# Red-Black Trees

Autumn 2011

# Red-Black Trees

- *Red-black trees*:
  - Binary search trees augmented with node color
  - Operations designed to guarantee that the height $h = O(\lg n)$
- First: describe the properties of red-black trees
- Then: prove that these guarantee $h = O(\lg n)$
- Finally: describe operations on red-black trees

# Red-Black Properties

- The *red-black properties*:
  1. Every node is either red or black
  2. Every leaf (NULL pointer) is black
     - Note: this means every "real" node has 2 children
  3. If a node is red, both children are black
     - Note: can't have 2 consecutive reds on a path
  4. Every path from node to descendent leaf contains the same number of black nodes
  5. The root is always black

# Red-Black Trees

- Put example on board and verify properties:
  1. Every node is either red or black
  2. Every leaf (NULL pointer) is black
  3. If a node is red, both children are black
  4. Every path from node to descendent leaf contains the same number of black nodes
  5. The root is always black
- *black-height*: # black nodes on path to leaf
  - Label example with *h* and bh values

# Height of Red-Black Trees

- *What is the minimum black-height of a node with height h?*
- A: a height-$h$ node has black-height $\geq h/2$
- Theorem: A red-black tree with $n$ internal nodes has height $h \leq 2 \lg(n + 1)$
- *How do you suppose we'll prove this?*

# RB Trees: Proving Height Bound

- Prove: $n$-node RB tree has height $h \leq 2 \lg(n+1)$
- Claim: A subtree rooted at a node $x$ contains at least $2^{bh(x)} - 1$ internal nodes
  - Proof by induction on height $h$
  - Base step: $x$ has height 0 (i.e., NULL leaf node)
    - *What is bh(x)?*

# RB Trees: Proving Height Bound

- Prove: $n$-node RB tree has height $h \leq 2 \lg(n+1)$
- Claim: A subtree rooted at a node $x$ contains at least $2^{bh(x)} - 1$ internal nodes
  - Proof by induction on height $h$
  - Base step: $x$ has height 0 (i.e., NULL leaf node)
    - *What is bh(x)?*
    - A: 0
    - So…subtree contains $2^{bh(x)} - 1$
      $= 2^0 - 1$
      $= 0$ internal nodes (TRUE)

# RB Trees: Proving Height Bound

- Inductive proof that subtree at node $x$ contains at least $2^{bh(x)} - 1$ internal nodes
  - Inductive step: $x$ has positive height and 2 children
    - Each child has black-height of bh($x$) or bh($x$)-1 *(Why?)*
    - The height of a child = (height of $x$) - 1
    - So the subtrees rooted at each child contain at least $2^{bh(x) - 1} - 1$ internal nodes
    - Thus subtree at $x$ contains
      $(2^{bh(x) - 1} - 1) + (2^{bh(x) - 1} - 1) + 1$
      $= 2 \bullet 2^{bh(x)-1} - 1 = 2^{bh(x)} - 1$ nodes

# RB Trees: Proving Height Bound

- Thus at the root of the red-black tree:

$n \geq 2^{\text{bh}(root)} - 1$ *(Why?)*

$n \geq 2^{h/2} - 1$ *(Why?)*

$\lg(n+1) \geq h/2$ *(Why?)*

$h \leq 2 \lg(n + 1)$ *(Why?)*

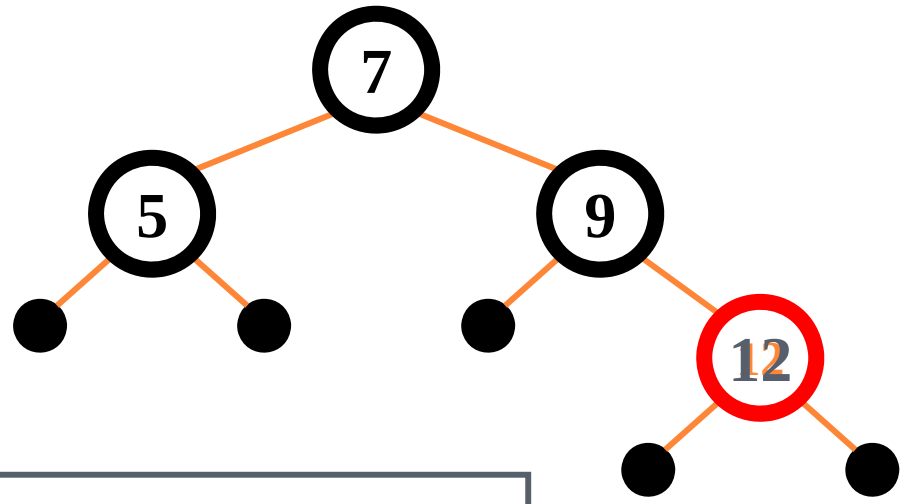
Thus $h = O(\lg n)$

# RB Trees: Worst-Case Time

- So we've proved that a red-black tree has O(lg *n*) height
- Corollary: These operations take O(lg *n*) time:
  - Minimum(), Maximum()
  - Successor(), Predecessor()
  - Search()
- Insert() and Delete():
  - Will also take O(lg *n*) time
  - But will need special care since they modify tree

# Red-Black Trees: An Example

- *Color this tree:*




Red-black properties:

