

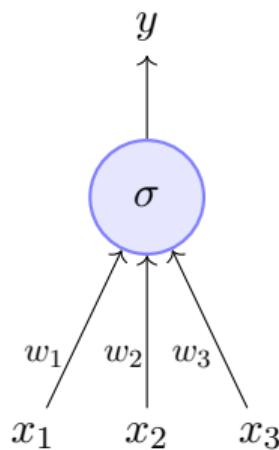
3.36pt

CS7015 (Deep Learning) : Lecture 2

McCulloch Pitts Neuron, Thresholding Logic, Perceptrons, Perceptron Learning Algorithm and Convergence, Multilayer Perceptrons (MLPs), Representation Power of MLPs

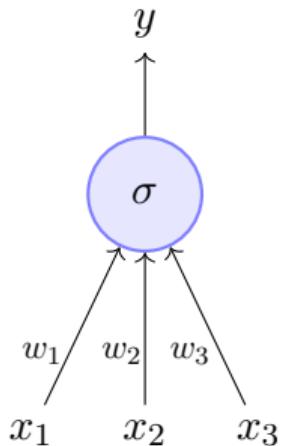
Mitesh M. Khapra

Department of Computer Science and Engineering
Indian Institute of Technology Madras



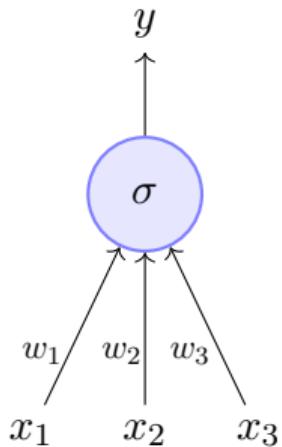
Artificial Neuron

- The most fundamental unit of a deep neural network is called an *artificial neuron*



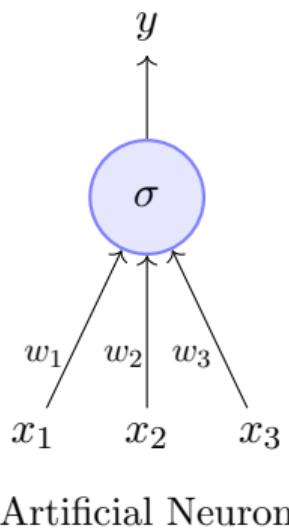
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- Why is it called a neuron ? Where does the inspiration come from ?

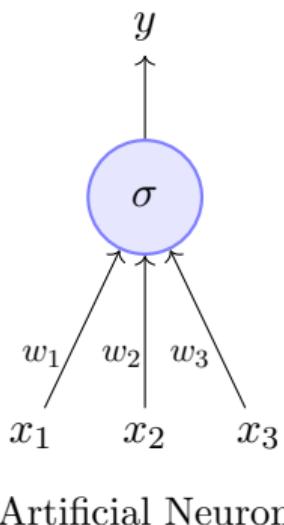


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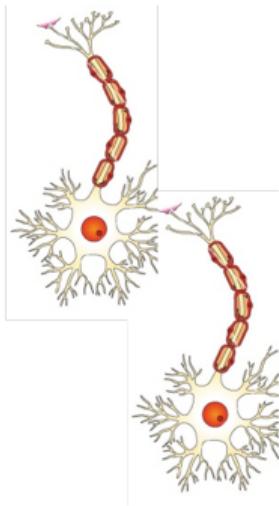


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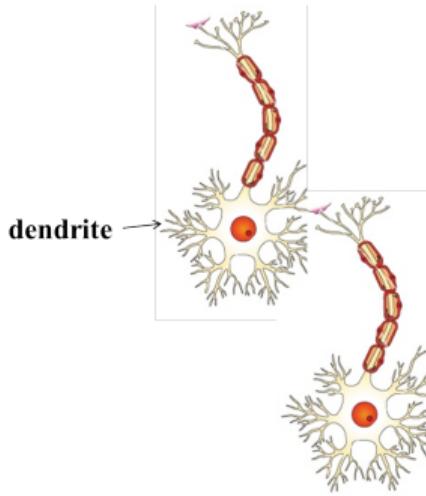
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- Why is it called a neuron ? Where does the inspiration come from ?
- The inspiration comes from biology (more specifically, from the *brain*)
- *biological neurons = neural cells = neural processing units*
- We will first see what a biological neuron looks like ...



Biological Neurons*

*Image adapted from

<https://cdn.vectorstock.com/i/composite/12,25/neuron-cell-vector-81225.jpg>

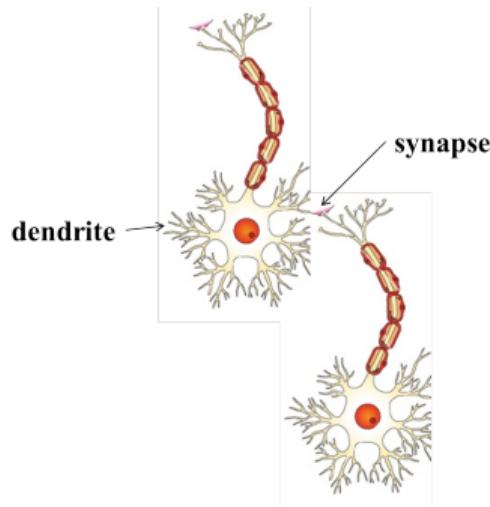


- **dendrite:** receives signals from other neurons

Biological Neurons*

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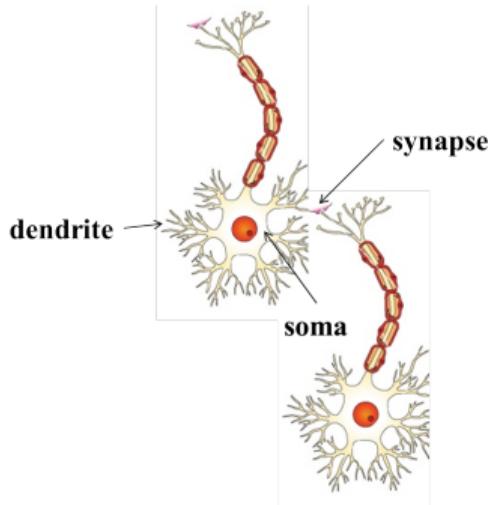
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Biological Neurons*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons

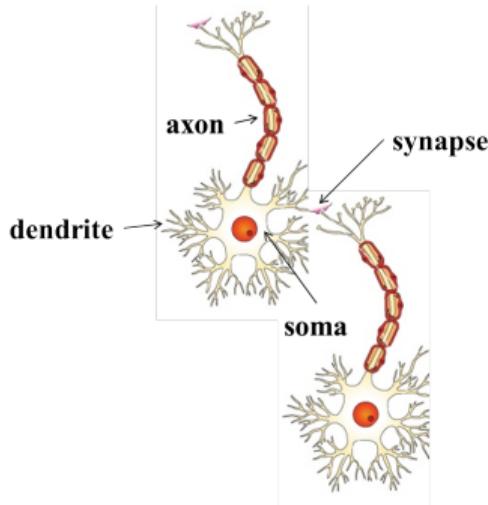
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- **dendrite:** receives signals from other neurons
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- **soma:** processes the information

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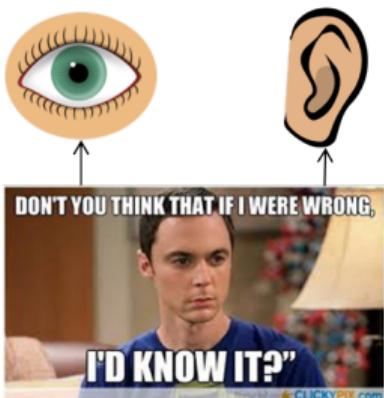
Biological Neurons*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons
- **soma:** processes the information
- **axon:** transmits the output of this neuron

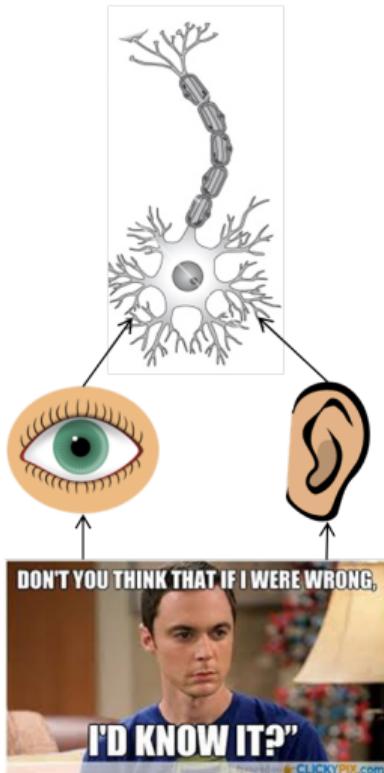
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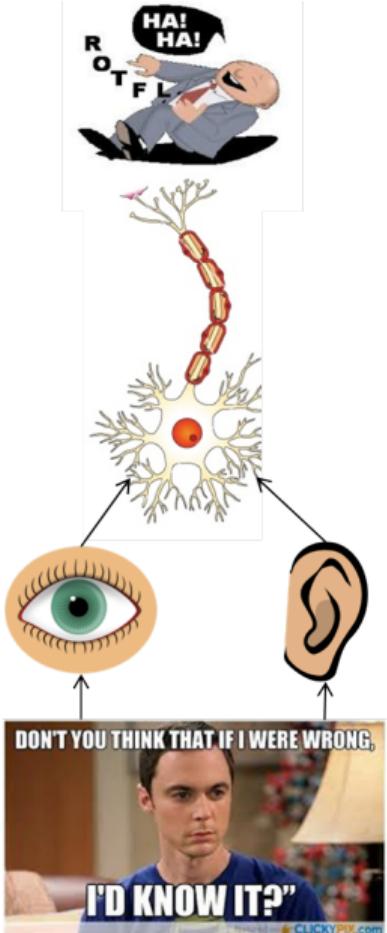
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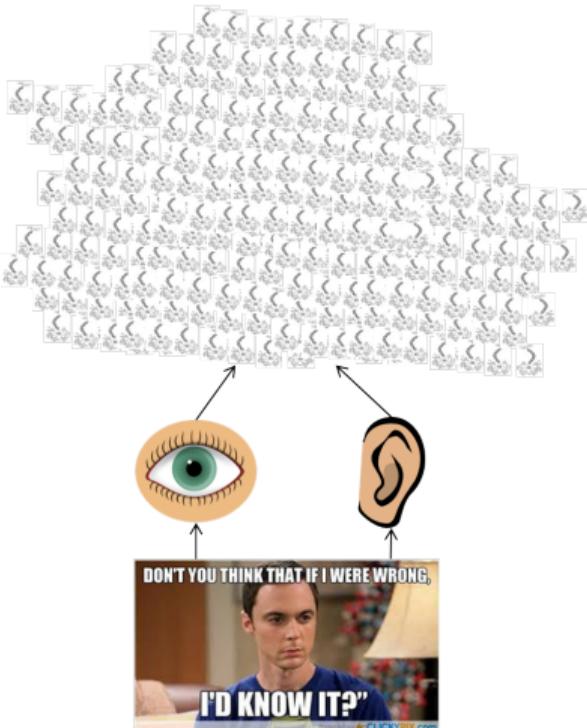
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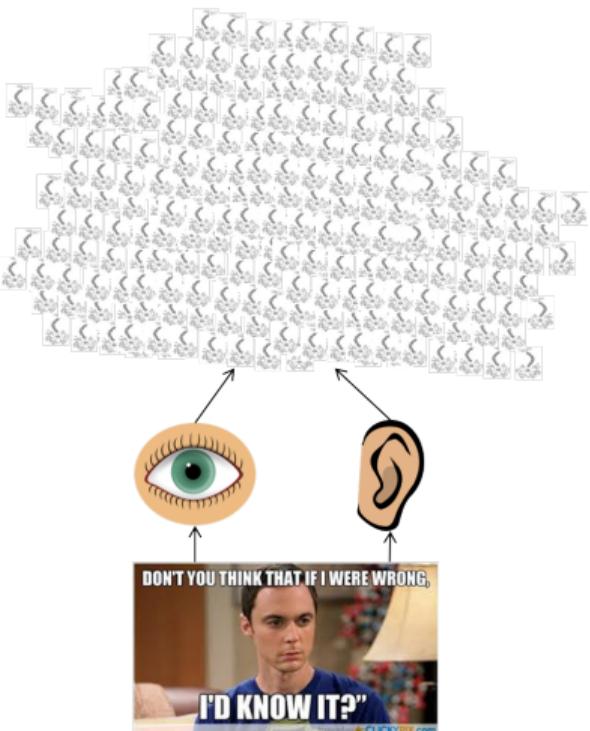


- Let us see a very cartoonish illustration of how a neuron works
- Our sense organs interact with the outside world
- They relay information to the neurons
- The neurons (may) get activated and produce a response (laughter in this case)

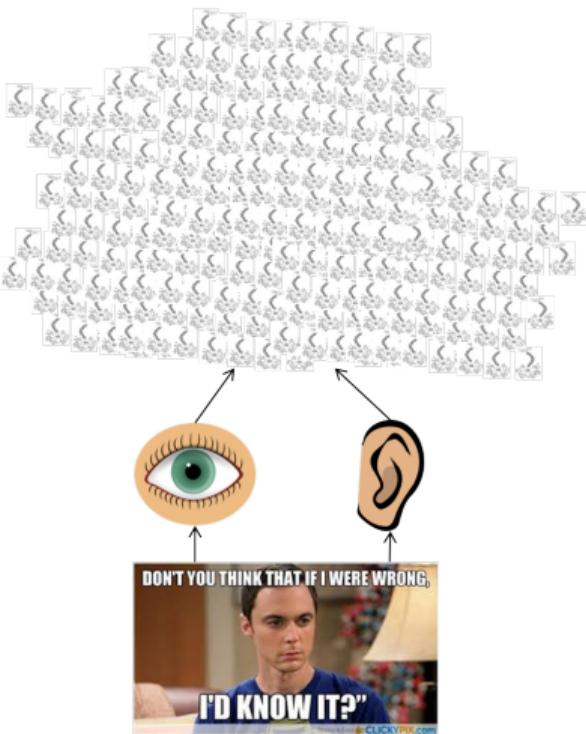
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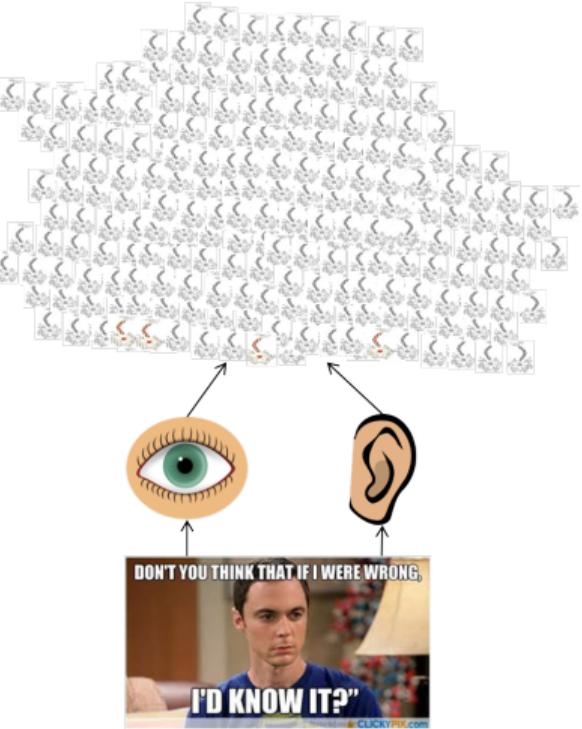


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- There is a massively parallel interconnected network of neurons

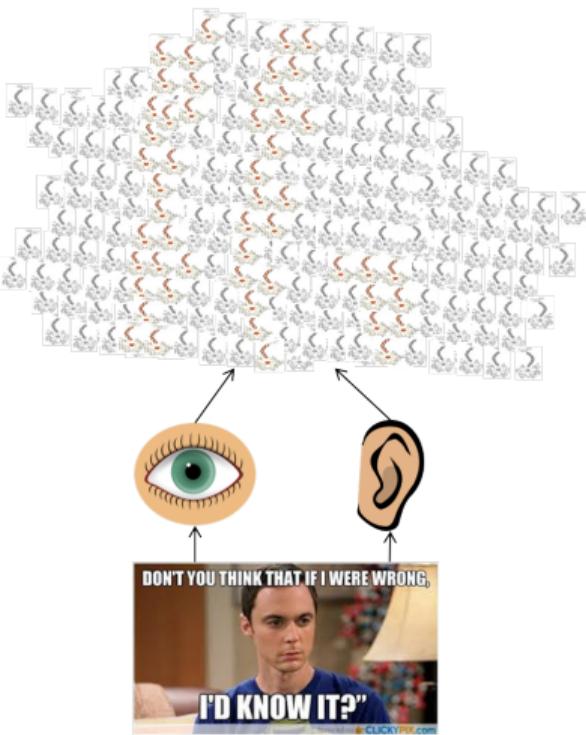


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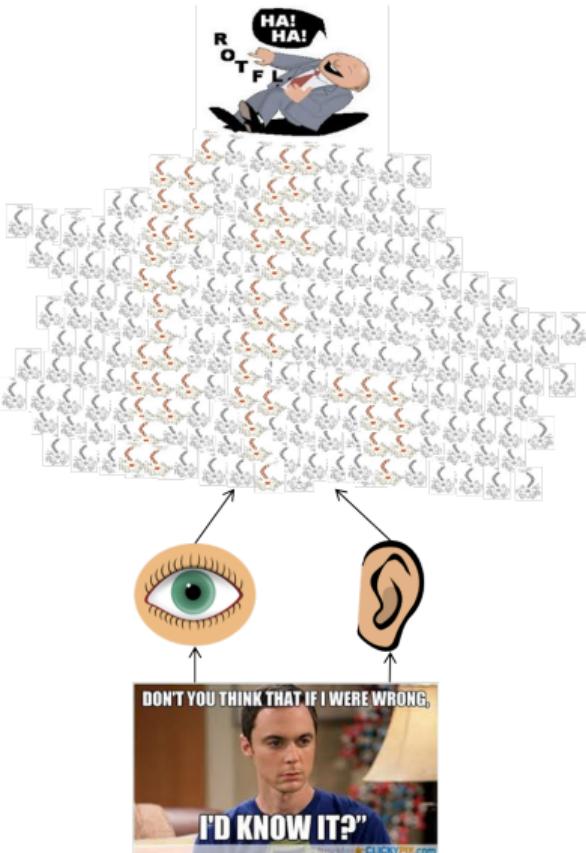




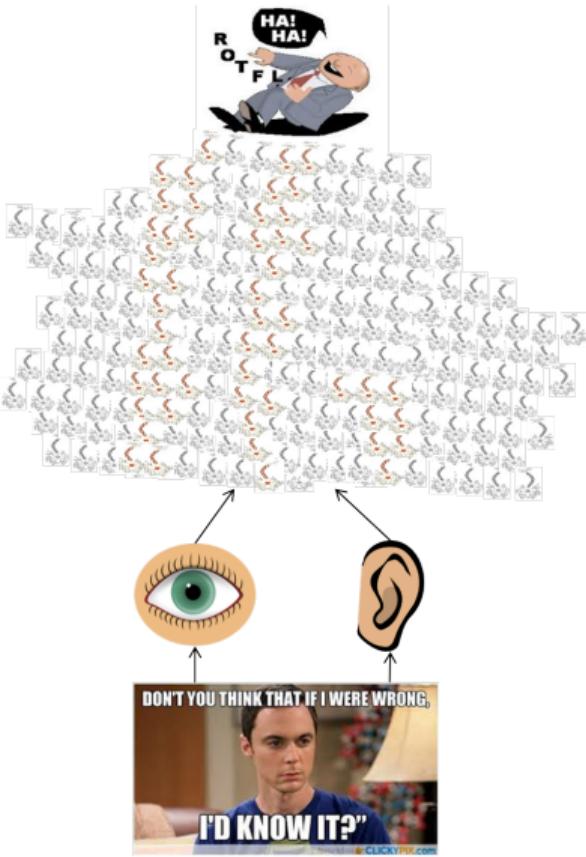
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- There is a massively parallel interconnected network of neurons
- The sense organs relay information to the lowest layer of neurons
- Some of these neurons may fire (in red) in response to this information and in turn relay information to other neurons they are connected to



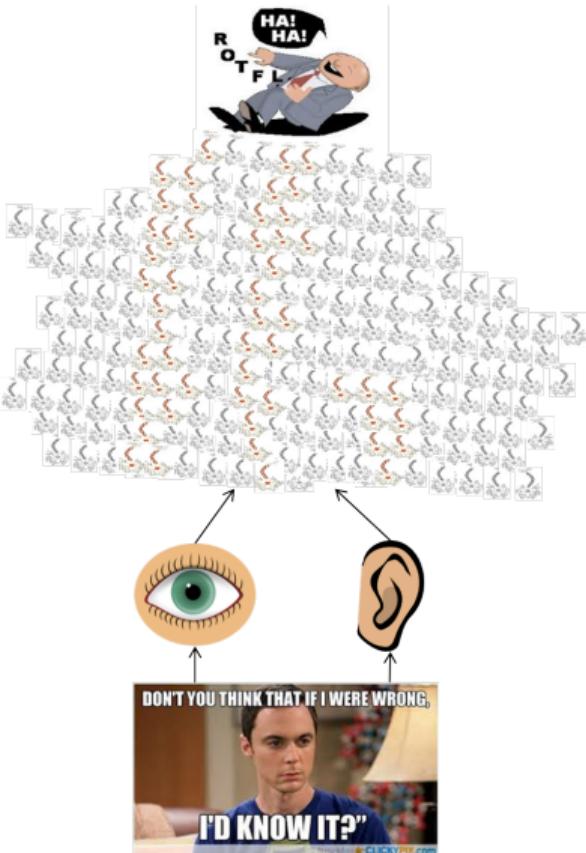
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- These neurons may also fire (again, in red) and the process continues



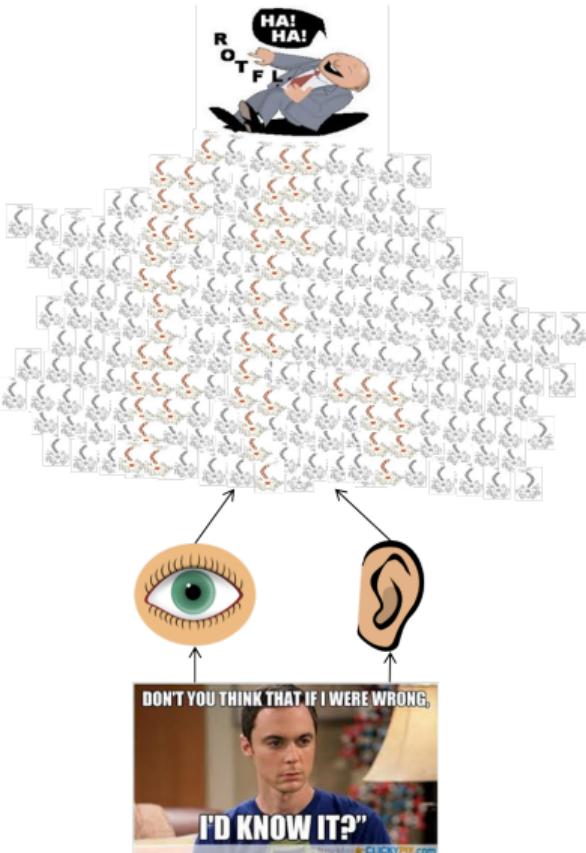
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- An average human brain has around 10^{11} (100 billion) neurons !

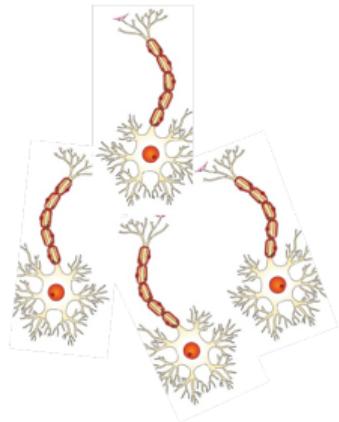


- This massively parallel network also ensures that there is division of work



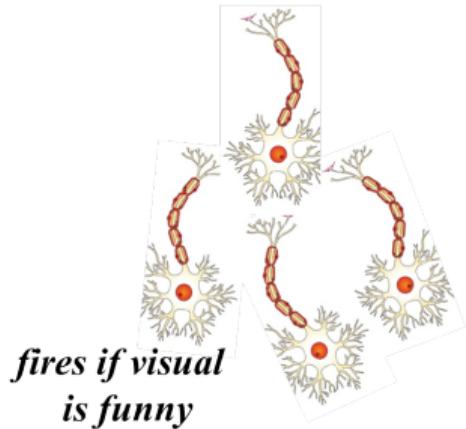
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A simplified illustration

*fires if at least
2 of the 3 inputs fired*

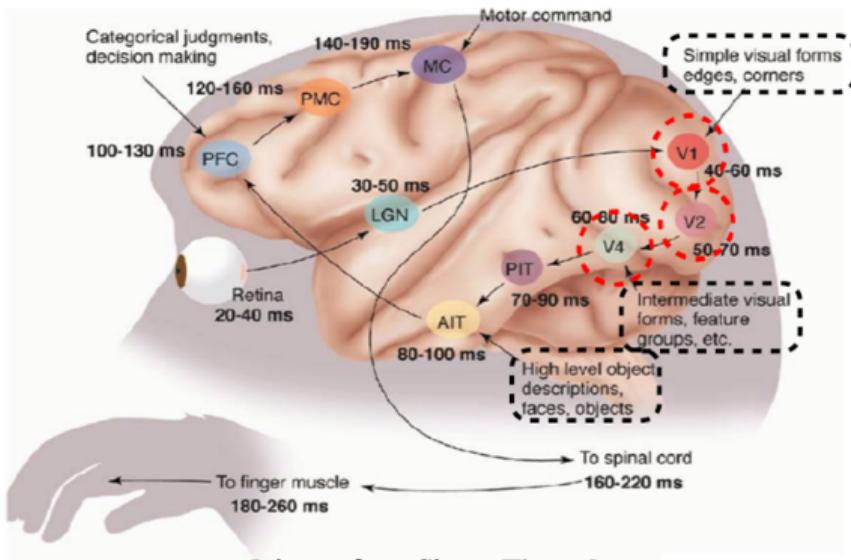


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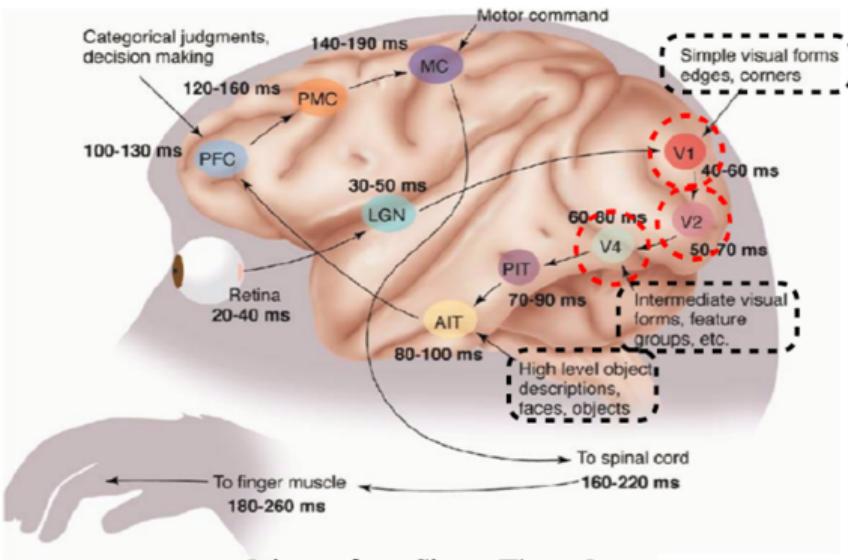
A simplified illustration

- The neurons in the brain are arranged in a hierarchy



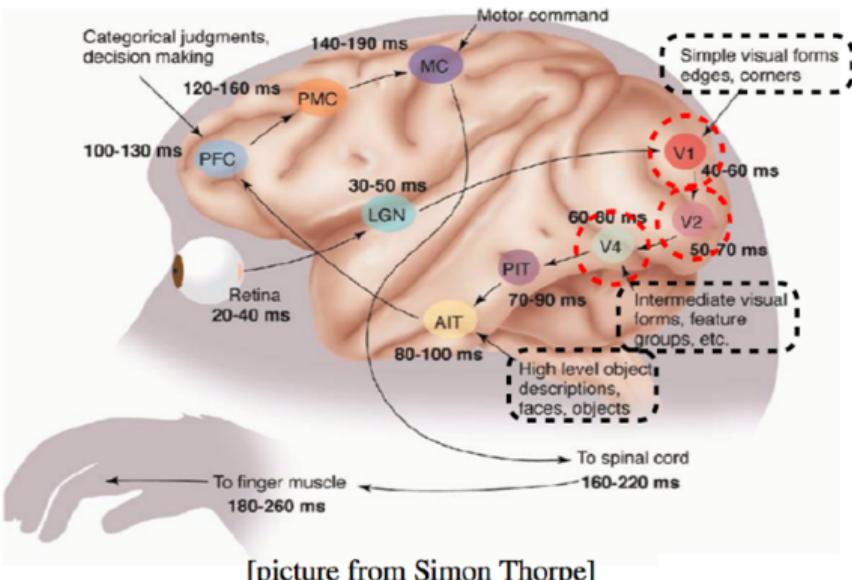
[picture from Simon Thorpe]

- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information



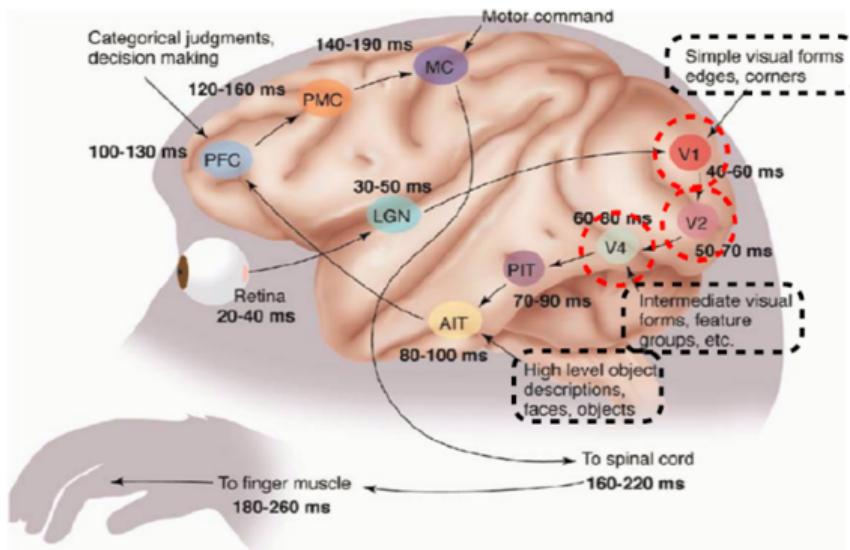
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- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information
- Starting from the retina, the information is relayed to several layers (follow the arrows)
- We observe that the layers *V1*, *V2* to *AIT* form a hierarchy (from identifying simple visual forms to high level objects)



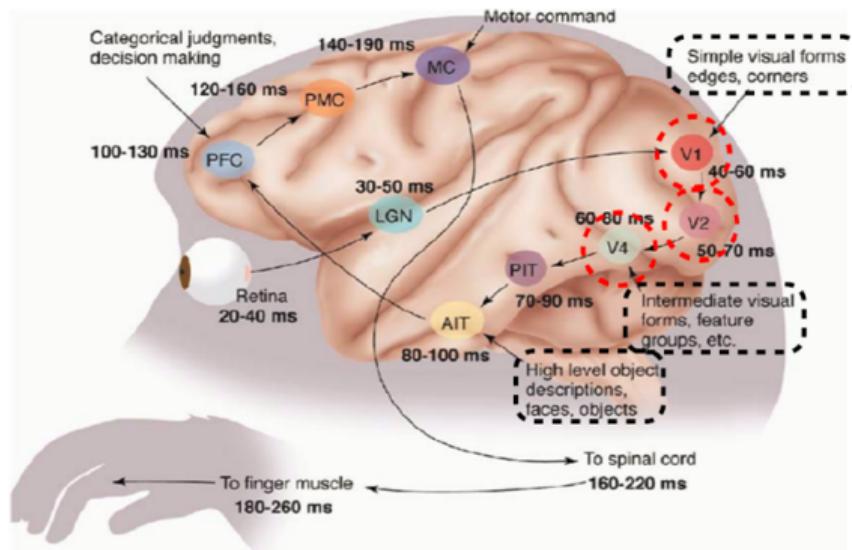
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Layer 1: detect edges & corners

Sample illustration of hierarchical processing
a

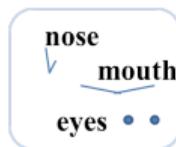
^aIdea borrowed from Hugo Larochelle's lectures



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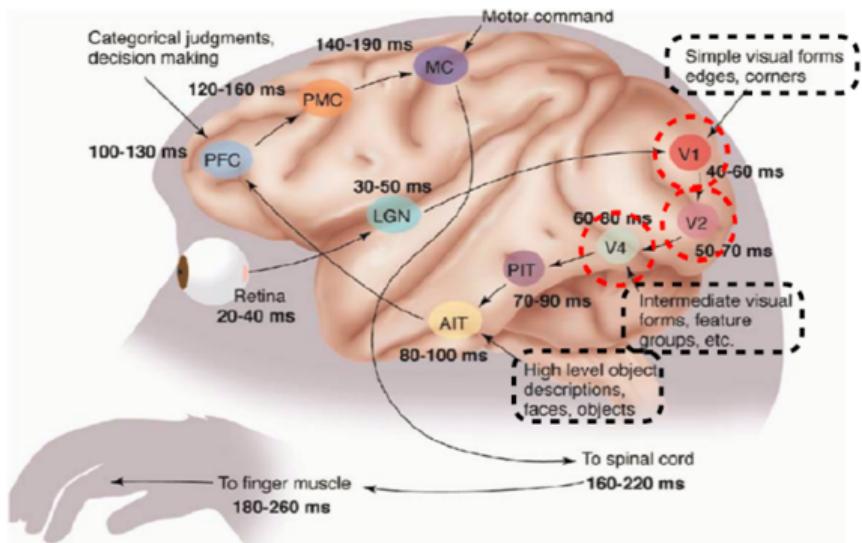
Layer 1: detect edges & corners



Layer 2: form feature groups

Sample illustration of hierarchical processing
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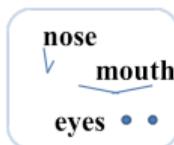
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Layer 1: detect edges & corners



Layer 2: form feature groups



Layer 3: detect high level objects, faces, etc.

