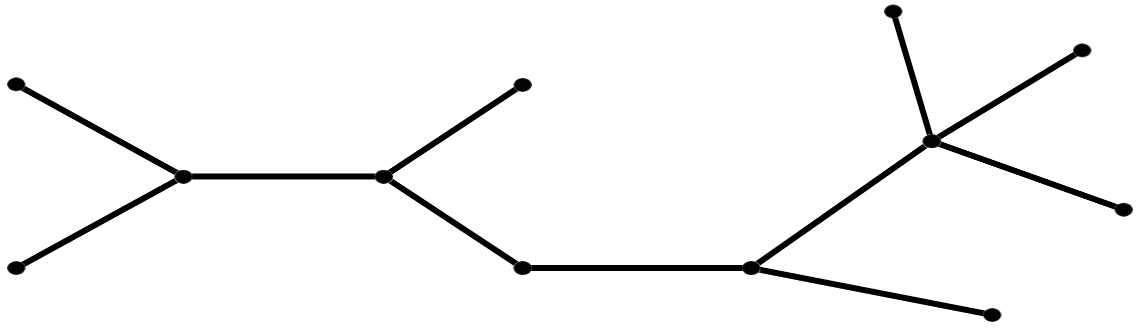
Minimum Spanning Tree

- Can there be more than one minimum spanning tree (MST) for an undirected graph?
    - Yes
- What happens if the graph is unweighted?
    - All spannings trees are minimum spanning trees

Optimal substructure

MST T:

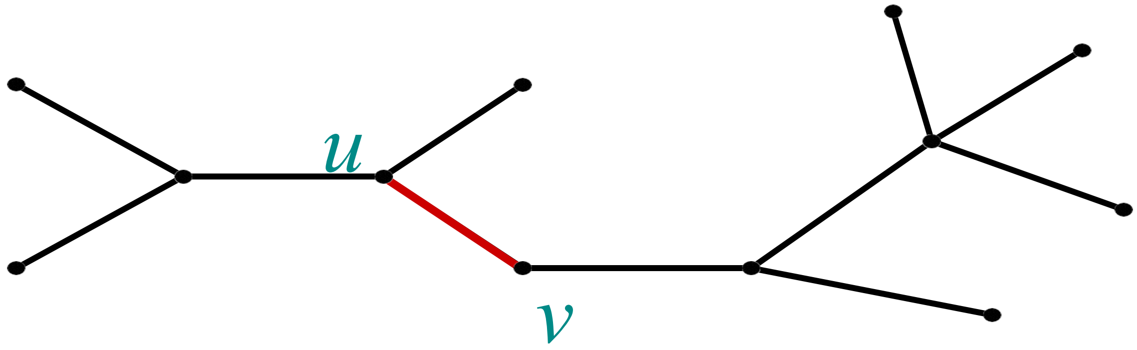
(Other edges of G are not shown.)



Optimal substructure

MST T:

(Other edges of G are not shown.)

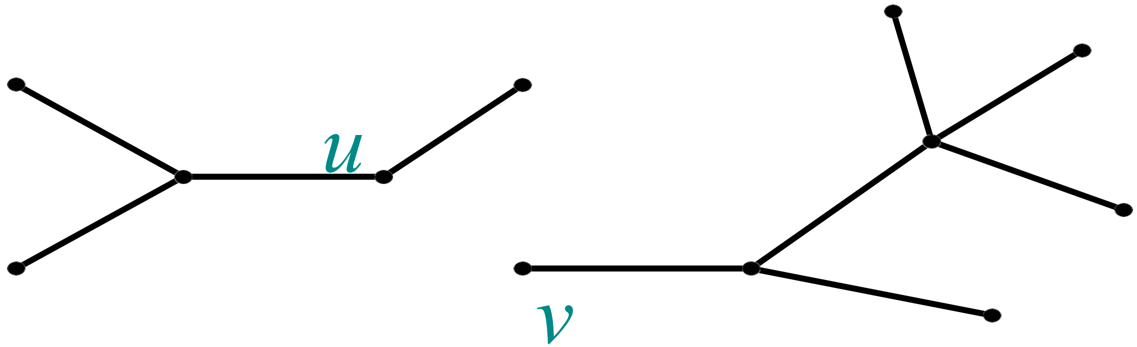


Remove any edge $(u, v) \in T$.

Optimal substructure

MST T:

(Other edges of G are not shown.)

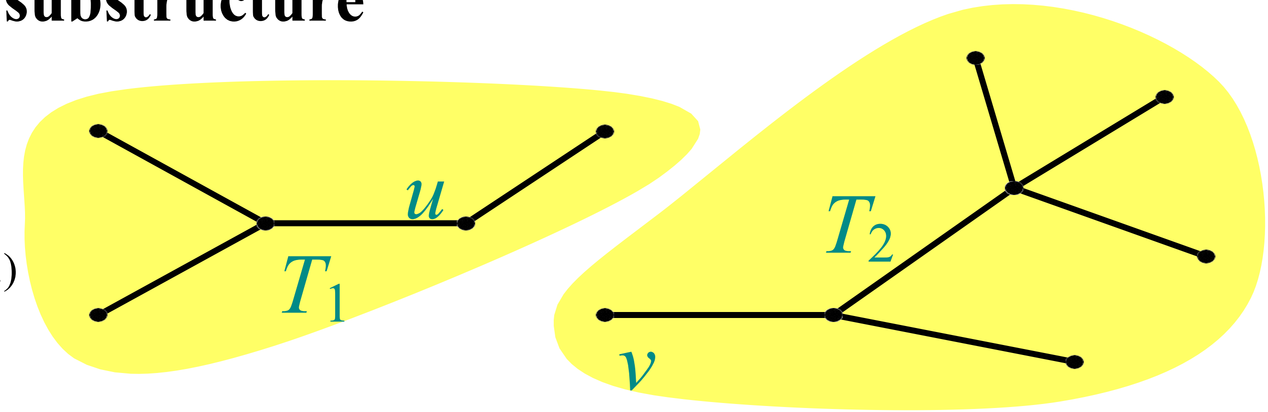


Remove any edge $(u, v) \in T$.

