Module 21.2: Intuition behind Variational Autoencoders

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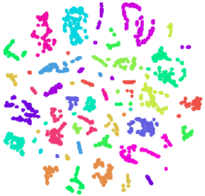

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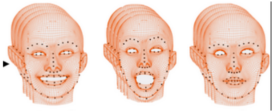

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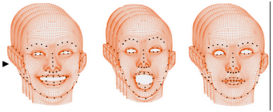

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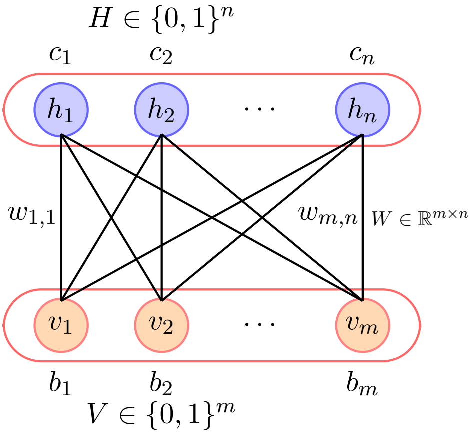
- In probabilistic terms we are interested in $P(z|X)$ and $P(X|z)$



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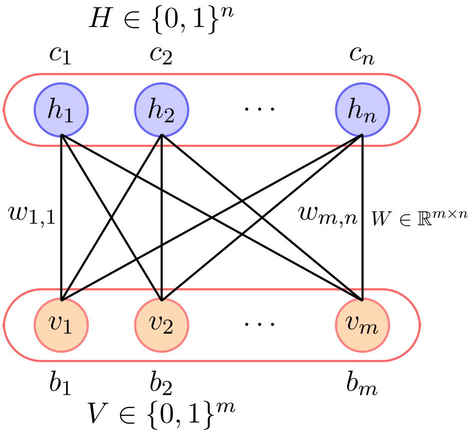

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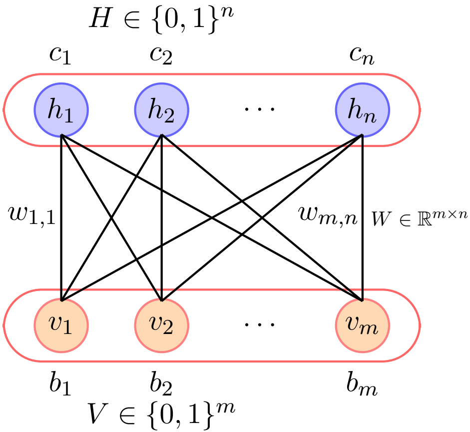

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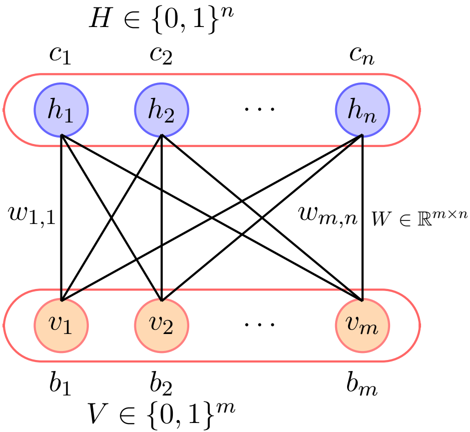

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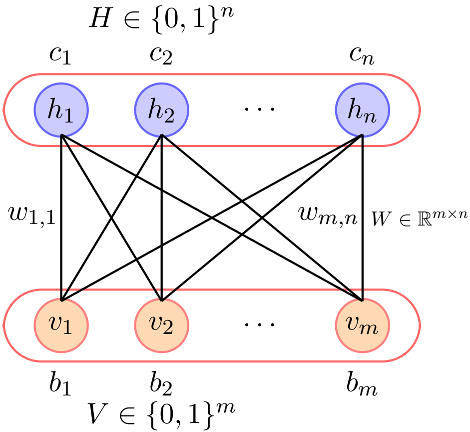

