Introduction to Algorithms

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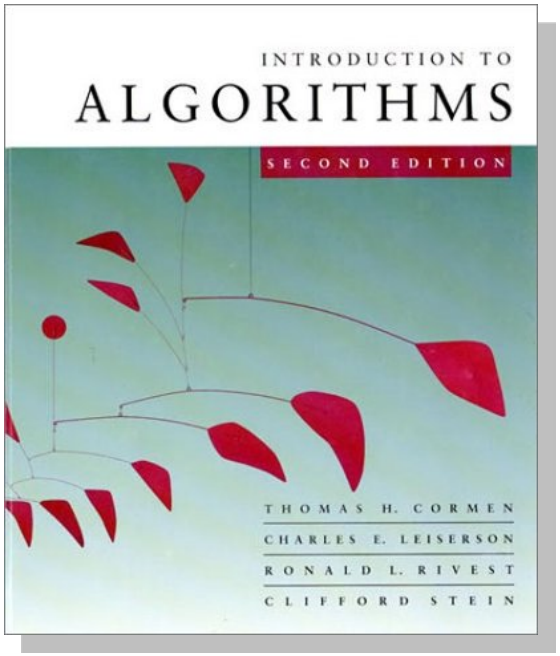

LECTURE 4

Quicksort

- Divide and conquer
- Partitioning
- Worst-case analysis
- Intuition
- Randomized quicksort
- Analysis

Prof. Charles E. Leiserson

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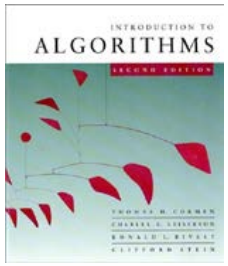

Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).

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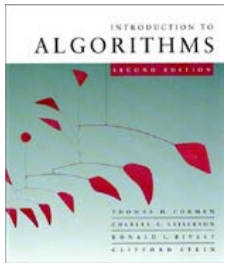

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

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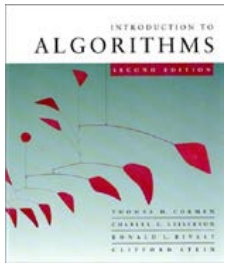
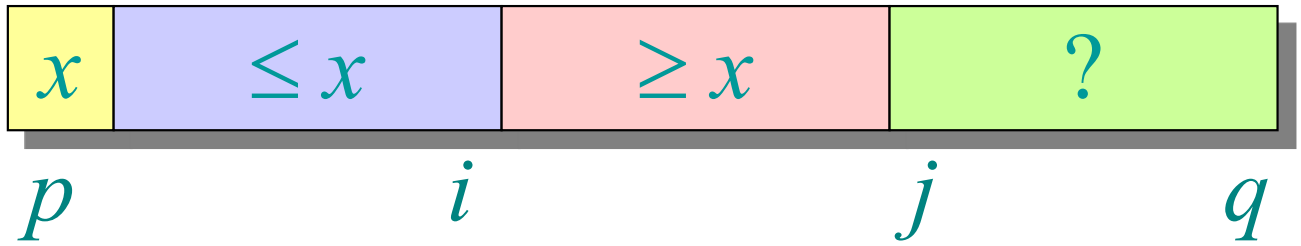


Partitioning subroutine

```
PARTITION(A, p, q)   ▷ A[p . . q]
    x ← A[p]         ▷ pivot = A[p]
    i ← p
    for j ← p + 1 to q
        do if A[j] ≤ x
                then  i ← i + 1
                      exchange A[i] ↔ A[j]
    exchange A[p] ↔ A[i]
    return i
```

Running time $= O(n)$ for n elements.

Invariant:



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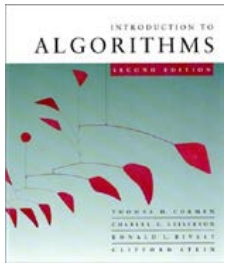

Example of partitioning

6	10	13	5	8	3	2	11

i *j*

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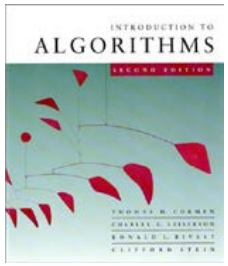

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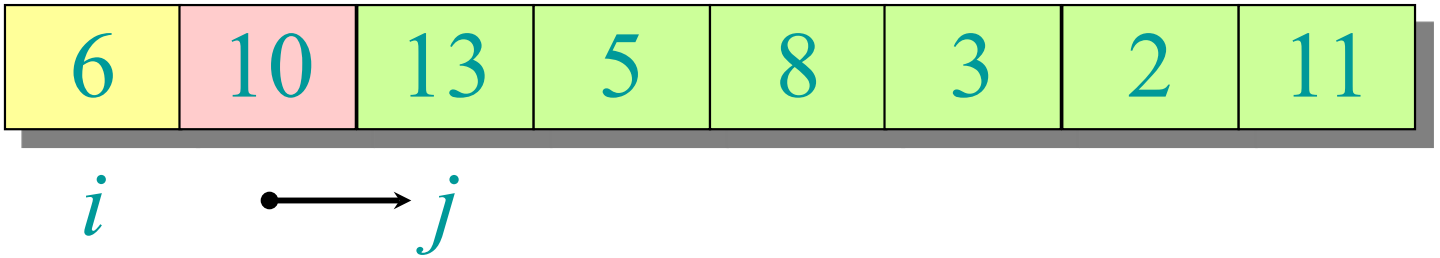

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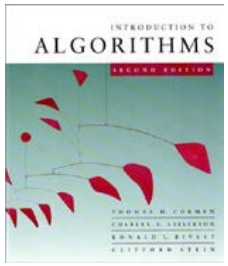

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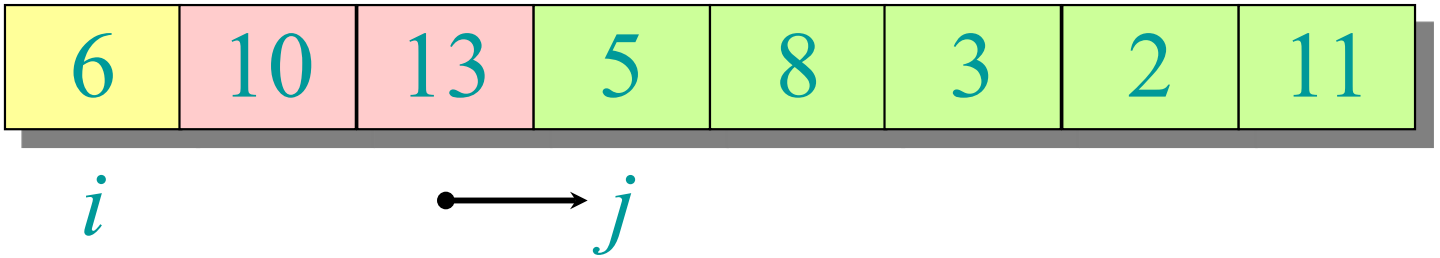

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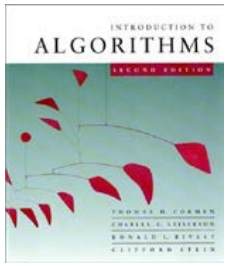

