COMP4033 Computer Graphics

Lecture 2: Review of Basic Math

Spring 2020

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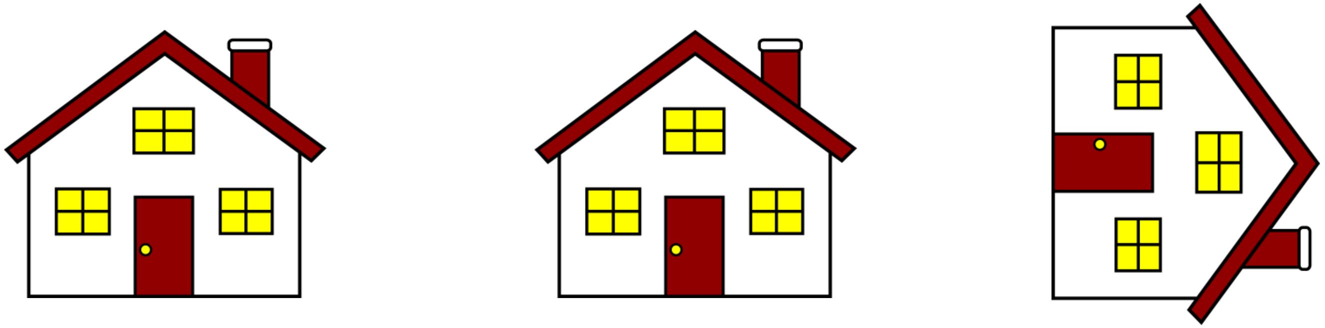
To Do

- Complete Assign 0
- Get help if issues with compiling, programming
- About first few lectures
  - Somewhat technical: core math ideas in graphics
  - Assign1 is simple (only few lines of code): Let you see how to use some ideas discussed in lecture, create images

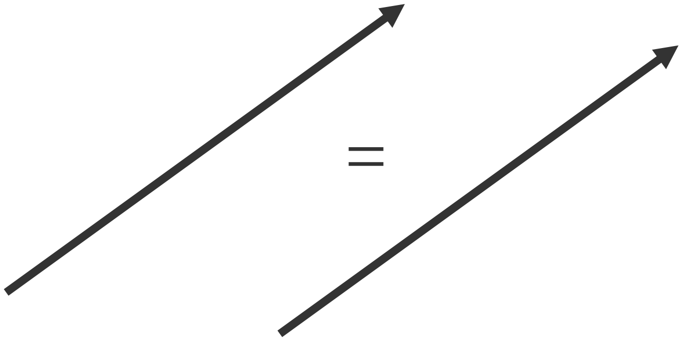
Motivation and Outline

- Many graphics concepts need basic math like linear algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)
  - E.g: a point is a vector, and an operation like translating or rotating points on object can be matrix-vector multiply



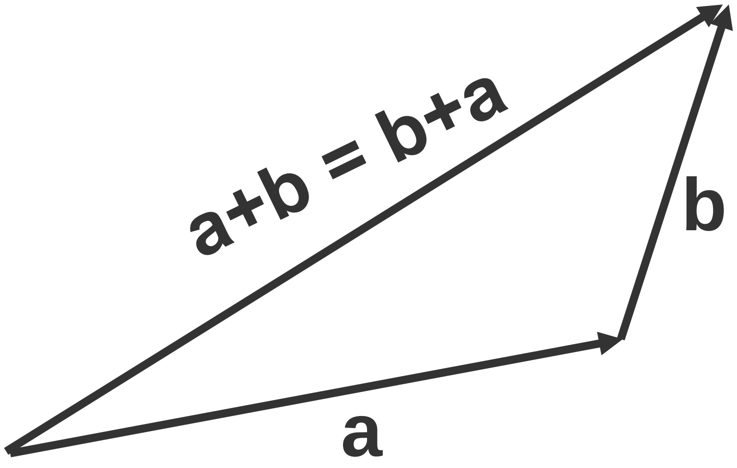
- Should be refresher on very basic material for most of you
  - Only basic math required

