Date: 14/06/21

Black Board

Design and Analysis of Algorithms

Topics:

-- **Probability Basics**

-- **Randomized Algorithms**

The Hire Assistant Example

Randomized Algo: an algorithm that uses a random step somewhere in its logic.

(Discrete prob)

Random Event (outcome): an outcome of a random experiment with an associate prob.

random experiment: Coin toss, throwing a dice, throwing two coins and taking sum. etc.

Random Variable: a variable containing the value representing an outcome of a random experiment.

e.g. $\underset{X}{\underline{\text{coin toss}}} \begin{cases} \to X=0 \to \text{tails} \\ \to X=1 \to \text{head} \end{cases}$

$\underset{X}{\underline{\text{dice throw}}} \begin{cases} \to 1 \\ 2 \\ 3 \\ \vdots \\ \to 6 \end{cases}$

<u>PDF</u>: a function mapping each value of a R.Vs to its corresponding prob.

<u>Pr</u>

<u>coin toss</u>:

$$\left.\begin{aligned} Pr\{X=0\} &= 0.5 \\ Pr\{X=1\} &= 0.5 \end{aligned}\right\} \text{Bernoulli R.V.}$$

$$\boxed{\sum_{x=1}^{n} Pr\{X=x\} = 1}$$

Uniform prob. Dist.

Every value of the R.V. is equally likely to occur.

e.g. throwing a fair dice. etc.

Expected value: mean, weighted average value of a R.V.

$$\boxed{E[X] = \sum_x Pr\{X=x\} \cdot x}$$

loaded coin.

$$Pr\{X=1\} = 0.8$$
$$Pr\{X=0\} = 0.2$$

$$E[X] = 0.8 \times 1 + 0.2 \times 0 = \underline{0.8}$$

which value of the r.v. I am likely to be closer to.

(usually)

E(X) tells us which value of X has a higher chance of occurring

$T(n)$, $E[T(n)] = O(n \lg n)$

average case analysis.

this algo is likely to take $n \lg n$.

Example:

$X_1 = 1$, heads coin 1

$X_1 = 0$, tails coin 1

$X_2 = 1$, heads coin 2

$X_2 = 0$, ~~heads~~ tails coin 2

$E[X_1 + X_2] :=$ # of heads in two tosses.

$Pr\{X_1 = 1\} = 0.7$ $\quad Pr\{X_1 = 0\} = 0.3$

$Pr\{X_2 = 1\} = 0.1$, $\quad Pr\{X_2 = 0\} = 0.9$

$0.9 \qquad\qquad 0.1$

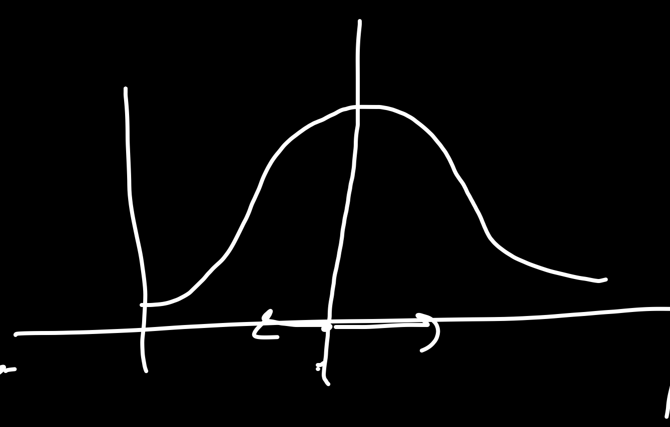
$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

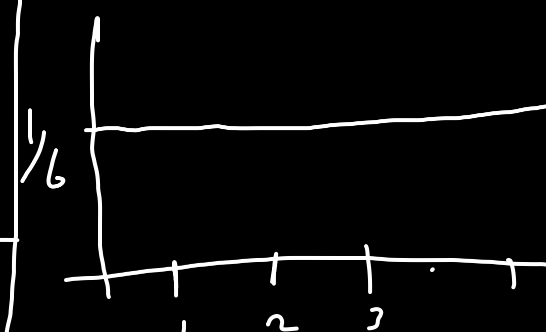
$$= [0.7 + 0.9]$$

$$E[X_1 + X_2] = \cancel{0.8} \quad 1.6$$

Average

⇒ likely value





Variance.