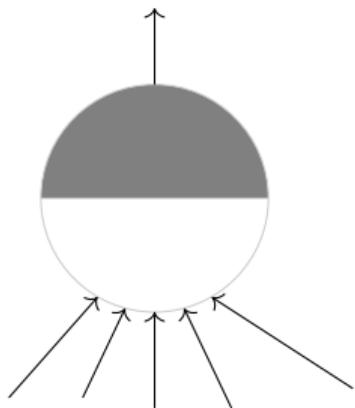
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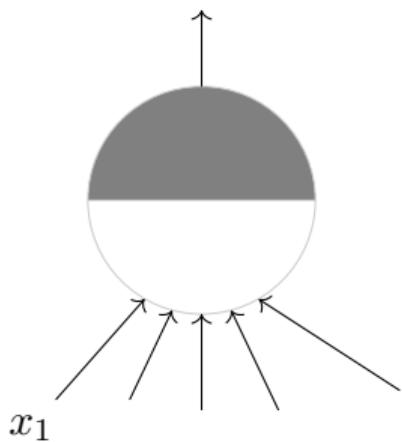
Disclaimer

- I understand very little about how the brain works !
- What you saw so far is an overly simplified explanation of how the brain works !
- But this explanation suffices for the purpose of this course !

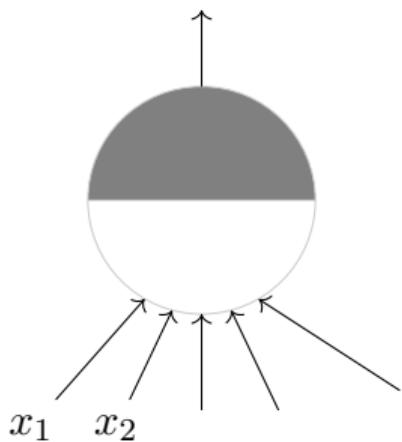
From biological units to computational units ...



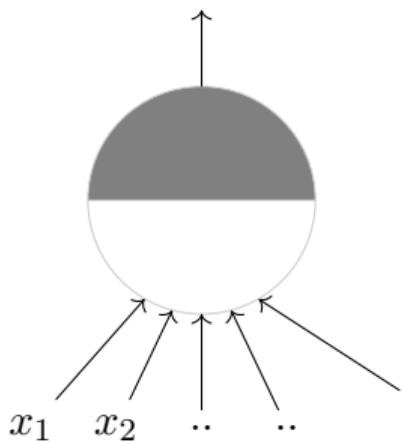
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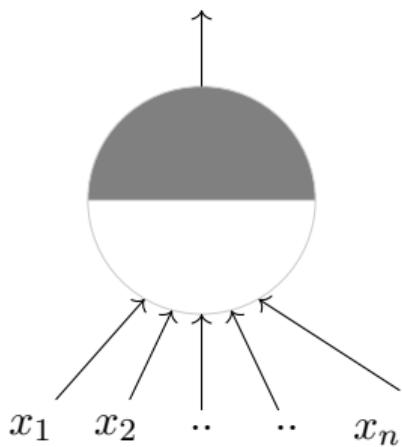
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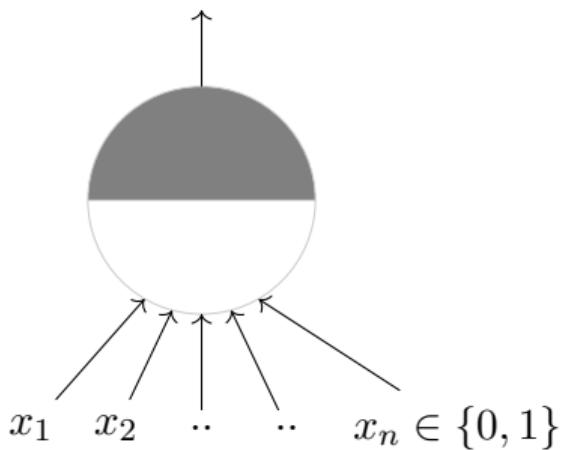
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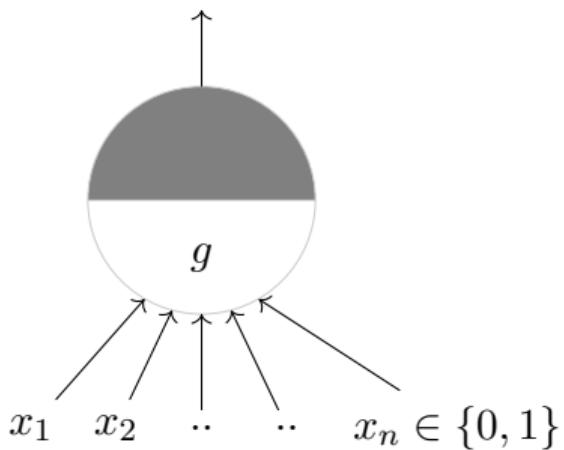
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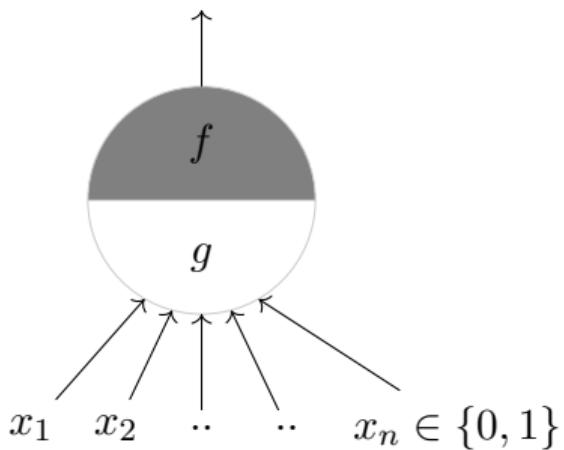
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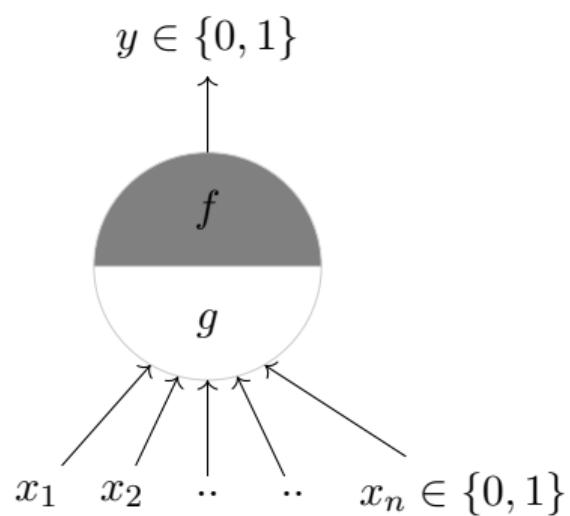
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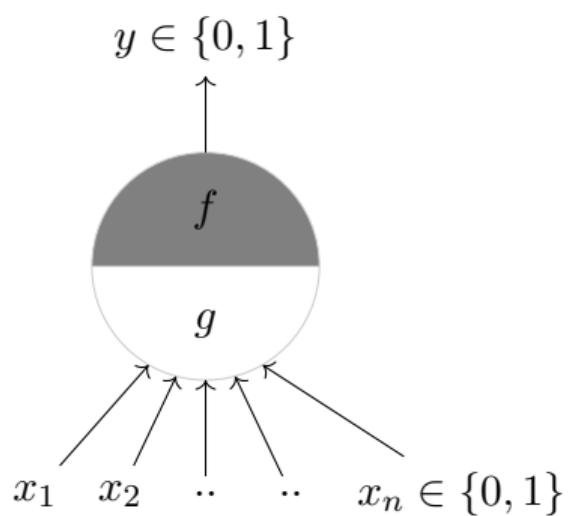
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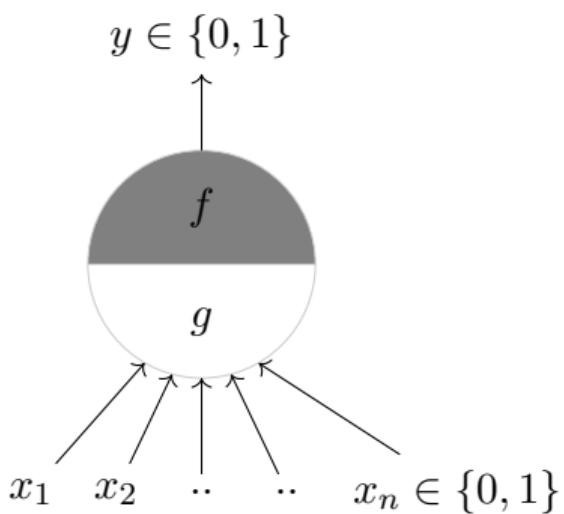
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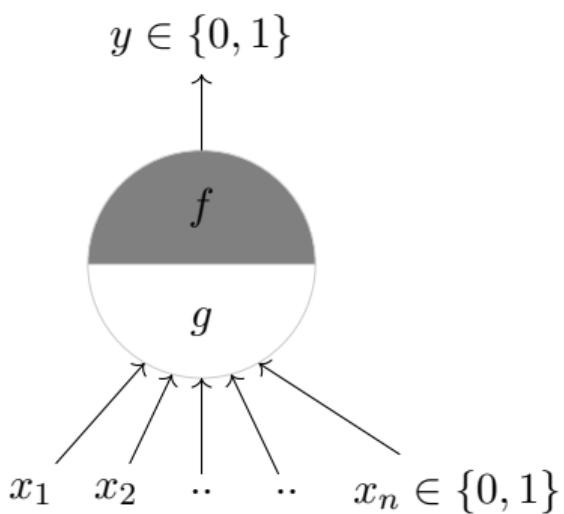
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- The inputs can be excitatory or inhibitory

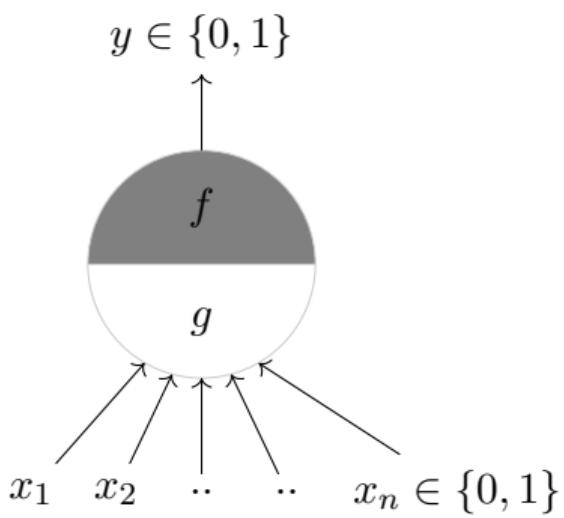


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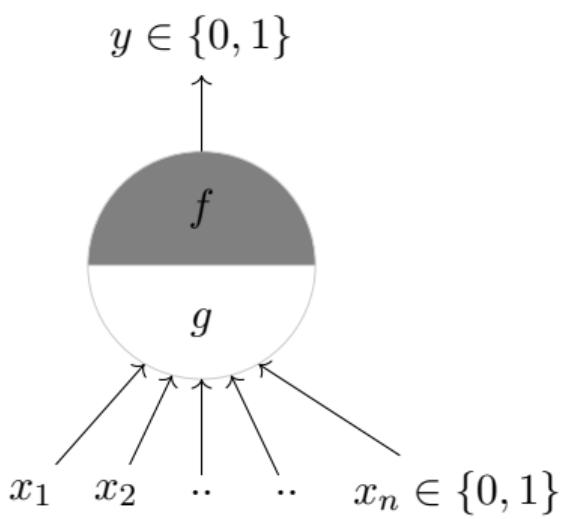
$$g(x_1, x_2, \dots, x_n) = g(\mathbf{x}) = \sum_{i=1}^n x_i$$



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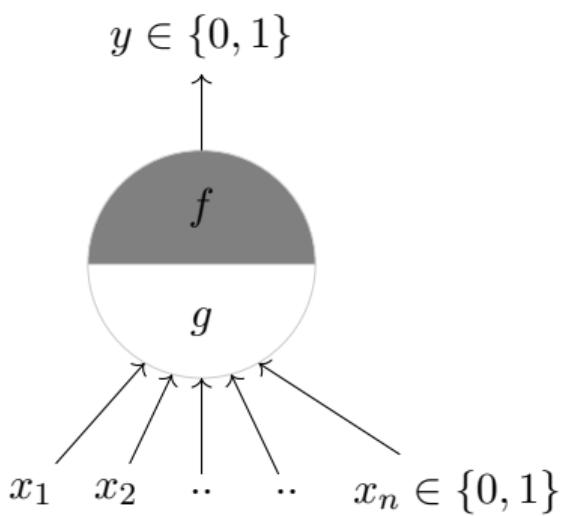
$$y = f(g(\mathbf{x})) = 1 \quad \text{if } g(\mathbf{x}) \geq \theta$$



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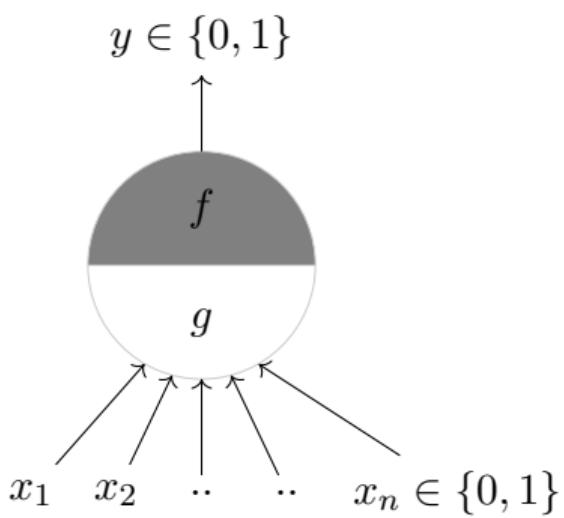


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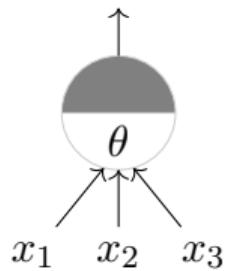
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- θ is called the thresholding parameter
- This is called Thresholding Logic

Let us implement some boolean functions using this McCulloch Pitts (MP) neuron

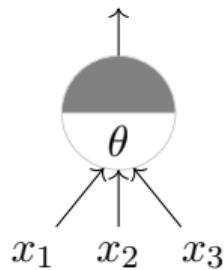
...

$$y \in \{0, 1\}$$



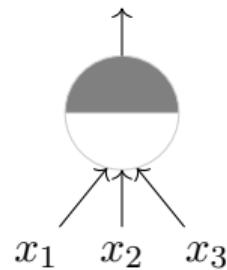
A McCulloch Pitts unit

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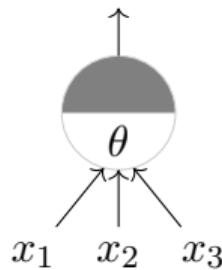
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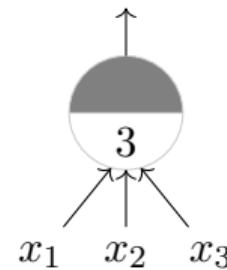
AND function

$$y \in \{0, 1\}$$



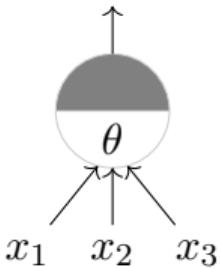
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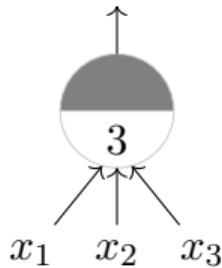
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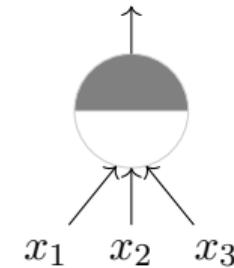
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



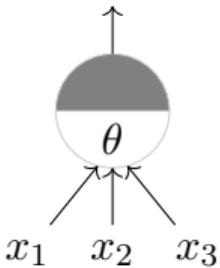
AND function

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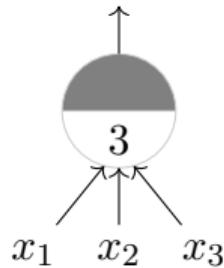
OR function

$$y \in \{0, 1\}$$



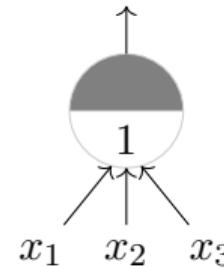
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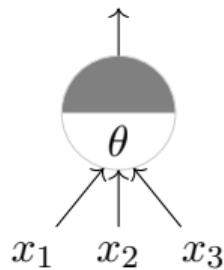
AND function

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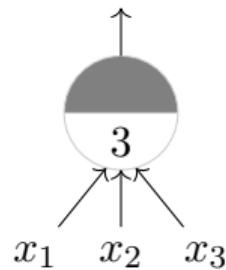
OR function

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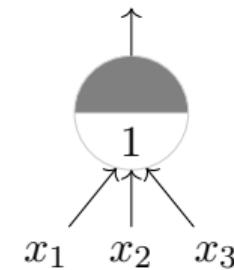
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



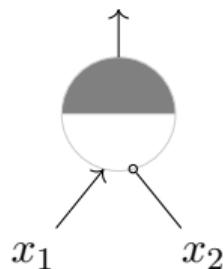
AND function

$$y \in \{0, 1\}$$



OR function

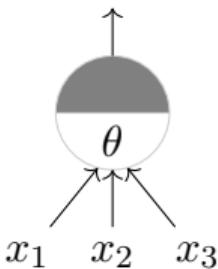
$$y \in \{0, 1\}$$



$$x_1 \text{ AND } !x_2 \text{ } a$$

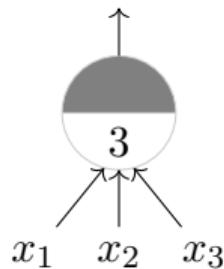
^acircle at the end indicates inhibitory input: if any inhibitory input is 1 the output will be 0

$$y \in \{0, 1\}$$



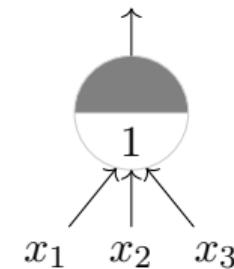
A McCulloch Pitts unit

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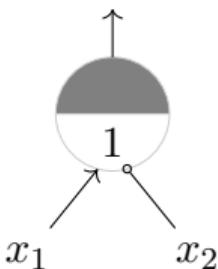
AND function

$$y \in \{0, 1\}$$



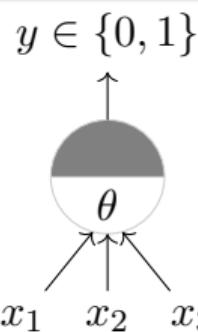
OR function

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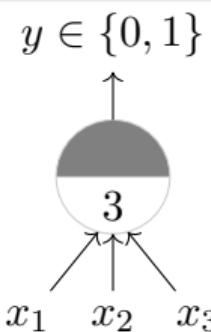


$$x_1 \text{ AND } !x_2 \text{ }^a$$

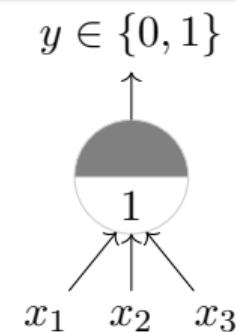
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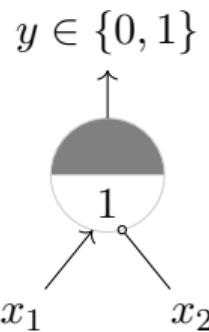
A McCulloch Pitts unit



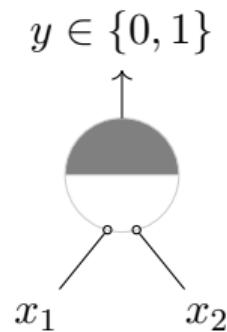
AND function



OR function



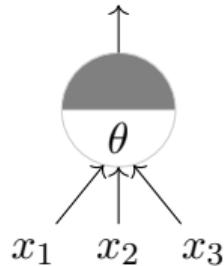
$x_1 \text{ AND } !x_2$ ^a



NOR function

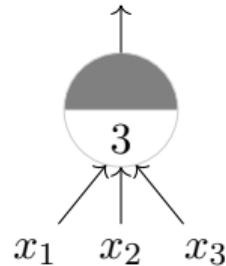
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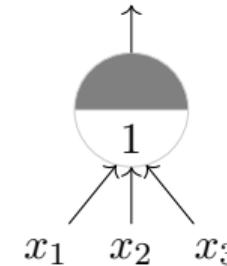
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



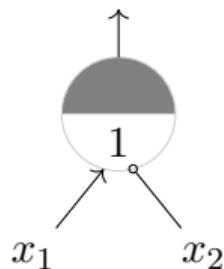
AND function

$$y \in \{0, 1\}$$



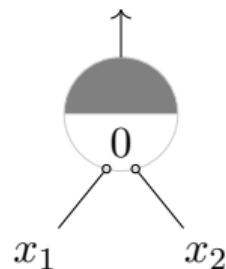
OR function

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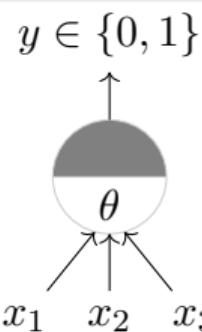
x_1 AND $!x_2$ ^a

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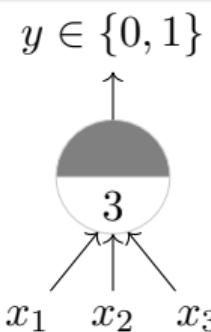


NOR function

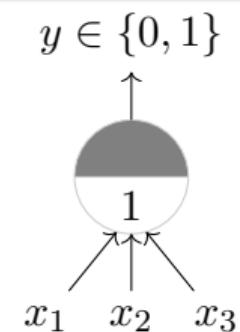
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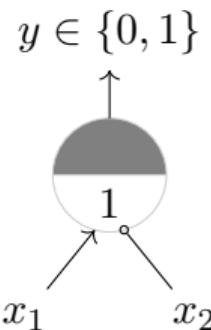
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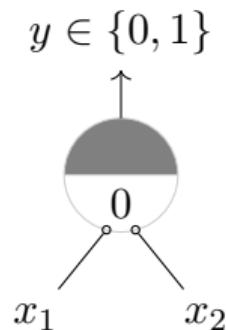
AND function



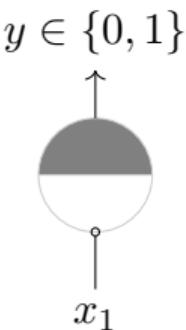
OR function



x_1 AND $!x_2$ ^a

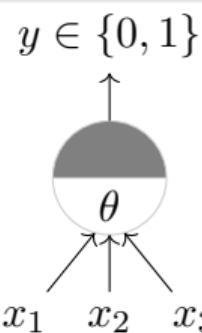


NOR function

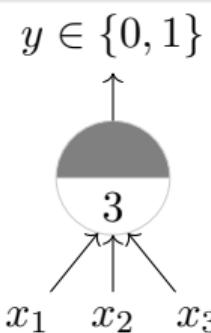


NOT function

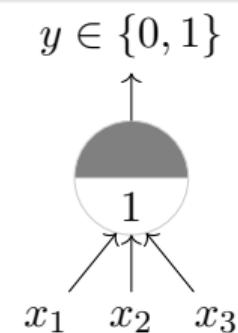
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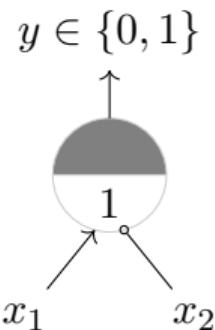
A McCulloch Pitts unit



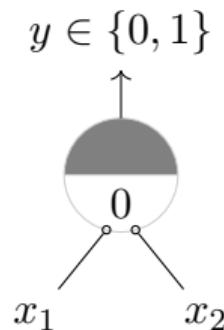
AND function



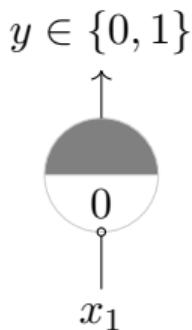
OR function



x_1 AND $!x_2$ ^a



NOR function



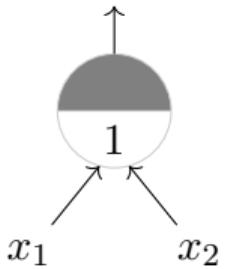
NOT function

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- Can any boolean function be represented using a McCulloch Pitts unit ?

- Can any boolean function be represented using a McCulloch Pitts unit ?
- Before answering this question let us first see the geometric interpretation of a MP unit ...

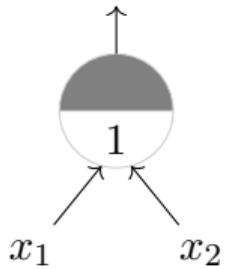
$$y \in \{0, 1\}$$



OR function

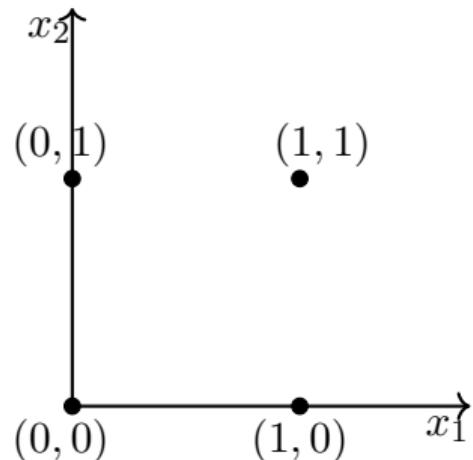
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$

$$y \in \{0, 1\}$$

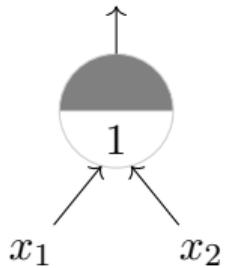


OR function

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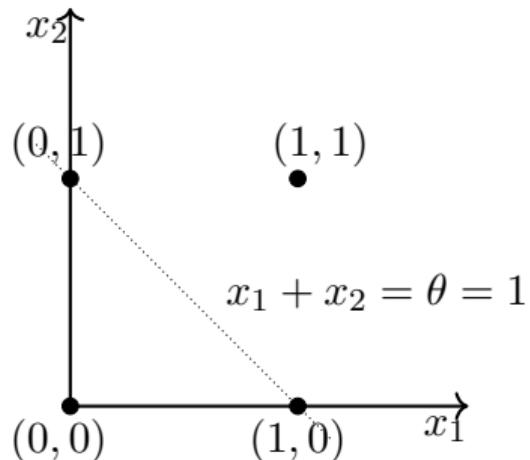


$$y \in \{0, 1\}$$

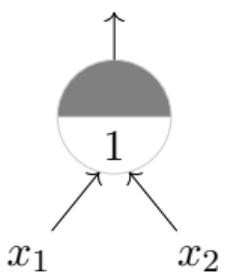


OR function

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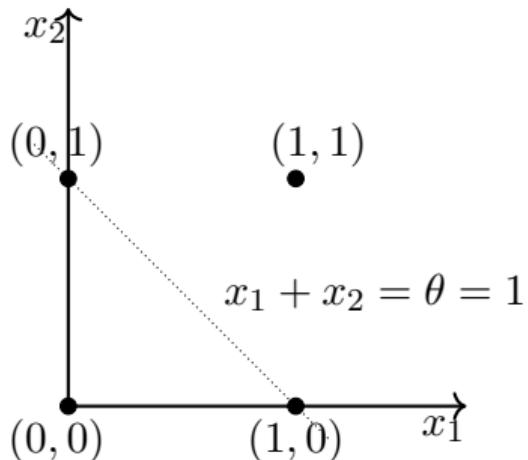
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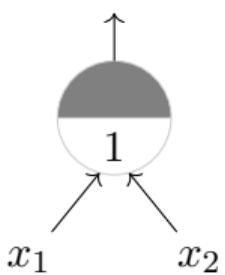
- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves

OR function

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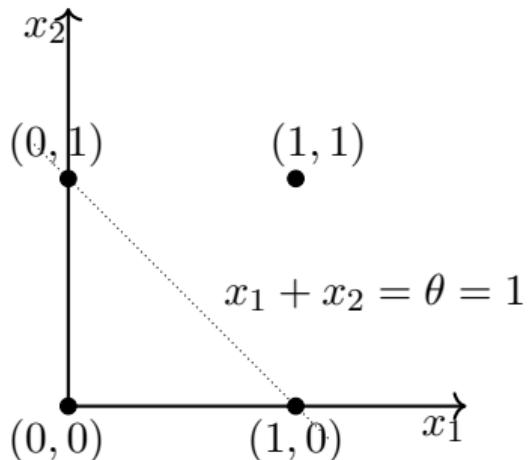
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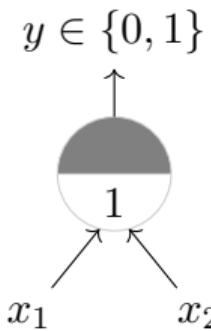


- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line $\sum_{i=1}^n x_i$ and points lying below this line