Vectors



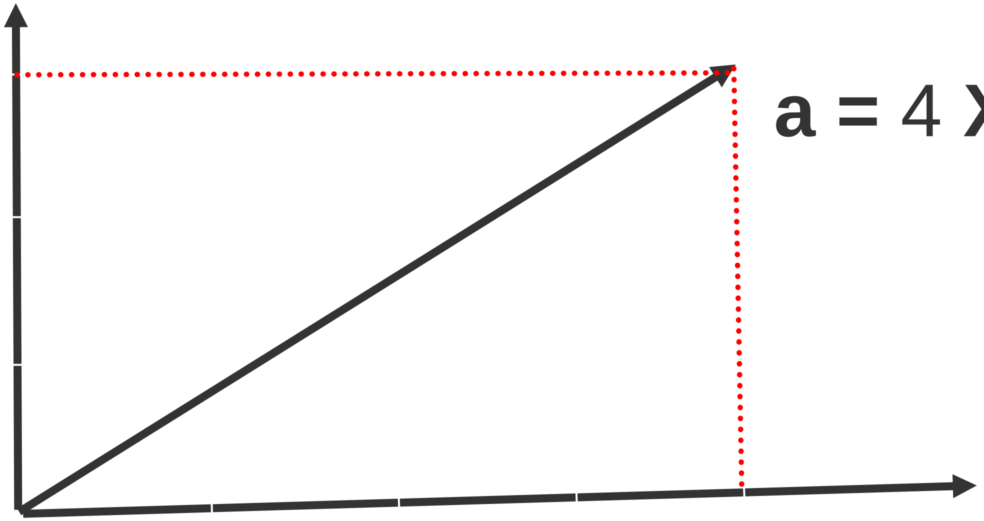
- Usually written as $\vec{a}$ or in bold. Magnitude written as $||\vec{a}||$
- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations

Vector Addition



- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

Cartesian Coordinates



$\mathbf{a} = 4\,\mathbf{X} + 3\,\mathbf{Y}$

- X and Y can be any (usually orthogonal ***unit***) vectors

$$\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \boldsymbol{a}^T = \begin{pmatrix} x & y \end{pmatrix} \qquad ||\boldsymbol{a}|| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$$