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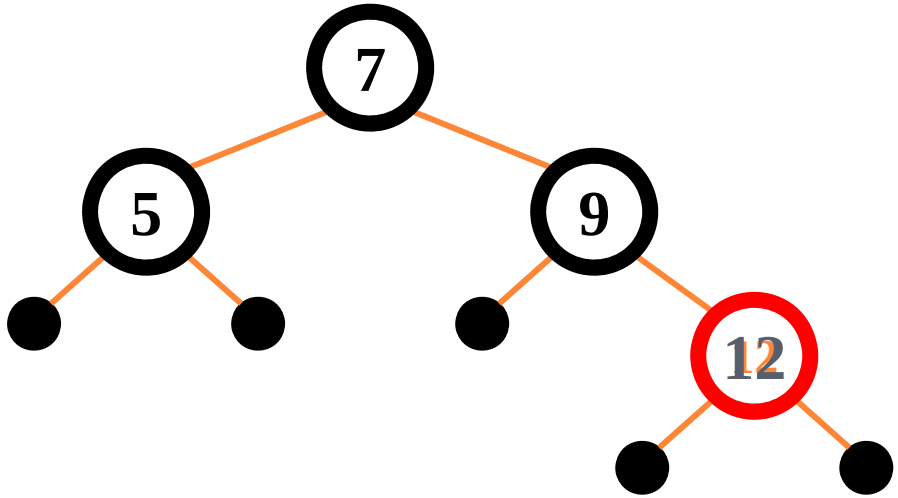
5. The root is always black

# Red-Black Trees: The Problem With Insertion

- *Insert 8:*
  - *Where does it go?*

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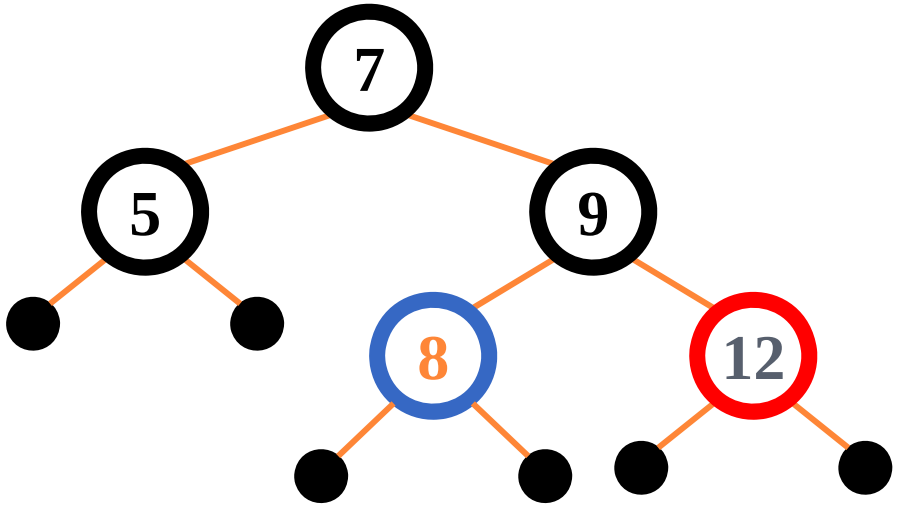
# Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*
  - *What color should it be?*

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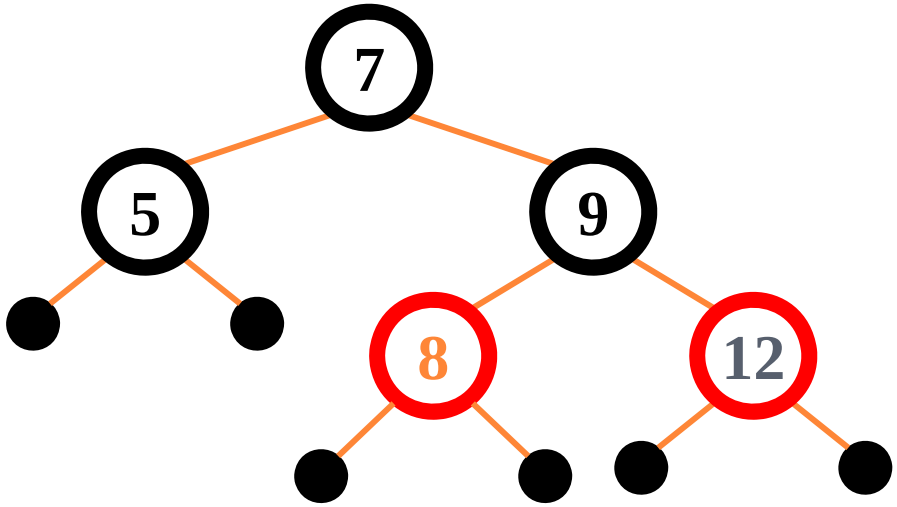


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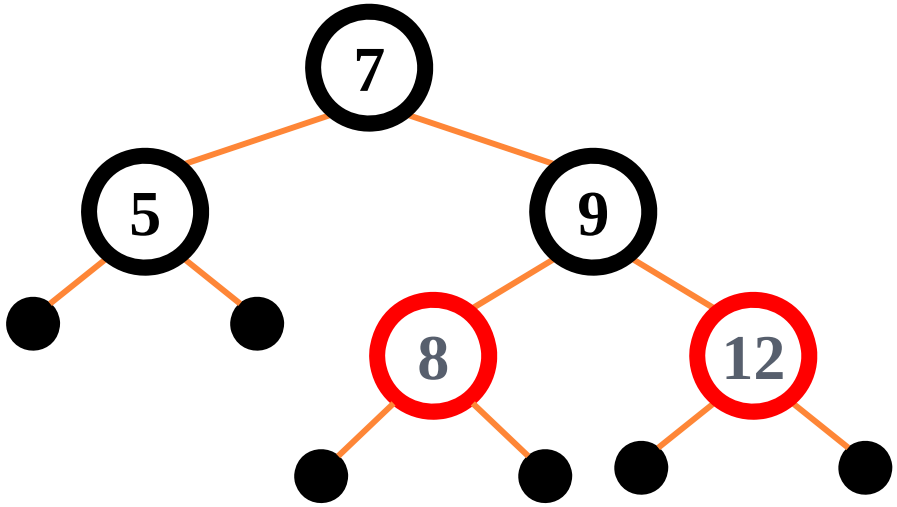
# Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*