Optimal substructure

MST T:

(Other edges of G are not shown.)



Remove any edge $(u, v) \in T$.

Then, T is partitioned into two subtrees T_1 and T_2.

Optimal substructure

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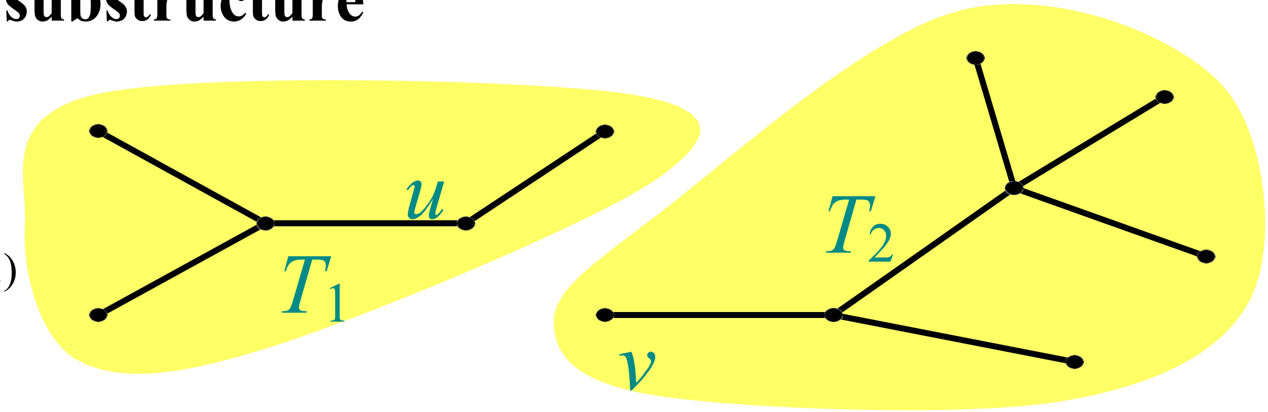


Remove any edge $(u, v) \in T$.

Then, T is partitioned into two subtrees T_1 and T_2.

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2.

Proof of optimal substructure

Proof. Cut and paste:

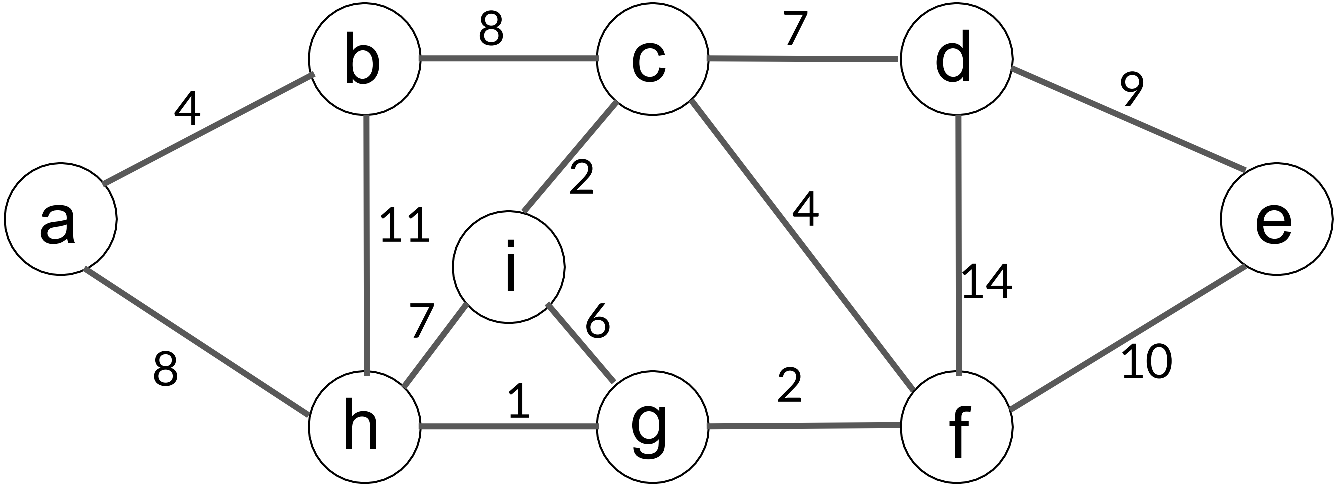
$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G.

Contradiction: since T was the minimum spanning tree for G

Kruskal Algorithm

T= Ø
Repeat
- find the least-weight edge (u,v) so that u and v are not connected in T
- add (u,v) to T



Kruskal Algorithm

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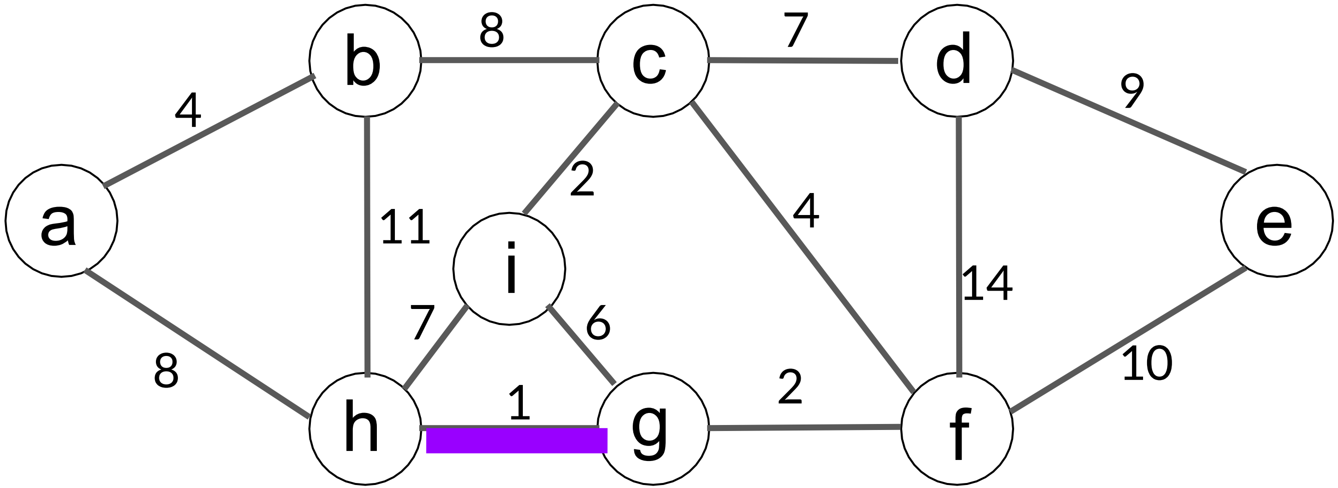