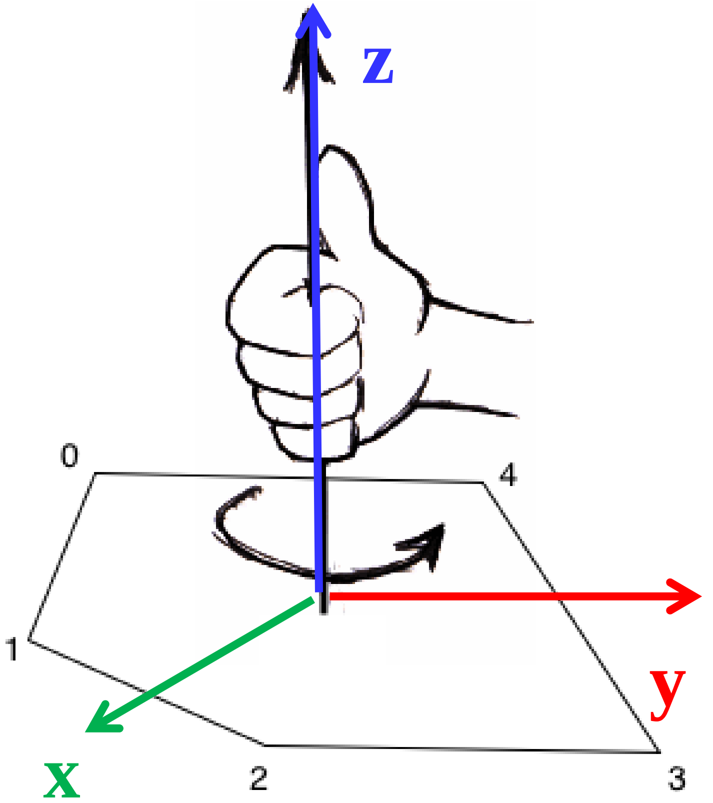
Vector Multiplication

- ***Dot product***
- Cross product
- Orthonormal bases and coordinate frames

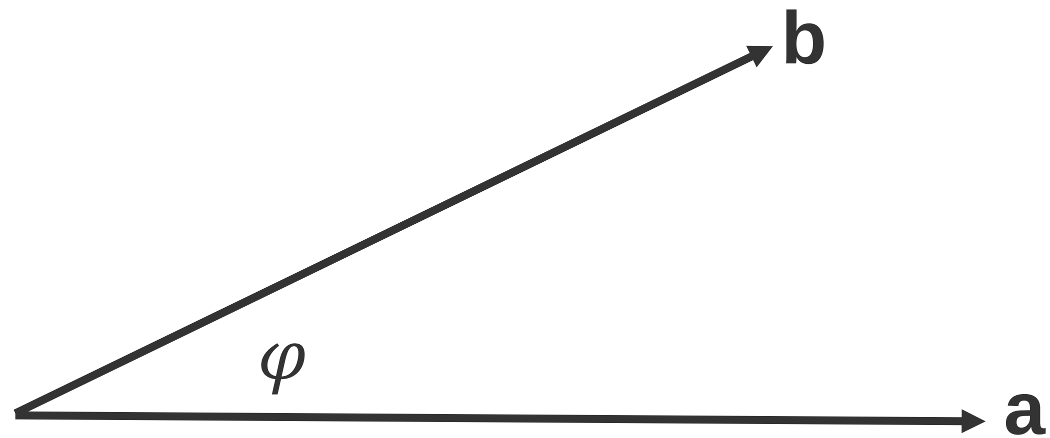
- Note: Some books talk about right and left-handed coordinate systems. We *always* use right-handed

Right hand role

- Four fingers extend in the order of the vertexes, in the example, it's 1->2->3->4.
- Thumb towards the front of the surface, in the example, it's upward.
  - The surface back: downward



Dot (scalar) product



$$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}=?$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$$

$$(k\boldsymbol{a})\cdot\boldsymbol{b}=\boldsymbol{a}\cdot(k\boldsymbol{b})=k(\boldsymbol{a}\cdot\boldsymbol{b})$$

$$\boldsymbol{a}\cdot\boldsymbol{b}=\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\varphi$$

$$\varphi=\cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right)$$

$$\begin{aligned} a \cdot b &= (x_a X + y_a Y) \cdot (x_b X + y_b Y) \\ &= x_a X \cdot x_b X + x_a X \cdot y_b Y \\ &\quad + y_a Y \cdot x_b X + y_a Y \cdot y_b Y \\ &= x_a x_b + y_a y_b \end{aligned}$$

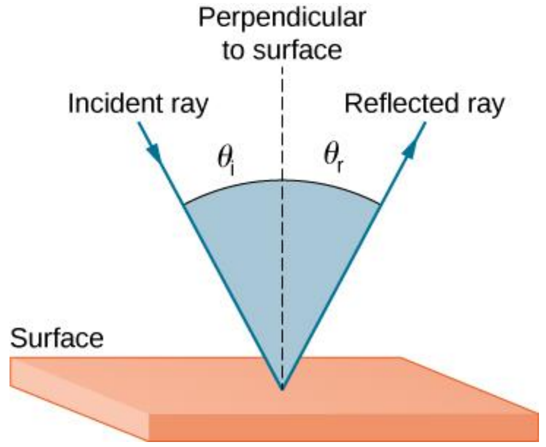
Dot product in Cartesian components

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ?$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