OR function

$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$

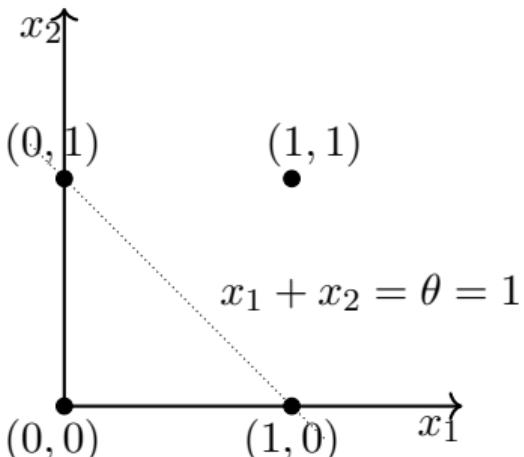


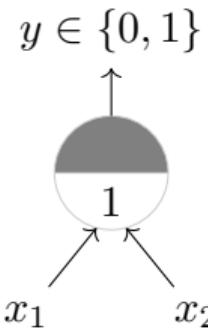


OR function

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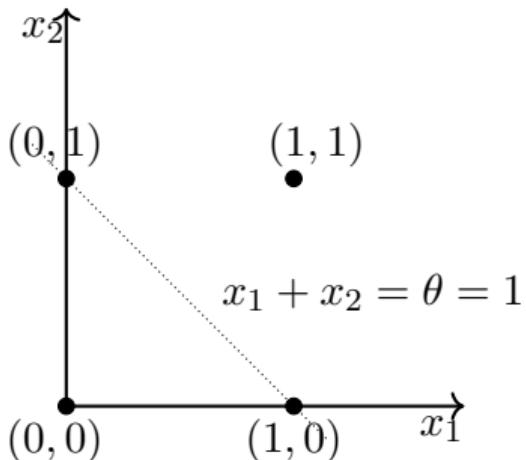
- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line $\sum_{i=1}^n x_i$ and points lying below this line
- In other words, all inputs which produce an output 0 will be on one side ($\sum_{i=1}^n x_i < \theta$) of the line and all inputs which produce a 1 will lie on the other side ($\sum_{i=1}^n x_i \geq \theta$) of this line





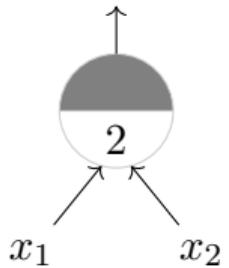
OR function

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- Let us convince ourselves about this with a few more examples (if it is not already clear from the math)

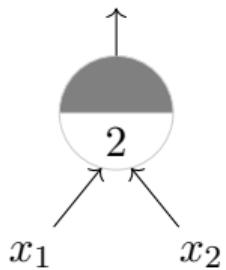
$$y \in \{0, 1\}$$



AND function

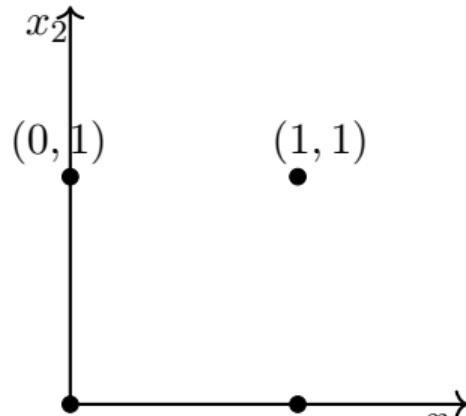
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 2$$

$$y \in \{0, 1\}$$

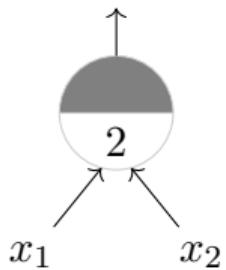


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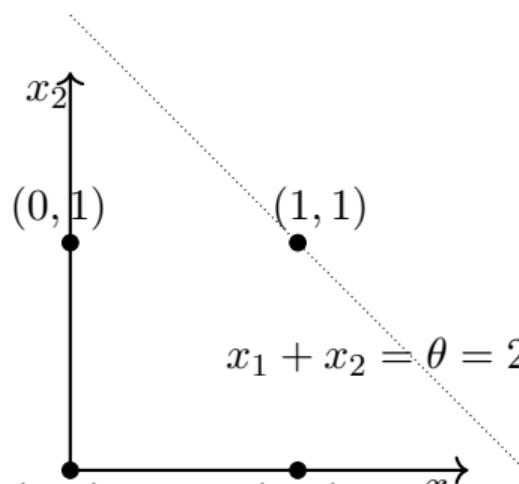


$$y \in \{0, 1\}$$

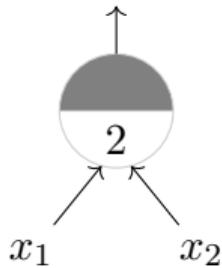


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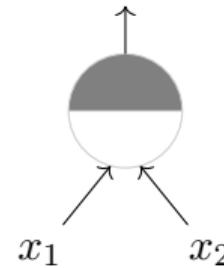
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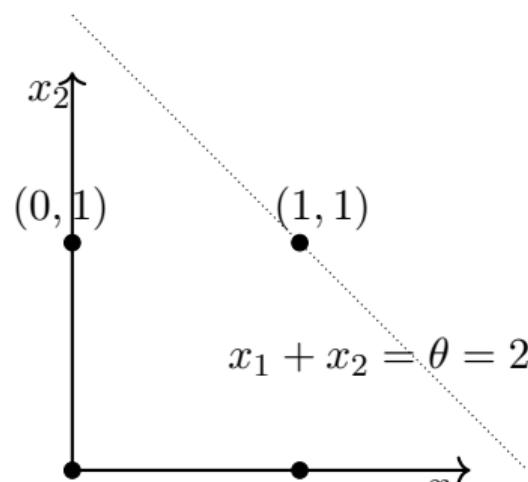
AND function

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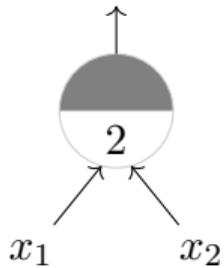
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Tautology (always ON)



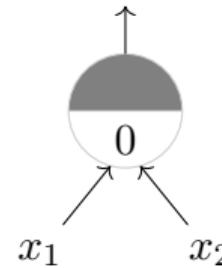
$$y \in \{0, 1\}$$



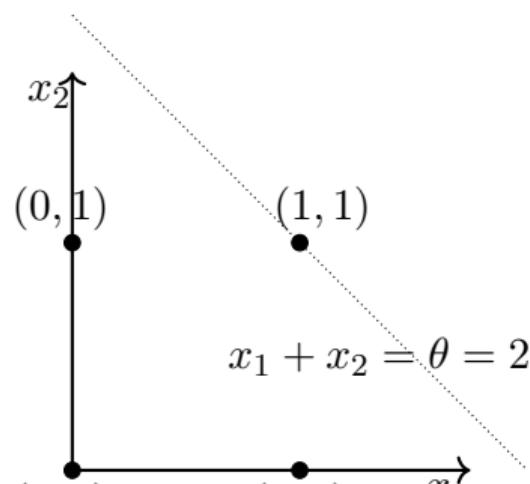
AND function

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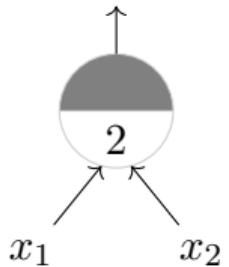
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Tautology (always ON)

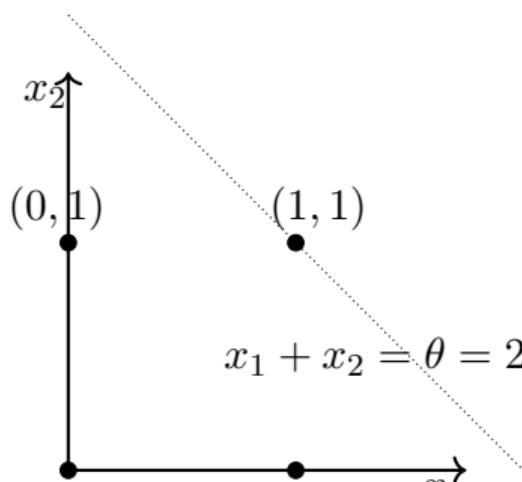


$$y \in \{0, 1\}$$

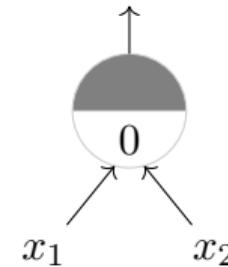


AND function

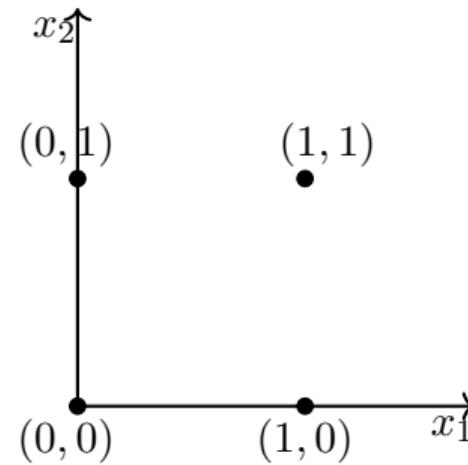
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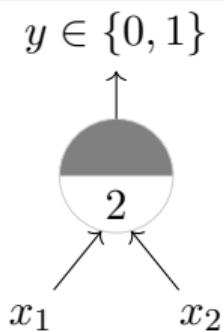


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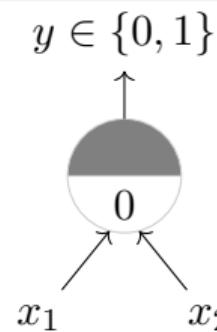
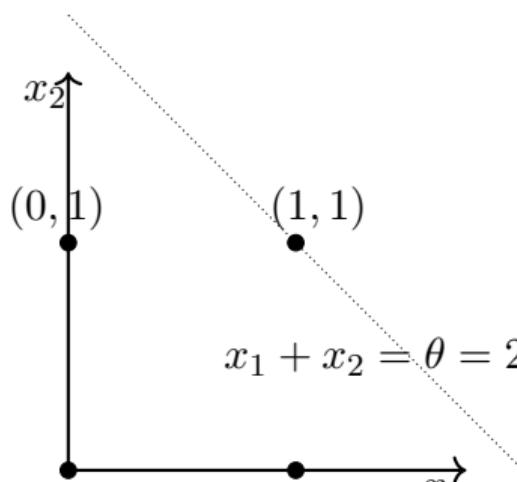
Tautology (always ON)



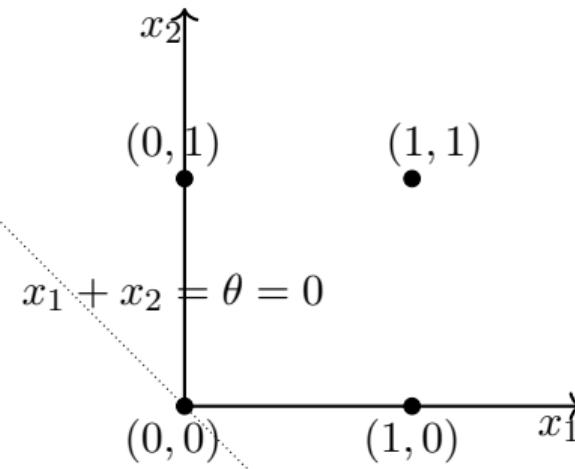


AND function

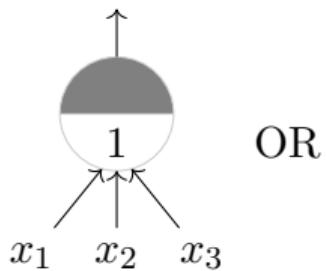
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 2$$



Tautology (always ON)

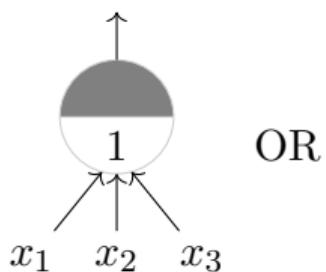


$$y \in \{0, 1\}$$



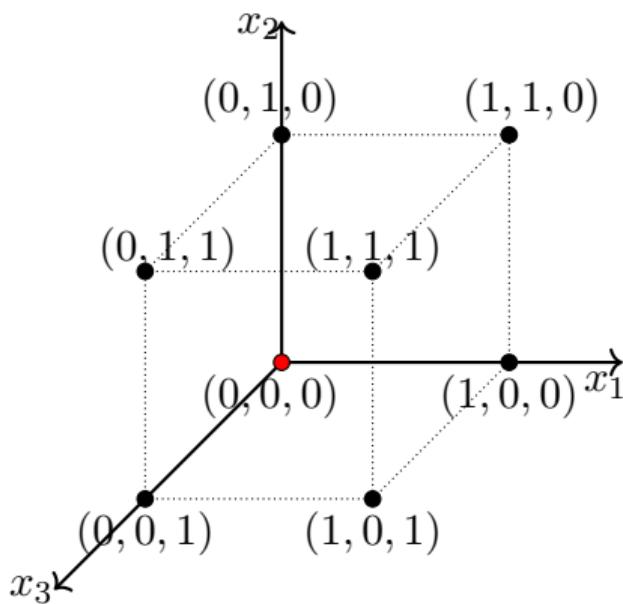
- What if we have more than 2 inputs?

$$y \in \{0, 1\}$$

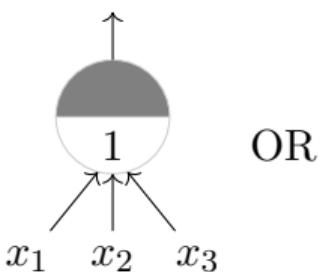


OR

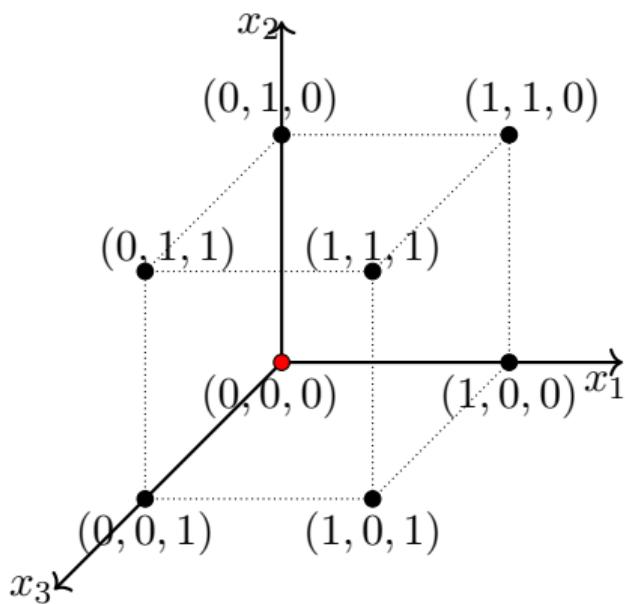
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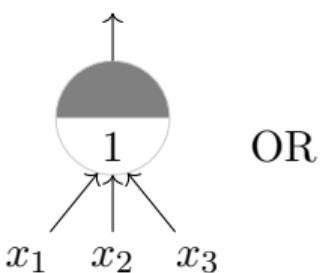
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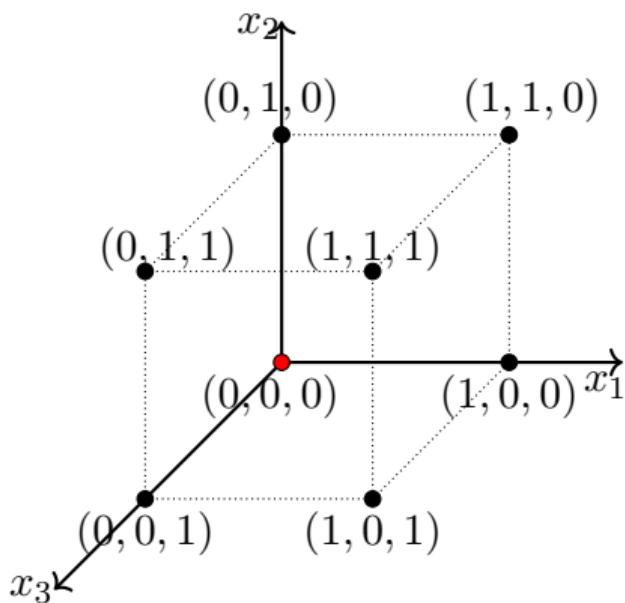
- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane



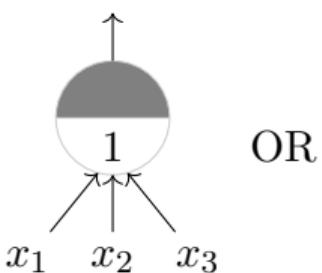
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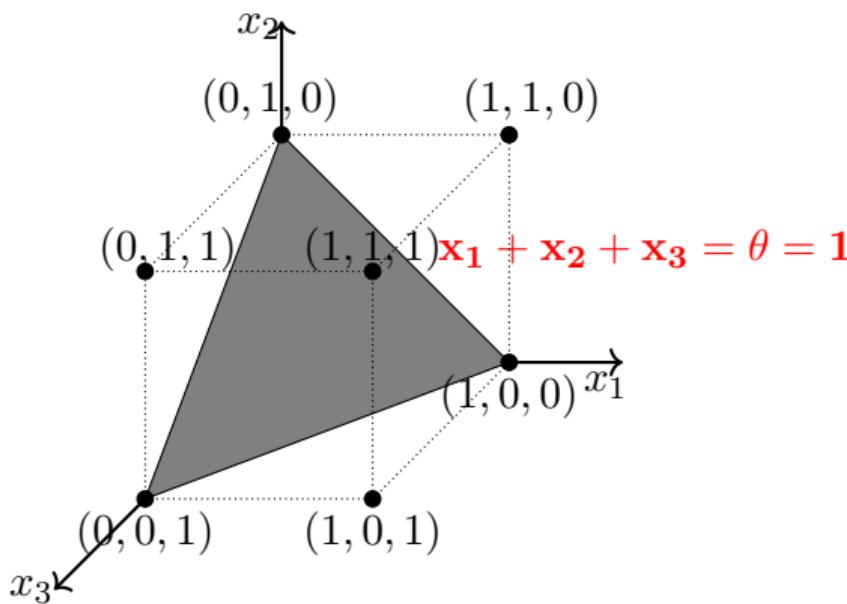
- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane
- For the OR function, we want a plane such that the point $(0,0,0)$ lies on one side and the remaining 7 points lie on the other side of the plane



$$y \in \{0, 1\}$$



- What if we have more than 2 inputs?
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- A single McCulloch Pitts Neuron can be used to represent boolean functions which are linearly separable

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- Linear separability (for boolean functions) : There exists a line (plane) such that all inputs which produce a 1 lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane)

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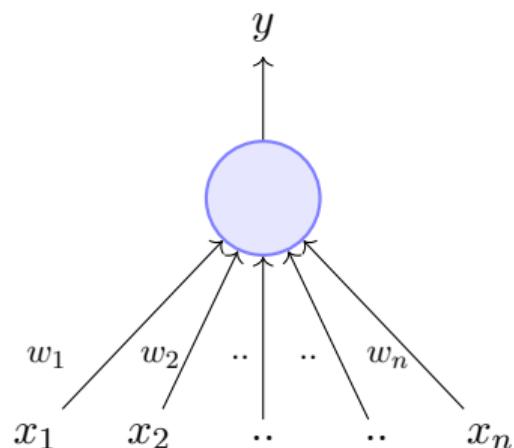
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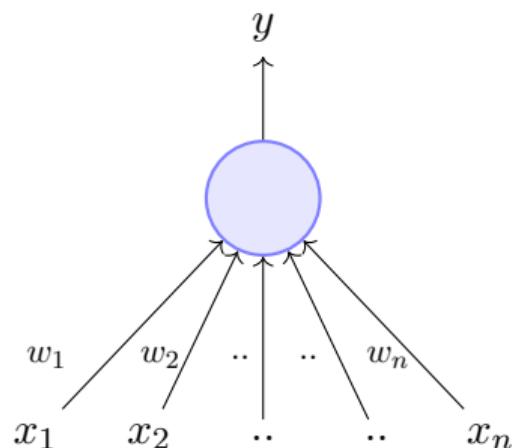
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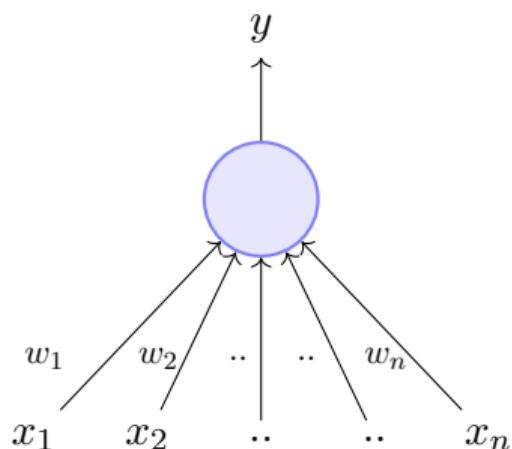
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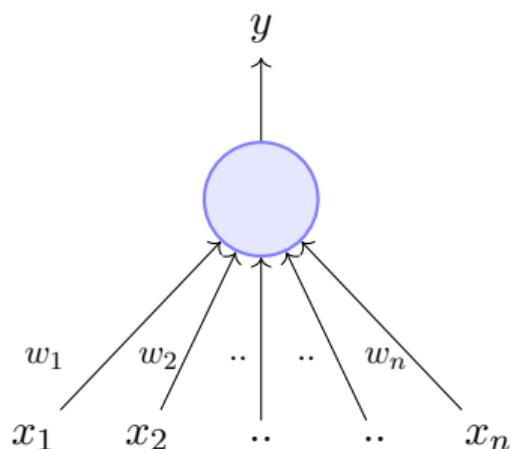
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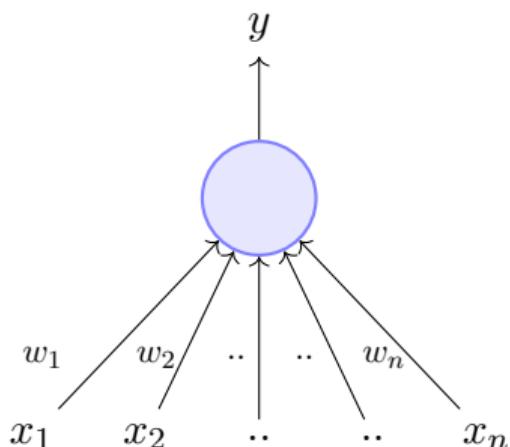
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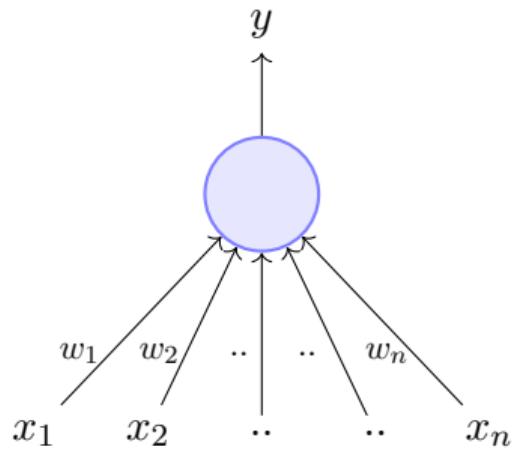
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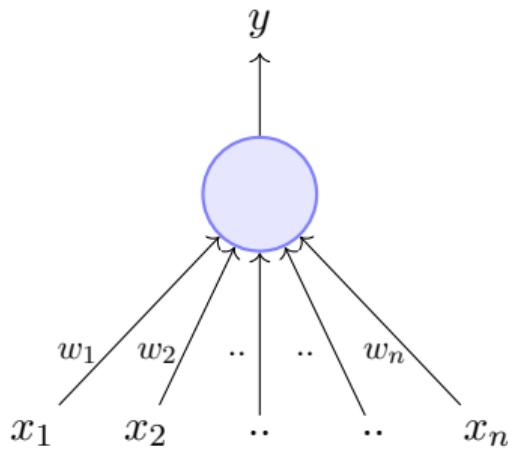


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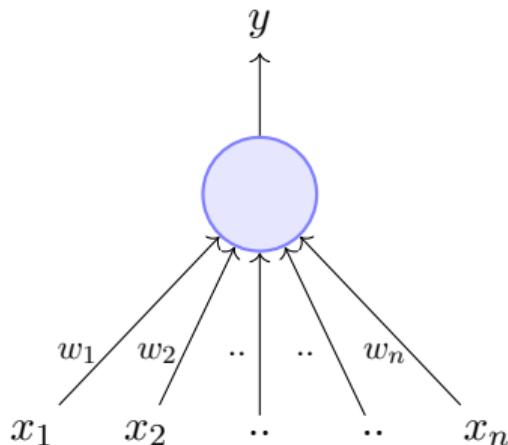


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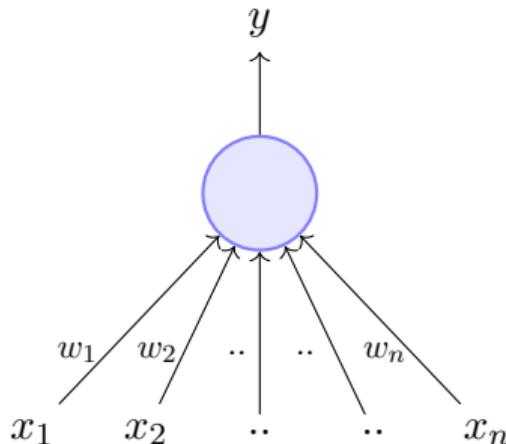


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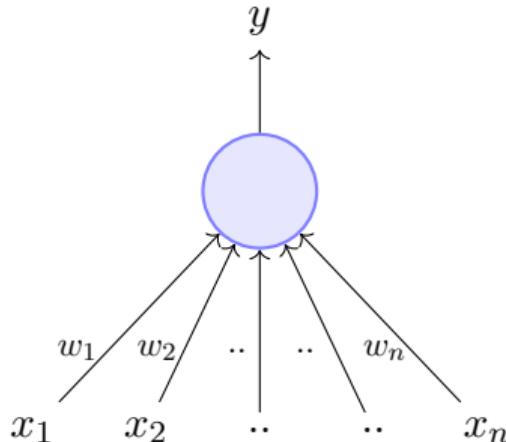
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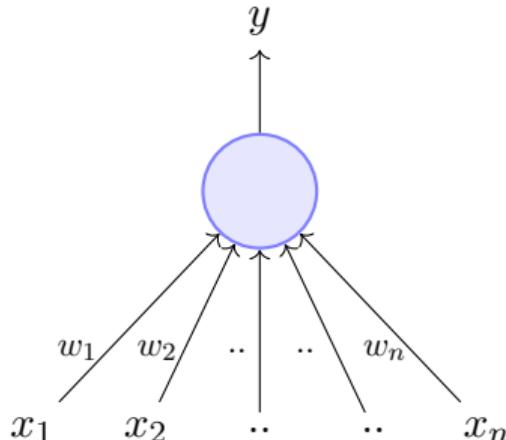


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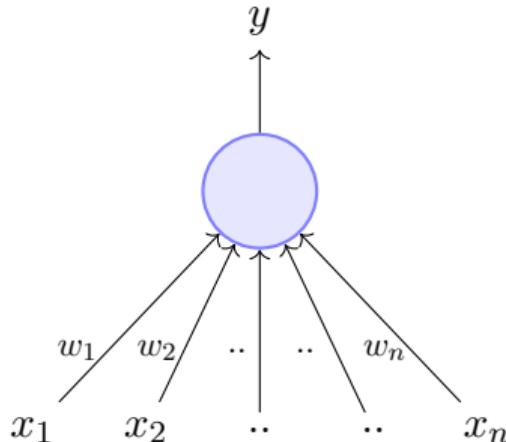
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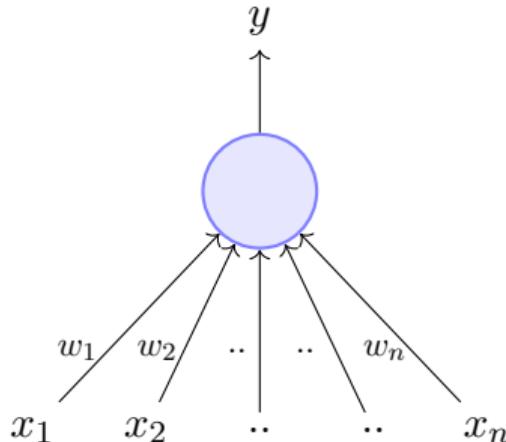
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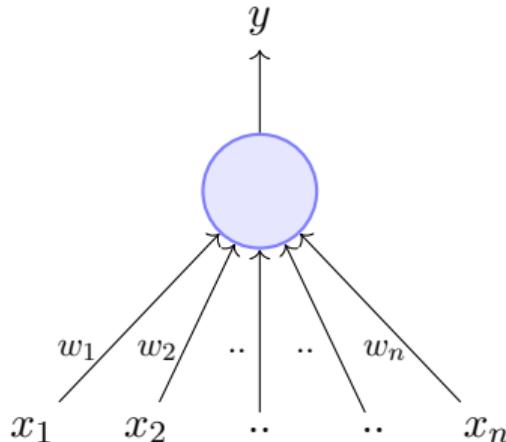
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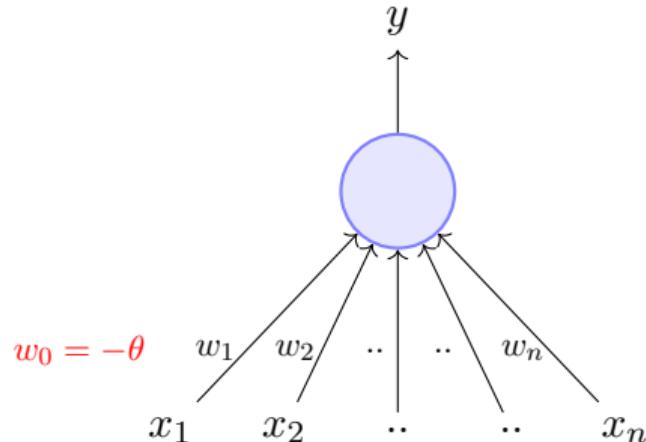
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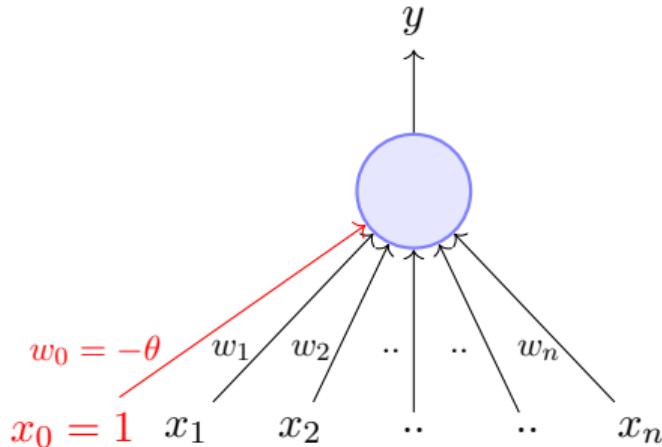
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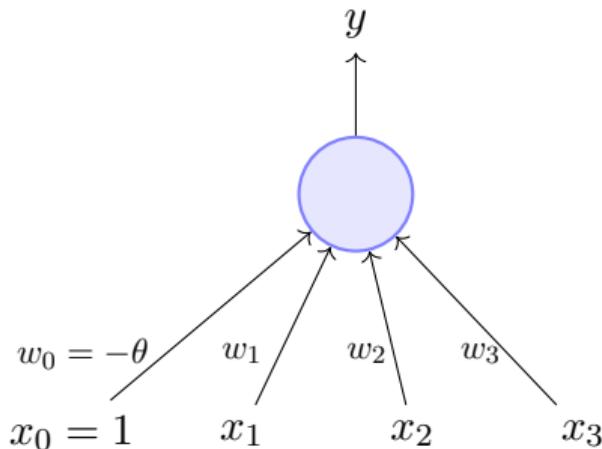
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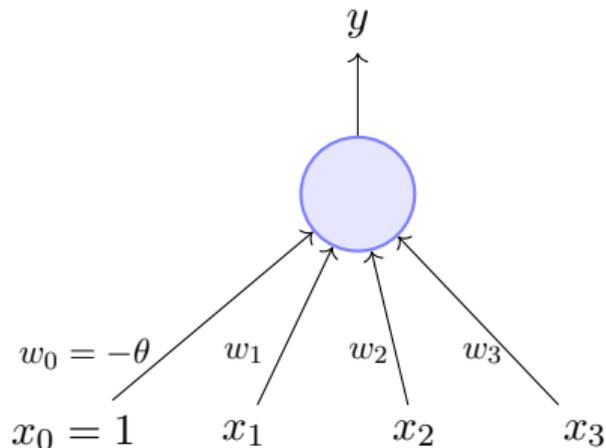
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We will now try to answer the following questions:

- Why do we need weights ?
- Why is $w_0 = -\theta$ called the bias ?



