Graph Algorithms

Depth-First Search (DFS)

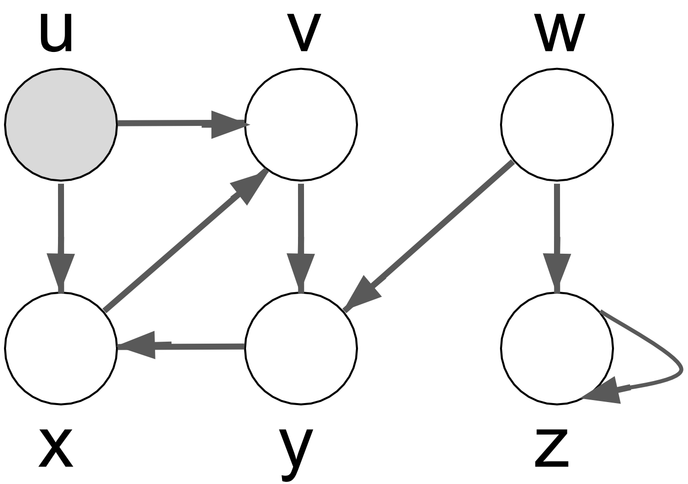
Depth-First Search (DFS)

- **Input**: A graph G = (V, E) and a vertex s
- **Output**:
  - Explores all the nodes in the graph reachable from **s**
  - generates a tree rooted at **s**
- Algorithm:
  - It starts from **s**, and follows the first path it finds and goes as deep as possible
  - During the execution of the algorithm, vertices are in one of the three following states:
    - UnDiscovered (white)
    - Discovered (gray)
    - Fully explored (black)
- Algorithm is implemented recursively
  - Can be implemented non-recursively using a stack

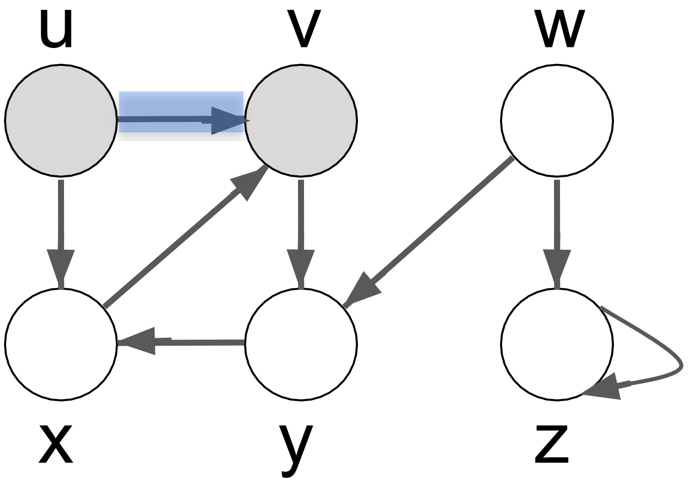
DFS: Basic Version

```
DFS(G)
    for each vertex u in G:
        color[u] = WHITE   #Mark w as undiscovered
    for each vertex u in G:
        if color[u] == WHITE   #if u is undiscovered
            DFS(G, u)
DFS(G, s)
    color[s] = GRAY    #discover s
    for each neighbor v of s:
        if (color[v] == WHITE)  #v is undiscovered
            DFS(G, v)
    color[v] = BLACK    #Mark s as fully-discovered(BLACK)
```