Dot product: some applications in CG

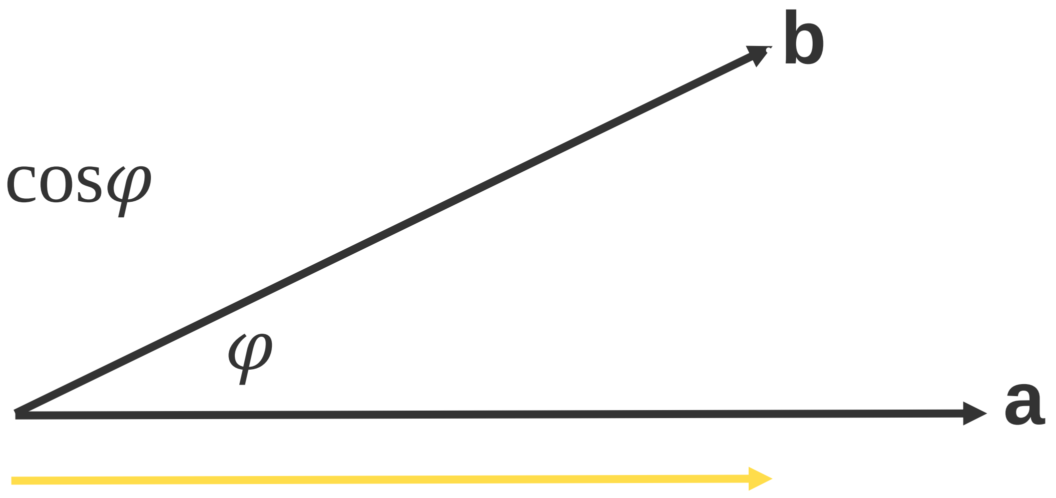
- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)



- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)

- Advantage: computed easily in cartesian components

Projections (of b on a)



$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\varphi$$

$$\|\boldsymbol{b} \rightarrow \boldsymbol{a}\| = ?$$

$$\|\boldsymbol{b} \rightarrow \boldsymbol{a}\| = \|\boldsymbol{b}\| \cos \boldsymbol{\varphi} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|}$$

$$\boldsymbol{b} \rightarrow \boldsymbol{a} = ?$$

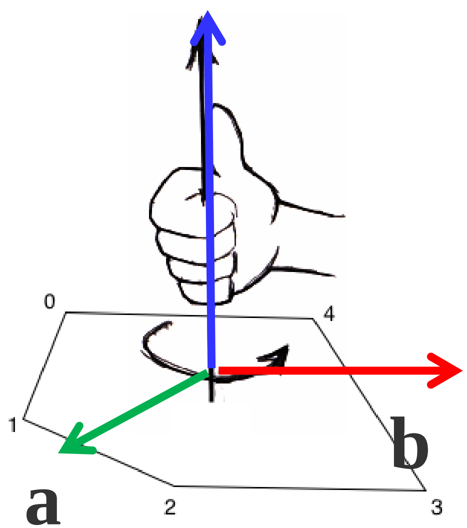
$$\boldsymbol{b} \rightarrow \boldsymbol{a} = \|\boldsymbol{b} \rightarrow \boldsymbol{a}\| \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

Vector Multiplication

- Dot product
- ***Cross product***
- Orthonormal bases and coordinate frames

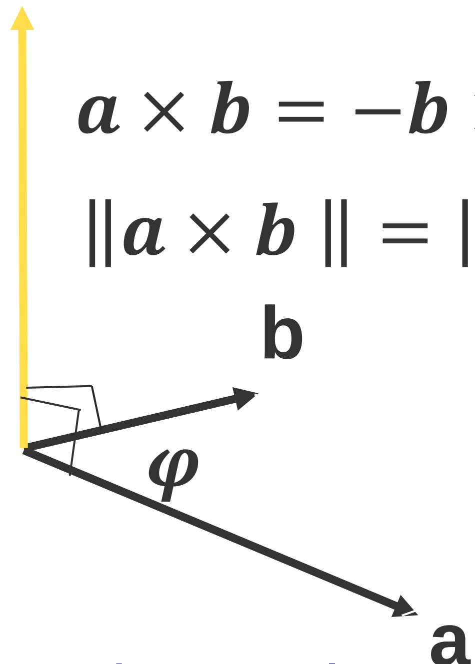
- Note: Some books talk about right and left-handed coordinate systems. We *always* use right-handed