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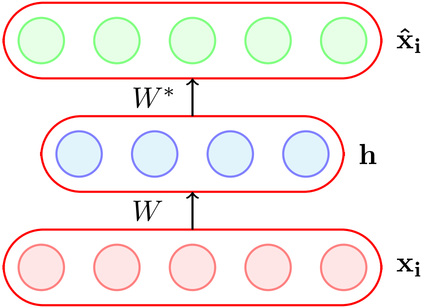
- **Approximation:** When using Contrastive Divergence, we approximate the expectation by a point estimate



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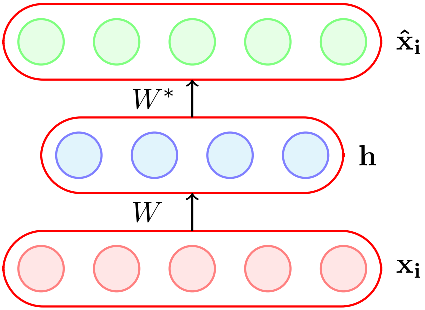

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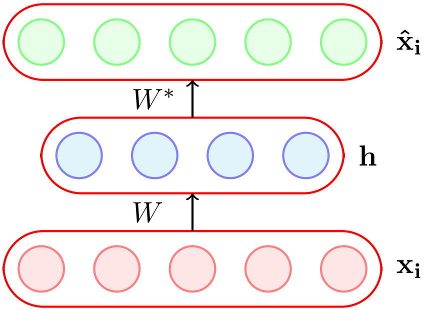

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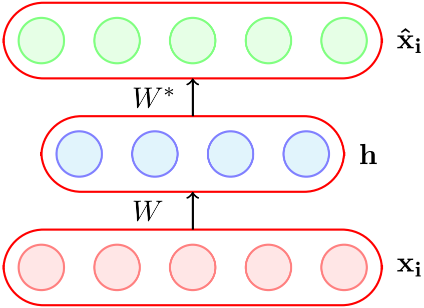
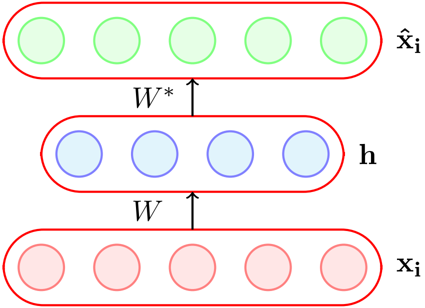
- In VAEs also we will start with the same objective function but use a different trick to solve the optimization problem (we will get there soon)



- Goal 1: Learn a distribution over the latent variables ($P(z|X)$)



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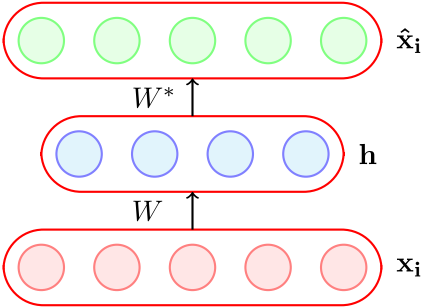

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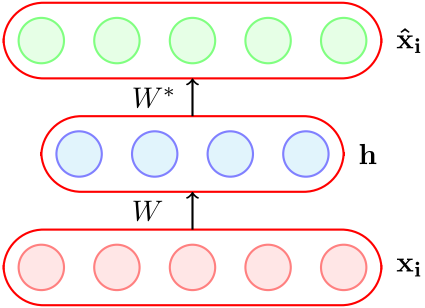

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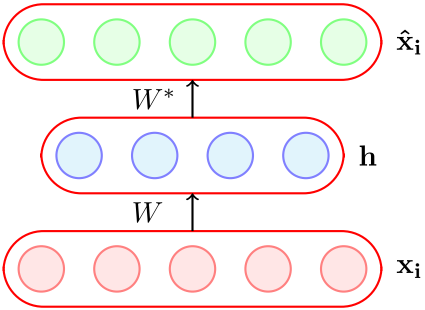