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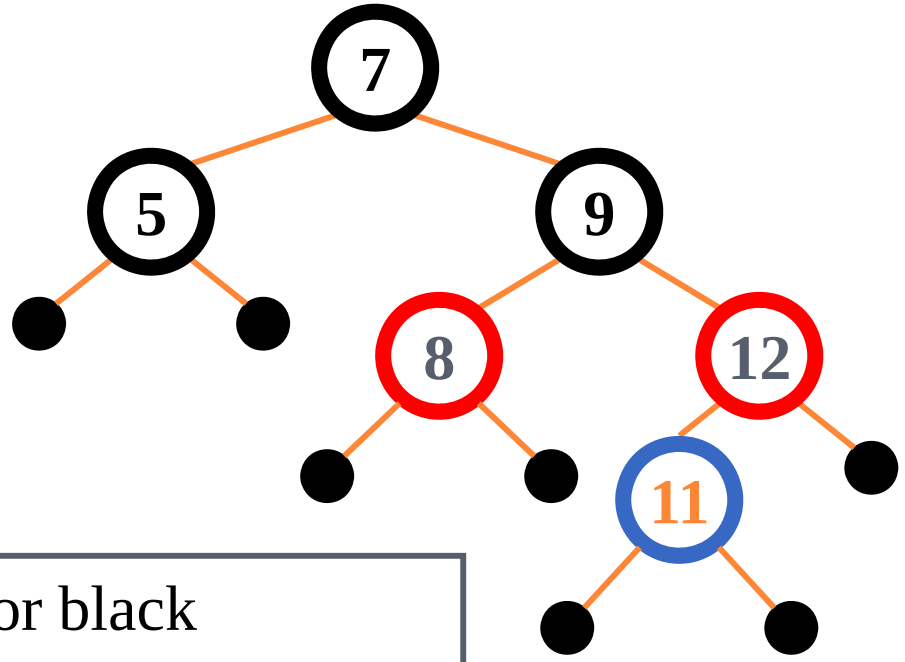
# Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*

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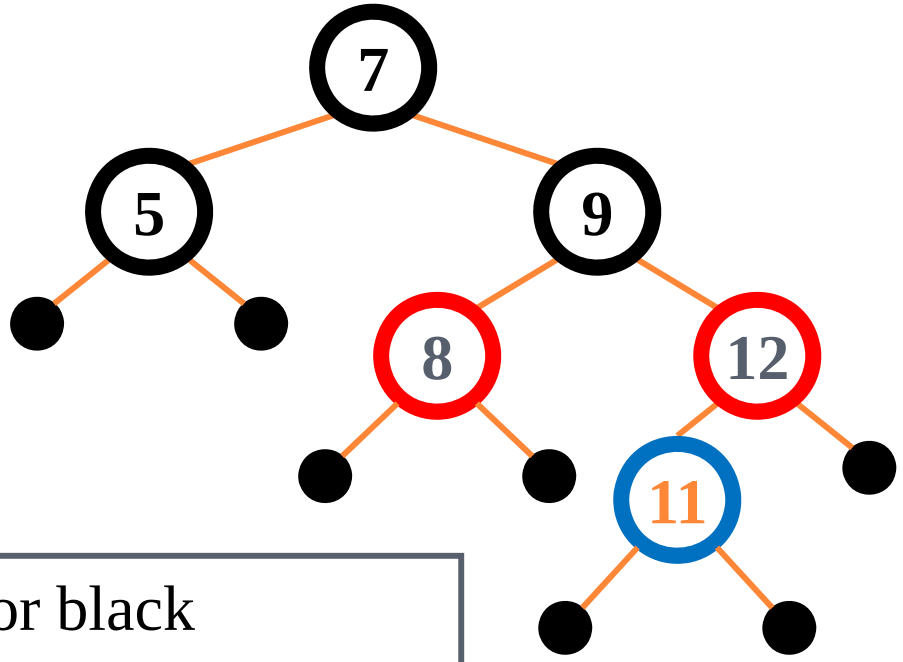
# Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#3)

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# Red-Black Trees: The Problem With Insertion

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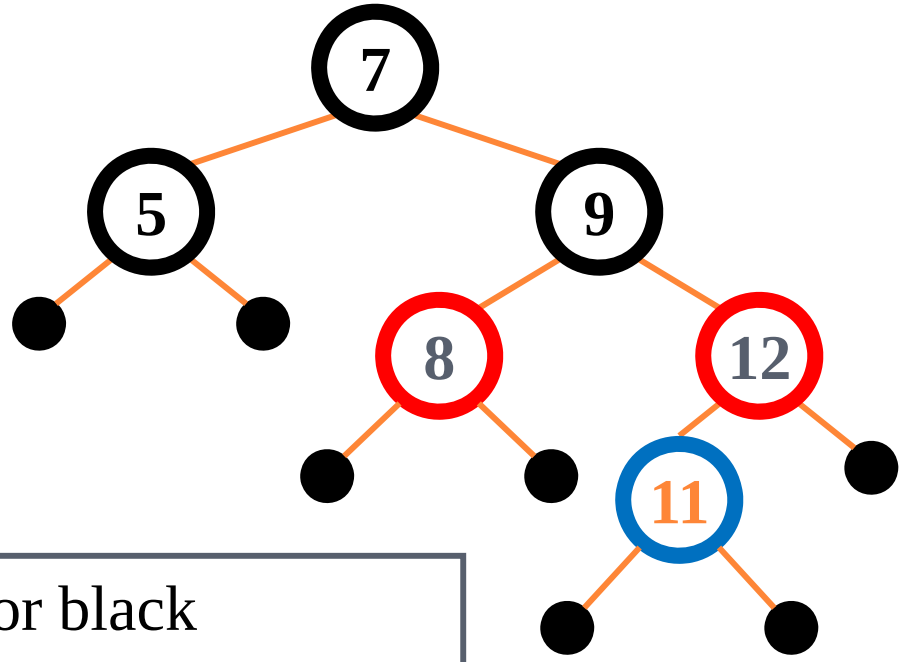
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#3)
    - Can't be black! (#4)