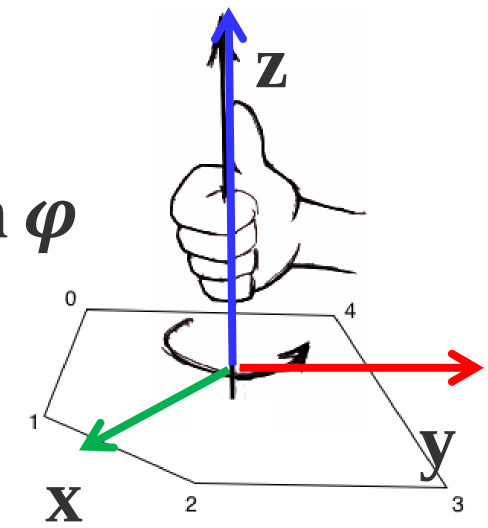
Cross (vector) product



$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin \boldsymbol{\varphi}$$





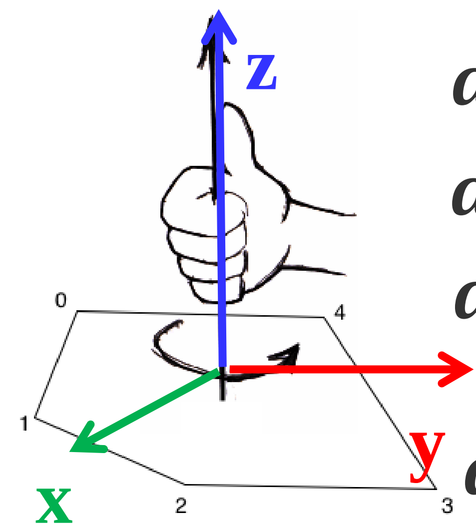
- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)
- x X y =? y X z =?

Cross product: Properties

$$\boldsymbol{x} \times \boldsymbol{y} = \boldsymbol{z}$$
$$\boldsymbol{y} \times \boldsymbol{x} = -\boldsymbol{z}$$

$$\boldsymbol{y} \times \boldsymbol{z} = \boldsymbol{x}$$
$$\boldsymbol{z} \times \boldsymbol{y} = -\boldsymbol{x}$$

$$\boldsymbol{z} \times \boldsymbol{x} = \boldsymbol{y}$$
$$\boldsymbol{x} \times \boldsymbol{x} = -\boldsymbol{y}$$



$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

$$\boldsymbol{a} \times (k\boldsymbol{b}) = k(\boldsymbol{a} \times \boldsymbol{b})$$

In class practise:

$$\boldsymbol{a} \times \boldsymbol{b} = (\boldsymbol{x_a X} + \boldsymbol{y_a Y} + \boldsymbol{z_a Z}) \times (\boldsymbol{x_b X} + \boldsymbol{y_b Y} + \boldsymbol{z_b Z})$$

Cross product: Cartesian formula?