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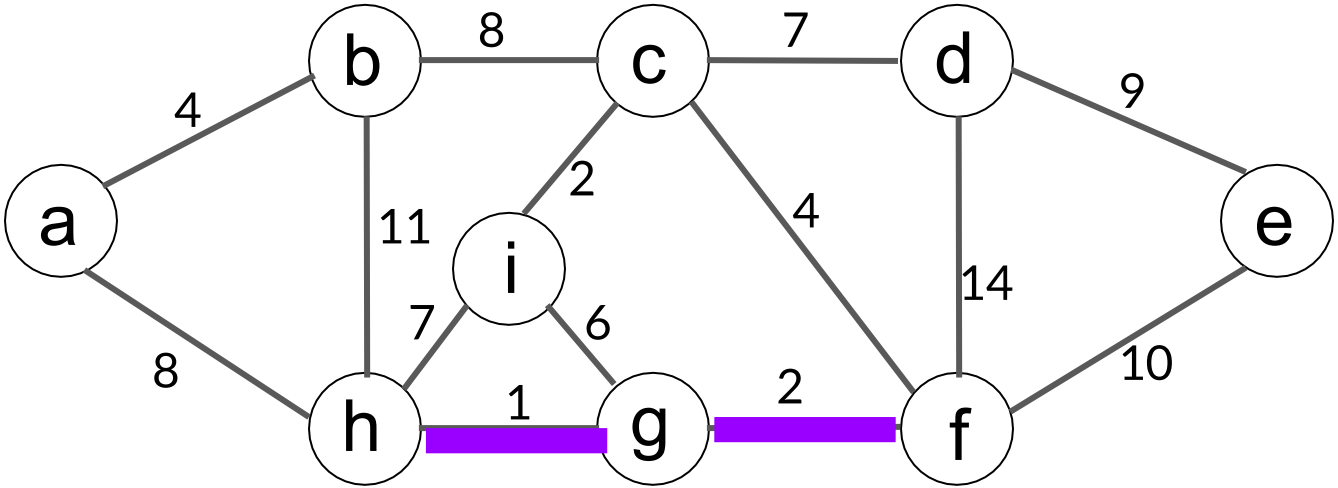

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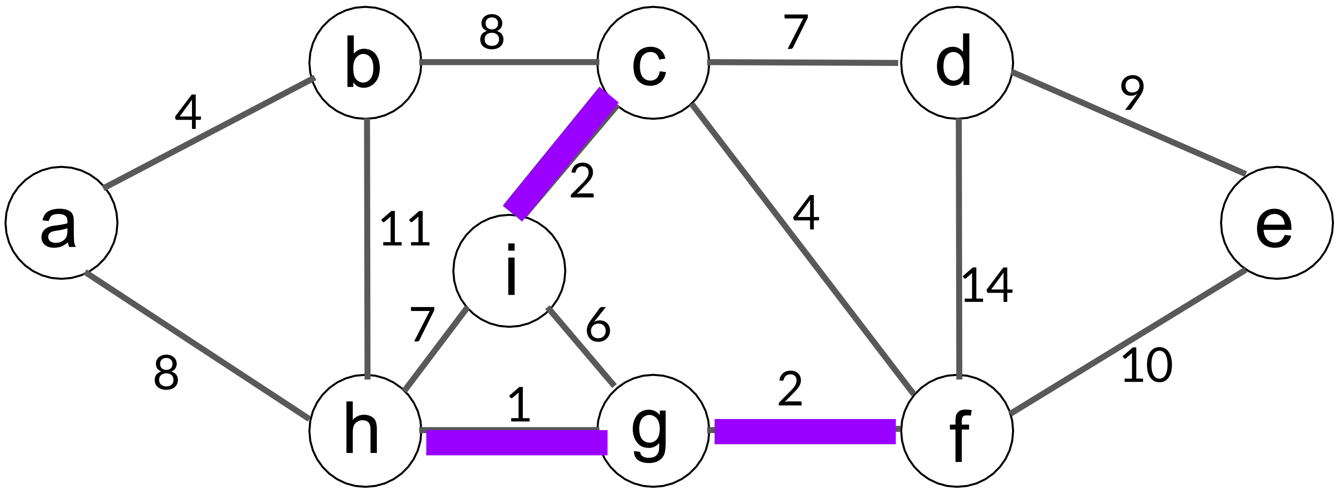

