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Course: Numerical Computing
Assignment: 2

$$I = \int_0^1 x^2 e^{-x} dx$$

1. Composite Trapezoidal Rule (T-Rule)

$h = 1 \quad N = 1$

$f_0 = 0 \; , \; f_1 = 0.36787944$

$$\frac{h}{2}(f_0 + f_1)$$

$$\frac{1}{2}(0.36787944)$$

$$= 0.18393972$$

$h = 0.5 \quad N = 2$

x	0	0.5	1
y	0	0.15163266	0.36787944

$$\frac{h}{2}(f_0 + 2f_1 + f_2)$$

$$\frac{0.5}{2}(0 + 0.30326533 + 0.36787944)$$

$$\frac{0.5}{2}(0.67114477)$$

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$= 0.16778619$

$h = 0.25 \quad N = 4$

x	0	0.25	0.5	0.75	1
y	0	0.04867505	0.15163266	0.26570619	0.36787944

$$\frac{h}{2}\left(f_0 + f_4 + 2(f_1 + f_2 + f_3)\right)$$

$$\frac{0.25}{2}\left(0 + 0.36787944 + 2(0.04867505 + 0.15163266 + 0.26570619)\right)$$

$$= \frac{0.25}{2}\left(0 + 0.36787944 + 2(0.4660139)\right)$$

$$= \frac{0.25}{2}(1.29990724)$$

$= 0.16248841$

$h = 0.125 \quad N = 8$

x	0	0.125	0.25	0.375	0.5	0.625	0.75
y	0	0.01378901	0.04867505	0.09665005	0.15163266	0.2090865	0.26570619

x	0.875	1
y	0.31915598	0.36787944

$$= \frac{h}{2}\left(f_0 + f_8 + 2(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7)\right)$$

$$= \frac{0.125}{2}\left(0 + 0.36787944 + 2(1.10469945)\right)$$

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$= 0.1610799$

Composite Simpson Rule (S-Rule)

$h = 0.5 \quad N = 1$

x	0	0.5	1
y	0	0.15163266	0.36787944

$= \frac{h}{3}\left(f_0 + f_2 + 4f_1\right)$

$= \frac{0.5}{3}\left(0 + 0.36787944 + 4(0.15163266)\right)$

$= 0.16240168$

$h = 0.25 \quad N = 2$

x	0	0.25	0.5	0.75	1
y	0	0.04867505	0.15163266	0.26570619	0.36787944

$\frac{h}{3}\left(f_0 + f_4 + 4(f_1 + f_3) + 2f_2\right)$

$\frac{0.25}{3}\left(0 + 0.36787944 + 4(0.31438123) + 2(0.15163266)\right)$

$= 0.16072248$

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$h = 0.125 \quad N = 4$

x	y
0	0
0.125	0.01378901
0.25	0.04867505
0.375	0.09665005
0.5 ~~0.625~~	0.15163266
0.625	0.2090865
0.75	0.26570619
0.875	0.31915998
1	0.36787944

$$\frac{h}{3}\left(f_0 + f_8 + 4(f_1+f_3+f_5+f_7) + 2(f_2+f_4+f_6)\right)$$

$$= \frac{0.125}{3}\left[0 + 0.36787944 + 4(0.63868555) + 2(0.4660139)\right]$$

$$= 0.1606139$$

2. The accumulated error in trapezoidal rule for interval of width h is given by

$$E = -\frac{(b-a)h^2}{12} f''(\eta), \quad a < \eta < b$$

$$|E| = \frac{h^2}{12} f''(\eta), \quad 0 < \eta < 1$$

$$f(x) = x^2 e^{-x}$$
$$f'(x) = 2xe^{-x} + (-x^2 e^{-x})$$
$$f'(x) = 2xe^{-x} - x^2 e^{-x}$$
$$f'(x) = e^{-x}(2x - x^2)$$
$$f''(x) = -e^{-x}(2x - x^2) + (2-2x)e^{-x}$$
$$f''(x) = e^{-x}(x^2 - 2x) + e^{-x}(2-2x)$$
$$f''(x) = e^{-x}(x^2 - 2x + 2 - 2x)$$
$$f''(x) = e^{-x}(x^2 - 4x + 2)$$

$$\frac{h^2}{12} e^{-1}(1^2 - 4 + 2) < \frac{h^2}{12} e^{-\eta}(\eta^2 - 4\eta + 2) < \frac{h^2}{12} e^{-0}(2)$$