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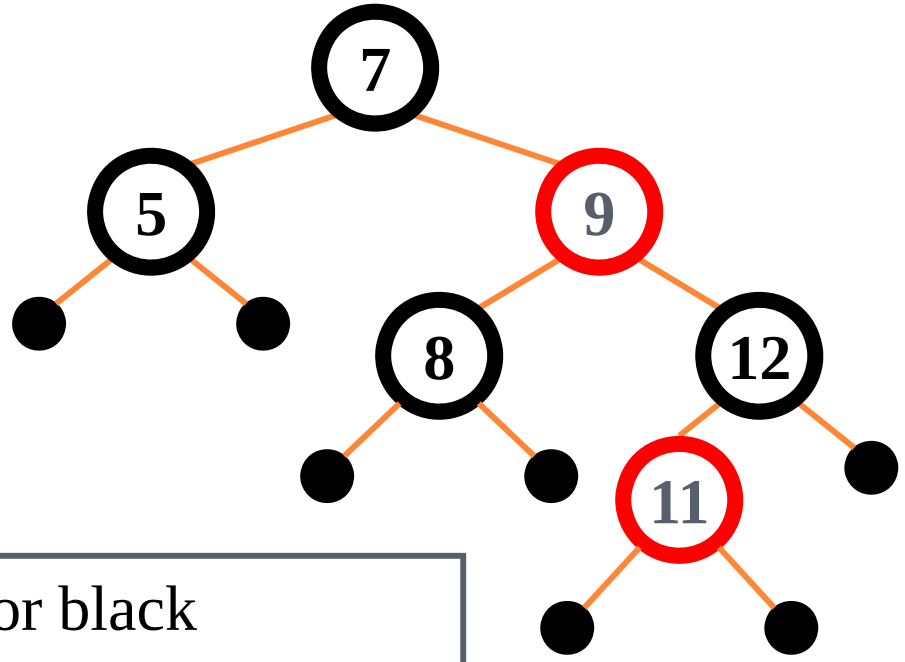
# Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Solution: recolor the tree

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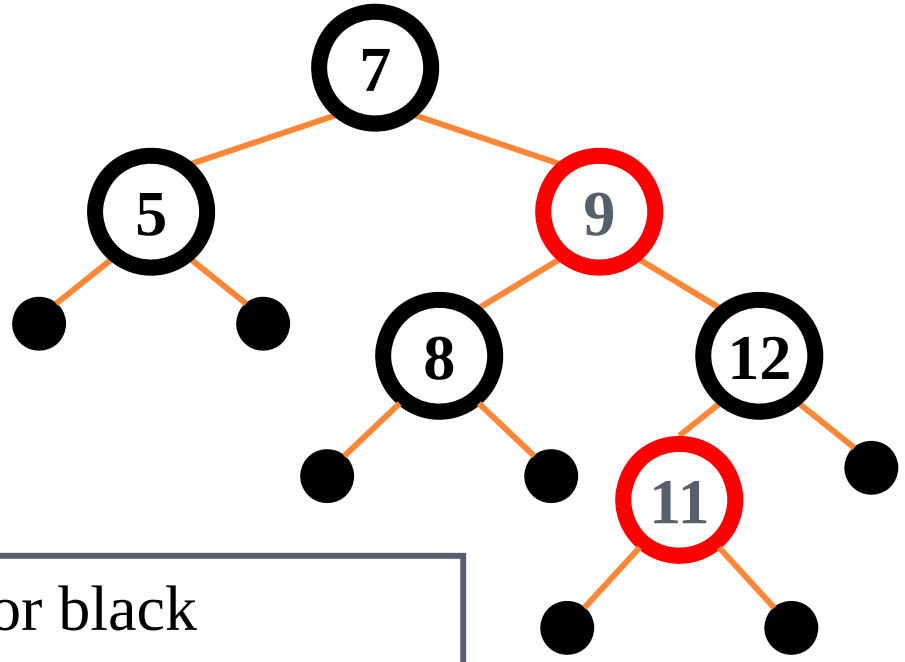
# Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*

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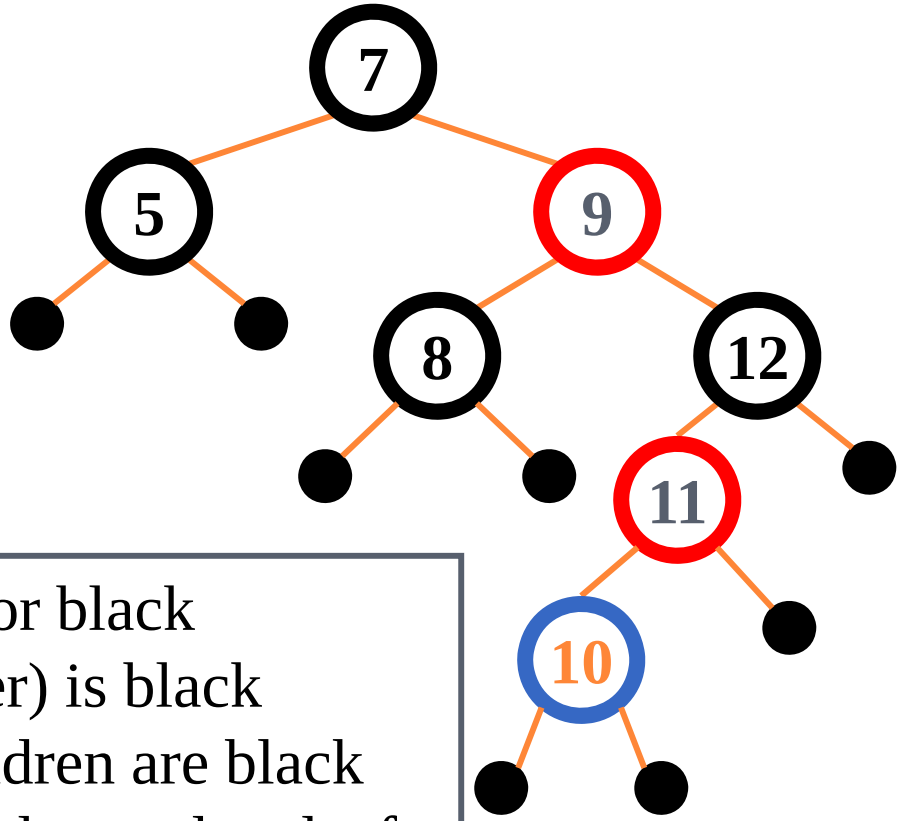
  - *Where does it go?*
  - *What color?*