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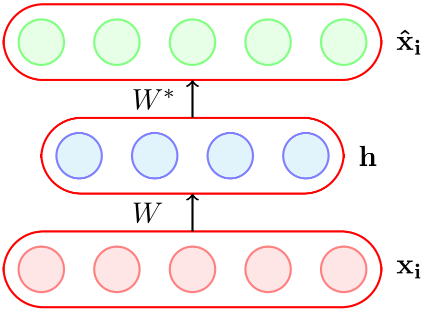
- In RBMs, we had assumed a certain structure (independencies) for the joint distribution whereas here we are assuming a certain family form for the distribution $P(z|X)$



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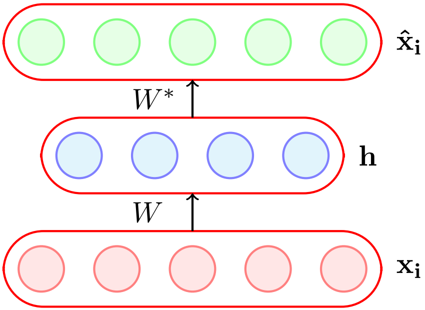

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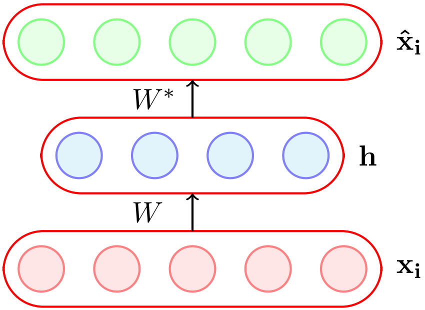

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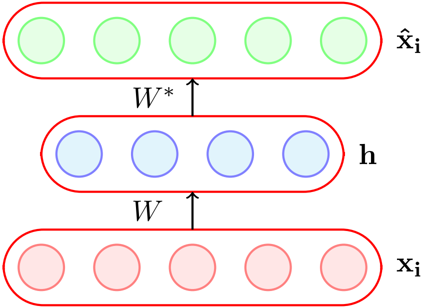

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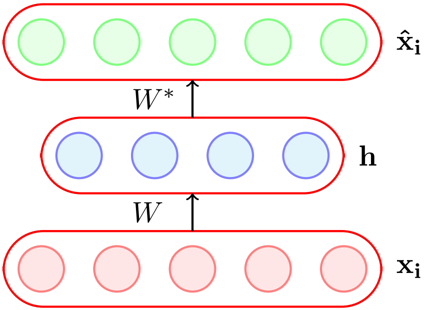

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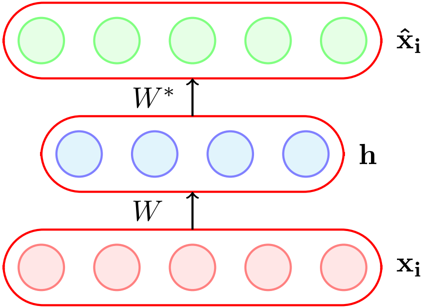
- Well, we could design the encoder to predict the mean and variance of $P(z|X)$



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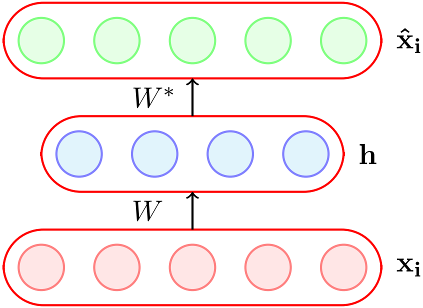