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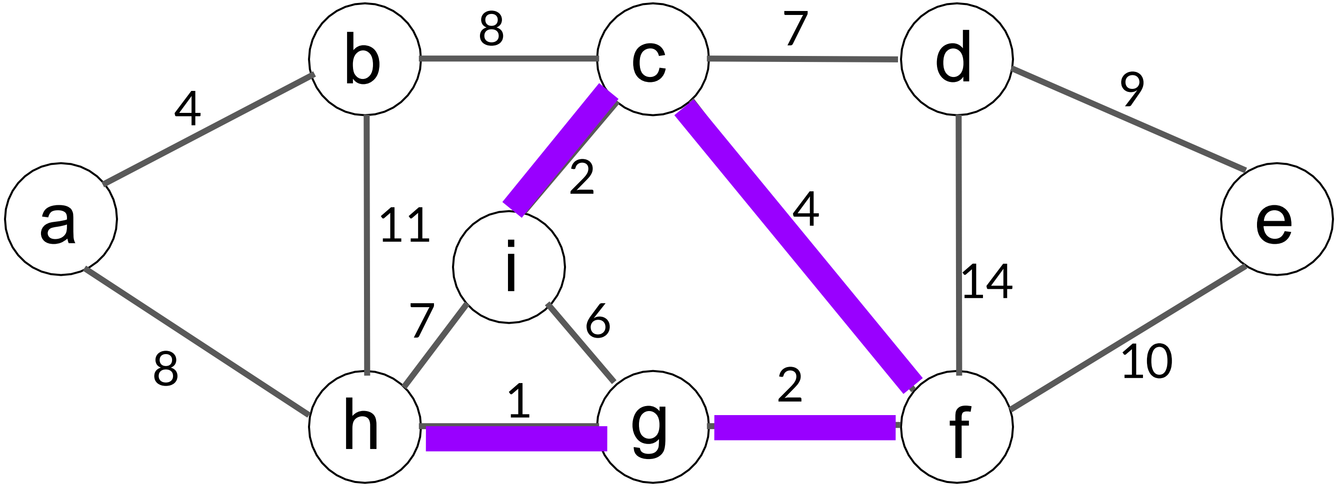

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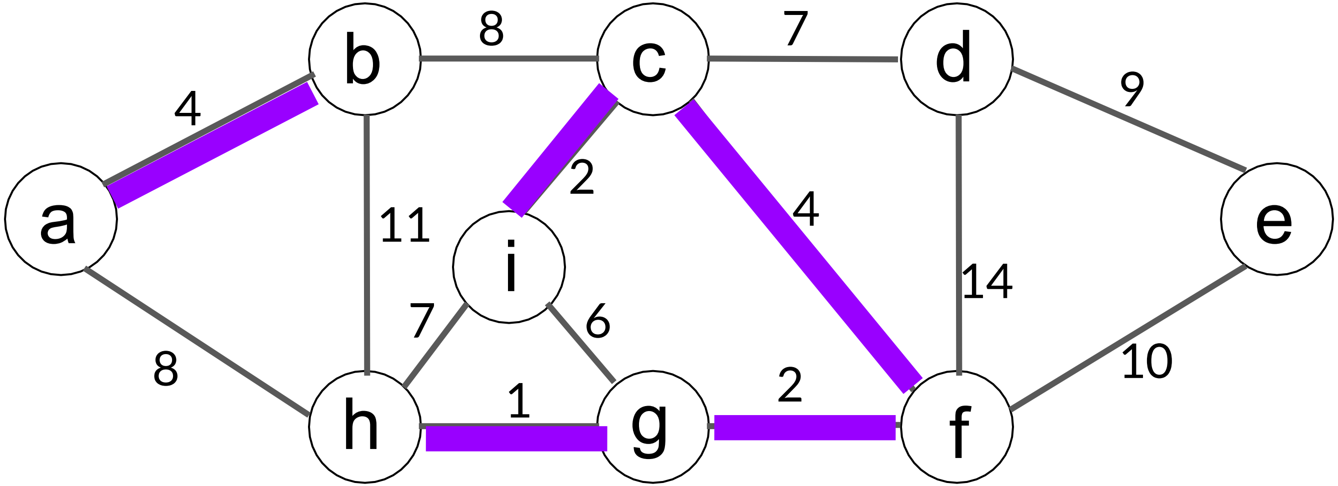

Kruskal Algorithm

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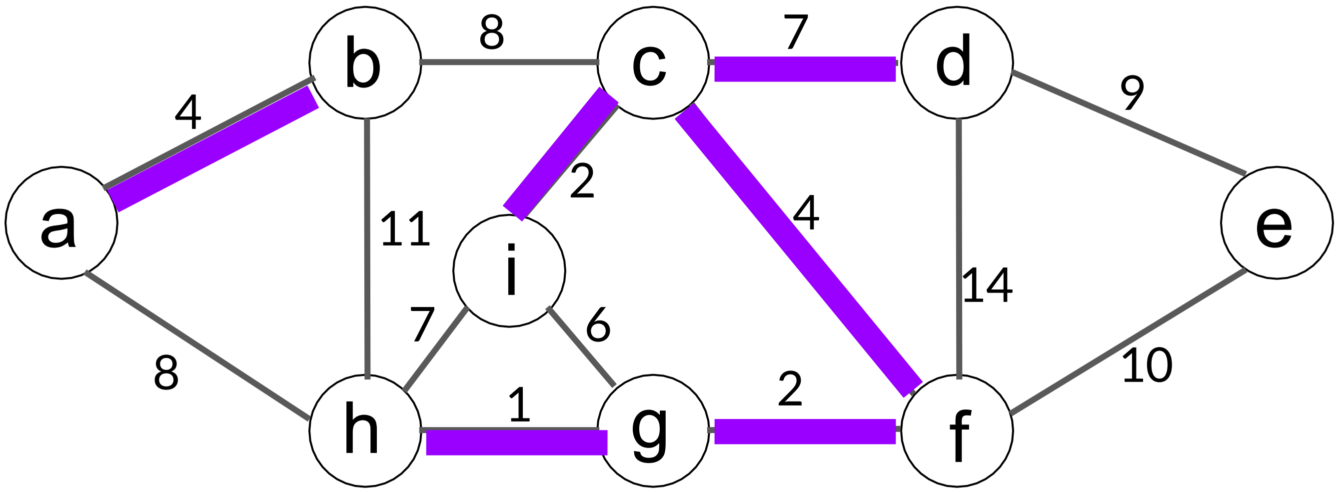