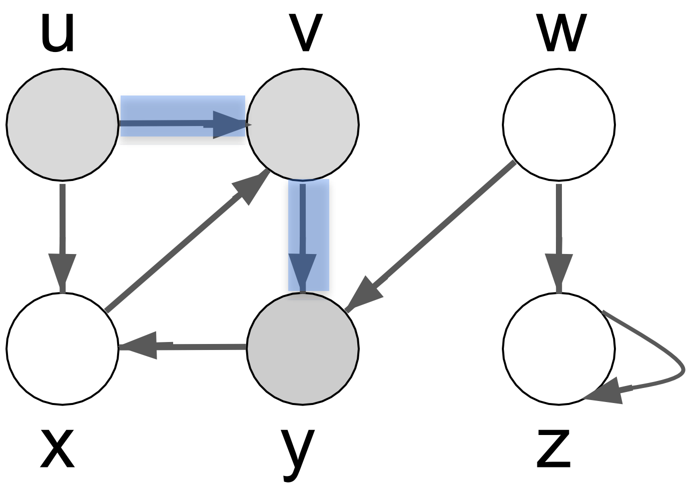
DFS Example



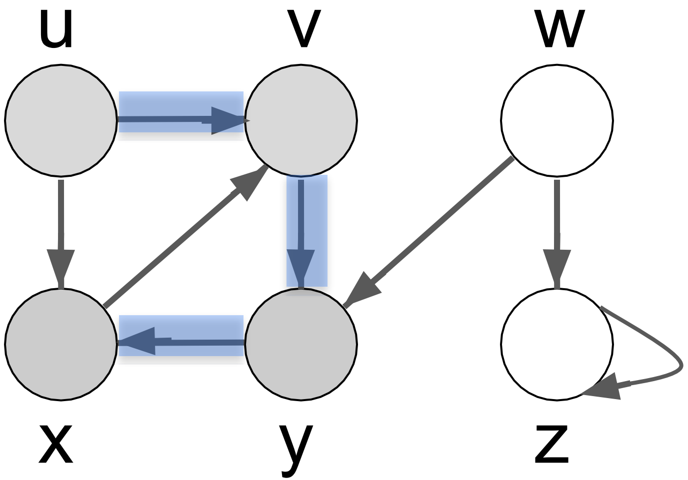
DFS Example



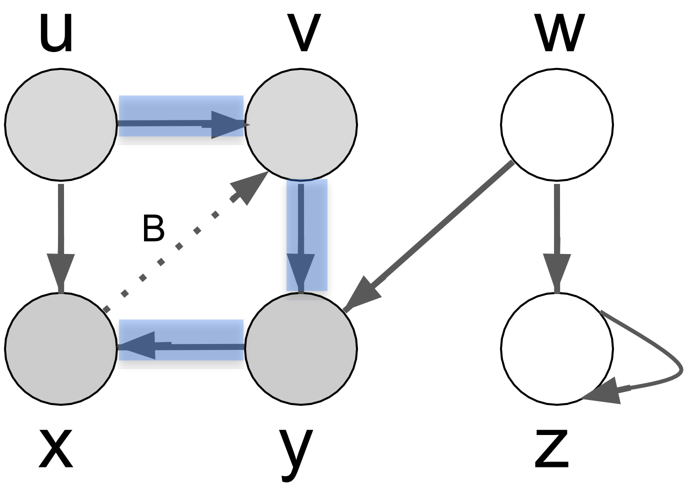
DFS Example



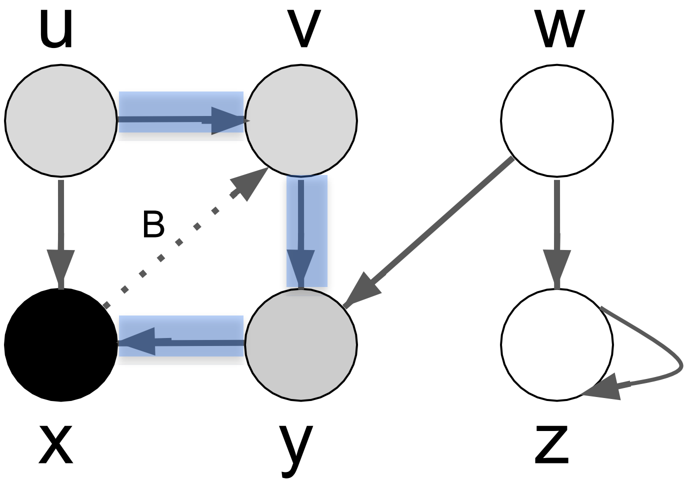
DFS Example



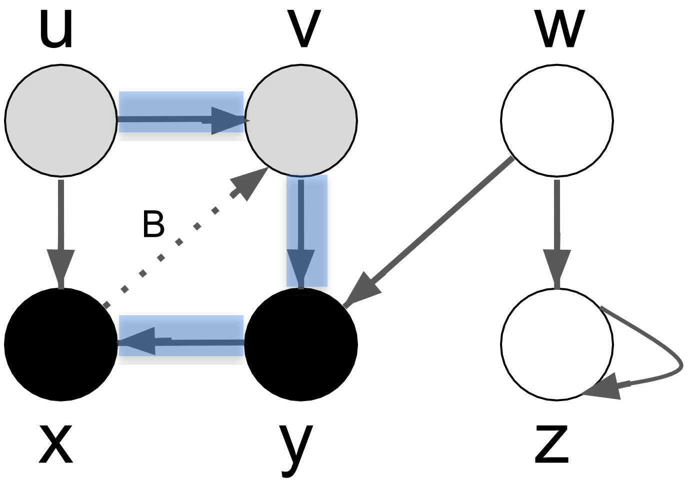
DFS Example



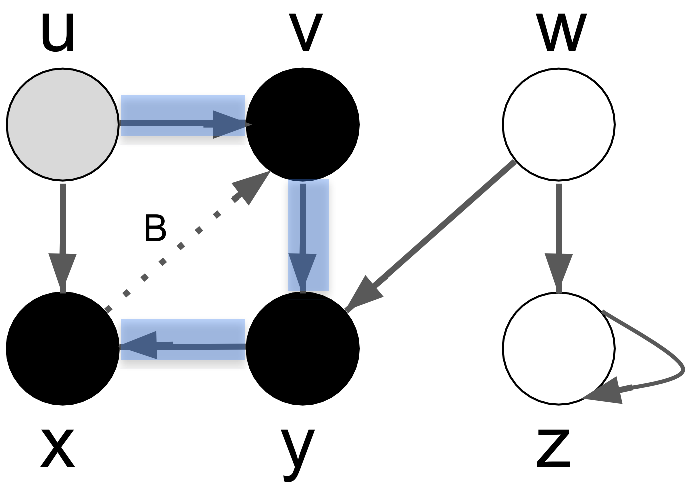
DFS Example



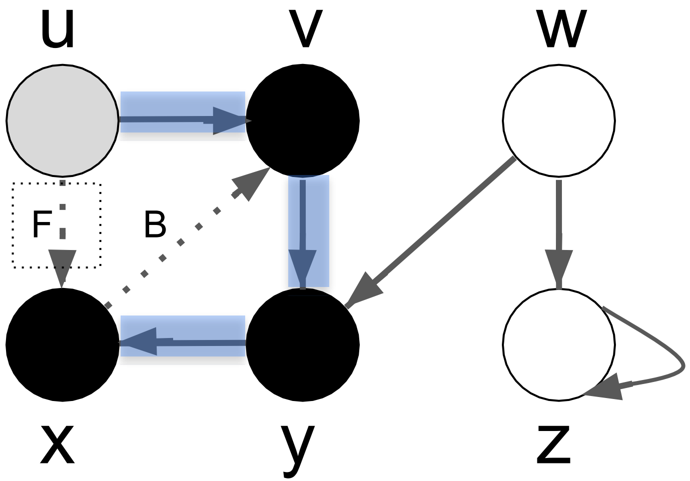
DFS Example



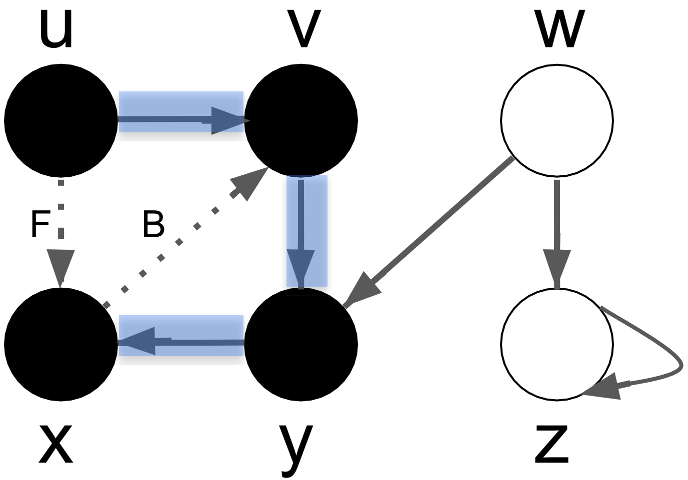
DFS Example



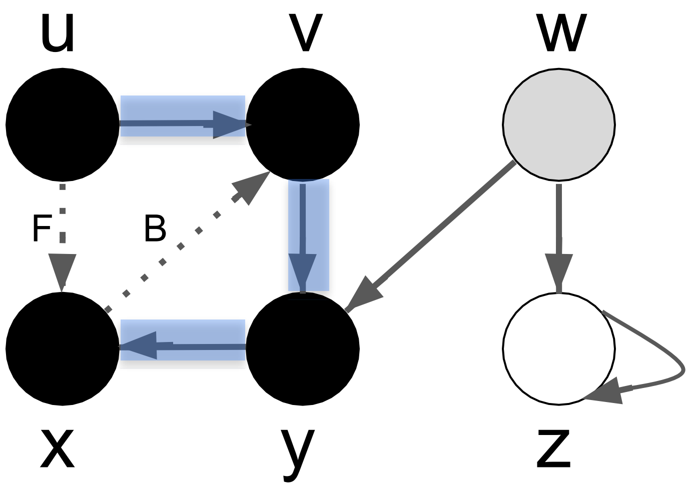
DFS Example



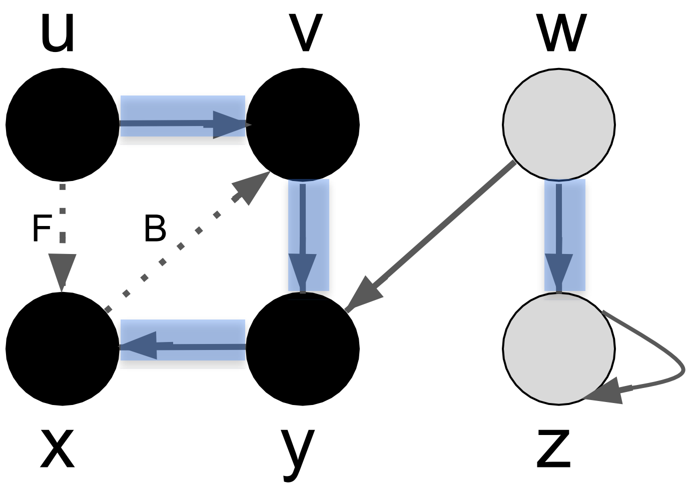
DFS Example



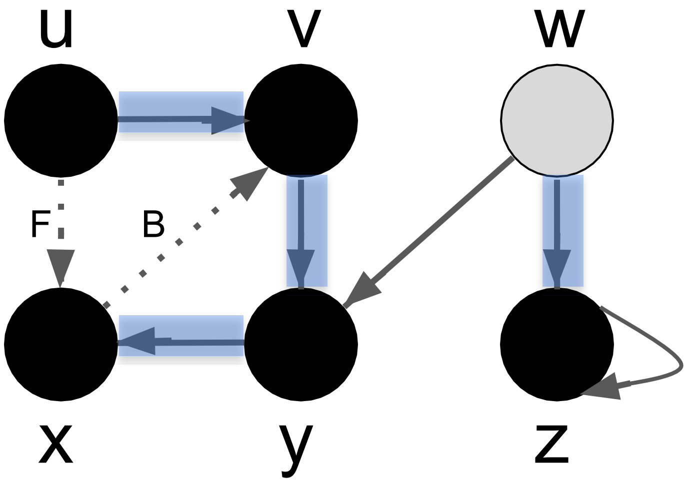
DFS Example



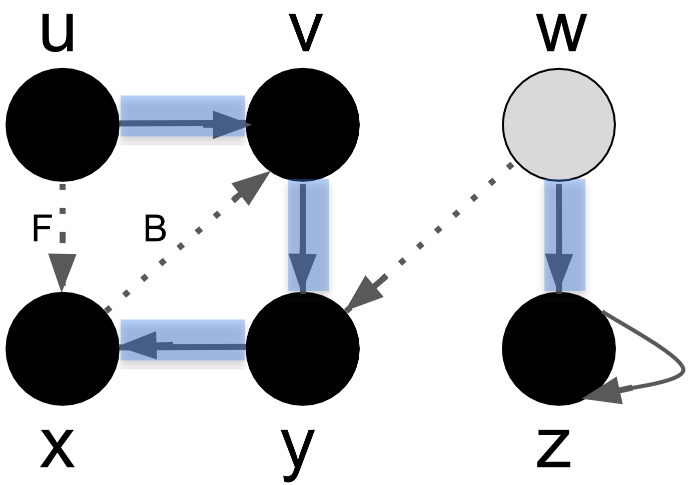
DFS Example



DFS Example



DFS Example



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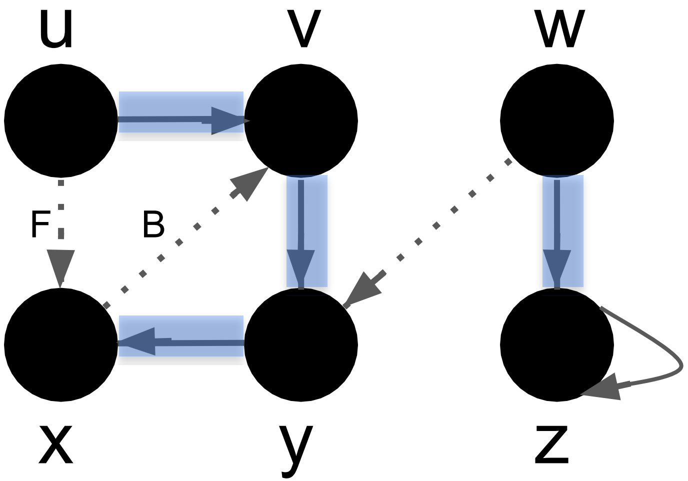


DFS: Basic Version

Θ(V)

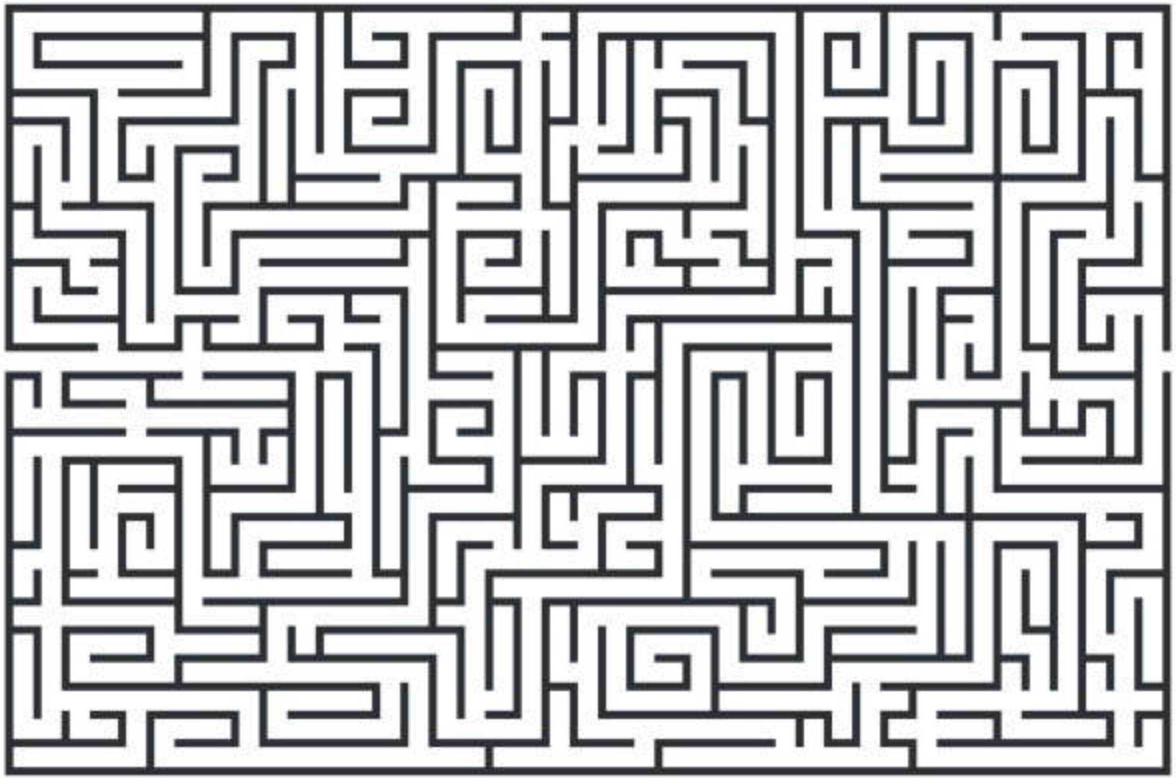
Θ(V)

Θ(E) in total:

- The loop iterates deg(u) times for each node s
- In total the runtime of for loop is Θ (∑deg(s)) = Θ(E)

```
DFS(G)
    for each vertex u in G:
          color[u] = WHITE  #Mark w as undiscovered
    for each vertex u in G:
         if color[u] == WHITE  #if u is undiscovered
              DFS(G, u)
DFS(G, s)
     color[s] = GRAY   #discover s
     for each neighbor v of s:
            if (color[v] == WHITE) #v is undiscovered
                 DFS(v)
     color[v] = BLACK   #Mark s as fully-discovered(BLACK)
```

DFS Application



DFS: Proof of correctness of DFS(G, s)

Lemma 1. There is a path from **s** to **v** if and only if **v** is discovered by DFS

- $\Leftarrow$ if **v** is discovered, then there is a path from **s** to **v**
  - **proof:** by induction on the number of vertices discovered
    - **Base case**: **s** is discovered
    - **Induction hypothesis**: Statement is true for the first i-1 vertices discovered
    - **Induction step**: Now, we need to prove it is true after i vertices are discovered. When the i-th vertex is discovered, it must have had a parent node that is already discovered. By induction hypothesis there is a path from s to the parent node (since it is one of those i-1 vertices). There is an edge connecting vertex i-1 and i. Therefore, there is a path from s to i
- $\Rightarrow$ if there is a path from **s** to **v**, then **v** is discovered
  - In other words, if **v** is not discovered then there is no path from **s** to **v**