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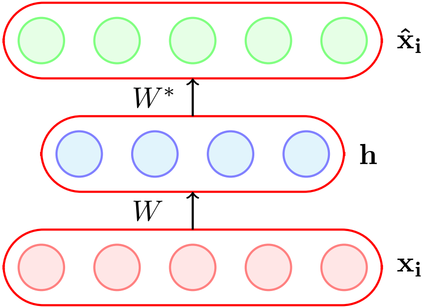

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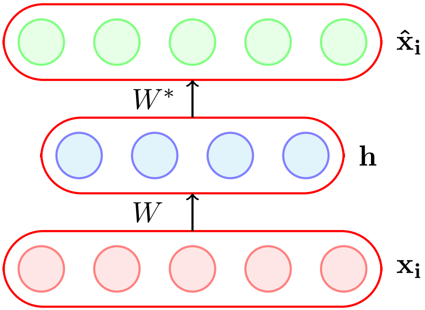

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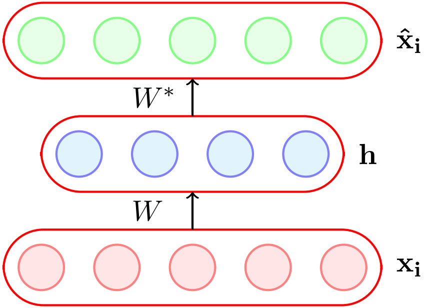

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- **Spoiler Alert:** We will be making some assumptions but to see why these assumptions make sense we will first revisit the concept of latent variables

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- Of course, we are assuming that we have enough latent variables to capture all characteristics of handwritten digits

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- With this intuition, we will now look at the full picture and discuss the objective function

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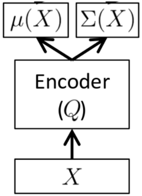

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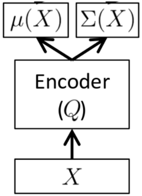

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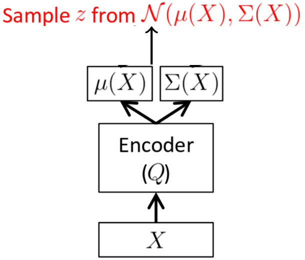

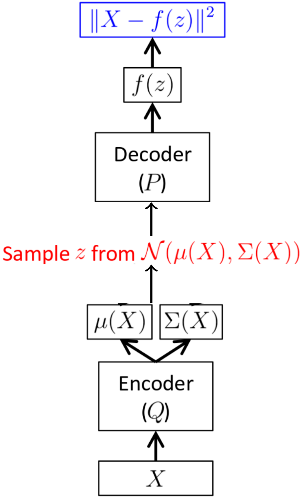

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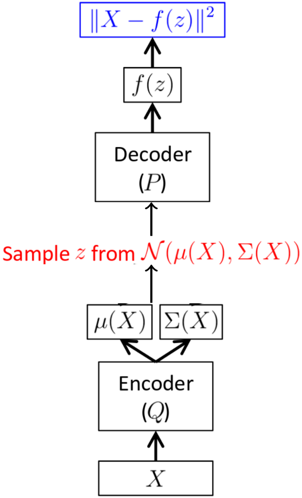
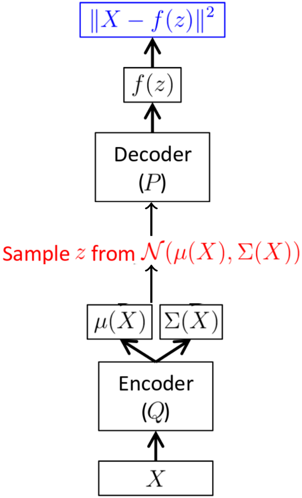
- The job of the decoder is to then reconstruct X from this sampled z



