目录

第一部分　Enumeration

1　Basic Enumeration: the twelvefold way

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	$m!\left\{ {n \atop m} \right\}$
n identical balls, m distinct bins	$\left(\binom{m}{n}\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m}\left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$

columns are also expressed as: arbitrary, injection(one-to-one), surjetion(onto).

scenario for row 2: m 个海盗分赃 n 个金币

scenario for row 3: n 个海盗坐上 m 个快艇

scenario for row 4: n 个金币装上 m 个快艇

1.1　(1,1)(2,2)(3,2)(4,2) are trivial

(recall) Binomial Therom: (x is used as a *formal variable*)

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

Multinomial Therom:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{m_1+\cdots+m_k=n}\binom{n}{m_1,\ldots,m_k}x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}$$

multinomial coefficient: assign n distinct balls to k distinct bins with the i-th bin receiving m_i balls

$$\binom{n}{m_1,\ldots,m_k} = \frac{n!}{m_1!m_2!\cdots m_k!}$$

1.2 (1,2) lower factorial

n-permutation $\pi \in [m]^n$of distinct elements: $(m)_n = m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$

1.3 (2,3) k-composition

The solutions to $x_1 + x_2 + \cdots + x_k = n$ in positive integers (k-composition of n): $\binom{n-1}{k-1}$. 证明用插板法

The solutions to $x_1 + x_2 + \cdots + x_k = n$ in nonnegative integers (weak k-composition of n): $\binom{n+k-1}{k-1}$. 证明：先给每个区补一个 1

The solutions to $x_1 + x_2 + \cdots + x_k \leq n$ in nonnegative integers: $\binom{n+k}{k}$. 证明：补上一个 $x_{k+1} = n - \sum_{i=1}^{k} x_i \geq 0$ 转化成为 weak k-composition

1.4 (2,1) multiset

$\left(\!\binom{n}{k}\!\right)$ is the number of k-combinations of an n-set with repetitions.$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

证明：k-multiset on $S = \{x_1, x_2, \ldots, x_n\}$, $m(x_1) + m(x_2) + \cdots + m(x_n) = k \quad m(x_i) \geq 0$

1.5 (3,3)(1,3)(3,1) Stirling number of the second kind

$\left\{ n \atop k \right\}$ the number of k-partitions of an n-set. It's hard to give a determinant.

recursive form:$\left\{ n \atop k \right\} = k\left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$ 证明：分 k 份，第 n 个插入任意一份 + 第 n 个单独一份

Bell number: $B_n = \sum_{k=1}^{n} \left\{ n \atop k \right\}$ the total number of partitions of an n-set.

By PIE, we can obtain: $\left\{ n \atop m \right\} = \frac{1}{m!}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^n$ 证明：把坏事件 A_i 定义为 $f : [n] \to [m]\backslash\{i\}, \quad i \in [m]$

1.6 (4,3)(4,1) partion of a number

$p_k(n)$ number of partitions of n into k parts.

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

recursive form: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ 证明：$x_k = 1$ 和 $x_k > 1$

$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}$, $p_k(n) = \sum_{j=1}^{k} p_j(n-k)$

2 generating functions

应对的问题：根据递归式求闭合式。核心：利用了形式化变量 x 作为桥梁。

2.1 generatingfunctionology

1. Recurrence: 根据问题定义写出递归式 $\alpha_n = c_1\alpha_{n-1} + c_2\alpha_{n-2} + \cdots$

2. Manipulation: 把 $G(x) = \sum_{n\geq 0} \alpha_n x^n$ 带入递归式，并整理成 $G(x) = f(x)G(x) + g(x)$ 的形式

3. Solving(expanding): 把 $G(x) = g(x)/(1 - f(x))$ 右边展开成 $G(x) = \sum_{n\geq 0} \beta(n)x^n$ 的形式，$\beta(n)$ 即是 α_n 的闭合式

2.2 properties

2.2.1 manipulation

$G(x) = \sum_{n\geq 0} g_n x^n$, $F(x) = \sum_{n\geq 0} f_n x^n$

shift: $x^k G(x) = \sum_{n\geq k\geq 0} g_{n-k} x^n$

addition: $F(x) + G(x) = \sum_{n\geq 0}(f_n + g_n)x^n$

convolution: $F(x)G(x) = \sum_{n\geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$ (shift is a special case of convolution)