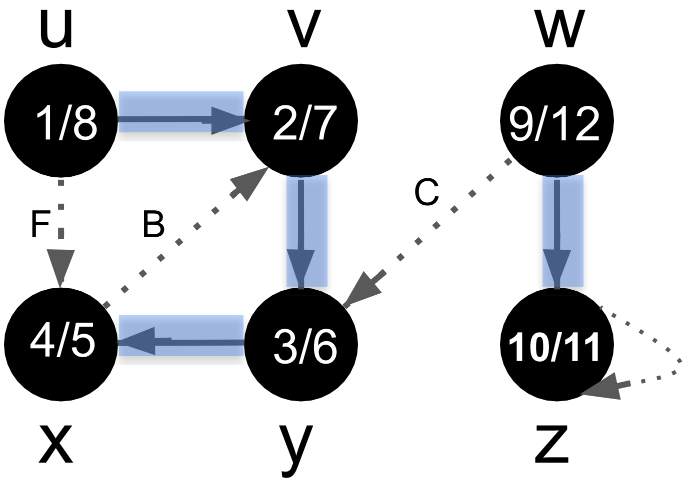
DFS vs. BFS

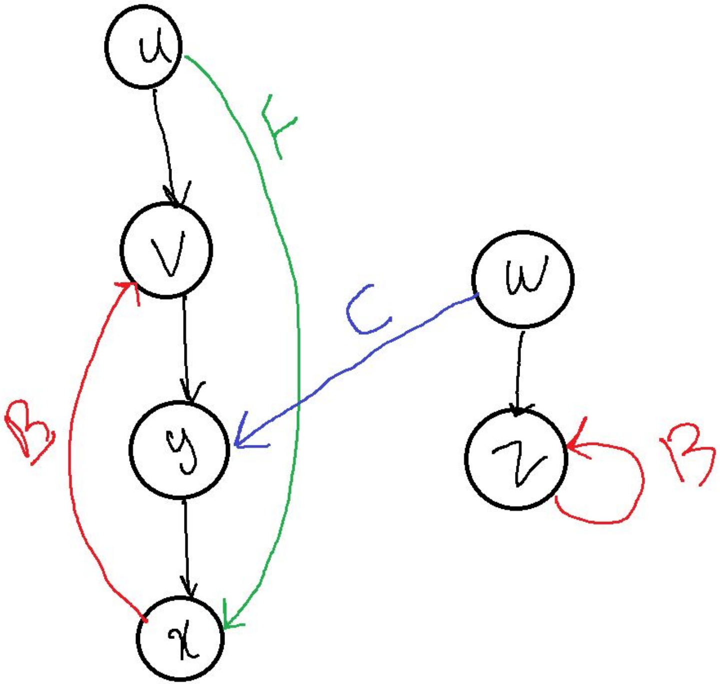
- Similarities
  - Both BFS and DFS could be used to
    - Check graph connectivity
    - Find the connected components containing s
      - Both start at an arbitrary node and explores the whole connected component
    - Check s-t connectivity
  - Both BFS(G, s) and DFS(G, s) generate a tree rooted at s
- Differences
  - BFS(G, s) finds the shortest path from s to all vertices reachable from s, but DFS(G, s) does not
  - BFS is implemented using a queue, but not DFS

DFS Tree / Forest

- Similar to BFS, we can construct a DFS tree
    - It could be used to find the path from s to all the reachable vertice
- When a vertex v is first discovered when exploring vertex u, we say vertex u is the parent of vertex v
- The edges (v, parent(v)) form a tree, and we can use them to find a path to s.
- A graph could have many different DFS trees depending on the order of exploring the neighbors of vertices

DFS Example





The same graph on the left: re-drawn to specify different classes of edges: B: Backward, F:Forward, C:Cross, and tree edges

DFS: classification of edges in the graph

- DFS classifies the edges of the input Graph G=(V, E).
- In a directed graph, an edge (u, v) is in one of the following classes:
  - Tree edge
    - If it is in the depth-first forest generated by the algorithm
  - Back edge
    - If **v** is ancestor of **u** in a depth-first tree
  - Forward edge
    - If **v** is a descendant of **u** in a depth-first tree
  - Cross edges
    - If it is not in any of the above category. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees

Discovery and finishing time

- We modify the algorithm to record the time when a vertex is first visited and the time when its exploring is finished
- These information will be very useful in design and analysis of algorithms

DFS: Enhanced Version

```
time =0
DFS(G)
    for each vertex u in G:
        Color[u] = WHITE
        Pred[u] = NULL
    time =0
    for each vertex u in G:
        if color[u] == WHITE
            DFS(G, u)
DFS(G, u)
    time = time+1
    discover[u] = time
    color[u] = GRAY
    for each neighbor v of u
        if (color[v] == WHITE)
            pred[v] = u
            DFS(G, v)
    color[u] = BLACK
    time = time + 1
    finish[u] = time
```

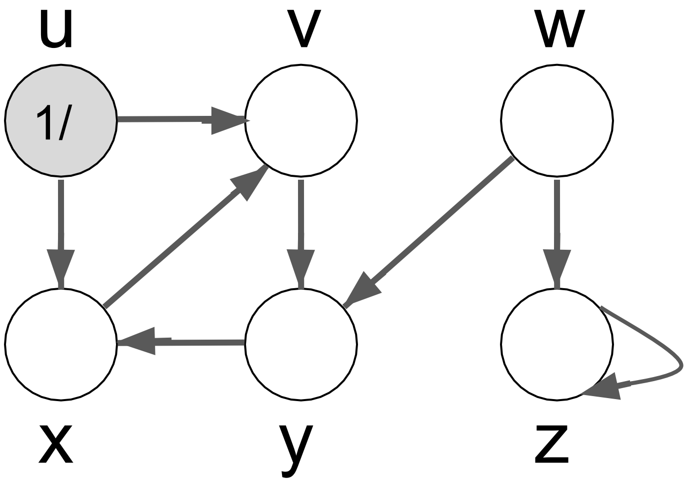
Enhanced DFS

- keeps track of whether a node is undiscovered, discovered, fully explored.
- keeps track of the time it started and finished with a vertex

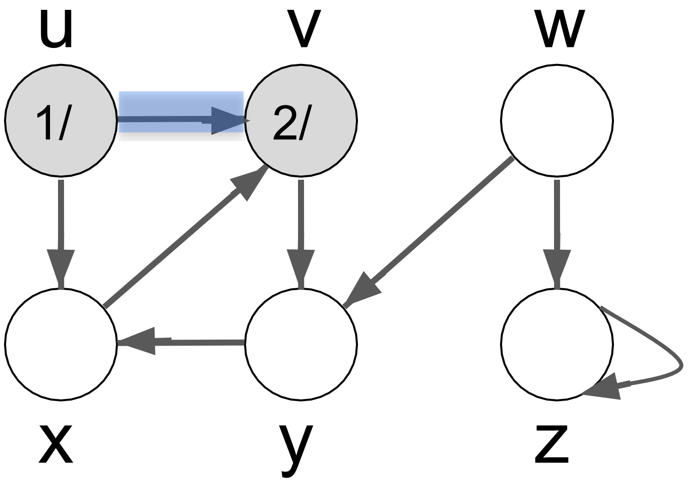
Output:

- a depth-first forest
- Timestamp each vertex:
  - d[u], when vertex u was discovered
  - f[u]: when u was fully explored (all its neighbors are discovered)