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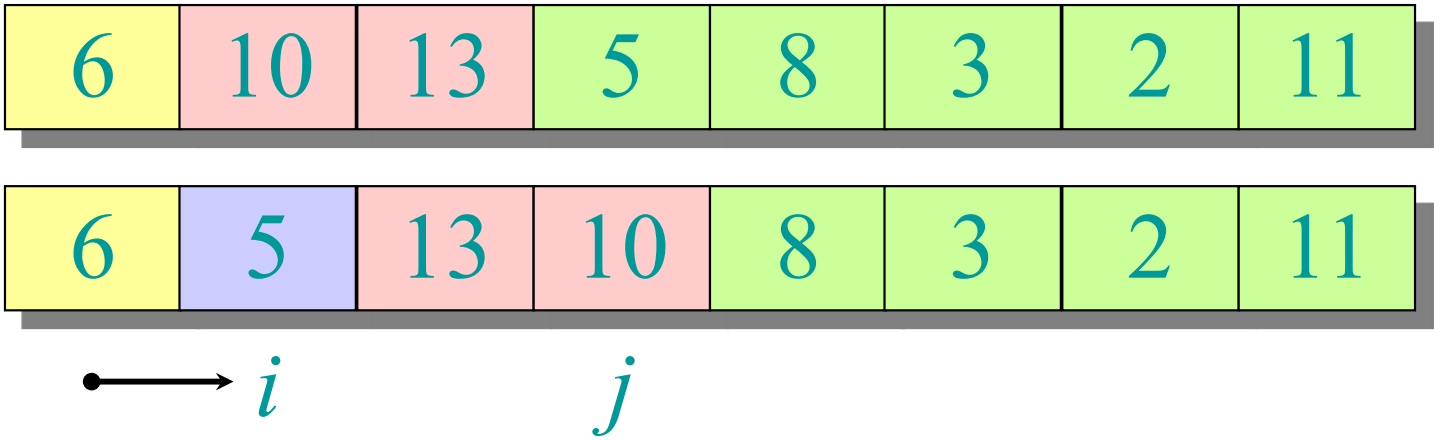

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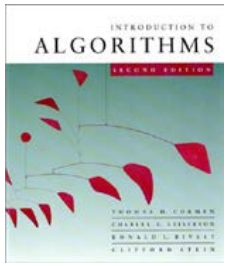

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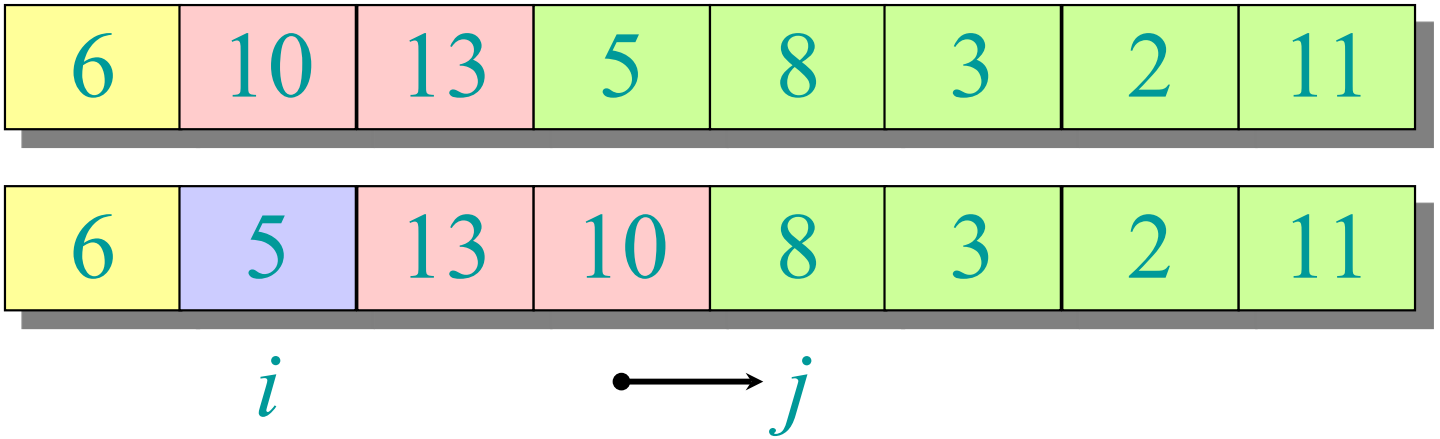

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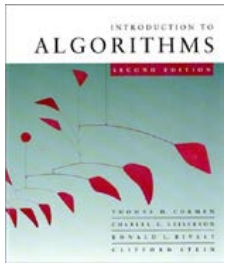

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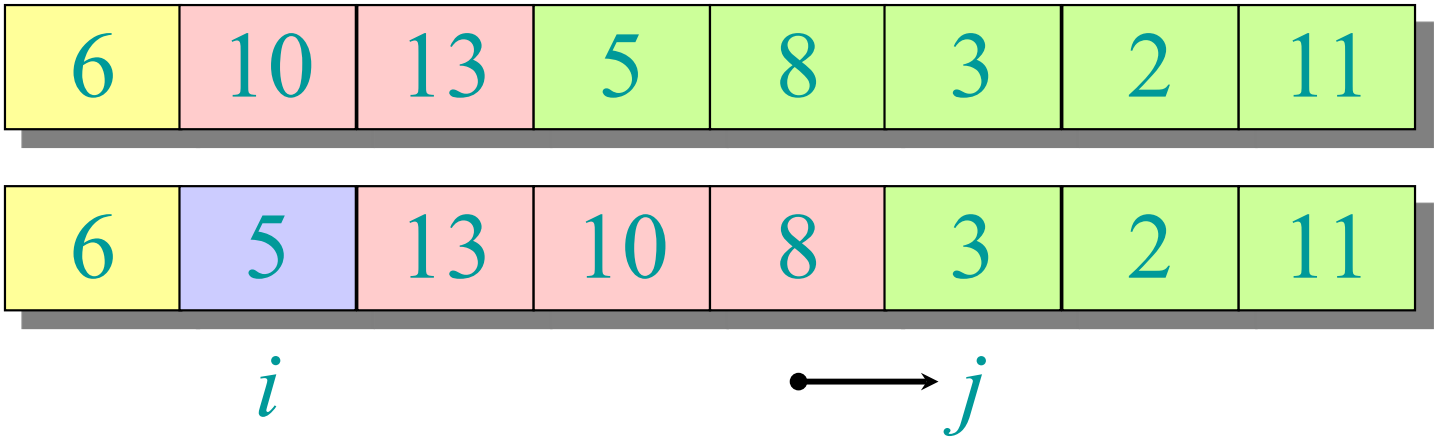

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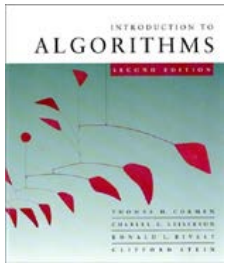

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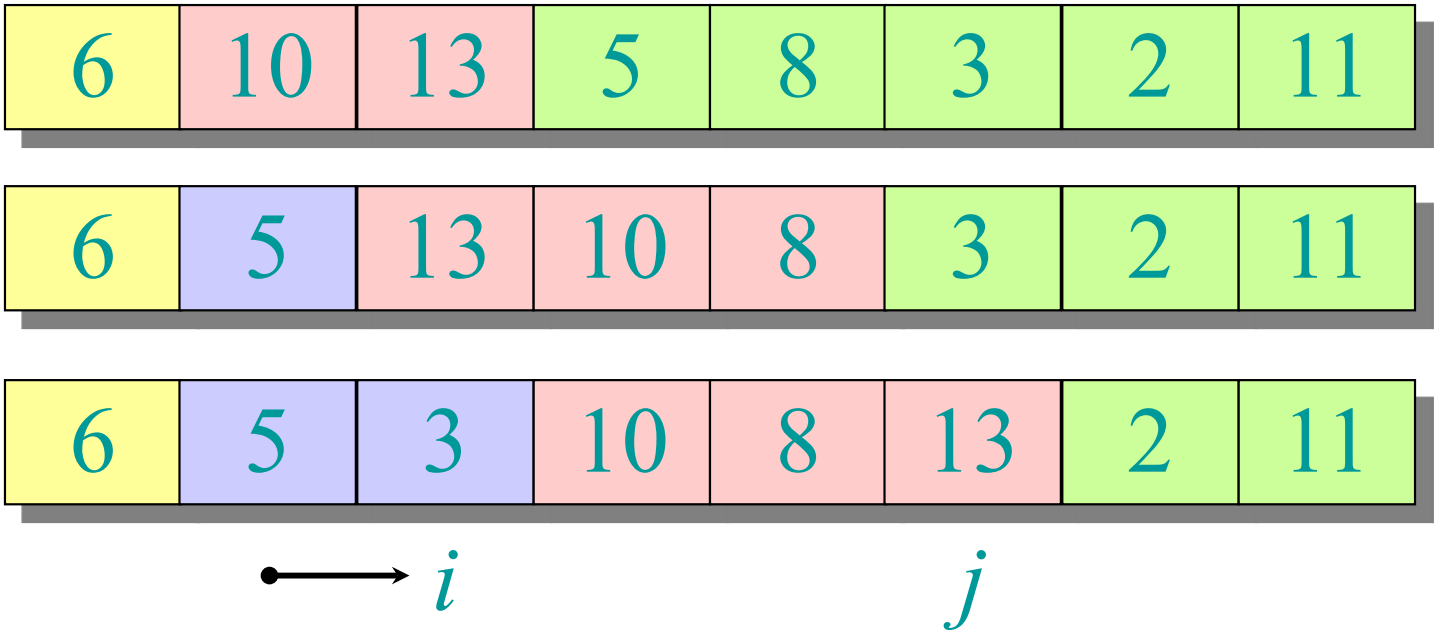