differentiation: $G'(x) = \sum_{n\geq 0}(n+1)g_{n+1}x^n$

从左到右很自然，然而应用的重点在于从右凑到左。

2.2.2 expanding

Taylor's expansion:

$$G(x) = \sum_{n\geq 0} \frac{G^{(n)}(0)}{n!} x^n$$

Geometric sequence:

$$\frac{a}{1 - bx} = a \sum_{n\geq 0} (bx)^n$$

Newton's formula: (其中 α 可以是分数、负数)

$$(1 + x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!}$$

2.3 application: Catelan number

2.3.1 examples

(Although the following are all equivalent, most of these examples are hard to be converted to another...give up trying and have a good day ^ ^)

- the number of expressions containing n pairs of parentheses which are correctly matched;
- the number of different ways n + 1 factors can be completely parenthesized
- the number of full binary trees with n + 1 leaves

- the number of monotonic paths along the edges of a grid with n Œ n square cells, which do not pass above the diagonal
- the number of stack-sortable permutations of {1, ..., n}
- ...

2.3.2 recursive form

$C_0 = 1$, for $n \geq 1$,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

It can be directly derived from the "full binary trees with n + 1 leaves" example by recursively removing the root.

2.3.3 closed form

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

which can be proved by generating function method.

3 Principle of Inclusion-Exclusion (PIE)

3.1 principle

Let $A_1, A_2, \ldots, A_n$ be a family of subsets of U. Then the number of elements of U which lie in none of the subsets $A_i$is

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} |A_I|, \quad A_I = \bigcap_{i \in I} A_i$$

with the convention that $A_\emptyset = U$.

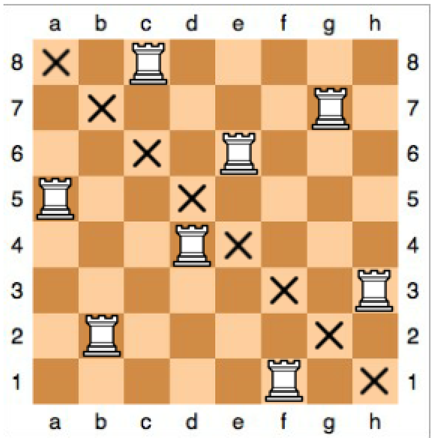
定义出坏事件 A_i，即可计算出坏事件一个都不发生的种类数。通常相同数量坏事件的交集大小 $|A_I|$ 也相同，使得上式容易计算。

3.2 application

3.2.1 The number of derangements

The number of derangements $\approx \frac{n!}{e}$ 证明：坏事件 A_i 为 the set of permutations with fixed point i

3.2.2 Permutations with restricted positions



(Interpreted in chess game)

B: a set of marked positions in an $[n] \times [n]$ chess board.

N_0: the number of ways of placing n non-attacking rooks on the chess board such that none of these rooks lie in B.

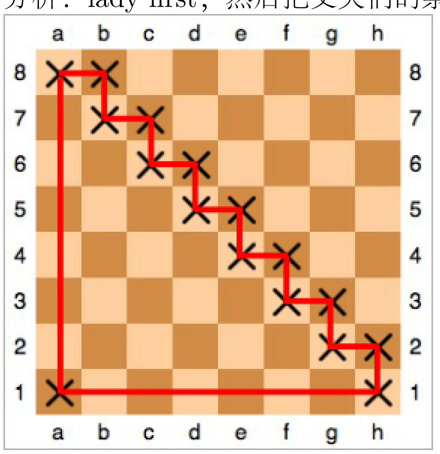
r_k: number of ways of placing k non-attacking rooks on B

$$N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$$

3.2.3 Problème des ménages

圆桌上坐 n 对夫妻，男女交替坐，夫妻不能挨着坐，有几种排座方法？

分析：lady first，然后把丈夫们的禁忌位置画在棋盘上，计算出 r_k 就完事了。



Lemma 1: The number of ways of choosing k non-consecutive objects from a collection of m objects arranged in a line is $\binom{m-k+1}{k}$. 证明：先取 k 个出来插到 $m-k+1$ 个缝隙中

