Given $|\alpha\rangle = (i+3)|0\rangle + 7|1\rangle$ and $|\beta\rangle = 2|0\rangle + 7i|1\rangle$.

a) $\langle\alpha|\beta\rangle$ [3 Marks]

$$\begin{aligned}
\langle\alpha|\beta\rangle &= ((-i+3)\langle 0| + 7\langle 1|)(2|0\rangle + 7i|1\rangle) \\
&= (-i+3)\times 2 + 7\times 7i \\
&= -2i + 6 + 49i \\
&= 6 + 47i
\end{aligned}$$

b) $|\alpha\rangle\langle\beta|$ [3 Marks]

$$\begin{aligned}
|\alpha\rangle\langle\beta| &= ((i+3)|0\rangle + 7|1\rangle)(2\langle 0| - 7i\langle 1|) \\
&= ((2i+6)|0\rangle\langle 0| + (7-21i)|0\rangle\langle 1| + 14|1\rangle\langle 0| - 49i|1\rangle\langle 1| \\
&= \begin{pmatrix} 2i+6 & 7-21i \\ 14 & -49i \end{pmatrix}
\end{aligned}$$

c) $\langle\alpha|\langle\beta|$ [3 Marks]

$$\begin{aligned}
\langle\alpha|\langle\beta| &= ((-i+3)\langle 0| + 7\langle 1|)(2\langle 0| - 7i\langle 1|) \\
&= (-2i+6)\langle 0|\langle 0| + (-7-21i)\langle 0|\langle 1| + 14\langle 1|\langle 0| - 49i\langle 1|\langle 1| \\
&= \begin{bmatrix} -2i+6 & -7-21i & 14 & -49i \end{bmatrix}
\end{aligned}$$

1) Write matrix: $3\left|001\right\rangle\left\langle 01\right|+2\left|011\right\rangle\left\langle 10\right|+5\left|100\right\rangle\left\langle 11\right|+9\left|110\right\rangle\left\langle 00\right|$ [3 Marks]

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2) Find tensor product of $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & i & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ [3 Marks]

$$\begin{aligned} A \otimes B &= \begin{pmatrix} 3B & 2B & 1B \\ 0B & iB & 7B \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 & 2 & 4 & 1 & 2 \\ 9 & 12 & 6 & 8 & 3 & 4 \\ 0 & 0 & i & 2i & 7 & 14 \\ 0 & 0 & 3i & 4i & 21 & 28 \end{pmatrix} \end{aligned}$$

3) Find norm of the $\left|\alpha\right\rangle = (i+3)\left|0\right\rangle + 7\left|1\right\rangle$ [3 Marks]

$$\begin{aligned} \sqrt{\left\langle\alpha\right|\left|\alpha\right\rangle} &= \sqrt{(-i+3)\left\langle 0\right| + 7\left\langle 1\right|)(i+3)\left|0\right\rangle + 7\left|1\right\rangle} \\ &= \sqrt{10+49} = \sqrt{59} \end{aligned}$$

i) Show that vectors are not orthogonal $|\alpha\rangle = (i+3)|0\rangle + 7|1\rangle$ and $|\beta\rangle = 2|0\rangle + 7i|1\rangle$ [3 Marks]

$$\begin{aligned}\langle\alpha|\beta\rangle &= ((-i+3)\langle 0| + 7\langle 1|)(2|0\rangle + 7i|1\rangle) \\ &= (-i+3)\times 2 + 7\times 7i \\ &= -2i + 6 + 49i \\ &= 6 + 47i\end{aligned}$$

As their inner product is not equal to zero hence no orthogonal.

ii) Find tensor product of $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & i & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ [3 Marks]

$$\begin{aligned}A \otimes B &= \begin{pmatrix} 3B & 2B & 1B \\ 0B & iB & 7B \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 & 2 & 4 & 1 & 2 \\ 9 & 12 & 6 & 8 & 3 & 4 \\ 0 & 0 & i & 2i & 7 & 14 \\ 0 & 0 & 3i & 4i & 21 & 28 \end{pmatrix}\end{aligned}$$

iii) Calculate $\langle\psi|\langle\phi|$, when $|\psi\rangle = (2+i)|00\rangle + 7|11\rangle$ and $|\phi\rangle = 3i|101\rangle + (2-i)|110\rangle + 3|111\rangle$ [3 Marks]

$$\begin{aligned}\langle\psi|\langle\phi| &= ((2-i)\langle 00| + 7\langle 11|)(-3i\langle 101| + (2+i)\langle 110| + 3\langle 111|) \\ &= (-3-6i)\langle 00101| + 5\langle 00110| + (6-2i)\langle 00111| - 21i\langle 11101| + (14+7i)\langle 11110| + 21\langle 11111|\end{aligned}$$

d) Calculate $|\psi\rangle|\phi\rangle$, when $\langle\psi| = (2+5i)\langle 00| + 7\langle 11|$ and $|\phi\rangle = 5i|101\rangle + (2-i)|110\rangle + 3|111\rangle$ [3 Marks]

$$\begin{aligned}|\psi\rangle|\phi\rangle &= ((2-5i)|00\rangle + 7|11\rangle)(5i|101\rangle + (2-i)|110\rangle + 3|111\rangle) \\ &= (10i+25)|00101\rangle + (-1-12i)|00110\rangle + (6-15i)|00111\rangle + \\ &\quad 35i|11101\rangle + (14-7i)|11110\rangle + 21|11111\rangle\end{aligned}$$

e) Find tensor product of $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & i & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ [3 Marks]

$$\begin{aligned}A \otimes B &= \begin{pmatrix} 3B & 2B & 1B \\ 0B & iB & 7B \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 & 2 & 4 & 1 & 2 \\ 9 & 12 & 6 & 8 & 3 & 4 \\ 0 & 0 & i & 2i & 7 & 14 \\ 0 & 0 & 3i & 4i & 21 & 28 \end{pmatrix}\end{aligned}$$

f) Write in Dirac's notation $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & i \\ 1 & 0 \end{pmatrix}$

$$|00\rangle\langle 0| + 2|00\rangle\langle 1| + 3|01\rangle\langle 0| + 4|01\rangle\langle 1| + i|10\rangle\langle 1| + |11\rangle\langle 0|$$