- Why are we trying to implement boolean functions?

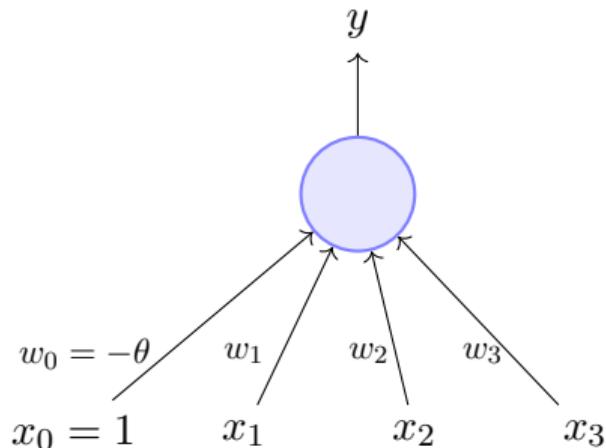


- Why are we trying to implement boolean functions?
- Consider the task of predicting whether we would like a movie or not

$x_1 = \text{isActorDamon}$

$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

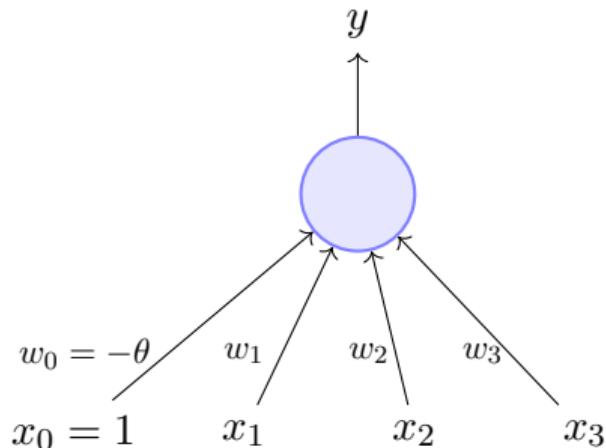


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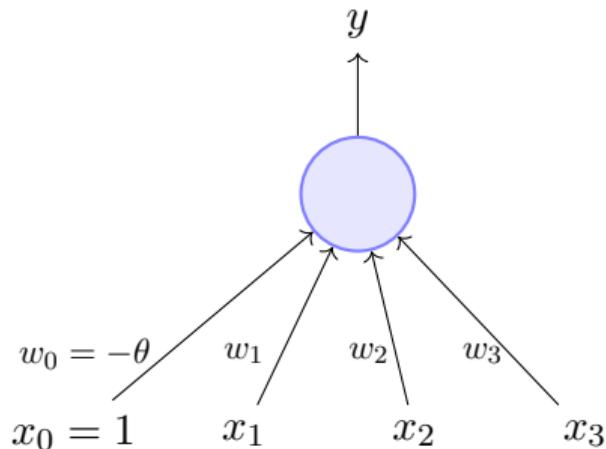


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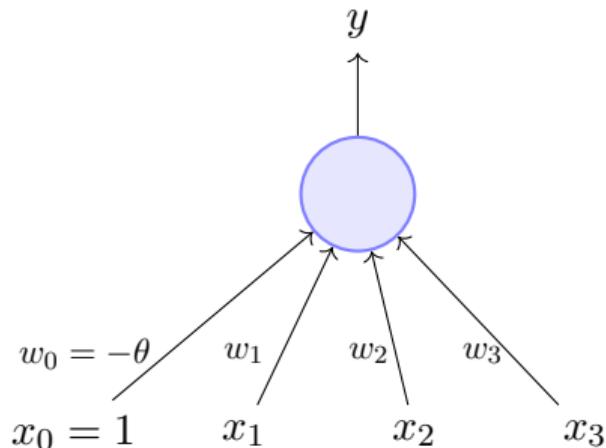


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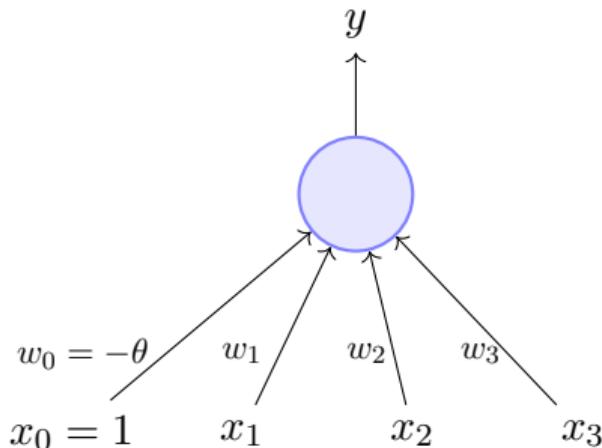
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- Given the following weight assignments $w_0 = -2$, $w_1 = 1$, $w_2 = 1$, $w_3 = 2$, which input configurations will result in an output of 1 (watch

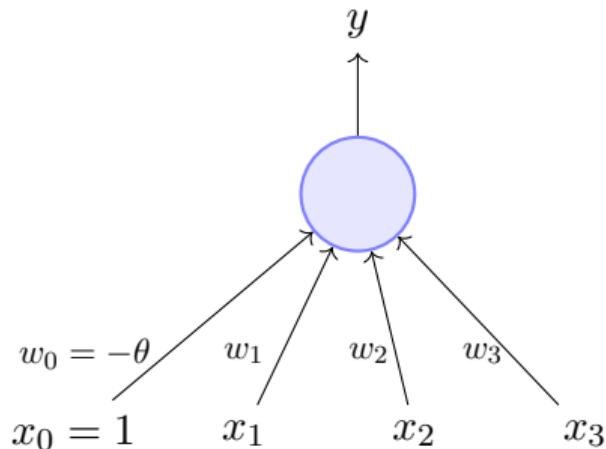


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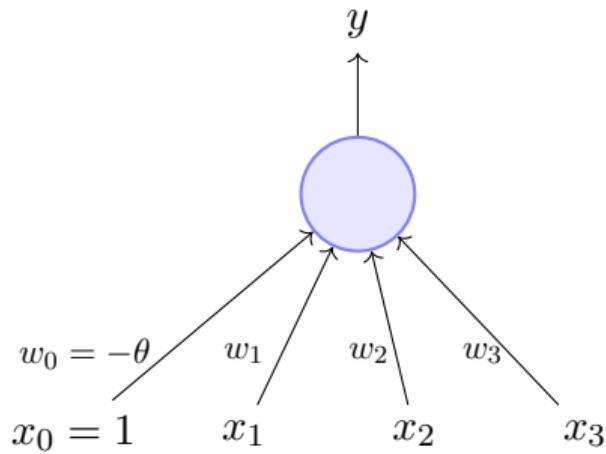


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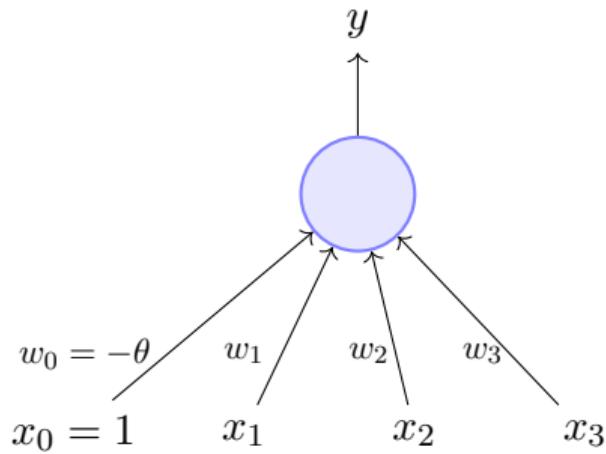


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- The weights (w_1, w_2, \dots, w_n) and the bias (w_0) will depend on the data (viewer history in this case)

What kind of functions can be implemented using the perceptron? Any difference from McCulloch Pitts neurons?

McCulloch Pitts Neuron

(assuming no inhibitory inputs)

$$y = 1 \quad if \sum_{i=0}^n x_i \geq \theta$$

$$= 0 \quad if \sum_{i=0}^n x_i < \theta$$

Perceptron

$$y = 1 \quad if \sum_{i=0}^n \textcolor{red}{w_i} * x_i \geq \theta$$

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- Then what is the difference ? The weights (including threshold) can be learned and the inputs can be real valued
- We will first revisit some boolean functions and then see the perceptron learning algorithm (for learning weights)

x_1	x_2	OR
0	0	

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0	0	0

x_1	x_2	OR	
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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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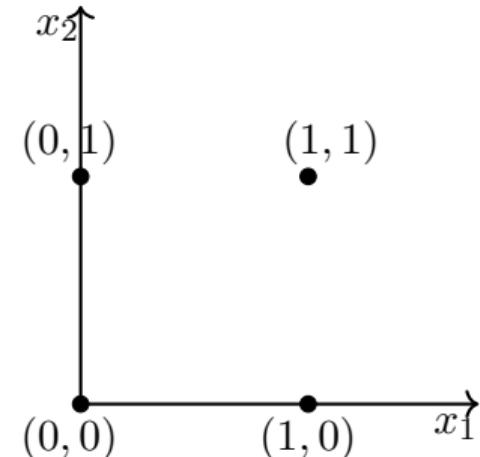
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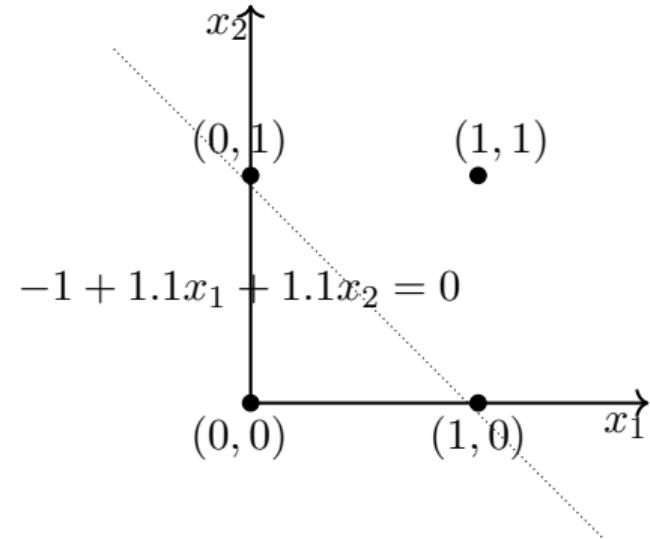
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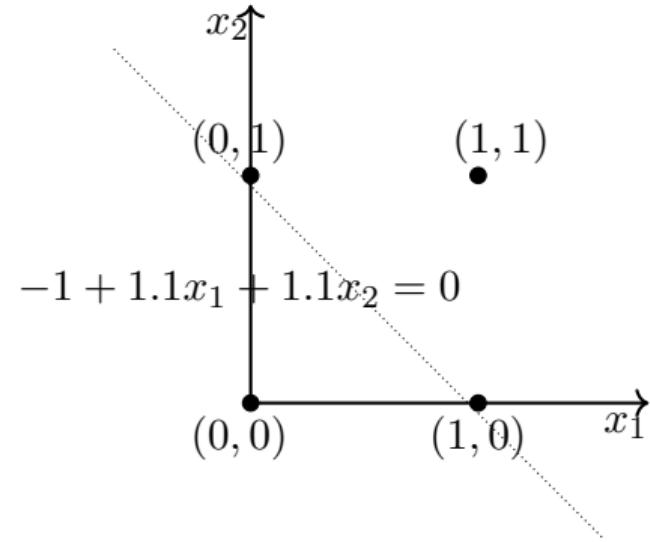
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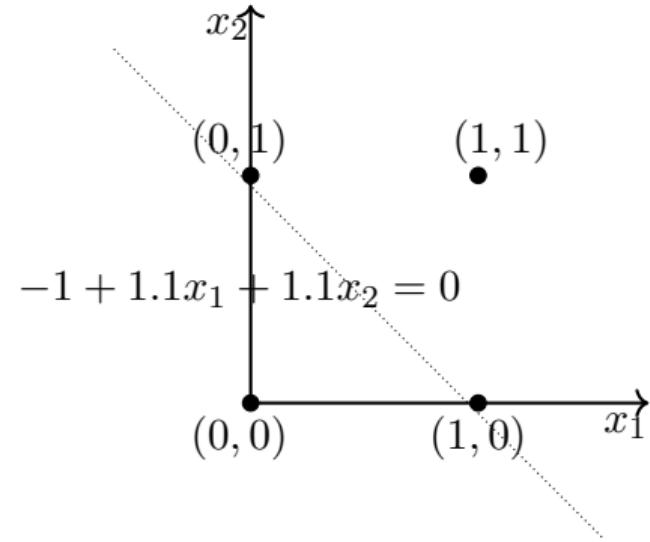
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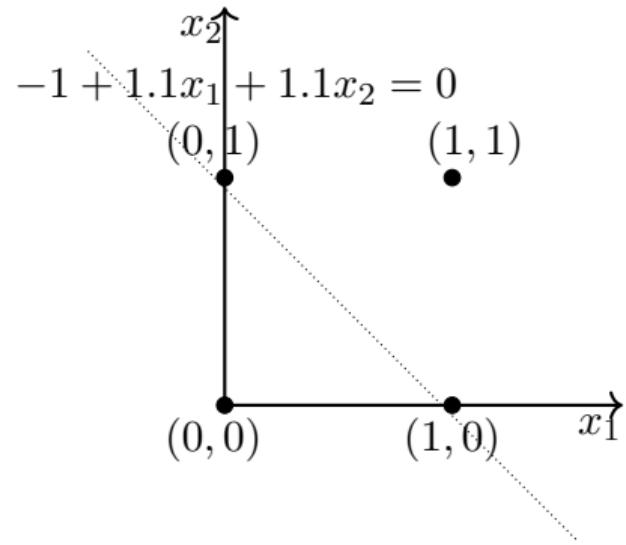
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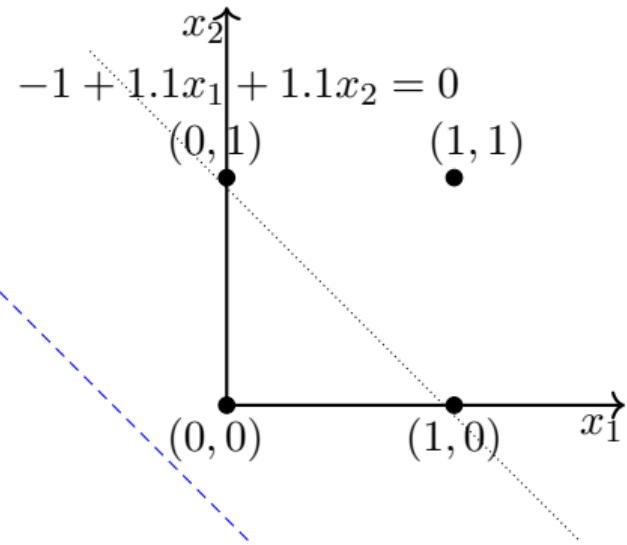
- Note that we can come up with a similar set of inequalities and find the value of θ for a McCulloch Pitts neuron also (Try it!)

Let us inspect this a bit more carefully ...

- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2

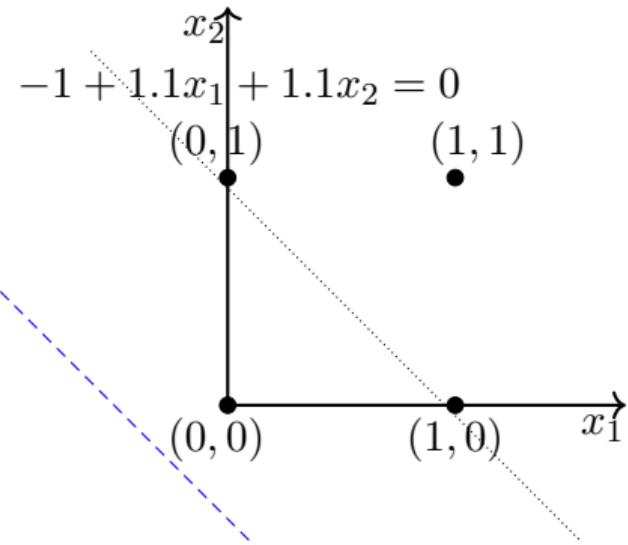


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- Say, $w_1 = -1, w_2 = -1$



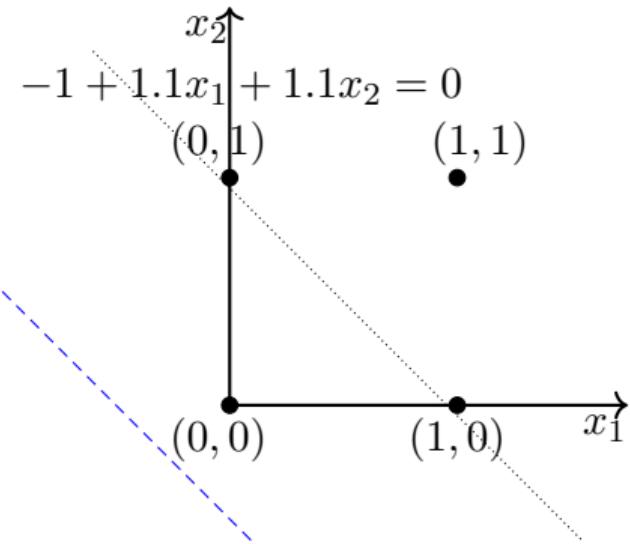
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- What is wrong with this line?



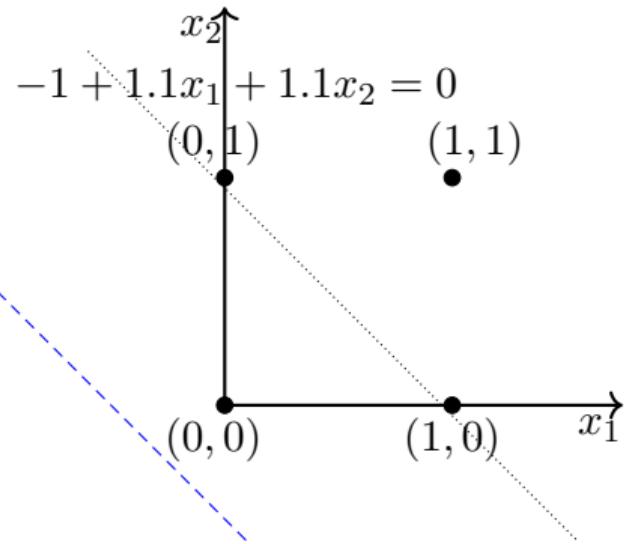
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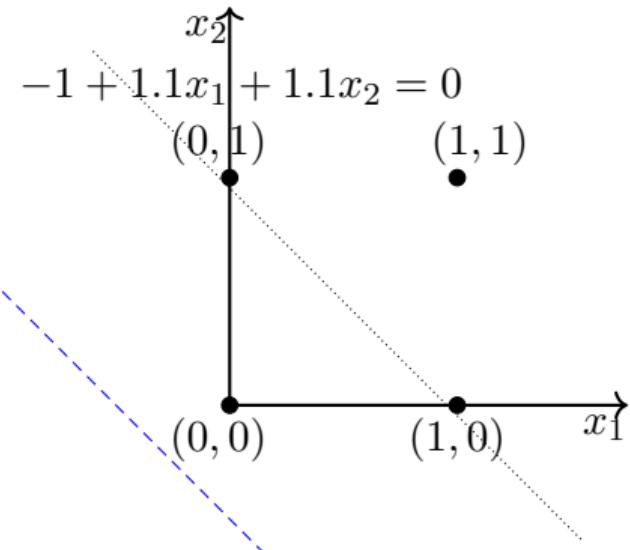
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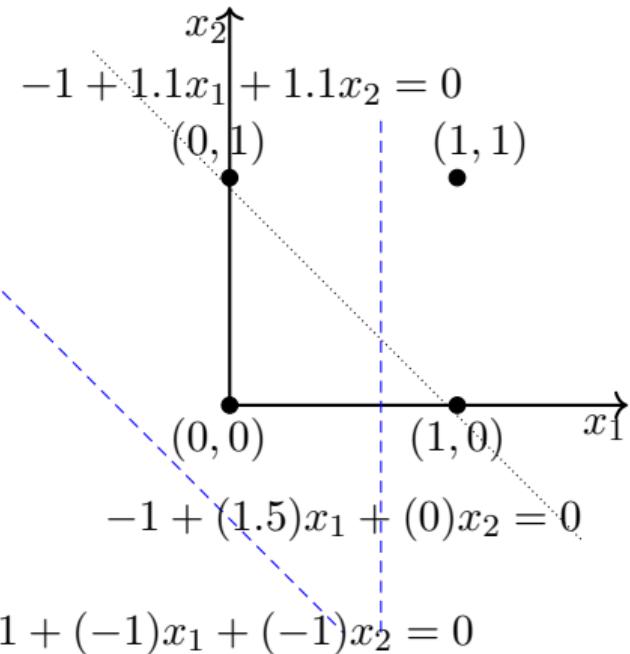
w_1	w_2	errors
-1	-1	1



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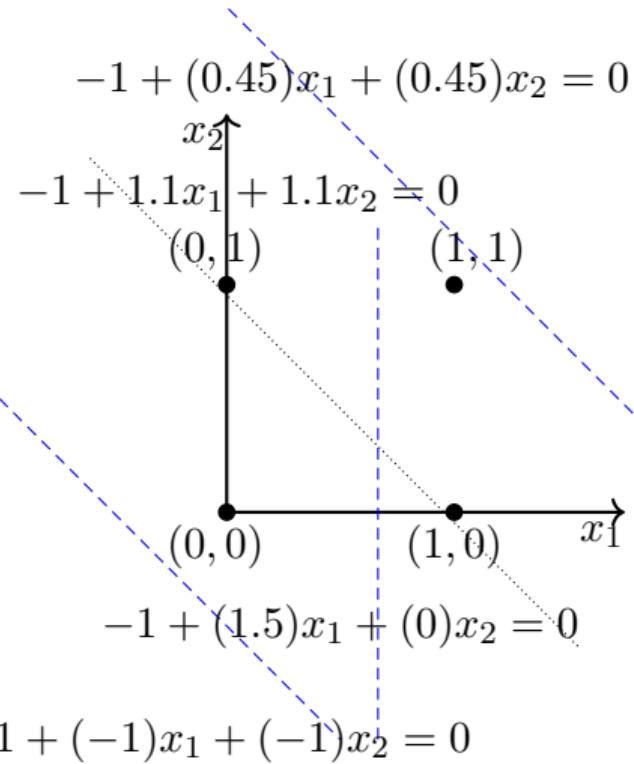
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1.5	0	1



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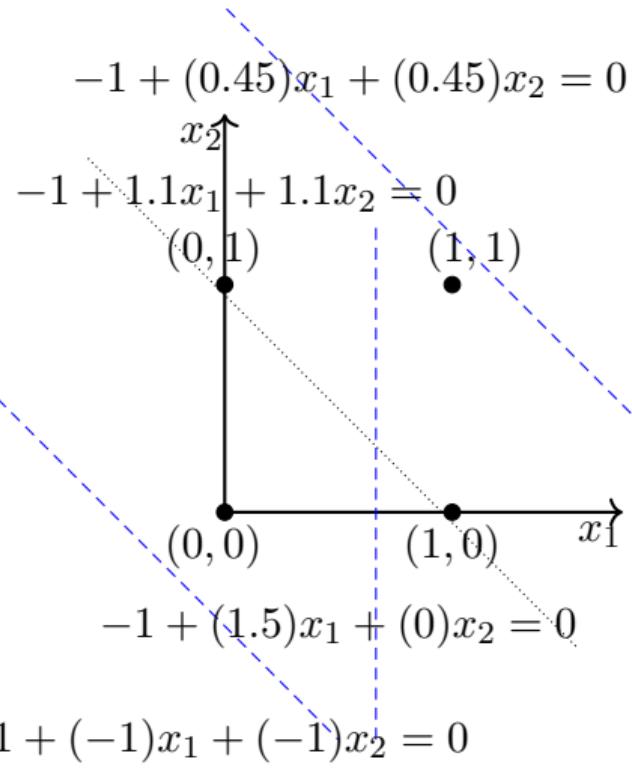
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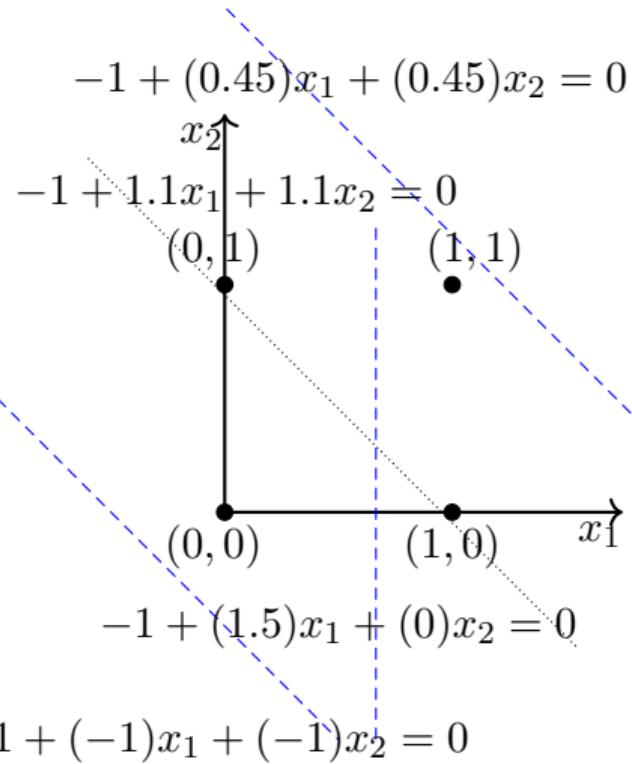
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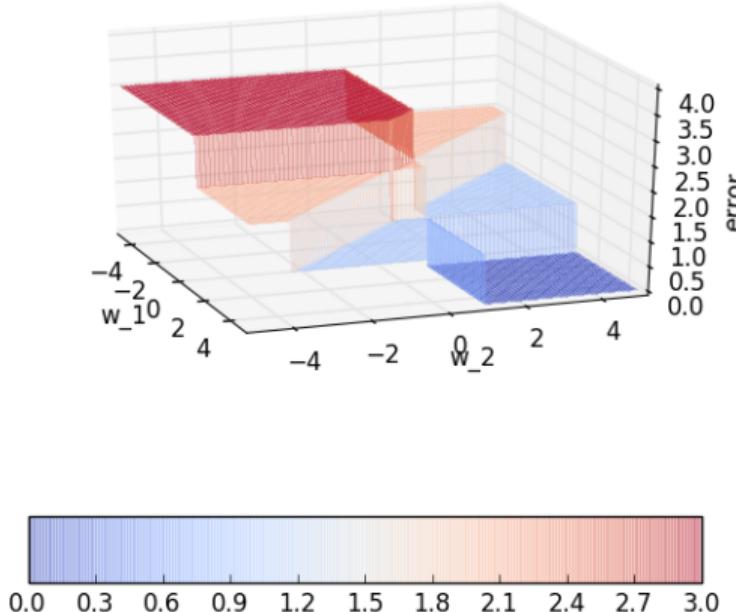
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- Let us plot the error surface corresponding to different values of w_0, w_1, w_2



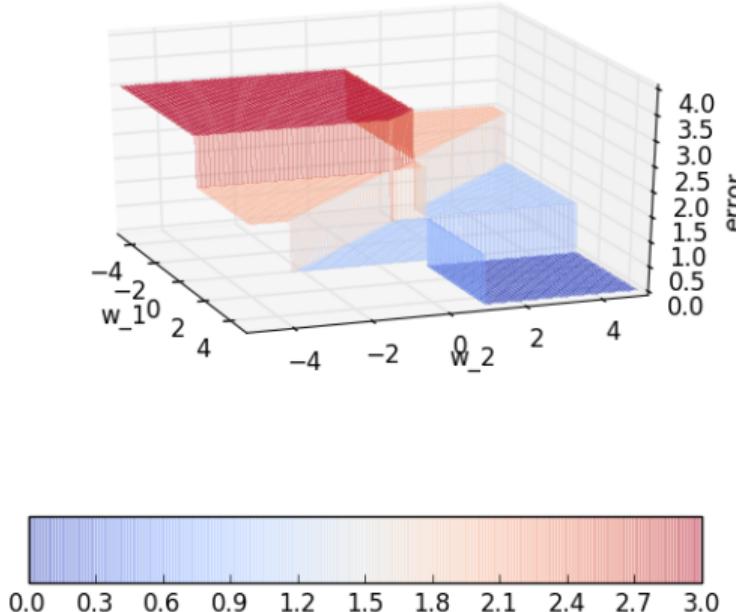
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- For the OR function, an error occurs if $(x_1, x_2) = (0, 0)$ but $-w_0 + w_1 * x_1 + w_2 * x_2 \geq 0$ or if $(x_1, x_2) \neq (0, 0)$ but $-w_0 + w_1 * x_1 + w_2 * x_2 < 0$



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- Apart from implementing boolean functions (which does not look very interesting) what can a perceptron be used for ?
- Our interest lies in the use of perceptron as a binary classifier. Let us see what this means...

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$x_1 = \text{isActorDamon}$

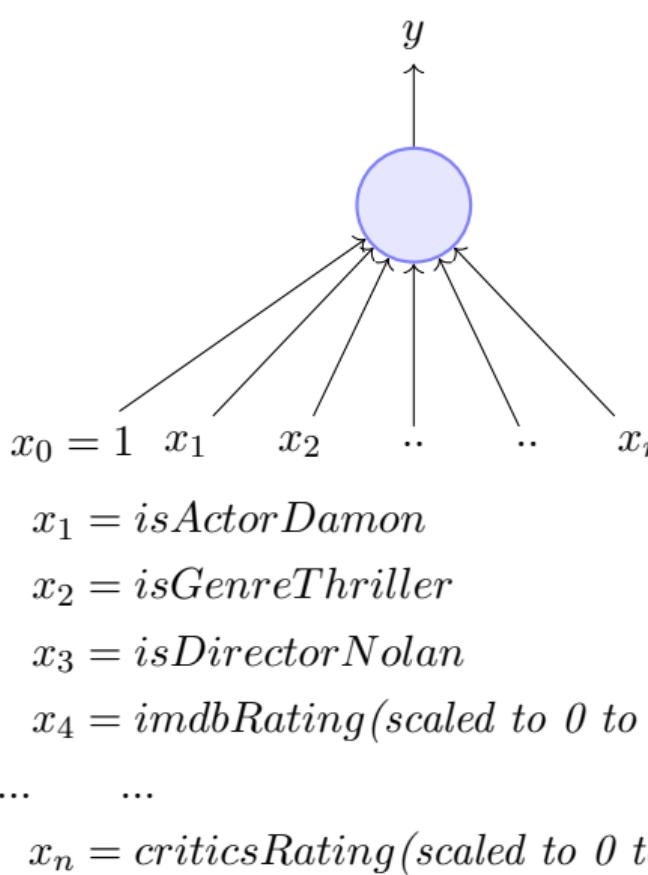
$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

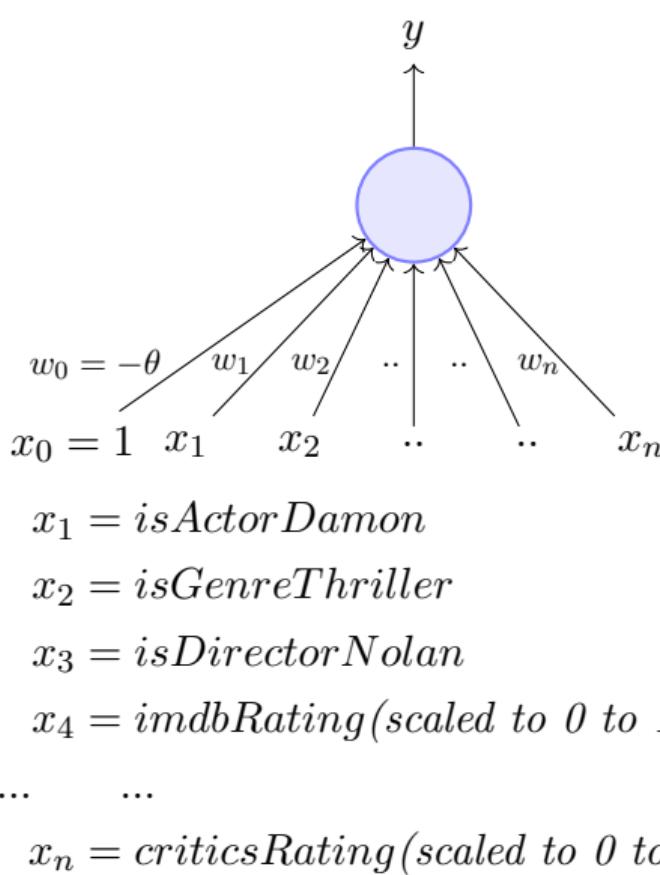
$x_4 = \text{imdbRating}(\text{scaled to 0 to 1})$

... ...