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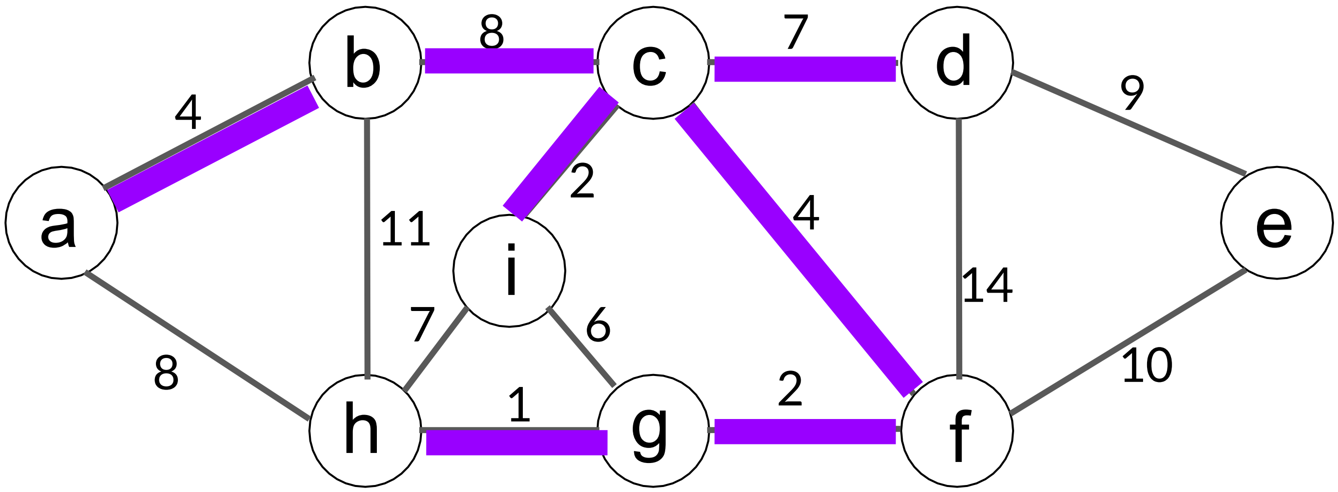

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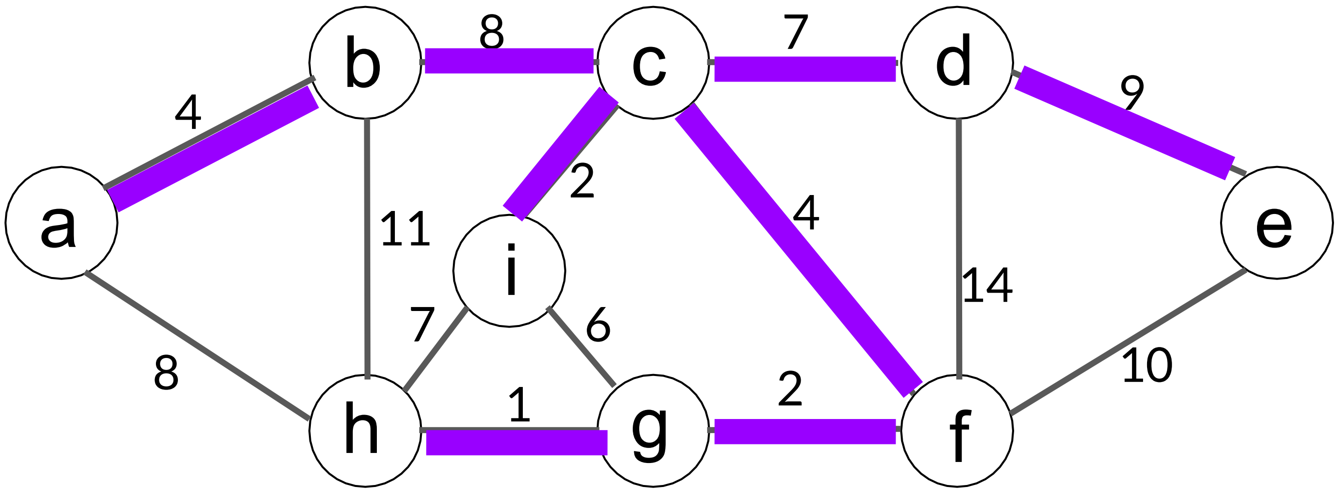
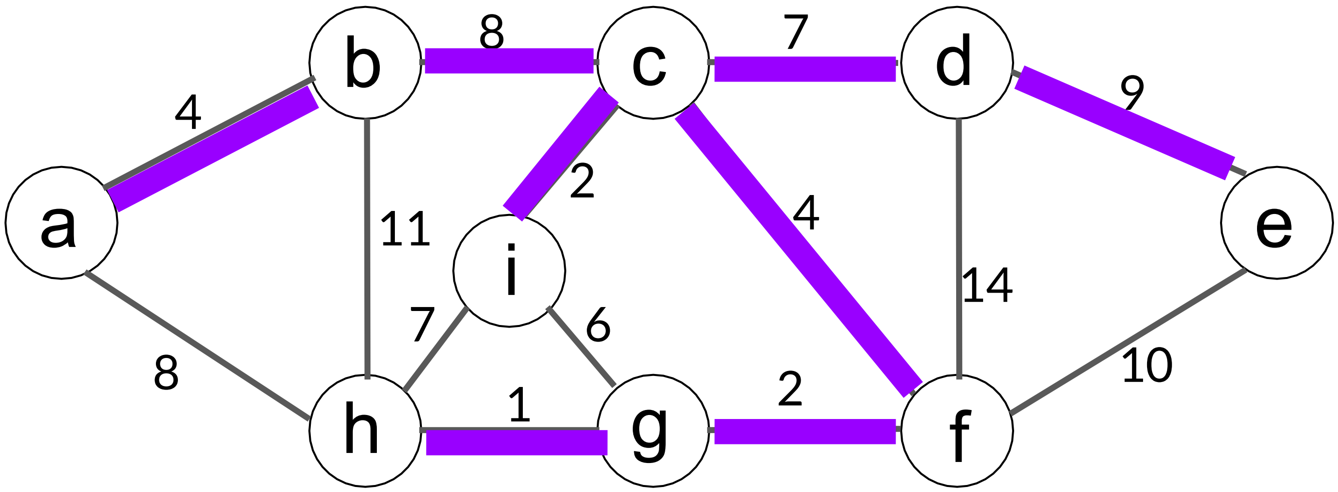


Kruskal Algorithm

Another way to look at Kruskal algorithm:

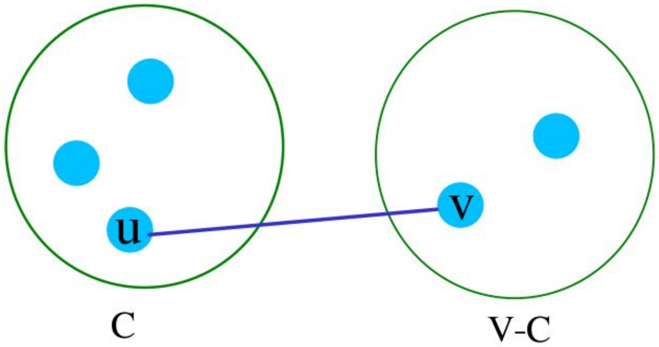
- At each step, the algorithm merges two connected component



Graph Cuts

In a graph G = (V, E)

- A **cut** is a partition of the vertices of the graph into two sets C, V-C
  - We show it by (C, V-C)
  - $C \subseteq V$.
- An edge (u,v) crosses the cut (C, V-C) if exactly one of u, v is in C
- If G is connected, then at least one edge crosses every cut



Tree Facts

- A tree of n vertices has n-1 edges
- There is a unique path between any two vertices in a tree
- If T is a tree and an edge $e \notin T$ is added to T, then the resulting graph contains a unique cycle C
- If $e' \in C$ then $T \cup \{e\} \setminus \{e'\}$ is a tree
  - If you add an edge e to a tree and this creates a cycle C, then removing any other edge $e' \in C$ will break the cycle and produce a tree
  - Proof in the next slide

Tree Facts

Theorem. Let T be a tree and $e=(u, v) \notin T$. The graph $T \cup \{e\}$ contains a cycle. For any edge $e'=(x, y)$ on the cycle, the graph $T' = T \cup \{e\} - \{e'\}$ is a tree.

Proof.

- $|T'| = |T| + 1 - 1 = |T| = |V| - 1 \rightarrow$if T' is connected, then it is a tree. Why?
  - $e \notin T$ and $e' \in T \cup \{e\}$
- Proving T' is connected
  - Consider any $s, t \in V$. Since T is connected, there is some path from s to t in T.
    - If that path does not cross (x, y), or if (x, y) = (u, v), then this path is also a path from s to t in T', so s and t are connected in T'.
    - If the path from s to t crosses (x, y). Assume WLOG that the path starts at s, goes to x, crosses (x, y), then goes from y to t. Since (u, v) and (x, y) are part of the same cycle, we can modify the original path from s to t so that instead of crossing (x, y), it goes around the cycle from x to y. This new path is then a path from s to t in T', so s and t are connected in T'. Thus any arbitrary pair of nodes are connected in T', so T' is connected.

Kruskal Algorithm

Proof of correctness (feasibility)

Proof by induction

Kruskal Algorithm: Proof of optimality

- **T**: MST found by Kruskal Algorithm
- **M**: optimal MST

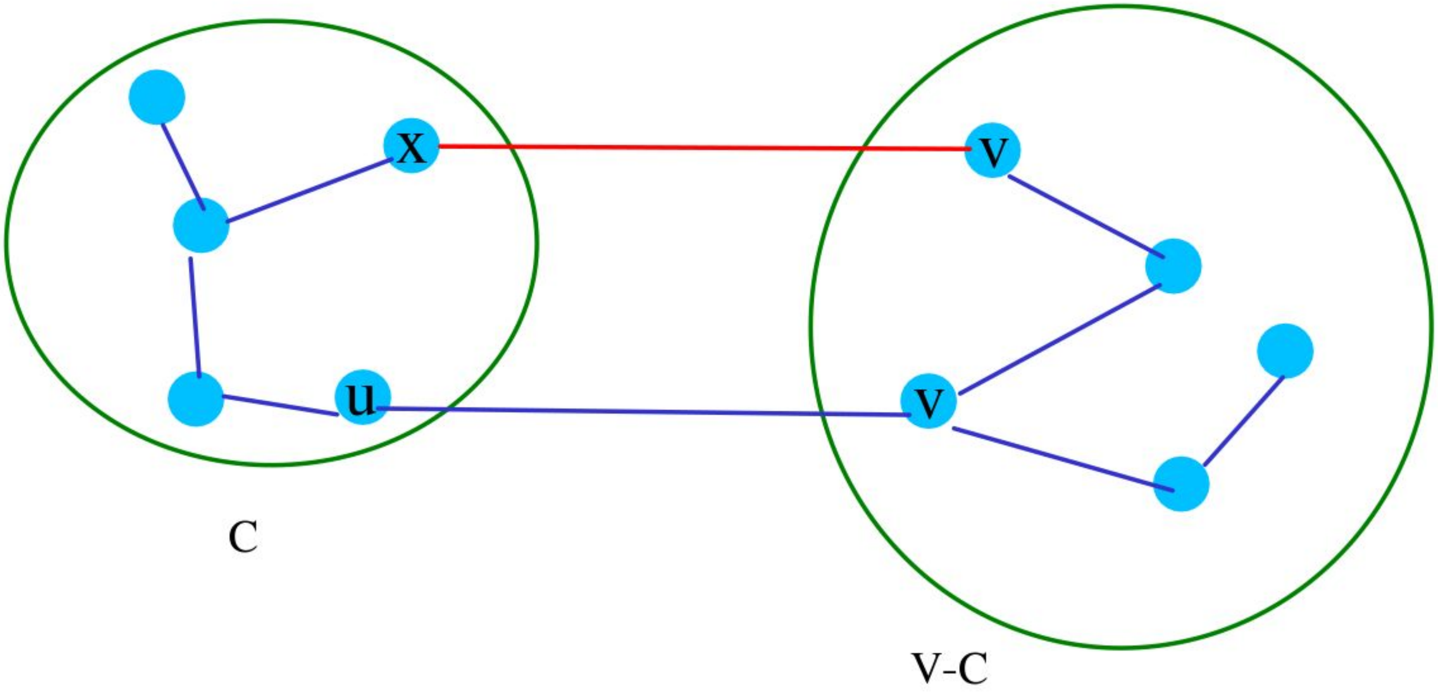
Proof by contradiction. Suppose T ≠ M.