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} x & y & z \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{pmatrix} \boldsymbol{y_a z_b - y_b z_a} \\ \boldsymbol{z_a x_b - z_b x_a} \\ \boldsymbol{x_a y_b - y_b x_a} \end{pmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{A}^* b = \begin{pmatrix} 0 & -\boldsymbol{z_a} & \boldsymbol{y_a} \\ \boldsymbol{z_a} & 0 & -\boldsymbol{x_a} \\ -\boldsymbol{y_a} & \boldsymbol{x_a} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x_b} \\ \boldsymbol{y_b} \\ \boldsymbol{z_b} \end{pmatrix}$$

Dual matrix of vector a

Vector Multiplication

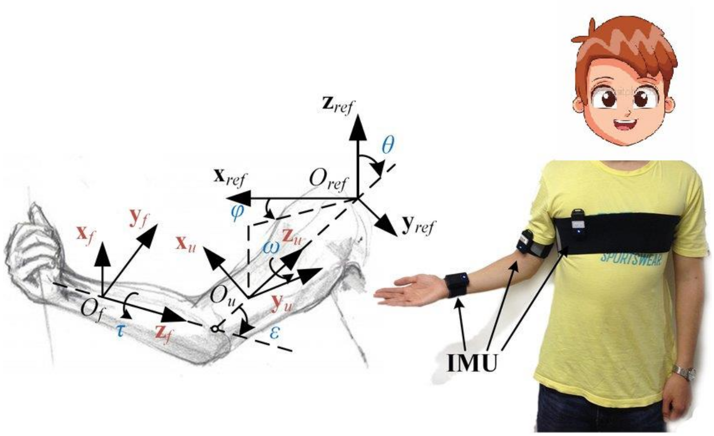
- Dot product
- Cross product
- ***Orthonormal bases and coordinate frames***

- Note: book talks about right and left-handed coordinate systems. We *always* use right-handed

Orthonormal bases/coordinate frames

- Important for representing points, positions, locations

- Often, many sets of coordinate systems (not just X, Y, Z)
  - Global, local, world, model, parts of model (head, hands, …)



- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Coordinate Frames

- Any set of 3 vectors (in 3D) so that

$$\|u\| = \|v\| = \|w\| = 1$$

$$u \cdot v = v \cdot w = w \cdot u = 0$$

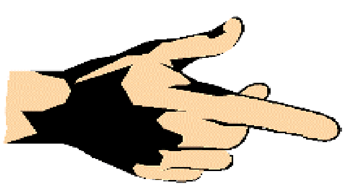
$$w = u \times v$$

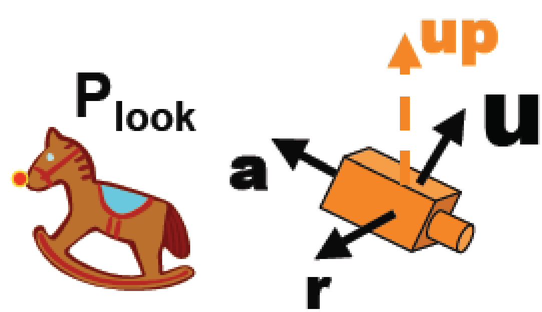
$$p = (p \cdot u)u + (p \cdot v)v + (p \cdot w)w$$

Constructing a coordinate frame

- Often, given a vector **a** (viewing direction) want to construct an orthonormal basis
- Need a second vector **up** (up direction of camera)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into)

$$\mathbf{a} = P_{eye} - P_{look}$$





$$\mathbf{r} = \mathbf{up} \times \mathbf{a}$$

$$\mathbf{u} = \mathbf{a} \times \mathbf{r}$$

Constructing a coordinate frame?

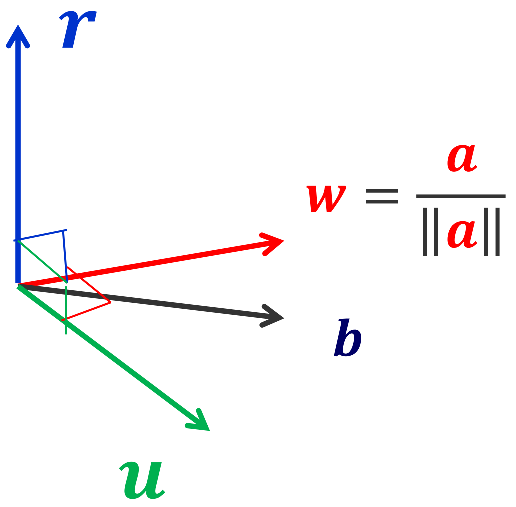
We want to associate **w** with **a**, and **b** with **up**