$x_n = \text{criticsRating}(\text{scaled to 0 to 1})$



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- In other words, we want the perceptron to find the equation of this separating plane (or find the values of $w_0, w_1, w_2, \dots, w_m$)

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$\mathbf{w} = \mathbf{w} + \mathbf{x}$;

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        | w = w + x ;  
    end  
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- Why would this work ?
- To understand why this works we will have to get into a bit of Linear Algebra and a bit of geometry...

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- Every point (\mathbf{x}) on this line satisfies the equation $\mathbf{w}^T \mathbf{x} = 0$

- We can thus rewrite the perceptron rule as

$$y = 1 \quad if \quad \mathbf{w}^T \mathbf{x} \geq 0$$

$$= 0 \quad if \quad \mathbf{w}^T \mathbf{x} < 0$$

- Consider two vectors \mathbf{w} and \mathbf{x}

$$\mathbf{w} = [w_0, w_1, w_2, \dots, w_n]$$

$$\mathbf{x} = [1, x_1, x_2, \dots, x_n]$$

$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^n w_i * x_i$$

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$$y = 1 \quad if \quad \mathbf{w}^T \mathbf{x} \geq 0$$

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- We are interested in finding the line $\mathbf{w}^T \mathbf{x} = 0$ which divides the input space into two halves
- Every point (\mathbf{x}) on this line satisfies the equation $\mathbf{w}^T \mathbf{x} = 0$
- What can you tell about the angle (α) between \mathbf{w} and any point (\mathbf{x}) which lies on this line ?

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- The angle is 90° ($\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$)

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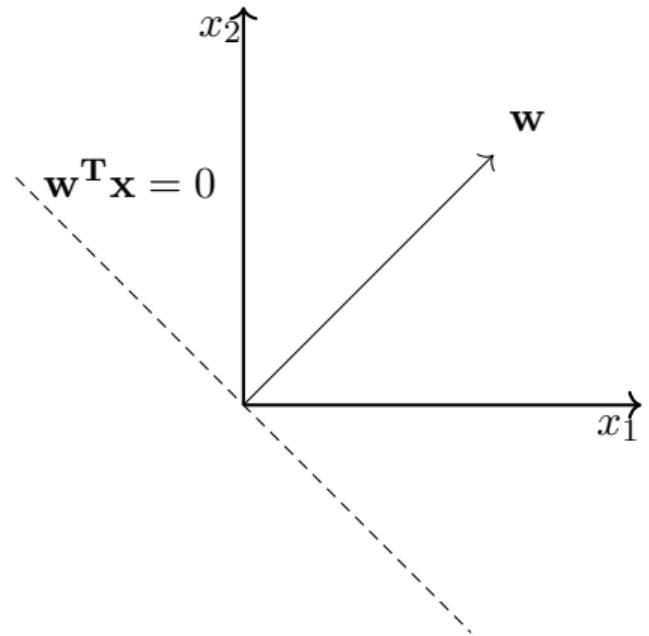
$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^n w_i * x_i$$

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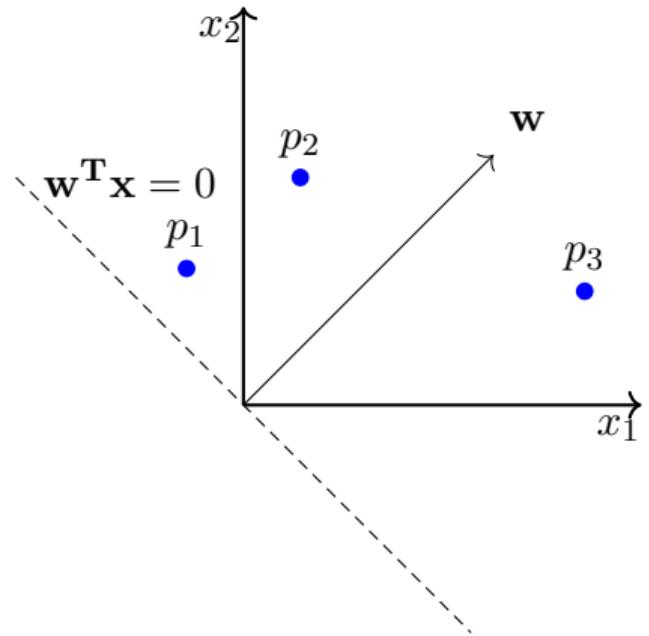
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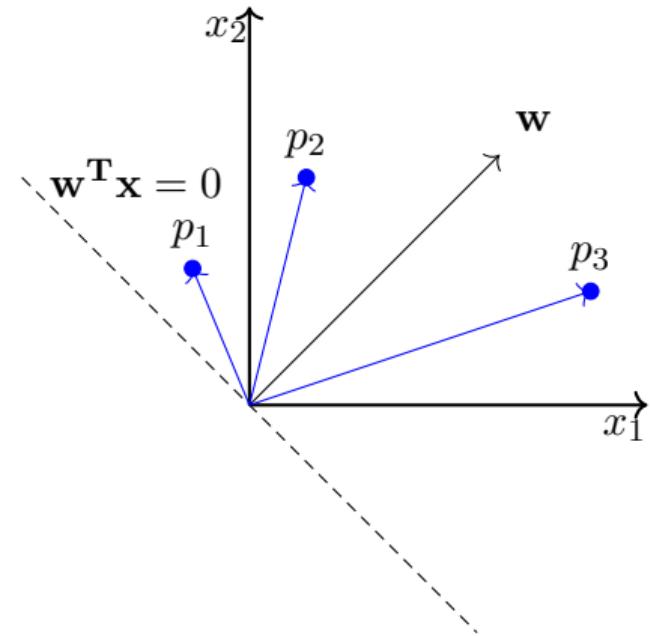
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- The angle is 90° ($\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$)
- Since the vector \mathbf{w} is perpendicular to every point on the line it is actually perpendicular to the line itself



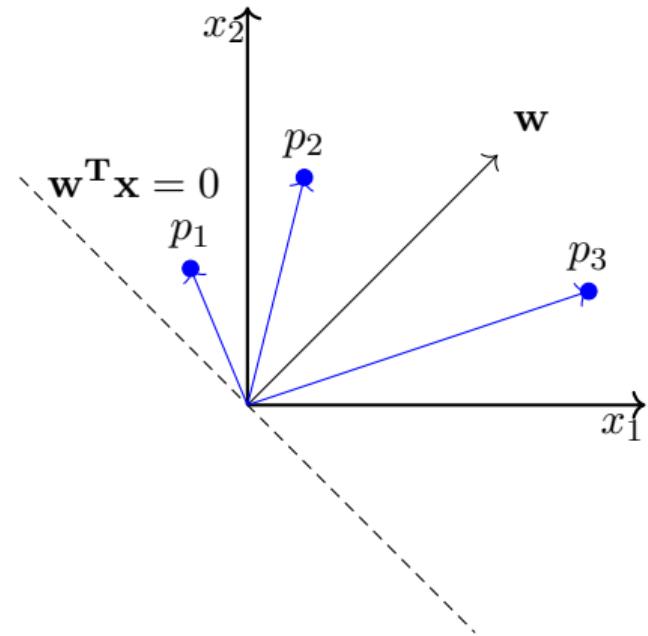
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)



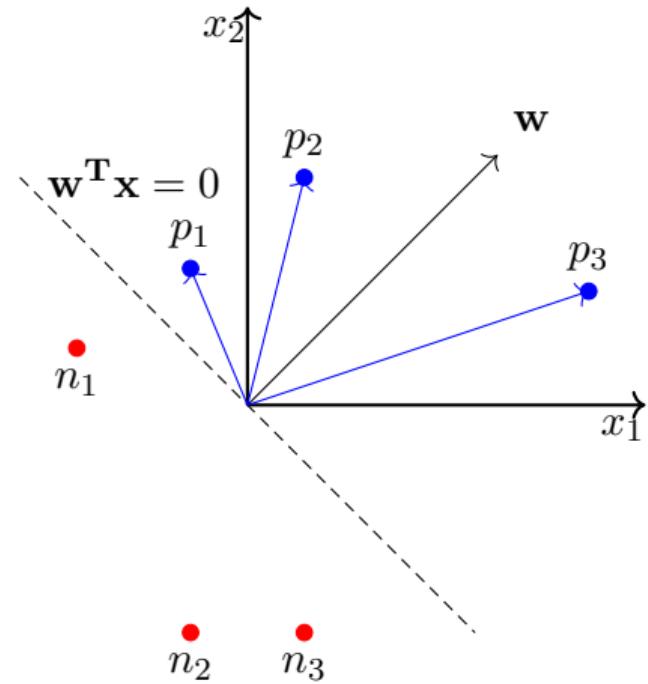
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
- What will be the angle between any such vector and \mathbf{w} ?



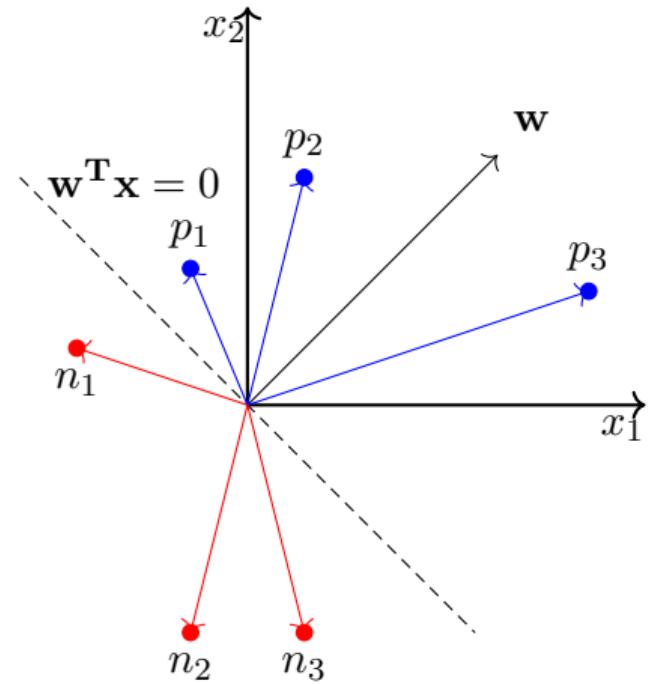
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
- What will be the angle between any such vector and \mathbf{w} ? Obviously, less than 90°



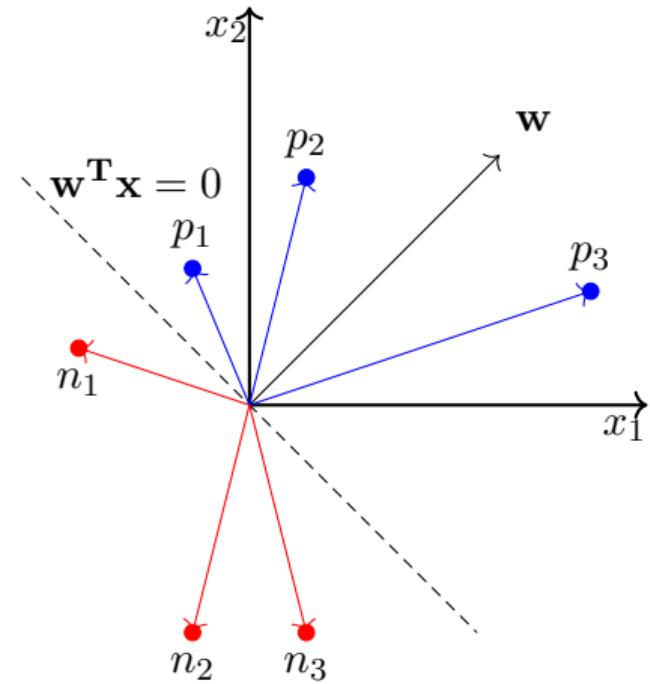
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
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- What about points (vectors) which lie in the negative half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} < 0$)



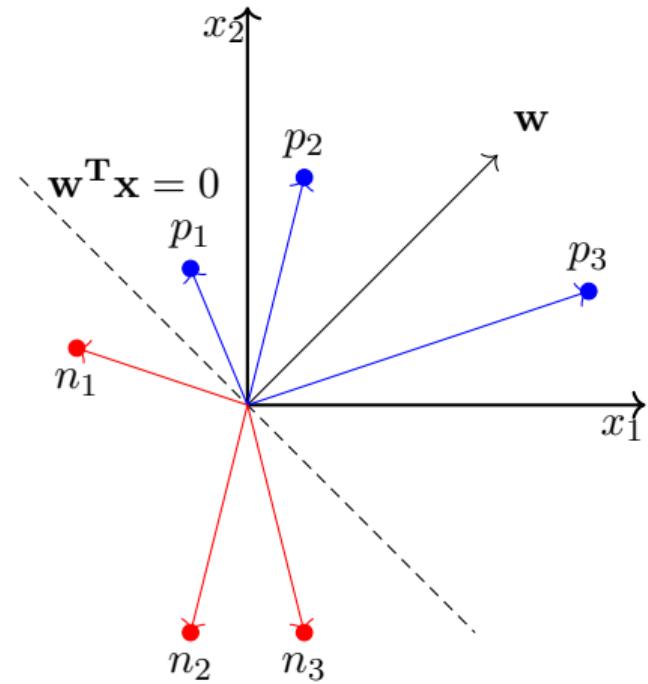
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- What will be the angle between any such vector and \mathbf{w} ? Obviously, greater than 90°
- Of course, this also follows from the formula ($\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$)



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- Of course, this also follows from the formula ($\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$)
- Keeping this picture in mind let us revisit the algorithm



Algorithm: Perceptron Learning Algorithm

```
P ← inputs with label 1;  
N ← inputs with label 0;  
Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
    if x ∈ P and w.x < 0 then  
        | w = w + x ;  
    end  
    if x ∈ N and w.x ≥ 0 then  
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    end  
end  
//the algorithm converges when all the  
inputs are classified correctly
```

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

Algorithm: Perceptron Learning Algorithm

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 $P \leftarrow$  inputs with label 1;  
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- For $\mathbf{x} \in P$ if $\mathbf{w} \cdot \mathbf{x} < 0$ then it means that the angle (α) between this \mathbf{x} and the current \mathbf{w} is greater than 90°

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} + \mathbf{x}$

$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$

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$$\begin{aligned}\cos(\alpha_{new}) &\propto \mathbf{w}_{new}^T \mathbf{x} \\ &\propto (\mathbf{w} + \mathbf{x})^T \mathbf{x}\end{aligned}$$

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$$\begin{aligned} \cos(\alpha_{new}) &\propto \mathbf{w}_{new}^T \mathbf{x} \\ &\propto (\mathbf{w} + \mathbf{x})^T \mathbf{x} \\ &\propto \mathbf{w}^T \mathbf{x} + \mathbf{x}^T \mathbf{x} \\ &\propto \cos\alpha + \mathbf{x}^T \mathbf{x} \end{aligned}$$

$$\cos(\alpha_{new}) > \cos\alpha$$

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$$\cos(\alpha_{new}) > \cos\alpha$$

- Thus α_{new} will be less than α and this is exactly what we want

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$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$

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- For $\mathbf{x} \in N$ if $\mathbf{w} \cdot \mathbf{x} \geq 0$ then it means that the angle (α) between this \mathbf{x} and the current \mathbf{w} is less than 90° (but we want α to be greater than 90°)
- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} - \mathbf{x}$

$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$
$$\propto (\mathbf{w} - \mathbf{x})^T \mathbf{x}$$