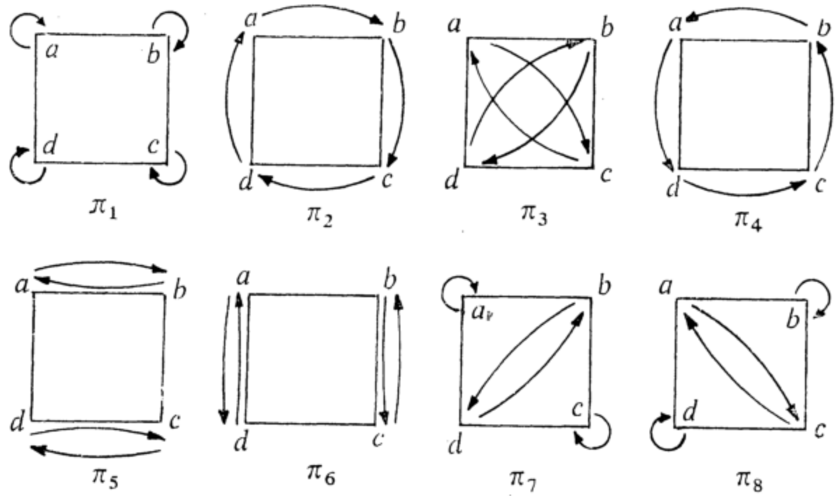
Lemma 2: The number of ways of choosing k non-consecutive objects from a collection of m objects arranged in a circle is $\frac{m}{m-k}\binom{m-k}{k}$. 证明：双重计数，A：选出环上 k 个不相邻的涂成红色，剩下挑出一个涂成蓝色；B：挑一个涂成蓝色，拔下来，环就变成线了，根据 Lemma 1 从剩下的 m-1 个中选 k 个涂红色

$r_k = \frac{2n}{2n-k}\binom{2n-k}{k}$

4 Pólya's theory of counting

应对的问题：存在由对称形成的等价类的情况下应该如何计数？

可爱的栗子：(给含有如下对称的正方形顶点涂黑白色)



Pólya 三部曲：

- 定义对称构成的群 G

可以写出所有群元素（如栗子里那样），也根据生成元来定义。如转动 $r(i) = (i+1) \mod n$，反转 $\rho(i) = n - i - 1$ 等等。注意一个生成元可以生成不止一个群元素。接下来求出 $|G|$，栗子里 $|G| = 8$

- 写出 cycle index P_G

什么是 cycle？对于排列（变换）$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$，经过 cycle decomposition 可以得到两个 cycle：(13)(254)。栗子里箭头形成的环就是 cycle。

通过形式化生成元 x 写出 cycle 的 structure representation $x_{\text{cycle 的长度}}^{\text{cycle 的个数}}$。栗子里的八个 structure representation 就是 $x_1^4, x_4^1, x_2^2, x_4^1, x_2^2, x_2^2, x_1^2x_2^1, x_1^2x_2^1$。

$P_G = \frac{1}{|G|}\sum$(cycle structure representation)。栗子里 $P_G = \frac{1}{8}(x_1^4 + 2x_4^1 + 3x_2^2 + 2x_1^2x_2^1)$

- 得到 pattern inventory

把 $x_i = \sum_{c\in colors} c^i$ 带入 P_G，展开以后的多项式系数就是想要的结果。比如栗子里是涂成黑白两色，那么 $P_G = \frac{1}{8}((b+w)^4 + 2(b^4+w^4) + 3(b^2+w^2)^2 + 2(b+w)^2(b^2+w^2)) = b^4 + b^3w + 2b^2w^2 + bw^3 + w^4$，涂两黑两白有两种涂法，其余都只有一种。

第二部分　Existence

5　Existence by counting

Shannon's circuit lower bound：在 n 个输入下 $\frac{2^n}{3n}$ 个逻辑门能组合出的不同的电路数小于总的布尔函数数量 2^n，因此*存在*一个布尔函数由这么多个门实现不了。

handshaking lemma: $\sum_{v\in V} d(v) = 2|E|$

Sperner's lemma: 思想：和是偶数，则*存在*偶数个奇数

6　The Pigeonhole Principle

定义萝卜和坑，如果萝卜比坑多，

1. *存在*一个坑至少俩萝卜；
2. 一个萝卜一个坑，*存在*一个萝卜在给定的坑外

6.1　Inevitable divisors

For any subset $S \subseteq \{1, 2, \ldots, 2n\}$ of size $|S| > n$, there are two numbers $a, b \in S$ such that $a|b$.