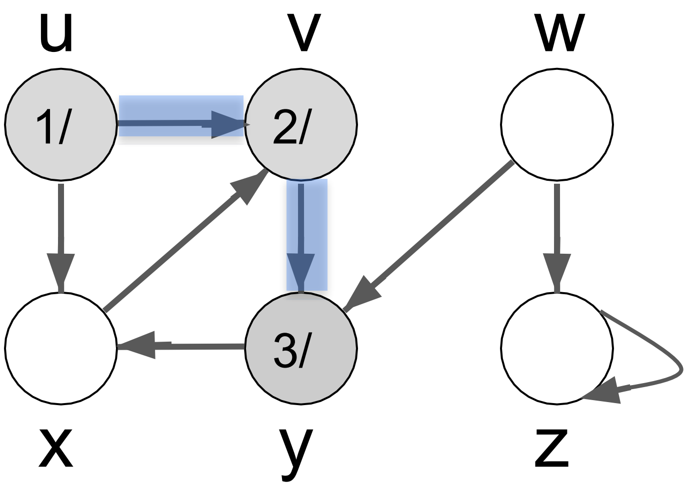
DFS Example



DFS Example



DFS Example



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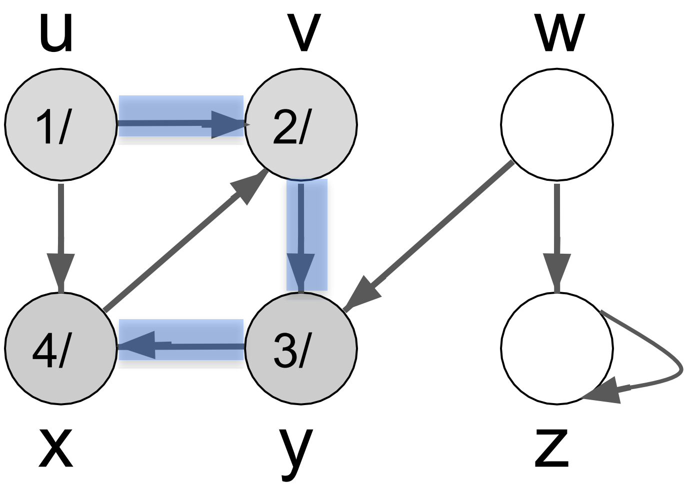

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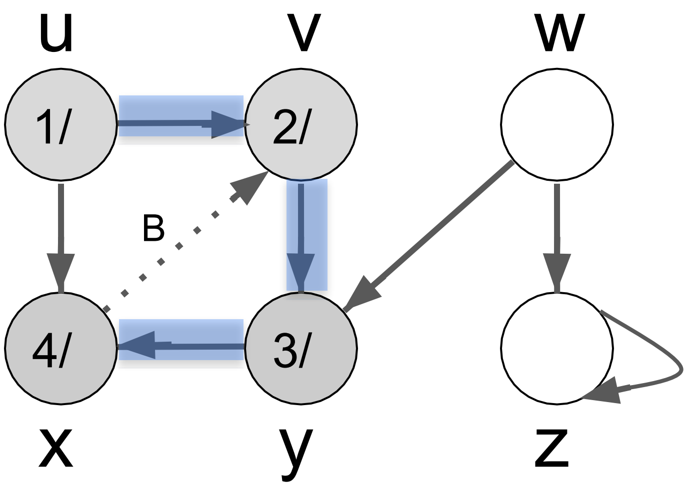

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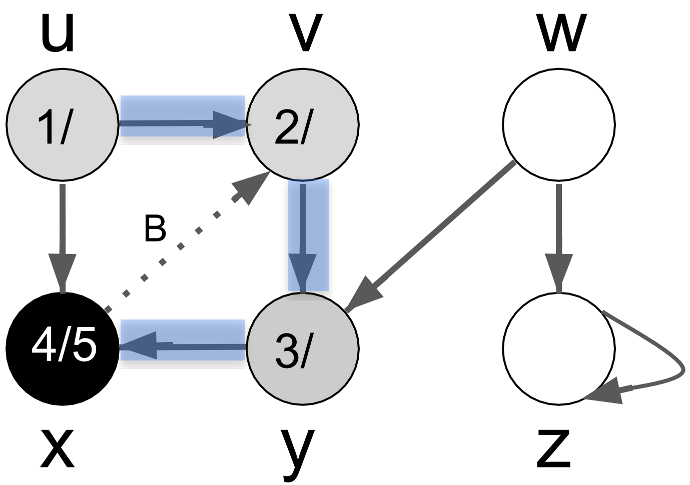

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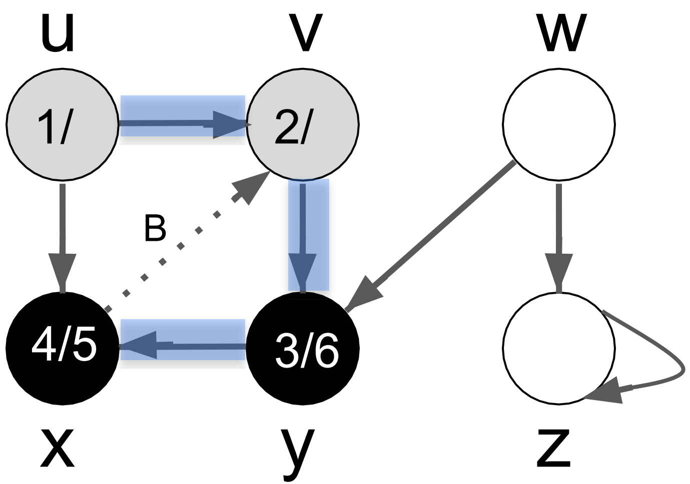

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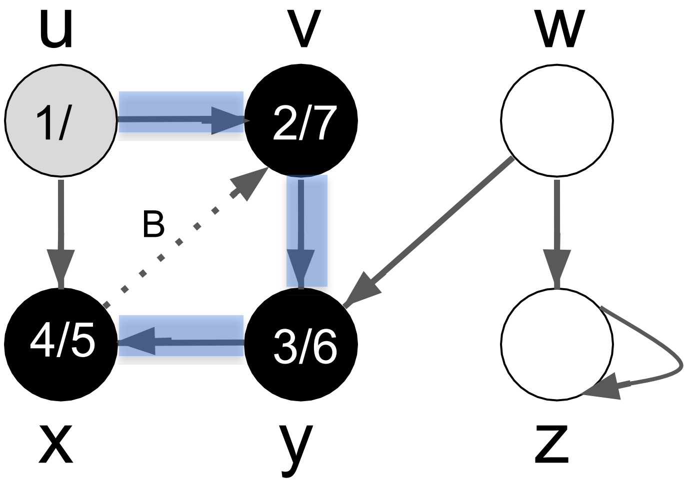

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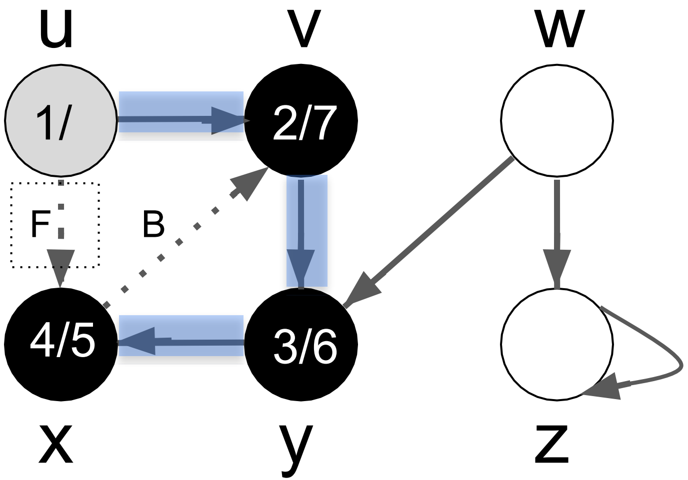