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

Algorithm: Perceptron Learning Algorithm

```
P ← inputs with label 1;  
N ← inputs with label 0;  
Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
    if x ∈ P and w.x < 0 then  
        | w = w + x ;  
    end  
    if x ∈ N and w.x ≥ 0 then  
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    end  
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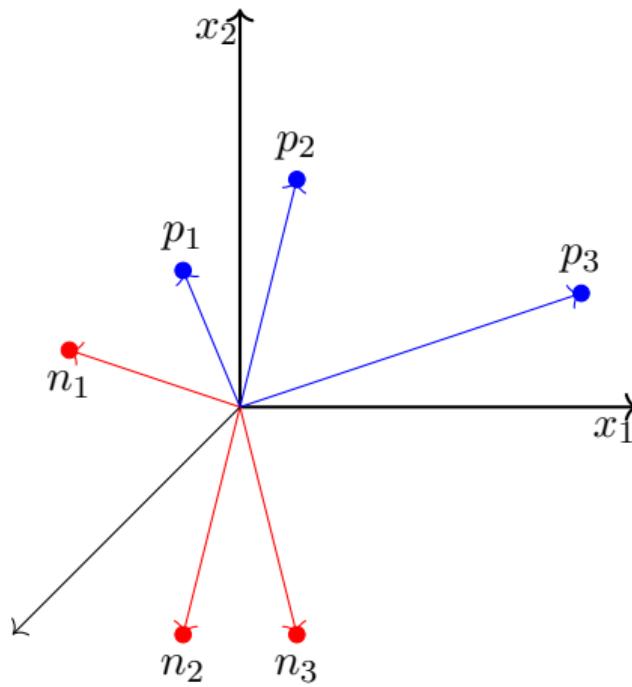
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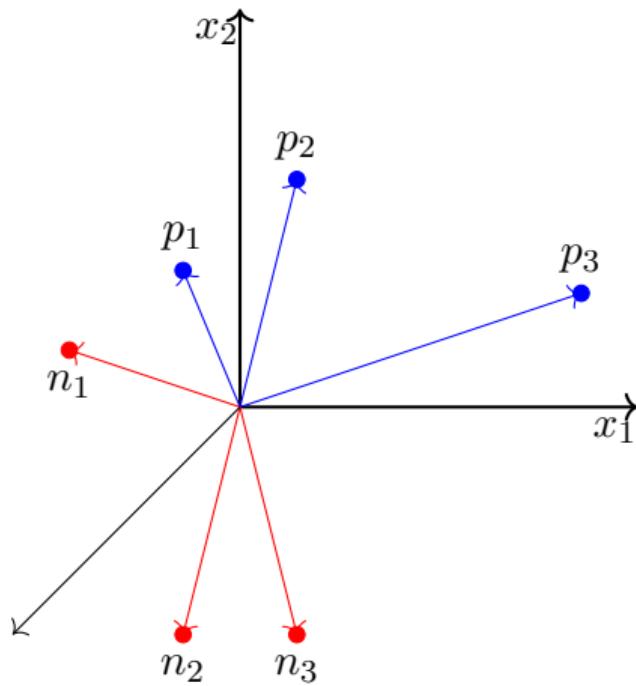
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- Thus α_{new} will be greater than α and this is exactly what we want

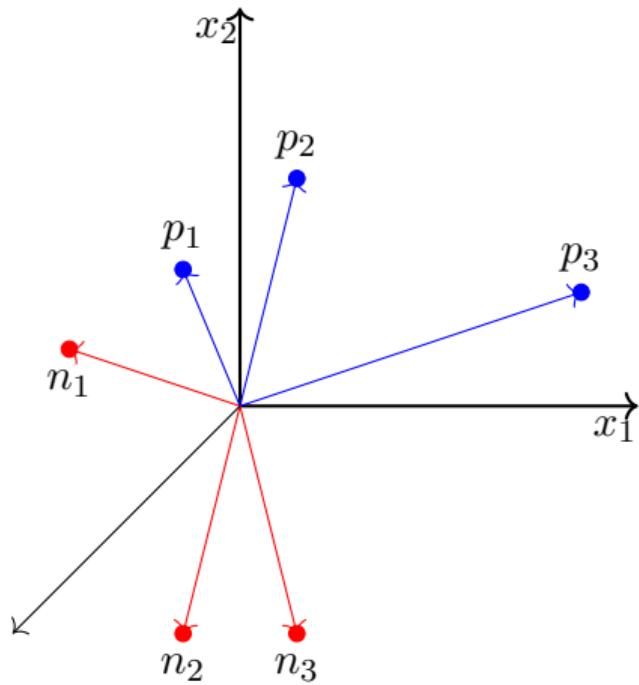
- We will now see this algorithm in action for a toy dataset



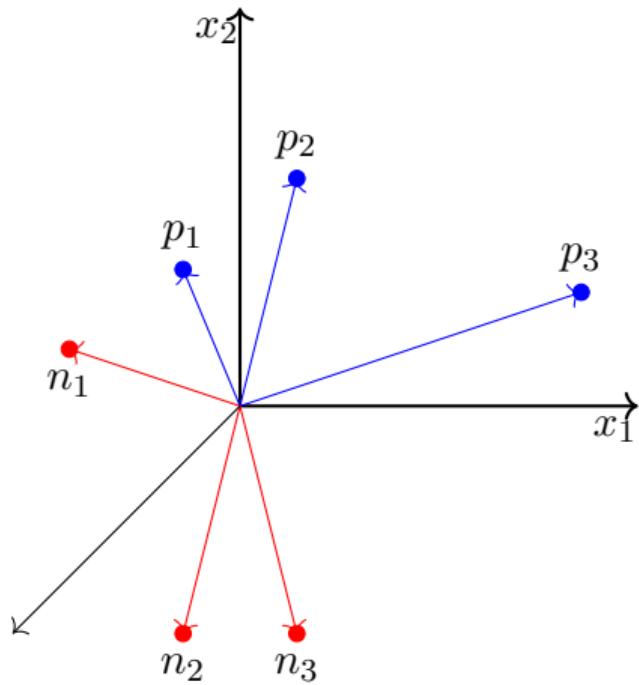
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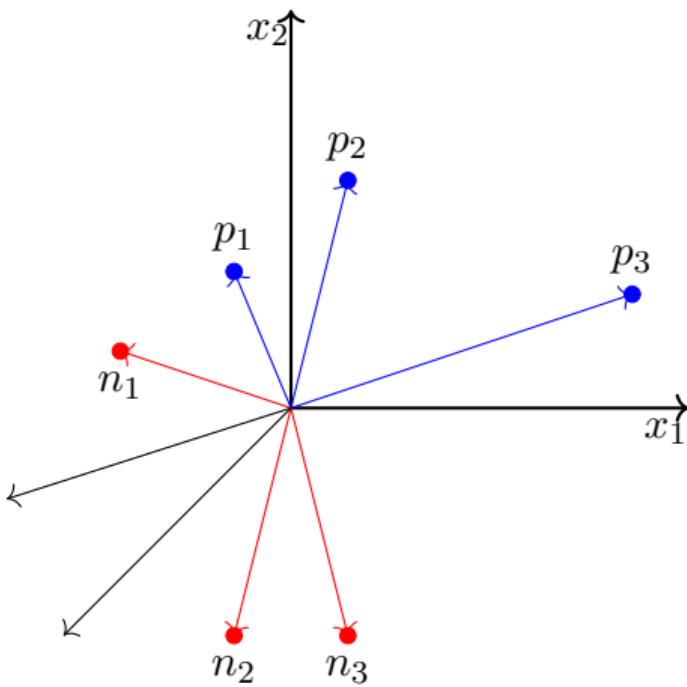
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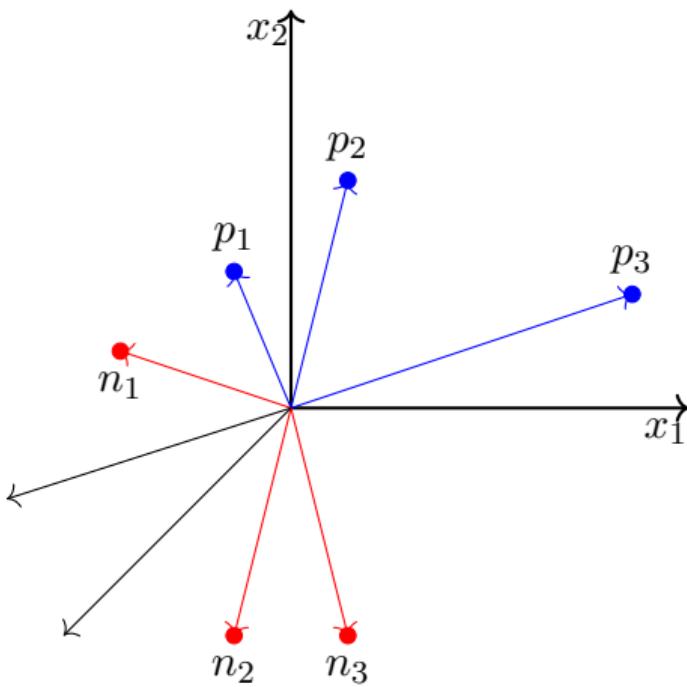
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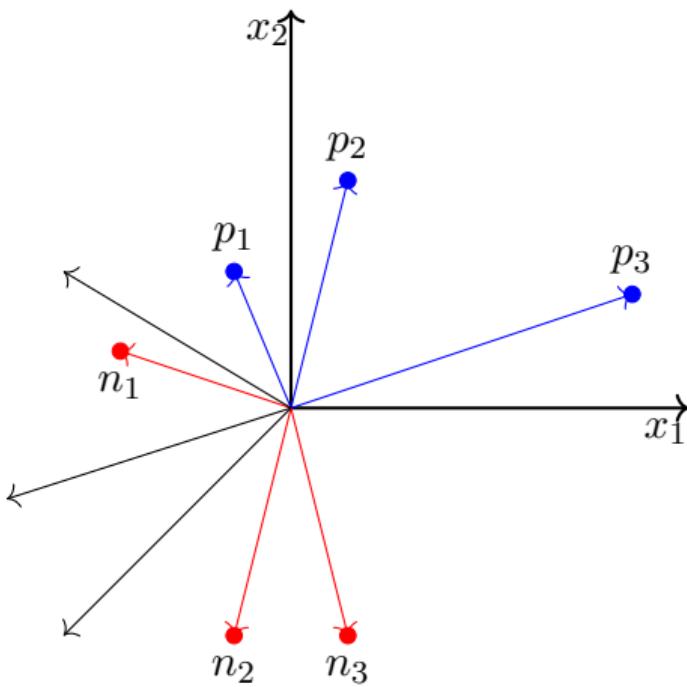
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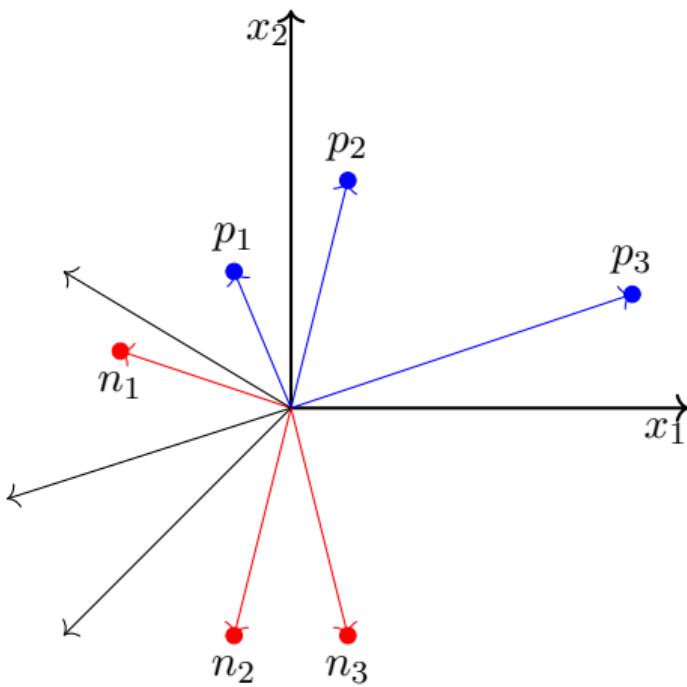
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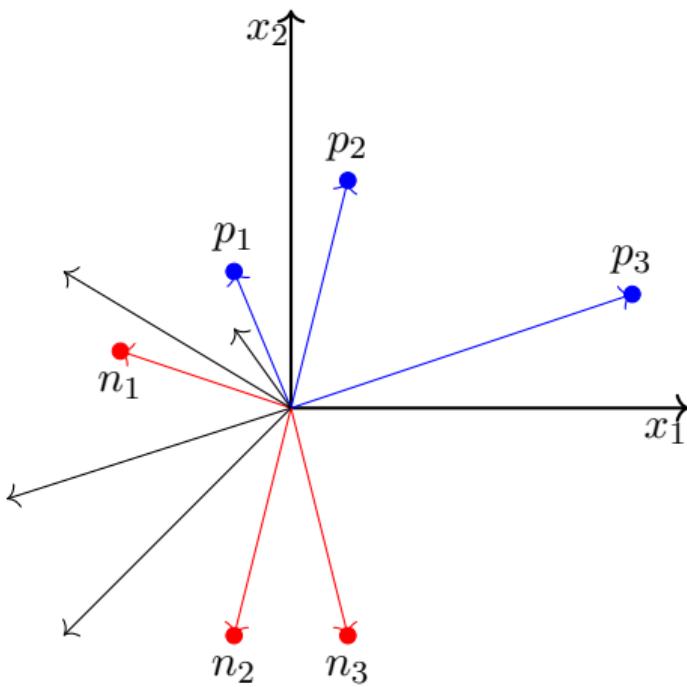
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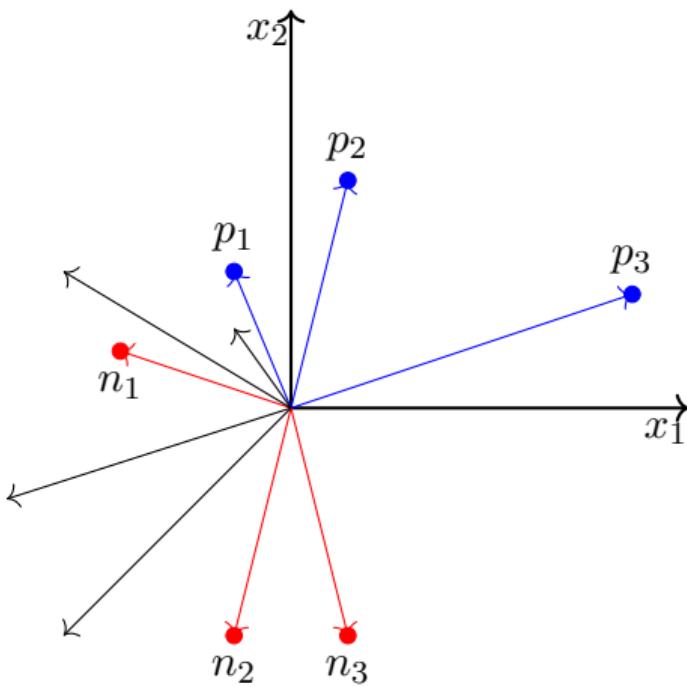
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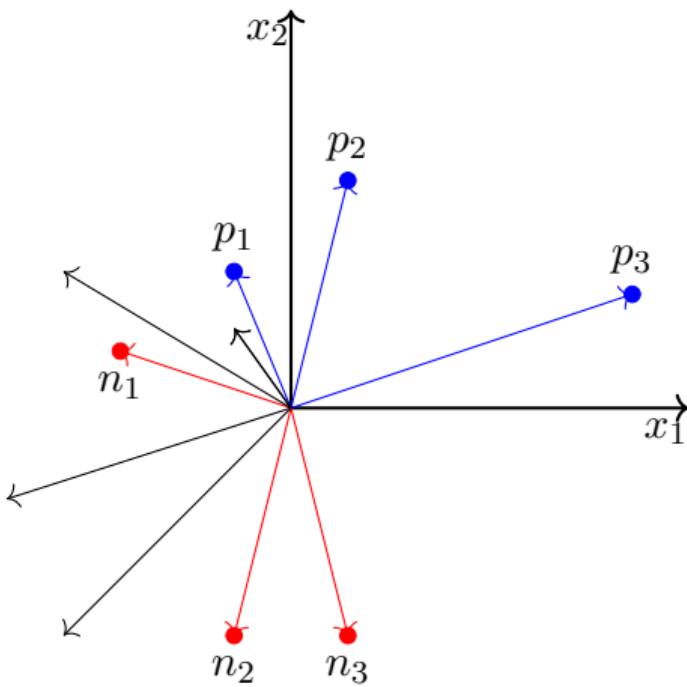
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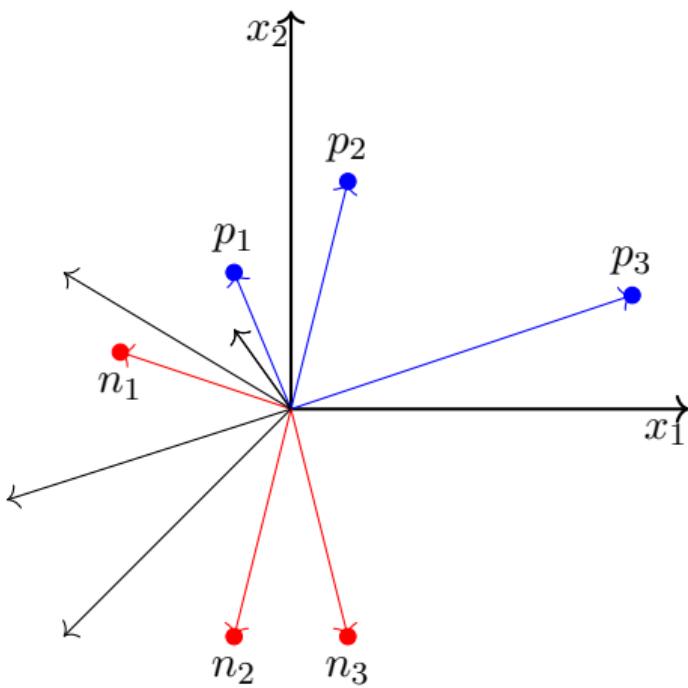
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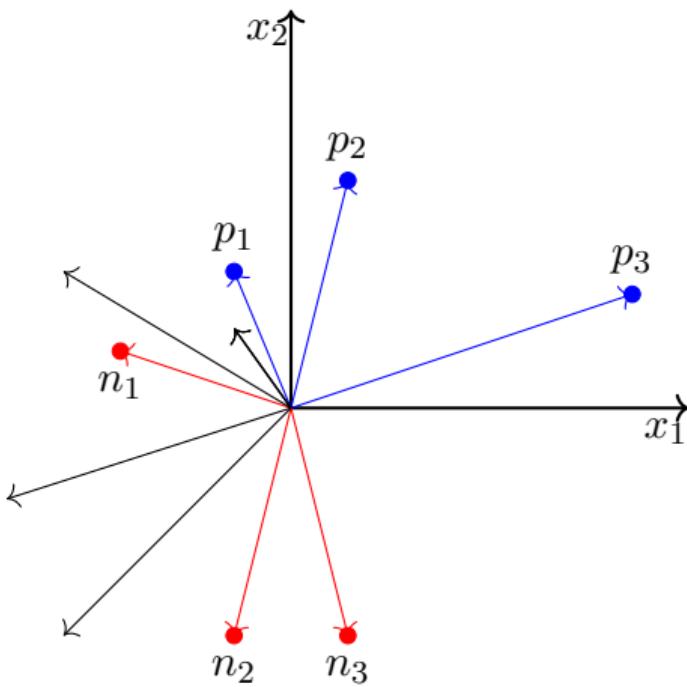
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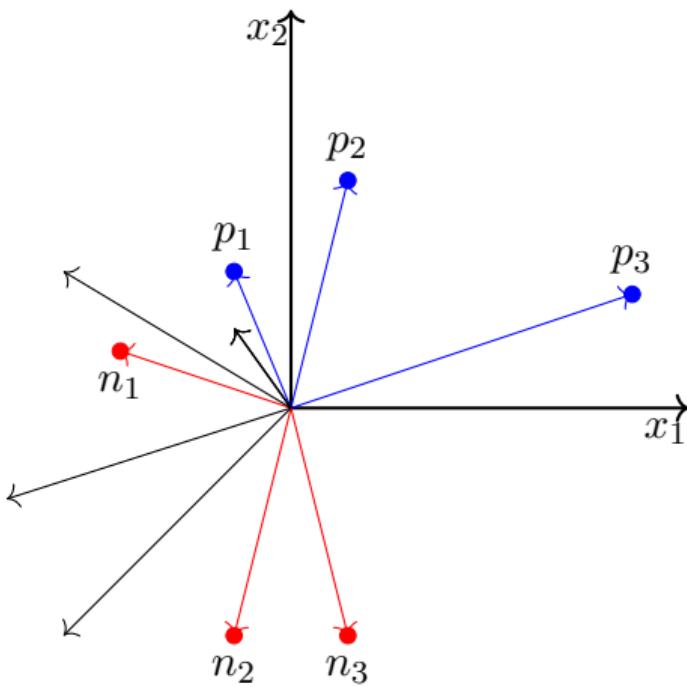
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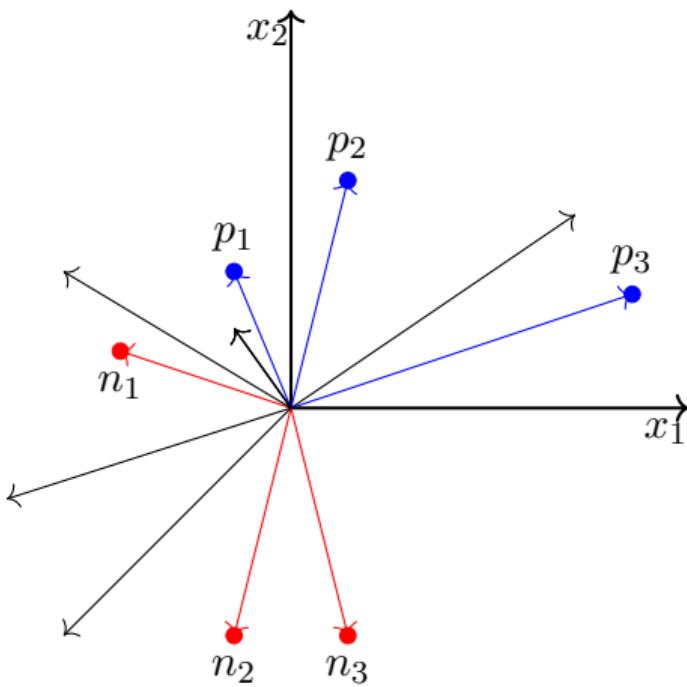
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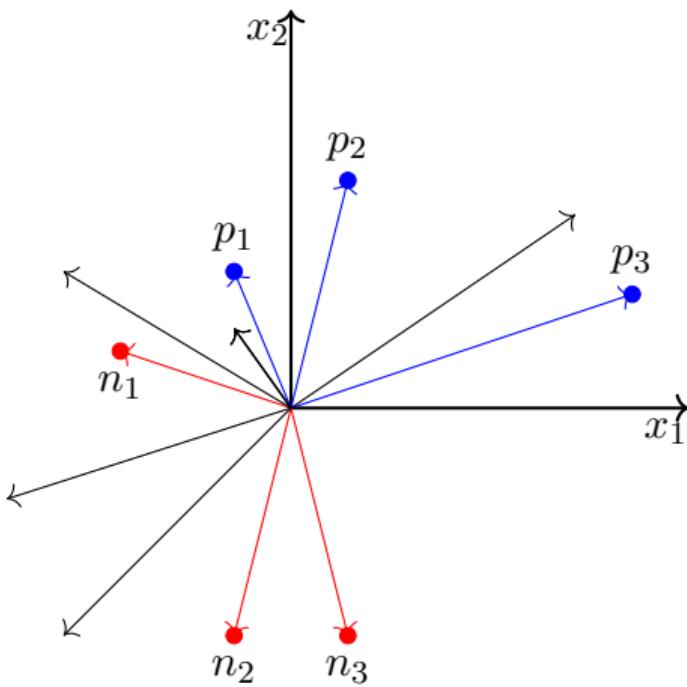
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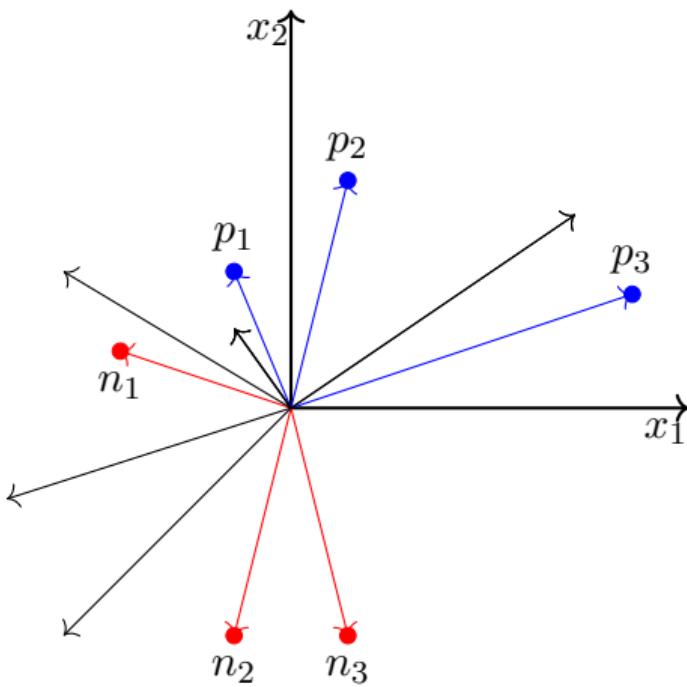
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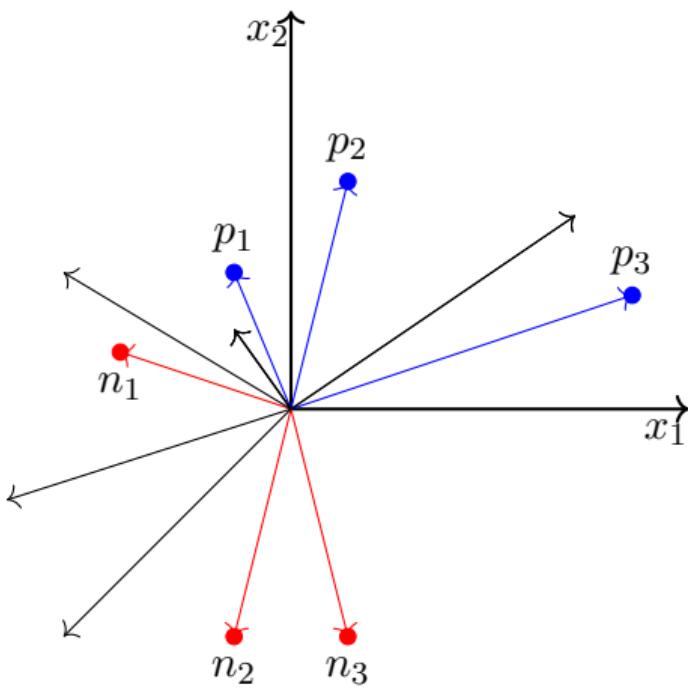
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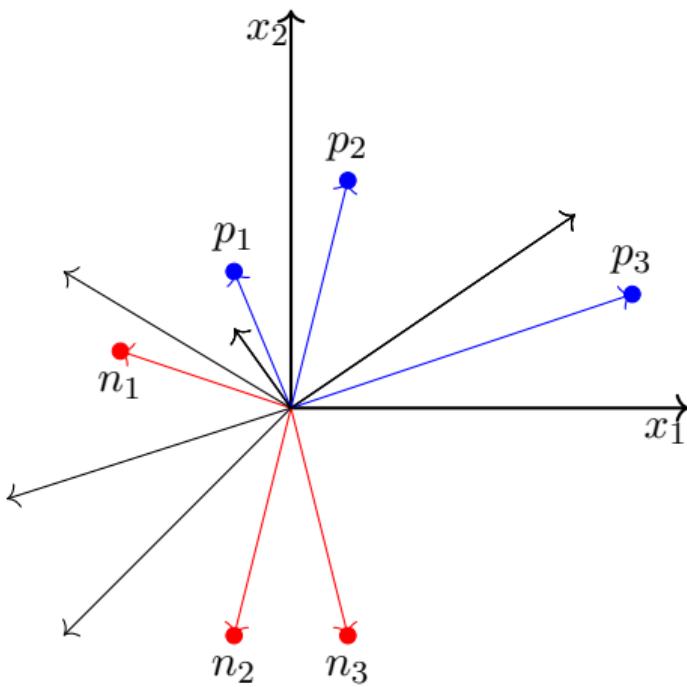
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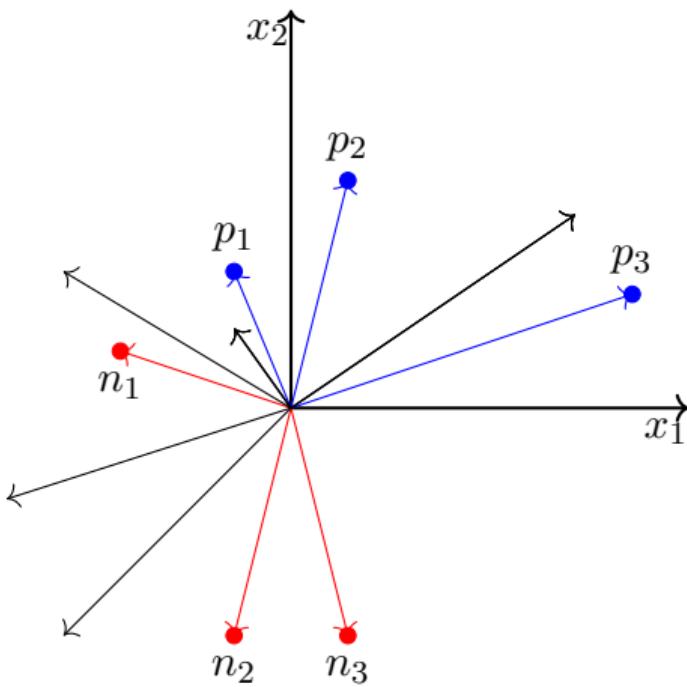
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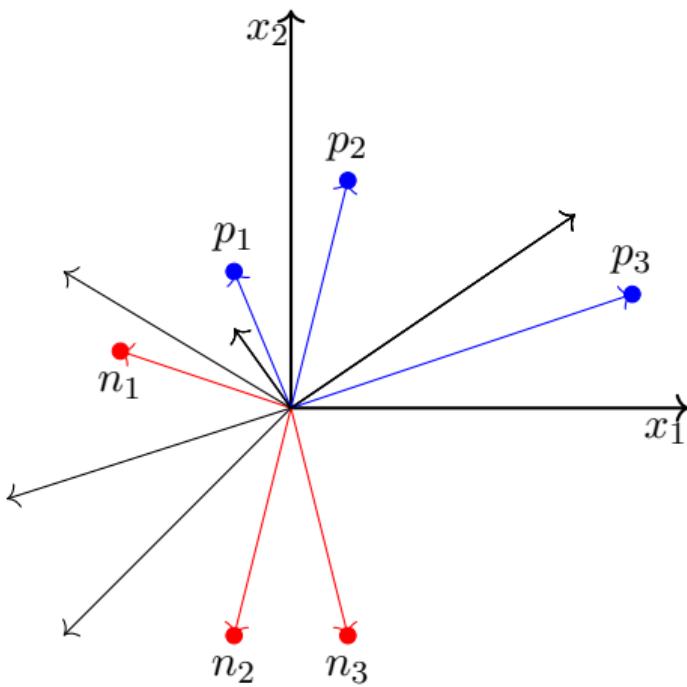
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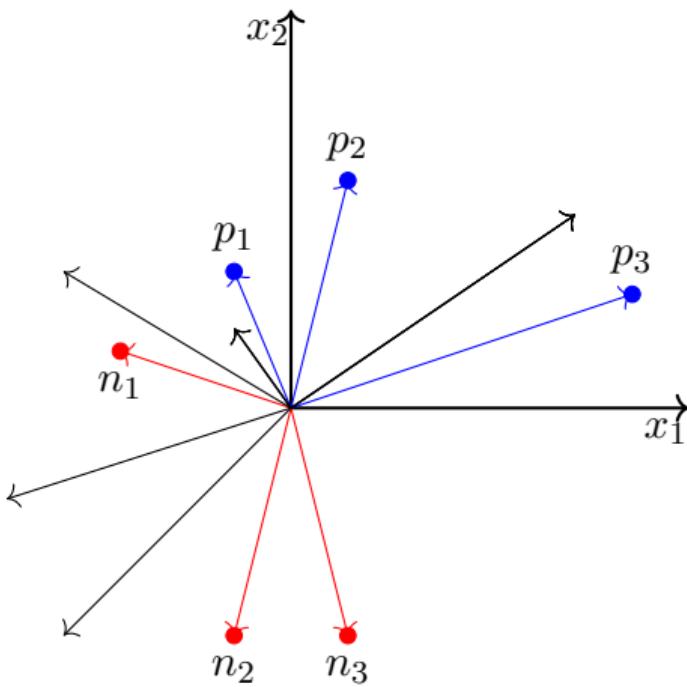
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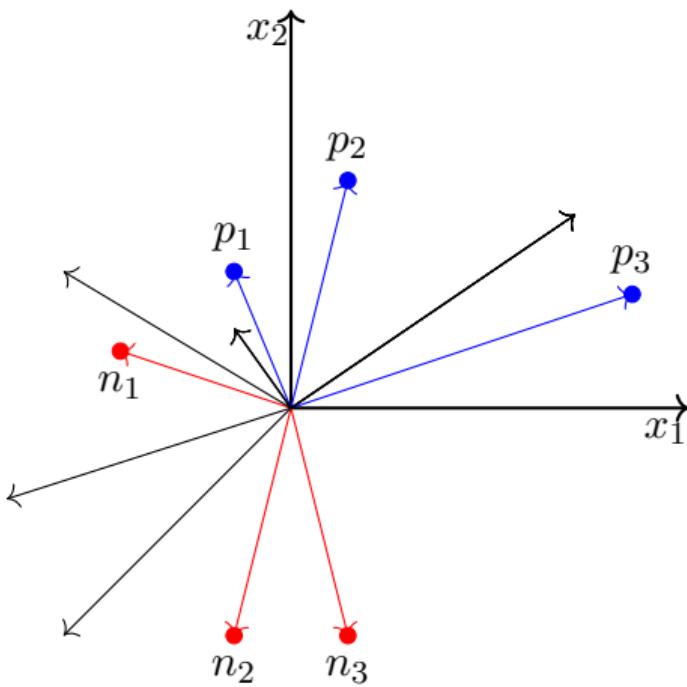
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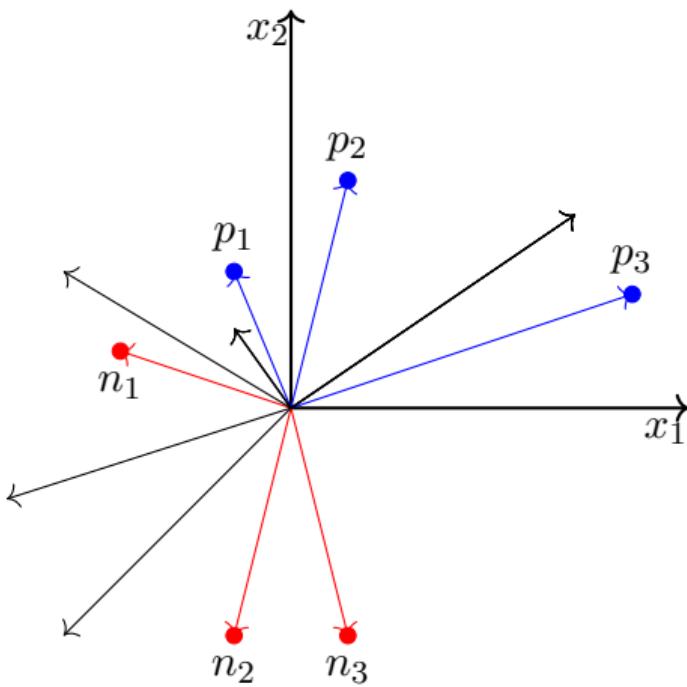
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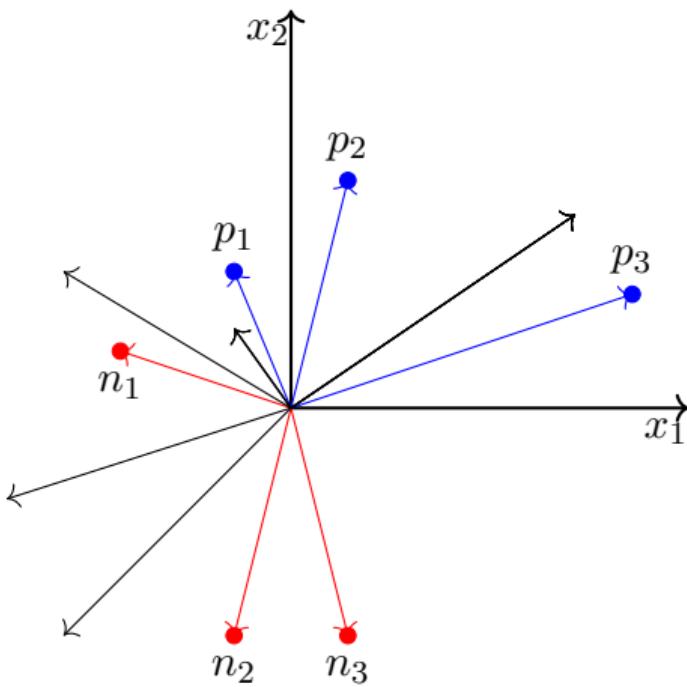
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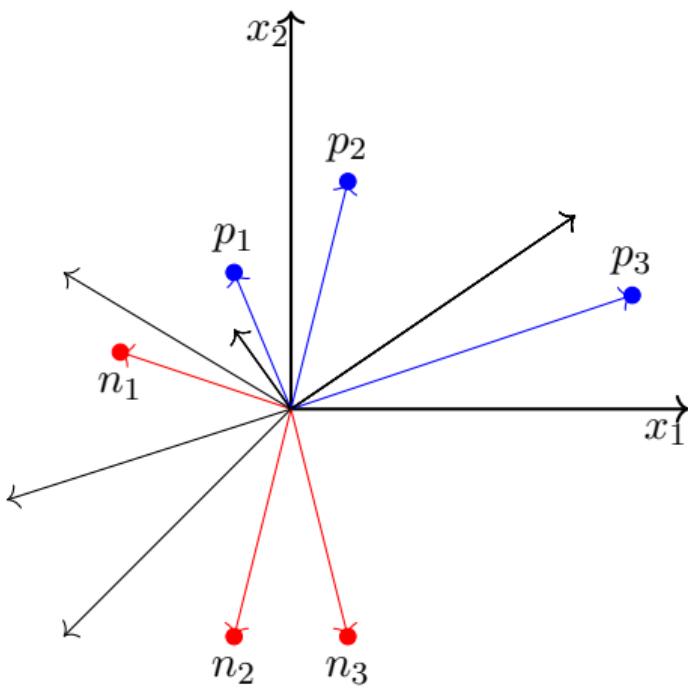
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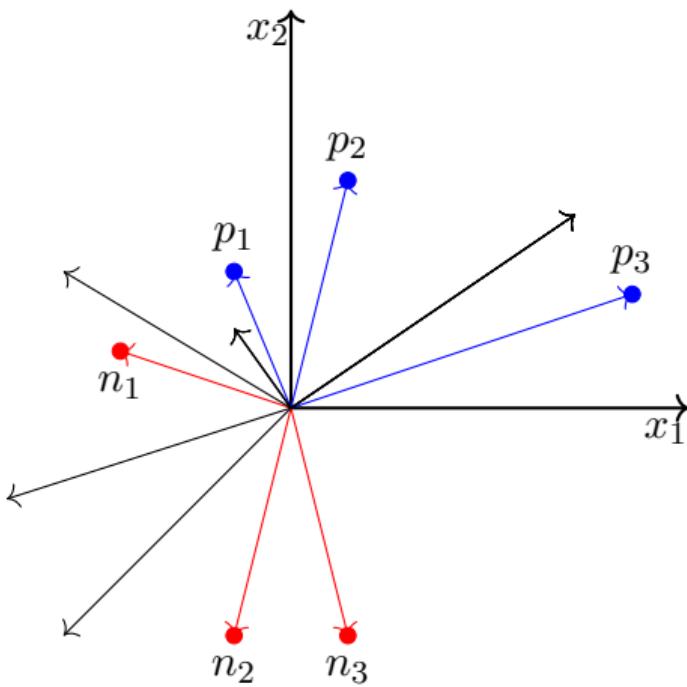
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- We now run the algorithm by randomly going over the points
- The algorithm has converged

- Now that we have some faith and intuition about why the algorithm works, we will see a more formal proof of convergence ...

Theorem

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Proof: On the next slide

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while !convergence do
    Pick random p  $\in P'$  ;
    if  $w^T p < 0$  then
        if  $p \in N$  then
            return w;
        else  $\leftarrow$  w +  $p$ ;
    end if
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- Let w^* be the normalized solution vector (we know one exists as the data is linearly separable)

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$$\begin{aligned} \text{Numerator} &= w^* \cdot w_{t+1} = w^* \cdot (w_t + p_i) \\ &= w^* \cdot w_t + w^* \cdot p_i \\ &\geq w^* \cdot w_t + \delta \quad (\delta = \min\{w^* \cdot p_i | \forall i\}) \\ &\geq w^* \cdot (w_{t-1} + p_j) + \delta \\ &\geq w^* \cdot w_{t-1} + w^* \cdot p_j + \delta \\ &\geq w^* \cdot w_{t-1} + 2\delta \\ &\geq w^* \cdot w_0 + (k)\delta \quad (\text{By induction}) \end{aligned}$$

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So far we have, $w^T \cdot p_i \leq 0$ (and hence we made the correction)

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$$\cos\beta \geq \frac{w^* \cdot w_0 + k\delta}{\sqrt{\|w_0\|^2 + k}}$$

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- $\cos\beta$ thus grows proportional to \sqrt{k}
- As k (number of corrections) increases $\cos\beta$ can become arbitrarily large
- But since $\cos\beta \leq 1$, k must be bounded by a maximum number
- Thus, there can only be a finite number of corrections (k) to w and the algorithm will converge!

Coming back to our questions ...

- What about non-boolean (say, real) inputs ?
- Do we always need to hand code the threshold ?
- Are all inputs equal ? What if we want to assign more weight (importance) to some inputs ?
- What about functions which are not linearly separable ?

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- Are all inputs equal ? What if we want to assign more weight (importance) to some inputs ? **A perceptron allows weights to be assigned to inputs**
- What about functions which are not linearly separable ? **Not possible with a single perceptron but we will see how to handle this ..**

- So what do we do about functions which are not linearly separable ?

- So what do we do about functions which are not linearly separable ?
- Let us see one such simple boolean function first ?

x_1	x_2	XOR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
1	1	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

$$w_0 + w_1 \cdot 0 + w_2 \cdot 1 \geq 0 \implies w_2 > -w_0$$

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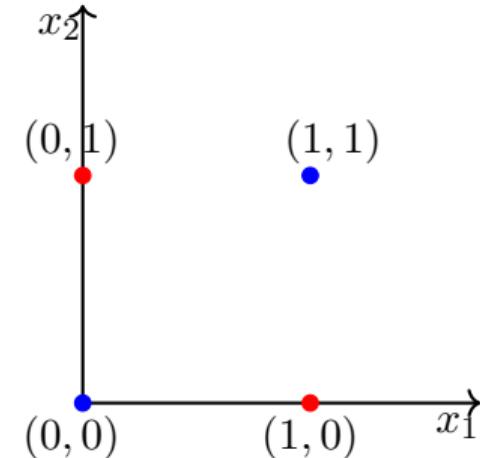
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- The fourth condition contradicts conditions 2 and 3