- Given this setup what should be the loss function for the decoder

$$||X - f(z; \theta)||^2$$



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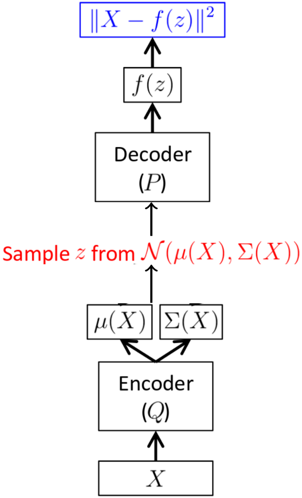

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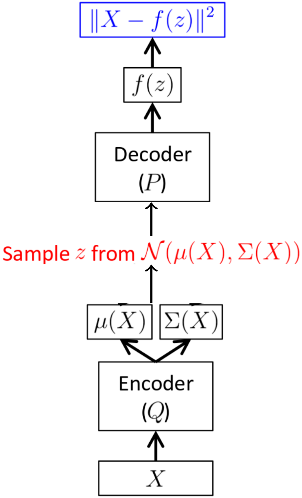

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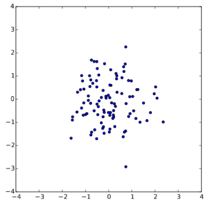
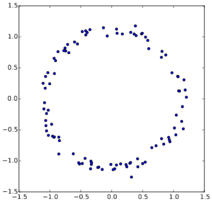
- Well if we knew the true $P(z)$ then we could have minimized the KL divergence between $Q(z|X)$ and $P(z)$
- But we don't know what $P(z)$ is so we will make an assumption that $P(z)$ is $N(0, I)$





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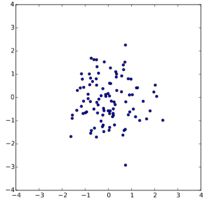

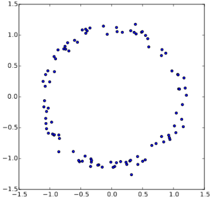

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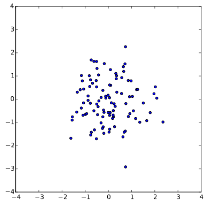

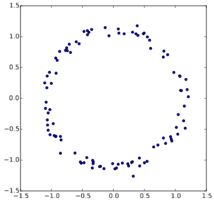

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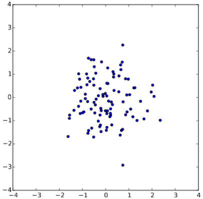

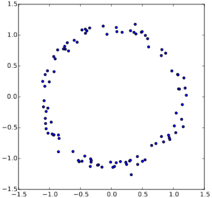

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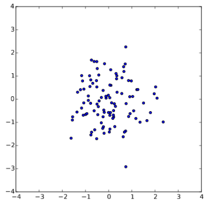
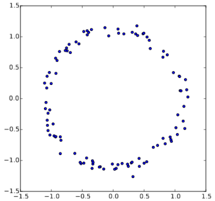
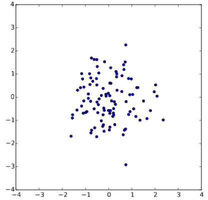
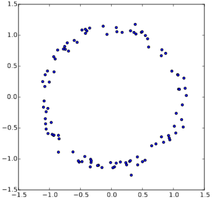
- Step 1: Start with a set of d variables that are normally distributed (that's exactly what we are assuming for $P(z)$)





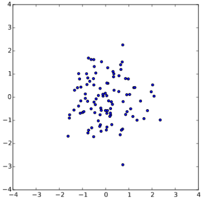
- Isn't this is very strong assumption ?
- For example, in the 2-dimensional case how can we be sure that P(z) is a normal distribution and not any other distribution
- The key insight here is that any distribution in d dimensions can be generated by the following steps
- Step 1: Start with a set of d variables that are normally distributed (that's exactly what we are assuming for $P(z)$)
- Step 2: Mapping these variables through a sufficiently complicated function (that's exactly what the first few layers of the decoder can do)