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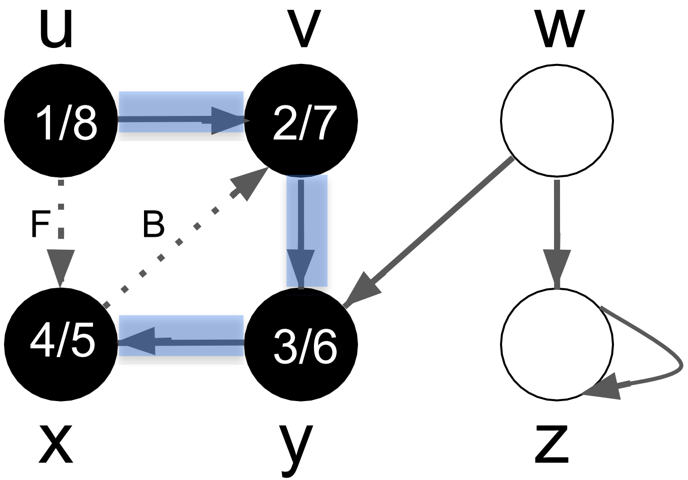
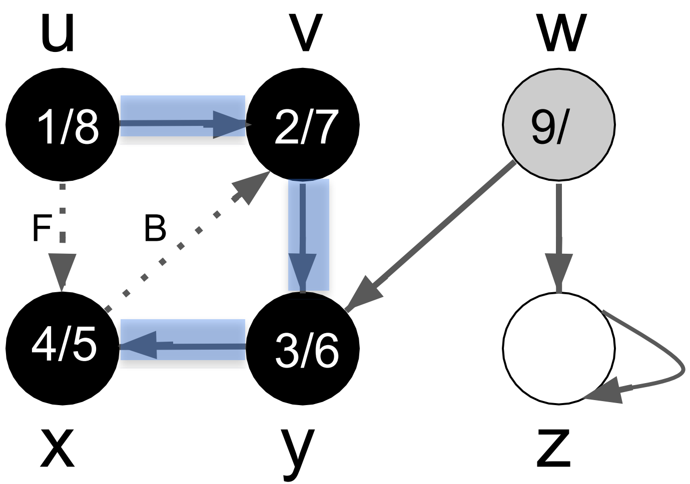


DFS Example



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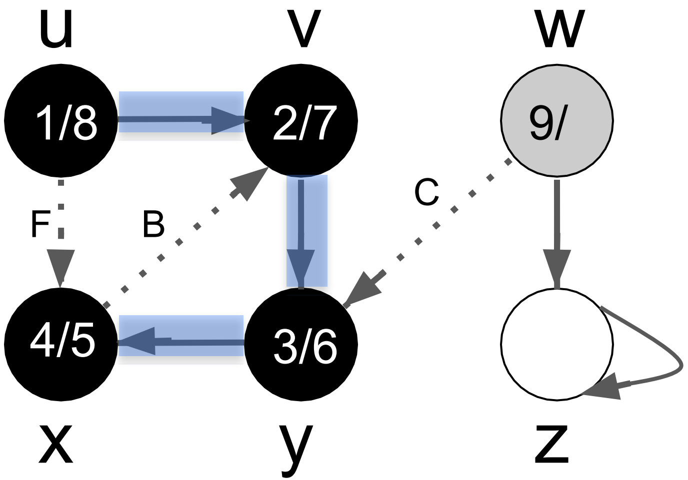

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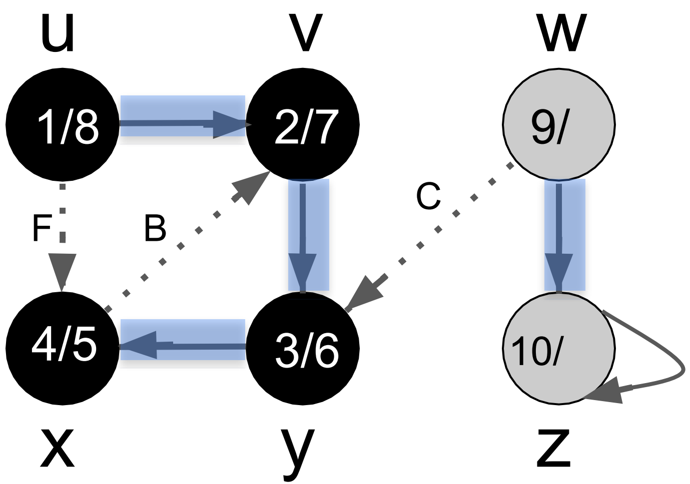

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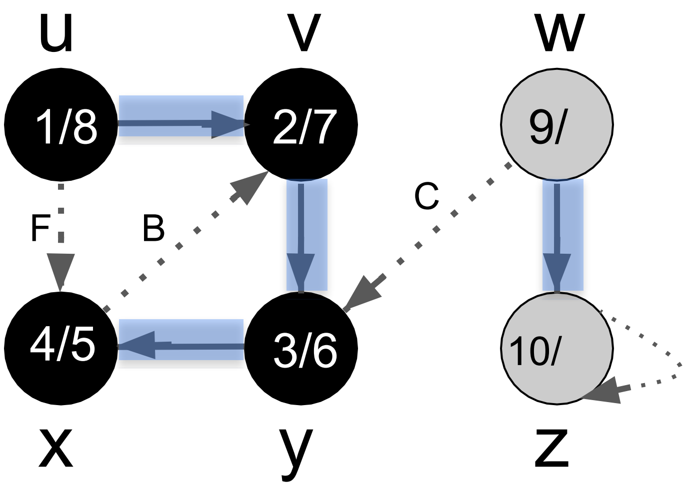

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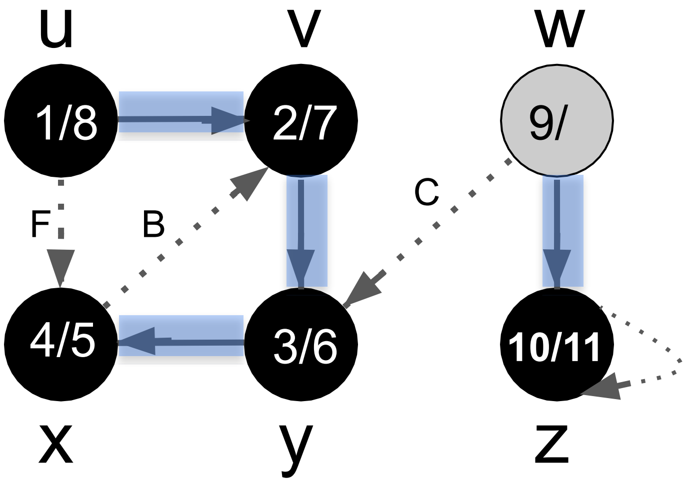