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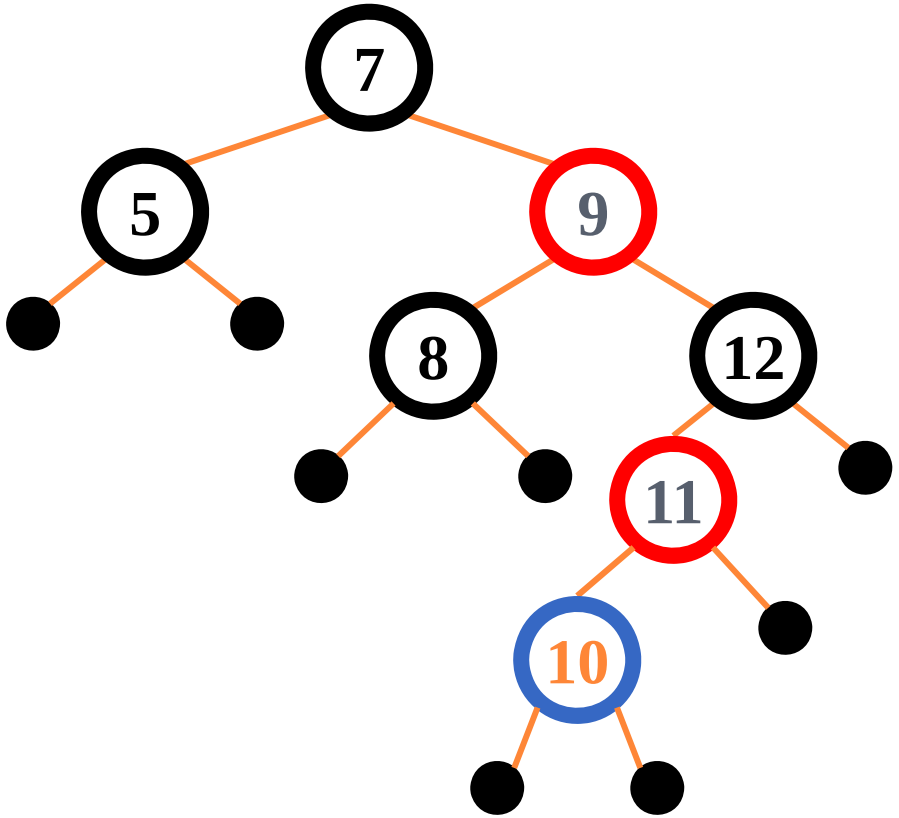
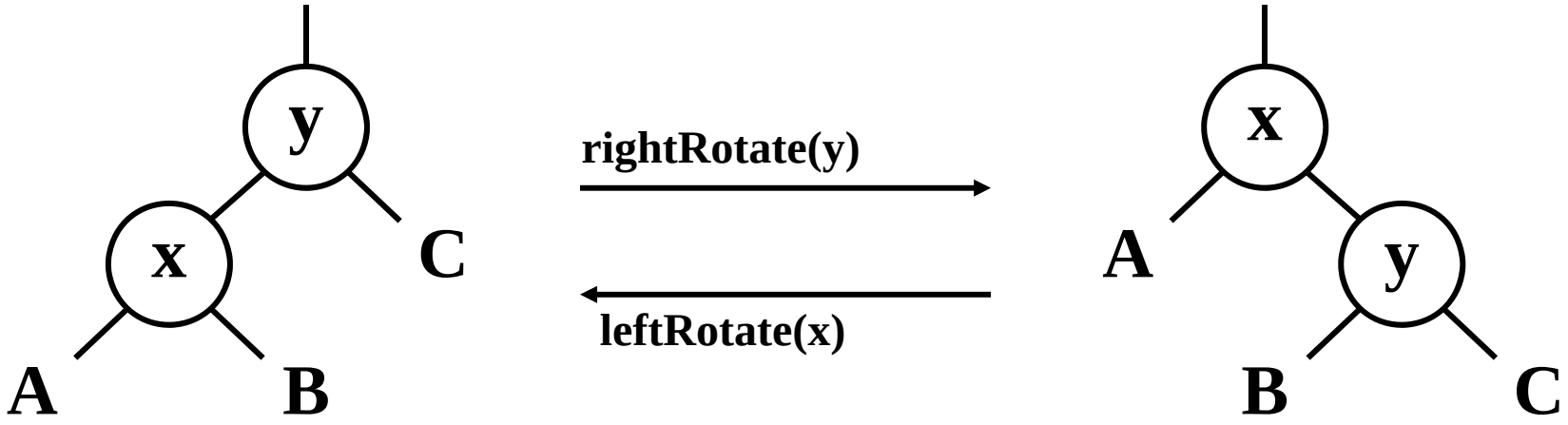
# Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*
    - A: no color! Tree is too imbalanced
    - Must change tree structure to allow recoloring
  - Goal: restructure tree in O(lg *n*) time