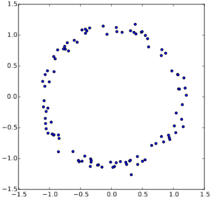




- In particular, note that in the adjoining example if z is 2-D and normally distributed then $g(z)$ is roughly ring shaped (giving us the distribution in the bottom figure)

$$g(z) = \frac{z}{10} + \frac{z}{||z||}$$





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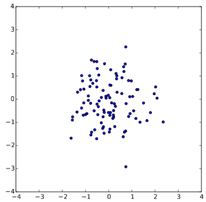
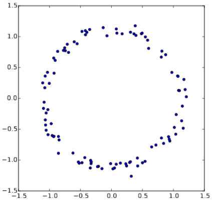
- In other words,

$$g(z) = \theta_1 z + \theta_2 z$$





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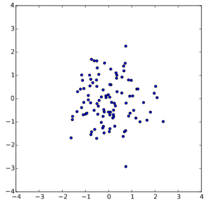
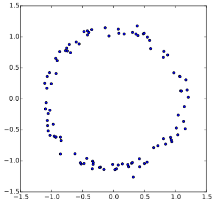
- The initial layers of the decoder could learns their weights such that the output is $g(z, \theta)$





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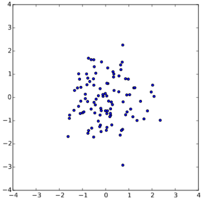
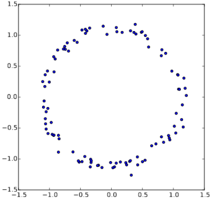
- The above argument suggests that even if we start with normally distributed variables the initial layers of the encoder could learn a complex transformation of these variables say $f(z, \theta_1)$ if required





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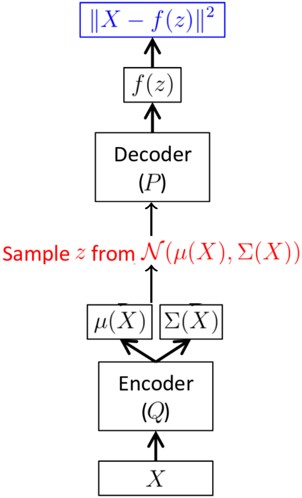
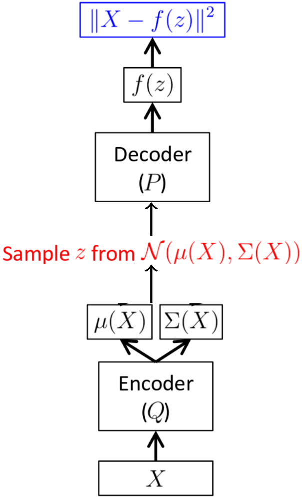
- The above argument suggests that even if we start with normally distributed variables the initial layers of the encoder could learn a complex transformation of these variables say $f(z, \theta_1)$ if required
- The objective function of the decoder will ensure that an appropriate transformation of z is learnt to reconstruct X



- So now that we are convinced that it is okay to assume P(z) is N(0, I) then the objective function of the encoder is straightforward



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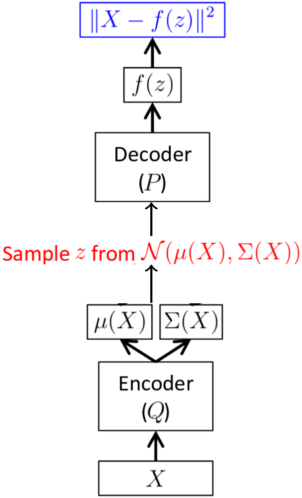
- We simply need to minimize $KL[N(\mu(X), \sigma(X)), N(0, I)]$