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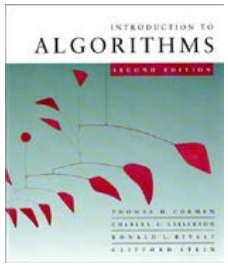

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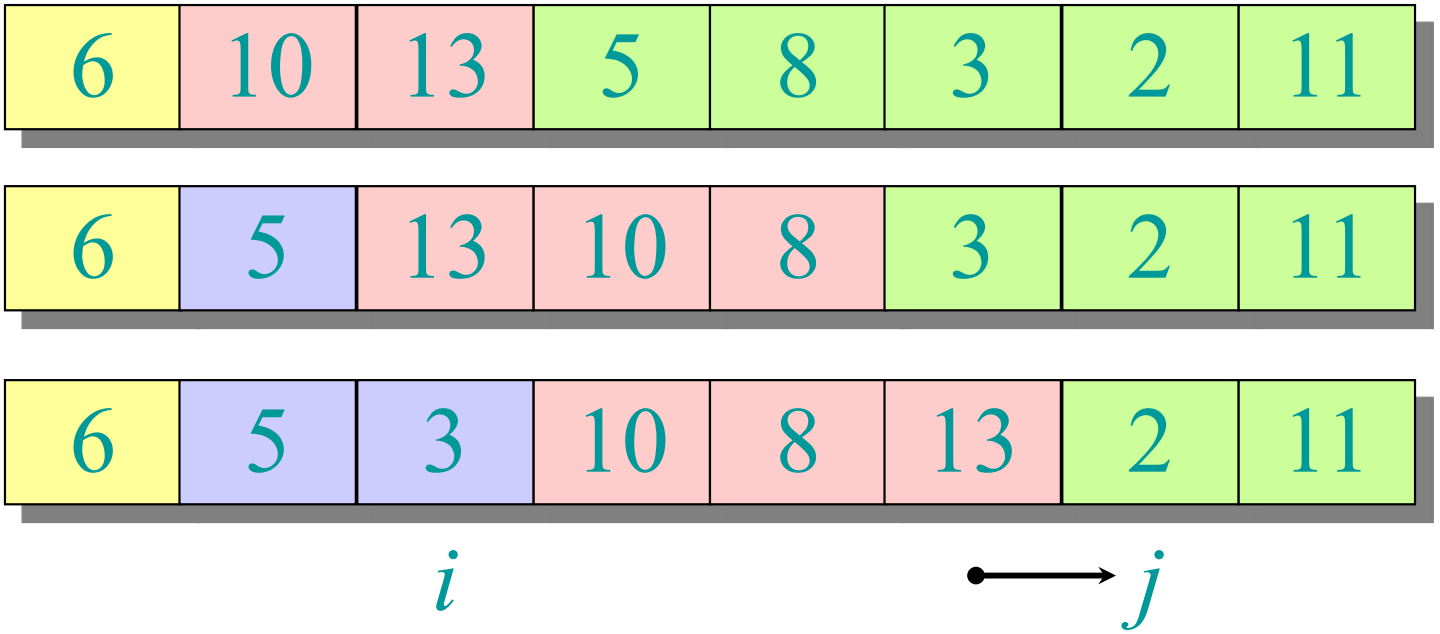

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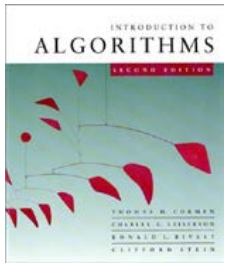

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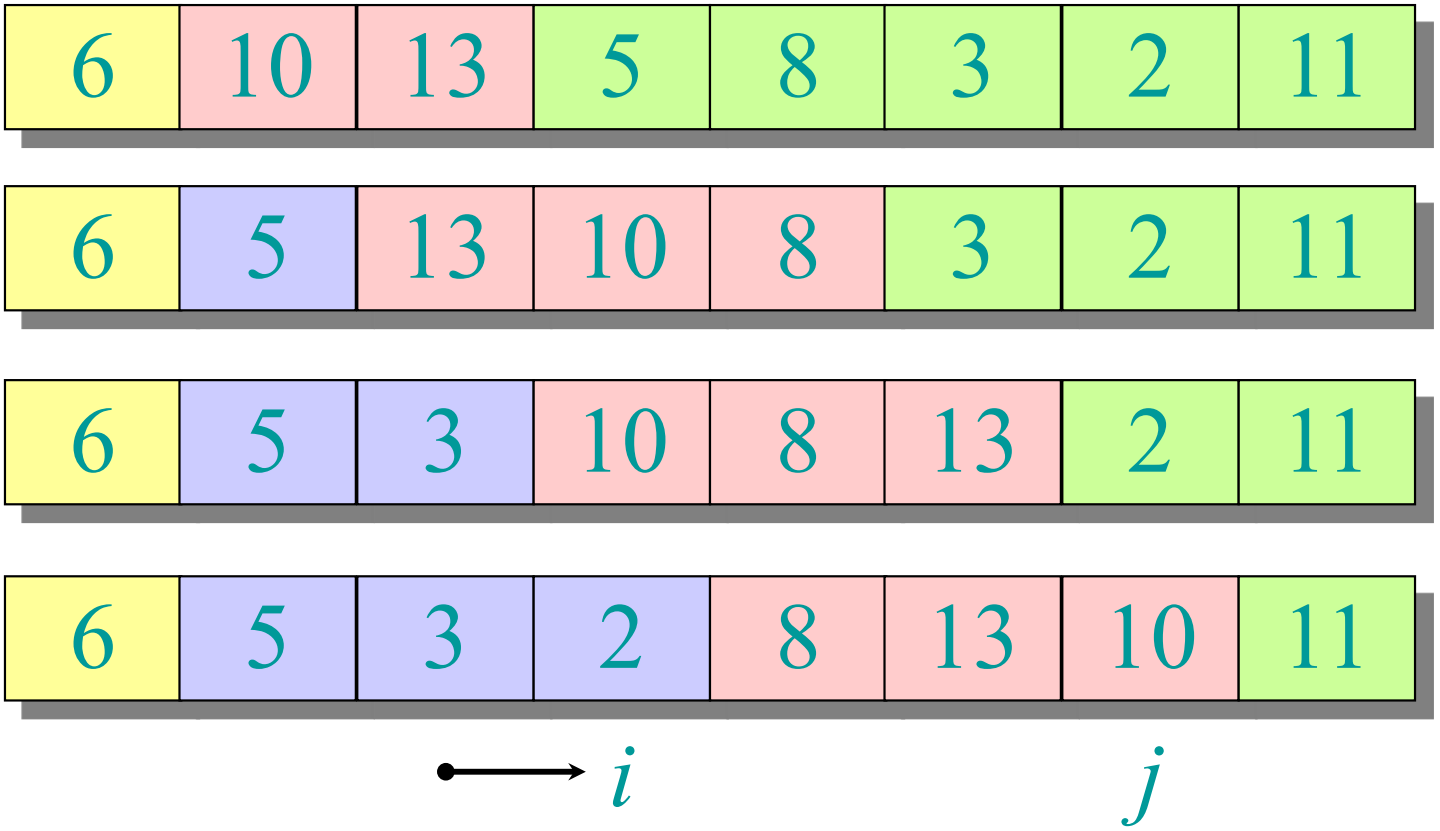