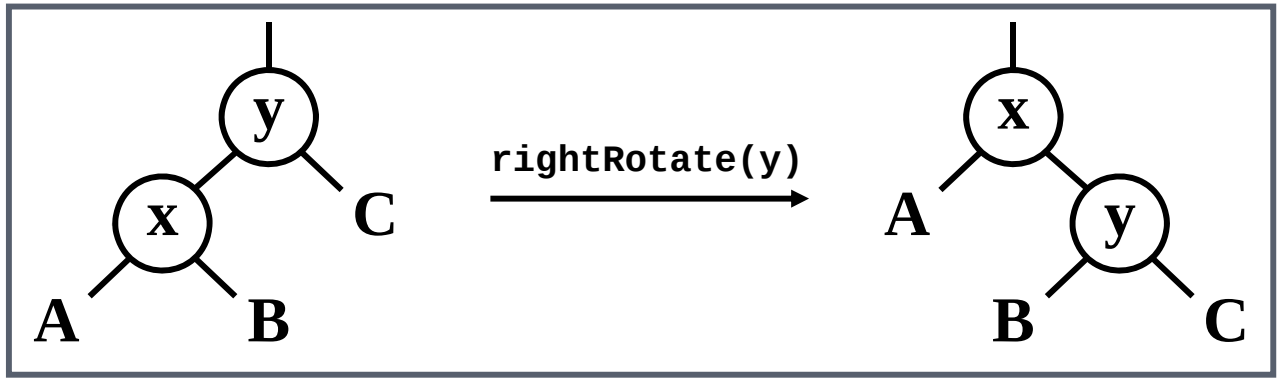
# RB Trees: Rotation

- Our basic operation for changing tree structure is called *rotation*:




- *Does rotation preserve inorder key ordering?*
- *What would the code for **rightRotate()** actually do?*

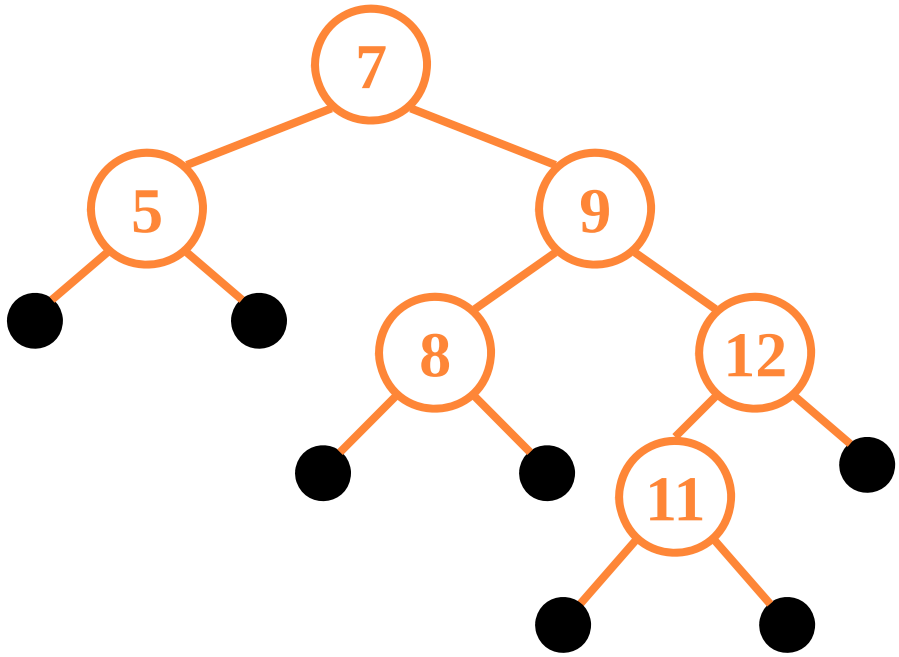
# RB Trees: Rotation




- Answer: A lot of pointer manipulation
  - *x* keeps its left child
  - *y* keeps its right child
  - *x*'s right child becomes *y*'s left child
  - *x*'s and *y*'s parents change
- *What is the running time?*

# Rotation Example

- Rotate left about 9:



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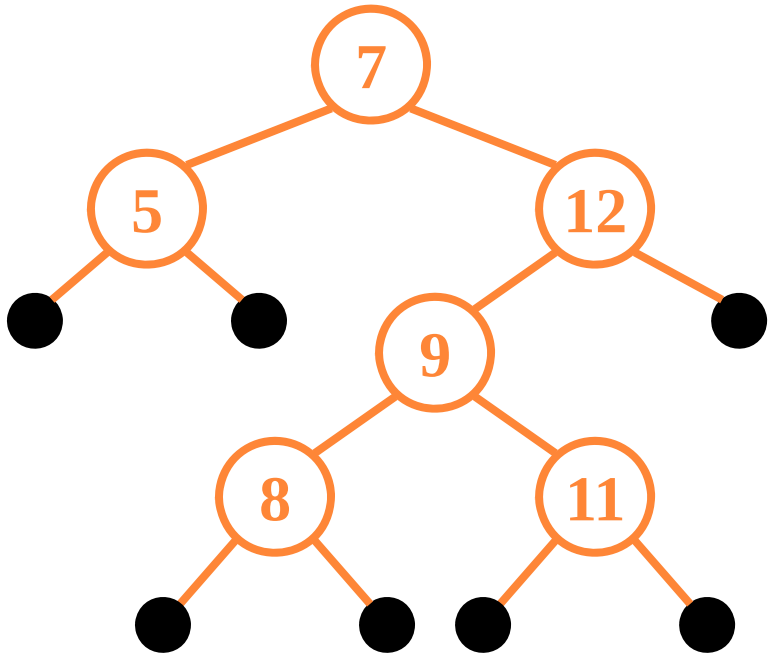

# Red-Black Trees: Insertion

- Insertion: the basic idea
  - Insert *x* into tree, color *x* red
  - Only r-b property 3 might be violated (if p[*x*] red)
    - If so, move violation up tree until a place is found where it can be fixed
  - Total time will be O(lg *n*)

```
rbInsert(x)
  treeInsert(x);
  x->color = RED;
  // Move violation of #3 up tree, maintaining #4 as invariant:
  while (x!=root && x->p->color == RED)
  if (x->p == x->p->p->left)
      y = x->p->p->right;
      if (y->color == RED)
          x->p->color = BLACK;
          y->color = BLACK;
          x->p->p->color = RED;
          x = x->p->p;
      else   // y->color == BLACK
          if (x == x->p->right)
              x = x->p;
              leftRotate(x);
          x->p->color = BLACK;
          x->p->p->color = RED;
          rightRotate(x->p->p);
  else    // x->p == x->p->p->right
      (same as above, but with
       "right" & "left" exchanged)
```