萝卜：每个数

坑：$C_m = \{2^k m \mid k \geq 0, 2^k m \leq 2n\}$，$m$ 为奇数，显然落在同一个坑里的数能整除，对 $2n$ 里的 n 个奇数就有 n 个不同的坑。

如果选出来萝卜数 $|S| > n$，那么*存在*俩萝卜落在一个坑里。

6.2　Monotonic subsequences

A sequence of more than mn different real numbers must contain either an increasing subsequence of length m + 1, or a decreasing subsequence of length n + 1.

intuition: the length of both the longest increasing subsequence and the longest decreasing subsequence cannot be small simultaneously.

萝卜：$(a_1, a_2, \ldots, a_N)$ the original sequence of $N > mn$ distinct real numbers

x_i: the length of the longest increasing subsequence ending at a_i;

y_i: the length of the longest decreasing subsequence starting at a_i.

坑：(x_i, y_i)

证明"一个萝卜一个坑"：$(x_i, y_i) \neq (x_j, y_j)$ whenever $i \neq j$

$N > mn$，说明*存在*一个萝卜在坑外

6.3　Dirichlet's approximation

Let x be an irrational number. For any natural number n, there is a rational number $\frac{p}{q}$ such that $1 \leq q \leq n$ and $\left|x - \frac{p}{q}\right| < \frac{1}{nq}$.

Let$\{x\} = x - \lfloor x \rfloor \in [0, 1)$denote the fractional part of the real number x.

萝卜：$n + 1$ 个数 $\{kx\}$，$k = 1, 2, \ldots, n + 1$

坑：n 个区间 $(0, \frac{1}{n}), (\frac{1}{n}, \frac{2}{n}), \ldots, (\frac{n-1}{n}, 1)$, x is irrational, so $\{kx\}$ cannot coincide with any endpoint of the above intervals

there *exist* $1 \le a < b \le n+1$, such that $|\{bx\} - \{ax\}| < \frac{1}{n}$. Therefore $|(b-a)x - (\lfloor bx \rfloor - \lfloor ax \rfloor)| \le \frac{1}{n}$

Let $q = b - a$ and $p = \lfloor bx \rfloor - \lfloor ax \rfloor$. We have$|qx - p| < \frac{1}{n}$ and $1 \le q \le n$.

7 The probabilistic method

7.1 principles and paradigms

7.1.1 lower bound of Ramsey number $R(k, k)$

Ramsey number $R(k, l)$: smallest integer n that in any 2-coloring K_n by red and blue, there is a red K_k or a blue K_l

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with two colors so that there is no monochromatic K_k subgraph.

概率定义：uniform random coloring of edges in K_n

坏事件：出现单色 K_k 子图

$\Pr($出现某一个坏事件$) = 2 \cdot 2^{-\binom{k}{2}}$

by union bound, $\Pr($至少一个坏事件发生$) \le \binom{n}{k} \cdot \Pr($出现某一个坏事件$) = \binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$

因此*存在*一种涂色使得坏事件一个都不发生。

7.1.2 Hamiltonian paths on tournament

There is a tournament on n players with at least $n!2^{-(n-1)}$Hamiltonian paths.

概率定义：uniform random tournament

指示变量 X_π：顺着某个排列 π 存在一条哈密尔顿回路

$E(X_\pi) = 2^{-(n-1)}$, $X = \sum X_\pi$, $E[X] = n!2^{-(n-1)}$ (by linearity of expectation)

因此*存在*一个竞赛图有不少于期望的哈密尔顿回路。

7.1.3 lower bound of independent sets

Let $G(V, E)$ be a graph on n vertices with m edges. Then G has an independent set with at least $\frac{n^2}{4m}$ vertices.