$T = e_1\ e_{2\,.}\ \ldots\ e_j\ \ldots\ e_n$

$M = e_1\ e_{2.}\ldots\ m_j\ \ldots\ m_n$

- T and M ar the same up to j-th edge. Suppose e_j=(u, v) is in T but not in M
- C: the connected component containing u when (u,v) was added to T
- When (u,v) was added, it was the least-cost edge crossing the cut (C, V-C)
  - (u, v) crosses the cut, since u and v were not connected when Kruskal's algorithm selected (u, v)
  - Kruskal algorithm select the least-cost edge crossing the cut
- **M** is a MST → There must be a path from u to v in **M**. This path begins in C and ends in V-C. → There must be an edge along that path where x in C and y in V-C. Since (u,v) is the least-code edge crossing (C, V-C) → **w(u,v) < w(x,y)**
- M’ = M-{(x,y)} U {(u,v)}. M’ is a spanning tree because it connects all vertices. Since (x,y) is on the cycle formed by adding (u, v)
- w(M’) = w(M) - w(x,y) + w(u,y) < w(M) → M’ is a MST → contradiction M was the optimal solution
- We used exchange argument
  - exchanging some part of the optimal solution with some part of the greedy solution improved the optimal solution →contradiction
- Note: here we are assuming the edge weights are **unique**, otherwise we do not reach a contradiction

Kruskal Algorithm: Proof of optimality



Kruskal Algorithm: Proof of optimality in general

- **T**: MST found by Kruskal Algorithm
- **M**: optimal MST

Proof. We will prove w(T) = w(M). If T = M, we are done. Otherwise T ≠ M, so T–M ≠ Ø.

- Suppose e_j=(u, v) is in T but not in M
- C: the connected component containing u when (u,v) was added to T
- When (u,v) was added, it was the least-cost edge crossing the cut (C, V-C)
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- w(M') = w(M) - w(x,y) + w(u,y) → w(M') **≤** w(M)
- M' is a MST → w(M) ≤ w(M') → w(M') = w(M)
- Note that |T – M'| = |T – M| – 1. Therefore, if we repeat this process once for each edge in T – M, we will have converted M into T while preserving w(M). Thus w(T) = w(M).
- We used exchange argument
  - exchanging one edge of M with one edge of T without increasing w(M)

Kruskal Algorithm: pseudocode

```
Kruskal(G)
    Sort the edges by non-decreasing weight e1 … em, w(ei) ≤ w(ei+1)
    T = ∅
    for each edge (u, v)
        if u and v are not connected by T
            T = T ∪ {(u,v)}
    return T
```

Kruskal Algorithm: pseudocode

O(E lg E) or O(E lg V)

```
Kruskal(G)
    Sort the edges by non-decreasing weight e1 … em, w(ei) ≤ w(ei+1)
    T = Ø
    for each edge (u, v)
        if u and v are not connected by T
            T = T ∪ {(u,v)}
    return T
```

O(E)

Use DFS → the runtime of DFS is O(V+E).
here , the runtime is O(V). why?

Runtime: O(VE)

Can we do better?

Kruskal Algorithm: A better implementation

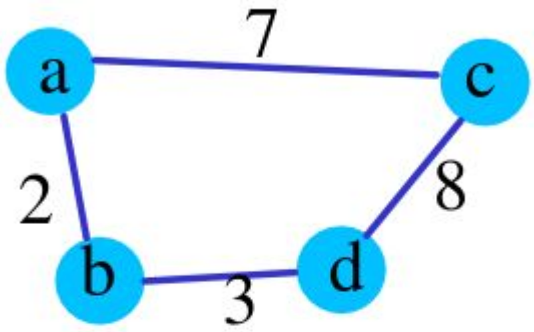
- Union-find data structure:
  - Represents a **partition** of set S= $\{e_1, e_2, \ldots, e_n\}$ into **disjoint subsets**
    - Initially n disjoint subsets $S_i = \{e_i\}$
  - a collection of disjoint sets $\{S_1, S_2, \ldots, S_k\}$
  - Each element of data belong to exactly one set
  - Each set is identified by a representative (some member of the set)
    - Specifies which set an element belongs to
- Operations of **union-find** data structure
  - **Make-set**(x): Create a set containing one element, x
  - **union**(x, y): unites the sets containing x and y into one set
  - **find**(x): returns a pointer to the representative of the set containing x

Kruskal Algorithm using union-find data structure

```
Kruskal(G)
    Sort the edges by non-decreasing weight e₁ … eₘ, w(eᵢ) ≤ w(eᵢ₊₁)
    T = Ø
    S = union-find data structure
    for each v in V
      S.make-set(v)
    for each edge (u, v)
           if S.find(u) != S.find(v)
                T = T ∪ {(u,v)}
                S.union(u, v)
     return T
```

Kruskal Algorithm using union-find data structure

- Each graph node is initially in its own subset
- Add an edge → union two subsets
- An edge **creates a cycle iff** its endpoints are in the **same subset**

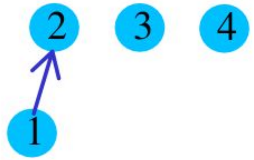


First implementation

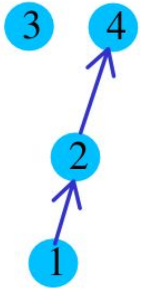
- Suppose we are partitioning set $\{1, \ldots, n\}$ into subsets $\boldsymbol{S_1}, \ldots, \boldsymbol{S_n}$
- Represent the partition as a **forest** of **trees**
  - Initially one single-node tree per subset
  - Each node has a **parent pointer**
- $Find(i)$ returns the **root** of the tree containing **element $\boldsymbol{i}$**
- $Union(i,j)$ makes one root the parent of the other
- Problem:
  - Long paths → slow find