Case 1

Case 2

Case 3

```
rbInsert(x)
  treeInsert(x);
  x->color = RED;
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  while (x!=root && x->p->color == RED)
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      y = x->p->p->right;
      if (y->color == RED)
          x->p->color = BLACK;
          y->color = BLACK;
          x->p->p->color = RED;
          x = x->p->p;
      else   // y->color == BLACK
          if (x == x->p->right)
              x = x->p;
              leftRotate(x);
          x->p->color = BLACK;
          x->p->p->color = RED;
          rightRotate(x->p->p);
  else    // x->p == x->p->p->right
      (same as above, but with
       “right” & “left” exchanged)
```

Case 1: uncle is RED

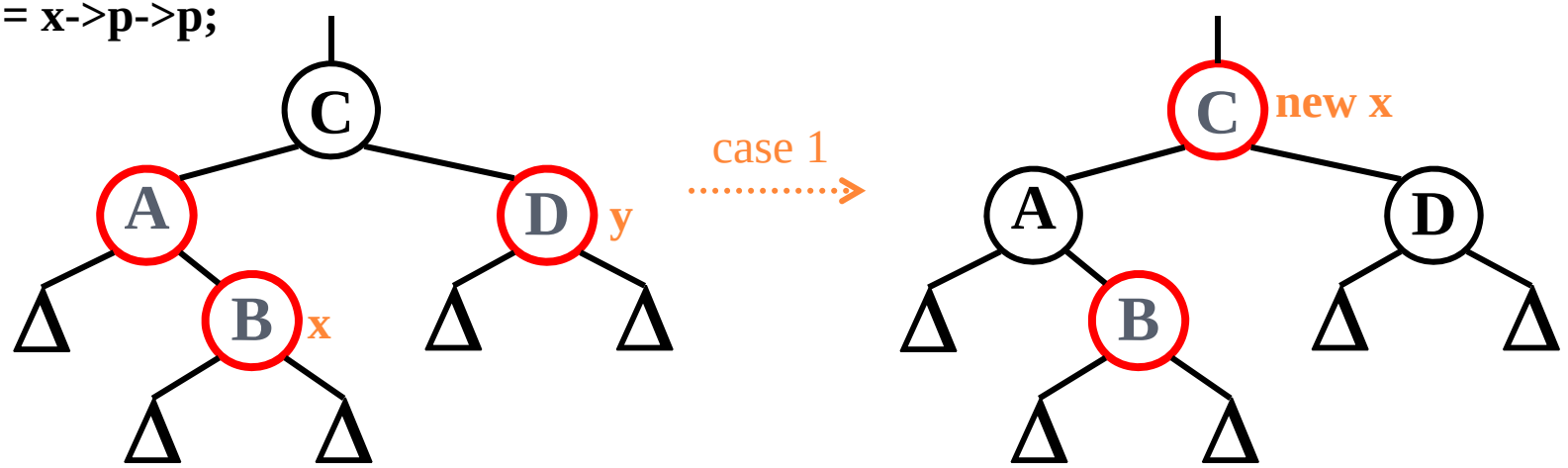
Case 2

Case 3

# RB INSERT: CASE 1

```
if (y->color == RED)
    x->p->color = BLACK;
    y->color = BLACK;
    x->p->p->color = RED;
    x = x->p->p;
```

- Case 1: “uncle” is red
- In figures below, all Δ’s are equal-black-height subtrees




Change colors of some nodes, preserving #4: all downward paths have equal b.h.
The while loop now continues with x’s grandparent as the new x