S : sample vertices with probability p. 指示变量 X_v：点 v 是否被选进 S

$E(|S|) = E(X) = E(\sum_{v \in V} X_v) = np$

坏事件 Y_{uv}：一条边的两个顶点 u 和 v 都被选进了 S. $E(Y_{uv}) = p^2$, $E(Y) = \sum E(Y_{uv}) = mp^2$

S 里有一条边就删掉一个点，剩下的肯定是独立集，$E(X - Y) = np - mp^2 \ge \frac{n^2}{4m}$

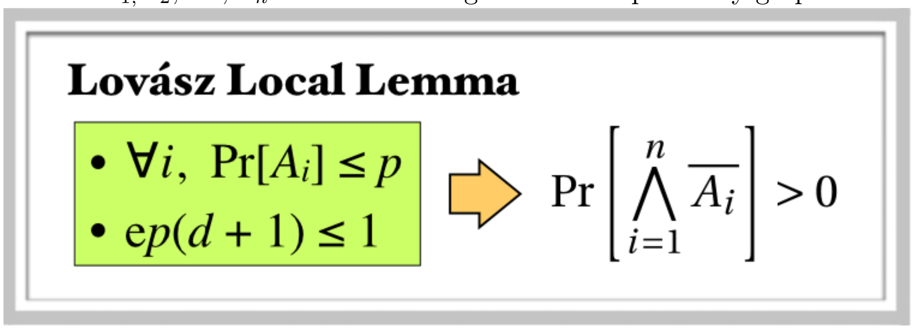
因此*存在*一个独立集包含比期望多的点

7.1.4 Coloring large girth graph

skipped.

7.2 Lovász Local Lemma

events: $A_1, A_2, \ldots, A_n$ d: maximun degree of the dependency graph



解读：每个坏事件概率不太大，并且坏事件不太相关，那么它们可能一个都不发生

- another lower bound of Ramsey number $R(k,k)$

$R(k,k) \geq Ck2^{k/2}$ for some constant C>0.
概率定义：uniform random coloring of edges in K_n
坏事件 A_s: S forms a monochromatic K_k. $\Pr[A_S] = 2^{1-\binom{k}{2}} = p$
分析相关性：A_s and A_T are dependent $\Leftrightarrow |S \bigcap T| \geq 2$. $d \leq \binom{k}{2}\binom{n}{k-2}$
$n = Ck2^{k/2}$, $ep(d+1) \leq 1$

第三部分 Extremal Combinatorics

8 Extremal graph theory

8.1 Mantel's Theorem

If $G(V,E)$ has $|V| = n$ and is triangle-free, then $|E| \leq \frac{n^2}{4}$.
Extremal graph: complete balanced bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$

- 证明 1：induction + pigeonhole principle

equivalent form: Any$G(V,E)$ with $|V| = n$ and $|E| > \frac{n^2}{4}$ must have a triangle.
basis: $n \leq 3$, trivial. induction: 选出中间有边的两个点，剩下 n-2 个点要么已经有三角形，要么必然和这俩形成三角形

- 证明 2：Cauchy-Schwarz inequality

intuition: For any edge $uv \in E$, no vertex can be a neighbor of both u and v
$\rightarrow$for any edge $uv \in E$, $d_u + d_v \leq n$. $\rightarrow \sum_{uv \in E}(d_u + d_v) \leq n|E|$.
$n|E| \geq \sum_{uv \in E}(d_u + d_v) = \sum_{v \in V} d_v^2$ (d_v被加了d_v次),

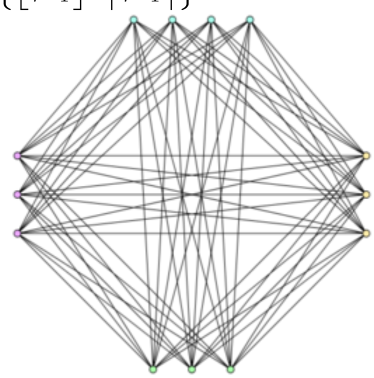
$\geq \frac{\left(\sum_{v\in V} d_v\right)^2}{n}$(Cauchy-Schwarz: $|\sum_{i=1}^{n} x_i\bar{y}_i|^2 \leq \sum_{j=1}^{n}|x_j|^2\sum_{k=1}^{n}|y_k|^2$.)
$= \frac{4|E|^2}{n}$(Euler'sequality: $\sum_{v\in V} d_v = 2|E|$)