low → mean happen with high prob.

$$1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) \;\cdots\; + 6\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\left(1 + 2 + 3 + 4 + 5 + 6\right) = \frac{21}{6} = \frac{7}{2} = 3.5$$

The Hire Assistant Example

$\rightarrow O(n)$ rand c++

RandomShuffle(A, n)

HireAssistant (A, n)

bestScore = 0
bestCand = 0
For each c = 1 to n
  s = interview(A, c) $\rightarrow c_i$
  IF s > bestScore
    bestScore = s
    bestCand = c
    hire(A, c) $\rightarrow c_h$

return (bestCand, bestScore)

$T(n)$

If the nth candidate is best so far he is hired.

$c_i \ll c_h$

$$Cost(n) = O(c_i n + c_h n)$$

worst case

Worst case scenario: when the CVs are ordered in increasing order of quality.

When the CVs' order is randomized, then every case is equally likely, because all $n!$ permutations of the CVs is equally likely.

A B C
A C B
B C A
B A C
C B A
C A B

$1/6$

$1/6$

$\frac{1}{6}$

⋮

$\frac{1}{52!}$ card shuffling.

Every order of CVs ~~is~~ occurs with prob. $\frac{1}{n!}$

Cost(n)

how is cost(n) impacted by randomshuffle

$$\text{cost}(n) = O(c_i n + c_h n) \rightarrow$$ just one of $n!$ orders.

$N_h = x_1 + x_2 + x_3 + x_4 + x_5$

$x_1 = 1$

$x_2 = 1$

$x_3 = 0$

$x_4 = 1$

$x_5 = 0$

1	2	3	4	5
3	5	2	6	4
H	H		H	

$$\boxed{\text{Cost}(n) = O(c_i n + c_h N_h)}$$

N_h = number of ppl hire

↳ is a random variable.

$N_h = \{1, 2, \dots, n\}$

$$E[\text{cost}(n)] = E[c_i n + c_h N_h]$$

$$= E[c_i n] + E[c_h N_h]$$

$$\boxed{E[\text{cost}(n)] = c_i n + c_h E[N_h]}$$

What is $E[N_h]$ expressed in terms of n?

Indicator R.V. $X_i := \begin{cases} 0 & \text{candidate } i \text{ was not hired} \\ 1 & \text{candidate } i \text{ was hired} \end{cases}$

$$\boxed{N_h = \sum_{i=1}^{n} X_i}$$

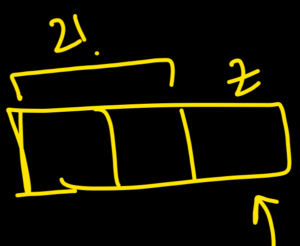
$$E[N_h] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

apply the def of x. value

$$E[X_i] = \Pr\{X_i=0\}\cdot 0 + \Pr\{X_i=1\}\cdot 1$$

$$\boxed{E[X_i] = \Pr\{X_i=1\}}$$

Prob that the i^{th} candidate was hired.



$$\frac{2!}{3!} = \frac{1}{3}$$

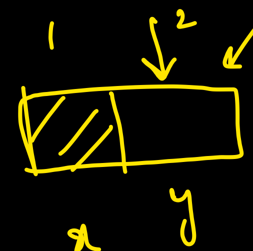
4 5

x y

x, y, z

$z > x, y$

$y > x$



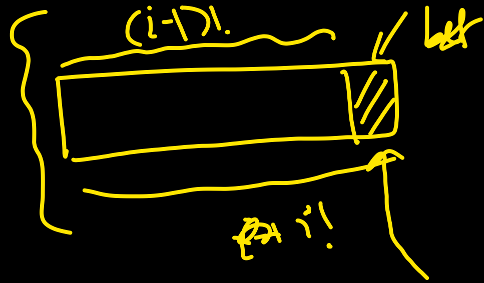
$$\Pr\{X_1=1\} = 1$$

$$\Pr\{X_2=1\} = \frac{1}{2}$$

$$\Pr\{X_3=1\} = \frac{1}{3}$$

$$\vdots$$

$$\Pr\{X_i=1\} = \frac{1}{i}$$



$$\frac{(i-1)!}{i!} = \frac{1}{i}$$

$$E[N_n] = \sum_{i=1}^{n} \frac{1}{i} \longrightarrow 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} = H_n$$