First implementation

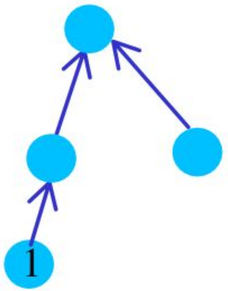
find(1)->1, find(2)->2



Union(1,2): parent[1] = 2
find(4)->4, find(1)->2



Union(4,2): parent[2] = 4
find(3)->3, find(1)->4



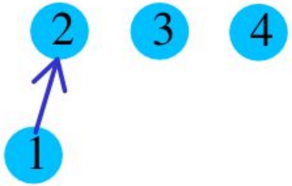
Union(3,4): parent[3] = 4

Union-find with union by rank

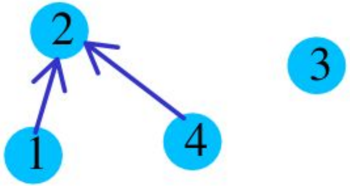
- Keep track of **heights** of trees
- Make **root with greater height** be the **parent**
    - Union of two trees with height h has height h + 1
    - Union of tree with height h and tree with height < h has height h

Union-find with union by rank

find(1)->1, find(2)->2



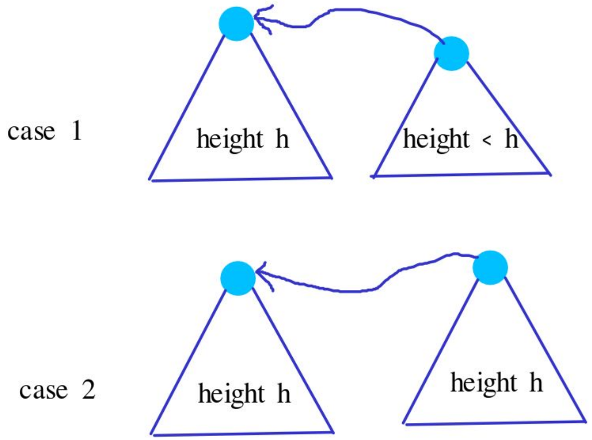
Union(1,2): same height, parent[1] = 2
find(4)->4, find(1)->2



Union(4,2):2's height is greater: parent[4] = 2

Runtime of Union-find with union by rank

- Each tree of height h contains at least 2^h nodes
- Proof by induction.
    - Base case: trees with height 0 have $2^0 = 1$ node
    - I.H.: a tree of height h contains at least 2^h nodes
    - Induction step: Having I.H, we want to show a tree of height h+1 contains at least 2^{h+1} nodes.
    - Case 1: Union of trees of height h and height < h
        - Left tree hast $\geq 2^h$ nodes
        - result has height h and $\geq 2^h$ nodes
    - Case 2: Union of trees of same height
        - each tree has $\geq 2^h$ nodes.
        - Result has height h+1 and $\geq 2^h + 2^h$ nodes
            - $2^h + 2^h = 2^{h+1}$

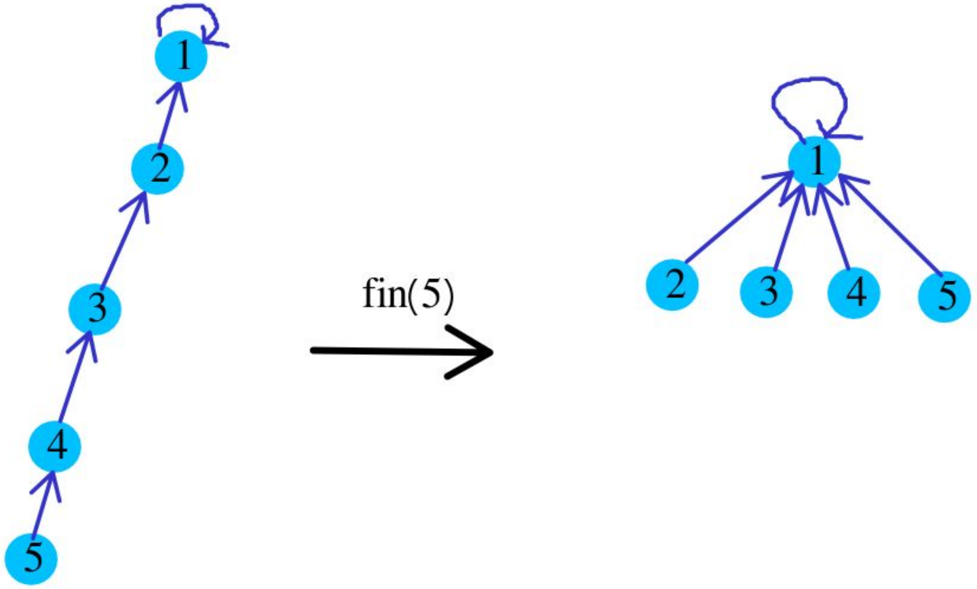


Runtime of Union-find with union by rank

- Each tree of height h contains at least 2^h nodes
- There are only n nodes in the graph
- Therefore the height is at most log n
- The longest path in the union-find first is log n
- So all union-find operations run in Θ (log n) time

Union-find by rank and **path compression**

- A recursive logic is used to achieve path compression with each call to the find operation.
- The Union operation may increase the height of the trees
- The find operation tries to reduce the height at each call and to achieve flatter trees
- The flatter the trees, lower is the complexity of find and union operations.




```
def find(x):
    if x != parent[x]:
        parent[x] = find(parent[x]) # path compression during find
    return parent[x]
```

Kruskal Algorithm

O(E lg E) or O(E lg V)

```
Kruskal(G)
    Sort the edges by non-decreasing weight e_1 … e_m, w(e_i) ≤ w(e_{i+1})
    T = Ø
    for i = 1 to m
        if e_i does not make a cycle with T
            T = T ∪ {e_i}
    return T
```

O(E)

Can be done in O(alpha(E+V)) using union-find data structure

Runtime: O(E log V)

Can we do better?

Acknowledgement

The slides of the following course:

https://web.stanford.edu/class/archive/cs/cs161/cs161.1138/

And the slides of several previous CS 341@waterloo especially Trevor’s Brown slides