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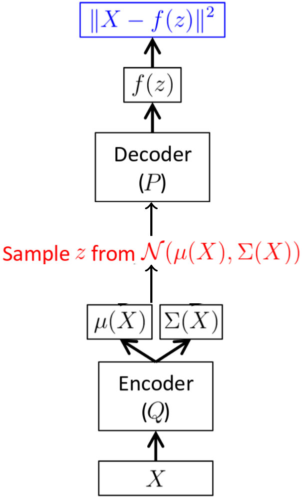
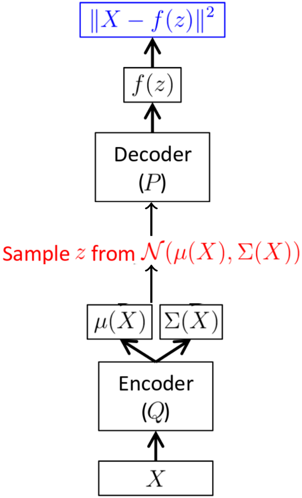
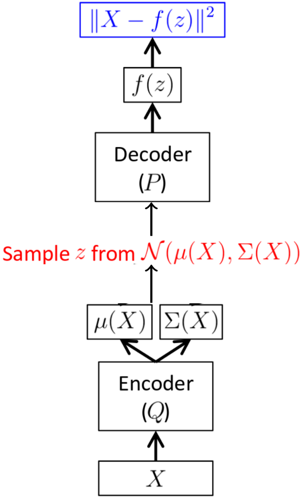
- That completes the full picture and we will summarize the discussion on the next slide



- Encoder: Generates the μ and σ of $Q(z|X)$



- Encoder: Generates the μ and σ of $Q(z|X)$
- Decoder: Samples from $Q(z|X)$ and reconstructs X



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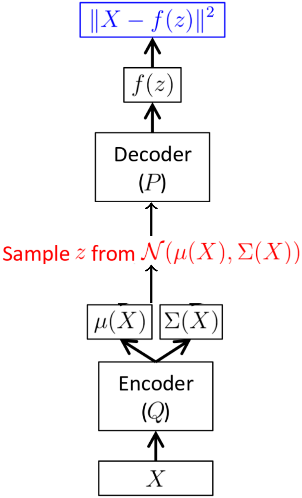

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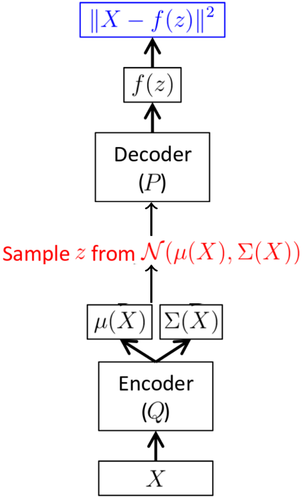

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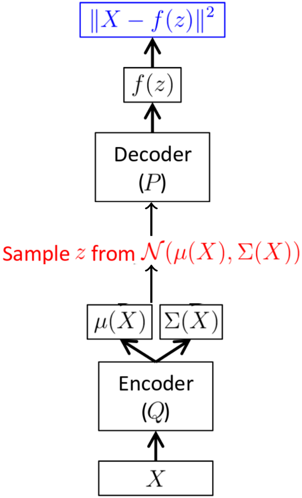
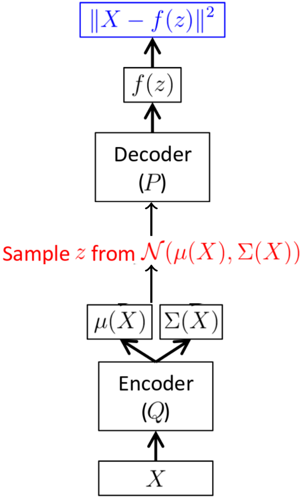
- There are still a few pieces missing and we will get back to them later



- This was a very simplistic (non-rigorous) introduction to VAEs



- This was a very simplistic (non-rigorous) introduction to VAEs
- We will now do a more rigorous discussion on the Math behind VAEs