$$H_n = O(\lg n)$$

$$E[N_n] = O(\lg n)$$

$$\boxed{E[\mathrm{cost}(n)] = O(c_i n + c_h \lg n)}$$

256

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Upper Bound on the n^{th} Harmonic Number

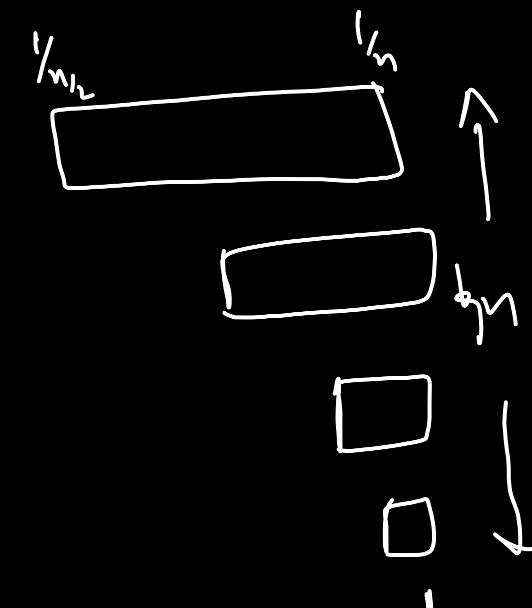
assume $n = 2^K$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le O(\lg n)$$

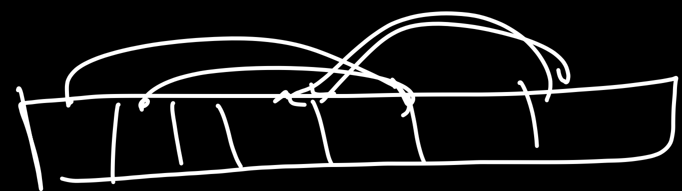
$$= 1 + \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) + \left(\frac{1}{8}+\frac{1}{9}\cdots+\frac{1}{15}\right) + \cdots + \left(\cdots \frac{1}{n}\right)$$

$$\le 1 + \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\cdots\frac{1}{8}\right) + \cdots + \left(\cdots \frac{1}{n/2}\right)$$

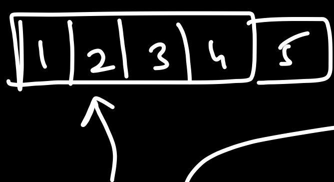
$$\le 1 + 1 + \cdots + 1$$



Random Shuffle

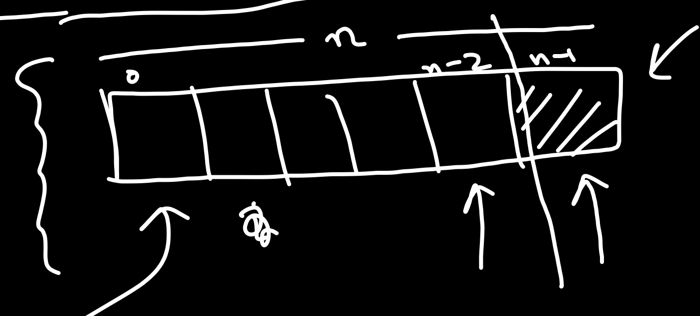


all $n!$ ~~numbers~~ orders should be equally likely



Equivalently: Every number has an equal chance of ending up at any slot.

$\frac{1}{n!}$



i = rand() % n
j = n-1
$\boxed{\text{swap(A[i], A[j])}}$

i = rand() % n-1
j --
swap(A[i], A[j])
⋮

O(n)

n swaps. n times loop