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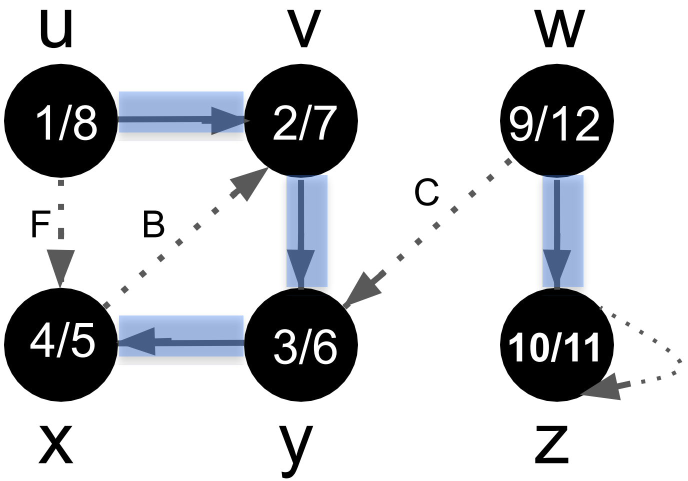
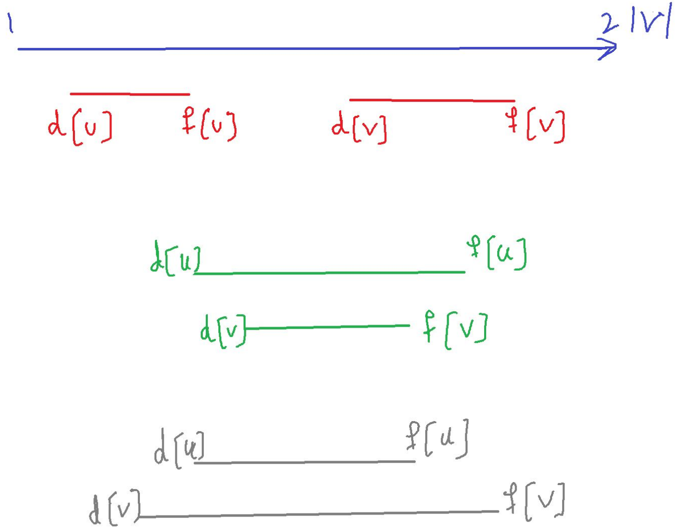
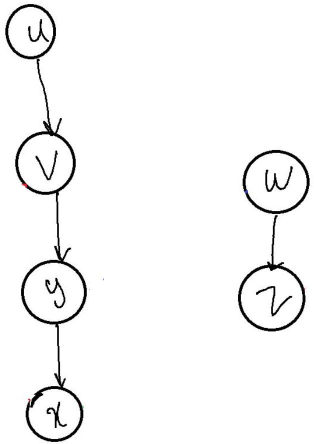
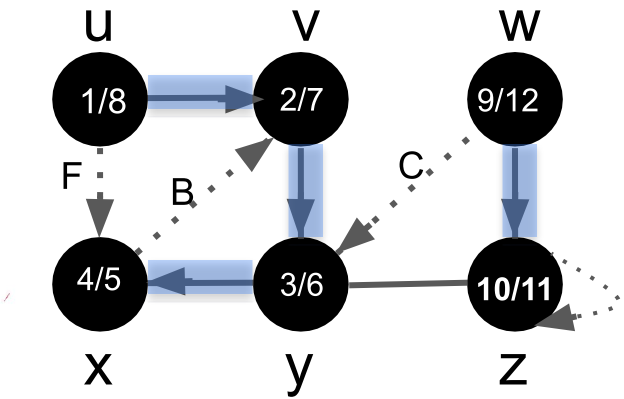


DFS properties: Parenthesis theorem

- In any DFS of a graph G, for any two vertices **u** and **v**, the intervals [d[u], f[u]] anad [d[v], f[v]] are either nested or disjoint
  - If two intervals overlap, then one is nested within the other: no partial overlap
  - Vertex corresponding to smaller interval is a descendant of the vertex corresponding to the larger one

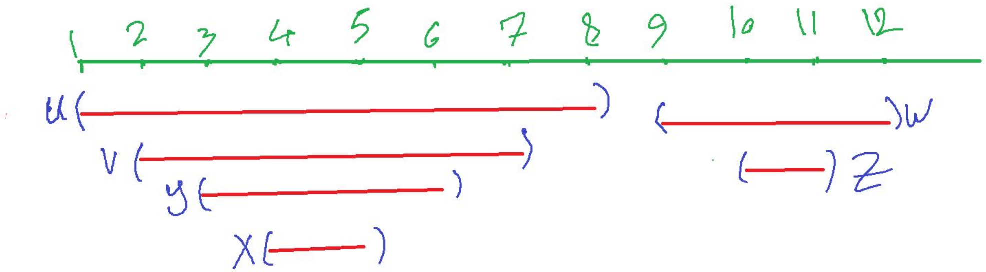


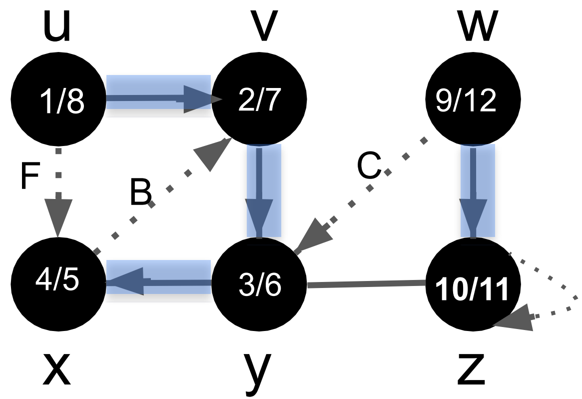




DFS properties

- Why is it called *parentheses* theorem
  - discovery time of a vertex: (
  - finish time of a vertex:)
  - The history of discoveries and finishing times makes a well-formed expression
    - Parenthesis are properly nested





DFS Properties

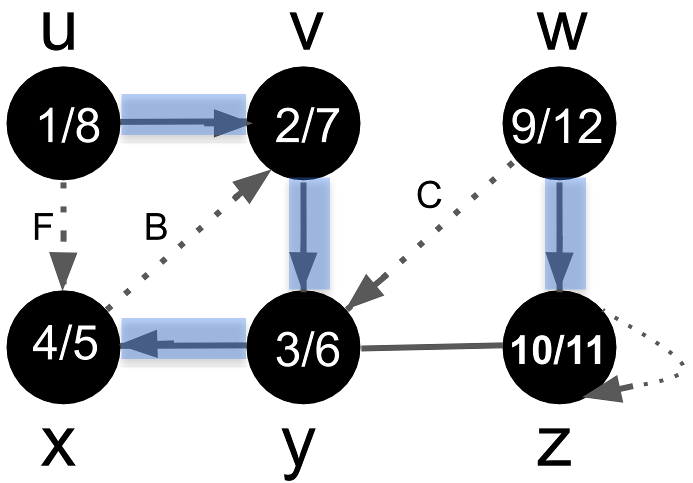
- Vertex v is a proper descendant of vertex u in the depth-first forest of graph G if and only if $d[u] < d[v] < f[v] < f[u]$
  - Follows from parenthesis theorem

DFS: Classification of edges: directed graph

- When we first explore an edge (u, v), the color of vertex v determine edge class
- WHITE: a tree edge
- GRAY: a back edge
- BLACK: a forward or cross edge
  - d[u] < d[v]
    - Forward edge
  - d[u] > d[v]
    - Cross edge

DFS: Classification of edges: undirected graph

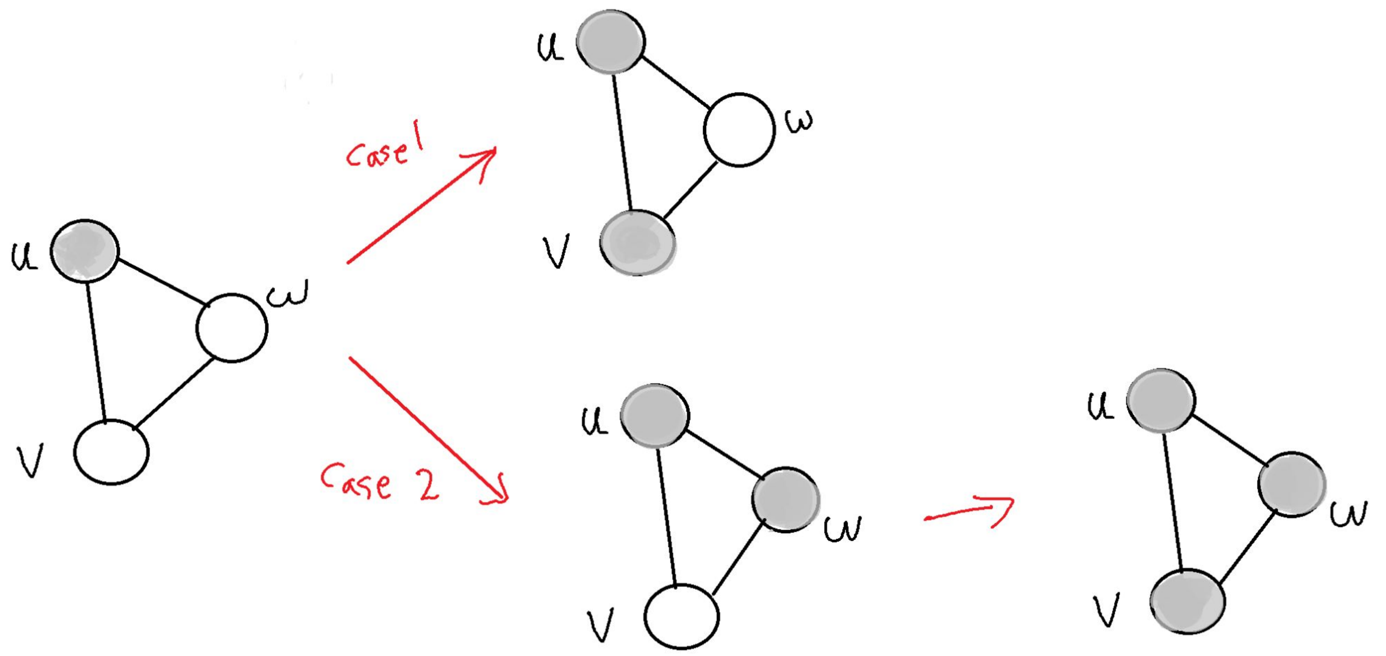
- In a DFS of an undirected graph G, every edge of the tree:
  - Tree edge
  - Back edge
- There are no cross or forward edges