- 证明 3：inequality of the arithmetic and geometric mean

Let A be the largest independent set in G and let $\alpha = |A|$, and take $B = V \setminus A$ and let $\beta = |B|$
lemma 1: triangle-free ⇒all v's neighbors form an independent set ⇒$d(v) \leq \alpha$
lemma 2: A's an independent set ⇒ all edges have at least one endpoint in B ⇒ $|E| \leq \sum_{v\in B} d_v$
by the inequality of the arithmetic and geometric mean, $E \leq \sum_{v\in B} d_v \leq \alpha\beta \leq (\frac{\alpha+\beta}{2})^2 = \frac{n^2}{4}$

8.2 Forbidden Cliques: Turán's Theorem

(generalization of the Mantel's Theorem, if triangle is seen as 3-clique)
Let $G(V,E)$be a graph with $|V| = n$. If G has no r-clique, $r \geq 2$, then $|E| \leq \frac{r-2}{2(r-1)}n^2$.
Extremal graph: Turán graph $T(n, r-1)$, a complete multipartite graph $K_{n_1,n_2,\ldots,n_{r-1}}$ with $n_i \in \left\{\left\lfloor\frac{n}{r-1}\right\rfloor, \left\lceil\frac{n}{r-1}\right\rceil\right\}$ for every i



- 证明 1：induction

intuition: If the number of edges is maximized, G has (r 1)-cliques.
A : a (r-1)-clique. $B = V \setminus A$. obviously $|A| = r-1$, $|B| = n-r+1$
$|E| = |E(A)| + |E(B)| + |E(A,B)|$
$|E(A)| = \binom{r-1}{2}$
$|E(B)| \leq \frac{r-2}{2(r-1)}(n-r-1)|$ (induction hypothesis)
$|E(A,B)| \leq (|A|-1)|B| = (r-2)(n-r+1)$ (avoid r-cliques while maximizing the number of edges)
$|E| \leq \frac{r-2}{2(r-1)}n^2$.

- 证明 2：weight shifting

给每个顶点一个非负的归一化权重 $w_v \geq 0, \sum_{v\in V} w_v = 1$
给图打个分：$S = \sum_{uv\in E} w_u w_v$
不管边的关系直接平均分配权重 $w_v = \frac{1}{n}$，得分 $S = \sum_{uv\in E} w_u w_v = \frac{|E|}{n^2}$.
这个得分不一定是最大的，注意到得分也可以写成 $S = \frac{1}{2}\sum_{u\in V} w_u W_u$, W_u 为 u 邻居的总权重，那么对于两个不相邻的点，把权重从 W 小的完全转移到 W 大的得分就会提高，W 相等转移完了得

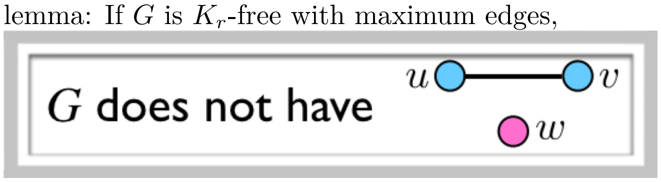
分也不变，所以最终权重会富集到一个 clique 上，由于 G 最多只有 (r-1)clique，因此最高可能的得分 $S \leq \binom{r-1}{2}\frac{1}{(r-1)^2} = \frac{r-2}{2(r-1)}$. 因此 $\frac{|E|}{n^2} \leq \frac{r-2}{2(r-1)}$