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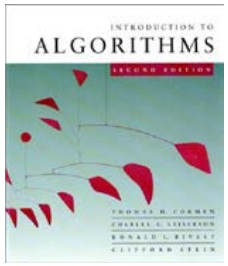

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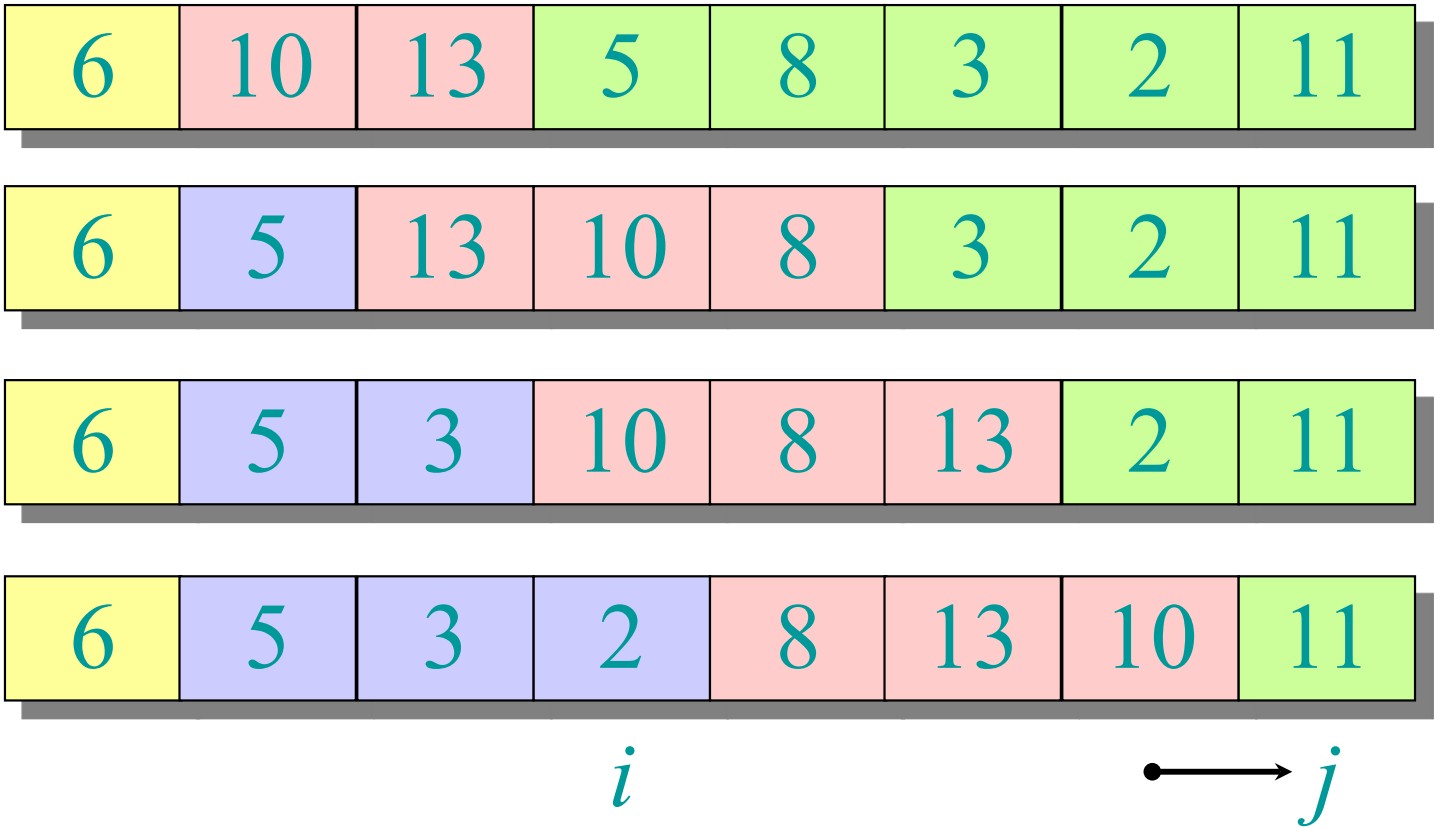

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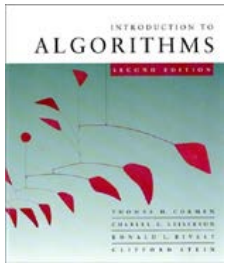

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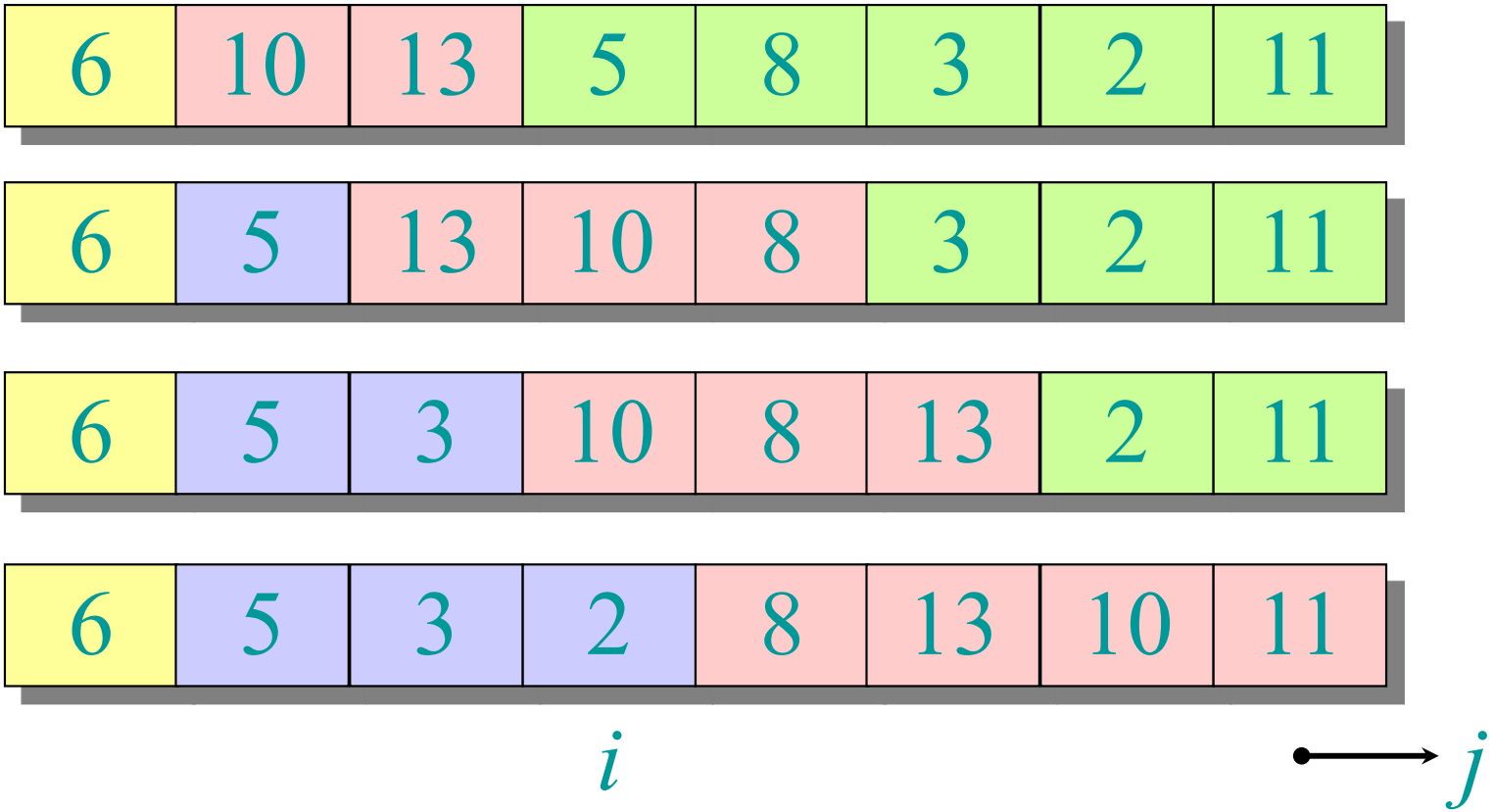