- But **a** and **up** are neither orthogonal nor unit norm
- And we also need to find **r, u.**
- **w, r, u forms a coordinate frame.**

1. $\boldsymbol{w} = \dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$

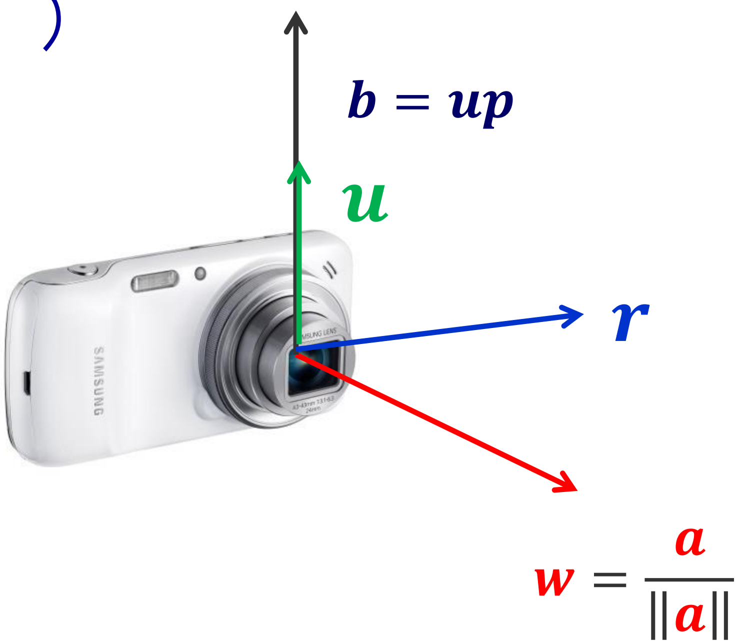
2. $\boldsymbol{r} = \dfrac{\boldsymbol{b} \times \boldsymbol{w}}{\|\boldsymbol{b} \times \boldsymbol{w}\|}$

3. $\boldsymbol{u} = \dfrac{\boldsymbol{a} \times \boldsymbol{r}}{\|\boldsymbol{a} \times \boldsymbol{r}\|}$