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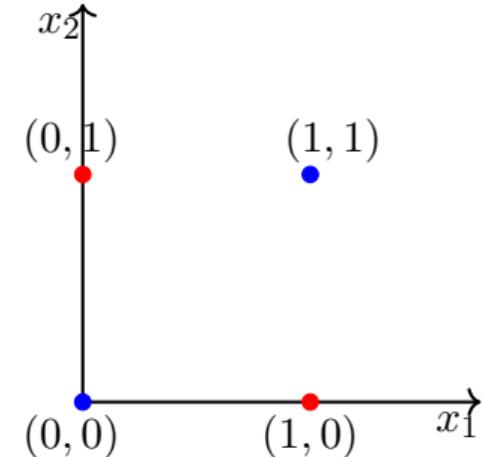
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- Hence we cannot have a solution to this set of inequalities

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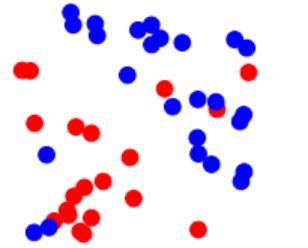
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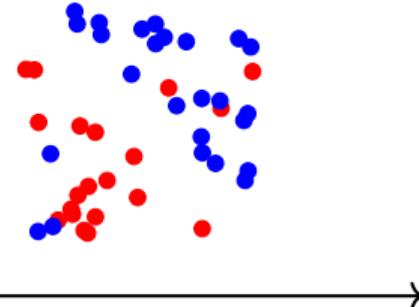
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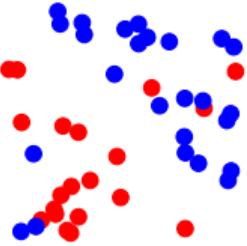
- And indeed you can see that it is impossible to draw a line which separates the red points from the blue points



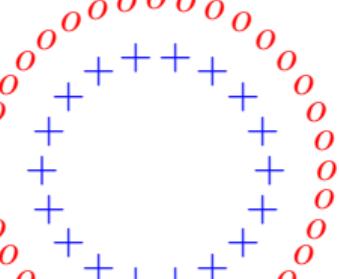
- Most real world data is not linearly separable and will always contain some outliers

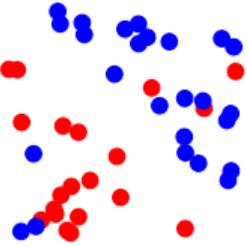


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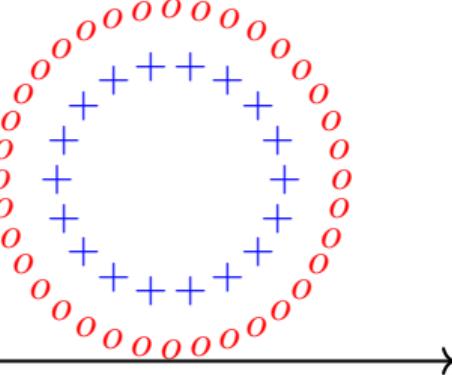


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- Most real world data is not linearly separable and will always contain some outliers
- In fact, sometimes there may not be any outliers but still the data may not be linearly separable
- We need computational units (models) which can deal with such data
- While a single perceptron cannot deal with such data, we will show that a network of perceptrons can indeed deal with such data



- Before seeing how a network of perceptrons can deal with linearly inseparable data, we will discuss boolean functions in some more detail ...

- How many boolean functions can you design from 2 inputs ?

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- Let us begin with some easy ones which you already know ..

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x_1	x_2
0	0
0	1
1	0
1	1

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x_1	x_2	f_1
0	0	0
0	1	0
1	0	0
1	1	0

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x_1	x_2	f_1
0	0	0
0	1	0
1	0	0
1	1	0

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0	0	0	0
0	1	0	0
1	0	0	0
1	1	0	1

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x_1	x_2	f_1	f_2	f_8
0	0	0	0	0
0	1	0	0	1
1	0	0	0	1
1	1	0	1	1

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x_1	x_2	f_1	f_2	f_3	f_8
0	0	0	0	0	0
0	1	0	0	0	1
1	0	0	0	1	1
1	1	0	1	0	1

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0	0	0	0	0	0	0
0	1	0	0	0	0	1
1	0	0	0	1	1	1
1	1	0	1	0	1	1

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0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1
1	0	0	0	1	1	0	1
1	1	0	1	0	1	0	1

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0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1
1	0	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	1

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1	0	0	0	1	1	0	0	1	1
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0	1	0	0	0	0	1	1	1	1	0
1	0	0	0	1	1	0	0	1	1	0
1	1	0	1	0	1	0	1	0	1	0

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0	0	0	0	0	0	0	0	0	0	1	1
0	1	0	0	0	0	1	1	1	1	0	0
1	0	0	0	1	1	0	0	1	1	0	0
1	1	0	1	0	1	0	1	0	1	0	1

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0	0	0	0	0	0	0	0	0	0	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0
1	0	0	0	1	1	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	0	1	0	1	0

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}
0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1

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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0

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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ?

- How many boolean functions can you design from 2 inputs ?
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x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}
- How many of these 2^{2^n} functions are not linearly separable ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

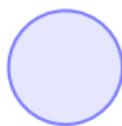
x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}
- How many of these 2^{2^n} functions are not linearly separable ? For the time being, it suffices to know that at least some of these may not be linearly inseparable (I encourage you to figure out the exact answer :-))

- We will now see how to implement **any** boolean function using a network of perceptrons ...

- For this discussion, we will assume True = +1 and False = -1

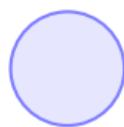
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons



x_1

x_2

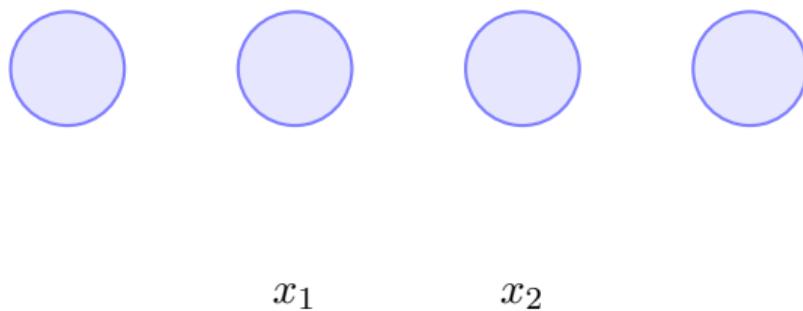
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- Each input is connected to all the 4 perceptrons with specific weights



x_1

x_2

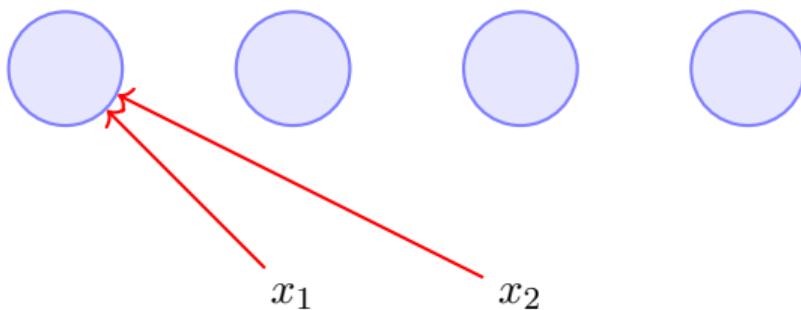
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red edge indicates $w = -1$

blue edge indicates $w = +1$

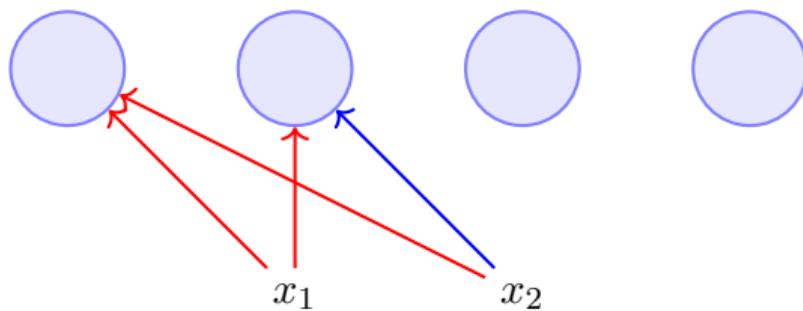
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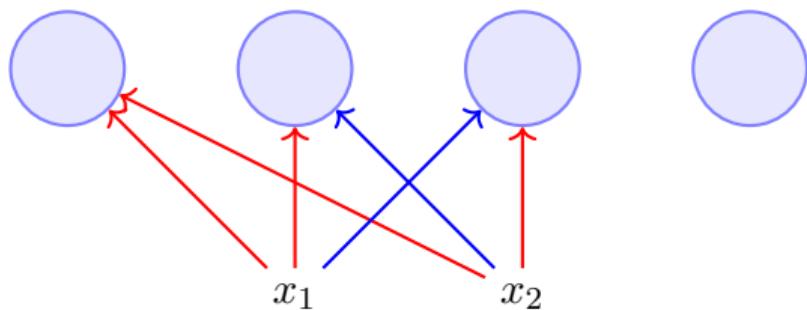
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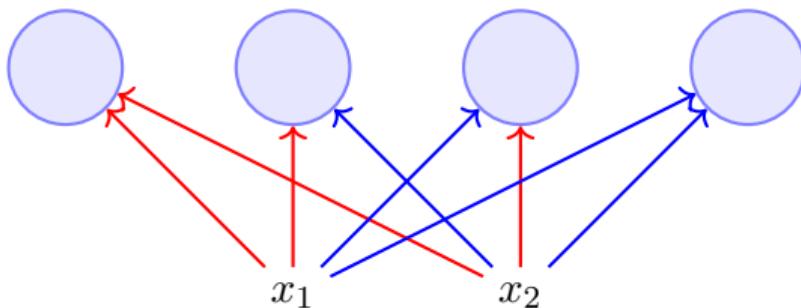
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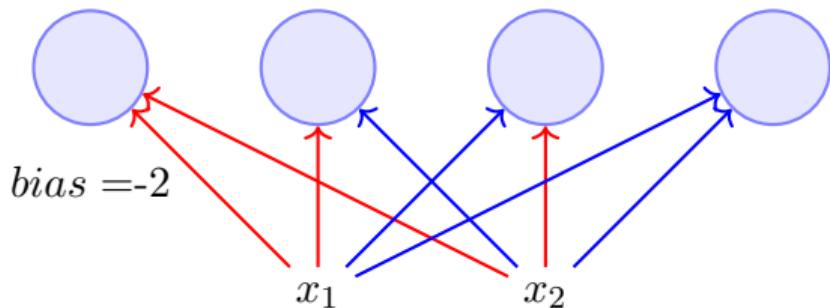
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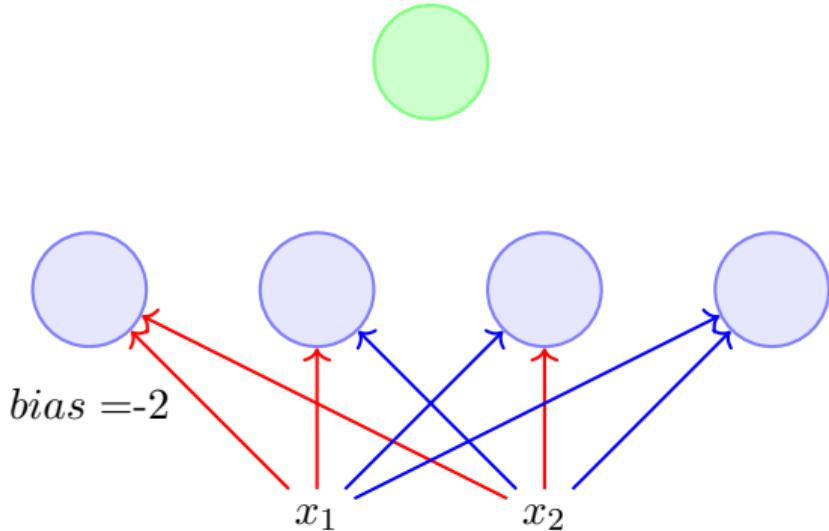
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)



red edge indicates $w = -1$

blue edge indicates $w = +1$

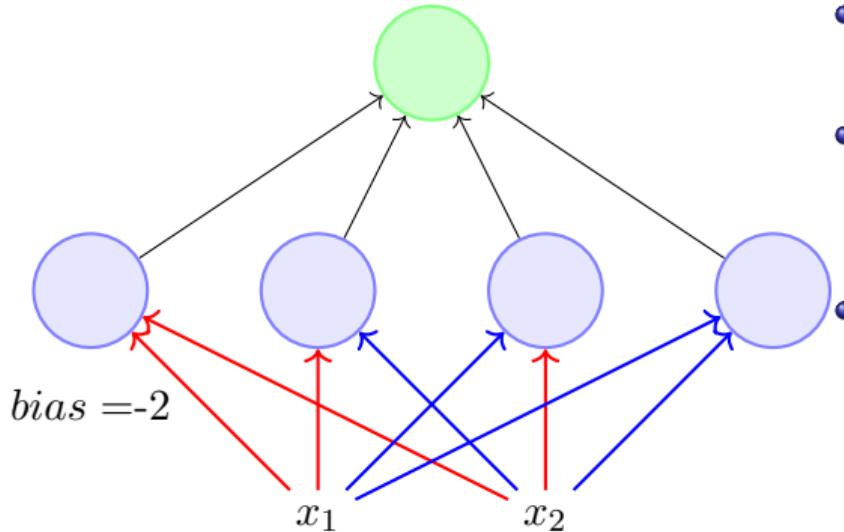
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- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



red edge indicates $w = -1$

blue edge indicates $w = +1$

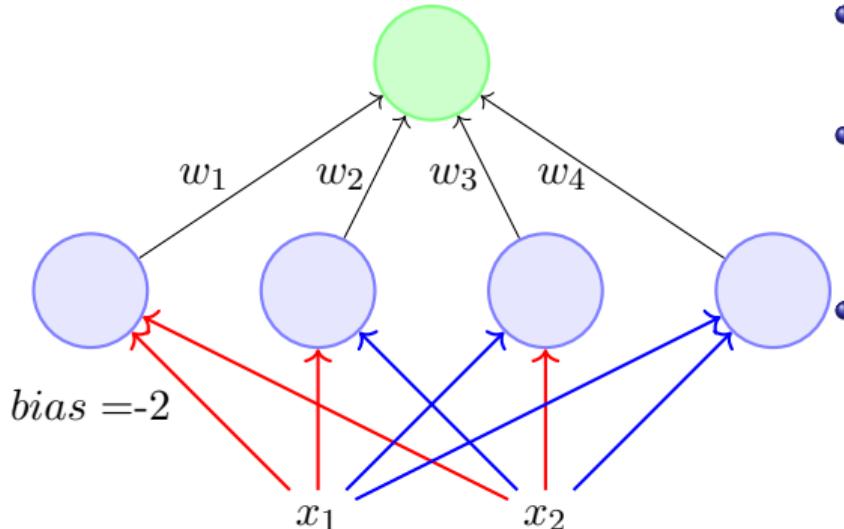
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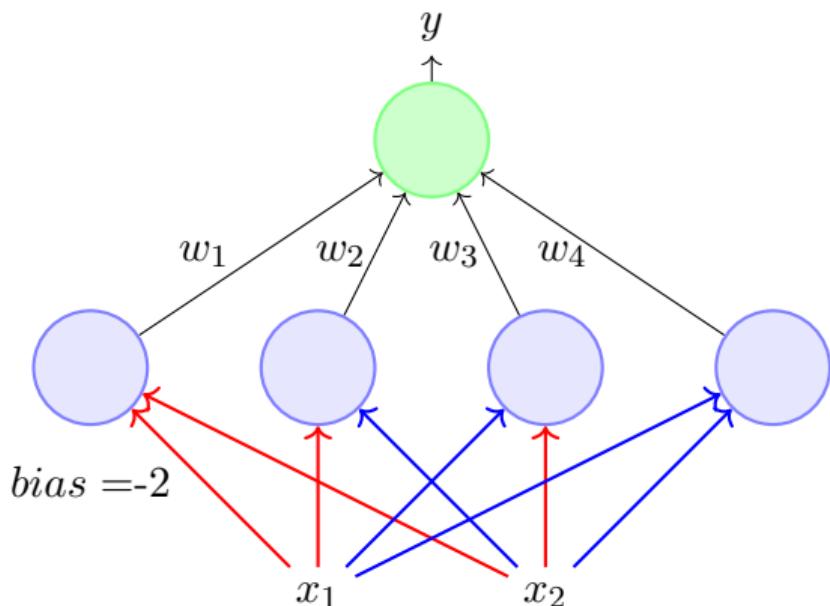
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 - Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)
 - The output of this perceptron (y) is the output of this network

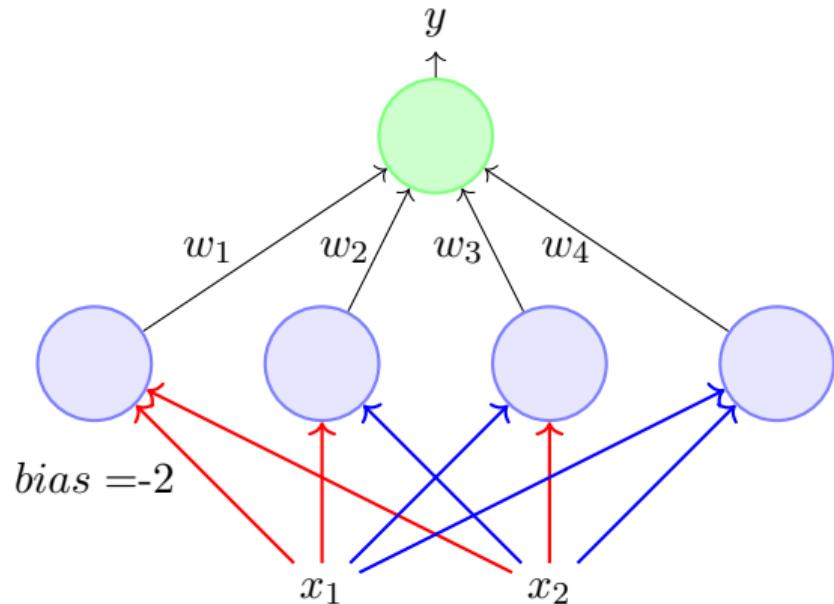


red edge indicates $w = -$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers

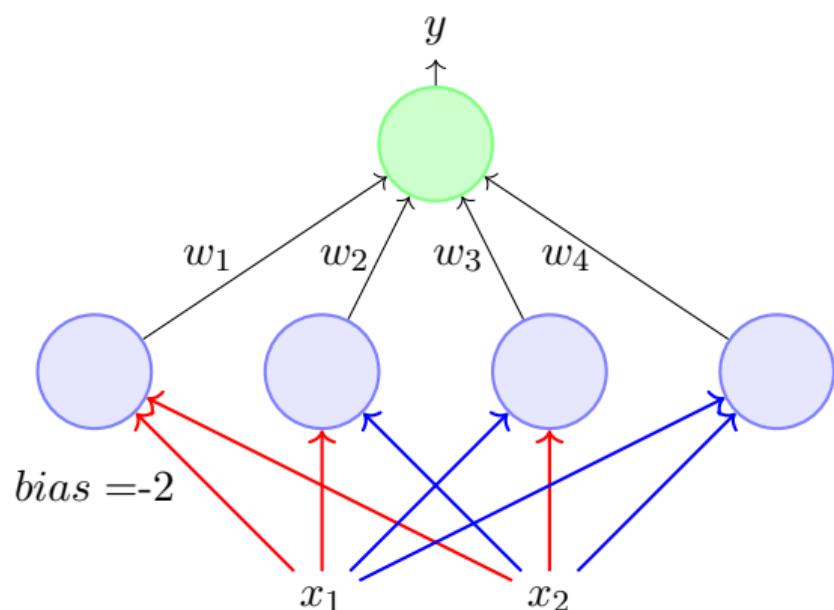


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**