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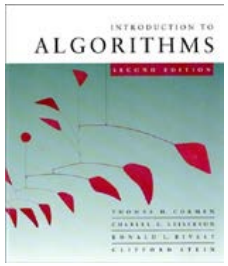

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11

i

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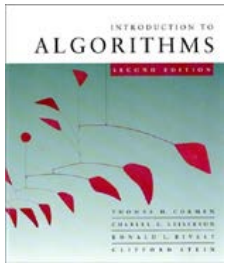

Pseudocode for quicksort

```
QUICKSORT(A, p, r)
    if p < r
        then q ← PARTITION(A, p, r)
             QUICKSORT(A, p, q–1)
             QUICKSORT(A, q+1, r)
```

Initial call: QUICKSORT(*A*, 1, *n*)

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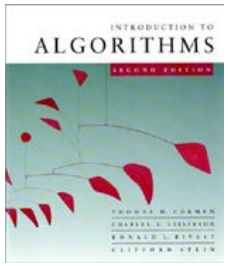

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

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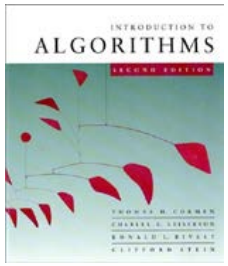

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

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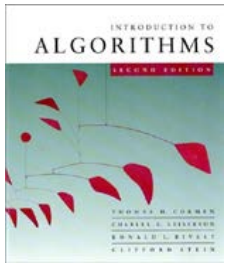

Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$

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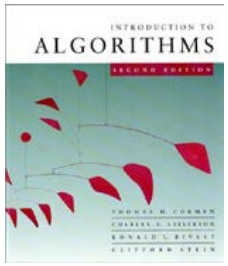

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$T(n)$

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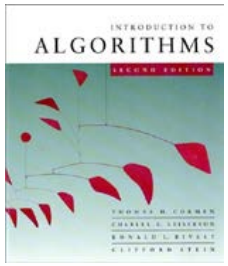

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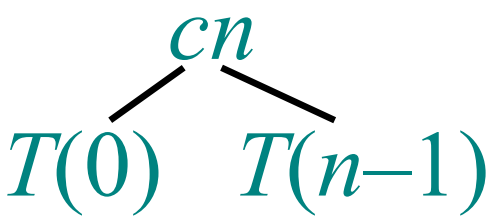

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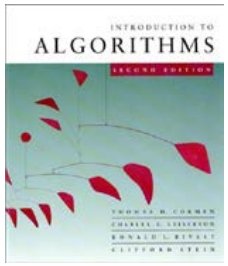

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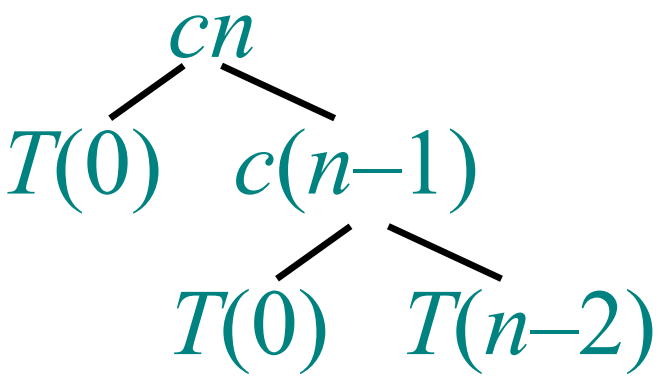

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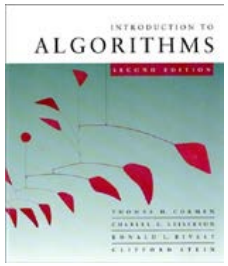

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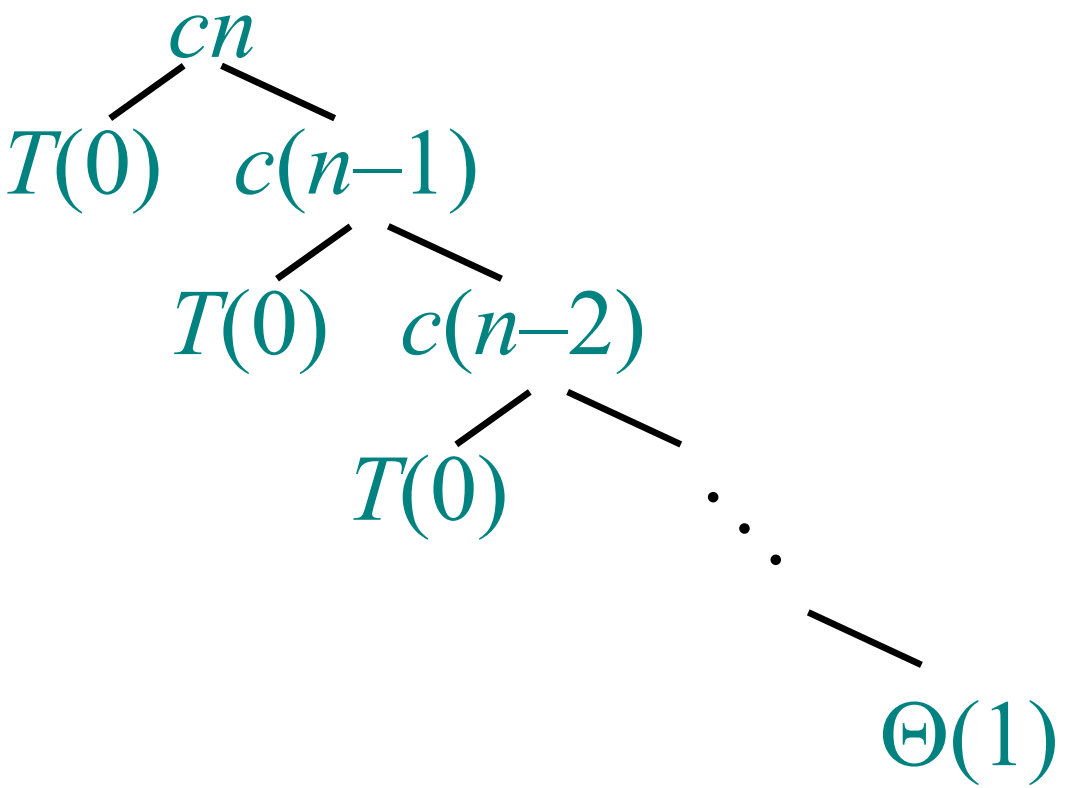

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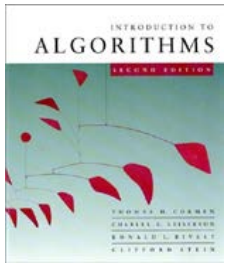