to get an accuracy of 0.1×10^{-10} let lower bound

$$\left|\frac{h^2 2e^{-0}}{12}\right| \leq \left|0.1 \times 10^{-10}\right|$$

$$h^2 \leq \frac{10^{-11} \times 6}{e^{-0}}$$

$h = 0.000007745$

$\cancel{0.0000180664}$

$\cancel{55368}$ $N = 129099$

The accumulated error in simpson rule for interval of width h is given by

$$E = \frac{-(b-a)h^4}{180} f^{iv}(\eta) \qquad a<\eta<b$$

$$|E| = \frac{h^4 \cdot f^{iv}(\eta)}{180}$$

$$f''(x) = e^{-x}(x^2-4x+2)$$

$$f'''(x) = -e^{-x}(x^2-4x+2) + e^{-x}(2x-4)$$

$$f'''(x) = e^{-x}(-x^2+4x-2+2x-4)$$

$$f'''(x) = e^{-x}(-x^2+6x-6)$$

$$f^{iv}(x) = -e^{-x}(-x^2+6x-6) + e^{-x}(-2x+6)$$

$$f^{iv}(x) = e^{-x}(x^2-6x+6-2x+6)$$

$$f^{iv}(x) = e^{-x}(x^2-8x+12)$$

$$\frac{h^4 e^{-1}(6)}{180} < \frac{h^4 e^{-1}(\eta^2-8\eta+12)}{180} < \frac{h^4 e^{0}(12)}{180}$$

To get same accuracy of

$$\frac{h^4 e^{-0}(12)}{180} \le 0.1\times10^{-10}$$

$$\sqrt[4]{h^4} \le \sqrt[4]{\frac{10^{-11}\times180}{12e^{-0}}}$$

$$h = \cancel{0.0098}\ \cancel{0.0053438}$$

$$N = \frac{1}{\cancel{0.0098}\ 0.01068} = \cancel{93.56}$$

$$h = 0.006999$$

$$N = 142$$

3. Romberg based trapezoidal rule

S. Size	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
h	0.1839397206			
		$I_T^1(h) = 0.1624016835$		
h/2	0.1677861928		0.1606105287	
		$I_T^1(h/2) = 0.1607224759$		0.1606028001
h/4	0.1624884051		0.1606029209	
		$I_T^1(h/4) = 0.1606103931$		
h/8	0.1610798961			

$$I_T^1(h) = \frac{4^1 I_T^0(h/2) - I_T^0(h)}{4-1}$$

$= 0.1624016835$

$$I_T^1(h/2) = \frac{4^1 I_T^0(h/4) - I_T^0(h/2)}{3}$$

$= 0.1607224759$

$$I_T^1(h/4) = \frac{4^1 I_T^0(h/8) - I_T^0(h/4)}{3}$$

$= 0.1606103931$

$$I_T^2(h) = \frac{4^2 I_T^1(h/2) - I_T^1(h)}{15}$$

$= 0.1606105287$

$$I_T^2(h/2) = \frac{4^2 (I_T^1(h/4)) - I_T^1(h/2)}{15}$$

$= 0.1606029209$

$$I_T^3(h) = \frac{4^3 I_T^2(h/2) - I_T^2(h)}{63}$$

$= 0.1606028001$

Romberg integration based on trapezoidal consumes four extrapolations and 10 function evaluations of 0.1×10^{-10}

S.Size	$O(h^4)$	$O(h^6)$	$O(h^8)$
$h/2$	$I_{h/2} = 0.1624016835$		
		$I_s'(h/2) = 0.1606105287$	
$h/4$	$I_{h/4} = 0.1607224759$		0.1606028001
		$I_s'(h/4) = 0.1606029209$	
$h/8$	$I_{h/8} = 0.1606103931$		

$$I_s'(h/2) = \frac{4^2 I_s^0(h/4) - I_s^0(h/2)}{15}$$

$$= 0.1606105287$$

$$I_s'(h/4) = \frac{4^2 I_s^0(h/8) - I_s^0(h/4)}{15}$$

$$= 0.1606029209$$

$$I_s^2(h/2) = \frac{4^3(I_s'(h/4)) - I_s'(h/2)}{63}$$

$$= 0.1606028001$$

Romberg integration based on Simpson consumes one extrapolation and four function evaluations to get accuracy of 0.1×10^{-10}.

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When we applied direct T-Rule we got accuracy upto 2 decimal place. However with Romberg extrapolation we were able to enhance our results upto 6 decimal places. When we applied direct S-Rule we got accuracy upto 4 decimal place. With romberg extrapolation we were able to increase it upto 6 decimal places. Romberg extrapolation allows us to get more accuracy by going in to higher order polynomials. ~~and~~ However the higher the order goes the more we need to reduce the step size.