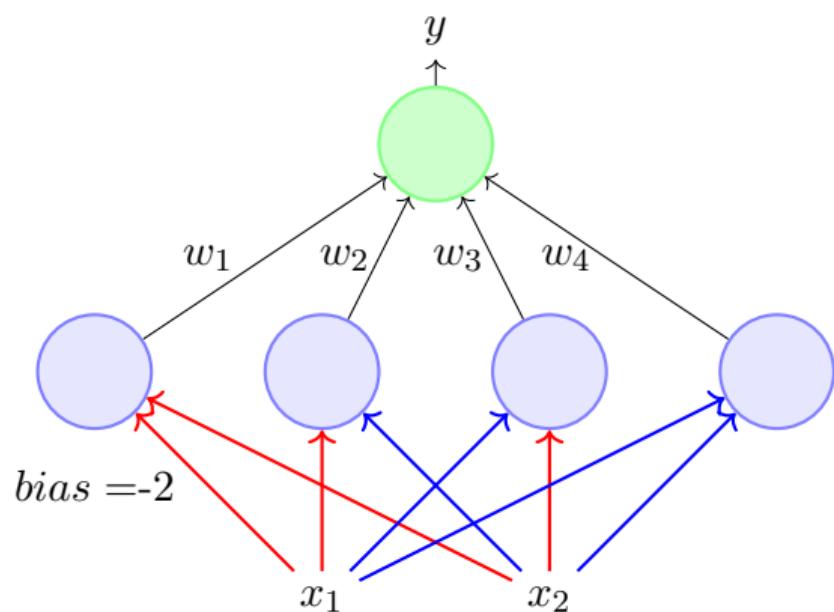


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**

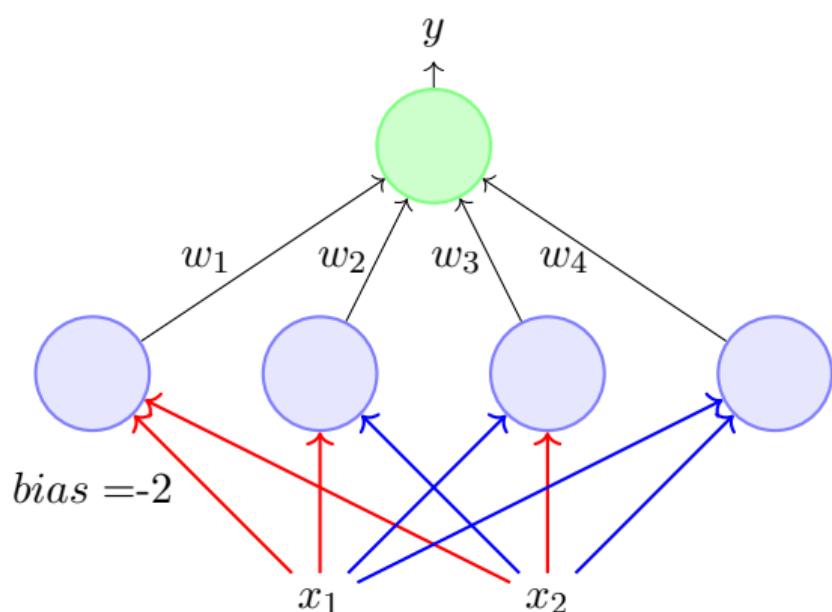


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**

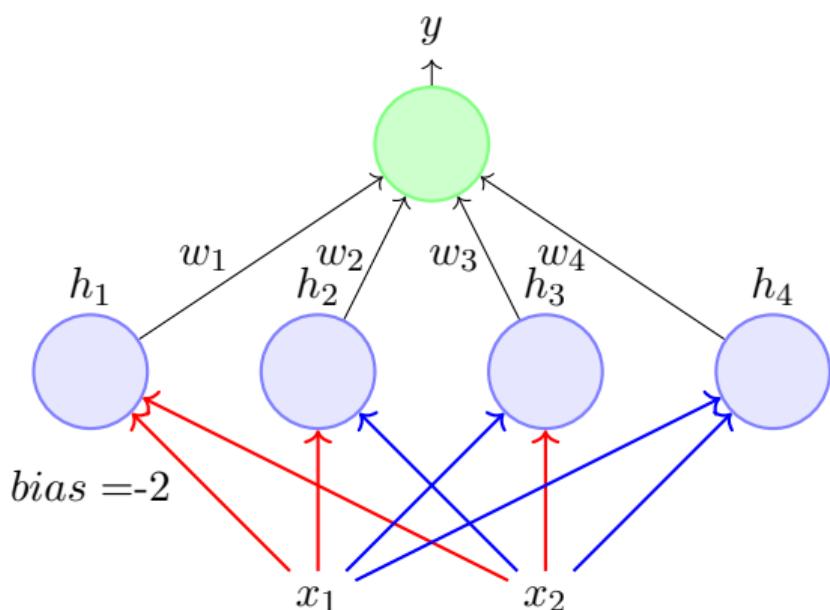


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

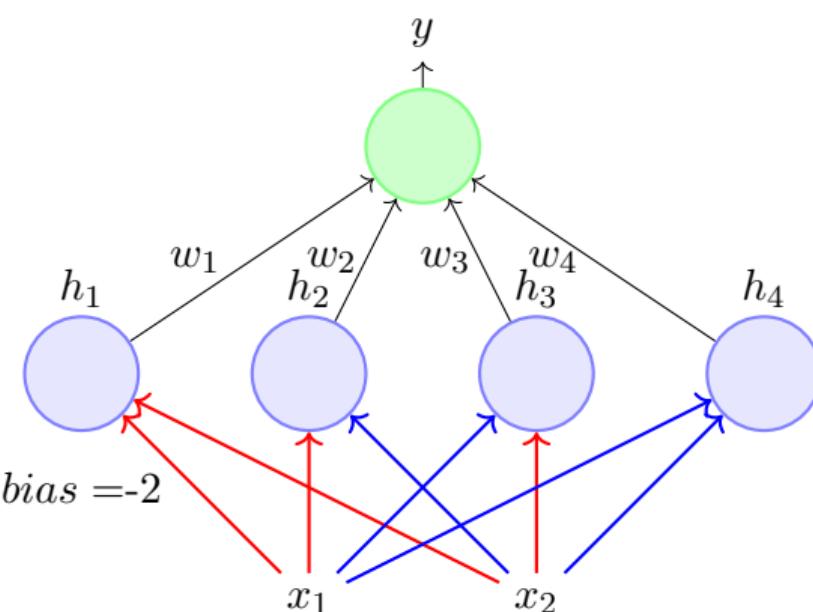
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- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**
- The outputs of the 4 perceptrons in the hidden layer are denoted by h_1, h_2, h_3, h_4



red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

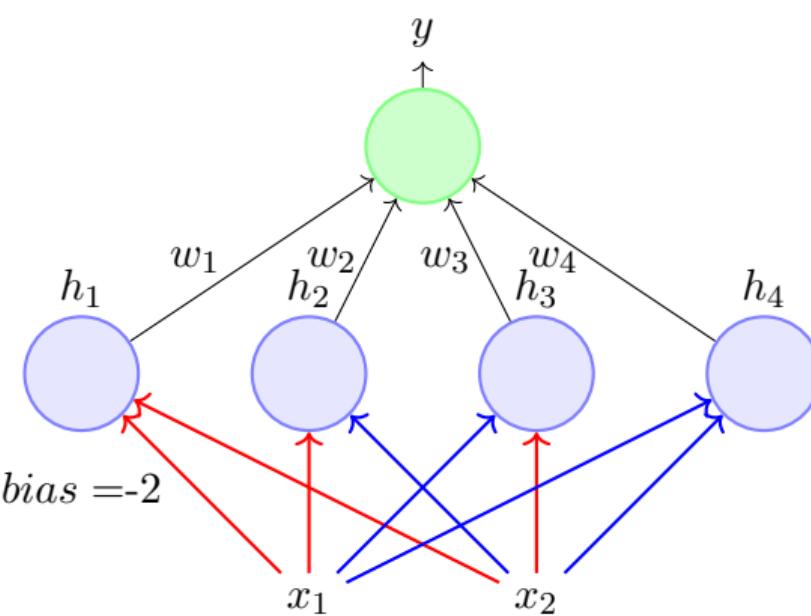


red edge indicates $w = -1$

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Terminology:

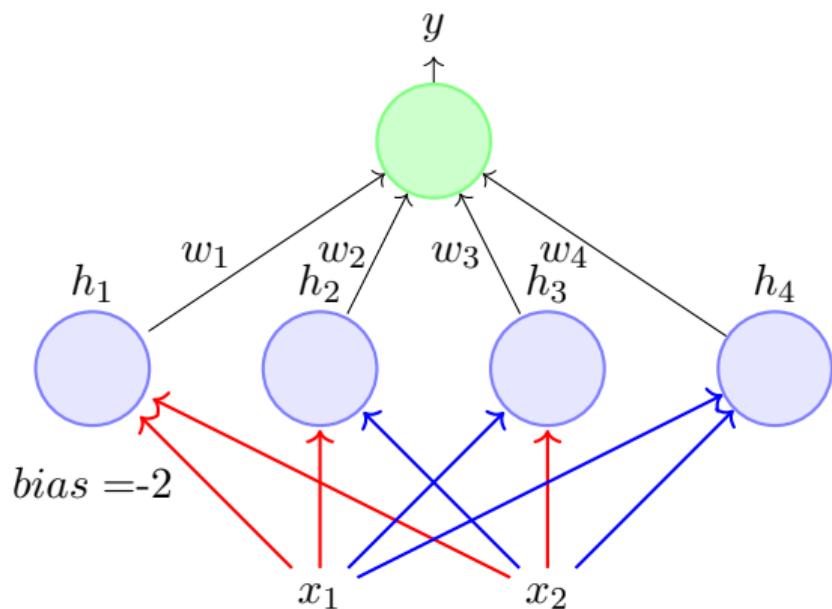


red edge indicates $w = -1$

blue edge indicates $w = +1$

- This network contains 3 layers
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- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**
- The outputs of the 4 perceptrons in the hidden layer are denoted by h_1, h_2, h_3, h_4
- The red and blue edges are called layer 1 weights
- w_1, w_2, w_3, w_4 are called layer 2 weights

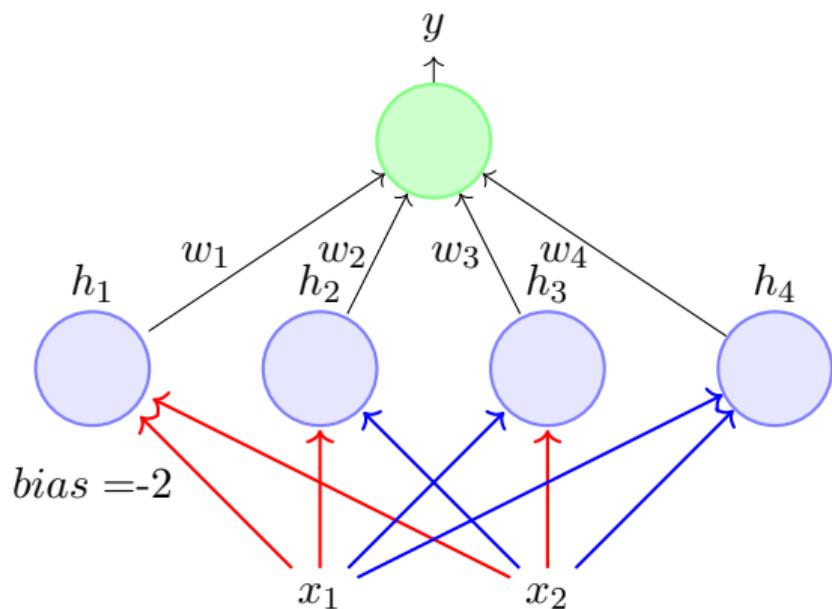
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !



red edge indicates $w = -1$

blue edge indicates $w = +1$

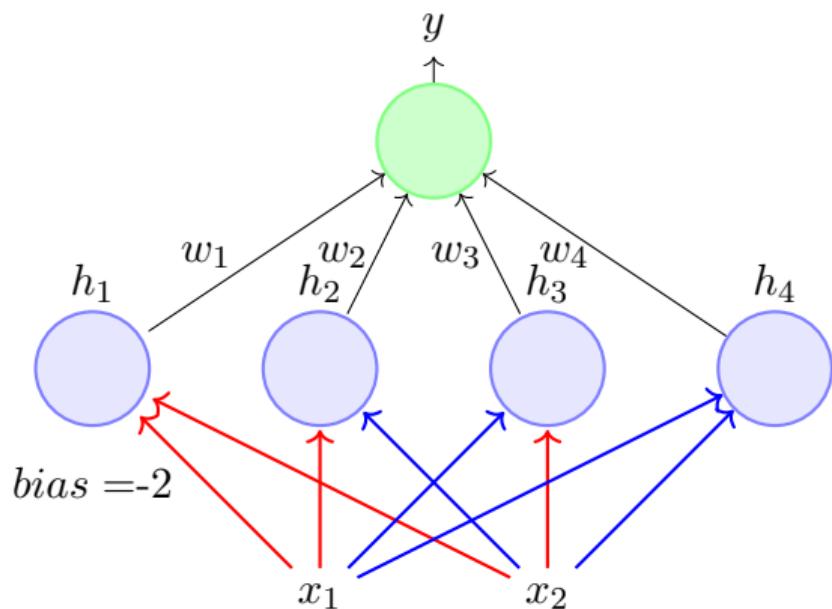
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- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network



red edge indicates $w = -1$

blue edge indicates $w = +1$

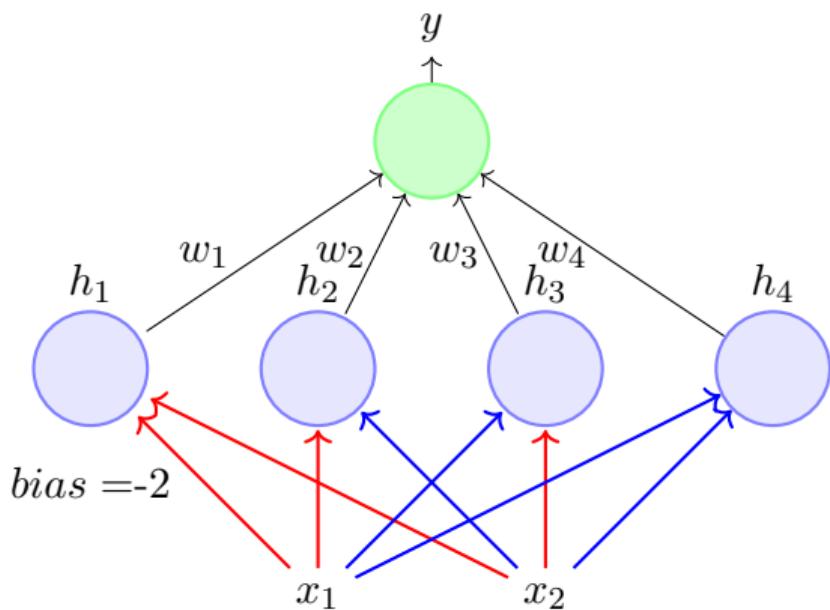
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- Astonishing claim!



red edge indicates $w = -1$

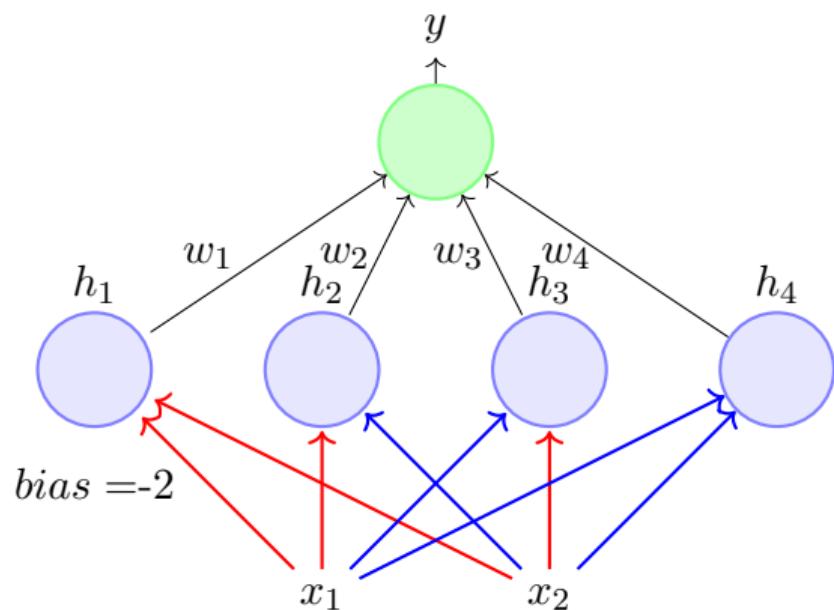
blue edge indicates $w = +1$

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- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on



red edge indicates $w = -1$

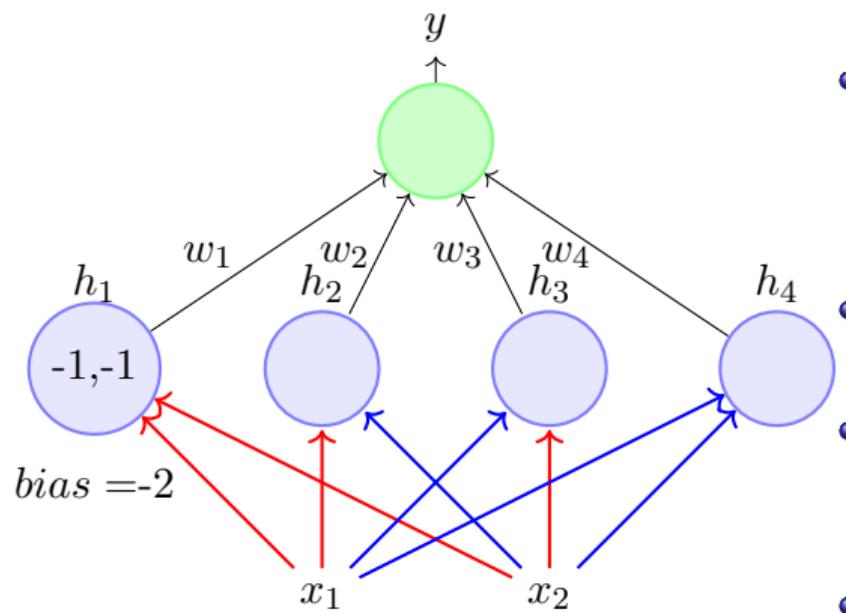
blue edge indicates $w = +1$



red edge indicates $w = -1$

blue edge indicates $w = +1$

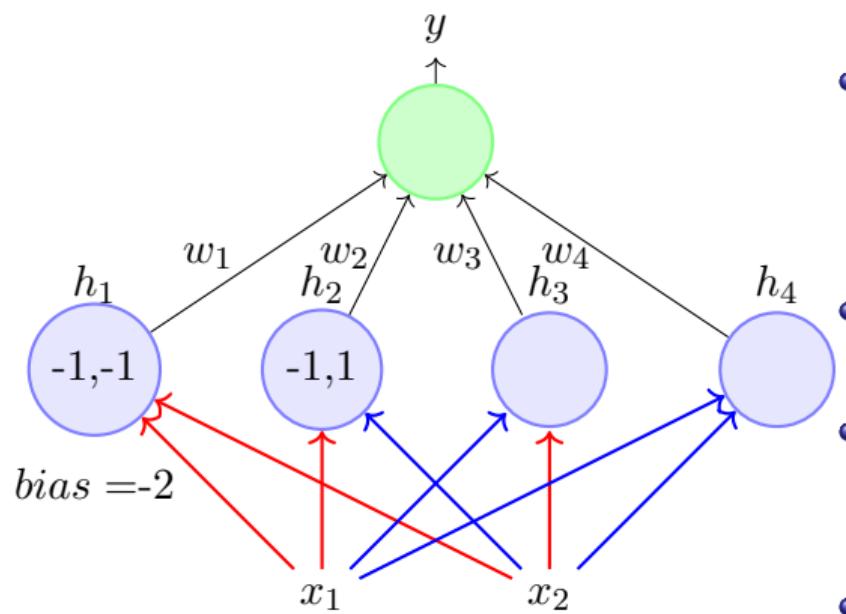
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- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)



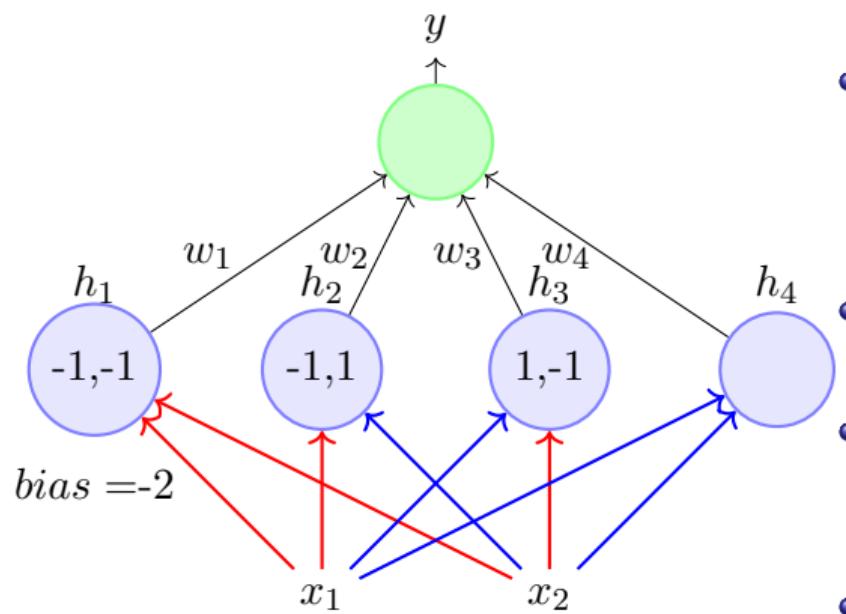
red edge indicates $w = -1$

blue edge indicates $w = +1$

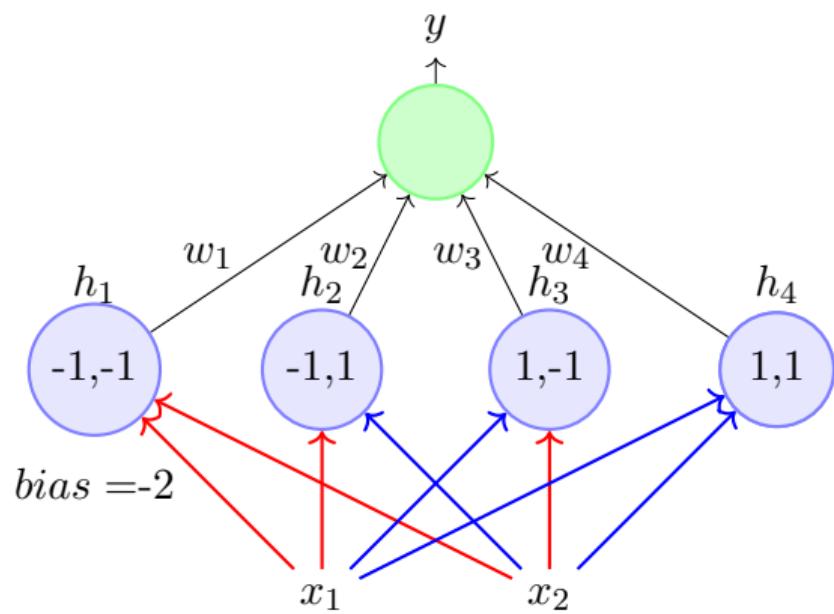
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- the first perceptron fires for $\{-1,-1\}$



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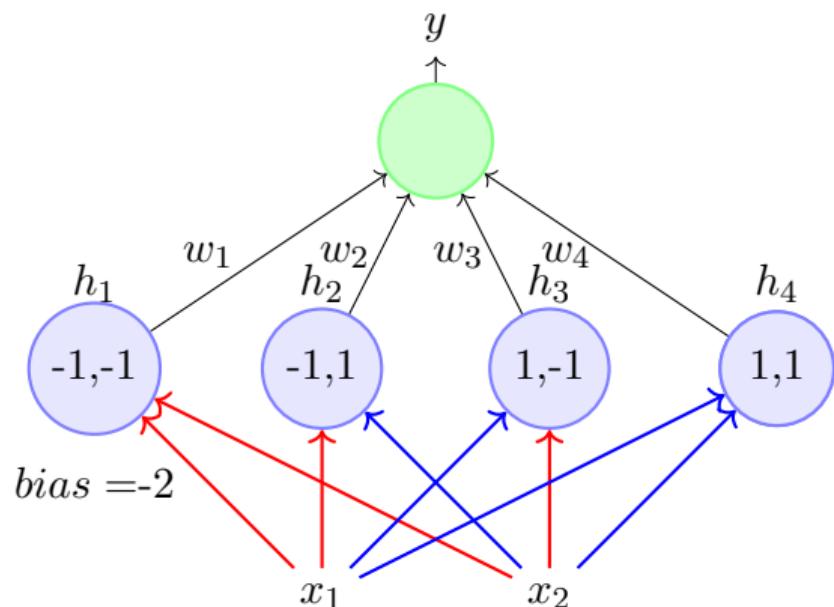
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- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the third perceptron fires for $\{1, -1\}$



red edge indicates $w = -1$

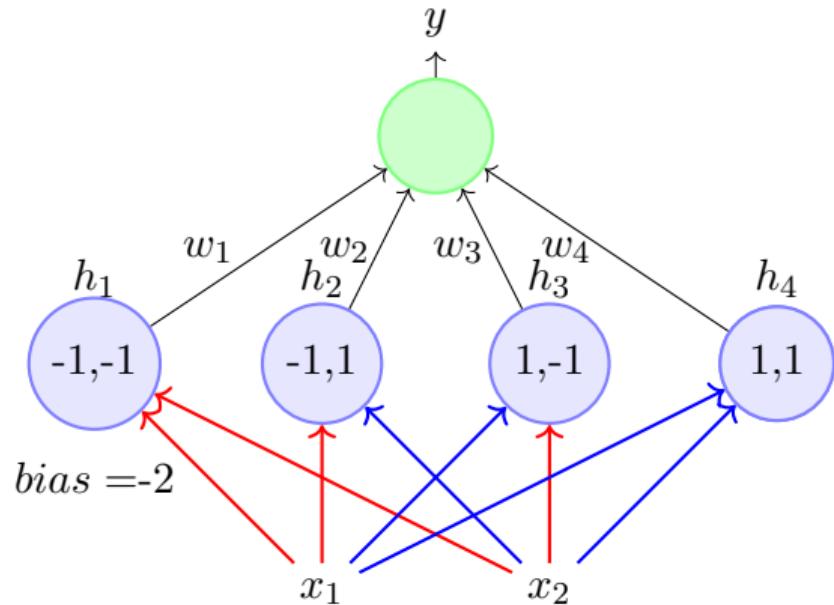
blue edge indicates $w = +1$

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- the fourth perceptron fires for $\{1,1\}$



- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- Let us see why this network works by taking an example of the XOR function

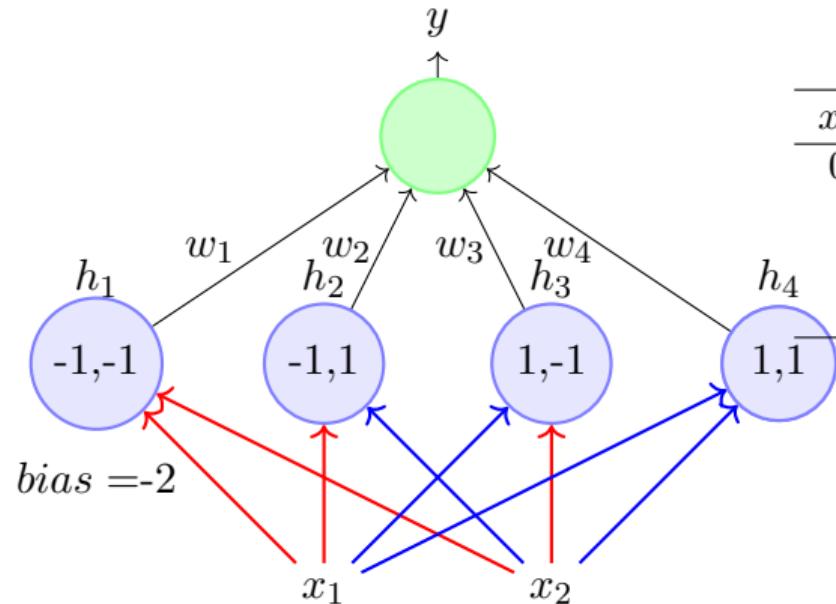
- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)



red edge indicates $w = -1$

blue edge indicates $w = +1$

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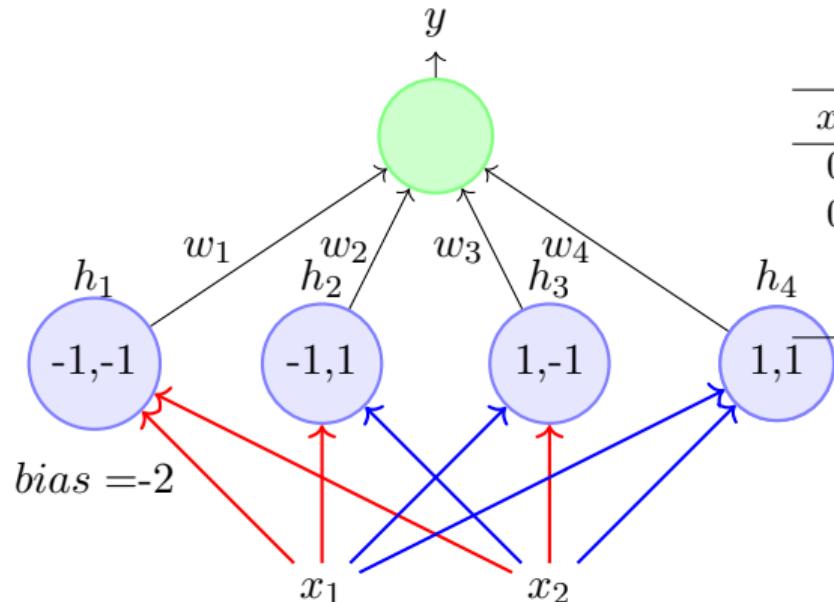


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x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1

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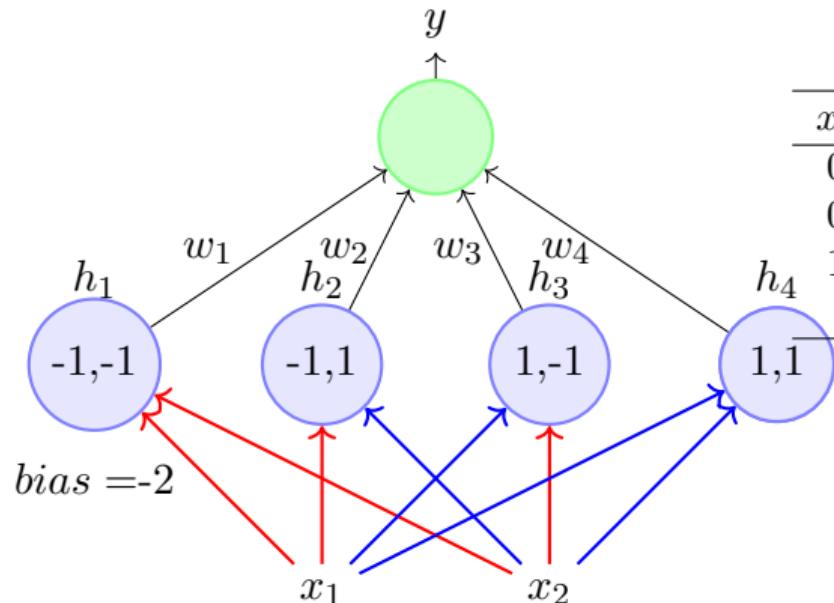


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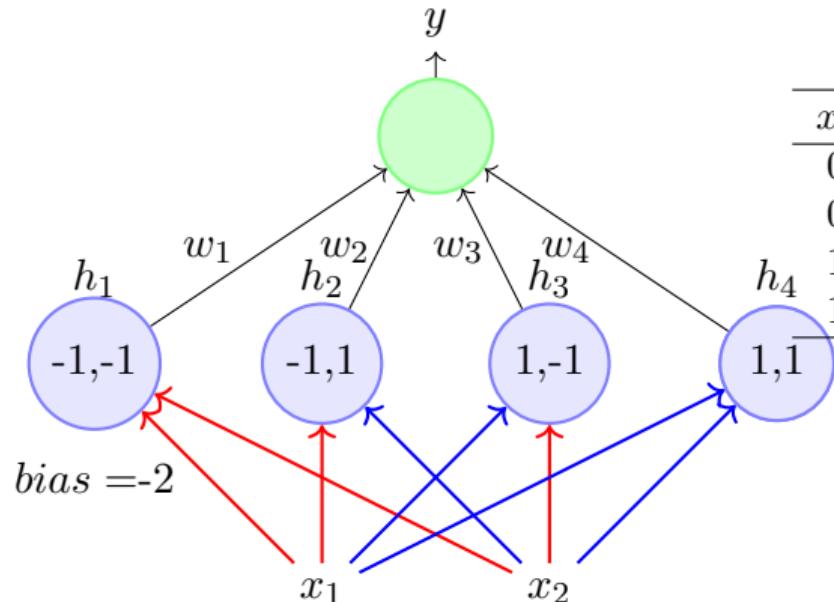


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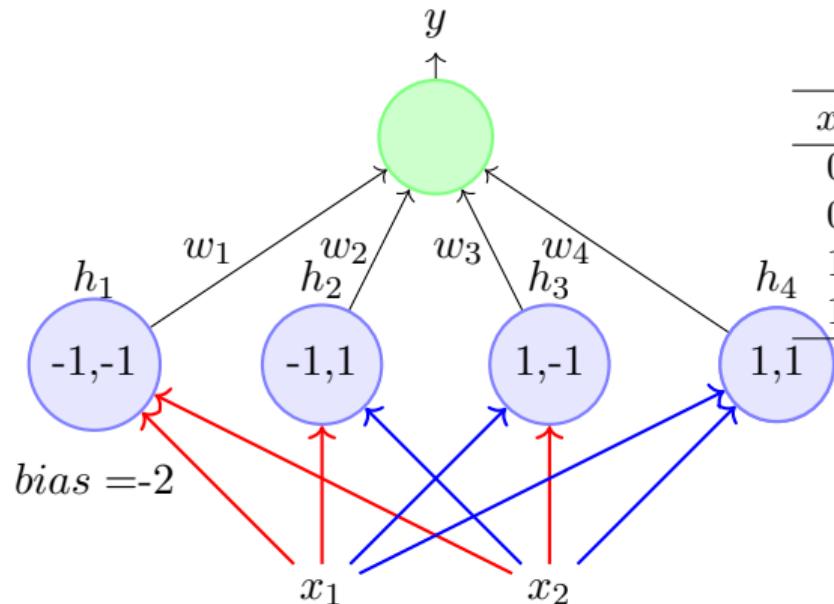


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1	1	0	0	0	0	1	w_4

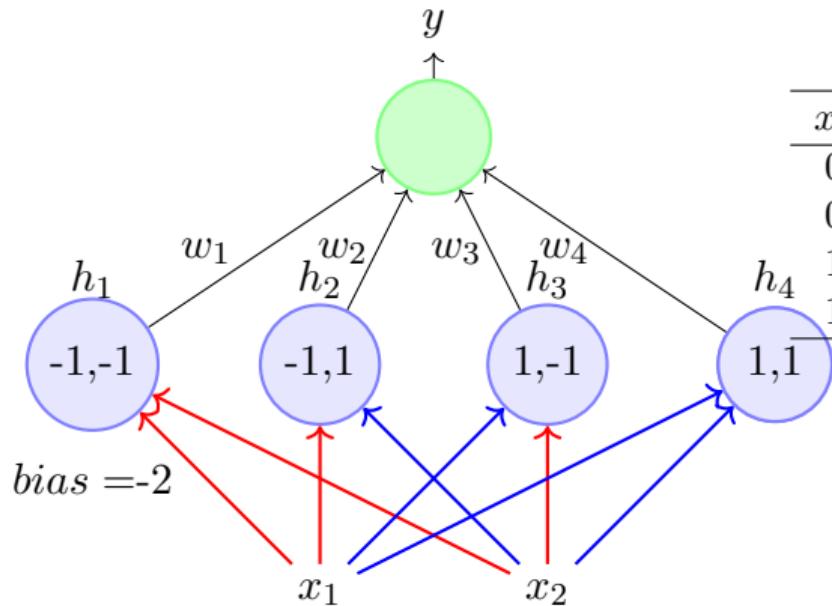
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- This results in the following four conditions to implement XOR: $w_1 < w_0, w_2 \geq w_0, w_3 \geq w_0, w_4 < w_0$

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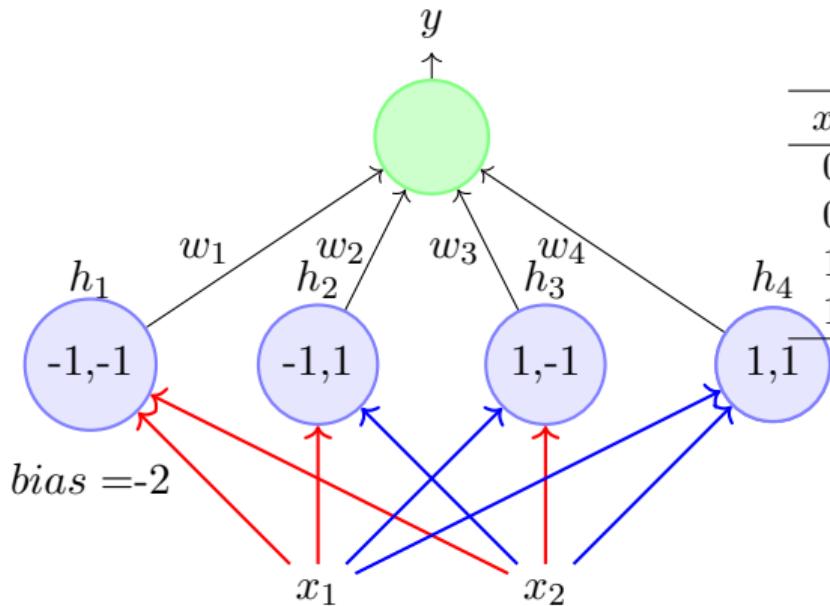


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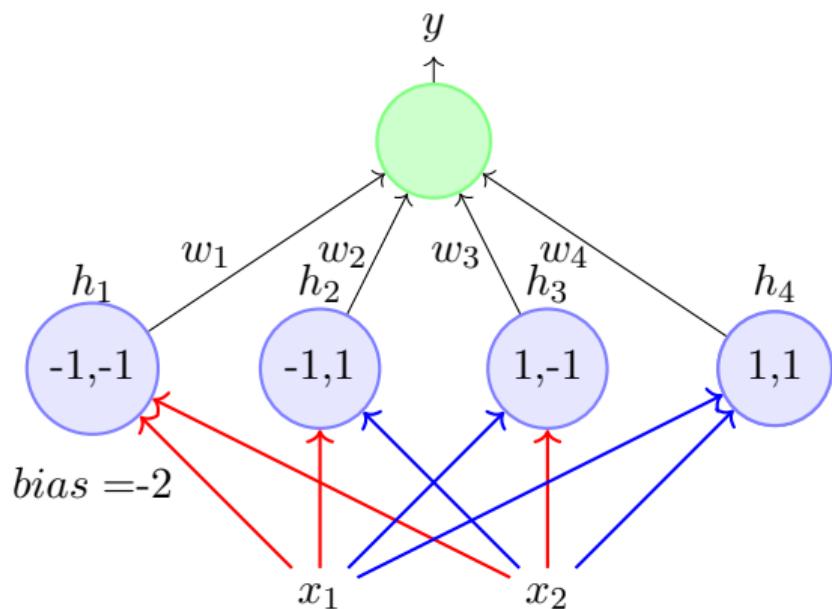
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- Unlike before, there are no contradictions now and the system of inequalities can be satisfied
- Essentially each w_i is now responsible for one of the 4 possible inputs and can be adjusted to get the desired output for that input

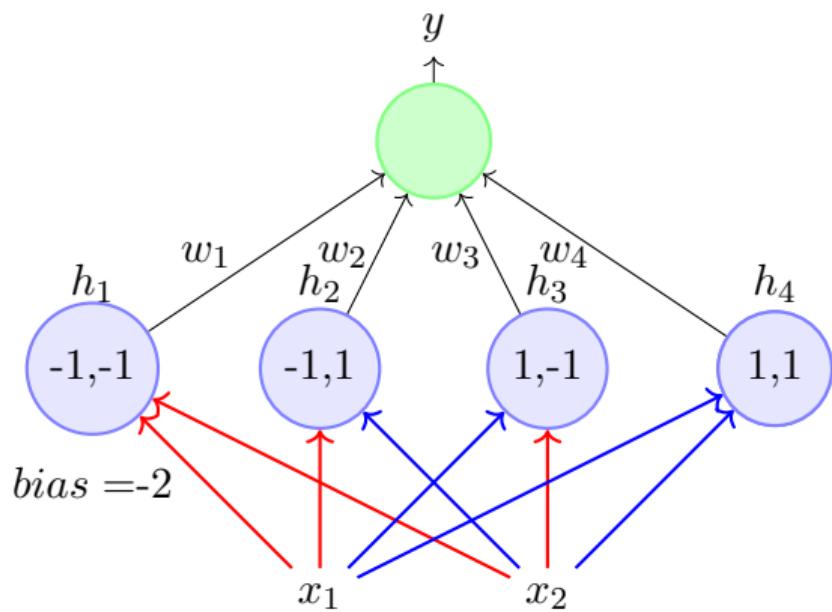
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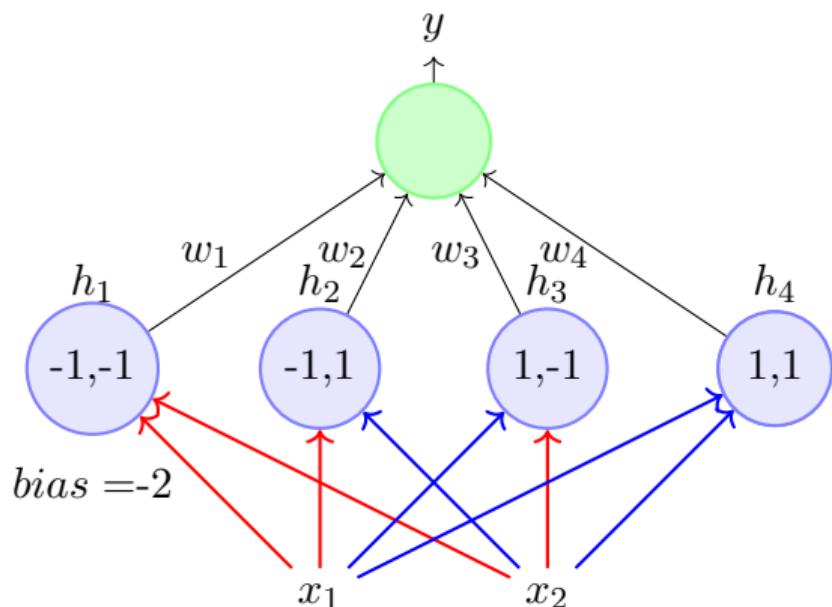
- It should be clear that the same network can be used to represent the remaining 15 boolean functions also
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- Try it!

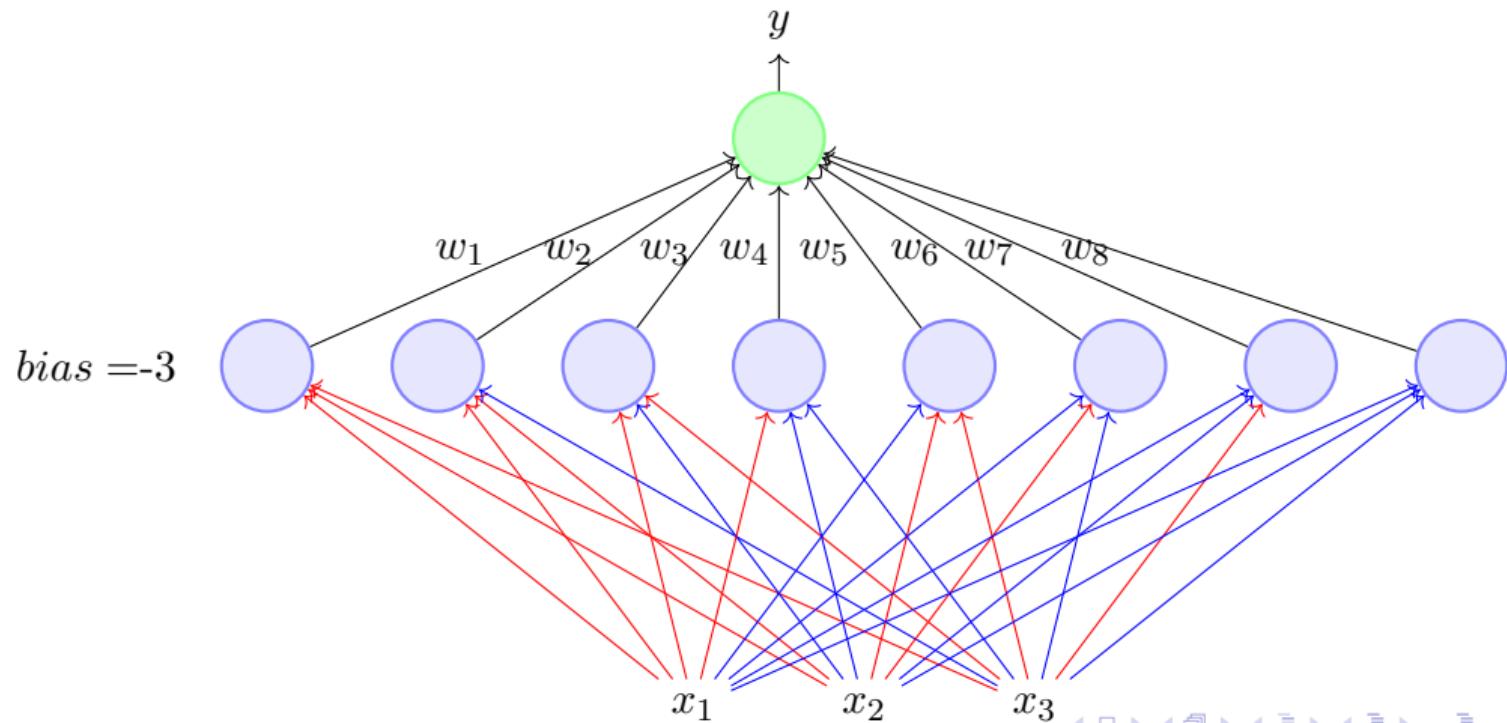


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- What if we have more than 3 inputs ?

- Again each of the 8 perceptrons will fire only for one of the 8 inputs
- Each of the 8 weights in the second layer is responsible for one of the 8 inputs and can be adjusted to produce the desired output for that input



- What if we have n inputs ?

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Catch: As n increases the number of perceptrons in the hidden layers obviously increases exponentially

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- Specifically, it tells us that a MLP with a single hidden layer can represent **any** boolean function

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- Enough about boolean functions!
- What about arbitrary functions of the form $y = f(x)$ where $x \in \mathbb{R}^n$ (instead of $\{0, 1\}^n$) and $y \in \mathbb{R}$ (instead of $\{0, 1\}$) ?
- Before answering the above question we will have to first graduate from *perceptrons* to *sigmoidal neurons* ...

Pre-requisites for Lecture 3 and Lecture 4

- Please do Assignment 1 sincerely (you will be able to understand the next two lectures better if you do so!)
- Some more reading material (if needed) will be posted on moodle