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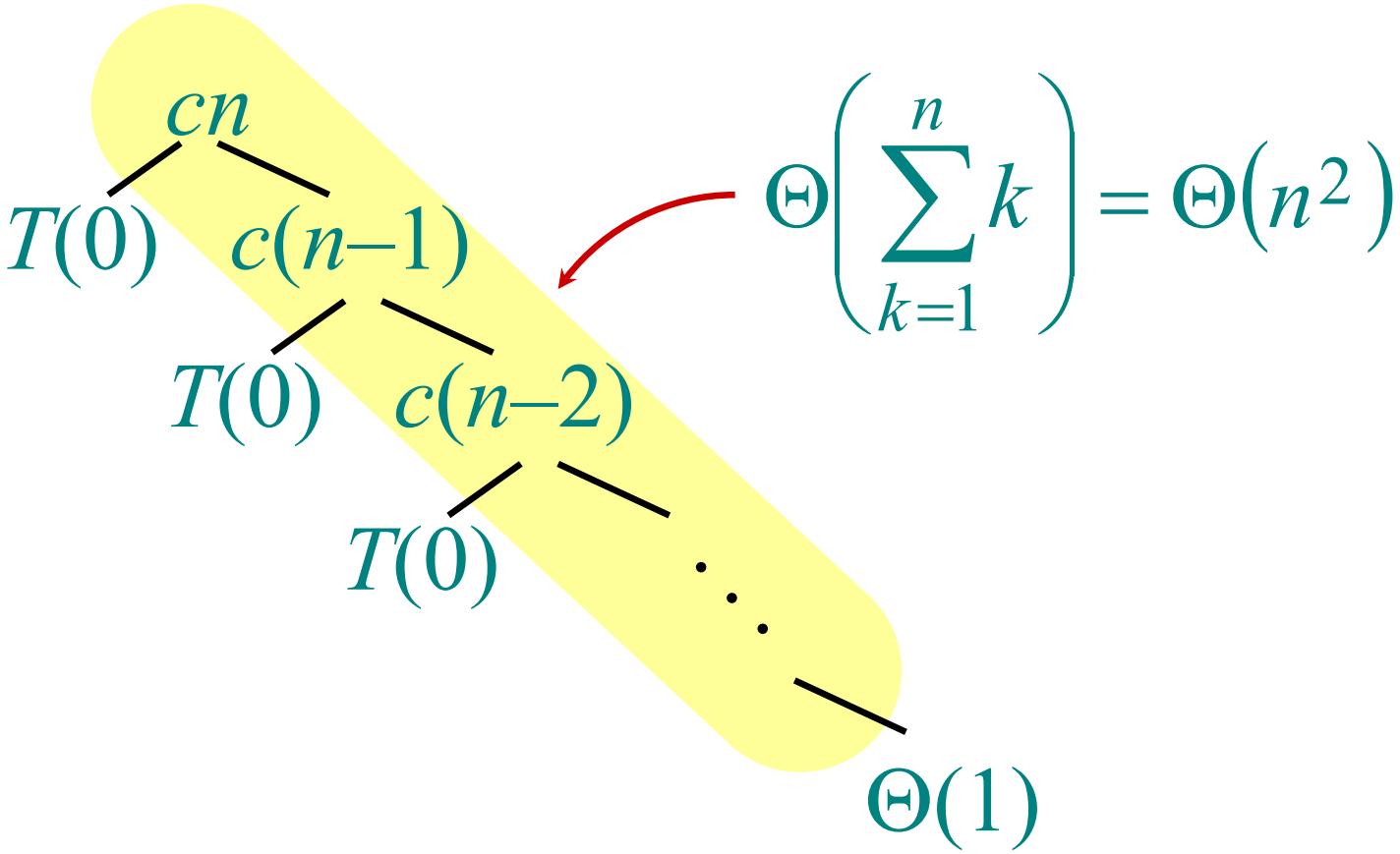

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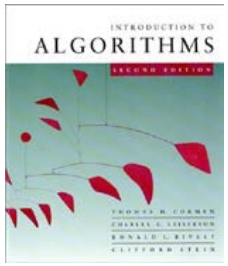

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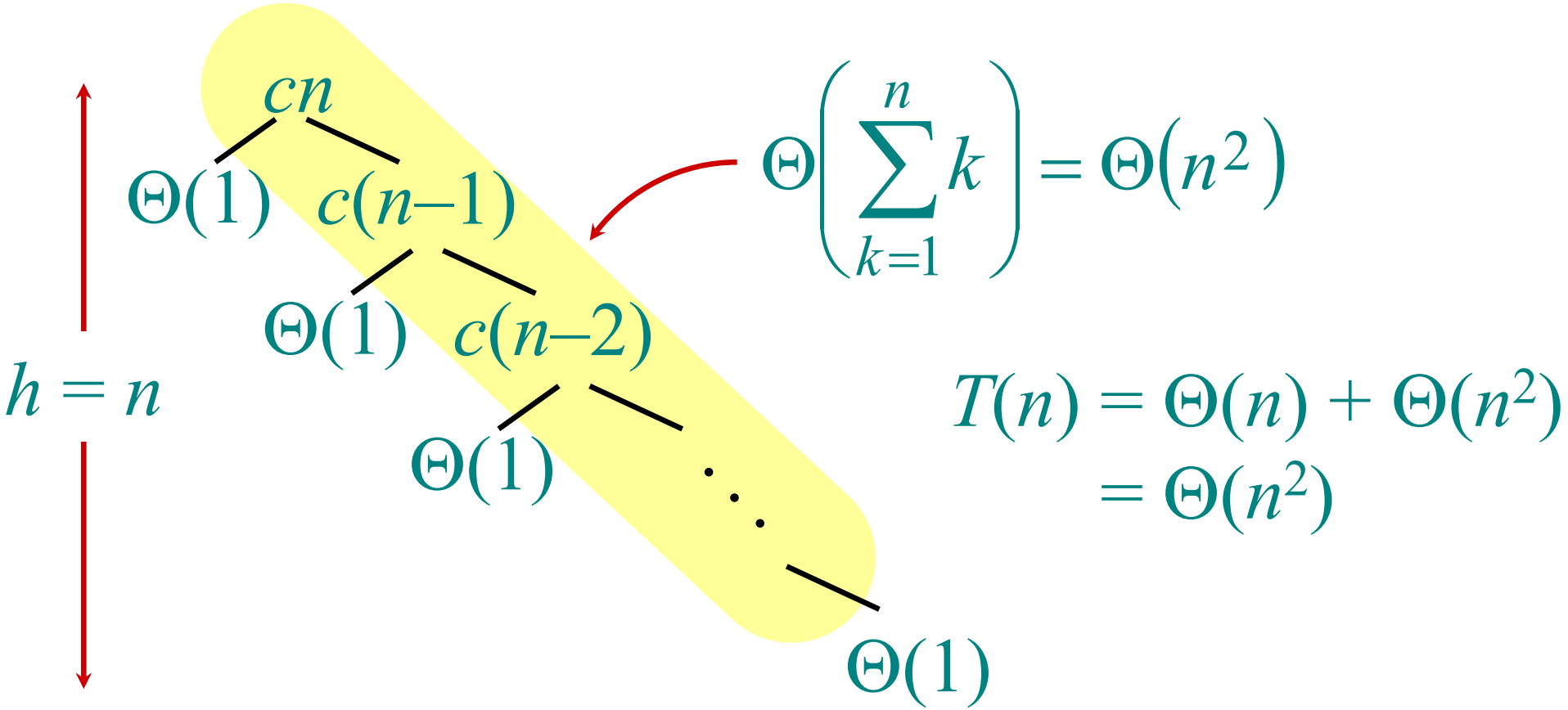

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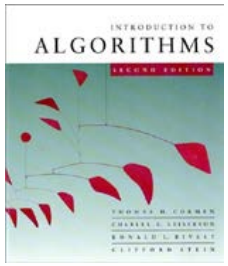

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \lg n) \qquad \text{(same as merge sort)}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

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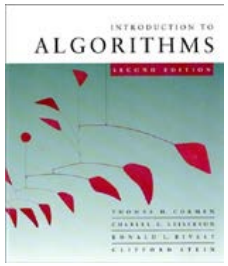