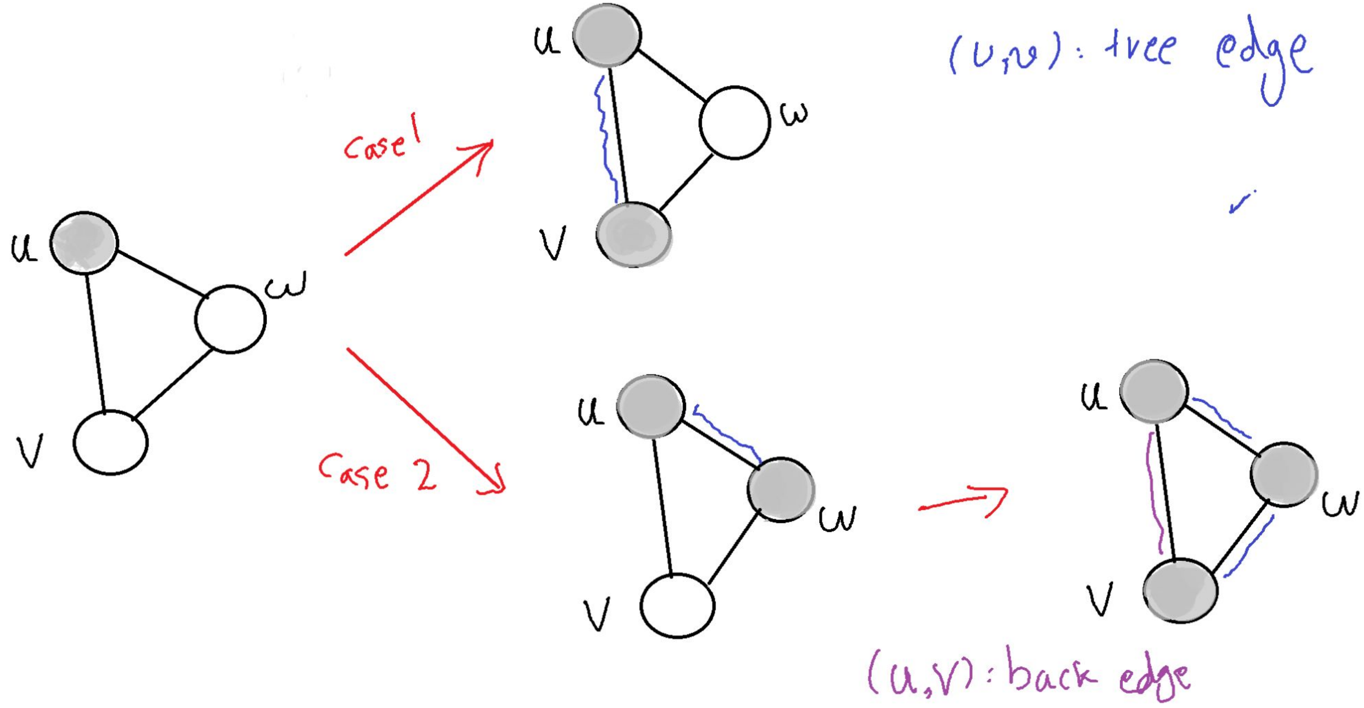
DFS: Classification of edges: undirected graph

- In a DFS of an undirected graph G, every edge of the tree:
  - Tree edge
  - Back edge
- There are no cross or forward edges
- Proof. Let (u, v) be an edge in G and assume WLOG d[u] < d[v]
  - d[u] < d[v]
    - u becomes gray first, vertex v is finished before u
    - Once u becomes grays there will be two possibilities
      - The neighbor v is discovered → (u,v) becomes a tree edge
      - Some other neighbor w will be discovered and that neighbor discovers v → (u, v) becomes a back edge

DFS: Classification of edges: undirected graph

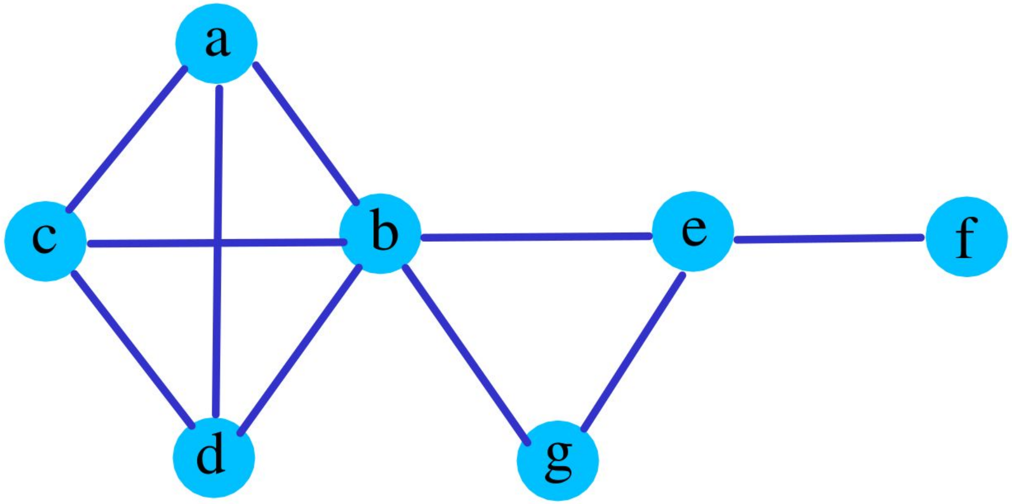


DFS: Classification of edges: undirected graph



Cut vertices and Cut Edges

- Definitions
  - A vertex v is a cut vertex if $G - v$ is not a connected graph.
  - An edge e is a cut edge if $G - e$ is not a connected graph.
- Example:
  - Cut vertices: b, e
  - Cut edges: (e, f)



Cut vertices and Cut Edges

- Problem. Designing an algorithm to identify cut vertices
  - **Input:** Graph G = (V, E)
  - **Output**: identify all cut vertices and cut edges of G

- Brute-force solution:
  - for all v in V
    - Compute $G - \{v\}$, check whether $G - \{v\}$ connected or not
  - Time: O(n (n+m))
  - n=|V|
  - m=|E|

Application of DFS

Detecting cycles in directed graphs

Lemma. A directed graph has a (directed) cycle iff DFS has a back edge.