# RB INSERT: CASE 1

```
if (y->color == RED)
    x->p->color = BLACK;
    y->color = BLACK;
    x->p->p->color = RED;
    x = x->p->p;
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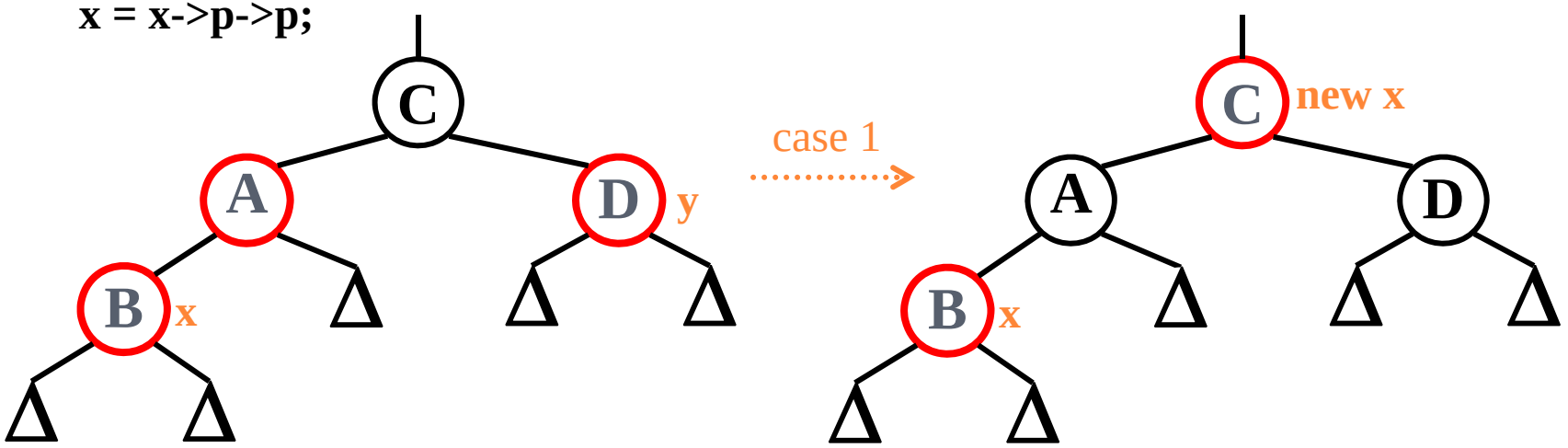
- Case 1: “uncle” is red
- In figures below, all Δ’s are equal-black-height subtrees

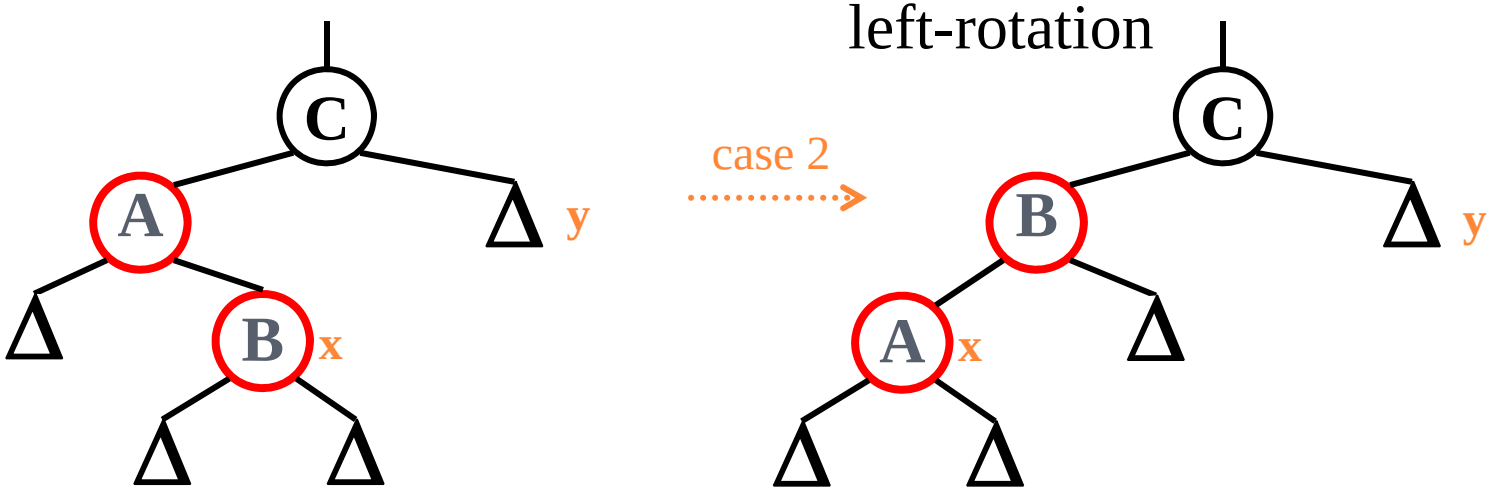



Same action whether x is a left or a right child

# RB INSERT: CASE 2

```
if (x == x->p->right)
    x = x->p;
    leftRotate(x);
// continue with case 3 code
```

- Case 2:
  - "Uncle" is black
  - Node *x* is a right child
- Transform to case 3 via a left-rotation




Transform case 2 into case 3 (x is left child) with a left rotation
This preserves property 4: all downward paths contain same number of black nodes

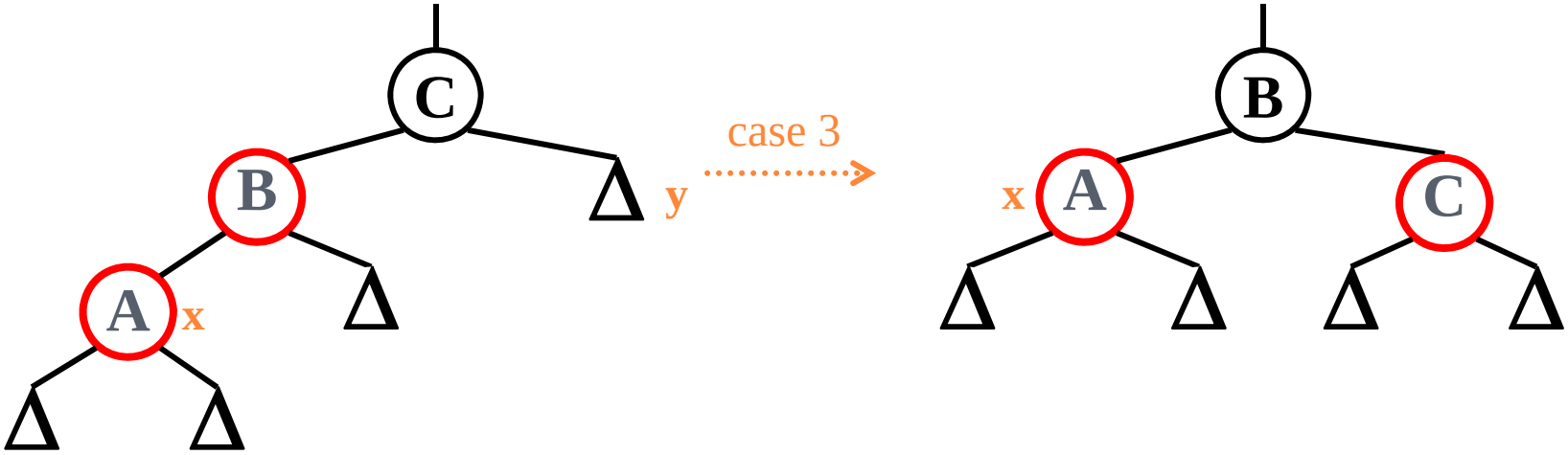
# RB INSERT: CASE 3

**x->p->color = BLACK;**

**x->p->p->color = RED;**

**rightRotate(x->p->p);**

- Case 3:
  - "Uncle" is black
  - Node *x* is a left child
- Change colors; rotate right




Perform some color changes and do a right rotation

Again, preserves property 4: all downward paths contain same number of black nodes

# RB INSERT: CASES 4-6

- Cases 1-3 hold if *x*'s parent is a left child
- If *x*'s parent is a right child, cases 4-6 are symmetric (swap left for right)