Analysis of “almost-best” case

$T(n)$

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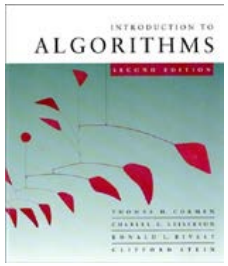
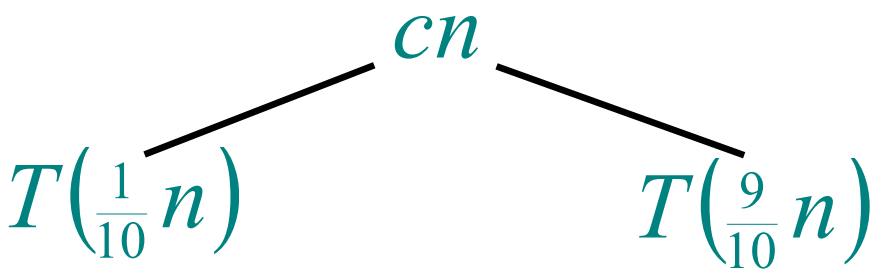


Analysis of "almost-best" case



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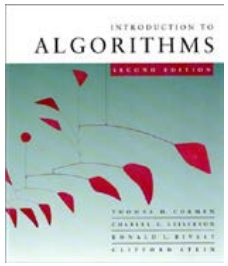

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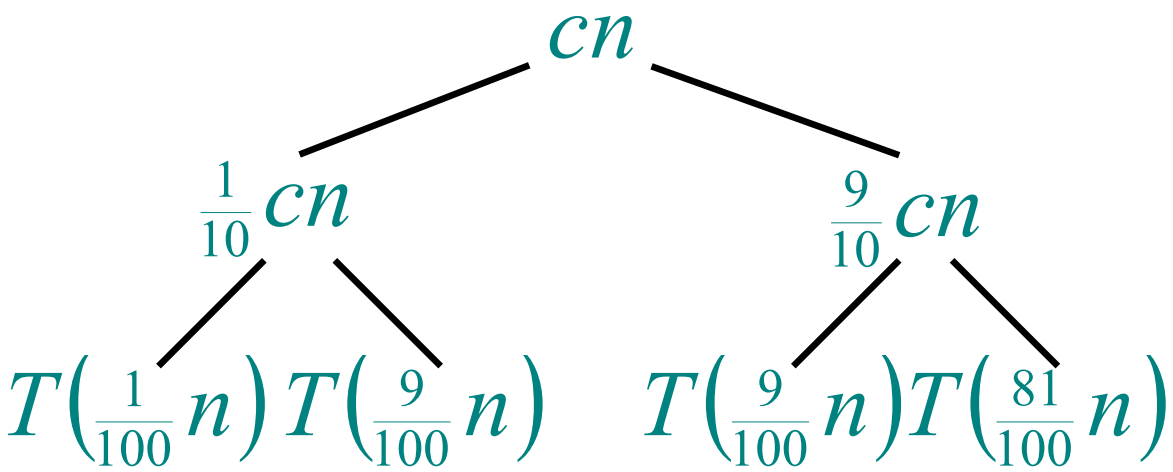

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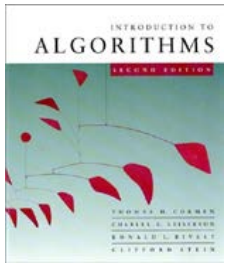

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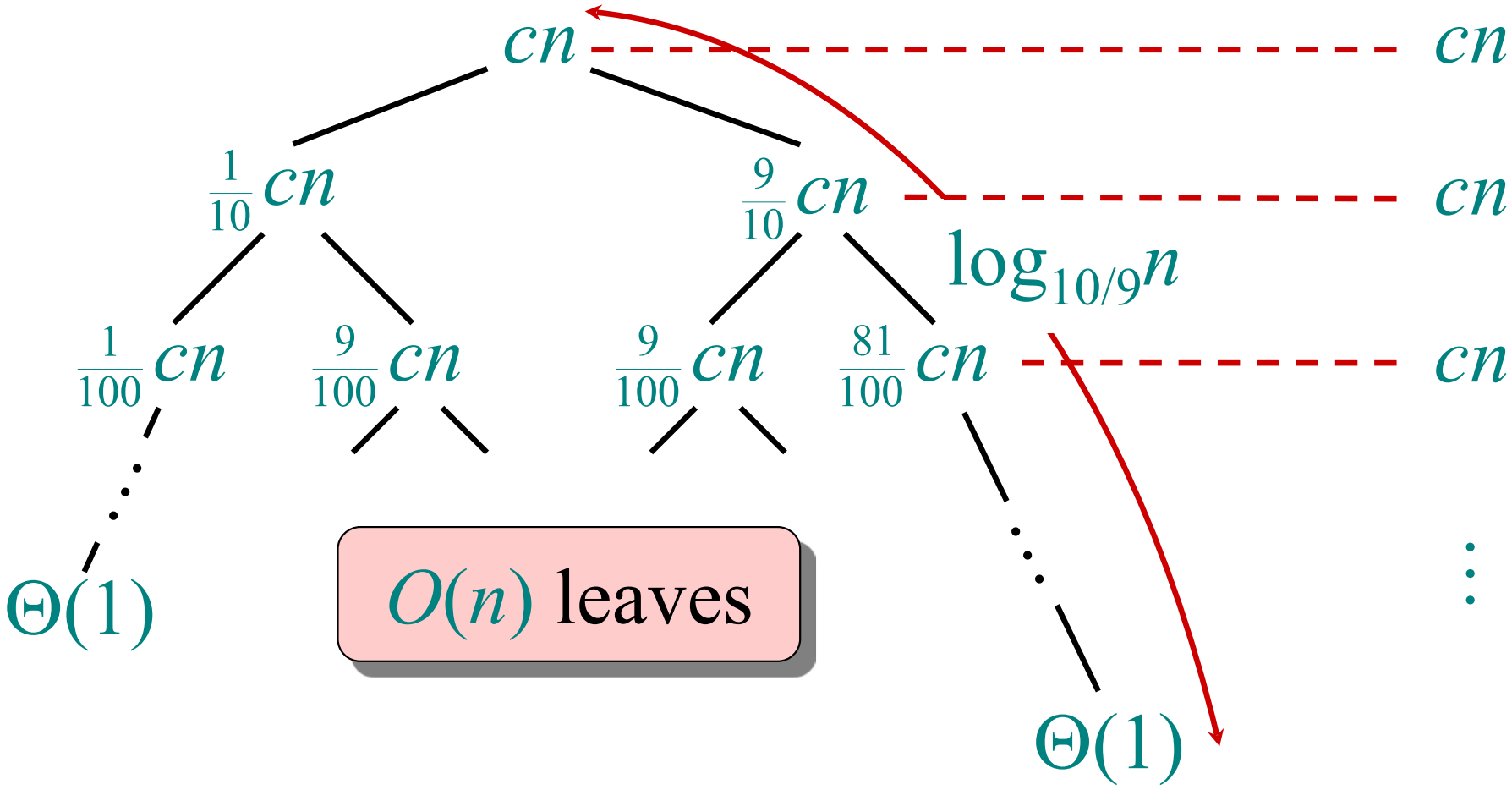

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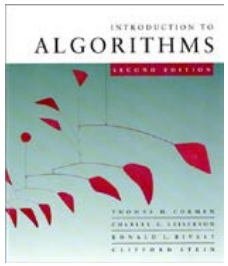