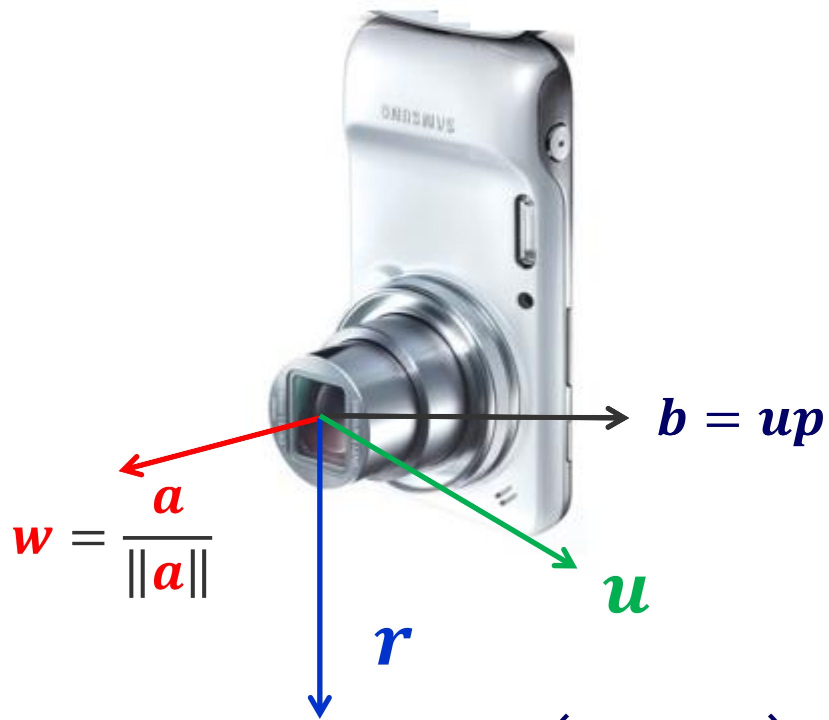


Constructing a camera coordinate frame

- up can be chosen from (0, 1, 0), (0, -1, 0), (1, 0, 0), (-1, 0, 0), (0, 0, -1), (0, 0, 1)



up=(0, 1, 0)



up=(1, 0, 0)

The camera coordinate frame is attached to the camera.

Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale
- Array of numbers ($m \times n$ = m rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition, multiplication by a scalar simple: element by element

Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 9 & 27 & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & 12 \end{pmatrix}$$

- Element (i,j) in product is dot product of row i of first matrix and column j of second matrix

Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

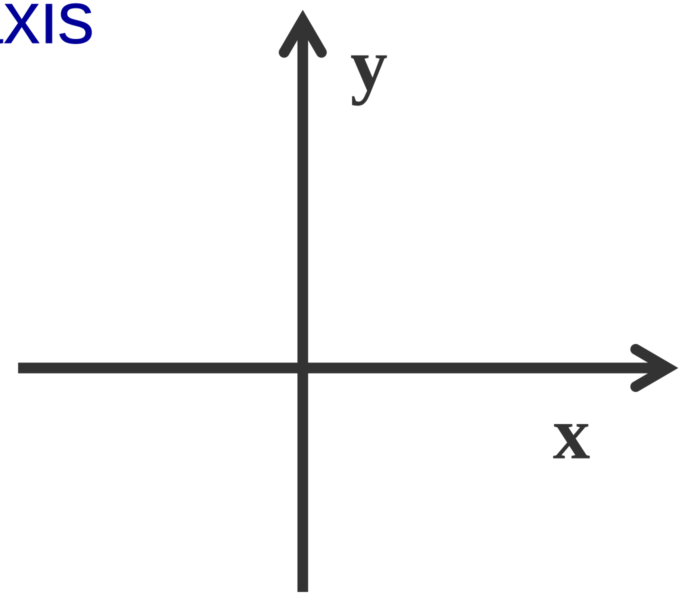
- Not even legal!
- Non-commutative (AB and BA are different in general)
- Associative and distributive
  - A(B+C) = AB + AC
  - (A+B)C = AC + BC

Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix ($m \times 1$)

- E.g. 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$



Transpose of a Matrix (or vector?)

$$\begin{pmatrix} 1 & 5 & 0 \\ 3 & 2 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- **$(AB)^T = B^T A^T$**

Identity Matrix and Inverses

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = A^{-1}A = I$

Vector multiplication in Matrix form

- Dot product?

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_c \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

- Cross product?

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{A}^* b = \begin{pmatrix} 0 & -\boldsymbol{z_a} & \boldsymbol{y_a} \\ \boldsymbol{z_a} & 0 & -\boldsymbol{x_a} \\ -\boldsymbol{y_a} & \boldsymbol{x_a} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x_b} \\ \boldsymbol{y_b} \\ \boldsymbol{z_b} \end{pmatrix}$$

Dual matrix of vector a

Writing assign 1

1. Prove: $$a \cdot b = \|a\|\|b\|\cos\varphi$$

2. Check: $$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} x & y & z \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{pmatrix} \boldsymbol{y_a z_b - y_b z_a} \\ \boldsymbol{z_a x_b - z_b x_a} \\ \boldsymbol{x_a y_b - y_b x_a} \end{pmatrix}$$

3. Please draw a, up, r, u vector on the below pictures, and specify which up you choose.

a)

b)

c)

d)