- 证明 3：the probabilistic method + Cauchy-Schwarz inequality

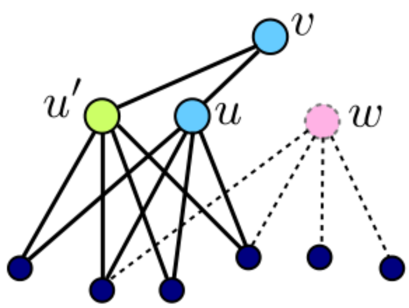
  lemma: $\omega(G) \geq \sum_{v\in V} \frac{1}{n-d_v}$. ($\omega(G)$: the number of vertices in a largest clique)
  概率定义：random permutation of vertices in V
  构建 clique S：从随机排列中一个个抽取，如果能和 S 中已有的点组成 clique 那么就放进 S
  指示变量 X_v：被选进 S 为 1，否则为 0
  $E[X_v] = \Pr[v \in S] \geq \frac{1}{n-d_v}$ 如果点排在所有非邻居之前，那么肯定会被选进 S
  $E[|S|] = \sum_{v\in V} E[X_v] \geq \sum_{v\in V} \frac{1}{n-d_v}$. 那么最大的 $|S| = \omega(G)$ 肯定不小于期望，引理得证。
  by Cauchy-Schwarz inequality:
  $n^2 = (\sum_{v\in V} \sqrt{n-d_v} \cdot \frac{1}{\sqrt{n-d_v}})^2 \leq \sum_{v\in V}(n-d_v)\sum_{v\in V}\frac{1}{n-d_v} \leq \omega(G)\sum_{v\in V}(n-d_v) \leq (r-1)(n^2 - 2|E|)$

- 证明 4：vertex duplication

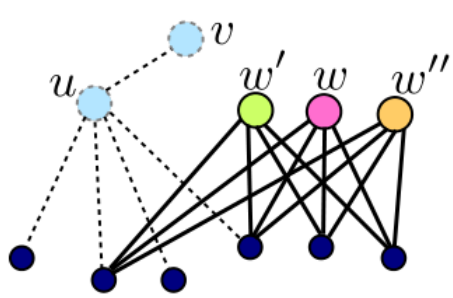
  lemma: If G is K_r-free with maximum edges,



  Case I. $d_w < d_u$or $d_w < d_v$. Without loss of generality, suppose that $d(w) < d(u)$



  duplicate u, delete w, still K_r-free, $|E'| = |E| + d_u - d_w > |E|$
  Case.II $d_w \geq d_u \wedge d_w \geq d_v$



  delete u,v, duplicate w twice, still K_r-free, $|E'| = |E| + 2d_w - (d_u + d_v - 1) > |E|$
  the lemma is proved.

$u \nsim v$ 形成了等价关系（传递性 + 自反性 + 对称性，传递性可以通过引理说明：u,w 之间没边，w,v 之间没边，u,v 之间肯定就没边），等价关系就可以把 V 划分成为若干个等价类，满足类内不相邻，类外都相邻。因此当边最多的时候，必然形成 Turán graph。

8.3 Forbidden cycles

(generalization of the Mantel's Theorem, if triangle is seen as 3-cycle)

Let $G(V,E)$ be a graph on n vertices. If girth$g(G) \geq 5$ then $|E| \leq \frac{1}{2}n\sqrt{n-1}$.

即既没有三角形也没有四边形。

选取一个中心点 u，$v_1, v_2, \ldots, v_d$ 是 u 的邻居，$S_1, S_2, \ldots, S_d$ 是邻居的邻居的集合（不包括 u）

没有三角形 $\Rightarrow v_1, v_2, \ldots, v_d$ 彼此不是邻居 $\Leftrightarrow S_i \cap \{u, v_1, v_2, \ldots, v_d\} = \emptyset$

没有四边形 $\Rightarrow v_1, v_2, \ldots, v_d$ 没有共同的邻居 $\Leftrightarrow S_i \cap S_j = \emptyset$

即上面提到的东西都没有重叠，由 $\{u, v_1, v_2, \ldots, v_d\} \cup S_1 \cup S_2 \cup \cdots \cup S_d \subseteq V$ 可以得到

$(d+1) + |S_1| + |S_2| + \cdots + |S_d| = (d+1) + (d(v_1) - 1) + (d(v_2) - 1) + \cdots + (d(v_d) - 1) \leq n \Rightarrow \sum_{v:v\sim u} d(v) = (d(v))^2 \leq n-1$

by Cauchy-Schwarz inequality:

$n(n-1) \geq \sum_{v\in V} d(v)^2 \geq \frac{\left(\sum_{v\in V} d(v)\right)}{n} = \frac{4|E|^2}{n}$

8.4 the fundamental theorem of extremal graph theory: The ErdsStone theorem

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

corollary:

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

$\mathrm{ex}(n,H)$: the largest number of edges that a graph $G \not\supseteq H$ on n vertices can have

K_s^r : the complete r-partite graph with s vertices in each class

$\chi(G)$: the chromatic number of G, the smallest number of colors that one can use to color the vertices so that no adjacent vertices have the same color.