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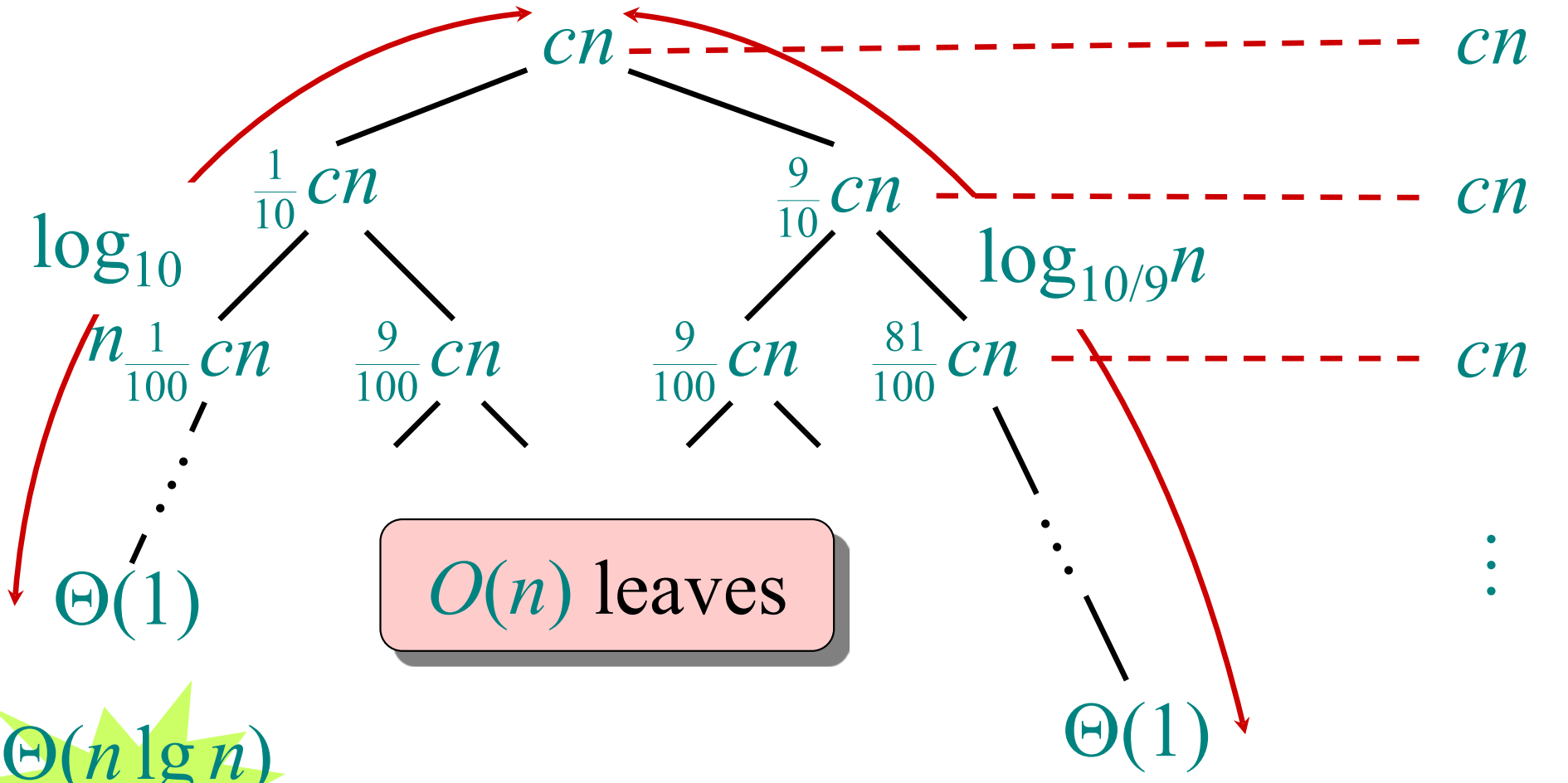

$\Theta(n \lg n)$

Lucky!

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

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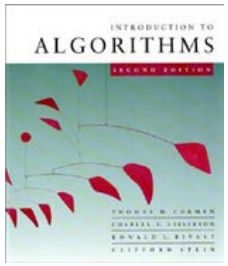

More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$$L(n) = 2U(n/2) + \Theta(n) \quad \textbf{\textit{lucky}}$$
$$U(n) = L(n-1) + \Theta(n) \quad \textbf{\textit{unlucky}}$$

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

Lucky!

How can we make sure we are usually lucky?

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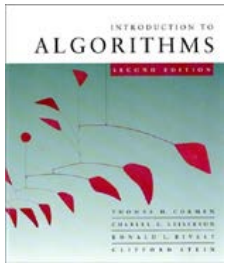

Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

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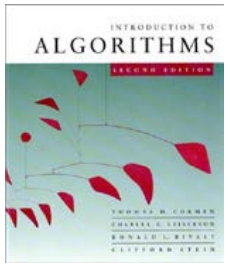

Randomized quicksort analysis

Let $T(n) =$ the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

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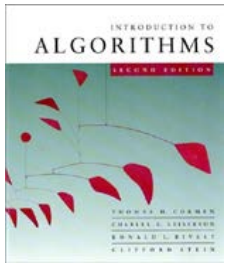

Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$

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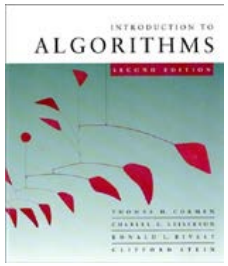

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

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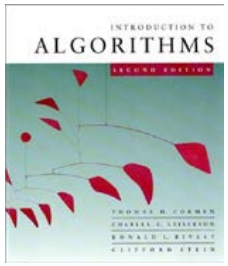

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Linearity of expectation.

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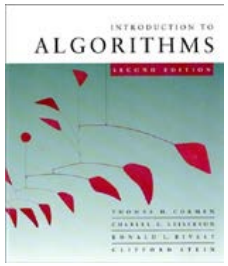

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\right] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

Independence of X_k from other random choices.

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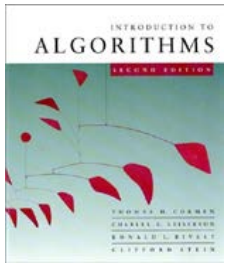

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\right] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(k)\right] + \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(n-k-1)\right] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

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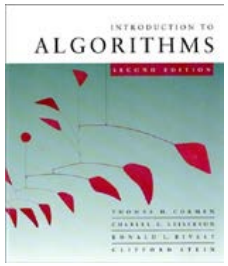

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\right] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(k)\right] + \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(n-k-1)\right] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E\left[T(k)\right] + \Theta(n)$$

Summations have identical terms.

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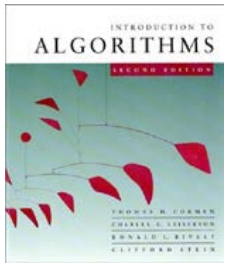

Hairy recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a\,n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a\,n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\displaystyle\sum_{k=2}^{n-1} k \lg k \le \tfrac{1}{2} n^2 \lg n - \tfrac{1}{8} n^2$ (exercise).

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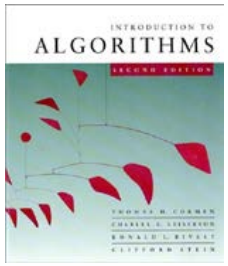

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

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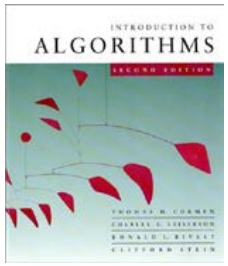

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.

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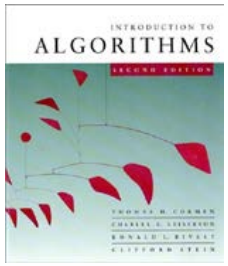

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired – residual***.

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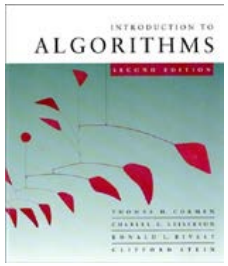

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an \lg n ,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

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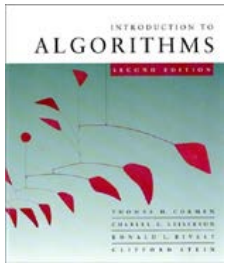

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

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