9 Extremal set theory

9.1 Sunflower Lemma

A set family $\mathcal{F} \subseteq 2^X$ is a sunflower of size r with a core $C \subseteq X$ if $\forall S, T \in \mathcal{F}$ that $S \neq T$, $S \cap T = C$. (注意核心也可以是空集)

Sunflower Lemma: Let $\mathcal{F} \subseteq \binom{X}{k}$. If $|\mathcal{F}| > k!(r-1)^k$, then $\mathcal{F}$ contains a sunflower of size r.

注意到这个界和 X 无关，即 k 元集多到一定程度的时候，必然包含一定规模的太阳花。

证明：induction on k ，核心技巧：把 k 元集降成 k-1 元集

选出 set family $\mathcal{G} \subseteq \mathcal{F}$ ，包含最多的 disjoint set。如果 $|\mathcal{G}| \geq r$，那么已经组成了空核心的 r-太阳花。

如果 $|\mathcal{G}| \leq r$，把里面所有点组合成集合 $Y = \bigcup_{S \in \mathcal{G}} S$，用反证法容易得到 Y 和 $\mathcal{F}$ 中所有集合相交。

我们想在 Y 中找到一个比较“流行”的点 y^*，包含它的集合数不比平均水平差，根据 Fpigeonhole principle 我们知道肯定存在这样的一个点。

平均水平：$\frac{|\mathcal{F}|}{|Y|} > \frac{k!(r-1)^k}{k(r-1)} = (k-1)!(r-1)^{k-1}$ （分子是前提条件，分母可以根据 Y 的定义得到）

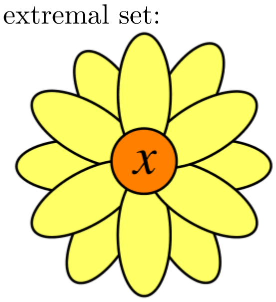
把包含这个点的集合都去掉这个点，得到包含 k-1 元集的 set family $\mathcal{H} = \{S \setminus \{y^*\} \mid S \in \mathcal{F} \wedge y^* \in S\}$，集合数量不低于平均水平。

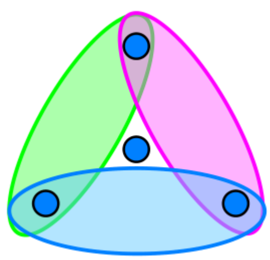
平均水平正好长得跟归纳假设一样，所以根据归纳假设，$\mathcal{H}$ 中包含一个 r-太阳花。把公共点 y^* 加回去即可得到 k 元集中的 r-太阳花

9.2 The ErdsKoRado Theorem

Let $\mathcal{F} \subseteq \binom{X}{k}$ where $|X| = n$ and $n \geq 2k$. If $\mathcal{F}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. (intersecting: for any $S, T \in \mathcal{F}$, $S \cap T \neq \emptyset$)

extremal set:





9.3 antichain

- Sperner's theorem

Let $\mathcal{F} \subseteq 2^X$ where $|X| = n$. If $\mathcal{F}$ is an antichain, then$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

- The LYM inequality (stronger)

$\sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$

10 Matching Theory

10.1 Hall's Theorem

A bipartite graph $G(U, V, E)$ has a matching of U if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$. ($N(S)$: set of vertices that are adjacent to one of the vertices in S.)

10.2 König-Egerváry theorem

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover

matching: $M \subseteq E$, no $e_1, e_2 \in M$ share a vertex

vertex cover: $C \subseteq V$, all $e \in E$ adjacent to some $v \in C$

matrix form: For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.

10.3 Dilworth's Theorem

Size of the largest antichain in the poset P = size of the smallest partition of P into chains.

poset: partially ordered set

chain: all pairs of elements in chain are comparable

antichain: all pairs of elements in antichain are incomparable.